Sangakuin Multiple Geometries: Examining Japanese Temple Geometry Beyond Euclid

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Sangaku in Multiple Geometries:
Examining Japanese Temple Geometries Beyond Euclid

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for the Murray State University Honors Diploma

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Abstract

When the country of Japan was closed from the rest of the world from 1603 until 1867 during the Edo period, the field of mathematics developed in a different way from how it developed in the rest of the world. One way we see this development is through the *sangaku*, the thousands of geometric problems hung in various Shinto and Buddhist temples throughout the country. Written on wooden tablets by people from numerous walks of life, all these problems hold true within Euclidean geometry. During the 1800s, while Japan was still closed, non-Euclidean geometries began to develop across the globe, so the isolated nation was entirely unaware of these new systems. Thus, we will explore the *sangaku* in two of the other well-known systems, namely the neutral and hyperbolic geometric systems. Specifically, we will highlight how these traditionally-solved problems change under the varying definitions of line parallelism.

Chapter 1

Introduction

What influences the way in which we view mathematics? Are there any social constructs that indicate our understanding? That is, is there a reason why our modern interpretation of mathematics is flooded with white, European men and what they discovered? Because of this, we tend to ignore results found from anywhere else from the world and anyone other than white men. What happens when we erase this presupposition – are there new findings? Do we perhaps even find a different understanding from mathematics, aside from just a subject that seems unreasonably focused in school?

One explanation as to what influences of interpretation of math can be found when we examine the amount of cross-cultural encounters a society has. The amount of cross-cultural encounters a society has is correlated to the perspective of math. We can see this when examining the 1700 and 1800s and the development of mathematics at this time. The European mathematics was influenced heavily by relatively recent math found by the Islamic Golden Age, such as the concept of zero. algebra, or different astrological findings. However, there was one country isolated from the world that had a Chinese foundation of math, Japan. From the early 1600s up until the mid 1800s, during what is now called the Edo period, Japan was isolated from the world, as Fukagawa and Rothman mention in *Sacred Mathematics* [4]. Because of this isolation, Japanese culture was allowed to grow and expand in truly unique ways, ways which are visible through the art produced during this period. One such type of artistic expression is the *sangaku* (also written *san gaku*, which literally translates as 'calculation tablets'), various tablets found at Shinto shrines and Buddhist temples. Combining art and geometry, these tablets offer us a direction to answering the aforementioned questions.

Japan and Europe developed their mathematics in vastly different ways. Japan took a more artistic approach to mathematics, seen in the development of *wasan* – traditional Japanese mathematics – and the *sangaku* tablets themselves. However, Europe took a limited artistic approach to mathematics during the same time. From the Father of Calculus to Euler, math was used to explain natural phenomena, such as gravity or analyzing numbers. Furthermore, Europe started to examine Euclid's Postulates in new ways during the 1800s, developing different parallel line cases and non-Euclidean geometries [13]. What if both *wasan* and European math grew together, allowing more people to not only learn more about math, but also to learn it in different ways? We take a look into this hypothetical by combining *sangaku* and non-Euclidean geometries.

We will first discuss the intricacies of *wasan* in Chapter 2, in order to lay the groundwork for the context of the tablet problems. In Chapter 3, we will take a look at the non-Euclidean geometric developments that occurred outside Japan during the Edo period. Lastly, in Chapter 4 we will combine the contents of our previous work to solve different *sangaku* problems in non-Euclidean geometries, with a specific focus on the Hyperbolic Poincaré Disk model.

Chapter 2

What is *Sangaku*?

“There is good when obeying the soul of mathematics and suffering when not.” This quote from Takebe Katahiro’s book *Fukyuu tetsujustu* (*Inductive Mathematics*) as stated in [10] summarizes the Edo perspective on math. During the Edo period (1603-1868 CE), math was recreational, synonymous to our modern ideas of music, art, sports, puzzles, board games, and even some video games -– depending on what type is being played, of course. Because of this, many Japanese people were mathematically literate, being able to use the abacus and to multiply numbers. The origin of Edo Japanese math, called *wasan*, originated from Yoshida Mitsuyoshi’s book, *Jinkō-ki*, written in 1627 near the start of the Edo period. This book, which we will discuss the content shortly, was so monumental and influential for the nation that almost every single family owned a copy of it [4]. As mathematics during the Edo culture was seen as a recreational activity, those living in Japan simply used it to pass the time, gaining some mathematical knowledge in the process. As with many other things, “just because you do not need something in life, it does not mean there is no point in doing it,” [10] such is the philosophy of studying mathematics during the Edo period in Japan. For more on the history of Edo Japan, please watch “Japan: Memoirs of a Secret Empire” [3] to gain an understanding of society during the era.

2.1 Pre-Edo Math

In order to further understand the importance of the *Jinkō-ki*, we need to first understand the build up to the *Jinkō-ki*, the start of traditional Japanese mathematics. In the 700s CE during the Nara period, China and Japan had numerous interactions with each other, which led to many Chinese ideologies travelling over to Japan. Chinese math books were the most influential items in the development of the Japanese understanding of math. Among all the early Chinese books, the *Zhou bi suan jing*, (*The Arithmetical Classic of the Gnomon and the Circular Path of Heaven*), which dealt heavily in astronomy and dating the calendar, was the earliest written — completed around first century BCE, as Swetz and Katz share [11]. The importance of the *Zhou bi* was its inclusion of fractions, the discussion of multiplication and division, and even the implicit use of square roots [4]. Another valuable Chinese book for Japanese math was the *Jiu zhang Suanshu* (*Nine Chapter on the Mathematical Art*), which contained well over 200 problems relating to mathematical fields such as surveying, taxation, and engineering. The methodology of the *Jiu zhang Suanshu* also implemented fractions, in conjunction with geometry, arithmetic progression, and the Pythagorean Theorem. Naturally, as these books entered Japan, they were used, but nothing was expanded until Yoshida wrote the *Jinkō-ki*.

One of the more impressive developments that came to fruition from the Chinese works was the creation of the *San Hakase*, the Department of Arithmetic Intelligence, established during the Nara period. As the name suggests, this agency dealt with many arithmetic happenstances that occurred in normal life for the Japanese people; they helped farmers, especially with measuring fields, and even helped levy taxes. The Department consisted of directors, subdirectors, officers, clerks, and assistants, with the a grand total of 1,330 people involved. Separated into seven groups, the *San Hakase* toured villages to ensure taxes were paid. With each position came a different allocation of sake, rice, and salt, but despite the arithmetic chore it was to perform this caluation for each individual, each person received the correct amounts. This was the extent of Japanese math up until the 1600s with the rise of the Edo period.

2.2 Edo Japan and the *Jinkō-ki*

Traditional Japanese math began its development as the Edo period arrived. Hideyoshi Toyotomi, who was allied originally with the first shogun of the era Tokugawa Ieyasu, decided to invade Korea after unifying Japan in 1591. As Japanese math had remained stagnant up until this point, invading Korea gave Japan a new mathematic instrument, the abacus, known as the *soroban* in Japanese. This instrument's popularization was aided by Mōri Shigeyoshi, the first identifiably Japanese mathematician, who owned the Chinese book *Suanfa Tong Zong* (*Systematic Treatise on Arithmetic*). He soon thereafter wrote *Warizansyo* (*Division Using Soroban*), finishing the book by 1622. Although he had great knowledge of the Chinese book, Mōri was not the only mathematician to study the *Suanfa Tong Zong*; throughout the 1620s, Yoshida Mitsuyoshi closely studied the book, eventually modifying the problems and adding illustrations. This book, the *Jinkō-ki* (*Large and Small Numbers*) came to be in 1627, the first Japanese-published complete mathematics book. This publication marks the point where traditional Japanese mathematics takes off, developing into *wasan*. Without Mōri and Yoshida, it would not have developed in the way that it did, as Yoshida quickly became a student of Mōri's and just as rapidly learned of everything his teacher could inform. *Wasan*, thanks to the *sakoku* decree, was allowed to develop in truly unique ways, which helped to differentiate from the European mindset.

As a book, Yoshida's *Jinkō-ki* not only represents *wasan*, but it also is a glimpse into the Edo era: the second edition was printed in four colors and shows the artistic expression found from the period. This chosen coloration aided immensely in its rapid spread throughout the country, since the book was visually appealing. What exactly made *Jinkō-ki* so monumental? As most introductory math books nowadays do, the *Jinkō-ki* begins by introducing the number system, increasing in ones, tens, hundreds, thousands, even denominations up to 10^{16}. One caveat, though, was that the term changes on each increased digit location. For example, in the Edo book, *oku* represented 100,000, *chō* denoted 1,000,000, etc. In comparison to our modern system, which refers to each group of numbers as, for example, thousands, ten thousands, and hundred thousands, the numbers used in the *Jinkō-ki* were not set up as *jū sen* (ten thousand) but instead understood as its own entitity, *man*. In traditional Japanese mathematics, we would not under 10,540 as ten thousand five hundred forty, but instead we would view it as one *man* five *hyaku* four *jū*. Similarly, 4,231 is four *sen* two *hyaku* three *jū* one. Edo Japanese people did not view a hundred thousand as a hundred thousands, but instead as a singular hundred thousand. The same was true for decimals, where each digit placement was assigned a different name.

Following the number system, Yoshida wrote about different ways of measuring: length, weight, area, volume, and even special rulers, for both cloth and carpentry, as well as identifying important weights of gold and silver. All of these measurements would have been useful for the average farmer, worker, or merchant. *Jinkō-ki* even discusses multiplication and division, the latter of which was extremely useful when working alongside an abacus. The book concludes with applications to the real world and challenge problems to enhance one's mathematical abilities. *Idai* are challenge problems where the author gave no answer to entice people to solve for themselves, but they were a common feature of arithmetic books. As we will discover shortly, *sangaku* emulated this, producing a method called *idai keishō*. *Idai keishō* is the main idea from where *sangaku* comes. Someone would create an "unsolved" problem for someone else to prove, but the author would provide the steps or the formula used in order to provide a small amount of guidance.

Even within the *Jinkō-ki*, *idai keishō* aided people with arithmetic, but algebra was a different story entirely. Algebra problems were only solvable using the *sangi*, still in use despite its initial Japanese use in the 700s, and Chinese methods called *tianyuanshu*, or "heavenly element." Enter Sawaguchi Kazuyuki's *Kokon sanpōki* (*Old and New Mathematics*), written in 1671. This book also engaged in *idai keishō*, presenting fifteen multivariate problems at then end. The answers were not known until 1674, when Seki Takakazu wrote *Hatsubi sanpō* (*Detailed Mathematics*), and suddenly the answers for all fifteen problems were given. Seki used a special notation called *bōshohō* (point counting method), where he used alphanumeric symbols to solve multivariate problems, therefore creating a long hand calculation, *hissan*. See Figure 2.2.1 for a visualization of *bōshohō* using *wasan*. Both the implementation of *hissan* and *bōshohō* gave Japan the ability to move past Chinese math and truly develop and advanced *wasan*. Seki was such a talented mathematician that he possibly discovered the Bernoulli numbers before Jacob Bernoulli did, as propositioned in *Wasan* [10]. Because of his mathematical prowess, Seki, as many other Japanese mathematicians of the time did, had numerous protégés in school, many of whom started their own math schools containing various levels, from an elementary school equivalent up to even college level math. One notable student is Katahiro Takebe, who plainly described the view *wasan* has on math in *Fukyū testujutsu*; "[t]here is good when obeying the soul of mathematics and suffering when not." [10]

2.3 Development and Globalization of Sangaku

Sangaku are visualizations of *wasan* and *idai keishō*, where people present findings for the public to enjoy the math presented, to emulate the result, and to encourage others to develop new *sangaku*. They were presented on an average of 153 cm by 69.4 cm wood tablets and were typically hung in Buddhist temples and Shinto shrines. This

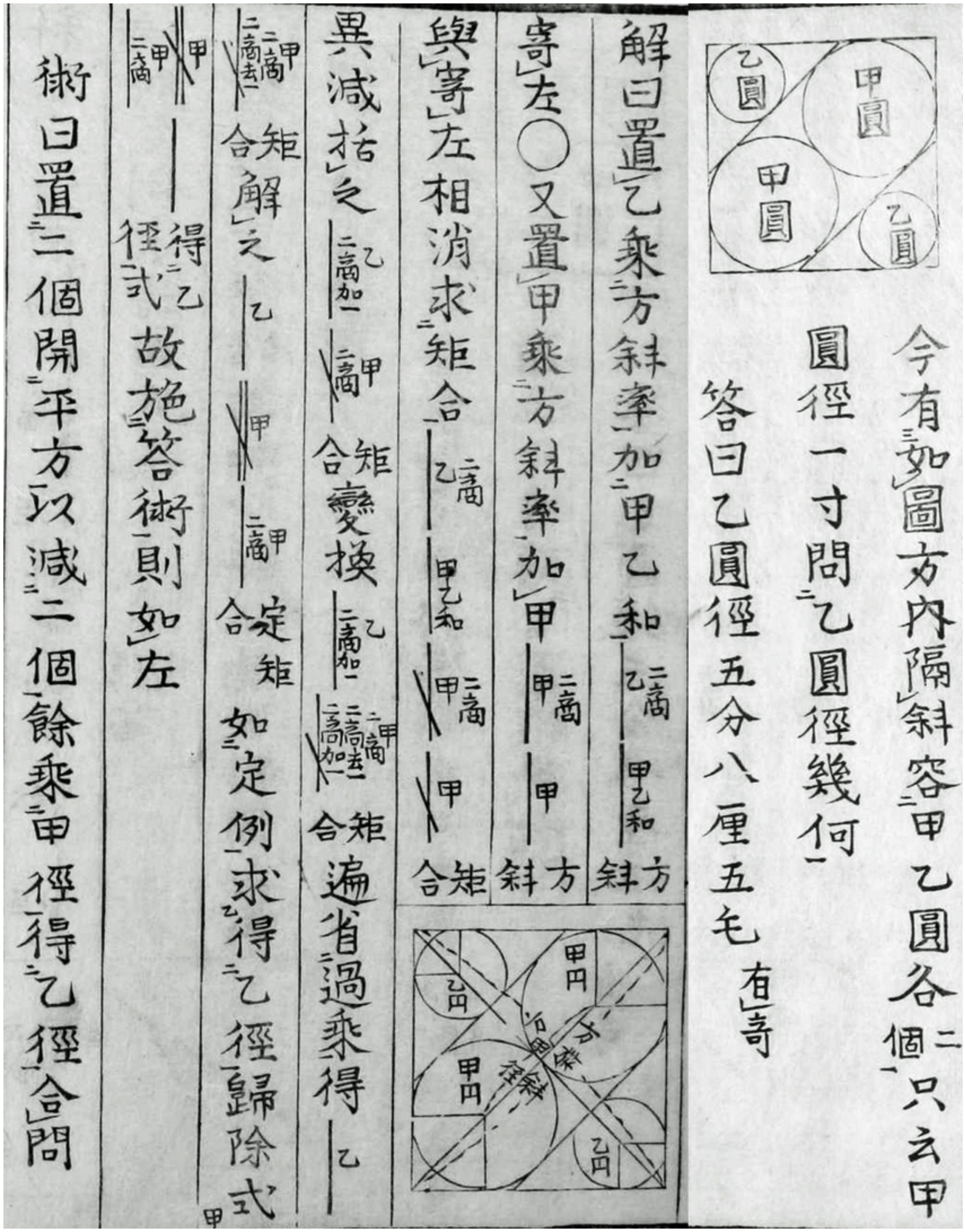


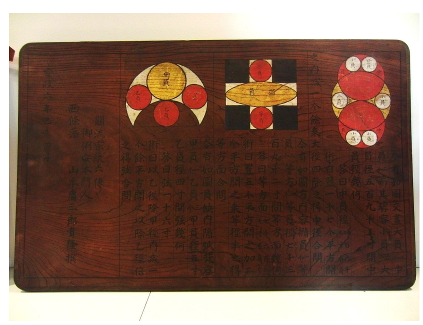
Figure 2.2.1: *Bōshohō* from Sakabe Kohan's *Sanpo Tenzan Shinanroku* [7].

tradition originated from the hanging of *ema*, the drawing of horses for deities as many poor people could not afford to offer the deities an actual animal. Many people even consider *sangaku* as mathematical *ema*, as they are offered in a similar fashion.

Each problem on a *sangaku*, no matter the difficulty, followed a similar formula; first was the question itself, followed by the answer, and the formula used completed the individual problem. As with all *idai keishō*, the solving method was given so others could get to the same conclusion and extrapolate upon the problem, creating a new one. However, the *sangaku* were unique in that each tablet did not contain only one problem; they would have a large variety of problems, of all difficulties. As many were offered from schools called *juku*, there was no set theme, topic, or principle to an individual *sangaku* tablet.

The lack of theme was aided by the demographics of class; different social classes attended the *juku*. Often the samurai class with the equivalent of a masters' degree taught reading, writing, and the *soroban*, and even farmers attended at a low price to know how to tend to their fields. About 80,000 *juku* were in Japan by the end of the Edo period, so we can assume that math was widely known. A *sangaku* problem proposed by a child could be located right next to a graduate level problem. The content of a tablet was based on the establishment or person writing it, not so much keeping certain topics together. Thus, mathematicians were not the only people to create problems; children, women, everyone could create a *sangaku* if they so desired. Both Irie Shinjun's Katayama Hiko shrine *sangaku* and the Konnoh Hachimangu shrine *sangaku* are surviving tablets and can be seen in Figure 2.3.1.

(a) Katayama Hiko *Sangaku* tablet [4].



(b) *Sangaku* to Konnoh Hachimangu shrine [5].

Figure 2.3.1: Two *Sangaku* tablets from the 1800s.

We give most of our credit to modern day global recognition of *sangaku* to one man, Yamaguchi Kanzan (1781-1850), a mathematician from Suibara who traveled throughout the country to record numerous *sangaku* over the course of eleven years, taking six journeys between the years 1817 and 1828. These journeys, appropriately called "*sangaku* pilgrimages" were common during the 1800s. People would travel around the country to learn more and contribute to the betterment of mathematics through *idai keishō*. How were these pilgrimages even that common; certainly, travel, especially walking across a whole country, was not common, right? The opposite is true for the Edo period; travelling was the main interaction people had. In fact, the shogun, who lived in the capital Edo, the namesake for the time period, summoned all the *daimyo*, leaders of the prefectures, to the capital every other year. Thus, this yearly travel to and from the city allowed for paths to develop, along with sightseeing locations for commoners. Yamaguchi is one of our sole first-hand accounts on *sangaku*, and in Chapter 7 of [4], we notice that he was well-known for his mathematical prowess. His journal, which contains close to 700 pages, mentions problems from eighty-seven tablets, of which only two survived, thanks to rotting or other destruction; in fact, he tried to publish the journal as *Syuyuu Snapō*, (*Travel Mathematics*). Although the book was never published, the original diary is a cultural asset in Agano, even though Yamaguchi's words have not been translated into modern Japanese. *Sacred Mathematics* [4] shares his third pilgrimage, starting and ending in Edo from 1820-1822.

Despite their cultural importance, *sangaku* never made their way to the global view. In conjunction with the falling out of *wasan* after the Edo period, these tablets dwindled in number over time, even if people still wrote and dedicated math problems in the 1900s. The world's first exposure to *sangaku* was in 1989 through *Japanese Temple Problems: San Gaku* by David Pedoe and Fukagawa Rothman. Rosalie Hosking mentions in an article that this did not provide era-accurate solutions to the numerous *sangaku* problems shown [7]. She includes describing *bōshohō* in more detail, defining certain rules, demarcations, and even solves a problem in the traditional method, truly showing us a difference in mathematical ideology between the modern day and Edo period Japan. As Hosking shares, "careful attention must be given to finding and translating solutions to *sangaku* in the literature in order to accurately represent and understand the methods used in Edo Japan" [7].

Chapter 3

Fundamentals of Geometry

In this section, we will be discuss relevant geometric foundations, and differentiate them based on which geometries the specific definition, theorem, or axiom falls under.

3.1 Neutral Geometry and the Parallel Postulates

Ever since Euclid published his *Elements* around 300 BCE, his 467 propositions of planar and solid geometry both aided geometric developments and acted as the foundation of geometry, with some of the propositions acting as axioms, as noted in *Roads to Geometry* [13]. One of these, appropriately dubbed Euclid's Fifth Postulate, seemed complex to geometers even into the 1800s:

Postulate 3.1.1 (Euclid's Fifth)**.** *If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.*

This wording, aside from being extremely long winded, is difficult to interpret. To put it plainly, if we have a line intersect two other lines, and if the sum of the interior angles is less than $180°$, then the two lines intersect. We can see a visualization of this

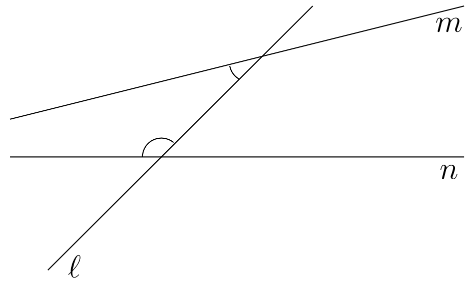


Figure 3.1.1: Euclid's Fifth Postulate in action: line ℓ intersects lines m and n. The marked interior angles have sum less than 180°, meaning m and n will intersect.

in Figure 3.1.1. The two "horizontal" lines by Euclid's Fifth Postulate should intersect because together the two interior angles are less than 180°. Despite the elusivity of a concrete proof, mathematicians still operated under the assumption that there was a sole interpretation of this, resulting in the titular Euclidean geometry.

Since his fifth postulate is his only postulate defining parallel lines – or, rather, what makes lines not parallel – examining geometry without making commitments on Euclid's Fifth Postulate is called **Neutral geometry**. All of the following definitions are all used within Neutral geometry, as they do not assume anything about parallel lines.

Definition 3.1.2. Given a point O and a positive real number r, the **circle** with center O and radius r is defined to be the set of all points P such that the distance from O to P is r,

$$C(O, r) = \{P \mid OP = r\}.$$

Definition 3.1.3. A line ℓ is **tangent** to a circle if ℓ intersects the circle in exactly one point. If P is said point, then ℓ is tangent to the circle at P.

In Figure 3.1.2 we see a circle O of radius r, tangent to line ℓ at point P. Additionally, we note that there are an infinite amount of tangent lines to a circle. Should we take a finite amount of tangents, we encounter more shapes.

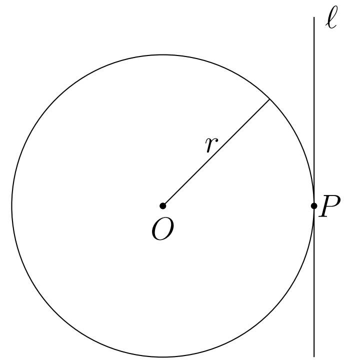


Figure 3.1.2: A circle, $C(O, r)$ with center O and radius r. Line ℓ is tangent with the circle at point P.

Definition 3.1.4. Suppose $P_1, P_2, \ldots, P_n$ are n distinct points such that no three of the points are colinear. If any two of the segments $\overline{P_1P_2}, \overline{P_2P_3}, \ldots, \overline{P_nP_1}$ are either disjoint or share an endpoint, the points $P_1, P_2, \ldots, P_n$ and those segments are a **polygon**, denoted by $P_1P_2 \ldots P_n$. The points $P_1, \ldots, P_n$ are called the vertices of the polygon and the segments are called the sides. A polygon with n sides is called an n-sided polygon or an n-gon.

Definition 3.1.5. If all the segments in a polygon are congruent to each other and all the angles are congruent to each other, then the shape is a **regular polygon**.

We can see two examples of polygons in Figure 3.1.3, the first of which is not a regular 5-gon, which we call a pentagon. The second polygon is a regular 8-gon, called an octagon.

Definition 3.1.6. A circle that contains all vertices of a polygon is said to **circumscribe** the polygon. The circle is called the circumcircle and its center is called the circumcenter of the polygon. If the polygon has a circumcircle, we say that the it can be circumscribed.

Definition 3.1.7. A circle that is drawn inside and is tangent to all the sides of a polygon is called an **incircle**. The center of an incircle is called the **incenter** and lies where the angle bisectors meet.

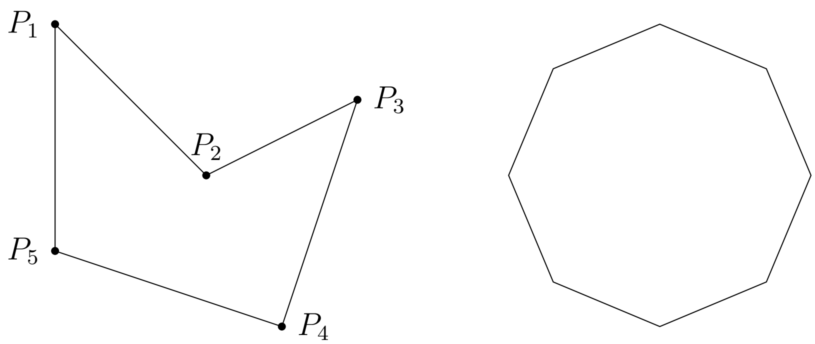


Figure 3.1.3: On the left is a 5-gon with vertices P_1, P_2, P_3, P_4, and P_5. On the right is a regular 8-gon.

Definition 3.1.8. A line that divides an angle into two equal angles is called the **angle bisector**.

Definition 3.1.9. A line ℓ that intersects another line m at 90° is called a **perpendicular** line. If ℓ divides m into two equal segments, ℓ is also called a **perpendicular bisector**.

We can see an example of circumscribing and inscribing a circle or polygon in Figure 3.1.4. Additionally, the red line in the figure is an angle bisector of one of the triangle's angle. Furthermore, the same line is a perpendicular of the opposite leg of the triangle.

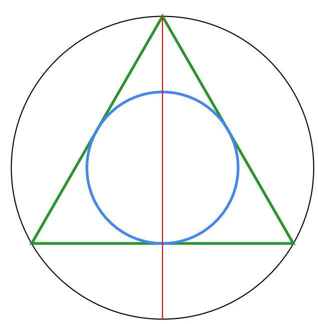


Figure 3.1.4: An **inscribed circle** (blue) and **circumscribed** triangle (green) with a red **angle bisector**.

3.2 Regarding Triangles

Many *sangaku* problems deal with triangles, so, to aid our understanding, we will first mention notations and also address an important theorem for our solutions. When referring to a triangle, such as the one in Figure 3.2.1, lower case letters will denote one of the three sides. The angles that are opposite a side will share the same letter but will be capitalized. For example, the angle opposite side a we will call angle A. If more than one angle exists at a vertex, we will assume the standard three-letter notation for an angle. The triangle itself will be named by the three vertices.

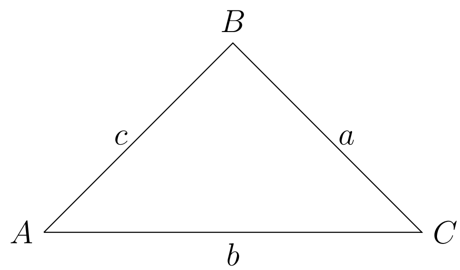


Figure 3.2.1: Triangle $\triangle ABC$ has sides a, b, and c, with corresponding angles A, B, and C.

If we find that the lengths of sides a and b are the same, then we have a special case of triangle, an Isosceles triangle.

Theorem 3.2.1 (Isosceles Triangle Theorem). *If two sides of an isosceles triangle are equal then the angles opposite to the equal sides will also have the same measure.*

Thus, for the case where $a = b$ as mentioned, we would also say that $A = B$. Another case for the Isosceles Triangle Theorem exists when all sides are equal to each other. We find that all three angles A, B, and C are all the same.

3.3 Euclidean Geometry

The Euclidean geometric system is the geometry with which people are most familiar. This is what is typically covered in high school geometry courses, and our basic understanding on triangles, squares, and other shapes are all based on the Euclidean Parallel Postulate, a natural conclusion of the original interpretation of Euclid's Fifth Postulate.

Definition 3.3.1 (Euclidean Parallel Postulate)**.** Through a point P, there can be drawn at most one line m that is parallel to any line ℓ that does not contain P.

Because this is the geometric system with which most people are familiar, there are many conclusions that we must note. Some of the more important Euclidean results are:

1. There exists a triangle in which that sum of the measures of the interior angles is $180°$.
2. Every triangle can be circumscribed, and the center of the circumscribing circle is the concurrence point of the perpendicular bisectors of two of the sides of the triangle.
3. The sum of the measures of the interior angles of a triangle is constant for all triangles.
4. A rectangle exists.

See [13] for more information regarding Euclidean geometry, and the conclusions made from the parallel postulate. Noting these five Euclidean results, there are *sangaku* problems that we can quickly state or categorize as solely Euclidean.

3.4 Elliptic Parallel Postulate

Our investigation will not ultilize Elliptic geometry, but we include this for completeness. The first of the non-Euclidean geometries we will cover is the Elliptic Parallel Postulate. Developed by Bernhard Riemann in 1854, this system has the second of the three cases of parallel lines; in Elliptic Geometry, parallel lines do not exist. As 3.4.1 states:

Definition 3.4.1 (Elliptic Parallel Postulate)**.** Through any point in the (Elliptic) plane, there exist no lines parallel to a given line.

But what does this geometry look like? Explaining the differences between Elliptic and Euclidean geometry is more difficult than simply showing different models. The model of Elliptic geometry is a globe, as we see in Figure 3.4.1. When travelling across continents, as an example, we see this in the Euclidean shortest path is not the actual shortest path. This is why pilots fly close to the North Pole when flying from the US to a different location in the Northern Hemisphere. Planes travel along a line, defined as great circles, the largest possible circle one could draw on the sphere – the same size as the Equator.

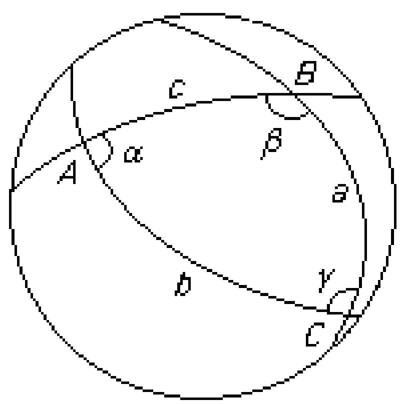


Figure 3.4.1: Elliptic geometric model with a triangle [8].

3.5 Hyperbolic Parallel Postulate

The Hyperbolic Parallel Postulate concludes our cases for parallel lines, which requires that multiple parallel lines exist through one point, set up by the Hyperbolic Parallel Postulate 3.5.1:

Postulate 3.5.1 (Hyperbolic Parallel Postulate)**.** *The measure of the angle of parallelism, the minimum angle for two lines to intersect, for a line ℓ and a point P that is not on ℓ is less than* $90°$*.*

This postulate gives us a theorem to easily analyze what occurs in this geometric system:

Theorem 3.5.2. *Given a line ℓ and a point P not on ℓ, there are at least two lines through P that are parallel to ℓ.*

Hyperbolic geometry came into existence in 1868 with Eugenio Beltrami's paper "Essay on the Interpretation of non-Euclidean Geometry," which answered questions proposed by János Bolyai and Nikolai Lobachevski in the 1830s, showing consistency in hyperbolic geometry [13]. Similar to elliptic geometry, the existence of multiple parallel lines gives us differing results from the Euclidean Geometric system, including the invalidation of the existence of rectangles. Additional information regarding models of Hyperbolic geometries can be found in [2].

Much like with Elliptic geometry, seeing a visualization of Hyperbolic geometry makes understanding incredibly easier. Unlike Elliptic, however, models for this system are not as simple as a sphere or a globe. Since this geometry is innately more difficult to understand, mathematicians have struggled to create a perfect model. Henri Poincaré gave us the "disk," model, called the Poincaré Disk model, as shown in Figure 3.5.1.

Similarly to the notion that Euclidean geometry only pertains to one case of parallel lines, so does the angle sum in Euclidean geometry. Specifically for Hyperbolic geometry,

Theorem 3.5.3 (Triangle Angle Sum Theorem: Hyperbolic)**.** *For any Hyperbolic triangle $\triangle ABC$, the angle sum of $\triangle ABC$, denoted as S_T, is strictly less than* $180°$*. Notionally,*

$$S_T = m\angle A + m\angle B + m\angle C < 180°.$$

Corollary 3.5.4 (Polygon Angle Sum: Hyperbolic). *For a Hyperbolic polygon, the angle sum for the polygon with n sides, denoted as S_P is less than the number of sides minus 2 times 180°. Written out, this is seen as*

$$S_P < (n-2) * 180°$$

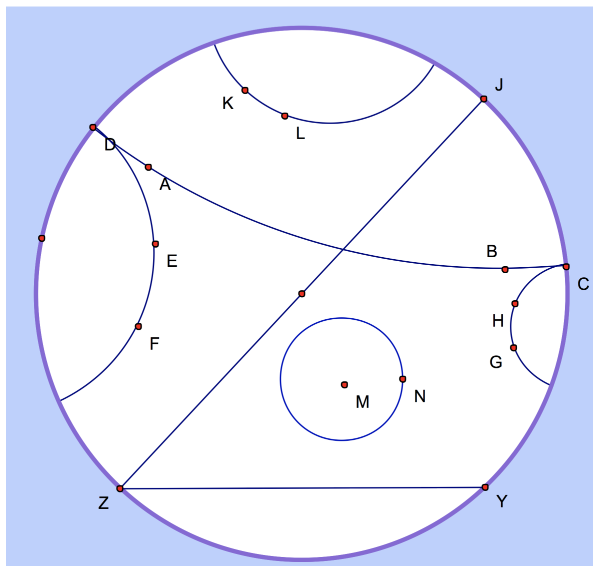


Figure 3.5.1: Poincaré Disk model with lines and circles [6].

Figure 3.5.2 shows us visual comparisons between Elliptic, Euclidean, and Hyperbolic geometry, as well as providing an example of the plane.

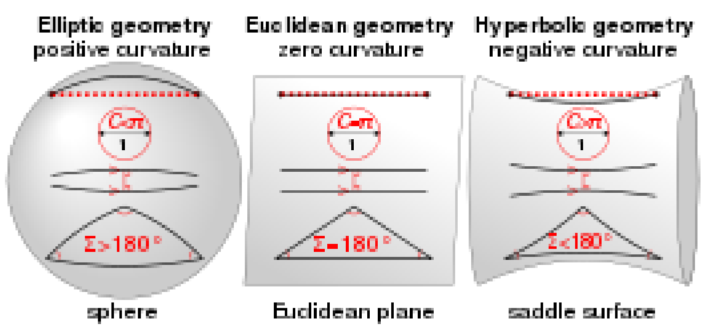
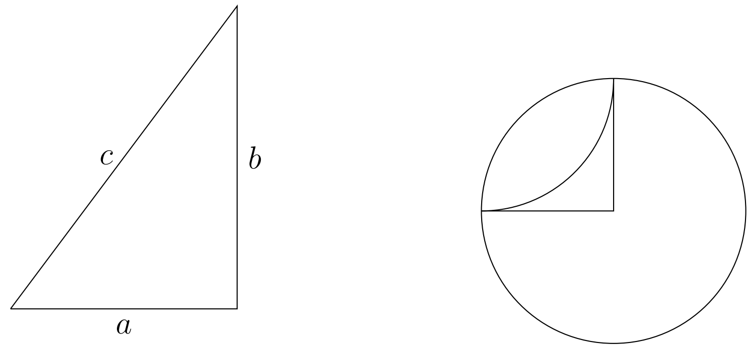


Figure 3.5.2: Euclidean, Elliptic, and Hyperbolic geometries [12].



(a) A Euclidean right triangle. (b) A Hyperbolic right triangle.

Figure 3.6.1: Right triangles as seen in different geometries.

3.6 On the Pythagorean Theorem

As mentioned in Chapter 2, the majority of *wasan* had its roots in Chinese mathematics, among which books was the *Zhou bi Suan Phan*, where the Pythagorean Theorem was the highlight. Because of this, many *sangaku* utilize the Pythagorean Theorem. As we are taught in school, we know one form of this theorem, but we also need different formulas for our different geometries. These formulas and more information regarding the specifics of the geometries, can be found in [1].

Theorem 3.6.1 (Pythagorean Theorem: Euclidean)**.** *In a right triangle ΔABC as drawn in Figure 3.6.1a, the length of the hypotenuse squared is equal to the length of the base squared plus the length of the height squared. That is,*

$$a^2 + b^2 = c^2. \tag{3.6.1}$$

In essence, the Pythagorean Theorem states that there exists a relationship between the hypotenuse of a right triangle to the other two sides. We can translate this into both Elliptic and Hyperbolic, though we exclude the Elliptic variant, as our analysis will only pertain to Hyperbolic geometry. The Hyperbolic version of the Pythagorean Theorem is stated in Theorem 3.6.2. Unlike the Elliptic counterpart, where the value is simpler to interpret, ρ is more abstract and harder to define succinctly, but it is a positive constant dependent upon the specific geometry one is working with (see [1]). The value of ρ is associated with the curvature of a line in the model, and in the Poincaré Disk, this value is equal to 1.

Theorem 3.6.2 (Pythagorean Theorem: Hyperbolic)**.** *In a triangle ΔABC as seen in Figure 3.6.1b, the hyperbolic cosine of the length of one side divided by rho, ρ, is equal to the hyperbolic cosine of the length of another side divided by rho times the hyperbolic cosine of the last side divided by rho. To put it more simply,*

$$\cosh\left(\frac{c}{\rho}\right) = \cosh\left(\frac{a}{\rho}\right)\cosh\left(\frac{b}{\rho}\right). \tag{3.6.2}$$

3.7 Law of Cosines

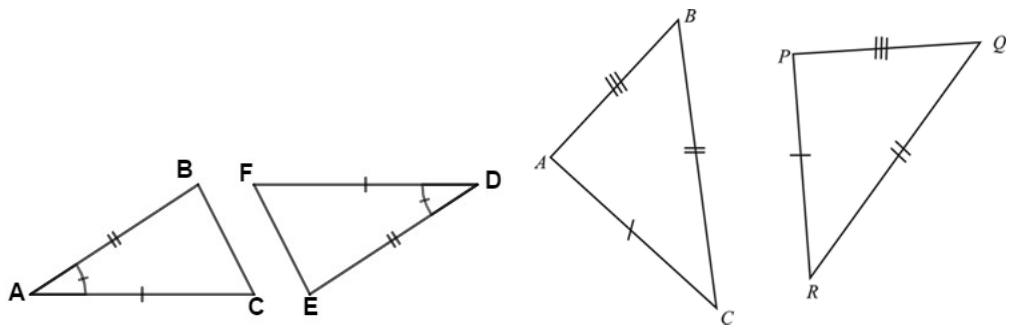
Another valuable asset for solving numerous *sangaku* – albeit from a modern setting – is the Law of Cosines. This is especially valuable when trying to find lengths between radii when a triangle is formed, as seen in one of the *sangaku* we analyze in Chapter 4. Each geometry has a unique formula for the Law of Cosines. For Hyperbolic geometry, the Law of Cosines is the following:

Law 3.7.1 (Hyperbolic Law of Cosines)**.** *Any triangle within the Poincaré disk model, where a, b, and c are the lengths of the triangle's sides and A denotes $m\angle BAC$. satisfies*

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(A).$$

3.8 Law of Sines

Just as with the Law of Cosines, the Law of Sines is another very helpful tool for us to utilize, as many *sangaku* deal with the relationship between angles of a triangle



(a) SAS Congruence [9]. (b) SSS Congruence [14].

Figure 3.9.1: Two kinds of triangle congruence.

with the length of the corresponding side. Additionally, though each geometry has a formula for this law, the formula differs slightly between the three types. Notably, for Hyperbolic geometry, this relationship between angles and lengths is as follows:

Law 3.8.1 (Law of Sines: Hyperbolic)**.** *Any right triangle in the Poincaré Disk model satisfies*

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}. \tag{3.8.1}$$

3.9 Congruence

When we construct our proofs, we will be using many properties of triangles, so establishing certain congruences is vital. All of the following triangle congruence theorems are all valid within neutral geometry, allowing them for all geometries.

Theorem 3.9.1 (Side Angle Side (SAS) Triangle Congruence)**.** *If two triangles exist such that two neighboring sides and the angle between them in one triangle are respectively congruent to neighboring sides and the angle between them in the other triangle, then the two triangles are congruent.*

We see an example of the Side Angle Side congruence in Figure 3.9.1a, where $\overline{DE} \cong \overline{AB}$, $\overline{AC} \cong \overline{DF}$, and $\angle A \cong \angle D$. We find that $\Delta ABC \cong \Delta DEF$.

Theorem 3.9.2 (Side Side Side (SSS) Triangle Congruence)**.** *If two triangles are such that the sides of each triangle are respectively congruent with the sides of the other triangle, then the triangles are congruent.*

We see an example of the Side Side Side congruence in Figure 3.9.1b, where $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, and $\overline{AC} \cong \overline{RP}$. We find that $\Delta ABC \cong \Delta PQR$.

Chapter 4

Examining Sangaku Outside of Euclid

Before examining the *sangaku*, certain disclaimers must be made. As mentioned in Section 2.3, the perspective presented this chapter is not an example of how traditional Japanese mathematicians would have solved the problems. We will not use *bōshohō*, meaning all mathematical ideas will be from a 21st century American perspective. Thus, to find a more accurate methodology, please refer to Hosking's article "Solving *Sangaku*" [7]. Additionally, the development of non-Euclidean geometry hadn't been developed until the mid 1800s in Europe, just before the end of the Edo era. Had these geometric systems reached Japan during this time of closure, perhaps then some *sangaku* might have been modeled in Hyperbolic or Elliptic geometries. Since time did not work that way, however, this examination is purely hypothetical. Additionally, the original *sangaku* did not have an official "name" given to them, but for simplicity's sake and to reduce the use of the word *sangaku*, we will be given the problems names, based on either the original proposer or the problem itself.

Another important question to ask is what is our methodology? That is, how are we going to approach each problem? Solving these we have two separate goals:

first, to prove the problem can work in Neutral geometry and, second, to calculate an answer. After the calculation, we reflect back on what is known, comparing the Euclidean and Hyperbolic results. Thus, each of the problems we encounter will have four sections, an introduction, the Euclidean result, a Neutral proof, and the Hyperbolic calculation with a short comparison. However, before we start solving, let us first see which problems cannot be done outside of Euclidean geometry, which we do in 4.1.

Additionally, we add some conventions that we use throughout our analysis. First, when showing or describing a circle, the center of the circle is how we derive both its name and radius. For example, a circle with center called "r" will have a radius of length r. Furthermore, not all of the original *sangaku* had centers in the circles, but we add them to clarify radii and circles.

4.1 Sangaku only for Euclid

As mentioned in Section 3.3, there are many assumptions we make within Euclidean geometry, including the existence of a rectangle. For that reason alone, we can clearly see that any *sangaku* that contains a square is only valid within the Euclidean geometry. Thus, a *sangaku* proposed by thirteen-year-old Satō Naosue in 1847 cannot be done in a non-Euclidean setting. The problem is as follows:

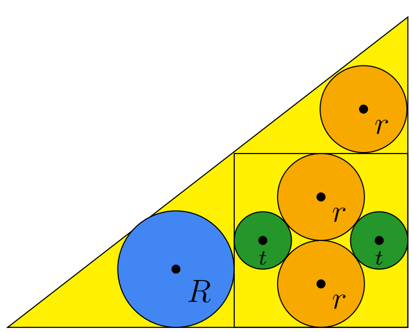


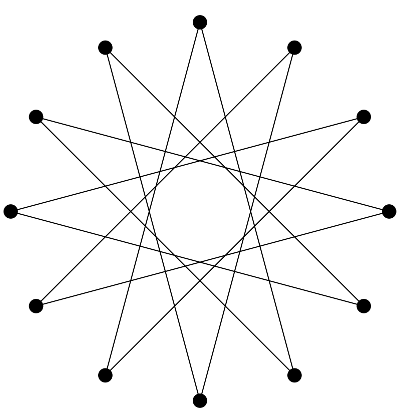
Figure 4.1.1: *Sangaku* proposed by Satō Naosue.

Satō's *Sangaku* *Two circles of radius r and two of radius t are inscribed in a square, as shown in Figure 4.1.1. The square itself is inscribed in a large right triangle and, as illustrated, two circles of radii R and r are inscribed in the small right triangles outside the square. Show* *that* $R = 2t$.

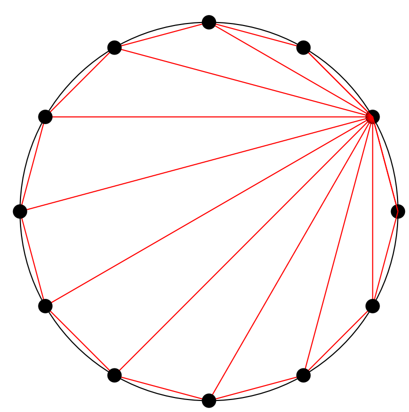
Our issue when trying to view this problem arises within the description, specifically where it calls for a "square." However, as mentioned in Section 3.3, squares only exist within Euclidean geometry. Thus, we can simply say that Sato's *Sangaku* is unsolvable in non-Euclidean geometry.

Another claim resulting from the Euclidean Parallel Postulate is the all triangles can be circumscribed by a circle. A further conclusion from this is that no shapes can be circumscribed by a circle outside of Euclidean geometry, as we can divide the shape into smaller triangles. One example of a *sangaku* that results from a circumscribed shape is seen from none other than Yamaguchi Kanzan's 1819 diary.

Twelve-Pointed Star *Sangaku*. *We stick pins into the position of each vertex of a regular dodecagon (twelve-sided shape), as shown in Figure 4.1.2a. Then we take the string of length* $l = 150$ *cm and wrap it around the pins, as shown. This forms a small regular dodecahedron in the center. Find the length of the side* s *of the small dodecahedron.*



(a) The Twelve-Pointed Star *Sangaku*.



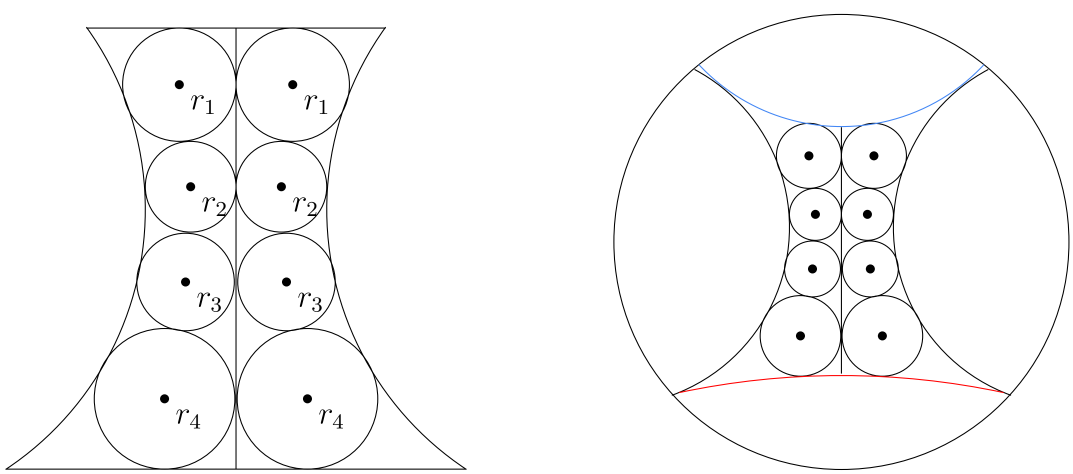
(b) Inscribed triangles in the Twelve-Pointed Star *Sangaku*.

Figure 4.1.2: Twelve-Pointed Star *Sangaku* and inscribed triangles in the star.

Although initially this construction does not seem to violate circumscribed triangles, we can quickly see this is not the case. We may create a circle through each of the "pins" and quickly discover that the dodecagon is circumscribed. Since we can create a regular dodecagon by constructing ten adjacent triangles, we find that this triangle is circumscribed, and we cannot guarantee this exists within non-Euclidean geometry. Figure 4.1.2b shows us these circumscribed triangles, colored in red, within the circumscribed dodecagon.

Both of these *sangaku* deal with shapes that we deal with commonly in geometry, such as circles, triangles, and squares. This is not the case for all problems, however. Some *sangaku*, such as the next one, deal with shapes and items that were commonly used in Edo Japan, such as a *gunpai*, a fan used for referees in sumo wrestling.

The Sugano Teizou *Gunpai Sangaku*. *A chain of four circles of radii r_1, r_2, r_3, and r_4 are touched on one side by the line l and on the other side by a circle arc of radius r, which in sumo wrestling is called the "gunpai" ("umpire's fan"). See Figure 4.1.3a. Find r_4 in terms of r_1, r_2, and r_3.*



(a) Euclidean *Gunpai*.

(b) Potential *Gunpai* in Hyperbolic geometry.

Figure 4.1.3: *Gunpai* in Euclidean and Hyperbolic geometries.

This is not a shape we deal with normally outside of sumo, but we sadly find this also only works within Euclidean geometry. Because we are not given lengths for the bordering lines, we cannot tell where to place the fan in the Poincaré Disk. As we see in Figure 4.1.3b, we can place the *gunpai* in the center, but we find the blue and red lines at the top and bottom of the drawing have different curvatures. Moving the *gunpai* to a different location in the model would result in differently curved lines. Thus, in order to utilize the Pythagorean Theorem properly and solve the problem similarly to Fukagawa and Rothman [4], we need more information given in order to help us fixate a location to place the fan, such as a given length of one of the lines. When we are given distances, we find that *sangaku* are easier, if not explicitly possible, to solve.

4.2 What *Sangaku* Work Then?

Now that we have seen a few examples of *sangaku* that can only be done within Euclidean geometry, let us examine problems that are solvable outside of this standard geometric system. We will specifically be looking at three *sangaku* problems, two of which are found on the Katayama Hiko tablet from Figure 2.3.1a and one found on another – the Kishi Mitsutomo *Juku Sangaku*.

4.2.1 Katayama Hiko Circle *Sangaku*

Our first example is from the Katayama Hiko shrine *sangaku*, which we can see in Figure 2.3.1a in the bottom row second from the right, and this problem only contains circles [4].

Katayama Hiko Circle (KHC) *Sangaku*. *A circle of radius r inscribes three circles of radius t, the centers of which form an equilateral triangle of side $2t$ as shown in Figure 4.2.1. Find t in terms of r.*

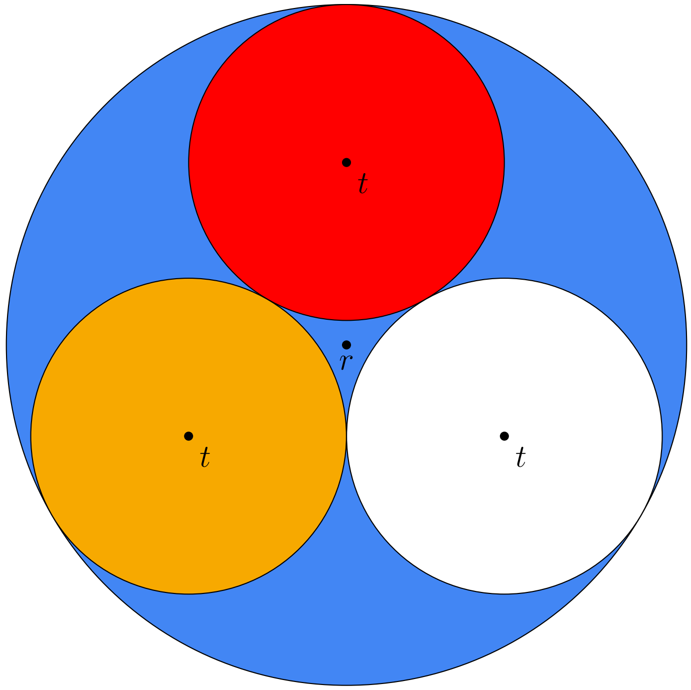


Figure 4.2.1: Katayama Hiko Circle *Sangaku*.

KHC Euclidean Solution. From the creator of the problem, we are given the solution $t = \dfrac{r}{.464}$, which is a Euclidean result. Let us begin our investigation with Neutral geometry to show that the given circular construction is allowable outside of Euclidean postulates.

KHC Proposition. *The Katayama Hiko Circle Sangaku satisfies the necessary requirements of Neutral geometry.*

Proof. Let $\triangle ABC$ be the equilateral triangle created from the given *sangaku*, where A, B, and C are the centers of the smaller circles. Also, let r and t denote the radii of the outer and inner circles, respectively. As defined by the problem, $\triangle ABC$ is an equilateral triangle with sides of length $2t$, so the measures of all three angles in this triangle are the same. We connect vertices A, B, and C to the center of the largest circle, meeting at the center R, creating three triangles, $\triangle ABR$, $\triangle BCR$, and $\triangle CAR$, as seen in Figure 4.2.2. As the radius of the large circle is distance r and the smaller circles are radius t, $\overline{AR} \cong \overline{BR} \cong \overline{CR} = r - t$. By the Side Side Side Congruence of Triangles given in Theorem 3.9.2, all three triangles $\triangle ABR \cong \triangle CBR \cong \triangle BAR$. Thus, we find the Katayama Hiko Circle problem satisfies Neutral geometry. $\square$

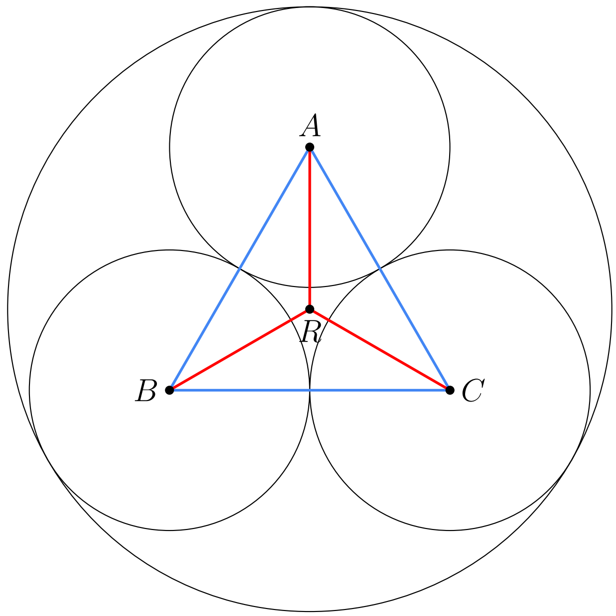


Figure 4.2.2: Auxiliary lines drawn for the Katayama Hiko circle problem.

KHC Hyperbolic Calculation. We can find the lengths of r and t by using the Hyperbolic Law of Cosines, Law 3.7.1. In order to use this, we must first consider the angle measures and plug in values accordingly. Using the Poincaré model, Theorem 3.5.3 tells us that any Hyperbolic triangle must have less than 180°. Thus, the measure of the angles in $\triangle ABC$ are all less than 60° as all three angles in the triangle must be equal. Since each of the smaller triangles are congruent isosceles triangles, we can consider take $\triangle BCR$ without loss of generality. We then can easily find that the $m\angle B < 30°$. Hence, using the Hyperbolic Law of Cosines 3.7.1, we have

$$\cosh(r-t) < \cosh(r-t)\cosh(2t) - \sinh(r-t)\sinh(2t)\cos(30°).$$

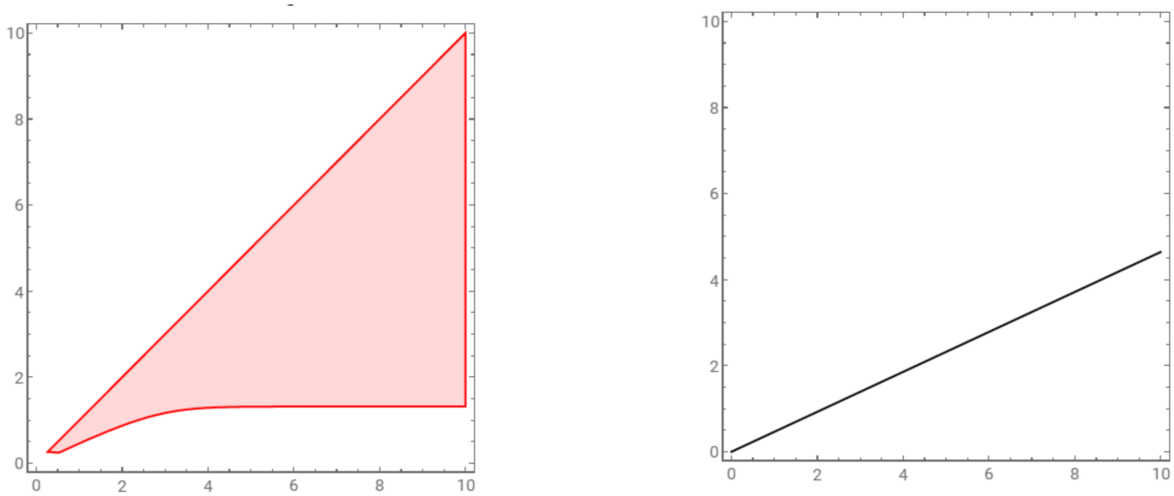
Using basic hyperbolic identities, we have this is equivalent to

$$\tanh(r-t) < \frac{\cosh(2t)-1}{\sinh(2t)\cos(30°)},$$

and clever algebraic manipulation gives us

$$\tanh(r-t) < \frac{2\sqrt{3}}{3}\tanh(t). \tag{4.2.1}$$

As the solution set to this equation is difficult to envision, we now consider the plot of this inequality to better understand the relationship between r and t. This graph, along with the graph of the Euclidean solution noted above are shown in Figure 4.2.3.



(a) Graph of Equation 4.2.1.

(b) Graph of Euclidean result.

Figure 4.2.3: Graphs corresponding to the Katayama Hiko Circle problem, where t is the vertical axis and r the horizontal

From these graphs, we note a few important facts. First, we note that the region for the Hyperbolic region has a lower bound as r approaches infinity, to a value of about $t = 1.31696$. We see that this value for t comes close to the limit even before $r = 4$. Another important conclusion comes from a comparison between the Hyperbolic and Euclidean results. The result for the Euclidean variant of the Katayama Hiko Circle is a valid solution for the Hyperbolic version, although the inverse is not true. Equation 4.2.1 has a looser bound than its Euclidean counterpart does, which only works for a linear relationship between r and t.

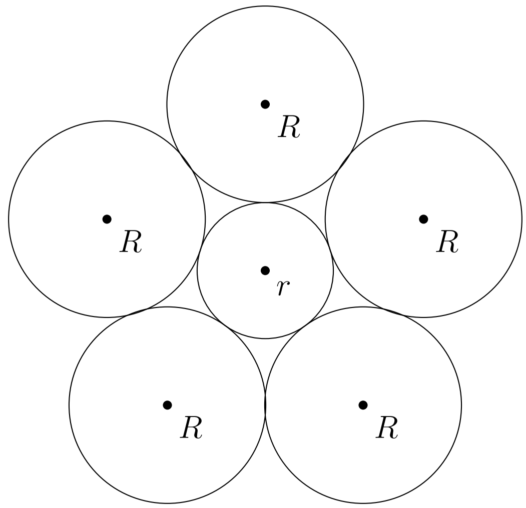


Figure 4.2.4: Problem from the Kishi Mitsutomo school's *sangaku*.

4.2.2 Kishi Mitsutomo School *Sangaku*

Next, let us investigate a *sangaku* problem from a *juku*, the Kishi Mitsutomo School. Though the *sangaku* itself exists at the Kitano shrine in Fujioka city, pictures of the tablet are elusive. The problem we choose reads as follows:

Kishi Mitsutomo School (KMS) *Sangaku*. *As shown in Figure 4.2.4, a circle of radius r is surrounded by a loop of five equal circles of radius R. Find r in terms of R.*

KMS Euclidean Result. The author gives us that, in Euclidean geometry, we are given the answer of $r = \dfrac{7R}{10}$, so let us proceed and show the validity of the posed question in Neutral geometry.

KMS Proposition. *The Kishi Mitsutomo School Sangaku satisfies the necessary requirements of Neutral geometry.*

Proof. Let a circle with radius r and center F be surrounded by five circles of radius R with centers A, B, C, D, and E, where each of these circles of radius R are tangent to two outer circles as well as the inner circle. we know that $r < R$ in order to

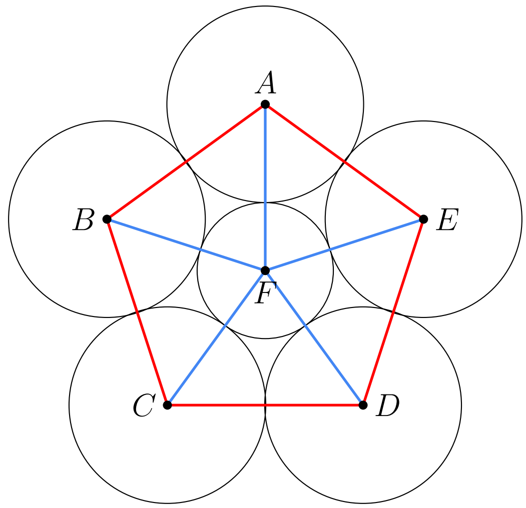


Figure 4.2.5: Auxiliary lines drawn for the Kishi Mitsutomo problem.

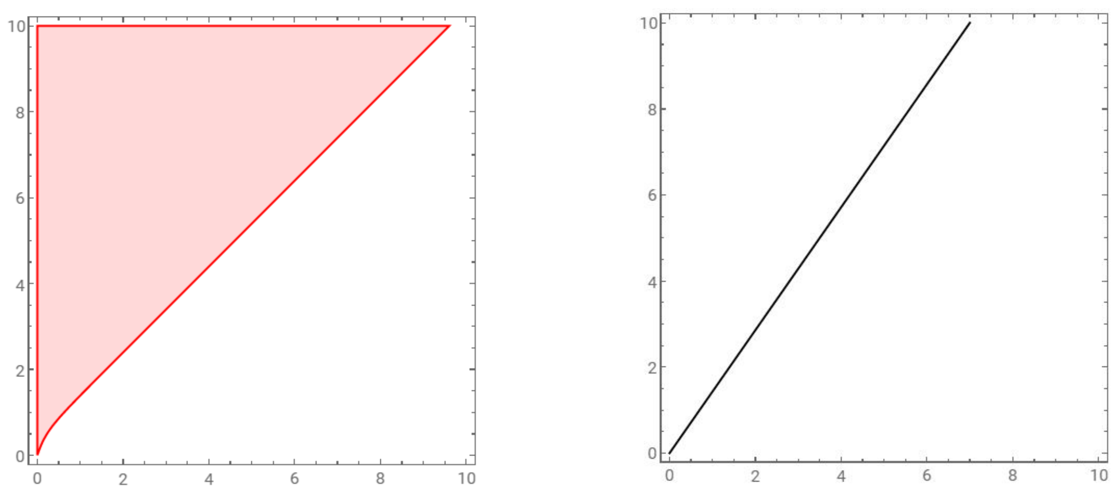
satisfy this construction. Connect all centers of each circles to its neighboring circles, creating lines through each tangent point of each circle. Call this polygon pentagon $ABCDE$ and note that F is connected to each vertex of the pentagon, as we seen drawn in Figure 4.2.5. We note that the lengths of these red sides are all $2R$ and the blue sides are all $R + r$. Thus, $ABCDE$ is comprised of five triangles, $\triangle ABF$, $\triangle BCF$, $\triangle CDF$, $\triangle DEF$, and $\triangle EAF$, with all equal sides. From the Side Side Side Congruence Theorem 3.9.2, we see that $\triangle ABF \cong \triangle BCF$. Using the same logic, we find that $\triangle BCF \cong \triangle CDF$. If we continue this, we find that, when we select two of these triangles, the two chosen triangles are congruent to each other. Thus, $ABCDE$ is a regular pentagon, and we find that this *sangaku* construction is valid for neutral geometry. □

KMS Hyperbolic Calculation. In calculating the length within Hyperbolic geometry, we need to work with the angle sum of a polygon. From Corollary 3.5.4, we see each angle in the pentagon, then, is less than 104°. Therefore, we note that the angle measurement of two matching angles in each isosceles triangle is less than 54°, since the angles in pentagon $ABCDE$ are being bisected by the sides of the triangles. The third angle in each triangle, the vertex angle, is strictly 72°, as the circle – which has 360° – is divided into fifths. Therefore, without loss of generality, let us use $\triangle ABF$ to find a relationship between R and r. The length $R + r$, which corresponds to the length of the legs of $\triangle ABF$, can be found using the Hyperbolic Law of Sines 3.8.1. As this side's corresponding angle is less than 54°, then we find that

$$\frac{\sin 54^\circ}{\sinh(r+R)} > \frac{\sin 72^\circ}{\sinh 2R}$$

Algebraically manipulating this inequality, we can rewrite the equation to give us a helpful model in showcasing the relationship between r and R. Specifically, this equation can be written as

$$\frac{2\sin 54^\circ}{\sin 72^\circ} > \frac{\sinh r}{\sinh R} + \frac{\cosh r}{\cosh R}. \tag{4.2.2}$$



(a) Graph of the final equation for the Kishi Mitsutomo problem.

(b) Euclidean result of Kishi Mitsutomo.

Figure 4.2.6: Graphs regarding the Kishi Mitsutomo *Sangaku*, where R is the vertical axis and r the horizontal.

Let us consider the graph of this equation, just as we did in the last example.

From Figure 4.2.6a, we see the relationship between r and R appears to be bounded from below by a positive linear relationship after approximately $r = 2$. We note that $R > r$ at every possible solution point, which makes arithmetic and geometric sense, as we expect the circles with radius R to be larger than the circle with radius r even in Hyperbolic geometry. Comparing the Euclidean and Hyperbolic results in Figure 4.2.6b, we notice that just like our previous *sangaku*, the Euclidean result is a potential solution for the Hyperbolic result.

4.2.3 Katayama Hiko Triangle *Sangaku*

The previous two *sangaku* we have investigated in Hyperbolic geometry only contain circles, so are there other constructions of problems that we can solve in this geometry? The answer is yes. We investigate a *sangaku* containing a triangle in our next problem, where we return to the same tablet as our first problem, the Katayama Hiko *sangaku*, seen in Figure 2.3.1a, borrowed from *Sacred Mathematics* [4]:

Katayama Hiko Triangle (KHT) *Sangaku*. *A circle of radius r is inscribed in an isosceles triangle with sides $a = 12$ and $b = 10$ as shown in Figure 4.2.7. Find r.*

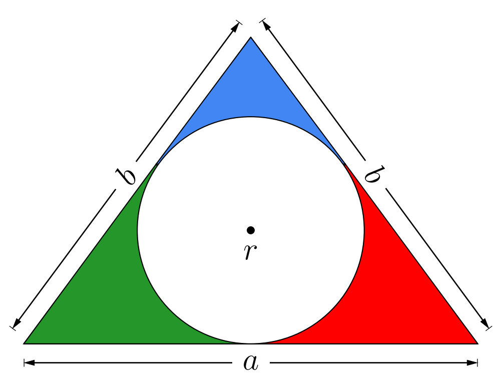


Figure 4.2.7: The Inscribed Circle *sangaku* from the Katayama Hiko tablet.

KHT Euclidean Solution. The tablet tells us that $r = 3$ when using Euclidean geometry. Now let us show that we can find this length neutrally.

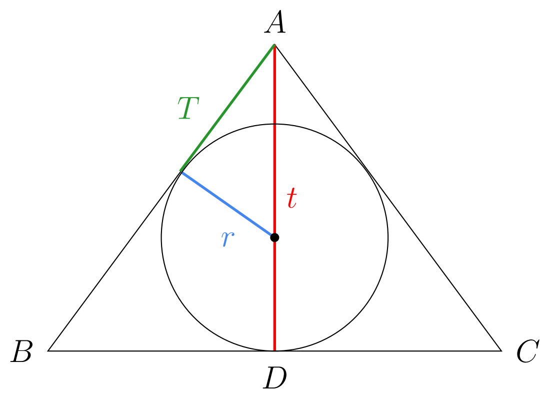


Figure 4.2.8: Auxiliary lines drawn on the Katayama Hiko Triangle problem.

KHT Proposition. *The Katayama Hiko Triangle Sangaku satisfies the necessary requirements of Neutral geometry.*

Proof. Construct the triangle and circle as defined, with $\triangle ABC$ having three sides of lengths 10, 12, and 10 for $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ respectively. Divide $\triangle ABC$ into two triangles by drawing the angle bisector of A, which we call t. Denote the location where the angle bisector intersects the base of the triangle as point D. By the definition of an incircle, we see that the center of the circle lies on h. Using the Isosceles Triangle Theorem (Theorem 3.2.1), we $\angle ABD \cong \angle ACD$. As $\overline{AB} \cong \overline{AC}$ from the Isosceles Triangle Theorem, both $\angle BDA$ and $\angle CDA$ are right angles. Similarly, $\overline{AD}$ acts as the perpendicular bisector of $\overline{BC}$, so $\overline{BD} = \overline{DC} = \dfrac{BC}{2}$. Because of Theorem 3.9.1, the Side Angle Side Congruence, both $\triangle ABD$ and $\triangle ACD$ are congruent. Thus, we find that the Katayama Hiko Triangle problem satisfies Neutral geometry. $\square$

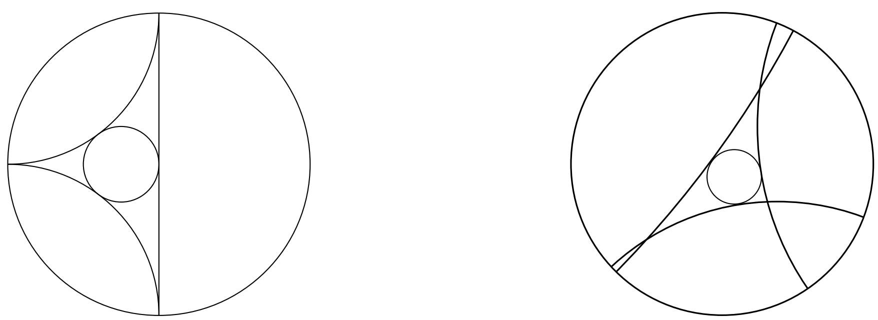
KHT Hyperbolic calculation. Before we can calculate r, we must first actually see the length r in the figure somewhere. We can draw the radius from the center of the circle inside either of $\triangle ABD$ or $\triangle ACD$, so without loss of generality, we choose $\triangle ABD$, and draw a radius of length r to the tangent point along $\overline{AB}$, denoting this length to A as T. Now, we have a right triangle with sides of length r, T and $t - r$, with $t - r$ acting as the hypotenuse. Now, we may use the Hyperbolic Pythagorean Theorem, Theorem 3.6.2, and find that:

$$\cosh(t - r) = \cosh r \cosh T.$$

Applying hyperbolic identities, we have this equivalent to

$$r = \operatorname{arctanh}\left(\frac{\cosh t - \cosh T}{\sinh t}\right) \tag{4.2.3}$$

and we conclude that the size of the radius depends upon the length of the angle bisector. We see this through Figure 4.2.9, where we see two of many different sizes for the circle.

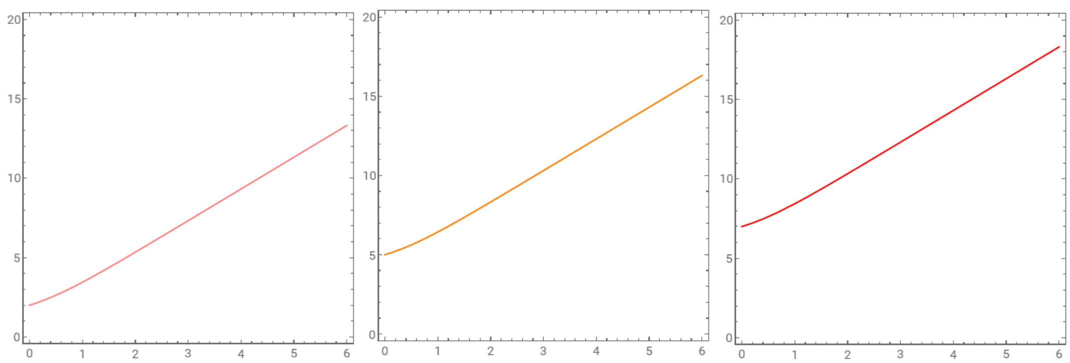


(a) One possible construction of Katayama Hiko Triangle.

(b) Another model for the Katayama Hiko Triangle.

Figure 4.2.9: Two hyperbolic visualizations of Katayama Hiko Triangle *sangaku*.

Thus, in attempting to find values for which r is valid, we will fix T and plot r vs t. Since T is a length along $\overline{AB}$, we know that $T < 10$, as that is the length of $\overline{AB}$. We turn our attention to Figure 4.2.10, where three separate values for T are explored: $T = 2$, $T = 5$, and $T = 7$, respectively. Notably, each graph shows us that r is seemingly linear, with a minimum value of whatever T is. In comparison to our Euclidean result, $r = 3$, we find that for our three T samples, when $r = 3$, t is about 7.3 for $T = 2$, 10.3 when $T = 5$, and 12.3 at $T = 7$. Thus, in order to keep our Euclidean result valid in Hyperbolic geometry, we must change the triangle itself to fully circumscribe the circle, a notable difference from our previous two *sangaku*. Figure 4.2.9 shows us how changing the triangle changes the radius of the circle. Another difference between this problem and the other two are that we have lines instead of regions for the radius.



(a) Graph of Equation 4.2.3 for $T = 2$.

(b) Graph of Equation 4.2.3 for $T = 5$.

(c) Graph of Equation 4.2.3 for $T = 7$.

Figure 4.2.10: Graphs showing r (the horizontal axis) in terms of t for $T = 2, 5$, and 7, where t is the vertical axis.

4.3 Conclusion

When we examined *sangaku* outside of Euclidean geometry, we noticed two main points. First, the Euclidean answers the original authors gave are also potential solutions of our Hyperbolic inequalities or equations at the end of calculating. Second, we saw that all the possible results allowed for more than than just the simple answer given. We can use these findings to learn of what makes a culture interpret math in a specific way. Additionally, we can look at different assumptions made. Should we ever apply modern geometric findings to more *sangaku*, similar results would be found. This usage of math problems as an offering gives us a new perspective into the differences between how Europeans of the time and Japanese people of the time treated mathematics as a whole, even utilizing gender and age. As time progresses, hopefully *sangaku* will be given more credit to the wonderful geometric findings we can see throughout these tablets.

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