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An extension to the planar Markus–Yamabe Jacobian conjecture

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Abstract. We extend the planar Markus–Yamabe Jacobian conjecture to differential systems having Jacobian matrix with eigenvalues with negative or zero real parts.

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1 Introduction

Let

$$\dot{X} = F(X), \qquad X \in \mathbb{R}^n, \qquad F \in C^1(\mathbb{R}^n, \mathbb{R}^n) \tag{1.1}$$

be a first order differential system. Let us denote by $J_F(X)$ the Jacobian matrix of $F(X)$. If O is a critical point of (1.1) and the eigenvalues of $J_F(O)$ have negative real parts, then O is asymptotically stable [2]. In particular, all orbits starting close enough to O tend asymptotically to O.

In [7] the question was raised, whether $J_F(X)$ having eigenvalues with negative real parts for every $X \in \mathbb{R}^n$ imply O to be globally asymptotically stable, i. e. whether all orbits in $\mathbb{R}^n$ tend asymptotically to O. Such a problem was named *Markus–Yamabe Jacobian conjecture* and several results were obtained under various additional hypotheses. A key step was made in [8], where it was proved that under Markus–Yamabe hypotheses, for planar systems the global asymptotic stability of O is equivalent to the injectivity of $F(X)$. Such a result led to study the problem applying methods previously used to study injectivity. The Markus–Yamabe Jacobian conjecture was solved in the positive in [4–6] for planar systems, and was proved to have negative answer in higher dimensions [1,3]. The three approaches proposed in in [4–6] first prove the injectivity of $F(X)$, then as a consequence get the global asymptotic stability. Actually, in all such papers injectivity is proved under much weaker hypotheses than that of negative real parts. In fact, it is sufficient to assume that the Jacobian matrix has nowhere real positive eigenvalues.

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Such general results did not lead to similarly general results in the study of the systems dynamics. This is likely due to the fact that accepting the possibility of eigenvalues with different real parts (positive, zero or negative) at different points of the plane does not allow to apply the procedure developed in [8] to establish the equivalence of injectiviy and global asymptotic stability. On the other hand, eigenvalues with zero real parts are compatible with asymptotic stability, even if not sufficient to imply it.

In this paper we assume $J_F(X)$ to be non-singular and have eigenvalues with non-positive real parts for all $X \in \mathbb{R}^2$. Differently from the classical case, in this case a system does not necessarily have a globally asymptotically stable critical point. If a critical point exists, we prove that either such a system has a global center, or there exists a globally asymptotically stable compact set. We show by an example that such a global attractor is not necessarily a critical point. If the system is analytic the conclusion can be sharpened, proving that either there exists a global center, or a globally asymptotically stable critical point. Our results follow from Olech approach to global attractivity [8] and Fessler theorem about global injectivity [4].

2 Results

We consider maps $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, $F(x,y) = (P(x,y), Q(x,y))$. We denote partial derivatives by subscripts. Let

$$J_F(x,y) = \begin{pmatrix} P_x(x,y) & P_y(x,y) \\ Q_x(x,y) & Q_y(x,y) \end{pmatrix}.$$

be the Jacobian matrix of F at (x,y). We denote by $D(x,y) = \det J_F(x,y) = P_x(x,y)Q_y(x,y) - P_y(x,y)Q_x(x,y)$ its determinant and by $T(x,y) = P_x(x,y) + Q_y(x,y)$ its trace. $T(x,y)$ is the divergence of the vector field $F(x,y)$.

In what follows we consider the differential system associated to F:

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y). \end{cases} \tag{2.1}$$

We denote by $\phi(t,x,y)$ the local flow defined by (2.1). We say that a critical point O of (2.1) is a *center* if it has a punctured neighbourhood filled with non-trivial cycles surrounding O. The largest connected set N_O filled with such cycles is called *period annulus of* O. If $N_O = \mathbb{R}^2 \setminus \{O\}$, then O is said to be a *global center*. We say that a critical point O of (2.1) is *asymptotically stable* if it is stable and attractive [2]. In this case we denote by A_O its *attraction region*. If $A_O = \mathbb{R}^2$ then O is said to be *globally asymptotically stable*.

In the proof of Theorem 2.2 we repeatedly use F injectivity. We report here the theorem applied, proved in [4].

Theorem 2.1. *Let $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ be such that:*

1) *$D(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$;*

2) *there is a compact set $K \subset \mathbb{R}^2$ such that $J_F(x,y)$ has no real positive eigenvalues for any $(x,y) \notin K$.*

Then F is injective.

For the sake of simplicity, without loss of generality from now on we assume $O = (0,0)$. The hypotheses we consider rely only on derivatives properties, hence they do not change after a translation. We set

$$T_- = \{(x,y) : T(x,y) < 0\},$$

and denote by $\overline{T_-}$ its closure. We denote by μ the 2-dimensional Lebesgue measure.

Theorem 2.2. *Assume $D(x,y) > 0$ and $T(x,y) \leq 0$ for all $(x,y) \in \mathbb{R}^2$. Let O be a critical point of (2.1). Then:*

i) O is a center if and only if it has a neighbourhood U_O such that $T(x,y)$ vanishes identically on U_O; in such a case (2.1) is Hamiltonian on all of N_O; if, additionally, F is analytic, then the system is Hamiltonian and O is a global center.

ii) O is asymptotically stable if and only if it belongs to $\overline{T_-}$; in such a case O is globally asymptotically stable.

iii) If $T(x,y)$ does not vanish identically, then there exists a globally asymptotically stable compact set M.

Proof. *i*.1) We claim that if O is a center, then $T(x,y)$ vanishes identically on N_O. By absurd, assume $T(x^*,y^*) < 0$ for some $(x^*,y^*) \in N_O$. By continuity there exists a neighbourhood U^* of (x^*,y^*) such that $T(x,y) < 0$ for all $(x,y) \in U^*$. Let γ^* be the cycle passing through (x^*,y^*) and Δ^* the bounded planar region having γ^* as boundary. Δ^* is invariant, hence $\mu(\Delta^*) = \mu(\phi(t,\Delta^*))$ for all $t \in \mathbb{R}$. By Liouville's theorem one has

$$0 = \frac{d}{dt}\mu(\phi(t,\Delta^*)) = \int_{\phi(t,\Delta^*)} T(x,y)\,dx\,dy < 0,$$

because $T(x,y) < 0$ on $\phi\,(t,\Delta^* \cap U^*)$, contradiction.

i.2) Vice-versa, assume $T(x,y)$ to vanish identically on a neighbourhood U_O of O. Then the system is Hamiltonian on a simply connected neighbourhood $V_O \subset U_O$. Let $H(x,y)$ be its Hamiltonian function. One has

The Hessian matrix of $H(x,y)$ is

$$J_F(x,y) = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix} = \begin{pmatrix} Q_x & Q_y \\ -P_x & -P_y \end{pmatrix}. \tag{2.2}$$

The Hessian determinant is $H_{xx}H_{yy} - H_{xy}H_{yx} = P_xQ_y - P_yQ_x = D(x,y) > 0$, hence $H(x,y)$ has a minimum at O. As a consequence, O is a center.

i.3) If additionally F is analytic, then also $T(x,y)$ is analytic. If it vanishes in a neighbourhood of O then it vanishes on all of $\mathbb{R}^2$, hence the system is Hamiltonian on all of $\mathbb{R}^2$. We claim that N_O is unbounded. In fact, let us assume by absurd N_O is bounded, hence also ∂N_O is bounded. By F injectivity [4], ∂N_O contains no critical points, hence by Poincaré–Bendixson theorem ∂N_O is a non-trivial cycle. One can consider the Poincaré map defined on a section Σ of ∂N_O. Such a map is analytic and coincides with the identity map on $\Sigma \cap N_O$, hence it coincides with the identity map on all of Σ. As a consequence every orbit meeting $\Sigma \cap \partial N_O$ is a cycle, hence ∂N_O is contained in the period annulus, contradicting the fact that it is the boundary of N_O. Moreover, every connected components of ∂N_O is unbounded. In fact, if a connected components of ∂N_O was bounded, then by its invariance and by Poincaré–Bendixson theorem either it would be a cycle or it would contain a critical point. The former case has already been considered above, the latter one can be excluded by the injectivity of F.

i.4) In order to prove that O is a global center we use again the injectivity of F. For $\varepsilon > 0$ let B_ε be the open disk of radius $\varepsilon > 0$ centered at O. F is a diffeomorphism, hence the anti-image $D_\varepsilon = F^{-1}(B_\varepsilon)$ is an open neighbourhood of O. By construction and by the injectivity of F, D_ε contains all the points of $(x,y) \in \mathbb{R}^2$ such that $|F(x,y)| < \varepsilon$, hence for all $(x,y) \notin D_\varepsilon$ one has $|F(x,y)| \geq \varepsilon$. Let us choose ε small enough such that $\partial N_O \cap D_\varepsilon = \emptyset$. Let ∂N_O^u be an unbounded component of ∂N_O. Then working as in [8], since $T(x,y) \leq 0$ and $|F(x,y)| \geq \varepsilon$ outside D_ε, one proves that every orbit starting close enough to ∂N_O^u is unbounded too, hence it is not a cycle, contradicting the fact that ∂N_O^u is in the boundary of N_O. As a consequence $\partial N_O = \emptyset$ and $N_O = \mathbb{R}^2 \setminus \{O\}$.

ii) Assume O to be asymptotically stable and A_O its region of attraction. By hypothesis, in every neighbourhood of O there are points such that $T(x,y) < 0$, and by continuity this occurs in an open subset of A_O. If by absurd A_O is bounded, then by its invariance, for all t

$$0 = \frac{d}{dt}\,\mu(\phi(t, A_O)) = \int_{\phi(t,A_O)} T(x,y)\,dx\,dy < 0,$$

contradiction. Hence A_O is unbounded. Assume by absurd there exists a bounded connected component ∂A_O^b of ∂A_O. As above, by Poincaré–Bendixson theorem either it is a cycle or contains a critical point. If it is a cycle, it cannot surround O, since in such a case A_O would be bounded. Hence it surrounds another critical point, violating F injectivity. The same violation would occur if ∂A_O^b contained a critical point. Then the argument proceeds as in point *i*.4), showing that $\partial A_O = \emptyset$ and $A_O = \mathbb{R}^2$.

Vice-versa, assume $O \in \overline{T_-}$. Then $T(x,y)$ does not vanish identically on any neighbourhood U_O of O, hence by point *i*) it is not a center. By the hypotheses on $D(0,0)$ and $T(0,0)$, O is a non degenerate elementary critical point of center-focus type, according to the real part of its eigenvalues. If such real parts are negative O is a focus, hence asymptotically stable. If such real parts are zero, one proves, as at the beginning of point *i*), that O cannot be accumulation point of cycles, hence it is asymptotically stable. Working as in point *i*.4) one proves that it is globally asymptotically stable.

iii) If $O \in \overline{T_-}$, then point *ii*) applies and one can take $M = \{O\}$.

If $O \notin \overline{T_-}$, it has a neighbourhood U_O where $T(x,y)$ vanishes identically, hence it is a center. We claim that N_O is bounded. In fact, if N_O is unbounded one can proceed as in point *i*.4), in order to prove that every orbit starting close enough to ∂N_O^u is unbounded, contradicting the fact that ∂N_O^u is part of the boundary. The boundedness of N_O implies the boundedness of ∂N_O, which is a cycle, by the absence of critical points on ∂N_O^u. Let us consider a section Σ of ∂N_O and its Poincaré map. Such a map is the identity on $\Sigma \cap N_O$, and has no fixed points on $\Sigma \setminus N_O$, otherwise there would be a cycle γ containing ∂N_O, $T(x,y)$ would vanish identically inside γ and every orbit inside γ would be a cycle, contradicting the fact that ∂N_O is the boundary of N_O. Hence the Poincaré map is strictly monotone, which implies either attractivity or repulsivity of ∂N_O. Repulsivity is not compatible with the sign of the divergence, hence ∂N_O is attractive, and $\overline{N_O}$ is asymptotically stable. Its global attractivity can be proved as in *i*.4) and *ii*), proving that the boundary of its region of attraction is empty. □

An example of globally asymptotically stable critical point belonging to $\overline{T_-}$ is the origin in the following differential system,

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x - y^3, \end{cases} \tag{2.3}$$

for which one has

$$J_F(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & -3y^2 \end{pmatrix}.$$

One has $D(x,y) = 1$, $T(x,y) = -3y^2 \leq 0$, hence T_- is x-axis.

If (2.1) is not analytic, then a center need not be global. We construct a system satisfying the hypotheses of Theorem 2.2, having a non-global center and a globally asymptotically stable compact set. Let $\alpha \in C^\infty(\mathbb{R}, \mathbb{R})$ be such that

$$\begin{cases} \alpha(r) = 0, & r \leq 1, \\ \alpha(r) > 0, & r > 1, \\ \alpha'(r) > 0, & r > 1. \end{cases}$$

Let us set $r = \sqrt{x^2 + y^2}$. The vector field defined by the system

$$\begin{cases} \dot{x} = y - x\,\alpha(r), \\ \dot{y} = -x - y\,\alpha(r). \end{cases} \tag{2.4}$$

Setting $cr = x, sr = y$, the Jacobian matrix of the vector field is

$$J_F(x,y) = \begin{pmatrix} -\alpha(r) - xc\alpha'(r) & 1 - xs\alpha'(r) \\ -1 - yc\alpha'(r) & -\alpha(r) - ys\alpha'(r) \end{pmatrix}.$$

Its determinant is $1 + \alpha^2(r) + r\alpha(r)\alpha'(r) > 0$ and its trace is $-2\alpha(r) - 2r\alpha'(r) \leq 0$. For $r \leq 1$ the trace is zero, for $r > 1$ the trace is negative. The system (2.4) is Hamiltonian for $r \leq 1$, with a center at O whose central region is the disk of radius 1 centered at O. Such a disk is a global attractor, since $\dot{r} < 0$ for $r > 1$.

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