The advantages of algebra for learning proof

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Abstract: Calls to expand proof beyond geometry are usually based on the general benefits of proving for students, but this essay argues that proof should be extended to algebra because the representational systems in algebra are well-suited for understanding the generality and deductive flow of proofs.

Keywords: reasoning, proof, algebra, symbolic notation

Reasoning-and-proving—the broad practice of investigating phenomena, making and justifying conjectures, critiquing arguments, and constructing a formal proof—has long been recognized as a cornerstone of mathematics. It has also been recommended as a cornerstone of mathematics education (e.g., NCTM, 2009), not only because of its importance in mathematics but also because it promotes deep learning. Many educators call for reasoning-and-proving across all courses (e.g., Stylianides, 2007; Thompson et al., 2012) and we agree reasoning-and-proving is valuable beyond geometry, but not just to increase students' exposure. We argue reasoning-and-proving should be extended to algebra because algebra has specific features that can support students' learning of reasoning-and-proving in ways geometry typically does not.

Our position stems from two difficulties students often have with reasoning-and-proving:

1. Students think a general argument only proves the result for a single case (usually the case used as the arbitrary representative in the argument); and
2. Students often get confused about the direction of a conditional statement and assume what they are trying to prove (or think all statements in a proof argument are independently true).

We think these difficulties are expected in geometry given its representation systems, whereas algebra's representation systems can spur explicit conversations about these aspects of reasoning-and-proving.

STUDENTS THINK GENERAL ARGUMENTS ONLY PROVE THE RESULT FOR A SINGLE CASE

Chazan (1993) interviewed geometry students and found that, even when they comprehended the steps of a deductive proof about angles in a parallelogram, they thought it was only a proof of the angles in *that particular* parallelogram. In geometry textbooks, there are general claims and particular claims, but students may not realize the important difference between them (Otten et al., 2014).

To illustrate how this confusion is linked to the representational systems in geometry, imagine a particular parallelogram. Imagine we wanted to explore some specific features of this shape, like its side lengths, height, or angles. We might label the parallelogram, producing an image like Figure 1.

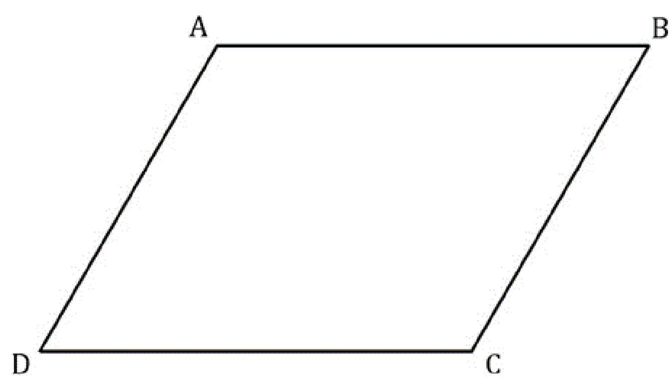

Figure 1. A particular parallelogram, or is it a generic instance representing all parallelograms?

Now, imagine we want to explore a conjecture about *all* parallelograms. We naturally draw an arbitrary parallelogram so we can tinker with it to test the conjecture (Cuoco et al., 2010), and we might label it so we can write down some statements. Although this is supposed to represent the set of all parallelograms, our image still looks basically like Figure 1.

Because of geometry's representational limitations, our generic parallelogram looks like a specific parallelogram, we just have to think about it in an entirely different way. Geometric technologies, thankfully, can allow us to drag shapes around to show shifts between infinitely-many specific triangles (Ng et al., 2020), but there still tends to be one shape displayed at a time and the generality has to occur mentally. Those comfortable with reasoning-and-proving can handle this shift from specific to generic quite fluently, but what about students learning this for the first time?

Algebra, in contrast, provides representations that explicitly distinguish between particular instances (10, -27, 1.0045) and generic representatives (x, y, n). Numerals and variables provide opportunities to explicitly see and discuss the difference in reasoning and can motivate both the usefulness of variables and the purpose of proof for establishing general truths or uncovering general structures.

STUDENTS GET CONFUSED BETWEEN HYPOTHESES, CONCLUSIONS, AND STATEMENTS ALONG THE WAY

Another phenomenon is students assuming the conclusion within their proof rather than deducing it, or thinking every step within the argument is independently true (Clements & Battista, 1992). For example, consider students who are proving that, in any parallelogram, the diagonals bisect one another. They may produce a two-column proof or some other record of their argument, but they are also likely to use something like Figure 2 as the central focus of their reasoning.

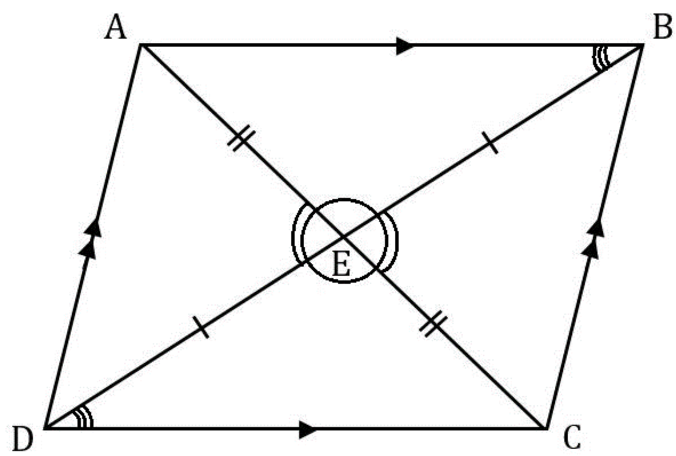

Figure 2. A parallelogram with several true properties marked.

Putting aside that this "arbitrary" parallelogram looks exactly like a specific parallelogram, students are also eventually faced with a situation where several true facts are marked in the diagram: ABCD is a parallelogram, there are two pairs of congruent angles, there are alternate-interior-angles that are also congruent, and, importantly, the diagonals AC and BD have been split into congruent pieces. Students, rightly, notice that all of these marked aspects are true. But students can get confused about the underlying logical dependency—some things are true only as a result of other things being true. Perhaps most important is the fact that the diagonals being bisected was only known to be true *based on* the original fact that ABCD was a parallelogram.

Students may have kept a separate record of what they started with and where they ended, but that record of dependencies is not visually encoded in Figure 2 itself even though the various true facets are encoded there (Dimmel & Herbst, 2015). Someone adept at reasoning-and-proving could discern the logical ordering, but what about students learning this for the first time? Or students who may have difficulty holding all those steps in their mind without the support of a clear visual aid? Furthermore, it does not help that Figure 2 could just as well be the marked diagram for the proof of the converse statement.

This logical confusion can be directly addressed in algebra where there are different representational norms. In algebra, we keep a written record of key transformations and manipulations. Consider the argument in Figure 3 that any monic quadratic equation can be written as a squared term plus a constant.

$$\begin{aligned} y &= x^2 + bx + c \\ &= x^2 + bx + \frac{b^2}{4} - \frac{b^2}{4} + c \\ &= \left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c \\ &= \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) \end{aligned}$$

Figure 3. An algebraic argument with a directed logical flow.

Like the parallelogram example, this representation includes several true things and excludes their justifications (but those justifications could be added if one wished, just as a two-column proof could be added to complement the markings in Figure 2). The key advantage in algebra is that the logical flow of statements is visually available. We can see that the standard coefficients of b and c were the starting point and that a chain of reasoning led to the form where the vertex $\left(-\frac{b}{2}, c - \frac{b^2}{4}\right)$ is visible. The statements in the middle are also true and may even be interesting on their own (e.g., where did the $\frac{b^2}{4}$ come from?), but for this argument we can see they are part of a chain of reasoning leading to a revealing form of the equation. Furthermore, if we looked at a converse argument, rewriting vertex form into standard form, we could see some connections in the underlying structure of quadratics, but the symbolic representation would make it clear we worked in the other direction.

This algebraic record of the logical flow of statements also lends itself to a discussion of the need for justifications. Whereas in geometry many students feel that all the features of the diagram are independently true (they are already there in the diagram from the start), in algebra we must do some work to get to each new statement and so each of those steps calls for a reason.

CONCLUSION

Many teachers may already be making connections to reasoning-and-proving in algebra, such as asking students to provide a justification for each step of an equation manipulation. We encourage teachers to build on these advantages of algebraic representations by (1) in algebra, calling out how a truth about specific numbers can be rewritten as a truth about a general class of numbers through the power of variables, and (2) in geometry, adopting some way to harken back to numbers versus variables when contrasting a specific shape with a generic shape that represents an infinite class. Overall, we hope that these ideas have motivated more conversations in algebra and pre-algebra about reasoning-and-proving as a way to represent generalization and deduction.

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