Spring 5-2022

A Direct Method for Modeling and Simulations of Elliptic and Parabolic Interface Problems

Kumudu Janani Gamage
Old Dominion University, kgama002@odu.edu

A DIRECT METHOD FOR MODELING AND SIMULATIONS OF ELLIPTIC AND PARABOLIC INTERFACE PROBLEMS

by

Kumudu Janani Gamage
B.S. October 2009, University of Kelaniya, Sri Lanka
M.S. August 2014, Middle East Technical University, Northern Cyprus
M.S. August 2016, Old Dominion University

A Dissertation Submitted to the Faculty of
Old Dominion University in Partial Fulfillment of the
Requirements for the Degree of

DOCTOR OF PHILOSOPHY

COMPUTATIONAL AND APPLIED MATHEMATICS

OLD DOMINION UNIVERSITY
May 2022

Approved by:

Yan Peng (Director)

Richard D. Noren (Member)

Ruhai Zhou (Member)

Zhilin Li (Member)

ABSTRACT

A DIRECT METHOD FOR MODELING AND SIMULATIONS OF ELLIPTIC AND PARABOLIC INTERFACE PROBLEMS

Kumudu Janani Gamage
Old Dominion University, 2022
Director: Dr. Yan Peng

Interface problems have many applications in physics. In this dissertation, we develop a direct method for solving three-dimensional elliptic interface problems and study their application in solving parabolic interface problems. As many of the physical applications of interface problems can be approximated with partial differential equations (PDE) with piecewise constant coefficients, our derivation of the model is focused on interface problems with piecewise constant coefficients but have a finite jump across the interface. The critical characteristic of the method is that our computational framework is based on a finite difference scheme on a uniform Cartesian grid system and does not require an augmented variable as in the augmented approach. So the implementation of the method is easier to understand for non-experts in the area. The discretization of the PDE uses the standard seven-point central difference scheme for grid points away from the interface and a twenty-seven-point compact scheme that considers the jump discontinuities in the solution, flux, and jump ratio for grid points near or on the interface. As a result, the developed model can obtain second-order accuracy globally for both the solution and the solution's gradient. Moreover, our numerical experiment indicates that eigenvalues of the coefficient matrix of the resulting linear system for the finite difference scheme are located in the left half-plane, implicating our method's stability. We have also developed a model for solving two and three-dimensional parabolic interface problems using the Crank-Nicolson scheme and some modifications into the direct immersed interface method (IIM). The developed model can accurately capture the discontinuities in the solution across the interface and achieve second-order accuracy for both the solution and the solution's gradient in both space and time.

ACKNOWLEDGEMENTS

First, I would like to offer my sincere gratitude to my supervisor Dr. Yan Peng for her patience, understanding, guidance, and support throughout my Ph.D.

I am also very much obliged to Prof. Zhilin Li from North Carolina State University for his continuous support, advice, and immense knowledge on my research. I also would like to thank Dr. Richard D. Noren and Prof. Ruhai Zhou for helpful advice and support over the courses, their friendship, encouragement and concern throughout my Ph.D. studies, and insightful comments on my dissertation. Furthermore, I owe particular debts of gratitude to Dr. Gordon Melrose for his continuous support in my teaching assistantship. Moreover, I would like to express my gratitude to our former Graduate Program Director, Dr. Raymond Cheng, for his helpful guidance and support over the Ph.D.

I also offer many thanks to all my classmates, fellow graduate students, faculty, and staff members of our Department for their kindness, friendship, support, and providing a stimulating environment throughout my Ph.D. I give special thanks to Degi Betcher, my international student advisor, for her enormous help during my stay at Old Dominion University.

My heartfelt thanks go to Hema aunty for her assistance, love, delicious food, and encouragement in the United States of America. I also would love to express my sincere gratitude for Ven. Bandarawela Seelaviveka, Sachith Dissanayaka, Kurnia Foe, and the Sri Lankan community in Norfolk for their constant friendship and help through tough times in my stay here in the United States America.

I also would like to express gratitude to all my teachers, especially Joy Malcolm, my Mathematics teacher, who helped me overcome challenges in Mathematics education and convinced me to choose the mathematics stream as my higher education path. I also would like to thank all my former professors at the University of Kelaniya, Sri Lanka, and the Middle East Technical University at Nothern Cyprus for their guidance, constant support in my undergraduate and master's degree, and preparing me so well for my Ph.D. journey. I wish them a long, happy and successful life.

I want to thank my husband, Manula Pathirana, for his support, love, and encouragement offered during the years. I also thank my loving son Maxwell and daughter Alexandria for bringing love and comfort to my life. Finally, I want to thank and dedicate this dissertation to my parents Gunadasa Gamage and Padumawathi Gamage, for their unconditional love, encouragement, and long-time confidence throughout my life and studies.

TABLE OF CONTENTS

Page

LIST OF TABLES

Table Page

LIST OF FIGURES

Figure Page

CHAPTER 1

INTRODUCTION

Interface problems have taken attention due to their wide range of applications in the real world. Examples of interface problems often seen in the real world are bubble formation in fluid dynamics [1], electromigration of voids [2], glacier prediction [3], growth of internal blood clots [4], heat propagation in different materials in thermodynamics, and Stefan problems and crystal growth [5] and many others.

Mathematically, interface problems usually lead to partial differential equations (PDE) whose input data are discontinuous or singular across the interfaces in the solution domain [6]. Due to the presence of these discontinuities or singularities across the interface, solutions to these problems may be non-smooth or even discontinuous across those interfaces [6]. Therefore, many standard numerical methods designed for smooth solutions work poorly for solving interface problems. Over the past years, various numerical methods for solving interface problems have been developed. In this dissertation, we study the accuracy, efficiency, and stability of immersed interface method (IIM) for solving elliptic and parabolic interface problems.

Below we discuss the two main problems we are interested in solving in this dissertation.

1.1 MODEL PROBLEMS

1.1.1 ELLIPTIC INTERFACE PROBLEMS

We are interested in solving elliptic interface problem of the form,

$$\nabla \cdot (\beta(\mathbf{x})\nabla u(\mathbf{x})) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega \backslash \Gamma, \tag{1}$$

$$[u](\mathbf{X}) = w(\mathbf{X}), \quad \mathbf{X} \in \Gamma, \tag{2}$$

$$[\beta u_{\boldsymbol{n}}](\mathbf{X}) = v(\mathbf{X}), \quad \mathbf{X} \in \Gamma, \tag{3}$$

with given boundary conditions on $\partial\Omega$, where, Γ is a smooth interface in the domain Ω and the interface Γ divides the domain Ω into two subdomains Ω^+ and Ω^- and therefore, $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. See Figure 1 for an illustration. $\mathbf{X}$ is a point on the interface Γ, $\mathbf{x}$ is a point in Ω and $\boldsymbol{n}$ is the unit outward normal vector to the interface at the point $\mathbf{X}$. The

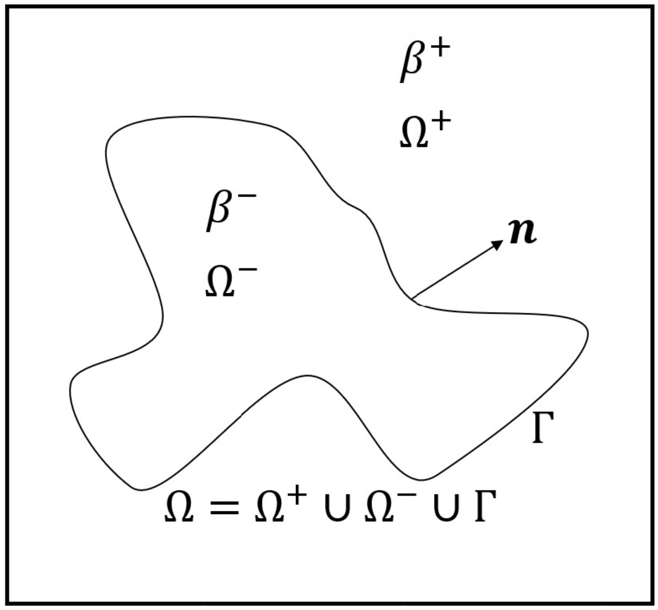


Fig. 1. A diagram of a rectangular domain Ω with a smooth interface Γ.

superscript $+$ or $-$ denotes the limiting value of a function from one side or the other of the interface. Here, $[u] = [u](\mathbf{X}) = u^+(\mathbf{X}) - u^-(\mathbf{X})$ is the jump in the solution at $\mathbf{X}$ and $u_{\boldsymbol{n}} = \boldsymbol{n}.\nabla u = \frac{\partial u}{\partial \boldsymbol{n}}$ is the normal derivative of solution u. Moreover, the coefficient β is defined as,

$$\beta(\mathbf{x}) = \begin{cases} \beta^-(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \Omega^-, \\ \beta^+(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \Omega^+. \end{cases} \tag{4}$$

In general, for these kinds of elliptic interface problems, if coefficient β is discontinuous or source term f is singular across the interface, solution or its derivatives across the interface may be non-smooth or discontinuous. For example, if coefficient β in the equation (1) is discontinuous with source term f being continuous across the interface, then the solution u and flux βu_n are continuous across the interface, making $[u] = 0$ and $[\beta u_{\boldsymbol{n}}] = 0$, however the derivative of the solution in the normal direction to the interface is discontinuous [7]. Moreover, if β is continuous, but source term f has a δ-function singularity along with the interface, then $[u] = 0$ and $[\beta u_n]$ will be no longer be zero; instead, it will be equal to the strength of the source term [7].

Since many applications of elliptic interface problems have a piecewise constant coefficient, our goal of the dissertation is to develop a model to solve three-dimensional elliptic interface problems of the form equation (1) with a piecewise constant coefficient.

1.1.2 PARABOLIC INTERFACE PROBLEMS

We are also interested in solving two and three-dimensional parabolic interface problems of the form,

$$u_t = \nabla \cdot (\beta(\mathbf{x}) \nabla u(\mathbf{x}, t)) - f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \backslash \Gamma, \tag{5}$$

$$[u](\mathbf{X}) = w(\mathbf{X}, t), \quad \mathbf{X} \in \Gamma, \tag{6}$$

$$[\beta u_{\boldsymbol{n}}](\mathbf{X}) = v(\mathbf{X}, t), \quad \mathbf{X} \in \Gamma. \tag{7}$$

1.2 REVIEW OF NUMERICAL METHODS FOR INTERFACE PROBLEMS

Interface problems often originate in fluid dynamics, electromagnetism, molecular biology, and material science. Consequently, one can find many different approaches to find the numerical solution of such problems in the literature [8–12].

To solve interface problems, we can use a body-fitted grid [13–15], or a Cartesian grid [10, 16, 17]. One of the many advantages of choosing a Cartesian grid over the body-fitted grid is that grid generation is easy. Moreover, if the Cartesian grid is used, we can take advantage of many available software packages such as fast Poisson solvers [18], the Clawpack [19] and the Amrclawpack [20], the level set method [21–23], the structured multigrid solvers [24, 25], the immersed boundary method [8], and many others [26].

Below is a review of a few popular methods that use finite difference methods for interface problems on a uniform Cartesian grid system.

1.2.1 POPULAR FINITE DIFFERENCE METHODS FOR INTERFACE PROBLEMS

1. The smoothing method

In this method, coefficients are smoothed by introducing a Heaviside function which is both continuous and smooth at some ϵ radius from the interface. We present the idea through a one-dimensional case.

Lets first assume that $\beta(x)$ has a finite jump at the interface $x = \alpha$. Then we can define,

$$\beta(x) = \begin{cases} \beta^-(x) & \text{if} \quad x < \alpha, \\ \beta^+(x) & \text{if} \quad x > \alpha. \end{cases} \tag{8}$$

Then, we can smooth out the β using,

$$\beta_\epsilon(x) = \beta^-(x) + (\beta^+(x) - \beta^-(x))H_\epsilon(x - \alpha), \tag{9}$$

where, $H_\epsilon(x)$ is the smoothed Heaviside function,

$$H_\epsilon(x) = \begin{cases} 0 & \text{if} \quad x < -\epsilon, \\ \frac{1}{2}(1 + \frac{x}{\epsilon} + \frac{1}{\pi}\sin\frac{\pi x}{\epsilon}) & \text{if} \quad |x| \leq \epsilon, \\ 1 & \text{if} \quad x > \epsilon. \end{cases} \tag{10}$$

and, ϵ is a small number that depends on the mesh size. The coefficient of the sine function in $H_\epsilon(x)$ definition is chosen so that the $H_\epsilon(x)$ is continuously differentiable at $x = \pm\epsilon$. The smoothing method is easy to implement for one-dimensional interface problems. However, it is not easy to implement in two and three-dimensions unless the interface is represented using a zero-level set of a Lipschitz continuous function. Furthermore, the method is not very accurate as it smooths the coefficient β resulting in smearing the solution at the interface [27].

2. The harmonic averaging method

In this method, accurate coefficients are calculated by taking the harmonic averages of them on a uniform grid system. The method is more accurate than the smoothing method. See examples in [28–30]. Consider the following one-dimensional model problem,

$$(\beta u_x)_x = f(x), \tag{11}$$

which can be discretized as,

$$\frac{\beta_{i+\frac{1}{2}}(u_{i+1} - u_i) - \beta_{i-\frac{1}{2}}(u_i - u_{i-1})}{h^2} = f(x_i), \tag{12}$$

where $h = x_i - x_{i-1}$ is the step size of a uniform grid in x-direction. If the coefficient β is smooth, we can define $\beta_{i+\frac{1}{2}} = \beta(x_{i+\frac{1}{2}})$, where $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$. If β is discontinuous in (x_{i-1}, x_{i+1}), then the harmonic average of β is given by,

$$\beta_{i+\frac{1}{2}} = (\frac{1}{h}\int_{x_i}^{x_{i+1}} \beta^{-1}(x) \quad dx)^{-1}. \tag{13}$$

The method is second-order accurate for one-dimensional elliptic problems with $[u]_\alpha = [\beta u_x]_\alpha = [f]_\alpha = 0$ in maximum norm, primarily due to the cancellation of errors. However, it does not give second-order accuracy for the two-dimensional problems in general as the cancellations of errors are unlikely to occur for arbitrary interfaces [27]. Furthermore, accurate computation of the integral close to the interface is not trivial for a three-dimensional problem with discontinuous β.

3. Peskin's immersed boundary (IB) method

Peskin [31] developed immersed boundary method (IB) to model blood flow in human heart. Later, this method has been used to study the interaction between fluid flows and elastic membranes and many others [32–42]. Such problems involves studying interface problems with singular source terms. In his method, he used a discrete delta function to distribute a singular source to nearby grid points. Vast number of discrete delta functions can be found in the literature. However, commonly used delta functions are the hat function,

$$\delta_\epsilon(x) = \begin{cases} \frac{(\epsilon-|x|)}{\epsilon^2} & \text{if} \quad |x| < \epsilon, \\ 0 & \text{if} \quad |x| \geq \epsilon, \end{cases} \tag{14}$$

and Peskin's original discrete cosine delta function,

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{4\epsilon}(1 + \cos(\frac{\pi x}{2\epsilon})) & \text{if} \quad |x| < 2\epsilon, \\ 0 & \text{if} \quad |x| \geq 2\epsilon. \end{cases} \tag{15}$$

Figure 2 illustrates the diagram of these two delta functions. For some one-dimensional interface problems, one can obtain the second-order accuracy by using the discrete hat function given by equation (14) [43]. However, solutions obtained by using the second discrete delta function are only first-order accurate. This method is popular because of its robustness and easiness of implementation. The discontinuity is spread over several grid cells using the discrete delta function, and it smears the numerical solution near the interface. For higher dimensions, the discrete delta function is the product of one-dimensional discrete delta functions with $\delta_\epsilon(x, y) = \delta_\epsilon(x)\delta_\epsilon(y)$ [26]. There have been many improvements to the original Peskin's method to obtain higher-order accuracy with a minimal distribution of discontinuities or singular source terms over the computational grid [44–46]. Also, the IB package written in Python and MATLAB is accessible in [47].

All the discussed methods above result in a solution to an interface problem which smears at the interface as their approaches require either smearing the discontinuities in coefficients

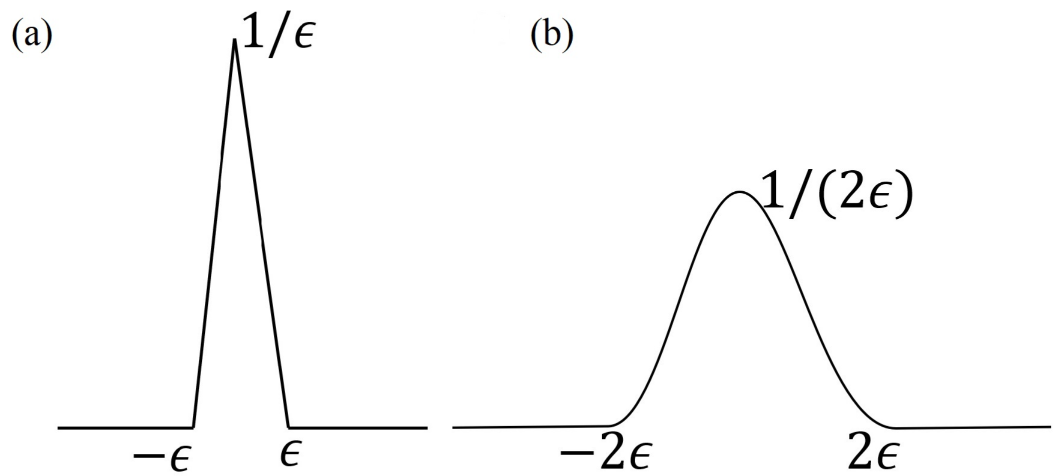


Fig. 2. Two particular discrete delta functions. (a) Discrete hat delta function, (b) Peskin's discrete delta function. This Figure is adapted from Ref. [6]

or singular sources. Consequently, these methods can get second-order accuracy using average norms such as L^1 or L^2 norm, but it is unlikely to get the same order of accuracy using point-wise norms as L^∞. This is because average norms usually do not reflect the errors correctly at or near the interfaces. Therefore we are interested in methods that give second-order accuracy using point-wise norms rather than the average norms.

With those issues we discussed in smooth interface methods, several sharp interface methods were developed. Among these methods, the immersed interface method (IIM) developed by Leveque and Li [48] has become more popular among the numerical community as it is the first sharp interface method that preserves the discontinuities in solution and the first second-order method for solving interface problems [49]. Also IIM has been applied to different linear and nonlinear problems including hyperbolic elliptic systems [50], elasticity systems [51, 52], Hele-Shaw flow [53], traffic flow [54], glacier prediction [3], simulations of porosity evolution in chemical vapor infiltration [55] and shape identification in inverse problems [56].

Below we discuss the main characteristics of IIM and its improvements over the past few years.

4. The immersed interface method (IIM)

Immersed interface method (IIM) [48] was primarily developed to solve linear elliptic and parabolic interface problems on a uniform Cartesian grid system. To preserve the discontinuities in solution or derivative across the interface, IIM uses the jump conditions across the interface by modifying the standard centered finite difference scheme only for the points closer to or at the interface. As the number of grid points that need this special attention is lower than the regular grid points, computational cost does not increase much due to these modifications.

Below we illustrate the key idea of IIM using a one-dimensional model problem with piecewise constant coefficient,

$$(\beta u_x)_x = f(x), \tag{16}$$

$$[u] = w, \quad [\beta u_{\boldsymbol{n}}] = v. \tag{17}$$

At regular grid points, whose three-point stencil do not cut through the interface, the finite difference scheme for IIM is the one with the standard three point stencil

$$\frac{(\beta_{i+\frac{1}{2}}(U_{i+1} - U_i) - \beta_{i-\frac{1}{2}}(U_i - U_{i-1}))}{h^2} = f_i, \tag{18}$$

where, $\beta_{i+\frac{1}{2}} = \beta(x(i + \frac{h}{2}))$, $f_i = f(x_i)$. But we have to find the finite difference scheme at irregular grid points, whose three-point stencil is cut through the interface. Let's take the case in which interface α lies in between the grid points x_i and x_{i+1} as an example. Then we can assume that the finite difference (FD) scheme for irregular grid points x_i and x_{i+1} are as follows,

$$\gamma_{i,1}U_{i-1} + \gamma_{i,2}U_i + \gamma_{i,3}U_{i+1} = f(x_i) + C_i, \tag{19}$$

$$\gamma_{i+1,1}U_i + \gamma_{i+1,2}U_{i+1} + \gamma_{i+1,3}U_{i+2} = f(x_{i+1}) + C_{i+1}. \tag{20}$$

Now we are going to explain how to find the coefficients, γ of the FD scheme in the equations (19) and (20). We do this by minimizing the magnitude of the local truncation error at each irregular grid point. First, let's write the local truncation error at the irregular point x_i,

$$T_i = \gamma_{i,1}u(x_{i-1}) + \gamma_{i,2}u(x_i) + \gamma_{i,3}u(x_{i+1}) - f(x_i) - C_i. \tag{21}$$

Then, do the Taylor expansion of $u(x_{i-1})$, $u(x_i)$ and $u(x_{i+1})$ at the interface α. The Taylor expansion of the irregular grid point x_i and x_{i+1} are given as follows,

$$u(x_{i+1}) = u^+(\alpha) + u_x^+(\alpha)(x_{i+1} - \alpha) + u_{xx}^+(\alpha)\frac{(x_{i+1} - \alpha)^2}{2} + O(h^3), \tag{22}$$

$$u(x_i) = u^-(\alpha) + u_x^-(\alpha)(x_i - \alpha) + u_{xx}^-(\alpha)\frac{(x_i - \alpha)^2}{2} + O(h^3). \tag{23}$$

Then the equation (22) is rewritten by replacing the limiting values from the plus side in terms of limiting values from the minus side using the jump relations (17) and the PDE (16). So the new expression for $u(x_{i+1})$ is given by,

$$\begin{aligned} u(x_{i+1}) = (u^-(\alpha) + w) + (\frac{v}{\beta^+} + \frac{\beta^- u_x^-(\alpha)}{\beta^+})(x_{i+1} - \alpha) \\ + \frac{\beta^- u_{xx}^-(\alpha)}{\beta^+} \frac{(x_{i+1} - \alpha)^2}{2} + O(h^3). \end{aligned} \tag{24}$$

From the PDE (16), when approaching α from left side,

$$\beta^- u_{xx}^- = f(\alpha). \tag{25}$$

So, using the equations (21)-(25), the local truncation error T_i at the irregular point x_i can be written as,

$$\begin{aligned} T_i = & (\gamma_{i,1} + \gamma_{i,2} + \gamma_{i,3}) u^-(\alpha) \\ & + ((x_{i-1} - \alpha)\gamma_{i,1} + (x_i - \alpha)\gamma_{i,2} + \frac{\beta^-}{\beta^+}(x_{i+1} - \alpha)\gamma_{i,3}) u_x^-(\alpha) \\ & + \frac{1}{2}((x_{i-1} - \alpha)^2\gamma_{i,1} + (x_i - \alpha)^2\gamma_{i,2} + \frac{\beta^-}{\beta^+}(x_{i+1} - \alpha)^2\gamma_{i,3}) u_{xx}^-(\alpha) - f(\alpha) - C_i + O(h). \end{aligned} \tag{26}$$

Since the interface is one dimension lower than the computational domain, it is enough to have $O(h)$ local truncation error at irregular grid points. So, by comparing the finite difference approximation with PDE from the minus side, we get,

$$\gamma_{i,1} + \gamma_{i,2} + \gamma_{i,3} = 0, \tag{27}$$

$$(x_{i-1} - \alpha)\gamma_{i,1} + (x_i - \alpha)\gamma_{i,2} + \frac{\beta^-}{\beta^+}(x_{i+1} - \alpha)\gamma_{i,3} = 0, \tag{28}$$

$$\frac{1}{2}(x_{i-1} - \alpha)^2\gamma_{i,1} + \frac{1}{2}(x_i - \alpha)^2\gamma_{i,2} + \frac{\beta^-}{2\beta^+}(x_{i+1} - \alpha)^2\gamma_{i,3} = \beta^-. \tag{29}$$

Once all these γ coefficients are found, one can find the correction term C_i as,

$$C_i = \gamma_{i,3}(w + (x_{i+1} - \alpha)\frac{v}{\beta^+}). \tag{30}$$

Similarly, we can compute the γ_{i+1}s by considering the local truncation error T_{i+1} at $x = x_{i+1}$. Then we can solve the linear system obtained from equations (18), (19) and (20), to approximate the solution for PDE (16) at all grid points.

Derivation of IIM for two and three dimensions can be done in a similar manner. As IIM integrates the jump discontinuities at the interface into its derivation, the method can achieve second-order accuracy for both the solution and for the gradient in L^∞ norm even if the interface is not aligned with the Cartesian grid points. When β is continuous, the resulting coefficient matrix of the FD scheme for two and three-dimensional elliptic interface problems is a block tridiagonal sparse matrix. Therefore, standard iterative methods such as SOR or the multigrid methods can be used to solve the system of linear equations efficiently [57]. However, for numerical examples with significant jump discontinuities in the coefficient β, the resulting linear system is usually ill-conditioned, and IIM may not converge or give accurate answers [57].

Due to the convergence issues described above in IIM for larger jump ratios, a fast immersed interface method (FIIM) is proposed to solve interface problems with piecewise constant coefficients but discontinuities at the interface. The following section introduces FIIM to solve interface problems with large ratios in the coefficient β.

5. The fast immersed interface method (FIIM)

In [17], a fast immersed interface method (FIIM), also called an augmented method, is proposed to solve the interface problems with piecewise constant coefficients. In this method, the elliptic equation is preconditioned before applying the original IIM. In addition, an unknown intermediate function for a jump in the normal derivative across the interface is introduced to take advantage of fast Poisson solvers on a rectangular region.

Below we illustrate the key idea of FIIM using a two-dimensional model problem with a piecewise constant coefficient.

Consider the following model problem,

$$\nabla \cdot (\beta(x,y)\nabla u) = f(x,y), \tag{31}$$

$$[u] = w, \quad [\beta u_{\boldsymbol{n}}] = v, \tag{32}$$

with specified boundary condition on $\partial\Omega$.

In this method, the coefficient β is given by,

$$\beta(x,y) = \begin{cases} \beta^- & \text{if} \quad (x,y) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x,y) \in \Omega^+. \end{cases} \tag{33}$$

Now as the coefficient β is peicewise constant, we can divide the equation (31) by β and reformulate the original equation into the following equivalent problem with an unknown augmented variable $[u_{\boldsymbol{n}}] = g$.

$$\begin{cases} \Delta u = \frac{f}{\beta^-} \quad \text{if} \quad (x,y) \in \Omega^-, \\ \Delta u = \frac{f}{\beta^+} \quad \text{if} \quad (x,y) \in \Omega^+, \end{cases} \tag{34}$$

$$[u] = w, \quad [u_{\boldsymbol{n}}] = g, \tag{35}$$

with the same boundary condition as in the original elliptic interface problem given. Then we can use standard five-point stencil for discretization of the equivalent problem together with some modification into the right hand side. The discrete form of the equation (31) is given as

$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} = \frac{f_{i,j}}{\beta_{i,j}} + C_{i,j}, \tag{36}$$

where $C_{i,j}$ is the correction term which depends on w and g.

We first have to select some control points $X_1, X_2, ..., X_{N_c}$ on the interface where N_c is the number of control points selected to apply the augmented method. let $\mathbf{W} = [W_1, W_2, ..., W_{N_c}]^T$ and $\mathbf{G} = [G_1, G_2, ..., G_{N_c}]^T$ be the discrete values of the jump conditions w and g at the control points. Let $B(\mathbf{W}, \mathbf{G})$ be a mapping from $\mathbf{W}$ and $\mathbf{G}$ to the correction term $C_{i,j}$. As $B(\mathbf{W}, \mathbf{G})$ is a linear combination of $\mathbf{W}$ and $\mathbf{G}$ at the control points, we can write

$$B(\mathbf{W}, \mathbf{G}) = B\mathbf{G} - B_1\mathbf{W}, \tag{37}$$

where, B and B_1 are matrices. Now the matrix form of the equation (36) can be written as

$$A\mathbf{U} + B\mathbf{G} \ = \mathbf{F} + B_1\mathbf{W} = \mathbf{F_1}, \tag{38}$$

where, $\mathbf{U}$ is the approximate solution to original problem and $\mathbf{F}$ is the vector representation of $\frac{f_{i,j}}{\beta_{i,j}}$. Now we need one more equation other than equation (38) to have a closed system since there are two unknown quantities $\mathbf{U}$ and $\mathbf{G}$ in equation (38). For that we can define a residual vector $\mathbf{R}$ at control points using the flux jump condition.

$$\mathbf{R}(\mathbf{G}) = [\beta \mathbf{U}_{\boldsymbol{n}}](\mathbf{G}) - \mathbf{V} = \beta^+ \mathbf{U}^+_{\boldsymbol{n}}(\mathbf{G}) - \beta^- \mathbf{U}^-_{\boldsymbol{n}}(\mathbf{G}) - \mathbf{V}, \tag{39}$$

where, $\mathbf{U}^+_{\boldsymbol{n}}$ and $\mathbf{U}^-_{\boldsymbol{n}}$ are the discrete approximations of the normal derivative from each side of the interface at control points. We find $\mathbf{G}$ such that $\mathbf{R}(\mathbf{G}) = 0$. Now for an approximate $\mathbf{G}$, we can obtain the solution $\mathbf{U}$ by solving equation (38). Then we can obtain $\mathbf{U}^+_{\boldsymbol{n}}$ and $\mathbf{U}^-_{\boldsymbol{n}}$ at control points by interpolating U_{ij}. For the interpolation, FIIM uses weighted least squares interpolation. As the interpolation is linear, normal derivative of the approximate solution at control points can be written as,

$$\frac{\partial \mathbf{U}}{\partial \boldsymbol{n}} = P^{\pm}\mathbf{U} + Q^{\pm}\mathbf{G} + S^{\pm}\mathbf{V} + T^{\pm} \ \mathbf{W}. \tag{40}$$

As vector $\mathbf{G}$ should satisfy the second interface condition $\beta^+\mathbf{U}_{\boldsymbol{n}}^+ - \beta^-\mathbf{U}_{\boldsymbol{n}}^- = \mathbf{V}$, we get the following matrix equation,

$$P\mathbf{U} + Q\mathbf{G} - S\mathbf{V} - T\mathbf{W} = 0, \tag{41}$$

where,
$P = \beta^+P^+ - \beta^-P^-$, $Q = \beta^+Q^+ - \beta^-Q^-$ and
$S = I + \beta^-S^- - \beta^+S^+$, $T = \beta^-T^- - \beta^+T^+$.
By combining the equations (38) and (41), we can get the following linear system

$$\begin{bmatrix} A & B \\ P & Q \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mathbf{F_1} \\ S\mathbf{V} + T\mathbf{W} \end{bmatrix}. \tag{42}$$

Now by eliminating $\mathbf{U}$ from equation (42), we can get a linear system for $\mathbf{G}$,

$$(Q - PA^{-1}B)\mathbf{G} = S\mathbf{V} + T\mathbf{W} - PA^{-1}\mathbf{F_1} = \mathbf{F_2}. \tag{43}$$

As $\mathbf{G}$ is defined only on the interface, corresponding coefficient matrix for $\mathbf{G}$ is only $N_c \times N_c$, which is much smaller than for $\mathbf{U}$. The generalized minimal residual (GMRES) iterative method is used to solve the Schur complement system for the augmented variable. FIIM has shown to be second-order accurate for both the solution and for solution gradient in maximum norm [17].

1.3 MOTIVATION AND DISSERTATION OUTLINE

Despite FIIM giving second-order accuracy for both solution and its gradient, its implementation is not trivial, and setting up the Schur complement system and solving it brings extra computational cost. Therefore, a new direct IIM method [58] for solving one and two-dimensional elliptic interface problems with variable coefficients without using an augmented variable was developed. As in FIIM, the PDE is reformulated at irregular grid points. They eliminated the need for an augmented variable using approximations for gradient and second-order derivatives at the interface as part of the finite difference scheme at the irregular grid points. As a result, the method is second-order accurate for both the solution and its gradient.

In this dissertation, we extend the Direct IIM method in [58] to three-dimensions to solve elliptic interface problems with piecewise constant coefficients followed by the convergence analysis. First, our study aims to obtain a direct method for solving the three-dimensional interface problems, which can achieve second-order accuracy in both solution and its gradient.

Then, we investigate the application of direct IIM to solve parabolic interface problems with piecewise constant coefficients by combining it with the Crank-Nicolson scheme.

The remaining part of the dissertation consists of five chapters. In chapter 2, we study how to reformulate the direct IIM method developed by [58] to solve two-dimensional elliptic interface problems with piecewise constant coefficients. The original method is applicable for any variable coefficients, including the piecewise constant coefficients. We want to adapt the model so that we can reduce the unnecessary complications in the implementation caused by the upwind type discretizations for having variable coefficients. We also run some numerical experiments for two-dimensional elliptic interface problems to demonstrate the accuracy of the direct IIM.

In chapter 3, we will develop a direct method for three-dimensional elliptic interface problems with piecewise constant coefficients but have a finite jump across the interface using finite difference discretization on a uniform Cartesian grid system. Here we explain in detail how to extend the two-dimensional model described in chapter 2 to three-dimensions. Also, we will numerically show that the convergence of the developed new model is second-order for both the solution and its gradient using some examples.

Chapter 4 develops a model for solving two-dimensional parabolic interface problems with piecewise constant but discontinuous coefficients. The model combines the Crank-Nicolson scheme with some modifications into the direct method described in chapter 2. We also investigate the order convergence and stability of the method by doing some numerical experiments.

Chapter 5 develops a model for solving three-dimensional parabolic interface problems with piecewise constant but discontinuous coefficients. The model combines the Crank-Nicolson scheme with some modifications into the direct method described in chapter 3. We also investigate the order of convergence and stability of the method by doing some numerical experiments.

Chapter 6 finalizes the dissertation by summarizing my results and outlines the future study areas.

CHAPTER 2

STUDY FOR TWO-DIMENSIONAL ELLIPTIC INTERFACE PROBLEMS WITH PIECEWISE CONSTANT COEFFICIENTS

In this chapter, we discuss how to adapt the original direct IIM in [58] to solve two-dimensional elliptic interface problems with piecewise constant coefficients. The original method in [58] is applicable for any variable coefficients, including the piecewise constant coefficients. We study how to reformulate the model to reduce the unnecessary complications in the implementation caused by the upwind type discretizations for having variable coefficients. We also do some numerical experiments for two-dimensional elliptic interface problems to demonstrate the accuracy of the direct IIM method.

2.1 ALGORITHM FOR TWO-DIMENSIONAL ELLIPTIC INTERFACE PROBLEMS WITH PIECEWISE CONSTANT COEFFICIENTS

We first assume that the coefficient β in the elliptic interface problem (1) is a positive piecewise constant, i.e.,

$$\beta(x,y) = \begin{cases} \beta^- & \text{if} \quad (x,y) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x,y) \in \Omega^+. \end{cases} \tag{44}$$

So that we can rewrite equation (1) as

$$\beta u_{xx} + \beta u_{yy} = f(x,y), \quad (x,y) \in \Omega \backslash \Gamma. \tag{45}$$

Since the coefficient β is a positive piecewise constant, we can divide it from both sides of the equation (45) to get the following equivalent problem:

$$u_{xx} + u_{yy} = \frac{f(x,y)}{\beta}, \quad (x,y) \in \Omega \backslash \Gamma, \tag{46}$$

with jump conditions across the interface Γ,

$$[u](\mathbf{X}) = w(\mathbf{X}), \quad [\beta u_{\boldsymbol{n}}](\mathbf{X}) = v(\mathbf{X}), \tag{47}$$

where $\mathbf{X} = (X, Y)$ is a point on the interface Γ.

For simplicity, we assume that the domain Ω is a square, say $[a, b] \times [a, b]$ and use a uniform grid with,

$$x_i = a + ih, i = 0, 1, ..., M; \quad y_j = a + jh, j = 0, 1, ..., M; \tag{48}$$

where $h = b - a/M$ is the step size in each direction. Interface Γ is represented by the zero level set of a Lipschitz continuous function $\phi(x, y)$

$$\Gamma = \{(x, y), \quad \phi(x, y) = 0, \quad (x, y) \in \Omega\}. \tag{49}$$

The position of any grid point relative to the interface Γ can be determined by using the level set function

$$\phi(x, y) < 0 \quad \text{for} \quad (x, y) \in \Omega^-, \tag{50}$$

$$\phi(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma, \tag{51}$$

$$\phi(x, y) > 0 \quad \text{for} \quad (x, y) \in \Omega^+. \tag{52}$$

In order to obtain the coefficients of the finite difference scheme at each grid point in the domain, we need to identify the regular and irregular grid points. Grid points whose five-point stencil cut through the interface are called the irregular grid points and those that are not are called the regular grid points. We can identify regular and irregular grid points using the level set function ϕ.

Let $\phi(x_i, y_j) = \phi_{i,j}$ be the level set grid function. At a grid point (x_i, y_j), we define,

$$\phi_{i,j}^{\max} = \max\{\phi_{i-1,j}, \phi_{i,j}, \phi_{i+1,j}, \phi_{i,j-1}, \phi_{i,j+1}\}, \tag{53}$$

$$\phi_{i,j}^{\min} = \min\{\phi_{i-1,j}, \phi_{i,j}, \phi_{i+1,j}, \phi_{i,j-1}, \phi_{i,j+1}\}. \tag{54}$$

A grid point (x_i, y_j) is called regular if $\phi_{i,j}^{\max}\phi_{i,j}^{\min} > 0$, otherwise it is called irregular. The interfacial points which cuts the grid line are called the control points. See Figure 3 for illustrations.

2.1.1 FINITE DIFFERENCE SCHEME FOR TWO-DIMENSIONAL INTERFACE PROBLEMS

Our intention is to obtain a finite difference scheme of the form

$$\sum_{m}^{n_s} \tau_m U_{i+i_m, j+j_m} = F_{ij}, \tag{55}$$

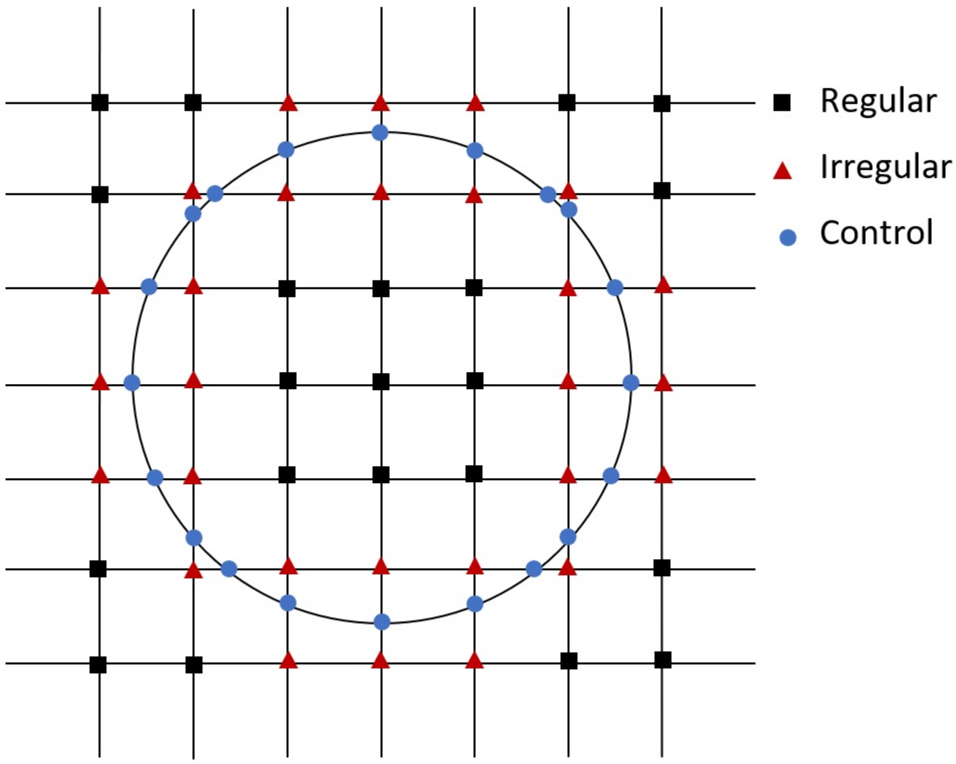


Fig. 3. Illustration of regular, irregular, and control points in a square domain with a circular interface. This Figure is adapted from Ref. [58].

at any grid point (x_i, y_j), where the summation is taken over n_s, the number of grid points centered at (x_i, y_j) and i_m and j_m takes values from $\{-1, 0, 1\}$. Here, $F_{ij} = F(x_i, y_j)$ is the right-hand side of the difference scheme which is computed as a contribution from source term, coefficient β and some correction terms. By finding proper coefficients τ_m, we desire the resulting finite difference scheme still be second-order accurate in both the solution and the gradient of the solution. Note that we have excluded the dependency of m on i and j for simplicity.

For a regular grid point (x_i, y_j), discretizeation of the equation (46) using the standard central difference scheme will result the standard five-point stencil of the form

$$\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} + \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} = \frac{f_{i,j}}{\beta_{i,j}}, \tag{56}$$

where $f_{i,j} = f(x_i, y_j)$, $\beta_{i,j} = \beta(x_i, y_j)$ and $U_{i,j}$ is the numerical solution at $u(x_i, y_j)$.

In order to derive the numerical scheme for irregular grid points, we assume that five-point stencil of the irregular grid point (x_i, y_j) is only cut through the right arm, say at $\mathbf{x}^* = (x_i^*, y_j^*) = (x_i^*, y_j)$. Without loss of generality, we also have assumed that $(x_i, y_j) \in \Omega^-$. See Figure 4 for an illustration.

Now we can consider the Taylor expansion of $u(x_{i-1}, y_j)$, $u(x_i, y_j)$ and $u(x_{i+1}, y_j)$ at the control point $\mathbf{x}^*$,

$$u(x_i, y_j) = u^-(x_i{}^*) + u_x^-(x_i - x_i{}^*) + u_{xx}^- \frac{(x_i - x_i{}^*)^2}{2} + O(h^3), \tag{57}$$

$$u(x_{i-1}, y_j) = u^-(x_i{}^*) + u_x^-(x_{i-1} - x_i{}^*) + u_{xx}^- \frac{(x_{i-1} - x_i{}^*)^2}{2} + O(h^3), \tag{58}$$

$$u(x_{i+1}, y_j) = u^+(x_i{}^*) + u_x^+(x_{i+1} - x_i{}^*) + u_{xx}^+ \frac{(x_{i+1} - x_i{}^*)^2}{2} + O(h^3). \tag{59}$$

In order to account for the jump discontinuities of the solution and its derivatives at the control point $\mathbf{x}^*$, we have to eliminate the values on the onside, say u^+, u_x^+, u_{xx}^+ in terms of other side u^-, u_x^-, u_{xx}^-. For example, u^+ can be written as

$$u^+ = [u] + u^-. \tag{60}$$

Now, we can obtain the central difference approximation for u_{xx} by combining the equations (57), (58), (59) and jump conditions in equation (47). Consider the discretization in the x-direction.

$$u_{xx}^- = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j} + C_{i,j}^x}{h^2}, \tag{61}$$

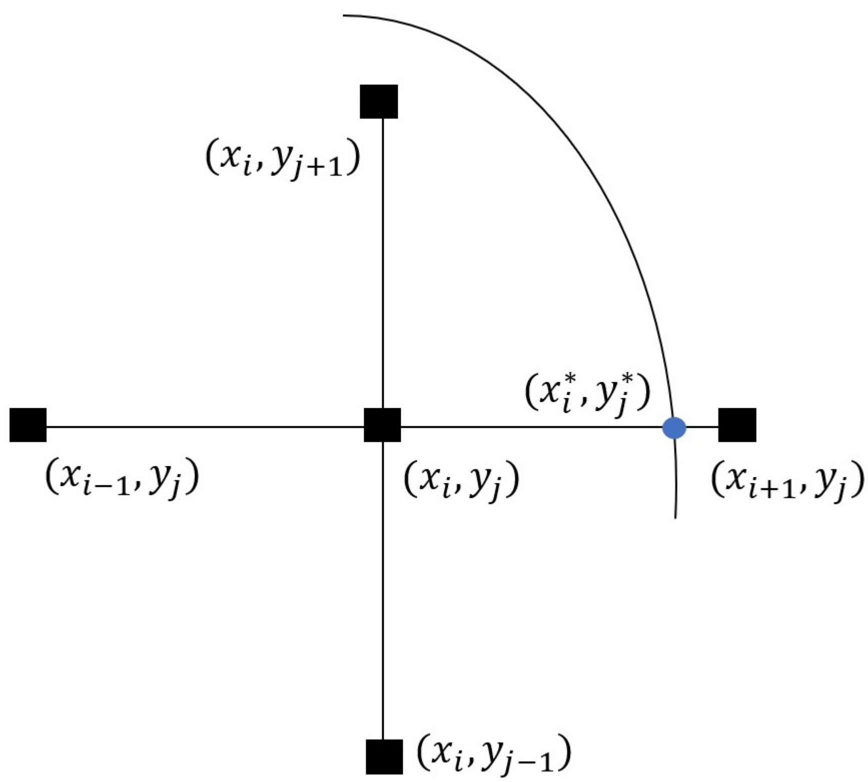


Fig. 4. A diagram of an irregular grid point whose five-point stencil only cuts through the right arm.

where, $C^x_{i,j}$ is the correction term in the x-direction. One can derive the formula for the correction term $C^x_{i,j}$ as,

$$C^x_{i,j} = -[u] - [u_x](x_{i+1} - x_i{}^*) - [u_{xx}]\frac{(x_{i+1} - x_i{}^*)^2}{2}. \tag{62}$$

Similarly, if the five-point stencil of the irregular grid point (x_i, y_j) also cuts from the top arm, say at $\mathbf{x}^{**} = (x_i^{**}, y_j^{**}) = (x_i, y_j^{**})$, we can obtain the central difference approximation for u_{yy} as,

$$u^-_{yy} = \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1} + C^y_{i,j}}{h^2}, \tag{63}$$

where,

$$C^y_{i,j} = -[u] - [u_y](y_{j+1} - y_j{}^{**}) - [u_{yy}]\frac{(y_{j+1} - y_j{}^{**})^2}{2}. \tag{64}$$

Note that, if the five-point stencil of the irregular point is only cut through the right arm, then $C^y_{i,j} = 0$. However, if the five-point stencil of the irregular point is cut more than once say from the right and the top, we have to correct the difference scheme both at the right and the top. So when that happens, the resulting finite difference scheme at the irregular grid point can be written as

$$\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} + \frac{C^x_{i,j}}{h^2} + \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{h^2} + \frac{C^y_{i,j}}{h^2} = \frac{f_{i,j}}{\beta_{i,j}}. \tag{65}$$

Now, you can see that, to obtain the difference scheme at irregular point (x_i, y_j), we need to know the jumps $[u]$, $[u_x]$, $[u_y]$, $[u_{xx}]$ and $[u_{yy}]$ at corresponding control points. Those jumps can be obtained by differentiating the known jumps $[u] = w$ and $[\beta u_{\boldsymbol{n}}]$ at the interface. To do so, it turns out that it would be very convenient if we do the local coordinate transformation at the control point $\mathbf{x}^*$ in the normal and tangential directions to the interface.

2.1.2 JUMP RELATIONS AND THE LOCAL COORDINATE TRANSFORMATION

First, let's assume that parametric representation of the interface Γ is given by

$$\Gamma = \{(x^*(s), y^*(s)), \quad x^*(s) \in C^2, \quad y^*(s) \in C^2\}, \tag{66}$$

where, s is an arc-length parameter. At a point (x^*, y^*) on the interface, let's define the local coordinate system in the normal and tangential direction as,

$$\xi = (x - x^*)\cos(\theta) + (y - y^*)\sin(\theta), \tag{67}$$

$$\eta = -(x - x^*)\sin(\theta) + (y - y^*)\cos(\theta), \tag{68}$$

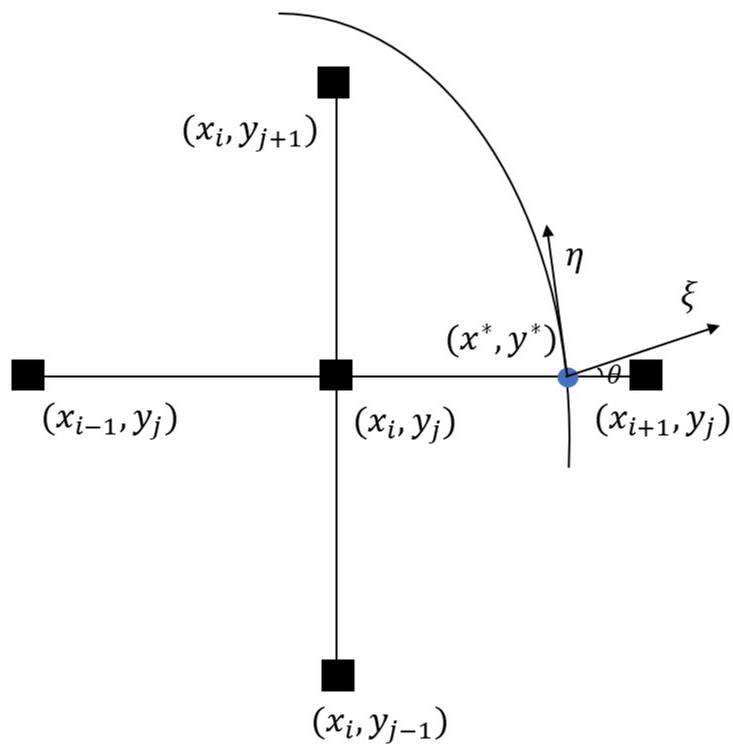


Fig. 5. A sketch of a two-dimensional local coordinates transformation. where, (x_i, y_j) is an irregular grid point, (x^*, y^*) is a control point on the x-axis, ξ and η are the normal and tangential direction at (x^*, y^*), and θ is the angle between x-axis and the normal direction.

where, θ is the angle between x-axis and the outward normal vector to the interface from Ω^+ direction. See Figure 5 for an illustration. In the neighborhood of the control point (x^*, y^*), the interface can be parameterized as,

$$\xi = \chi(\eta), \quad \text{with} \quad \chi(0) = 0, \quad \chi'(0) = 0. \tag{69}$$

Then curvature κ of the interface at (x^*, y^*) is $\chi''(0) = 0$.

As mentioned earlier, to derive the finite difference scheme at irregular grid points, we have to find the jumps $[u]$, $[u_x]$, $[u_y]$, $[u_{xx}]$ and $[u_{yy}]$ using the known jumps $[u]$ and $[\beta u_{\boldsymbol{n}}]$. To do so, we have to take the derivatives of the known jump conditions at the interface. To make those computations easier, we first consider the local coordinate transformation of the PDE.

Note that under the local coordinate transformation, PDE (46) remains the same. We would have a new notation for $u(x, y)$, $f(x, y)$ in the local coordinate transformation. Say, $\overline{u}(\xi, \eta) = u(x, y)$ and $\overline{f}(\xi, \eta) = f(x, y)$. For simplicity, we drop the bars and use the same notation in the local coordinate transformations as in the Cartesian coordinate system.

In local coordinates, we can write jump in the solution $[u] = w$ as

$$u^+(\chi(\eta), \eta) - u^-(\chi(\eta), \eta) = w. \tag{70}$$

Differentiating the equation (70) with respect to η gives

$$u_\xi^+ \chi' + u_\eta^+ - (u_\xi^- \chi' + u_\eta^-) = w_\eta = w', \tag{71}$$

or in compact form,

$$[u_\xi]\chi' + [u_\eta] = w'. \tag{72}$$

Differentiating the equation (72) with respect to η gives

$$[u_{\xi\xi}]\chi'^2 + 2[u_{\xi\eta}]\chi' + [u_\xi]\chi'' + [u_{\eta\eta}] = w''. \tag{73}$$

At a control point (x^*, y^*), $(\xi, \eta) = (0, 0)$, $\chi'(0) = 0$ and $\kappa = \chi''(0)$. Evaluating equations (72) and (73) at the control point (x^*, y^*),

$$[u_\eta] = w' = w_\eta, \tag{74}$$

$$[u_{\eta\eta}] = w'' - [u_\xi]\kappa = D_1. \tag{75}$$

Now, we can write $[\beta u_{\boldsymbol{n}}] = v$ in local coordinates as

$$[\beta u_\xi] = v. \tag{76}$$

Similarly taking the derivative of equation (76) with respect to η and evaluating the same at the control point (x^*, y^*), we get

$$[\beta u_{\xi\eta}] = [\beta u_\eta]\kappa + v' = D_2. \tag{77}$$

Now, we have all together five jump relations which are given by the equations (74)-(77) together with the jump in the solution $[u] = w$.

Also, from PDE (46) itself, we can obtain two more additional jump relations.

$$u_{xx}^+ + u_{yy}^+ = \frac{f^+}{\beta^+} = D_3, \tag{78}$$

$$u_{xx}^- + u_{yy}^- = \frac{f^-}{\beta^-} = D_4. \tag{79}$$

Now, we will do the coordinate transformation of jump relations (74)-(79) into Cartesian coordinates and express all limiting values in terms of u^-, u^-_x, u^-_y, u^-_{xx} and u^-_{xy}.

From jump condition $[u] = w$, we get

$$u^+ = u^- + w, \tag{80}$$

Using the coordinate transformation equations (67) and (68), the equations (76) and (74) become

$$u^+_x \cos(\theta) + u^+_y \sin(\theta) = \frac{v}{\beta^+} + \frac{\beta^-}{\beta^+}(u^-_x \cos(\theta) + u^-_y \sin(\theta)), \tag{81}$$

$$-u^+_x \sin(\theta) + u^+_y \cos(\theta) = w' - u^-_x \sin(\theta) + u^-_y \cos(\theta). \tag{82}$$

Then multiplying the equation (81) by $\sin(\theta)$ and equation (82) by $\cos(\theta)$, we get the following relation for u^+_y.

$$u^+_y = u^-_x(\rho - 1)sc + u^-_y(\rho s^2 + c^2) + cw' + \frac{sv}{\beta^+}, \tag{83}$$

where, $\rho = \frac{\beta^-}{\beta^+}$ is the jump ratio and s and c stands for $\sin(\theta)$ and $\cos(\theta)$ respectively.

Similarly, by multiplying equation (81) by $\cos(\theta)$ and the equation (82) by $-\sin(\theta)$ and adding them each other, we get a relation for u^+_x.

$$u^+_x = u^-_x(s^2 + \rho c^2) + u^-_y(\rho - 1)sc - sw' + \frac{cv}{\beta^+}. \tag{84}$$

From coordinate transformation of equation (75), we get

$$u^+_{xx}s^2 + u^+_{yy}c^2 - 2csu^+_{xy} = u^-_{xx}s^2 + u^-_{yy}c^2 - 2csu^-_{xy} + D_1. \tag{85}$$

Also, from coordinate transformation of equation (77), we get the following equation,

$$\beta^+(-scu^+_{xx} + (c^2 - s^2)u^+_{xy} + scu^+_{yy}) = \beta^-(-scu^-_{xx} + (c^2 - s^2)u^-_{xy} + scu^-_{yy}) + D_2. \tag{86}$$

By solving the linear equations (85), (86), (78) and equation (79), we can find u^+_{xx}, u^+_{xy}, u^+_{yy} and u^-_{yy}.

$$\begin{aligned} u^+_{xx} = {} & [s^4 + (4\rho - 2)s^2c^2 + c^4]u^-_{xx} + 2(\rho - 1)(s^3c - sc^3)u^-_{xy} + (s^2 - c^2)D_1 \\ & - (2sc/\beta^+)D_2 + c^2D_3 - [(2\rho - 1)s^2c^2 + c^4]D_4, \end{aligned} \tag{87}$$

$$\begin{aligned} u^+_{xy} = {} & 2(\rho - 1)(s^3c - sc^3)u^-_{xx} + (\rho s^4 - (2\rho - 4)s^2c^2 + \rho c^4)u^-_{xy} - 2scD_1 \\ & + [(c^2 - s^2)/\beta^+]D_2 + scD_3 - [\rho s^3c - (\rho - 2)sc^3]D_4, \end{aligned} \tag{88}$$

$$\begin{aligned} u^+_{yy} = {} & -[s^4 + (4\rho - 2)s^2c^2 + c^4]u^-_{xx} - 2(\rho - 1)(s^3c - sc^3)u^-_{xy} - (s^2 - c^2)D_1 \\ & + (2sc/\beta^+)D_2 + s^2D_3 + [(2\rho - 1)s^2c^2 + c^4]D_4, \end{aligned} \tag{89}$$

$$u^-_{yy} = -u^-_{xx} + D_4. \tag{90}$$

2.1.3 APPROXIMATION OF THE CORRECTION TERMS

In this section, we will discuss how to interpolate the correction terms $C^x_{i,j}$ and $C^y_{i,j}$ in the finite difference scheme for an irregular grid point.

For a given irregular grid point (x_i, y_j), we first select a point $\mathbf{x}^* = (x^*_i, y^*_j)$ on the interface Γ which is closest to (x_i, y_j).

Without loss of generality, let's first assume that (x_i, y_j) is an irregular point that lies inside the interface, and its five-point stencil is only cut through the right arm as in Figure 4.

As we are seeking the difference scheme at an irregular point to be in the form of equation (55), we expand each $u(x_{i+i_m}, y_{j+j_m})$ about the control point $\mathbf{x}^* = (x^*_i, y^*_j)$.

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}) = u^{\pm} + (x_{i+i_m} - x^*_i)u^{\pm}_x + (y_{j+j_m} - y^*_j)u^{\pm}_y + \frac{1}{2}(x_{i+i_m} - x^*_i)^2 u^{\pm}_{xx} \\ + \frac{1}{2}(y_{j+j_m} - y^*_j)^2 u^{\pm}_{yy} + (x_{i+i_m} - x^*_i)(y_{j+j_m} - y^*_j)u^{\pm}_{xy} + O(h^3), \end{aligned} \tag{91}$$

where plus or minus sign is chosen depending on whether (x_{i+i_k}, y_{j+j_k}) lies in Ω^+ or Ω^-. Note that in order to incorporate the discontinuances at the interface, we can rewrite right-hand side of the equation (91) by eliminating limiting values in plus side by limiting values in minus side using relations (80), (83), (84), (87), (88), (89) and (90). Then the Taylor series expansion in equation (91) will contain u^-, u^-_x, u^-_y, u^-_{xx}, and u^-_{xy}.

Now the equation (91) will be in the form,

$$u(x_{i+i_m}, y_{j+j_m}) = d^1_m u^- + d^2_m u^-_x + d^3_m u^-_y + d^4_m u^-_{xx} + d^5_m u^-_{xy} + d^6_m, \tag{92}$$

where $i_m, j_m = \{-1, 0, 1\}$. Here, the coefficients d^1_m through d^6_m are known. Now, we can see the correction term $C^x_{i,j}$ can also be reformulated as

$$C^x_{i,j} = e_1 u^- + e_2 u^-_x + e_3 u^-_y + e_4 u^-_{xx} + e_5 u^-_{xy} + e_6. \tag{93}$$

Interpolation for $C^y_{i,j}$ will be in the same form, only with coefficients changed. Here, the coefficients e_1 through e_6 in the equation (93) are also known.

We assume that the correction term $C^x_{i,j}$ can be approximated as

$$C^x_{i,j} = \sum_{m}^{n_s} \gamma_m U_{i+i_m, j+j_m} + \gamma_c. \tag{94}$$

As in [58], we take $n_s = 9$. See Figure 6 for an illustration of the compact nine-point stencil for the irregular point (x_i, y_j). By comparing equations (93) and (94), we get the following

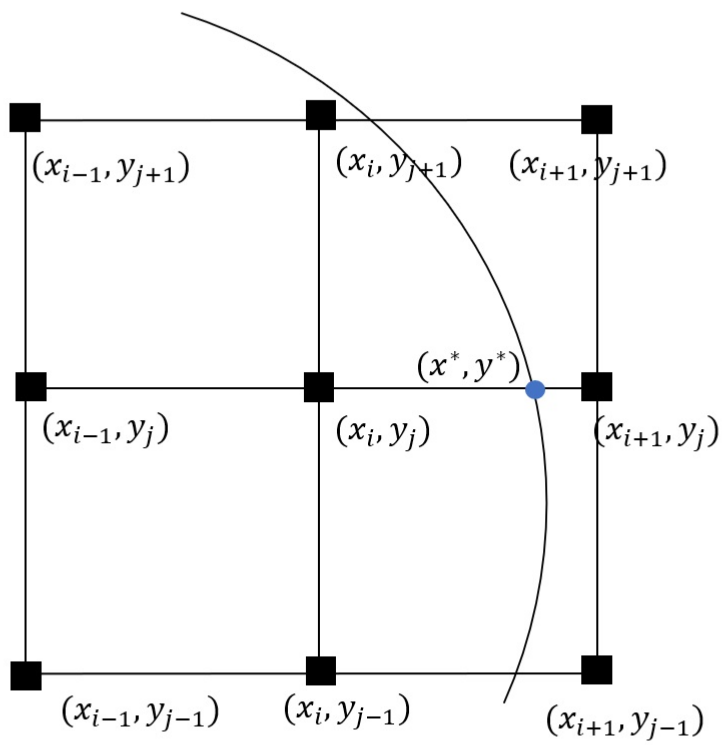


Fig. 6. A diagram of nine-point stencil for an irregular grid point (x_i, y_j) whose right arm cut through the interface at (x_i^*, y_j^*).

linear system

$$\begin{bmatrix} d_1^1 & d_2^1 & \cdots & d_3^1 & d_{n_s-1}^1 & d_{n_s}^1 \\ d_1^2 & d_2^2 & \cdots & d_3^2 & d_{n_s-1}^2 & d_{n_s}^2 \\ d_1^3 & d_2^3 & \cdots & d_3^3 & d_{n_s-1}^3 & d_{n_s}^3 \\ d_1^4 & d_2^4 & \cdots & d_3^4 & d_{n_s-1}^4 & d_{n_s}^4 \\ d_1^5 & d_2^5 & \cdots & d_3^5 & d_{n_s-1}^5 & d_{n_s}^5 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_{n_s-1} \\ \gamma_{n_s} \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}. \tag{95}$$

Here, we can see that the linear system (95) contains five equations with nine unknown γ coefficients. Since the number of unknowns in the linear system are greater than the number of equations, we have a underdetermined linear system. To solve the linear system, one can use the singular value decomposition (SVD) to get the minimum norm solution. Then we can find γ_c in equation (94) as follows

$$\gamma_c = e_6 - \sum_m d_m^6 \cdot \gamma_m. \tag{96}$$

At this stage, finite difference scheme at the irregular grid point (x_i, y_j) is fully determined. Notice that correction terms are only needed at the irregular grid points. Hence the additional cost of determining correction terms is only $O(N)$ rather than that for the linear system solver of the finite difference equations, which is $O(N^2)$.

Now, the linear system of the finite difference equations can be written as

$$A_h \mathbf{U} = \boldsymbol{F}, \tag{97}$$

where, $\mathbf{U}$ is the numerical solution of equation (45), A_h is the discrete elliptic operator and $\boldsymbol{F}$ is the modified right-hand side that is constructed from the source term f, boundary conditions and correction terms. For constant β, the matrix A_h becomes discrete Laplacian, and only right-hand side vector $\boldsymbol{F}$ needs to be changed due to corrections terms. Therefore, for constant β, the above system of equations can be easily solved using a fast-Poisson solver. However, when β is piecewise constant, A_h becomes the standard five-point Laplacian at regular grid point and compact nine-point scheme at irregular grid points. Therefore, linear system (97) can be solved using some iterative methods like GMRES or using some direct solvers.

In order to recover the gradient of the solution at the interface, one can use the following formula

$$U_{\boldsymbol{n}}^{-} = \cos(\theta)U_x^{-} + \sin(\theta)U_y^{-}, \tag{98}$$

$$U_{\boldsymbol{n}}^{+} = \rho\cos(\theta)U_x^{-} + \rho\sin(\theta)U_y^{-} + v/\beta^{+}, \tag{99}$$

$$U_{\tau}^{-} = -\sin(\theta)U_x^{-} + \cos(\theta)U_y^{-}, \tag{100}$$

$$U_{\tau}^{+} = -\sin(\theta)U_x^{-} + \cos(\theta)U_y^{-} + w', \tag{101}$$

where, $\boldsymbol{n}$ and τ represent the normal and tangential directions to the interface at some control point. Equation (98) was obtained using the coordinate transformation defined in the equation (67) and equation (99) was obtained using flux jump condition (76). The equations (100) and (101) come from the equation (74). In order to interpolate these gradients from the solution, we will again assume that interpolation scheme will be in the same form as equation (94). where the point (x_i, y_j) is chosen to be the closest irregular point to the control point that is considered. So for example, when approximating $U_{\boldsymbol{n}}^{-}$, we can still use the resulting linear system (95) but by changing the right-hand vector of the same replaced by vector $[0, \cos(\theta), \sin(\theta), 0, 0, 0]^T$ and for $U_{\boldsymbol{n}}^{+}$ by replacing right side vector vector into $[0, \rho\cos(\theta), \rho\sin(\theta), 0, 0, v/\beta^{+}]^T$.

2.1.4 OUTLINE OF THE ALGORITHM

In this section, the outline of the algorithm for solving two-dimensional elliptic interface problems is given.

Step 1: Immerse the irregular domain (interface) into a square domain $\Omega = [a, b] \times [a, b]$ and represent the interface using zero level set function.

Step 2: Determine regular and irregular grid points and the location of control points using the level set grid function ϕ.

Step 3: Apply the standard 5-point central difference scheme at regular grid points.

Step 4: Solve the underdetermined linear system given by the equation (95) to calculate γ's in the correction terms $C_{i,j}^{x}$ and $C_{i,j}^{y}$ at irregular grid points. Then, find the compact 9-points scheme at irregular points.

Step 5: Solve the system of linear equations (97).

Step 6: Recover the gradients of the solution using equations (98) to (101).

2.2 NUMERICAL EXAMPLES

Here we show some numerical examples and perform some error analysis to show the performance of direct IIM method to solve two-dimensional elliptic interface problems. The errors are illustrated in L^∞ norm and order of convergence of the method is calculated using

$$r = \frac{1}{\log(2)}\log\frac{\| E_{2h} \|_\infty}{\| E_h \|_\infty}. \tag{102}$$

In all listed tables in this section, N is the number of the grid points in each direction. $E(u)$ is the maximum norm error of the numerical solution. $E(u_{\boldsymbol{n}})$ and $E(u_\eta)$ are the maximum norm error in the normal and tangential direction of the computed solution.

Example 2.1

In this example, both the solution $u(x, y)$ and its flux have a jump discontinuity. The differential equation is given by

$$(\beta u_x)_x + (\beta u_y)_y = f, \tag{103}$$

on the domain $\Omega = [-1, 1] \times [-1, 1]$, where interface Γ is a circle represented by the zero level set of the function

$$\phi(x, y) = x^2 + y^2 - 0.25. \tag{104}$$

The source term f is defined as

$$f(x, y) = \begin{cases} 0 & \text{if} \quad (x, y) \in \Omega^-, \\ -2\beta^+ \sin(x)\cos(y) & \text{if} \quad (x, y) \in \Omega^+. \end{cases} \tag{105}$$

The coefficient β in equation (103) is a piecewise constant given by

$$\beta(x, y) = \begin{cases} \beta^- & \text{if} \quad (x, y) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x, y) \in \Omega^+. \end{cases} \tag{106}$$

The jump in solution $u(x, y)$ and it's flux are given by

$$[u] = \sin(x)\cos(y) - (x^2 - y^2), \tag{107}$$

$$[\beta u_{\boldsymbol{n}}] = \beta^+(2x\cos(x)\cos(y) - 2y\sin(x)\sin(y)) - 4\beta^-(x^2 - y^2). \tag{108}$$

Here, numerical experiments were conducted for different β jumps. And the boundary conditions were taken from the exact solution of

$$u(x, y) = \begin{cases} x^2 - y^2 & (x, y) \in \Omega^-, \\ \sin(x)\cos(y) & (x, y) \in \Omega^+. \end{cases} \tag{109}$$

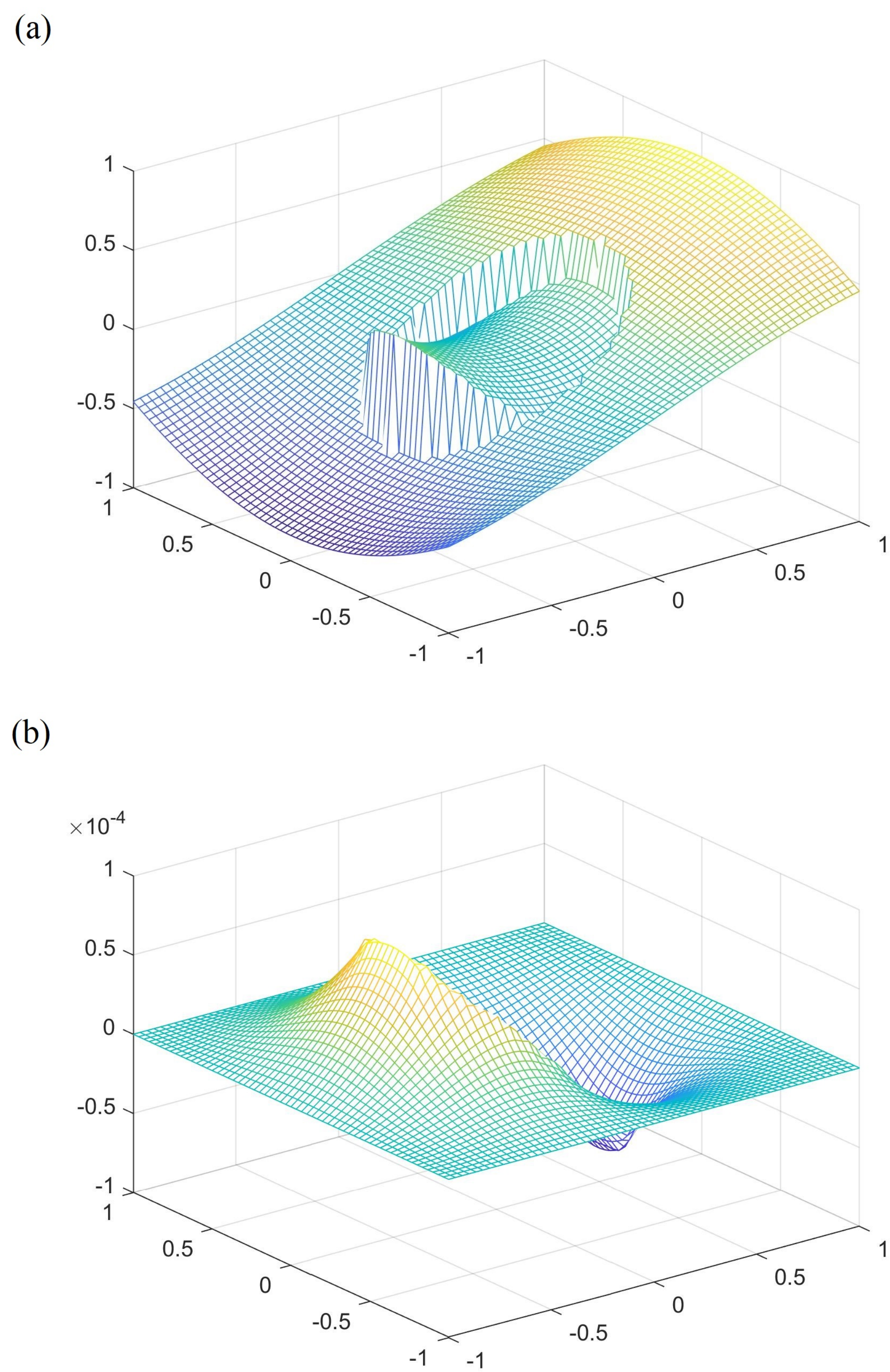


Fig. 7. (a) Plot of the numerical solution for Example 2.1, (b) Plot of the error of numerical solution with $N = 60$, $\beta^- = 1$ and $\beta^+ = 1000$.

Table 1
A grid refinement analysis for Example 2.1 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	r	$E(u_{\boldsymbol{n}})$	r	$E(u_\eta)$	r
20	1.41E-04		1.16E-02		6.22E-03	
40	3.01E-05	2.22	3.93E-03	1.57	1.42E-03	2.13
80	6.52E-06	2.21	1.41E-03	1.48	6.90E-04	1.04
160	1.46E-06	2.16	3.71E-04	1.92	2.56E-04	1.43
320	2.81E-07	2.38	6.72E-05	2.47	4.72E-05	2.44

Figure 7 shows a slice of the computed solution and its error distribution for Example 2.1 with a modest jump in β as 1000. In Tables 1 and 2, we present the numerical experiment results for grid refinement analysis of Example 2.1. Table 1 shows the results for $\beta^- = 1$ and $\beta^+ = 1$ and Table 2 shows it for $\beta^- = 1$ and $\beta^+ = 10000$. It can be seen that the solution and the solution's derivatives (normal and tangential) are on the average of second-order accuracy.

Table 2
A grid refinement analysis for Example 2.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	r	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order
20	6.59E-04		2.44E-02		7.08E-03	
40	1.85E-04	1.83	6.07E-03	2.01	5.03E-03	0.49
80	4.73E-05	1.96	1.28E-03	2.24	1.39E-03	1.86
160	1.21E-05	1.97	2.62E-04	2.29	4.39E-04	1.66
320	2.53E-06	2.25	4.30E-05	2.61	9.26E-05	2.25

Example 2.2

In this example, both the solution $u(x,y)$ and its flux have a jump discontinuity. The differential equation is given by

$$(\beta u_x)_x + (\beta u_y)_y = f, \tag{110}$$

on the domain $\Omega = [-1,1] \times [-1,1]$, where interface Γ is a circle represented by the zero level set of the function

$$\phi(x,y) = x^2 + y^2 - 0.25. \tag{111}$$

The source term f is defined as

$$f(x,y) = \begin{cases} -2\beta^- \cos(x+y) & \text{if} \quad (x,y) \in \Omega^-, \\ 6\beta^+(x+y) & \text{if} \quad (x,y) \in \Omega^+. \end{cases} \tag{112}$$

The coefficient β in equation (110) is a piecewise constant given by

$$\beta(x,y) = \begin{cases} \beta^- & \text{if} \quad (x,y) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x,y) \in \Omega^+. \end{cases} \tag{113}$$

The jump in solution $u(x,y)$ and it's flux are given by

$$[u] = \cos(x+y) - (x^3 + y^3), \tag{114}$$

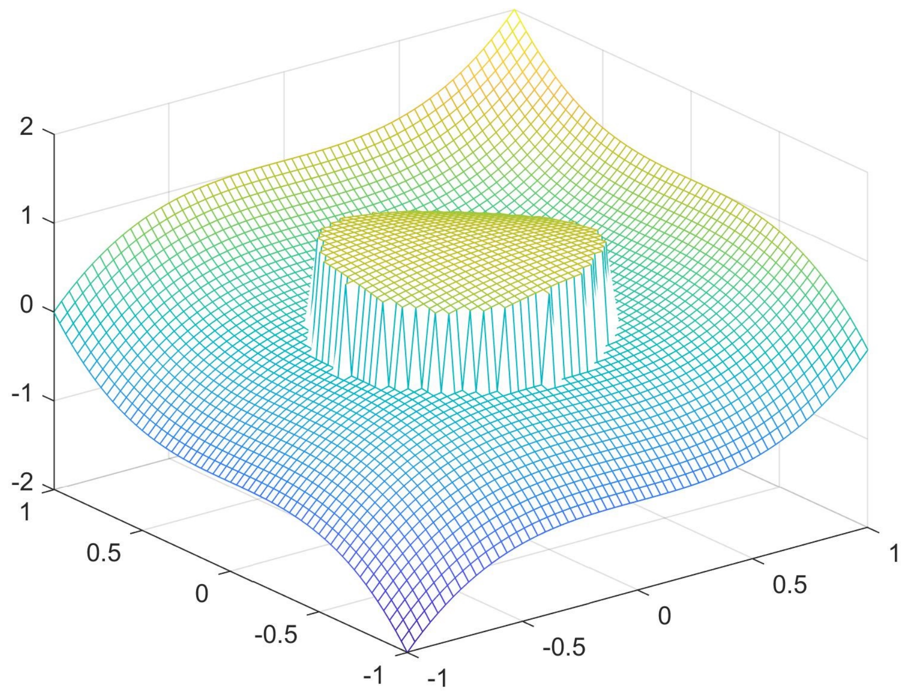
$$[\beta u_{\boldsymbol{n}}] = 6\beta^+(x^3 + y^3) + 2\beta^- \sin(x+y)(x+y). \tag{115}$$

The boundary conditions were taken from the exact solution of

$$u(x,y) = \begin{cases} \cos(x+y) & (x,y) \in \Omega^-, \\ x^3 + y^3 & (x,y) \in \Omega^+. \end{cases} \tag{116}$$

Figure 8 shows a slice of the computed solution and its error distribution for Example 2.2 with a modest jump in β as 1000. In Tables 3 and 4, we demonstrate the numerical experiment results for grid refinement analysis of Example 2.2. Table 3 shows the results for $\beta^- = 1$ and $\beta^+ = 1$ and Table 4 shows it for $\beta^- = 1$ and $\beta^+ = 10000$. It can be seen that the solution and the solution's derivatives (normal and tangential) are on the average of second-order.

(a)



(b)

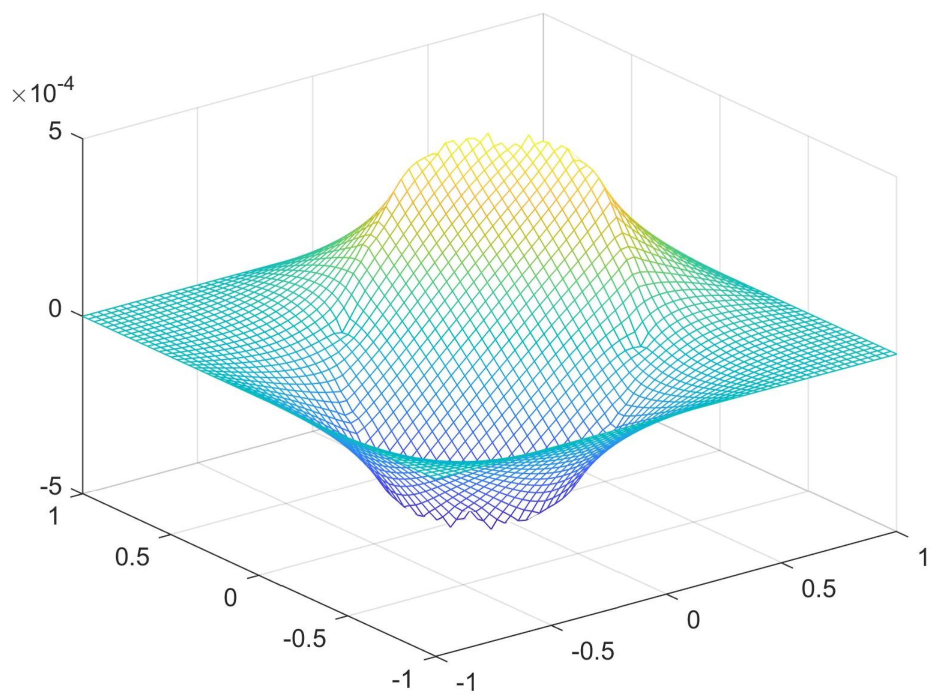


Fig. 8. (a) Plot of the numerical solution for Example 2.2, (b) Plot of the error of numerical solution with $N = 60$, $\beta^- = 1$ and $\beta^+ = 1000$.

Table 3
A grid refinement analysis for Example 2.2 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order
20	2.37E-03		6.43E-02		4.95E-02	
40	4.93E-04	2.26	1.60E-02	2.01	1.79E-02	1.46
80	1.03E-04	2.26	4.43E-03	1.85	4.10E-03	2.13
160	2.25E-05	2.19	1.16E-03	1.93	1.09E-03	1.91
320	4.79E-06	2.23	2.19E-04	2.41	2.69E-04	2.02

Table 4
A grid refinement analysis for Example 2.2 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order
20	5.01E-03		9.50E-02		5.17E-02	
40	1.21E-03	2.05	1.61E-02	2.56	2.87E-02	0.85
80	2.63E-04	2.20	4.04E-03	2.00	8.14E-03	1.82
160	6.54E-05	2.01	1.13E-03	1.84	2.06E-03	1.98
320	1.26E-05	2.37	2.06E-04	2.45	3.87E-04	2.41

CHAPTER 3

STUDY FOR THREE-DIMENSIONAL ELLIPTIC INTERFACE PROBLEMS WITH PIECEWISE CONSTANT COEFFICIENTS

In this chapter, we propose a direct method for solving three-dimensional elliptic interface problems. The model is an extension of the two-dimensional model described in chapter 2 to three-dimensions.

3.1 ALGORITHM DESCRIPTION FOR THREE-DIMENSIONAL INTERFACE PROBLEMS

Let's consider the following three-dimensional elliptic interface problem with a piecewise constant coefficient,

$$\beta(u_{xx} + u_{yy} + u_{zz}) = f(x, y, z), \quad (x, y, z) \in \Omega \backslash \Gamma, \tag{117}$$

$$[u](\mathbf{X}) = w(\mathbf{X}), \quad [\beta u_{\boldsymbol{n}}](\mathbf{X}) = v(\mathbf{X}). \tag{118}$$

For simplicity, we assume the domain Ω is a solid cube (See Figure 9), say $[a, b] \times [a, b] \times [a, b]$. For the discretization of the domain, consider the uniform grid system with

$$\begin{aligned} x_i &= a + ih, \quad i = 0, 1, ..., N, \\ y_j &= a + jh, \quad j = 0, 1, ..., N, \\ z_l &= a + lh, \quad l = 0, 1, ..., N, \end{aligned} \tag{119}$$

where $h = (b - a)/N$.

We intend to develop a finite difference scheme of the form

$$\sum_{m}^{n_s} \tau_m U_{i+i_m, j+j_m, l+l_m} = F_{ijl}, \tag{120}$$

at any grid point (x_i, y_j, z_l). where the summation is taken over n_s, the number of grid points centered at (x_i, y_j, z_l) and i_m, j_m and l_m take the values from $\{-1, 0, 1\}$. By finding the proper coefficients τ_m, we desire the resulting finite difference scheme to still be second-order accurate in both the solution and the gradient of the solution. Note that we have excluded the dependency of m on i, j and l for simplicity.

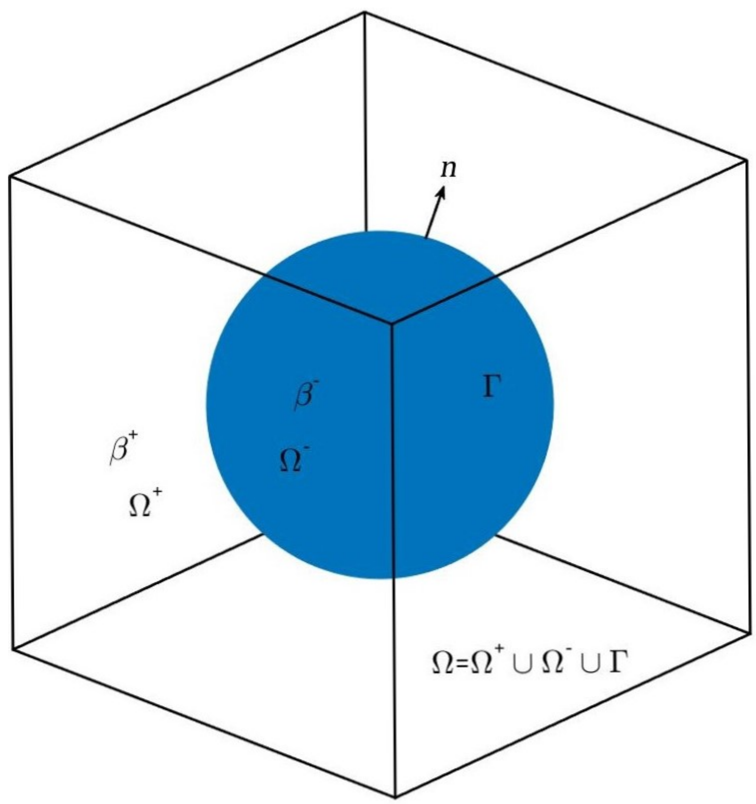


Fig. 9. A diagram of a cubic domain Ω with a smooth interface Γ. where $\boldsymbol{n}$ is the unit outward normal vector to the interface Γ.

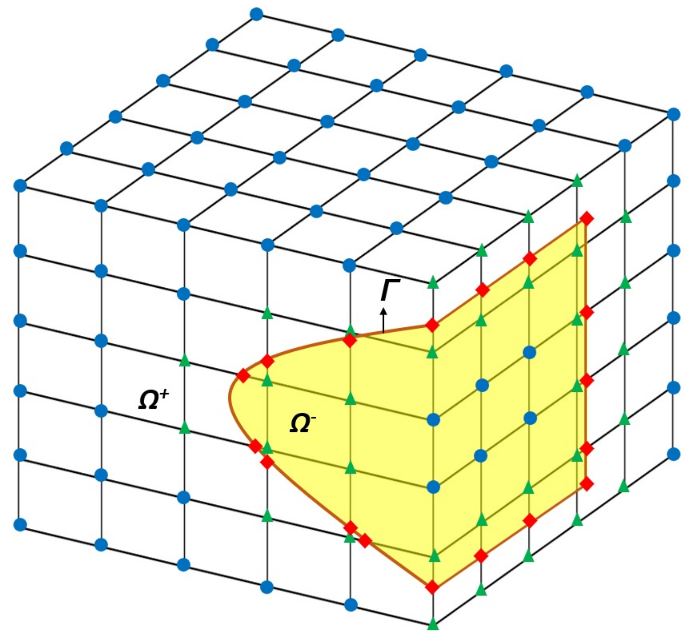


Fig. 10. A cross section illustration of domain Ω with a smooth interface Γ. where solid dots are regular grid points, solid triangles are irregular grid points and solid diamonds are the control points.

Also, we want to use the standard seven-point stencil at regular grid points. A point (x_i, y_j, z_l) is regular if the interface Γ does not cut the grid lines between any points in the standard seven-point stencil and is irregular otherwise. See Figure 10 for an illustration.

3.1.1 FINITE DIFFERENCE SCHEME FOR THREE-DIMENSIONAL INTERFACE PROBLEMS

For a regular grid point (x_i, y_j, z_l), after dividing both sides of the equation (117) by β, we can discretize the same using a standard central difference scheme, which has a seven-point stencil to get the finite difference scheme

$$\begin{aligned}\frac{U_{i-1,j,l}-2U_{i,j,l}+U_{i+1,j,l}}{h^2}+\\ \frac{U_{i,j-1,l}-2U_{i,j,l}+U_{i,j+1,l}}{h^2}+\\ \frac{U_{i,j,l-1}-2U_{i,j,l}+U_{i,j,l+1}}{h^2}\\ =\frac{f_{i,j,l}}{\beta_{i,j,l}},\end{aligned} \tag{121}$$

where $f_{i,j,l}=f(x_i,y_j,z_l)$, $\beta_{i,j,l}=\beta(x_i,y_j,z_l)$ and $U_{i,j,l}$ is the numerical solution at $u(x_i,y_j,z_l)$.

In order to derive the numerical scheme for irregular grid points, we do the Taylor expansion at a control point similar to the two-dimensional case and obtain the following resulting finite difference scheme

$$\begin{aligned}\frac{U_{i-1,j,l}-2U_{i,j,l}+U_{i+1,j,l}}{h^2}+C^x_{i,j,l}\\ +\frac{U_{i,j-1,l}-2U_{i,j,l}+U_{i,j+1,l}}{h^2}+C^y_{i,j,l}\\ +\frac{U_{i,j,l-1}-2U_{i,j,l}+U_{i,j,l+1}}{h^2}+C^z_{i,j,l}=\frac{f_{i,j,l}}{\beta_{i,j,l}}.\end{aligned} \tag{122}$$

where, $C^x_{i,j,l}$ is the correction term in the x-direction and so on. For a particular irregular grid point whose seven-point stencil cut through the right arm at the control point $\mathbf{x}^*=(x_i^*,y_j^*,z_j^*)$, the correction term $C^x_{i,j,l}$ is given by

$$C^x_{i,j,l}=\pm\frac{[u]}{h^2}\pm[u_x]\frac{(x_{i+1}-x^*)}{h^2}\pm[u_x]\frac{(x_{i+1}-x^*)^2}{2h^2}. \tag{123}$$

Here, plus or minus is chosen depending on which side the irregular grid point (x_i,y_j,z_l) lies. The correction terms in the y and z-directions also will be in the same format.

Now, you can see that, to obtain the difference scheme at irregular point (x_i,y_j,z_l), we need to know the jumps $[u]$, $[u_x]$, $[u_y]$, $[u_z]$, $[u_{xx}]$, $[u_{yy}]$ and $[u_{zz}]$. We must do this by differentiating the known jumps $[u]=w$ and $[\beta u_{\boldsymbol{n}}]$ at the interface. So to do so, we will first perform a local coordinate transformation into the normal and tangential direction to interface Γ at the control point $\mathbf{x}^*=(x_i{}^*,y_j^*,z_l^*)$ as we did for the two-dimensional case.

3.1.2 LOCAL COORDINATE TRANSFORMATION

At a given point $(x^*,y^*,z^*)\in\Gamma$, let ξ be the normal direction of Γ, η and ζ be two orthogonal directions tangential to Γ. Then, local coordinates in the normal and the tangential

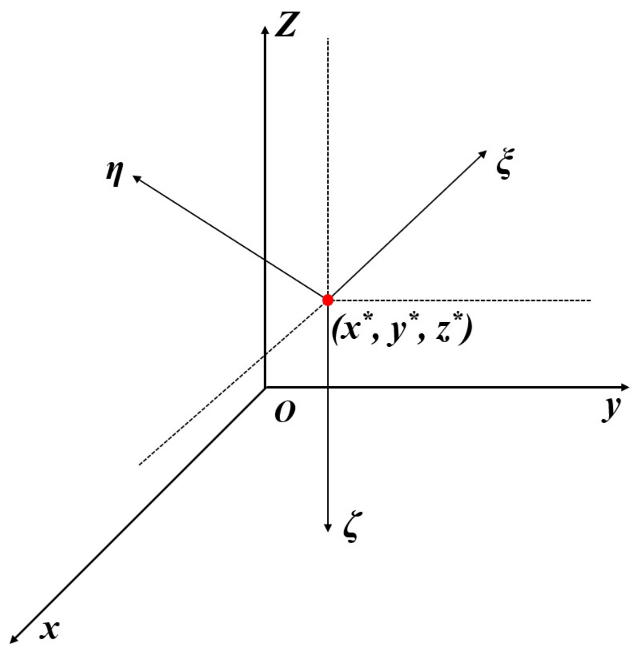


Fig. 11. A sketch of a three-dimensional local coordinate transformation.

directions are

$$\begin{aligned} \xi &= (x - x^*)\alpha_{x\xi} + (y - y^*)\alpha_{y\xi} + (z - z^*)\alpha_{z\xi}, \\ \eta &= (x - x^*)\alpha_{x\eta} + (y - y^*)\alpha_{y\eta} + (z - z^*)\alpha_{z\eta}, \\ \zeta &= (x - x^*)\alpha_{x\zeta} + (y - y^*)\alpha_{y\zeta} + (z - z^*)\alpha_{z\zeta}, \end{aligned} \tag{124}$$

where $\alpha_{x\xi}$ is the directional cosine between the x-axis and ξ and and so on. See Figure 11 for an illustration. The three-dimensional coordinate transformation given by the equation (124) can be written in the matrix form

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = A \begin{bmatrix} x - x^* \\ y - y^* \\ z - z^* \end{bmatrix}, \tag{125}$$

where,

$$A = \begin{bmatrix} \alpha_{x\xi} & \alpha_{y\xi} & \alpha_{z\xi} \\ \alpha_{x\eta} & \alpha_{y\eta} & \alpha_{z\eta} \\ \alpha_{x\zeta} & \alpha_{y\zeta} & \alpha_{z\zeta} \end{bmatrix}. \tag{126}$$

Then, local coordinate transformation of first derivative of any differentiable function $q(x, y, z)$ can be written as

$$\begin{bmatrix} \overline{q}_\xi \\ \overline{q}_\eta \\ \overline{q}_\zeta \end{bmatrix} = A \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}. \tag{127}$$

Also, local coordinate transformation of a second derivative of any differentiable function $q(x, y, z)$ can be written as

$$\begin{bmatrix} \overline{q}_{\xi\xi} & \overline{q}_{\xi\eta} & \overline{q}_{\xi\zeta} \\ \overline{q}_{\eta\xi} & \overline{q}_{\eta\eta} & \overline{q}_{\eta\zeta} \\ \overline{q}_{\zeta\xi} & \overline{q}_{\zeta\eta} & \overline{q}_{\zeta\zeta} \end{bmatrix} = A \begin{bmatrix} q_{xx} & q_{xy} & q_{xz} \\ q_{yx} & q_{yy} & q_{yz} \\ q_{zx} & q_{zy} & q_{zz} \end{bmatrix} A^T, \tag{128}$$

where A^T is the transpose of the matrix A. Note that under the local coordinate transformation, the PDE equation (117) remains the same. Therefore, we drop the bars for simplicity.

3.1.3 LOCAL COORDINATE TRANSFORMATION IN TERMS OF LEVEL SET FUNCTION

At a given point (x^*, y^*, z^*) on the interface, let ξ to be parallel to the normal direction of the interface pointing outward, i.e.

$$\xi = \frac{\nabla \phi}{|\nabla \phi|} = \frac{(\phi_x, \quad \phi_y, \quad \phi_z)^T}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}, \tag{129}$$

and η and ζ are in the tangent plane passing through the point (x^*, y^*, z^*). However unlike the two-dimensional case, we have to choose two tangential directions. In practice, we choose the first tangential direction as

$$\begin{aligned} &\text{if} \quad \phi_x^2 + \phi_y^2 \neq 0; \quad && \eta = \frac{(\phi_y, \quad -\phi_x, \quad 0)^T}{\sqrt{\phi_x^2 + \phi_y^2}}, \\ &\text{otherwise} && \eta = \frac{(\phi_z, \quad 0, \quad -\phi_x)^T}{\sqrt{\phi_x^2 + \phi_z^2}}, \end{aligned} \tag{130}$$

and the corresponding second tangential direction as

$$\begin{aligned} &\text{if} \quad \phi_x^2 + \phi_y^2 \neq 0; \quad && \zeta = \frac{\mathbf{s}}{|\mathbf{s}|}, \quad \text{where} \quad \mathbf{s} = (\phi_x\phi_z, \quad \phi_y\phi_z, \quad -\phi_x^2 - \phi_y^2)^T, \\ &\text{otherwise} && \zeta = \frac{\mathbf{t}}{|\mathbf{t}|} \quad \text{where} \quad \mathbf{t} = (-\phi_x\phi_y, \quad \phi_x^2 + \phi_z^2, \quad -\phi_y\phi_z)^T. \end{aligned} \tag{131}$$

3.1.4 COMPUTING THE PRINCIPAL CURVATURES USING THE LEVEL SET FUNCTION

In the neighborhood of a control point $\mathbf{x}^* = (x^*, y^*, z^*)$, interface can be parameterized as

$$\xi = \chi(\eta, \zeta), \quad \text{with} \quad \chi(0,0) = 0, \quad \chi_\eta(0,0) = 0, \quad \chi_\zeta(0,0) = 0. \tag{132}$$

In order to compute the derivatives of jumps at the interface, we need to obtain the second-order tangential derivatives $\chi_{\eta\eta}$, $\chi_{\zeta\zeta}$, $\chi_{\eta\zeta}$ of χ at $\mathbf{x}^*$.

On the interface, $\phi(\chi(\eta,\zeta), \eta, \zeta) = 0$. First, lets consider the implicit differentiation of ϕ in terms of η and ζ respectively.

$$\phi_\eta + \phi_\xi \chi_\eta = 0, \tag{133}$$

$$\phi_\zeta + \phi_\xi \chi_\zeta = 0. \tag{134}$$

Then, differentiating the equations (133) and (134) by η and ζ we get

$$\begin{aligned} \phi_{\eta\eta} + \phi_{\eta\xi}\chi_\eta + (\phi_{\xi\eta} + \phi_{\xi\xi}\chi_\eta)\chi_\eta + \phi_\xi \chi_{\eta\eta} &= 0, \\ \phi_{\eta\zeta} + \phi_{\eta\xi}\chi_\zeta + (\phi_{\xi\zeta} + \phi_{\xi\xi}\chi_\zeta)\chi_\eta + \phi_\xi \chi_{\eta\zeta} &= 0, \\ \phi_{\zeta\zeta} + \phi_{\zeta\xi}\chi_\zeta + (\phi_{\xi\zeta} + \phi_{\xi\xi}\chi_\zeta)\chi_\zeta + \phi_\xi \chi_{\zeta\zeta} &= 0. \end{aligned} \tag{135}$$

Since, $\chi_\eta(0,0) = 0$ and $\chi_\zeta(0,0) = 0$ on the interface, we get

$$\begin{aligned} \chi_{\eta\eta} &= -\phi_{\eta\eta}/\phi_\xi, \\ \chi_{\zeta\zeta} &= -\phi_{\zeta\zeta}/\phi_\xi, \\ \chi_{\eta\zeta} &= -\phi_{\eta\zeta}/\phi_\xi, \end{aligned} \tag{136}$$

where,

$$\begin{bmatrix} \phi_\xi \\ \phi_\eta \\ \phi_\zeta \end{bmatrix} = A \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}. \tag{137}$$

3.1.5 LOCAL COORDINATE TRANSFORMATION OF JUMP CONDITIONS

As we pointed out earlier, in order to find the jump conditions $[u]$, $[u_x]$, $[u_y]$, $[u_z]$, $[u_{xx}]$, $[u_{yy}]$ and $[u_{zz}]$ at a control point $\mathbf{x}^* = (x^*, y^*, z^*)$, it is convenient to differentiate the jump conditions $[u] = w$ and $[\beta u_{\boldsymbol{n}}] = v$ along the interface and then do the coordinate transformation.

Now, let's first differentiate $[u] = w$ with respect to η and ζ respectively,

$$[u_\xi]\chi_\eta + [u_\eta] = w_\eta, \tag{138}$$

$$[u_\xi]\chi_\zeta + [u_\zeta] = w_\zeta. \tag{139}$$

Then, differentiate the equation (138) with respect to η and we get the following,

$$\chi_\eta \frac{\partial [u_\xi]}{\partial \eta} + \chi_{\eta\eta}[u_\xi] + \chi_\eta [u_{\eta\xi}] + [u_{\eta\eta}] = w_{\eta\eta}, \tag{140}$$

Differentiating the equation (138) with respect to ζ gives,

$$\chi_\eta \frac{\partial [u_\xi]}{\partial \zeta} + \chi_{\eta\zeta}[u_\xi] + \chi_\zeta [u_{\eta\xi}] + [u_{\eta\zeta}] = w_{\eta\zeta}. \tag{141}$$

Differentiating the equation (139) with respect to ζ gives,

$$\chi_\zeta \frac{\partial [u_\xi]}{\partial \zeta} + \chi_{\zeta\zeta}[u_\xi] + \chi_\zeta [u_{\zeta\xi}] + [u_{\zeta\zeta}] = w_{\zeta\zeta}. \tag{142}$$

Now, we need to differentiate $[\beta u_{\boldsymbol{n}}] = v$. Before, we do that, lets express the unit normal vector of the interface with respect to χ,

$$\boldsymbol{n} = \frac{(1, -\chi_\eta, -\chi_\zeta)}{\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2}}. \tag{143}$$

Now, we rewrite the flux jump condition $[\beta u_{\boldsymbol{n}}] = v$ using the local coordinate transformations as follows,

$$[\beta(u_\xi - u_\eta\chi_\eta - u_\zeta\chi_\zeta)] = v(\eta, \zeta)\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2}. \tag{144}$$

After differentiating the equation (144) with respect to η, we get,

$$\begin{aligned}[\beta(u_{\xi\xi}\chi_\eta + u_{\xi\eta} - \chi_\eta \frac{\partial u_\eta}{\partial \eta} - \chi_\zeta \frac{\partial u_\zeta}{\partial \eta} - u_\eta\chi_{\eta\eta} - u_\zeta\chi_{\eta\zeta})] \\ = v_\eta\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2} + v\frac{\chi_\eta\chi_{\eta\eta}}{\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2}}.\end{aligned} \tag{145}$$

Likewise, differentiating the equation (144) with respect to ζ, we get,

$$\begin{aligned}[\beta(u_{\xi\xi}\chi_\zeta + u_{\xi\zeta} - \chi_\eta \frac{\partial u_\eta}{\partial \zeta} - \chi_\zeta \frac{\partial u_\zeta}{\partial \zeta} - u_\eta\chi_{\eta\zeta} - u_\zeta\chi_{\zeta\zeta})] \\ = v_\zeta\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2} + v\frac{\chi_\zeta\chi_{\zeta\zeta}}{\sqrt{1 + \chi_\eta^2 + \chi_\zeta^2}}.\end{aligned} \tag{146}$$

Since, $\chi_\eta(0,0) = \chi_\zeta(0,0) = 0$ on the interface, equations (138) to (146) reduced to following jump conditions.

$$[u] = w, \tag{147}$$

$$[u_\eta] = w_\eta, \tag{148}$$

$$[u_\zeta] = w_\zeta, \tag{149}$$

$$[\beta u_\xi] = v, \tag{150}$$

$$[u_{\eta\eta}] = -\chi_{\eta\eta}[u_\xi] + w_{\eta\eta} = D_1, \tag{151}$$

$$[u_{\zeta\zeta}] = -\chi_{\zeta\zeta}[u_\xi] + w_{\zeta\zeta} = D_2, \tag{152}$$

$$[u_{\eta\zeta}] = -\chi_{\eta\zeta}[u_\xi] + w_{\eta\zeta} = D_3, \tag{153}$$

$$[\beta u_{\xi\eta}] = \chi_{\eta\eta}[\beta u_\eta] + \chi_{\eta\zeta}[\beta u_\zeta] + v_\eta = D_4, \tag{154}$$

$$[\beta u_{\xi\zeta}] = \chi_{\eta\zeta}[\beta u_\eta] + \chi_{\zeta\zeta}[\beta u_\zeta] + v_\zeta = D_5. \tag{155}$$

Now, we have all together nine jump relations. Also, from PDE equation (117) itself, we can obtain two more additional jump relations,

$$u^+_{xx} + u^+_{yy} + u^+_{zz} = \frac{f^+}{\beta^+} = D_6, \tag{156}$$

$$u^-_{xx} + u^-_{yy} + u^-_{zz} = \frac{f^-}{\beta^-} = D_7. \tag{157}$$

Now, we will do the coordinate transformation of jump relations (147)-(157) into Cartesian coordinates and express all the limiting values from outside the interface in terms of u^-, u^-_x, u^-_y, u^-_z, u^-_{xx}, u^-_{xy}, u^-_{xz}, u^-_{yy} and u^-_{yz}. Derivation of the expression for u^+, u^+_x, u^+_y, u^+_z, u^+_{xx}, u^+_{xy}, u^+_{xz}, u^+_{yy}, u^+_{yz} and u^+_{zz} are given in the Appendix A. Notice that D_1 through D_5 can also be expressed in terms of u^-_x, u^-_y, u^-_z.

3.1.6 THE APPROXIMATION OF THE CORRECTION TERMS

In this section, we will discus how to interpolate the correction terms $C^x_{i,j,l}$, $C^y_{i,j,l}$ and $C^z_{i,j,l}$ from the Cartesian coordinate transformations of the jump relations given by equations (147) to (155) and the other two jump conditions equations (156) and (157) given by the PDE itself.

We follow the same approach as described in chapter 2 to approximate the correction terms. Given an irregular grid point (x_i, y_j, z_l), we first select a point $\mathbf{x}^* = (x^*_i, y^*_j, z^*_l)$ on the interface Γ near (x_i, y_j, z_l). Here we take this point as the control point closest to (x_i, y_j, z_l).

As we are seeking the difference scheme at the irregular point to be in the form of equation (120), we expand each $U_{i+i_m,j+j_m,l+l_m}$ about the control point $\mathbf{x}^* = (x_i^*, y_j^*, z_l^*)$. Where $m = 1, 2.., n_s$ and n_s is the number of grid points in the difference scheme. And i_m, j_m and l_m will take the values from $\{-1, 0, 1\}$. We will explain later how to choose n_s. Also note that m is really depended on i, j and l and we have omitted the dependency for simplicity.

Without loss of generality, let's first assume that (x_i, y_j, z_l) is an irregular point and seven-point stencil of (x_i, y_j, z_l) is only cut through the right arm and $(x_i, y_j, z_l) \in \Omega^-$. We will now consider the Taylor expansion of $u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m})$ about the control point $\mathbf{x}^* = (x_i^*, y_j^*, z_l^*)$.

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m}) &= u^\pm + (x_{i+i_m} - x_i^*)u_x^\pm + (y_{j+j_m} - y_j^*)u_y^\pm + (z_{l+l_m} - z_l^*)u_z^\pm \\ &+ \frac{1}{2}(x_{i+i_m} - x_i^*)^2 u_{xx}^\pm + \frac{1}{2}(y_{j+j_m} - y_j^*)^2 u_{yy}^\pm + \frac{1}{2}(z_{l+l_m} - z_l^*)^2 u_{zz}^\pm \\ &+ (x_{i+i_m} - x_i^*)(y_{j+j_m} - y_j^*)u_{xy}^\pm + (x_{i+i_m} - x_i^*)(z_{l+l_m} - z_l^*)u_{xz}^\pm \\ &+ (y_{j+j_m} - y_j^*)(z_{l+l_m} - z_l^*)u_{yz}^\pm + O(h^3), \end{aligned} \tag{158}$$

where plus or minus sign is chosen depending on whether $(x_{i+i_m}, y_{j+j_m}, z_{l+l_m})$ lies in Ω^+ or Ω^-. Using coordinate transformation of the nine interface relations (147) to (155) and other two jump relations given by equations (156) and (157), we can eliminate limiting values of plus side by limiting values of minus as explained in Appendix A. Then the Taylor series expansion of equation (158) will contain u^-, u_x^-, u_y^-, u_z^-, u_{xx}^-, u_{xy}^-, u_{xz}^-, u_{yy}^-, and u_{yz}^-.

Now the equation (158) will be in the form of,

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m}) &= c_m^1 u^- + c_m^2 u_x^- + c_m^3 u_y^- + c_m^4 u_z^- + c_m^5 u_{xx}^- \\ &+ c_m^6 u_{xy}^- + c_m^7 u_{xz}^- + c_m^8 u_{yy}^- + c_m^9 u_{yz}^- + c_m^{10}. \end{aligned} \tag{159}$$

Here, coefficients c_m^1 through c_m^{10} are known quantities. Now, we can see the correction term $C_{i,j,l}^x$ can be reformulated as,

$$C_{i,j,l}^x = a_1 u^- + a_2 u_x^- + a_3 u_y^- + a_4 u_z^- + a_5 u_{xx}^- + a_6 u_{xy}^- + a_7 u_{xz}^- + a_8 u_{yy}^- + a_9 u_{yz}^- + a_{10}. \tag{160}$$

Here, the coefficients a_1 through a_{10} are known. Interpolation for $C_{i,j,l}^y$ and $C_{i,j,l}^z$ will be the same form as for $C_{i,j,l}^x$, only with coefficients changed.

We assume that the correction terms $C_{i,j,l}^x$ can be approximated as follows,

$$C_{i,j,l}^x = \sum_m^{n_s} \gamma_m U_{i+i_m,j+j_m,l+l_m} + \gamma_c. \tag{161}$$

By comparing equations (160) and (161), we get the following linear system,

$$\begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_3^1 & c_{m-1}^1 & c_m^1 \\ c_1^2 & c_2^2 & \cdots & c_3^2 & c_{m-1}^2 & c_m^2 \\ & & \cdots & & & \\ c_1^8 & c_2^8 & \cdots & c_3^8 & c_{m-1}^8 & c_m^8 \\ c_1^9 & c_2^9 & \cdots & c_3^9 & c_{m-1}^9 & c_m^9 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_{m-1} \\ \gamma_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix}. \tag{162}$$

We can find γ_c in equation (161) as follows,

$$\gamma_c = a_{10} - \sum_{m}^{n_s} c_m^{10} \cdot \gamma_m. \tag{163}$$

Equation (162) gives a system of nine linear equations to determine the coefficients γ_m. We can anticipate a solvable system by choosing the number of grid points $n_s = 9$ in the stencil for an irregular grid point. However, the number of minimum grid points that should be involved to guarantee the existence of the solution to the above linear system is not theoretically studied. In our implementation, we take all 27 grid points in the cube centered at the irregular grid point (x_i, y_j, z_l). See Figure 12 for an illustration. That is $n_s = 27$. We have not faced any numerical convergence problem. We have an underdetermined linear system since the number of unknowns is greater than the number of equations. So, to solve the above linear system with 9 equations and 27 unknowns, one can use the singular value decomposition (SVD) to get the minimum norm solution. At this stage, finite difference scheme at the irregular grid point (x_i, y_j, z_l) is fully determined.

Now, linear system for finite difference equations can be written as,

$$A_h \mathbf{U} = \boldsymbol{F}. \tag{164}$$

Before we show some numerical results of the method for solving elliptic interface problems, let us explain further about the coefficient matrix (164). As we described earlier, coefficients τ_ms in the finite difference scheme (120) are obtained by adding the $27\gamma_m$ found by solving the underdetermined linear system equation (162) into the standard seven-point stencil which has the coefficients as same as the finite difference scheme coefficients of discrete Laplacian operator for Poisson equation in the absence of interface. Moreover, these $27\gamma_m$s

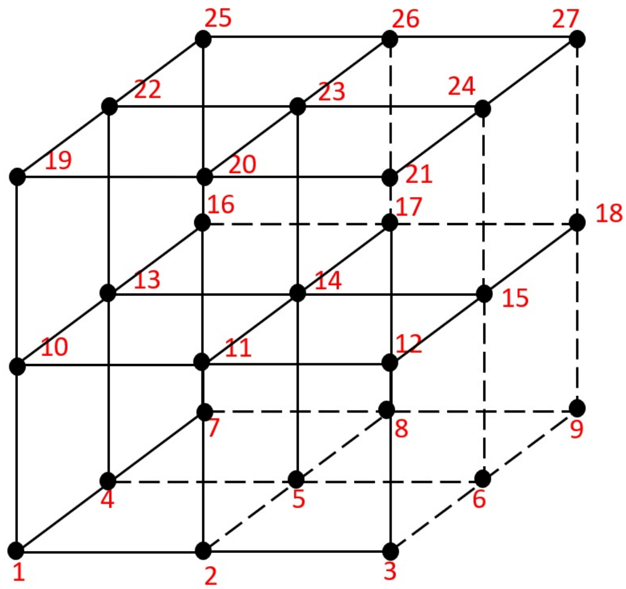


Fig. 12. The 27-point stencil for three-dimensional compact scheme.

are dependent on the location of the interface and the jump β in the PDE (117). The coefficients γ_ms in correction terms are $O(1/h^2)$ in general. However, they are far smaller than $1/h^2$ in magnitude from our calculation in numerical experiments. So, the larger weights of coefficients in the FD scheme (120) are still coming by the grid values of the seven-point stencil. Therefore, eigenvalues of the coefficient matrix will remain in the stability region. We will demonstrate the magnitude of eigenvalues and coefficients γ_ms for a particular example in section 3.2 for a problem with a modest jump in β.

In order to recover the gradient of the solution at the interface, one can use the following formula,

$$U_\xi^- = \alpha_{x\xi} U_x^- + \alpha_{y\xi} U_y^- + \alpha_{z\xi} U_z^-, \tag{165}$$

$$U_\xi^+ = \rho\alpha_{x\xi} U_x^- + \rho\alpha_{y\xi} U_y^- + \rho\alpha_{z\xi} U_z^- + v/\beta^+, \tag{166}$$

$$U_\eta^- = \alpha_{x\eta} U_x^- + \alpha_{y\eta} U_y^- + \alpha_{z\eta} U_z^-, \tag{167}$$

$$U_\eta^+ = \alpha_{x\eta} U_x^- + \alpha_{y\eta} U_y^- + \alpha_{z\eta} U_z^- + w_\eta, \tag{168}$$

$$U_\zeta^- = \alpha_{x\zeta} U_x^- + \alpha_{y\zeta} U_y^- + \alpha_{z\zeta} U_z^-, \tag{169}$$

$$U_\zeta^+ = \alpha_{y\zeta} U_x^- + \alpha_{y\zeta} U_y^- + \alpha_{z\zeta} U_z^- + w_\zeta, \tag{170}$$

where, ξ is the unit normal direction, and η and ζ are in the tangential directions to the interface at the control point $\mathbf{x}^* = (x_i^*, y_j^*, z_l^*)$.

The equations (165) and (166) were obtained using coordinate transformation of equation (150). And equations (167) and (168) were derived using coordinate transformation of equation (148). Also, the equations (169) and (170) were obtained by coordinate transformation of equation (149).

In order to interpolate these gradients from the solution, we will again assume that interpolation scheme will be in the same form as equation (161). where the point (x_i, y_j, z_l) is chosen to be the closest irregular point to the control point that is considered. For example, the resulting linear system is same as the equation (162) with right hand vector replaced by the vector $[0, \alpha_{x\xi}, \alpha_{y\xi}, \alpha_{z\xi}, 0, 0, 0, 0, 0, 0]^T$ for U_ξ^- and vector $[0, \rho\alpha_{x\xi}, \rho\alpha_{y\xi}, \rho\alpha_{z\xi}, 0, 0, 0, 0, 0, v/\beta^+]^T$ for U_ξ^+ calculation.

3.1.7 OUTLINE OF THE ALGORITHM

In this section, the outline of the algorithm for solving three-dimensional elliptic interface problems is given.

Step 1: Immerse the irregular domain (interface) into a cubic domain $\Omega = [a, b] \times [a, b] \times [a, b]$ and represent the interface using a zero level set function.

Step 2: Determine the regular and irregular grid points and the location of control points which are the intersection points of the interface and the grid lines using the level set grid function ϕ.

Step 3: Apply the standard 7-point central difference scheme at the regular grid points.

Step 4: Solve the underdetermined linear system given by equation (162) to calculate the correction terms $C^x_{i_k,j_k,l_k}$, $C^y_{i,j,l}$ and $C^z_{i,j,l}$ at irregular grid points. Then, find 27-point compact scheme at irregular points.

Step 5: Solve the system of linear equations given by equation (164).

Step 5: Recover the gradients of the solution.

3.2 NUMERICAL EXAMPLES

Here we show some numerical examples and perform error analysis to show the performance of the developed direct IIM to solve three-dimensional elliptic interface problems. For all examples here, errors are illustrated in L^∞ norm and order of convergence of the method is calculated using following formula,

$$r = \frac{1}{\log(2)} \log \frac{\| E_{2h} \|_\infty}{\| E_h \|_\infty}. \tag{171}$$

In all listed tables in this section, N is the number of grid points in each direction. $E(u)$ is the maximum norm error of the numerical solution. $E(u_{\boldsymbol{n}})$, $E(u_\eta)$ and $E(u_\zeta)$ are the maximum norm error in the normal and tangential directions of the computed solution respectively.

Example 3.1

In this example, both the solution $u(x, y, z)$ and its flux have a jump discontinuity. The differential equation is given by,

$$(\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z = f, \tag{172}$$

on the domain $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1]$, where interface Γ is a sphere represented by the zero level set of the function,

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 0.25. \tag{173}$$

The source term f is defined as,

$$f(x, y, z) = \begin{cases} 6\beta^- & \text{if} \quad (x, y, z) \in \Omega^-, \\ 6\beta^+ & \text{if} \quad (x, y, z) \in \Omega^+, \end{cases} \tag{174}$$

where,

$$\beta(x, y, z) = \begin{cases} \beta^- & \text{if} \quad (x, y, z) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x, y, z) \in \Omega^+. \end{cases} \tag{175}$$

The jump in solution $u(x, y, z)$ and it's flux are given by,

$$[u] = 10, \tag{176}$$

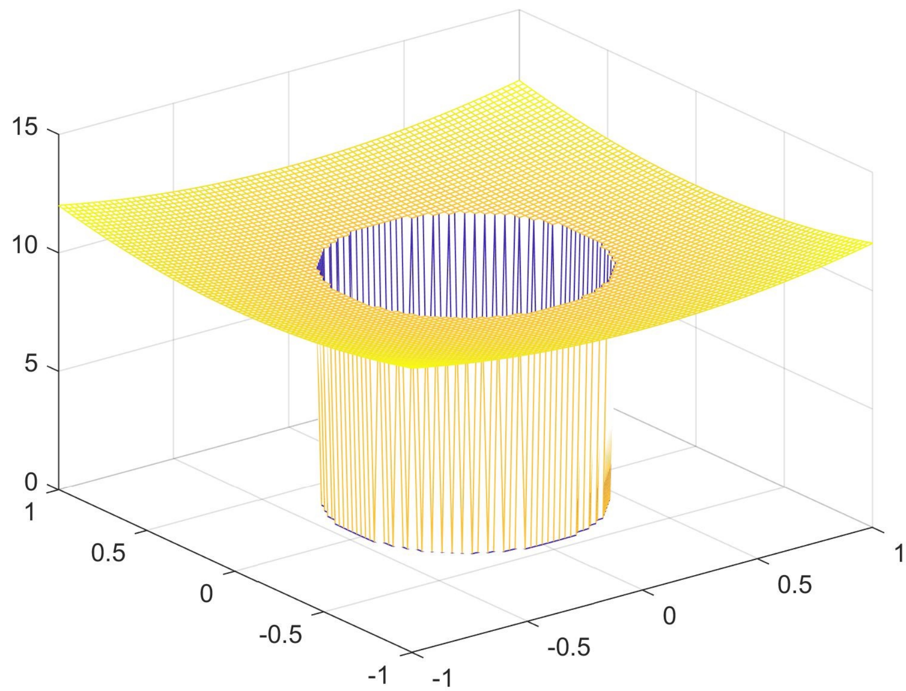
$$[\beta u_{\boldsymbol{n}}] = \beta^+ - \beta^-. \tag{177}$$

Boundary conditions were taken from the exact solution of,

$$u(x, y, z) = \begin{cases} x^2 + y^2 + z^2 & \text{if} \quad (x, y, z) \in \Omega^-, \\ x^2 + y^2 + z^2 + 10 & \text{if} \quad (x, y, z) \in \Omega^+. \end{cases} \tag{178}$$

We have tested different cases of jumps for β. Figure 13 shows the slice of a computed

(a)



(b)

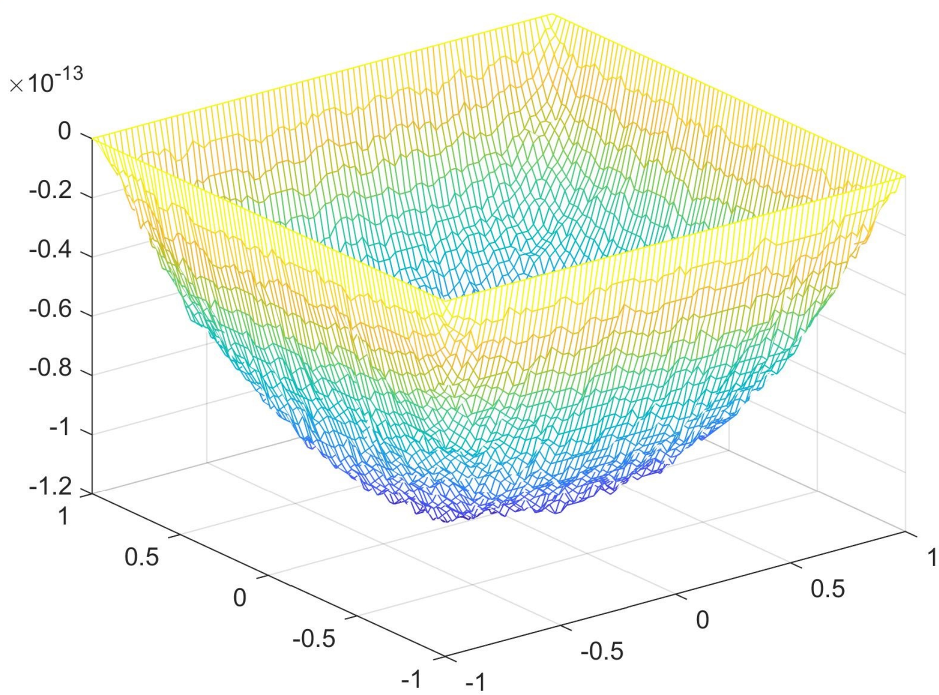


Fig. 13. (a) Plot of numerical solution for Example 3.1, (b) Plot of error of the numerical solution with $N = 80$, $\beta^{-} = 1$ and $\beta^{+} = 10$.

solution and the error of the computed solution. In Tables 5 and 6, we present the numerical experiment results for grid refinement analysis of Example 3.1. Table 5 shows the results for $\beta^- = 1$ and $\beta^+ = 1$ and Table 6 shows it for $\beta^- = 1$ and $\beta^+ = 10000$. The first column in the tables is the number of grid points N in each direction. The second column illustrates the maximum norm error of the numerical solution. The third, fourth, and five columns demonstrate the maximum norm error in the normal and tangential derivatives. Furthermore, the sixth column illustrates the number of control points involved in the computations. We can see the errors are in order of 10^{-12} to 10^{-15} in infinity norm. The errors are actually from round-off errors. This is because the solution is a quadratic function.

Table 5
A grid refinement analysis for Example 3.1 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	$E(u_{\boldsymbol{n}})$	$E(u_\eta)$	$E(u_\varsigma)$	N_b
10	8.81E-13	3.73E-13	3.32E-13	3.82E-13	248
20	4.30E-13	1.10E-14	1.18E-13	1.53E-13	828
40	8.10E-14	1.93E-15	4.92E-14	3.34E-14	3696
80	5.02E-14	1.54E-15	1.92E-14	1.68E-14	11628
160	2.00E-14	1.40E-15	8.26E-15	6.59E-15	49080

Table 6
A grid refinement analysis for Example 3.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	$E(u_{\boldsymbol{n}})$	$E(u_\eta)$	$E(u_\varsigma)$	N_b
10	8.88E-13	3.73E-13	3.32E-13	3.52E-13	248
20	4.60E-14	1.10E-13	1.18E-13	1.08E-13	828
40	1.16E-14	1.98E-14	1.92E-14	1.62E-14	3696
80	9.82E-15	2.64E-15	8.83E-15	8.63E-15	11628
160	6.77E-15	1.40E-15	6.36E-15	6.56E-15	49080

Example 3.2

In this example, both the solution $u(x, y, z)$ and its flux have a jump discontinuity. The differential equation is given by,

$$(\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z = f. \tag{179}$$

on the domain $\Omega = [-1, 1] \times [-1, 1] \times [-1, 1]$, where interface Γ is a sphere represented by the zero level set of the function,

$$\phi(x, y, z) = x^2 + y^2 + z^2 - 0.25, \tag{180}$$

The source term f is defined as,

$$f(x, y, z) = \begin{cases} 2\beta^- e^{r^2}(3 + 2r^2) & \text{if} \quad (x, y, z) \in \Omega^-, \\ 2\beta^+ e^{r^2}(3 + 2r^2) & \text{if} \quad (x, y, z) \in \Omega^+. \end{cases} \tag{181}$$

where,

$$\beta(x, y, z) = \begin{cases} \beta^- & \text{if} \quad (x, y, z) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x, y, z) \in \Omega^+. \end{cases} \tag{182}$$

The jump in solution $u(x, y, z)$ and it's flux are given by,

$$[u] = 1, \tag{183}$$

$$[\beta u_{\boldsymbol{n}}] = (\beta^+ - \beta^-)e^{0.25}. \tag{184}$$

Boundary conditions were taken from the exact solution of,

$$u(x, y, z) = \begin{cases} e^{r^2} & (x, y, z) \in \Omega^-, \\ e^{r^2} + 1 & (x, y, z) \in \Omega^+. \end{cases} \tag{185}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

We have tested different cases of jumps in β. Figure 14 shows a slice of the computed solution and error of the computed solution for Example 3.2 when $[\beta] = 100$. Tables 7 - 11 show the results of a grid refinement analysis for different jumps in β for Example 3.2. You can see the errors in solution and the errors in gradients are almost in second-order accuracy.

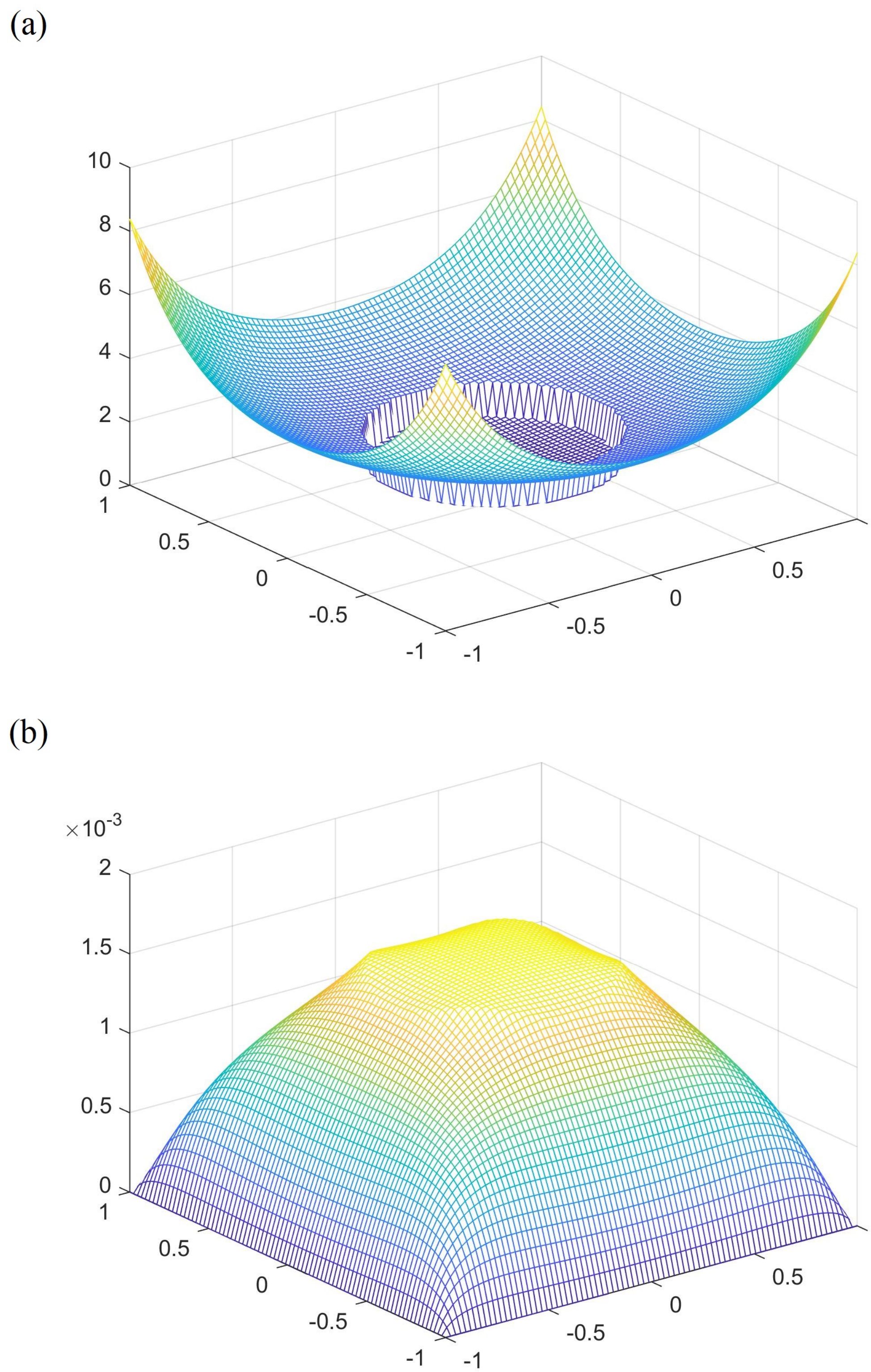


Fig. 14. (a) Plot of numerical solution for Example 3.2, (b) Plot of error of the numerical solution with $N = 80$, $\beta^{-} = 1$ and $\beta^{+} = 100$.

Table 7
A grid refinement analysis for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order	$E(u_\zeta)$	order
10	6.40E-02		0.134324		0.141663		0.14311	
20	1.83E-02	1.81	3.47E-02	1.95	4.08E-02	1.80	4.15E-02	1.79
40	4.85E-03	1.91	8.98E-03	1.95	1.12E-02	1.86	1.09E-02	1.93
80	1.25E-03	1.96	2.41E-03	1.90	2.79E-03	2.01	2.78E-03	1.97
160	3.16E-04	1.98	6.01E-04	2.00	7.05E-04	1.99	7.59E-04	1.87

Table 8
A grid refinement analysis for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 10$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order	$E(u_\tau)$	order
10	8.28E-02		0.11721		0.320327		0.325656	
20	2.34E-02	1.82	3.51E-02	1.74	7.45E-02	2.10	7.18E-02	2.18
40	6.20E-03	1.92	8.44E-03	2.06	1.75E-02	2.09	1.72E-02	2.06
80	1.58E-03	1.97	3.07E-03	1.46	5.87E-03	1.58	5.94E-03	1.54
160	4.01E-04	1.98	8.48E-04	1.86	1.53E-03	1.94	1.49E-03	2.00

Table 9
A grid refinement analysis for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 100$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order	$E(u_\tau)$	order
10	8.54E-02		1.23E-01		3.64E-01		3.73E-01	
20	2.40E-02	1.83	2.95E-02	2.06	8.97E-02	2.02	8.70E-02	2.10
40	6.32E-03	1.92	9.94E-03	1.57	2.50E-02	1.84	2.49E-02	1.81
80	1.61E-03	1.97	3.18E-03	1.64	9.32E-03	1.42	9.38E-03	1.41
160	4.08E-04	1.98	1.16E-03	1.45	2.26E-03	2.05	2.29E-03	2.04

Table 10
A grid refinement analysis for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 1000$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order	$E(u_\tau)$	order
10	8.58E-02		0.123625		0.363609		0.379305	
20	2.41E-02	1.83	2.97E-02	2.06	9.30E-02	1.97	8.98E-02	2.08
40	6.35E-03	1.93	8.68E-03	1.78	2.80E-02	1.73	2.80E-02	1.68
80	1.62E-03	1.97	3.81E-03	1.19	9.86E-03	1.50	9.92E-03	1.50
160	4.09E-04	1.98	1.43E-03	1.42	2.34E-03	2.08	2.37E-03	2.07

We also have done some analysis on eigenvalues of the coefficient matrix of the linear system of FD scheme (164) to understand the stability of the method. Figure 15 shows the eigenvalues of the coefficient matrix of FD scheme for Example 3.2 for the case of $[\beta] = 100$. Here we have taken the number of grid points in each direction of the computational domain as $N = 22$. You can see that all eigenvalues are negative, showing that the original method is stable for this example.

In Figure 16, we have demonstrated the values of γ_ms which are the coefficients of the underdetermined linear system (162) for Example 3.2 for computing the correction terms for a particular irregular grid point (x_i, y_j, z_l). We have labeled the 27 grid points in the FD scheme for this particular irregular grid point with red ink in the manner for which the lowest left corner grid point is number 1 and increase the label number by one as it goes to the right and up. Please refer the Figure 16 for an illustration.

Table 11
A grid refinement analysis for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	order	$E(u_{\boldsymbol{n}})$	order	$E(u_\eta)$	order	$E(u_\tau)$	order
10	8.58E-02		1.24E-01		3.64E-01		3.80E-01	
20	2.42E-02	1.83	2.98E-02	2.06	9.29E-02	1.97	8.97E-02	2.08
40	6.35E-03	1.93	8.29E-03	1.84	2.79E-02	1.73	2.80E-02	1.68
80	1.62E-03	1.97	3.87E-03	1.10	9.90E-03	1.50	9.96E-03	1.49
160	4.09E-04	1.98	1.40E-03	1.46	2.34E-03	2.08	2.38E-03	2.07

As we mentioned in the section 3.1.6, in order to find coefficients τ_ms of the FD scheme for a particular irregular point (Labeled as number 14), we have to add the coefficients of the standard seven-point stencil of the FD scheme for a Poisson equation (117) in the absence of an interface which takes $-6\ /\ h^2 \approx -661.00$ for the central point and $1\ /\ h^2 \approx 110.25$ for rest of the six grid points (labeled as number 5, 11, 13, 15, 17 and 23) to the γ_m we found in solving linear system (162). Note that we have taken the number of grid points in each direction of the computational domain as $N = 22$. We can see that larger weights of coefficients in the FD scheme are coming by values of the seven-point stencil, illustrating the reason that the original method always gives a stable solution.

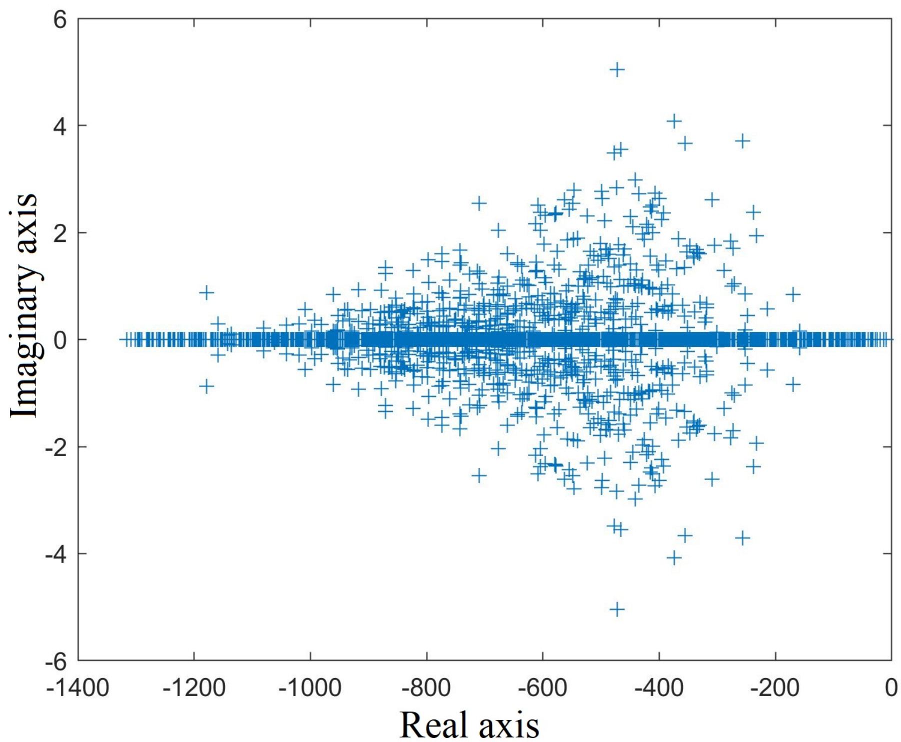


Fig. 15. The plot of eigenvalues of the coefficient matrix of the discrete elliptic operator for Example 3.2 with $N = 22$ for $\beta^- = 1$ and $\beta^+ = 100$.

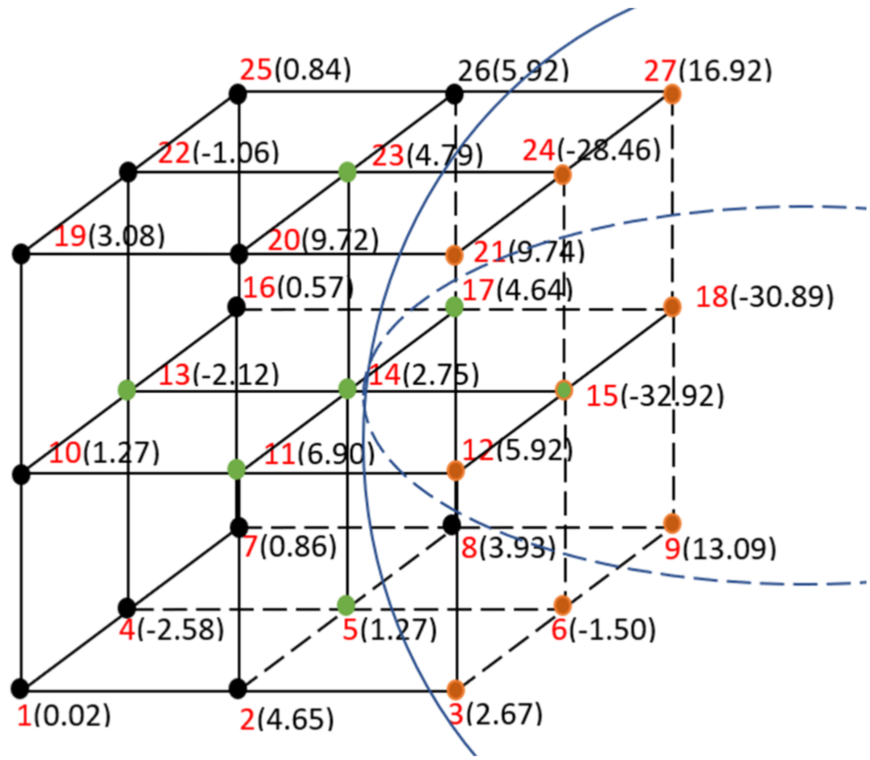


Fig. 16. The γ_m coefficients of a point (x_i, y_j, z_l) near the interface for Example 3.2 with $\beta^- = 1$ and $\beta^+ = 100$. where the standard seven-point stencil of the grid point (x_i, y_j, z_l), labeled as number 14 has demonstrated with solid green circles. The solid orange circles represent the grid points which are inside the sphere (the interface) and black ones are the rest of the grid points in the 27-point stencil which are outside the sphere. The standard seven-point stencil of (x_i, y_j, z_l) is only cut from the righthand side.

3.3 SUMMARY

In summary, we have developed a direct method for solving three-dimensional elliptic interface problems with piecewise constant coefficients. Here, we have studied the extension of the direct method for solving two-dimensional elliptic interface problems discussed in [58] to three-dimensions. We have derived an appropriate finite difference scheme with a standard seven-point stencil for regular grid point and a twenty-seven compact scheme for an irregular grid point. We have observed that the larger weights of the coefficients of the FD scheme for an irregular grid point are still coming from the original seven-point stencil, and the coefficients of the rest of the twenty points are much smaller. We have also studied the order of convergence of the developed method by doing grid refinement analysis for a few numerical examples. We have observed that the developed method is second-order accurate in both the solution and gradient of the solution. We also have shown that the method is always stable as the coefficient matrix of the linear system for the FD scheme has all the real parts of the eigenvalues that are negative.

CHAPTER 4

STUDY FOR SOLVING TWO-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS

In this chapter, we propose a direct method for solving two-dimensional parabolic interface problems with fixed interfaces.

Consider the following parabolic interface problem,

$$u_t = (\beta u_x)_x + (\beta u_y)_y - f(x, y, t), \quad (x, y) \in \Omega \backslash \Gamma, \tag{186}$$

$$[u] = w(:, t), \tag{187}$$

$$[\beta u_{\boldsymbol{n}}] = v(:, t), \tag{188}$$

with specified boundary and initial conditions.

When solving parabolic interface problems, numerical analysts prefer to use implicit over explicit methods as there are no time step restrictions to the implicit methods. So, we use the Crank-Nicholson scheme and some modifications to the direct IIM to solve parabolic interface problems to obtain second-order accuracy for the solution and its gradient in both time and space.

4.1 ALGORITHM FOR TWO-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS WITH PIECEWISE CONSTANT COEFFICIENTS

As in chapter 2 for solving two dimensional elliptic interface problems, we begin by reformulating the PDE by diving the both sides of the PDE (186) by β and get the following equivalent problem.

$$\frac{u_t}{\beta} = u_{xx} + u_{yy} - \frac{f(x, y, t)}{\beta}, \quad (x, y) \in \Omega \backslash \Gamma, \tag{189}$$

$$[u] = w(:, t), \tag{190}$$

$$[\beta u_{\boldsymbol{n}}] = v(:, t). \tag{191}$$

4.1.1 CRANK-NICOLSON SCHEME FOR TWO-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS

For any grid point (x_i, y_j), we can write the Crank-Nicolson scheme for equation (189) as,

$$\begin{aligned} \frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t \beta_{i,j}} &= \frac{1}{2}(\delta_x U_{i,j}^{n+1} + (C_{i,j}^{x})^{n+1} + \delta_y U_{i,j}^{n+1} + (C_{i,j}^{y})^{n+1} - \frac{f_{i,j}^{n+1}}{\beta_{ij}} \\ &+ \delta_x U_{i,j}^{n} + (C_{i,j}^{x})^{n} + \delta_y U_{i,j}^{n} + (C_{i,j}^{y})^{n} - \frac{f_{i,j}^{n}}{\beta_{i,j}}). \end{aligned} \tag{192}$$

where,

$$\delta_x U_{i,j}^{n} = \frac{U_{i-1,j}^{n} - U_{i,j}^{n} + U_{i+1,j}^{n}}{h^2}, \tag{193}$$

and correction term for a particular irregular grid point whose five-point stencil cut through the right arm at the control point $x^* = (x_i^*, y_j)$ can be written as,

$$C_{i,j}^{x} = \pm\frac{[u]}{h^2} \pm [u_x]\frac{(x_{i+1} - x^*)}{h^2} \pm [u_{xx}]\frac{(x_{i+1} - x^*)^2}{2h^2}. \tag{194}$$

Here, plus or minus is chosen depending on which side the irregular grid point lies. In a similar way, we can define the correction term $C_{i,j}^{y}$.

4.1.2 LOCAL COORDINATE TRANSFORMATION OF JUMP RELATIONS

To compute the correction terms $C_{i,j}^{x}$ and $C_{i,j}^{y}$, at irregular grid points, we need to find the values of $[u_x]$, $[u_y]$, $[u_{xx}]$ and $[u_{yy}]$ in terms of the given jumps $[u]$ and $[\beta u_{\boldsymbol{n}}]$. Required jump relations can be obtained by differentiating the given jumps $[u]$ and $[\beta u_{\boldsymbol{n}}]$ and using the PDE itself.

Similar to the two-dimensional elliptic interface problems in chapter 2, using the local coordinate system given by equations (67) and (68), we can obtain the following jump relations at the interface,

$$[u] = w, \tag{195}$$

$$[\beta u_{\boldsymbol{n}}] = v, \tag{196}$$

$$[u_\eta] = w_\eta, \tag{197}$$

$$[u_{\eta\eta}] = -\kappa[u_\xi] + w_{\eta\eta} = D_1, \tag{198}$$

$$[\beta u_{\eta\xi}] = \kappa[\beta u_\eta] + v_\eta = D_2. \tag{199}$$

Now from PDE itself we get the following relation,

$$\begin{aligned}[\beta(u_{\xi\xi}+u_{\eta\eta})] &= [f+u_t],\\ [\beta u_{\xi\xi}] &= -[\beta u_{\eta\eta}]+[f]+w_t.\end{aligned} \tag{200}$$

From coordinate transformation of equations (197)-(200), we get the following relations respectively,

$$-u_x^+ s + u_y^+ c = w_\eta - u_x^- s + u_y^- c, \tag{201}$$

$$u_x^+ c + u_y^+ s = \frac{v}{\beta^+} + \rho(u_x^- c + u_y^- s), \tag{202}$$

$$u_{xx}^+ s^2 - 2scu_{yx}^+ + u_{yy}^+ c^2 = u_{xx}^- s^2 - 2scu_{yx}^- + u_{yy}^- c^2 + D_1, \tag{203}$$

$$-scu_{xx}^+ + (c^2 - s^2)u_{xy}^+ + scu_{yy}^+ = -\rho scu_{xx}^- + \rho(c^2 - s^2)u_{xy}^- + \rho scu_{yy}^- + \frac{D_2}{\beta^+}, \tag{204}$$

$$u_{xx}^+ c^2 + 2scu_{yx}^+ + u_{yy}^+ s^2 = (\rho - s^2)u_{xx}^- + 2scu_{yx}^- + (\rho - c^2)u_{yy}^- + D_3, \tag{205}$$

where,

$$D_3 = -D_1 + \frac{[f]+w_t}{\beta^+}. \tag{206}$$

Now, from equation (195) and solving equations (201)-(205) to get u^+, u_x^+, u_y^+, u_{xx}^+, u_{xy}^+ and u_{yy}^+ in terms of u^-, u_x^-, u_y^-, u_{xx}^-, u_{xy}^- and u_{yy}^-, we get

$$u^+ = u^- + w, \tag{207}$$

$$u_x^+ = u_x^-(s^2 + \rho c^2) + u_y^-(\rho - 1)sc - sw_\eta + \frac{cv}{\beta^+}, \tag{208}$$

$$u_y^+ = u_x^-(\rho - 1)sc + u_y^-(\rho s^2 + c^2) + cw_\eta + \frac{sv}{\beta^+}, \tag{209}$$

$$\begin{aligned}u_{xx}^+ &= (c^2\rho + s^4 - c^2s^2 + 2c^2s^2\rho)u_{xx}^- - (2cs^3 - 2c^3s - 2cs^3\rho + 2c^3s\rho)u_{xy}^-\\ &\quad + (c^2\rho - c^4 + c^2s^2 - 2c^2s^2\rho)u_{yy}^- + s^2D_1 - \frac{2csD_2}{\beta^+} + c^2D_3,\end{aligned} \tag{210}$$

$$\begin{aligned}u_{xy}^+ &= -(2cs^3 - cs^3\rho + c^3s\rho - cs\rho)u_{xx}^- + (c^4\rho + s^4\rho + 4c^2s^2 - 2c^2s^2\rho)u_{xy}^-\\ &\quad - (2c^3s + cs^3\rho - c^3s\rho - cs\rho)u_{yy}^- - csD_1 + (c^2 - s^2)\frac{D_2}{\beta^+} + csD_3,\end{aligned} \tag{211}$$

$$\begin{aligned}u_{yy}^+ &= (\rho s^2 - s^4 + c^2s^2 - 2c^2s^2\rho)u_{xx}^- + (2cs^3 - 2c^3s - 2cs^3\rho + 2c^3s\rho)u_{xy}^-\\ &\quad + (\rho s^2 + c^4 - c^2s^2 + 2c^2s^2\rho)u_{yy}^- + c^2D_1 + \frac{2csD_2}{\beta^+} + s^2D_3,\end{aligned} \tag{212}$$

where $\rho = \beta^-/\beta^+$, $c = \cos(\theta)$ and $s = \sin(\theta)$. Here, β^- and β^+ are the limiting values of the coefficient β from inside and outside of the interface respectively. And from coordinate transformation of D_1 and D_2 are

$$\begin{aligned} D_1 = & -\kappa c(u_x^+ - u_x^-) - \kappa s(u_y^+ - u_y^-) + (s^2 w_{xx} - 2cs w_{xy} + c^2 w_{yy}) \\ & + \kappa(c w_x + s w_y), \end{aligned} \tag{213}$$

$$D_2 = \kappa c(\beta^+ u_y^+ - \beta^- u_y^-) - \kappa s(\beta^+ u_x^+ - \beta^- u_x^-) + (-s v_x + c v_y). \tag{214}$$

4.1.3 APPROXIMATION OF THE CORRECTION TERMS

In this section, we will discuss how to interpolate the correction terms $(C_{i,j}^x)^n$ and $(C_{i,j}^y)^n$. Our target is to approximate the correction terms so that the resulting Crank-Nicolson scheme for a two-dimensional parabolic interface problem with constant β would be the same as the one without an interface but with some slight modification to source term only.

So, we follow the same approach for obtaining the correction terms for two-dimensional elliptic interface problems. However, note that all our jump conditions are now time-dependent. Given an irregular grid point (x_i, y_j), we first select a control point $\mathbf{x}^* = (x^*, y^*)$ on the interface Γ which is closest to the irregular point (x_i, y_j).

First, let's assume that (x_i, y_j) is an irregular point that lies in Ω^- and it's five-point stencil is only cut through the right arm. Then in order to obtain the Crank-Nicolson scheme at the irregular point (x_i, y_j), we consider the Taylor expansion of all nine points of the form $u(x_{i+i_m}, y_{j+j_m})$ in the stencil of irregular grid point (x_i, y_j) at control point $\mathbf{x}^* = (x^*, y^*)$ at $t = t_n$.

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}) = & u^\pm + (x_{i+i_m} - x^*)u_x^\pm + (y_{j+j_m} - y^*)u_y^\pm + \frac{1}{2}(x_{i+i_m} - x^*)^2 u_{xx}^\pm \\ & + \frac{1}{2}(y_{j+j_m} - y^*)^2 u_{yy}^\pm + (x_{i+i_m} - x^*)(y_{j+j_m} - y^*)u_{xy}^\pm + O(h^3), \end{aligned} \tag{215}$$

where plus or minus sign is chosen depending on whether (x_{i+i_m}, y_{j+j_m}) lies in Ω^+ or Ω^-. Note that right-hand side of the above equation can be rewritten in terms of the limiting values u^-, u_x^-, u_y^-, u_{xx}^-, u_{xy}^- and u_{yy}^- by using the jump relations (207)-(212). where i_m, $j_m = \{-1, 0, 1\}$. So, we get a scheme,

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}) = & d_{i_m,j_m}^1 u^- + d_{i_m,j_m}^2 u_x^- + d_{i_m,j_m}^3 u_y^- + d_{i_m,j_m}^4 u_{xx}^- + d_{i_m,j_m}^5 u_{xy}^- + d_{i_m,j_m}^6 u_{yy}^- \\ & + d_{i_m,j_m}^7, \end{aligned} \tag{216}$$

where, the coefficients d_{i_m,j_m}^1 through d_{i_m,j_m}^6 are known time-independent quantities and coefficient d_{i_m,j_m}^7 is a time-dependent quantity. Now, we can see the correction term $(C_{i,j}^x)^n$ can also be reformulated as

$$(C_{i,j}^x)^n = b_1 u^- + b_2 u_x^- + b_3 u_y^- + b_4 u_{xx}^- + b_5 u_{xy}^- + b_6 u_{yy}^- + b_7, \tag{217}$$

where b_1 through b_6 are also known time-independent quantities, but b_7 is time-dependent.

We assume that the correction term $(C^x_{i,j})^n$ can be written using the nine-point stencil of the irregular grid point (x_i, y_j) as follows,

$$(C^x_{i,j})^n = \sum_{m}^{n_s} \alpha_m U^n_{i+i_m, j+j_m} + \alpha^x_c, \tag{218}$$

where, n_s is the number of grid points involved in finding correction term, which we took as nine, α_m and α^x_c depend on the location of the grid point relative to the control point $\mathbf{x}^*$, jump ratio ρ, and limiting values of β. Moreover, α_m is time-independent while α^x_c is time-dependent. Therefore, for each irregular grid point (x_i, y_j), the coefficient α_m need to be found only at once for each time $t = t_n$, but α^x_c should be updated. By comparing equations (216), (217) and (218), we get the following linear system,

$$\begin{bmatrix} d^1_{-1,-1} & d^1_{0,-1} & \cdots & d^1_{0,1} & d^1_{1,1} \\ d^2_{-1,-1} & d^2_{0,-1} & \cdots & d^2_{0,1} & d^2_{1,1} \\ d^3_{-1,-1} & d^3_{0,-1} & \cdots & d^3_{0,1} & d^3_{1,1} \\ d^4_{-1,-1} & d^4_{0,-1} & \cdots & d^4_{0,1} & d^4_{1,1} \\ d^5_{-1,-1} & d^5_{0,-1} & \cdots & d^5_{0,1} & d^5_{1,1} \\ d^6_{-1,-1} & d^6_{0,-1} & \cdots & d^6_{0,1} & d^6_{1,1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_8 \\ \alpha_9 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix}. \tag{219}$$

It seems we have to find nine unknown α coefficients. To find the αs, we have to solve a system of equations with six equations. We have an underdetermined linear system since the number of unknowns is greater than the number of equations. To solve the linear system, one can use the singular value decomposition (SVD) to get the minimum norm. We can find α^x_c as follows,

$$\alpha^x_c = b_7 - \sum_{m}^{n_s} d^7_{i_m, j_m} \cdot \alpha_m. \tag{220}$$

As α_m are time-independent, we only need to solve the linear system given by equation (219) at once. At this stage, Crank-Nicolson scheme at the irregular grid point (x_i, y_j) is fully determined. Now, linear system of the Crank-Nicolson scheme can be written as,

$$A\mathbf{U}^{n+1} = B\mathbf{U}^n + \mathbf{F}, \tag{221}$$

where, $\mathbf{U}^n$ is the numerical solution of the parabolic interface problem given by equation (186) at $t = t_n$, A and B are the coefficient matrix and $\mathbf{F}$ is a vector constructed from the source term f, boundary conditions and correction terms. For constant β, the matrix A and B are identical to the five-diagonal matrices for an equivalent problem without an interface, and the vector $\mathbf{F}$ is only needed to be changed due to corrections terms. However, when β is piecewise constant, A and B are nine-point banded matrices for irregular grid points and five-diagonal matrices for regular grid points.

In order to recover the gradient of the solution at the interface, we can still use the same formulas (98)-(101) in chapter 2.

4.1.4 AN OUTLINE OF ALGORITHM

In this section, an outline of the algorithm for solving two-dimensional parabolic interface problems is given.

Step 1: Immerse the irregular domain (interface) into a square domain and represent the interface using a zero level set function.

Step 2: Determine regular, irregular grid points and location of control points.

Step 3: Use standard 5-point central difference scheme at regular grid points for spatial discretization.

Step 4: Find α_m to get the time-independent part of the correction terms at grid points by solving the linear system equation (219).

Step 5: Find a 9-point compact scheme at each irregular point.

Step 6: Use Crank-Nicolson scheme for time discretization

Step 7: Calculate α_c which is the time-dependent part of the correction terms using equation (220) at each irregular grid point for each time step

Step 8: Solve the linear system $A\mathbf{U}^{n+1} = B\mathbf{U}^n + \mathbf{F}$ for each time step

Step 9: Recover the gradients of the solution using equations (98) to (101) in chapter 2.

4.2 NUMERICAL EXAMPLES

Here we do a numerical experiment and perform error analysis to show the performance of the developed method for solving parabolic interface problems. The errors are illustrated in L^∞ norm and order of convergence of the method is calculated using

$$r = \frac{1}{\log(2)} \log \frac{\| E_{2h} \|_\infty}{\| E_h \|_\infty}. \tag{222}$$

In all listed tables in this section, N is the number of the grid points in each direction. $E(u)$ is the maximum norm error of the numerical solution. $E(u_{\boldsymbol{n}})$ and $E(u_\eta)$ are the maximum norm error in the normal and tangential directions.

Example 4.1

In this example, both the solution $u(x, y, t)$ and its flux have a jump discontinuity. The differential equation is given by,

$$u_t = (\beta u_x)_x + (\beta u_y)_y - f, \tag{223}$$

on the domain $\Omega = [-1, 1] \times [-1, 1]$, where interface Γ is a circle represented by the zero level set of the function,

$$\phi(x, y) = x^2 + y^2 - 0.5^2. \tag{224}$$

The source term f is defined as,

$$f(x, y, t) = \begin{cases} 0 & \text{if} \quad (x, y) \in \Omega^-, \\ e^{-t}(x^2 - y^2) & \text{if} \quad (x, y) \in \Omega^+. \end{cases} \tag{225}$$

The coefficient β in equation (223) is a piecewise constant given by,

$$\beta(x, y) = \begin{cases} \beta^- & \text{if} \quad (x, y) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x, y) \in \Omega^+. \end{cases} \tag{226}$$

The jump in solution $u(x, y, t)$ and it's flux are given by,

$$[u] = e^{-t}(x^2 - y^2), \tag{227}$$

$$[\beta u_{\boldsymbol{n}}] = 4\beta^+ e^{-t}(x^2 - y^2). \tag{228}$$

Boundary and initial conditions were taken from the exact solution of,

$$u(x, y, t) = \begin{cases} 0 & \text{if} \quad (x, y) \in \Omega^-, \\ e^{-t}(x^2 - y^2) & \text{if} \quad (x, y) \in \Omega^+. \end{cases} \tag{229}$$

In Tables 12 - 14, we present the numerical experiment results for grid refinement analysis of Example 4.1 for different jumps in β at final time $T = 1$. In this study we use time step $\Delta t = h$. The second column in all the tables illustrates the maximum norm error of the numerical solution. The third column is the order convergence of the proposed method for spatial discretization. The fourth and fifth columns demonstrate the maximum norm error in the normal direction and its order convergence, respectively. Moreover, six and seven columns represent the maximum norm error in the tangential direction and order convergence. It can be seen that the solution and the solution's derivatives (normal and tangential) are on the average of second-order accuracy. Figure 17 shows a slice of the computed solution and error of the computed solution for Example 4.1 when $[\beta] = 1000$.

Table 12
A grid refinement analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
20	7.08E-06		1.83E-05		1.95E-05	
40	1.76E-06	2.01	4.52E-06	2.02	4.88E-06	2.00
80	4.42E-07	1.99	1.13E-06	1.99	1.21E-06	2.01
160	1.10E-07	2.00	2.84E-07	2.00	3.03E-07	2.00
360	2.18E-08	2.34	5.61E-08	2.34	5.96E-08	2.34

Table 13
A grid refinement analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 1000$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
20	9.67E-09		6.83E-08		6.21E-08	
40	2.46E-09	1.98	2.53E-08	1.44	3.02E-08	1.04
80	5.81E-10	2.08	6.99E-09	1.85	8.50E-09	1.83
160	1.57E-10	1.89	1.41E-09	2.31	3.02E-09	1.49
360	2.87E-11	2.45	7.74E-11	4.19	3.15E-10	3.26

We also did numerical experiments to find order convergence of the temporal discretization. In Tables 15 - 17, we demonstrate the temporal discretization analysis results for different jump ratio for Example 4.1 at $T = 2$. In these tables, Δt is the time step, $E(u)$, $E(u_n)$, $E(u_\eta)$ are in the maximum norm as explained previously. For this analysis, we kept the number of grid points in the computational domain as constant and set it to $N = 80$. It can be seen that the proposed method can also achieve average of second order in time for both solution and the solution's derivatives for the present parabolic interface problem.

Table 14
A grid refinement analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
20	9.80E-10		7.19E-09		6.51E-09	
40	2.66E-10	1.88	2.99E-09	1.27	3.51E-09	0.89
80	6.83E-11	1.96	1.13E-09	1.41	1.33E-09	1.40
160	1.59E-11	2.10	3.49E-10	1.69	4.96E-10	1.42
360	3.11E-12	2.35	8.06E-11	2.12	1.87E-10	1.41

Table 15
Temporal discretization analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 1$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
1/40	4.42E-07		1.13E-06		1.21E-06	
1/80	1.10E-07	2.00	2.88E-07	1.98	3.03E-07	2.00
1/160	2.76E-08	2.00	7.21E-08	2.00	7.58E-08	2.00
1/320	6.90E-09	2.00	1.80E-08	2.00	1.89E-08	2.00

Table 16
Temporal discretization analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 1000$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
1/40	5.81E-10		6.99E-09		8.50E-09	
1/80	1.94E-10	1.58	2.00E-09	1.81	2.83E-09	1.59
1/160	3.41E-11	2.51	2.62E-11	6.25	1.44E-10	4.29
1/320	8.35E-12	2.03	8.37E-13	4.97	3.35E-11	2.10

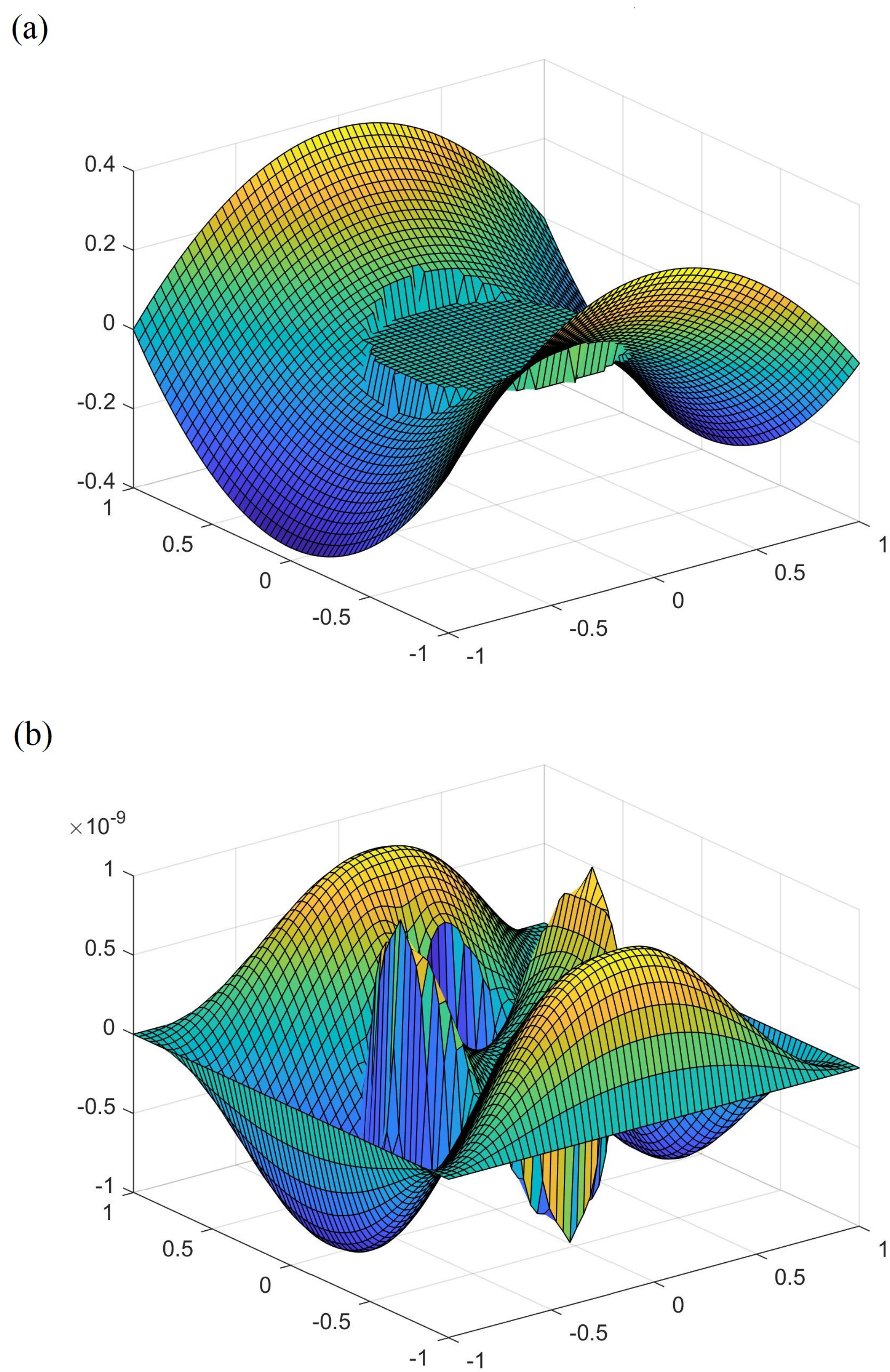


Fig. 17. (a) Plot of the numerical solution for Example 4.1, (b) Plot of the error of numerical solution with $N = 60$, $\beta^- = 1$ and $\beta^+ = 1000$ at $T = 1$.

Table 17
Temporal discretization analysis for Example 4.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r
1/40	6.83E-11		1.13E-09		1.33E-09	
1/80	2.14E-11	1.68	4.43E-10	1.35	4.96E-10	1.42
1/160	3.24E-12	2.72	6.41E-11	2.79	9.87E-11	2.33
1/320	8.10E-13	2.00	3.95E-12	4.02	6.09E-12	4.02

4.3 SUMMARY AND DISCUSSION

In this section, we present an algorithm for solving two-dimensional parabolic interface problems. Here we use the Crank-Nicolson scheme together with modified direct IIM to solve parabolic interface problems. The resulting method achieves second-order accuracy in both space and time for fixed interfaces for both solution and gradients. Moreover, the proposed method is unconditionally stable as it uses the Crank-Nicolson scheme for time discretization.

CHAPTER 5

STUDY FOR SOLVING THREE-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS

In this chapter, we propose a direct method for solving three-dimensional parabolic interface problems with fixed interfaces. The method uses the Crank-Nicholson scheme together with some modifications to the three-dimensional direct IIM that we developed in chapter 3.

Consider the following parabolic interface problem

$$u_t = (\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z - f(x, y, z, t), \quad (x, y, z) \in \Omega \backslash \Gamma, \tag{230}$$

$$[u] = w(:, t), \tag{231}$$

$$[\beta u_{\boldsymbol{n}}] = v(:, t), \tag{232}$$

with specified boundary and initial conditions.

5.1 CRANK-NICOLSON SCHEME FOR THREE-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS

For any grid point (x_i, y_j, z_l), the Crank-Nicolson scheme for equation (230) is as follows,

$$\begin{aligned} \frac{U_{ijl}^{n+1} - U_{ijl}^{n}}{\Delta t \beta_{ijl}} = \frac{1}{2}(&\delta_x U_{ijl}^{n+1} + (C_{ijl}^{x})^{n+1} + \delta_y U_{ijl}^{n+1} + (C_{ijl}^{y})^{n+1} + \delta_z U_{ijl}^{n+1} + (C_{ijl}^{z})^{n+1} \\ &- \frac{f_{ijl}^{n+1}}{\beta_{ijl}} + \delta_x U_{ijl}^{n} + (C_{ijl}^{x})^{n} + \delta_y U_{ijl}^{n} + (C_{ijl}^{y})^{n} + \delta_z U_{ijl}^{n} + (C_{ijl}^{z})^{n} - \frac{f_{ijl}^{n}}{\beta_{ijl}}). \end{aligned} \tag{233}$$

where,

$$\delta_x U_{ijl}^{n} = \frac{U_{i-1,j,l}^{n} - U_{i,j,l}^{n} + U_{i+1,j,l}^{n}}{h^2}, \tag{234}$$

and correction term for particular irregular grid point whose seven-point stencil cut through the right arm at the control point $\mathbf{x}^* = (x^*, y^*, z^*)$ is given by,

$$C_{ijl}^{x} = \pm\frac{[u]}{h^2} \pm [u_x]\frac{(x_{i+1} - x^*)}{h^2} \pm [u_{xx}]\frac{(z_{i+1} - z^*)^2}{2h^2}, \tag{235}$$

where, plus or minus sign is chosen depending on whether the irregular grid point (x_i, y_j, z_l) lies in plus or minus sign of the interface Γ.

5.1.1 LOCAL COORDINATE TRANSFORMATION OF JUMP RELATIONS

Now using the local coordinate transformation system (124) and by differentiating the jump relations $[u] = w$ and $[\beta u_{\boldsymbol{n}}] = v$ with respect to η and ζ, we can obtain

$$[u] = w, \tag{236}$$

$$[u_\eta] = w_\eta, \tag{237}$$

$$[u_\zeta] = w_\zeta, \tag{238}$$

$$[\beta u_\xi] = v, \tag{239}$$

$$[u_{\eta\eta}] = -\chi_{\eta\eta}[u_\xi] + w_{\eta\eta} = D_1, \tag{240}$$

$$[u_{\zeta\zeta}] = -\chi_{\zeta\zeta}[u_\xi] + w_{\zeta\zeta} = D_2, \tag{241}$$

$$[u_{\eta\zeta}] = -\chi_{\eta\zeta}[u_\xi] + w_{\eta\zeta} = D_3, \tag{242}$$

$$[\beta u_{\xi\eta}] = \chi_{\eta\eta}[\beta u_\eta] + \chi_{\eta\zeta}[\beta u_\zeta] + v_\eta = D_4, \tag{243}$$

$$[\beta u_{\xi\zeta}] = \chi_{\eta\zeta}[\beta u_\eta] + \chi_{\zeta\zeta}[\beta u_\zeta] + v_\zeta = D_5. \tag{244}$$

And from governing equation (230)

$$[\beta u_{\xi\xi}] = -[\beta u_{\eta\eta}] - [\beta u_{\zeta\zeta}] + [f] + w_t = D. \tag{245}$$

As similar to the two-dimensional case for solving parabolic interface problems, we do the coordinate transformation of all jump relations given by the equations (236)-(245) into Cartesian coordinates and express all limiting values u^+, u_x^+, u_y^+, u_z^+, u_{xx}^+, u_{xy}^+, u_{xz}^+, u_{yy}^+, u_{yz}^+, u_{zz}^+ in terms of u^-, u_x^-, u_y^-, u_z^-, u_{xx}^-, u_{xy}^-, u_{xz}^-, u_{yy}^-, u_{yz}^- and u_{zz}^-. Derivation of these expression are given in the Appendix B. Notice that D_1 through D_5 and can also be expressed in terms of u_x^-, u_y^-, u_z^-.

5.1.2 APPROXIMATION OF THE CORRECTION TERMS

In this section, we will discuss how to interpolate the correction terms $(C_{ijl}^x)^n$, $(C_{ijl}^y)^n$ and $(C_{ijl}^z)^n$.

We follow the same approach as we did in solving three-dimensional elliptic interface problems in chapter 3 to obtain the correction terms. However, note that all our jump conditions are now time-dependent. Given an irregular grid point (x_i, y_j, z_l), we first select a control point $\mathbf{x}^* = (x^*, y^*, z^*)$ on the interface Γ which is closest to the irregular point (x_i, y_j, z_l).

First, let's assume that (x_i, y_j, z_l) is an irregular point which lies in Ω^- and it's seven-point stencil is only cut through the right arm. Then in order to obtain the Crank-Nicolson scheme at the irregular point (x_i, y_j, z_l), we will consider the Taylor expansion of $u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m})$ about the control point $\mathbf{x}^* = (x^*, y^*, z^*)$ at $t = t_n$.

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m}) = {} & u^{\pm} + (x_{i+i_m} - x_i^*)u_x^{\pm} + (y_{j+j_m} - y_j)u_y^{\pm} + (z_{l+l_m} - z_l)u_z^{\pm} \\ & + \frac{1}{2}(x_{i+i_m} - x_i^*)^2 u_{xx}^{\pm} + \frac{1}{2}(y_{j+j_m} - y_j)^2 u_{yy}^{\pm} + \frac{1}{2}(z_{l+l_m} - z_l)^2 u_{zz}^{\pm} \\ & + (x_{i+i_m} - x_i^*)(y_{j+j_m} - y_j)u_{xy}^{\pm} + (x_{i+i_m} - x_i^*)(z_{l+l_m} - z_l)u_{xz}^{\pm} \\ & + (y_{j+j_m} - y_j^*)(z_{l+l_m} - z_l)u_{yz}^{\pm} + O(h^3), \end{aligned} \tag{246}$$

where plus or minus sign is chosen depending on whether $(x_{i+i_m}, y_{j+j_m}, z_{l+l_m})$ lies in Ω^+ or Ω^- and i_m, j_m, $l_m = \{-1, 0, 1\}$. Note that right-hand side of the equation (246) can be written in terms of the limiting values u^-, u_x^-, u_y^-, u_z^-, u_{xx}^-, u_{xy}^-, u_{xz}^-, u_{yy}^-, u_{yz}^- and u_{zz}^- by using the coordinate transformation of the jump relations given by the equations (236)-(245). So, we get a scheme,

$$\begin{aligned} u(x_{i+i_m}, y_{j+j_m}, z_{l+l_m}) = {} & c_m^1 u^- + c_m^2 u_x^- + c_m^3 u_y^- + c_m^4 u_z^- + c_m^5 u_{xx}^- + c_m^6 u_{xy}^- \\ & + c_m^7 u_{xz}^- + c_m^8 u_{yy}^- + c_m^9 u_{yz}^- + c_m^{10} u_{zz}^- + c_m^{11}. \end{aligned} \tag{247}$$

Here, the coefficients c_m^1 through c_m^{10} are known time-independent quantities. And the coefficient c_m^{11} is a time-dependent quantity. Now, we can see the correction term $(C_{ijl}^x)^n$ can also be reformulated as,

$$(C_{ijl}^x)^n = a_1 u^- + a_2 u_x^- + a_3 u_y^- + a_4 u_z^- + a_5 u_{xx}^- + a_6 u_{xy}^- + a_7 u_{xz}^- + a_8 u_{yy}^- + a_9 u_{yz}^- + a_{10} u_{zz}^- + a_{11}, \tag{248}$$

where a_1 through a_{10} are also known time-independent quantities, but a_{11} is a time-dependent quantity.

We assume that the correction term $(C_{ijl}^x)^n$ can be approximated as,

$$(C_{ijl}^x)^n = \sum_{m}^{n_s} \alpha_m U_{i+i_m, j+j_m, l+l_m}^n + \alpha_c^x. \tag{249}$$

By comparing the equations (247) and (248), we get the following linear system,

$$\begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_3^1 & c_{m-1}^1 & c_m^1 \\ c_1^2 & c_2^2 & \cdots & c_3^2 & c_{m-1}^2 & c_m^2 \\ & & \cdots & & & \\ c_1^8 & c_2^8 & \cdots & c_3^8 & c_{m-1}^8 & c_m^8 \\ c_1^9 & c_2^9 & \cdots & c_3^9 & c_{m-1}^9 & c_m^9 \\ c_1^{10} & c_2^{10} & \cdots & c_3^{10} & c_{m-1}^{10} & c_m^{10} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_{m-1} \\ \alpha_m \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \\ a_{10} \end{bmatrix}. \tag{250}$$

We can find α_c^x in equation (249) as,

$$\alpha_c^x = a_{11} - \sum_m^{n_s} c_m^{11} \cdot \alpha_m \tag{251}$$

Interpolation for $(C_{ijl}^y)^n$ and $(C_{ijl}^z)^n$ will be in the same form, only with coefficients changed.

As for the direct method for solving three-dimensional elliptic interface problems in Chapter 3, we take $n_s = 27$. That is, we will use the twenty-seven compact stencil as in Figure 12 illustrated in chapter 3. We have an underdetermined linear system since the number of unknowns is greater than the number of equations. So, to solve the linear system (250) with ten equations and 27 unknowns, one can use the singular value decomposition (SVD) to get the minimum norm solution.

As α_m's are time-independent, we only need to solve the linear system given by equation (250) at once. At this stage, the Crank-Nicolson scheme at the irregular grid point (x_i, y_j, z_l) is fully determined. Now, linear system of the Crank-Nicolson scheme can be written as,

$$A\mathbf{U}^{n+1} = B\mathbf{U}^n + \mathbf{F} \tag{252}$$

where, $\mathbf{U}^n$ is the numerical solution of the parabolic interface problem given by equation (230) at $t = t_n$, A and B are the coefficient matrix and $\mathbf{F}$ is a vector constructed from the source term f, boundary conditions and correction terms. For constant β, the matrix A and B are identical to the seven-diagonal matrices for an equivalent problem without an interface, and the right-hand sided vector $\mathbf{F}$ only need to be corrected. However, when β is piecewise constant, A and B are twenty-seven point banded matrices for irregular grid points and seven-diagonal matrices for regular grid points.

In order to recover the gradient of the solution at the interface, we can still use the formulas (165)-(170) in chapter 3.

5.1.3 AN OUTLINE OF ALGORITHM

In this section, the outline of the algorithm for solving three-dimensional parabolic interface problems is given.

Step 1: Immerse the irregular domain (interface) into a cubic domain and represent the interface using a zero level set function.

Step 2: Determine the regular and irregular grid points and the location of control points.

Step 3: Use standard 7-point central difference scheme at regular grid points for spatial discretization.

Step 4: Find α_ms to get the time-independent part of the correction terms at grid point by solving the linear system (250).

Step 5: Find 27-point compact scheme at each irregular point.

Step 6: Use Crank-Nicolson scheme for time discretization

Step 7: Calculate α_c which is the time-dependent part of the correction terms using equation (251) at each irregular grid point for each time step

Step 8: Solve the linear system $A\mathbf{U}^{n+1} = B\mathbf{U}^n + \mathbf{F}$ for each time step

Step 9: Recover the gradients of the solution using equations (165) to (170) in chapter 3.

5.2 NUMERICAL EXAMPLES

Here we do a numerical experiment and perform error analysis to show the performance of the Crank-Nicolson scheme together with the direct IIM method to solve parabolic interface problems. The errors are illustrated in L^∞ norm and order of convergence of the method is calculated using,

$$r = \frac{1}{\log(2)} \log \frac{\| E_{2h} \|_\infty}{\| E_h \|_\infty}. \tag{253}$$

In all listed tables in this section, N is the number of the grid points in each direction of the computational domain.

Example 5.1

In this example, both the solution $u(x, y, z, t)$ and its flux have a jump discontinuity. The differential equation is given by,

$$u_t = (\beta u_x)_x + (\beta u_y)_y + (\beta u_z)_z - f, \tag{254}$$

on the domain $\Omega = [-2,2] \times [-2,2] \times [-2,2]$. where the interface Γ is a sphere represented by the zero level set of the function as,

$$\phi(x,y,z) = x^2 + y^2 + z^2 - 1. \tag{255}$$

The source term f is defined as,

$$f(x,y,z,t) = \begin{cases} 0 & \text{if} \quad (x,y,z) \in \Omega^-, \\ e^{-t}(6\beta^+ + (x^2+y^2+z^2)) & \text{if} \quad (x,y,z) \in \Omega^+, \end{cases} \tag{256}$$

where, coefficient β is a piecewise constant,

$$\beta(x,y,z) = \begin{cases} \beta^- & \text{if} \quad (x,y,z) \in \Omega^-, \\ \beta^+ & \text{if} \quad (x,y,z) \in \Omega^+. \end{cases} \tag{257}$$

The jump in solution $u(x,y,z,t)$ and it's flux are given by,

$$[u] = e^{-t}, \tag{258}$$

$$[\beta u_{\boldsymbol{n}}] = \beta^+ e^{-t}. \tag{259}$$

Boundary and initial conditions were taken from the exact solution of,

$$u(x,y,t) = \begin{cases} 0 & \text{if} \quad (x,y,z) \in \Omega^-, \\ e^{-t}(x^2+y^2+z^2) & \text{if} \quad (x,y,z) \in \Omega^+. \end{cases} \tag{260}$$

In Tables 18 to 20, we present the numerical experiment results for grid refinement analysis of Example 5.1 for jump in β as 1, 1000 and 10000 respectively. In this study we use time step $\Delta t = h$ and final time $T = 2$. In all tables, $E(u)$ is the error in the computed solution in the maximum norm, $E(u_n)$ is the error in the normal derivative of the solution at the interface. $E(u_\eta)$ and $E(u_\zeta)$ are the error in the tangential derivative of the solution at the interface. It can be seen that the solution and the solution's derivatives (normal and tangential) are on the average of second-order accuracy. Figure 18 shows a slice of the computed solution and error of the computed solution for Example 5.1 when $[\beta] = 10000$.

Table 18
A grid refinement analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 1$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
10	5.03E-03		1.95E-03		4.82E-04		4.82E-04	
20	1.34E-03	1.90	4.56E-04	2.10	9.62E-05	2.32	1.02E-04	2.24
40	3.38E-04	1.99	1.14E-04	2.00	2.23E-05	2.11	2.43E-05	2.07
80	8.46E-05	2.00	2.86E-05	2.00	5.56E-06	2.00	6.21E-06	1.97

Table 19
A grid refinement analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 1000$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
10	1.50E-05		1.44E-05		6.61E-06		7.87E-06	
20	3.09E-06	2.28	6.72E-06	1.10	6.73E-06	0.03	6.73E-06	0.23
40	5.89E-07	2.39	2.41E-06	1.48	3.39E-06	0.99	3.40E-06	0.99
80	8.61E-08	2.77	5.13E-07	2.23	7.55E-07	2.17	7.55E-07	2.17

We also did numerical experiments to find the order convergence of the temporal discretization. In Tables 21 - 23, we show the numerical experiment results for temporal discretization of Example 5.1 with jump in β as 1, 1000 and 10000 at $T = 0.1$. For this experiment, we kept the number of grid points constant and used $N = 80$. Δt is the time step. It can be seen that the proposed method can also achieve an average of second-order in time for both solution and the solution's derivatives for the present parabolic interface problem.

Table 20
A grid refinement analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

N	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
10	1.54E-06		1.41E-06		7.41E-07		8.99E-07	
20	3.34E-07	2.20	6.99E-07	1.02	7.21E-07	0.04	7.21E-07	0.32
40	7.38E-08	2.18	2.84E-07	1.30	4.03E-07	0.84	4.03E-07	0.84
80	1.66E-08	2.16	8.10E-08	1.81	1.22E-07	1.72	1.21E-07	1.73

Table 21
Temporal discretization analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 1$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
1/40	2.34E-05		1.51E-05		2.86E-07		2.86E-07	
1/80	5.82E-06	2.01	3.80E-06	1.99	1.17E-07	1.29	1.17E-07	1.29
1/160	1.45E-06	2.00	9.55E-07	1.99	3.29E-08	1.83	3.29E-08	1.83
1/320	3.63E-07	2.00	2.39E-07	2.00	8.24E-09	2.00	8.25E-09	2.00

Table 22
Temporal discretization analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 1000$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
1/40	7.07E-08		7.84E-08		1.46E-07		1.31E-07	
1/80	1.98E-08	1.84	3.72E-08	1.07	5.47E-08	1.42	5.47E-08	1.26
1/160	4.18E-09	2.24	4.06E-10	6.52	2.52E-09	4.44	2.53E-09	4.43
1/320	1.04E-09	2.01	3.38E-11	3.59	1.90E-10	3.73	1.95E-10	3.70

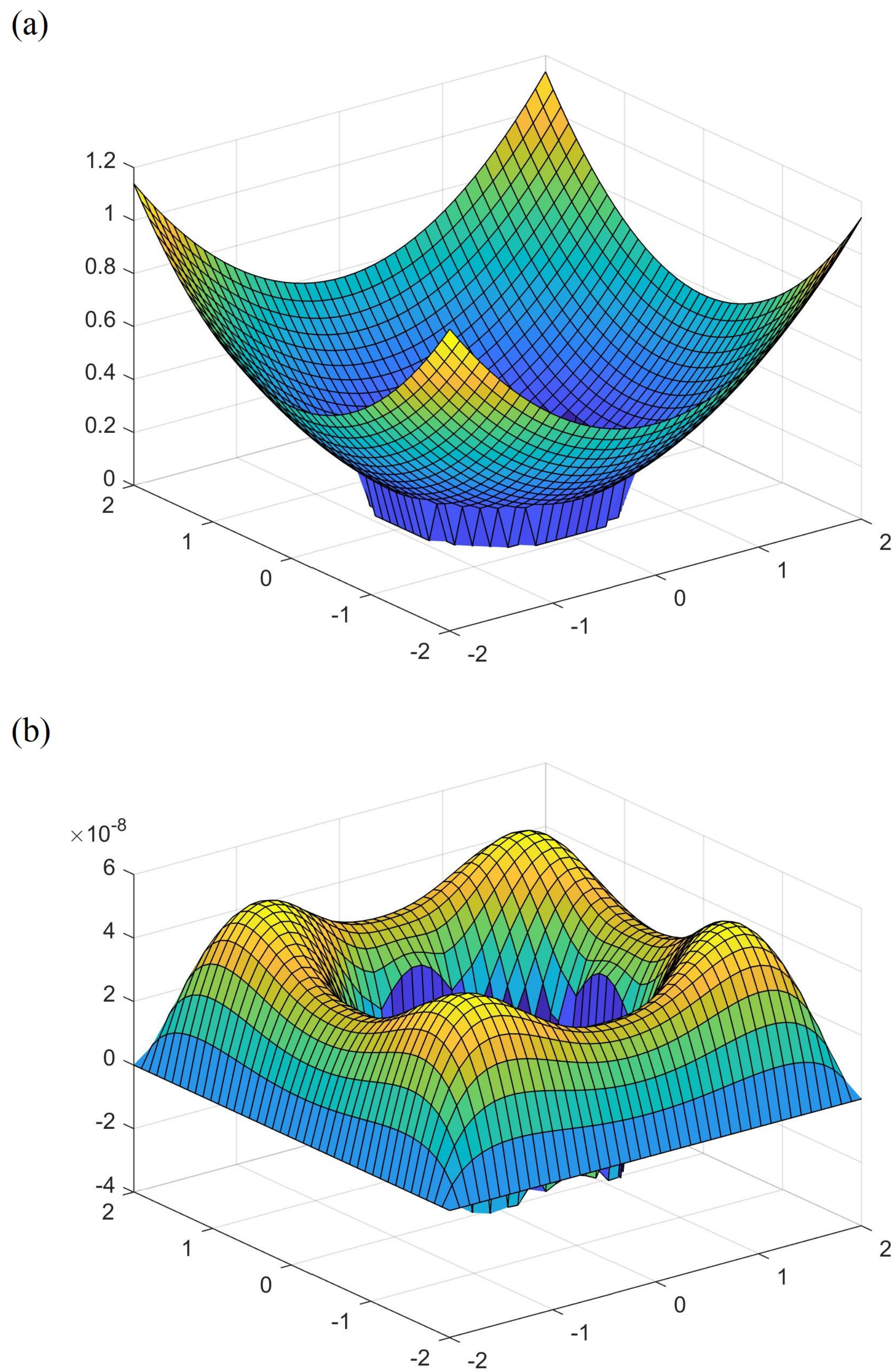


Fig. 18. (a) Plot of numerical solution for Example 5.1, (b) Plot of the error of numerical solution with $N = 60$, $\beta^- = 1$ and $\beta^+ = 10000$ at $T = 1$.

Table 23
Temporal discretization analysis for Example 5.1 with $\beta^- = 1$ and $\beta^+ = 10000$.

Δt	$E(u)$	r	$E(u_n)$	r	$E(u_\eta)$	r	$E(u_\zeta)$	r
1/40	6.88E-09		1.01E-08		1.65E-08		1.47E-08	
1/80	1.91E-09	1.85	6.08E-09	0.74	9.09E-09	0.86	9.09E-09	0.69
1/160	4.16E-10	2.20	1.17E-09	2.38	1.99E-09	2.19	2.00E-09	2.18
1/320	1.03E-10	2.01	1.14E-10	3.36	2.02E-10	3.30	2.03E-10	3.30

5.3 SUMMARY AND DISCUSSION

In this section, we present an algorithm for solving three-dimensional parabolic interface problems. Here we use the Crank-Nicolson scheme and modifications to the direct IIM developed in chapter 3 to solve parabolic interface problems. The resulting method achieves second-order accuracy in both space and time for both the solution and its derivatives at the interface. Furthermore, the proposed method is unconditional stable as it uses the Crank-Nicolson scheme for time discretization.

CHAPTER 6

CONCLUSIONS AND FUTURE WORK

In this dissertation, we have mainly studied developing a direct method for solving three-dimensional elliptic interface problems and its application for solving parabolic interface problems.

The developed direct IIM is an extension of the work in [58] for solving one and two-dimensional elliptic interface problems into three-dimensions. As many practical interface problems in the real world can be approximated with PDEs with piecewise constant coefficients, we focused our model derivation on interface problems with piecewise constant coefficients but have finite jumps across an interface. The derived model can obtain second-order accuracy globally for both the solution and the solution's gradient.

The prominent characteristic of the method is that our computational framework is based on finite difference schemes on uniform Cartesian grids. Moreover, it does not require an augmented variable as in FIIM. So the implementation of the method is easier to understand for non-experts in the area. As in the augmented method, we first divide the PDE in each subdomain by its coefficient β. In this way, we can use the standard seven-point stencil for the Poisson equation with the constant coefficient for most grid points in the computational domain with some modification only into the right-hand side of the equation. For grid points near or on the interface (irregular grid points), we use a twenty-seven-point compact scheme derived considering the jump discontinuities in the solution, flux, and jump ratio. The resulting linear system for the twenty-seven compact scheme leads to a twenty-seven block diagonal matrix. However, it is primarily a seven-block diagonal matrix as most grid points in the computational domain are regular. When the coefficient in the PDE is constant, the resulting finite-difference scheme is reduced to a standard seven-point central difference scheme. Therefore, many fast-Poisson solvers can be used to solve the linear system for this case. We have also conducted an eigenvalue analysis to study the method's stability for a modest jump ratio. Our numerical experiments indicate that eigenvalues of the coefficient matrix of the linear system for the FD scheme are located in the left half-plane, which implies the stability of our method. An important feature we see in the FD scheme is that larger weights of coefficients in the FD scheme for irregular grid points are still coming from the original seven-point stencil, and the coefficients of the rest of the twenty points are much smaller. The model was implemented using both MATLAB and FORTRAN routines. When solving the linear system in FORTRAN, we have used GMRES iterative method, while in Matlab, we used the "\" operator. However, we can solve the linear system using an appropriate multigrid solver to make the method more efficient in future studies. We also plan to extend our direct IIM to solve incompressible two-phase Navier-Stokes equations.

Our dissertation has also studied how to apply the direct IIM method to solve parabolic interface problems with piecewise constant coefficients. So we first derive an algorithm for solving two-dimensional parabolic interface problems. Later we developed the algorithm for solving three-dimensional parabolic interface problems. Both two and three-dimensional algorithms use the Crank-Nicolson scheme together with some modification of the direct IIM. Also, we have observed that both schemes can accurately capture the jumps in the solution across the interface. We have conducted numerical experiments to understand the order of convergence of the methods in both space and time. Both two and three-dimensional methods can achieve second-order accuracy for both the solution and its gradient in both space and time. Moreover, the proposed methods are unconditionally stable as they use the Crank-Nicolson scheme for time discretization.

REFERENCES

[1] R. J. LeVeque, Z. Li, Immersed interface methods for stokes flow with elastic boundaries or surface tension, SIAM J. Sci. Comput. 18 (1997) 709–735.

[2] Z. Li, H. Zhao, H. Gao, A numerical study of electro-migration voiding by evolving level set functions on a fixed cartesian grid, J. Comput. Phys. 152 (1999) 281–304.

[3] D. F. M. Li, Zhilin, J. T. Heine, A numerical method for diffusive transport with moving boundaries and discontinuous material properties, Int. J. Numer. Anal. Methods Geomech. 21 (1997) 653–662.

[4] A. L. Fogelson, A mathematical model and numerical method for studying platelet adhesion and aggregation during blood clotting, J. Comput. Phys. 56 (1984) 111–134.

[5] B. Soni, Zhilin Li, Fast and accurate numerical approaches for stefan problems and crystal growth, Numer. Heat Transf. B: Fundam. 35 (1999) 461–484.

[6] Li, Zhilin, The immersed interface method: A numerical approach for partial differential equations with interfaces, Ph.D. thesis, University of Washington, 1994.

[7] S. Deng, Immersed interface method for three dimensional interface problems and applications, Ph.D. thesis, North Carolina State University, 2001.

[8] C. S. Peskin, The immersed boundary method, Acta Numer. 11 (2002) 479–517.

[9] A. M. Roma, C. S. Peskin, M. J. Berger, An adaptive version of the immersed boundary method, J. Comput. Phys. 153 (1999) 509–534.

[10] Z. Li, The immersed interface method using a finite element formulation, Appl. Numer. Math. 27 (1998) 253–267.

[11] X.-D. Liu, R. P. Fedkiw, M. Kang, A boundary condition capturing method for poisson's equation on irregular domains, J. Comput. Phys. 160 (2000) 151–178.

[12] S. Yu, Y. Zhou, G. Wei, Matched interface and boundary (mib) method for elliptic problems with sharp-edged interfaces, J. Comput. Phys. 224 (2007) 729–756.

[13] I. Babuška, The finite element method for elliptic equations with discontinuous coefficients, Computing 5 (1970) 207–213.

[14] J. H. Bramble, J. T. King, A finite element method for interface problems in domains with smooth boundaries and interfaces, Adv. Comput. Math. 6 (1996) 109–138.

[15] Z. Chen, J. Zou, Finite element methods and their convergence for elliptic and parabolic interface problems, Numer. Math. 79 (1998) 175–202.

[16] Z. Li, T. Lin, X. Wu, New cartesian grid methods for interface problems using the finite element formulation, Numer. Math. 96 (2003) 61–98.

[17] Z. Li, A fast iterative algorithm for elliptic interface problems, SIAM J. Math. Anal. 35 (1998) 230–254.

[18] P. N. Swarztrauber, Fast poisson solver, In Studies in Numerical Analysis 24 (1984) 319–370.

[19] R. J. LeVeque, Clawpack and amrclaw— software for high-resolution godunov methods, in: 4-th ntl. Conf. on Wave Propagation, Citeseer, 1998.

[20] M. J. Berger, R. J. Leveque, Adaptive mesh refinement using wave-propagation algorithms for hyperbolic systems, SIAM J. Math. Anal. 35 (1998) 2298–2316.

[21] S. Osher, R. P. Fedkiw, Level set methods and dynamic implicit surfaces, volume 153, Springer, 2003.

[22] S. Osher, J. A. Sethian, Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations, J. Comput. Phys. 79 (1988) 12–49.

[23] J. A. Sethian, Level set methods and fast marching methods: evolving interfaces in computational geometry, fluid mechanics, computer vision, and materials science, volume 3, Cambridge university press, 1999.

[24] L. Adams, A multigrid algorithm for immersed interface problems, in: NASA Conference Publication, 1996, pp. 1–14.

[25] P. M. De Zeeuw, Matrix-dependent prolongations and restrictions in a blackbox multigrid solver, J. Comput. Appl. Math. 33 (1990) 1–27.

[26] Z. Li, An overview of the immersed interface method and its applications, Taiwan. J. Math. 7 (2003) 1–49.

[27] Z. Li, K. Ito, The immersed interface method: numerical solutions of PDEs involving interfaces and irregular domains, SIAM, 2006.

[28] J. B. Bell, C. N. Dawson, G. R. Shubin, An unsplit, higher order godunov method for scalar conservation laws in multiple dimensions, J. Comput. Phys. 74 (1988) 1–24.

[29] G. R. Shubin, J. B. Bell, An analysis of the grid orientation effect in numerical simulation of miscible displacement, Comput. Methods Appl. Mech. Eng. 47 (1984) 47–71.

[30] A. N. Tikhonov, A. A. Samarskii, Homogeneous difference schemes, USSR Comput. Math. and Math. Phys. 1 (1962) 5–67.

[31] C. S. Peskin, Numerical analysis of blood flow in the heart, J. Comput. Phys. 25 (1977) 220–252.

[32] K. M. Arthurs, L. C. Moore, C. S. Peskin, E. B. Pitman, H. Layton, Modeling arteriolar flow and mass transport using the immersed boundary method, J. Comput. Phys. 147 (1998) 402–440.

[33] R. P. Beyer Jr, A computational model of the cochlea using the immersed boundary method, J. Comput. Phys. 98 (1992) 145–162.

[34] D. C. Bottino, Modeling viscoelastic networks and cell deformation in the context of the immersed boundary method, J. Comput. Phys. 147 (1998) 86–113.

[35] C. S. Peskin, D. M. McQueen, A three-dimensional computational method for blood flow in the heart i. immersed elastic fibers in a viscous incompressible fluid, J. Comput. Phys. 81 (1989) 372–405.

[36] D. M. McQueen, C. S. Peskin, A three-dimensional computational method for blood flow in the heart. ii. contractile fibers, J. Comput. Phys. 82 (1989) 289–297.

[37] R. Cortez, L. Fauci, N. Cowen, R. Dillon, Simulation of swimming organisms: coupling internal mechanics with external fluid dynamics, Comput. Sci. Eng. 6 (2004) 38–45.

[38] R. Dillon, L. Fauci, A. Fogelson, D. Gaver III, Modeling biofilm processes using the immersed boundary method, J. Comput. Phys. 129 (1996) 57–73.

[39] L. J. Fauci, Interaction of oscillating filaments: a computational study, J. Comput. Phys. 86 (1990) 294–313.

[40] A. L. Fogelson, Continuum models of platelet aggregation: formulation and mechanical properties, SIAM J. Appl. Math. 52 (1992) 1089–1110.

[41] A. L. Fogelson, C. S. Peskin, A fast numerical method for solving the three-dimensional stokes' equations in the presence of suspended particles, J. Comput. Phys. 79 (1988) 50–69.

[42] D. Sulsky, J. Brackbill, A numerical method for suspension flow, J. Comput. Phys. 96 (1991) 339–368.

[43] R. P. Beyer, R. J. LeVeque, Analysis of a one-dimensional model for the immersed boundary method, SIAM J. Math. Anal. 29 (1992) 332–364.

[44] R. Cortez, M. Minion, The blob projection method for immersed boundary problems, J. Comput. Phys. 161 (2000) 428–453.

[45] A. K. Tornberg, B. Engquist, Regularization techniques for numerical approximation of pdes with singularities, J. Sci. Comput. 19 (2003) 527–552.

[46] A. K. Tornberg, B. Engquist, Numerical approximations of singular source terms in differential equations, J. Comput. Phys. 200 (2004) 462–488.

[47] N. A. Battista, W. C. Strickland, L. A. Miller, Ib2d: a python and matlab implementation of the immersed boundary method, Bioinspir. Biomim. 12 (2017) 036003.

[48] R. J. LeVeque, Z. Li, The immersed interface method for elliptic equations with discontinuous coefficients and singular sources, SIAM J. Math. Anal. 31 (1994) 1019–1044.

[49] K. Pan, Y. Tan, H. Hu, An interpolation matched interface and boundary method for elliptic interface problems, J. Comput. Appl. Math. 234 (2010) 73–94.

[50] Z. Li, D. Wang, J. Zou, Theoretical and numerical analysis on a thermo-elastic system with discontinuities, J. Comput. Appl. Math. 92 (1998) 37–58.

[51] Z. Li, X. Yang, An immersed finite element method for elasticity equations with interfaces, Contemp. Math. 383 (2005) 285–298.

[52] X. Yang, Immersed interface method for elasticity problems with interfaces, Ph.D. thesis, North Carolina State University, 2004.

[53] T. Y. Hou, Z. Li, S. Osher, H. Zhao, A hybrid method for moving interface problems with application to the hele–shaw flow, J. Comput. Phys. 134 (1997) 236–252.

[54] A. Wiegmann, K. P. Bube, The immersed interface method for nonlinear differential equations with discontinuous coefficients and singular sources, SIAM J. Math. Anal. 35 (1998) 177–200.

[55] S. Jin, X. Wang, Robust numerical simulation of porosity evolution in chemical vapor infiltration: Ii. two-dimensional anisotropic fronts, J. Comput. Phys. 179 (2002) 557–577.

[56] K. Ito, K. Kunisch, Z. Li, Level-set function approach to an inverse interface problem, Inverse Probl. 17 (2001) 1225.

[57] G. Chen, Accurate Gradient Computation for Elliptic Interface Problems with Discontinuous and Variable Coefficients, Ph.D. thesis, North Carolina State University, 2015.

[58] X. Chen, X. Feng, Z. Li, A direct method for accurate solution and gradient computations for elliptic interface problems, Numer. Algorithms 80 (2019) 709–740.

APPENDIX A

COORDINATE TRANSFORMATION OF JUMP RELATIONS FOR THREE-DIMENSIONAL ELLIPTIC INTERFACE PROBLEMS

Here we explain how to do the coordinate transformation of the jump relations (147)-(155) in chapter 3 for deriving the appropriate correction terms at an irregular grid point and obtain direct relations for all the limiting values of the u and its derivatives from outside the interface in terms of the limiting values from inside the interface.

First, from the definition of the jump relation (147) itself, we can obtain,

$$u^+ = u^- + w, \tag{261}$$

Then, by the coordinate transformation of jump relations (148)-(150), we get,

$$(u_x^+ - u_x^-)\alpha_{x\eta} + (u_y^+ - u_y^-)\alpha_{y\eta} + (u_z^+ - u_z^-)\alpha_{z\eta} = w_\eta, \tag{262}$$

$$(u_x^+ - u_x^-)\alpha_{x\zeta} + (u_y^+ - u_y^-)\alpha_{y\zeta} + (u_z^+ - u_z^-)\alpha_{z\zeta} = w_\zeta, \tag{263}$$

$$(u_x^+ - \rho u_x^-)\alpha_{x\xi} + (u_y^+ - \rho u_y^-)\alpha_{y\xi} + (u_z^+ - \rho u_z^-)\alpha_{z\xi} = v/\beta^+, \tag{264}$$

By solving the equations (262)-(264) for u_x^+, u_y^+ and u_z^+ in terms of u_x^-, u_y^- and u_z^-, one can get the following expressions,

$$\begin{aligned}
u_x^+ = &-(\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)u_x^-/r_1 \\
&- (\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)u_y^-/r_1 \\
&- (\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - \alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)u_z^-/r_1 \\
&+ (\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{y\zeta}\alpha_{z\eta})v/(\beta^+ r_1) \\
&- (\alpha_{y\xi}\alpha_{z\zeta} - \alpha_{z\xi}\alpha_{y\zeta})w_\eta/r_1 \\
&+ (\alpha_{y\xi}\alpha_{z\eta} - \alpha_{y\eta}\alpha_{z\xi})w_\zeta/r_1,
\end{aligned} \tag{265}$$

$$
\begin{aligned}
u_y^+ &= (\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\eta}\rho)u_x^-/r_1 \\
&+ (\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\rho)u_y^-/r_1 \\
&+ (\alpha_{x\eta}\alpha_{z\xi}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\rho)u_z^-/r_1 \\
&- (\alpha_{x\eta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{z\eta})v/(\beta^+ r_1) \\
&+ (\alpha_{x\xi}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{z\xi})w_\eta/r_1 \\
&- (\alpha_{x\xi}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{z\xi})w_\zeta/r_1,
\end{aligned} \tag{266}
$$

$$
\begin{aligned}
u_z^+ &= -(\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\rho + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\rho)u_x^-/r_1 \\
&- (\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\rho)u_y^-/r_1 \\
&+ (\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\rho)u_z^-/r_1 \\
&+ (\alpha_{x\eta}\alpha_{y\zeta} - \alpha_{x\zeta}\alpha_{y\eta})v/(\beta^+ r_1) - (\alpha_{x\xi}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{x\zeta})w_\eta/r_1 \\
&+ (\alpha_{x\xi}\alpha_{y\eta} - \alpha_{x\eta}\alpha_{y\xi})w_\zeta/r_1,
\end{aligned} \tag{267}
$$

where,

$$
r_1 = \alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}, \tag{268}
$$

By coordinate transformation of jump relations (151)-(155), we get,

$$
\begin{aligned}
&(u_{xx}^+ - u_{xx}^-)\alpha_{x\eta}^2 + (u_{yy}^+ - u_{yy}^-)\alpha_{y\eta}^2 + (u_{zz}^+ - u_{zz}^-)\alpha_{z\eta}^2 \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}(u_{xy}^+ - u_{xy}^-) + 2\alpha_{x\eta}\alpha_{z\eta}(u_{xz}^+ - u_{xz}^-) + 2\alpha_{y\eta}\alpha_{z\eta}(u_{yz}^+ - u_{yz}^-) = D1,
\end{aligned} \tag{269}
$$

$$
\begin{aligned}
&(u_{xx}^+ - u_{xx}^-)\alpha_{x\zeta}^2 + (u_{yy}^+ - u_{yy}^-)\alpha_{y\zeta}^2 + (u_{zz}^+ - u_{zz}^-)\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\zeta}\alpha_{y\zeta}(u_{xy}^+ - u_{xy}^-) + 2\alpha_{x\zeta}\alpha_{z\zeta}(u_{xz}^+ - u_{xz}^-) + 2\alpha_{y\zeta}\alpha_{z\zeta}(u_{yz}^+ - u_{yz}^-) = D2,
\end{aligned} \tag{270}
$$

$$
\begin{aligned}
&(u_{xx}^+ - u_{xx}^-)\alpha_{x\eta}\alpha_{x\zeta} + (u_{yy}^+ - u_{yy}^-)\alpha_{y\eta}\alpha_{y\zeta} + (u_{zz}^+ - u_{zz}^-)\alpha_{z\eta}\alpha_{z\zeta} \\
&+ (\alpha_{y\eta}\alpha_{x\zeta} + \alpha_{x\eta}\alpha_{y\zeta})(u_{xy}^+ - u_{xy}^-) + (\alpha_{z\eta}\alpha_{x\zeta} + \alpha_{x\eta}\alpha_{z\zeta})(u_{xz}^+ - u_{xz}^-) \\
&+ (\alpha_{z\eta}\alpha_{y\zeta} + \alpha_{y\eta}\alpha_{z\zeta})(u_{yz}^+ - u_{yz}^-) = D3,
\end{aligned} \tag{271}
$$

$$
\begin{aligned}
&(u_{xx}^+ - \rho u_{xx}^-)\alpha_{x\xi}\alpha_{x\eta} + (u_{yy}^+ - \rho u_{yy}^-)\alpha_{y\xi}\alpha_{y\eta} + (u_{zz}^+ - \rho u_{zz}^-)\alpha_{z\xi}\alpha_{z\eta} \\
&+ (\alpha_{y\xi}\alpha_{x\eta} + \alpha_{x\xi}\alpha_{y\eta})(u_{xy}^+ - \rho u_{xy}^-) + (\alpha_{z\xi}\alpha_{x\eta} + \alpha_{x\xi}\alpha_{z\eta})(u_{xz}^+ - \rho u_{xz}^-) \\
&+ (\alpha_{z\xi}\alpha_{y\eta} + \alpha_{y\xi}\alpha_{z\eta})(u_{yz}^+ - \rho u_{yz}^-) = D4/\beta^+,
\end{aligned} \tag{272}
$$

$$
\begin{aligned}
&(u_{xx}^+ - \rho u_{xx}^-)\alpha_{x\xi}\alpha_{x\zeta} + (u_{yy}^+ - \rho u_{yy}^-)\alpha_{y\xi}\alpha_{y\zeta} + (u_{zz}^+ - \rho u_{zz}^-)\alpha_{z\xi}\alpha_{z\zeta} \\
&+ (\alpha_{y\xi}\alpha_{x\zeta} + \alpha_{x\xi}\alpha_{y\zeta})(u_{xy}^+ - \rho u_{xy}^-) + (\alpha_{z\xi}\alpha_{x\zeta} + \alpha_{x\xi}\alpha_{z\zeta})(u_{xz}^+ - \rho u_{xz}^-) \\
&+ (\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{z\zeta})(u_{yz}^+ - \rho u_{yz}^-) = D5/\beta^+,
\end{aligned} \tag{273}
$$

We can solve the equations (269)-(273) together with the equations (156) and (157) in chapter 3 to get expressions for u^+_{xx}, u^+_{xy}, u^+_{xz}, u^+_{yy}, u^+_{yz}, u^+_{zz} in terms of u^-_{xx}, u^-_{xy}, u^-_{xz}, u^-_{yy}, u^-_{yz}.

Now we can write an expression for u^+_{xx} as,

$$\begin{aligned} u^+_{xx} = {} & C^{xx}_{xx}u^-_{xx} + C^{xx}_{xy}u^-_{xy} + C^{xx}_{xz}u^-_{xz} + C^{xx}_{yy}u^-_{yy} + C^{xx}_{yz}u^-_{yz} + C^{xx}_{D_1}D_1 + C^{xx}_{D_2}D_2 + C^{xx}_{D_3}D_3 + C^{xx}_{D_4}D_4 \\ & + C^{xx}_{D_5}D_5 + C^{xx}_{D_6}D_6 + C^{xx}_{D_7}D_7, \end{aligned} \tag{274}$$

where,

$$\begin{aligned} C^{xx}_{xx} = {} & (\alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} \\ & - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} + \alpha^3_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta} - \alpha^3_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} \\ & + \alpha_{x\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} \\ & - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} \\ & + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} - \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta} \\ & + \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} \\ & + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} + 3\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} \\ & + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} - 3\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta} + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\ & - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} \\ & + 3\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ & - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} \\ & + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta}\rho \\ & - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho + 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta}\rho \\ & + 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} \\ & + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\ & - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\ & + 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ & - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho \\ & - 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\ & + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\ & - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\ & - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2, \end{aligned} \tag{275}$$

$$
\begin{aligned}
C^{xx}_{xy} =& -(2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta} \\
& - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha_{z\eta} - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\
& + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta}\rho \\
& + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} \\
& - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
& - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
& + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\
& + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
& + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
& - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{276}
$$

$$
\begin{aligned}
C^{xx}_{xz} =& (2\alpha_{x\xi}\alpha_{x\eta}\alpha^3_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta} \\
& + 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha_{z\eta} + 2\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta} - 2\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
& + 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\rho \\
& - 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha_{z\eta}\rho - 2\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta}\rho + 2\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho \\
& - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + 4\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta} + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho \\
& - 2\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
& + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho - 4\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{277}
$$

$$\begin{aligned}
C^{xx}_{yy} =& -(2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} + 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} \\
&- 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho + 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
&+ 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{278}$$

$$\begin{aligned}
C^{xx}_{yz} =& (2\alpha_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^3_{z\eta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} \\
&- 2\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho \\
&+ 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + 4\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 4\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{279}$$

$$\begin{aligned}
C^{xx}_{D_1} =& -(\alpha_{x\xi}\alpha_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta} + \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{280}$$

$$\begin{aligned}
C^{xx}_{D_2} =& (\alpha_{x\xi}\alpha_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\xi}\alpha^3_{y\eta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^3_{z\eta} \\
&- \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi} + \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{281}$$

$$
\begin{aligned}
C^{xx}_{D_3} =& -(-2\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\eta}\alpha_{y\zeta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} \\
& + 2\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\eta}\alpha^2_{y\zeta} - 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\eta}\alpha^2_{z\zeta})/r_2,
\end{aligned} \tag{282}
$$

$$
\begin{aligned}
C^{xx}_{D_4} =& (-2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
& - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta})/(\beta^+ r_2),
\end{aligned} \tag{283}
$$

$$
\begin{aligned}
C^{xx}_{D_5} =& -(-2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& - 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& + 2\alpha_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} - 2\alpha_{x\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/(\beta^+ r_2),
\end{aligned} \tag{284}
$$

$$
\begin{aligned}
C^{xx}_{D_6} =& (\alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\zeta}\alpha_{z\xi}\alpha^3_{y\eta}\alpha^2_{z\zeta} - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& + 2\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
& + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} + 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} + \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} \\
& + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{285}
$$

$$
\begin{aligned}
C^{xx}_{D_7} =& -(\alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} \\
& - \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
& + 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
& + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
& - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{286}
$$

The expression for u^+_{xy} is,

$$
\begin{aligned}
u^+_{xy} =& C^{xy}_{xx}u^-_{xx} + C^{xy}_{xy}u^-_{xy} + C^{xy}_{xz}u^-_{xz} + C^{xy}_{yy}u^-_{yy} + C^{xy}_{yz}u^-_{yz} + C^{xy}_{D_1}D_1 + C^{xy}_{D_2}D_2 + C^{xy}_{D_3}D_3 + C^{xy}_{D_4}D_4 \\
& + C^{xy}_{D_5}D_5 + C^{xy}_{D_6}D_6 + C^{xy}_{D_7}D_7,
\end{aligned} \tag{287}
$$

where,

$$
\begin{aligned}
C_{xx}^{xy} =&(\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{z\zeta}^3 - \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{z\eta}^3 - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\zeta}^3 \\
&- \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3 + \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{z\zeta}^3\rho + \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{z\eta}^3\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta} + 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2 \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&- 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\zeta}^3\rho \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho - \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&- 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{288}
$$

$$
\begin{aligned}
C_{xy}^{xy} =&(\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^3 - \alpha_{x\zeta}^3\alpha_{y\eta}^3\alpha_{z\xi} + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3 \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{z\zeta}^3 + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2 \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3 - \alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2 - \alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3\rho - \alpha_{x\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{z\eta}^3\rho - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{z\zeta}^3\rho + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta} \\
&- 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3\rho - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\eta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3\rho + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{z\zeta}^3\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta}\rho - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}\rho + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&+ 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 3\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho - 3\alpha_{x\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{289}
$$

$$
\begin{aligned}
C_{xz}^{xy} =&(\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3 - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\\
&- \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3 + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\\
&- \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho + \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\\
&- \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\rho\\
&+ \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\rho - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\\
&- 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho\\
&- 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho\\
&- \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho\\
&- 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{290}
$$

$$
\begin{aligned}
C_{yy}^{xy} =&-(\alpha_{y\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3 - \alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^3\\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta}\\
&- \alpha_{y\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3\rho + \alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2 - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\\
&- \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 + 3\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\\
&+ 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^3\rho\\
&+ \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3\rho - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2\rho\\
&+ \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho\\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho\\
&- 3\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{291}
$$

$$
\begin{aligned}
C^{xy}_{yz} =&(\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} - \alpha^3_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta} + \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta} \\
&+ \alpha^2_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha_{z\eta} - \alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta} - \alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
&- \alpha_{z\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta}\rho + \alpha^3_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta} \\
&- 3\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 3\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\rho \\
&- \alpha^2_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha_{z\eta}\rho + \alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta}\rho + \alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
&- 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho - \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&+ \alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - \alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 3\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
&- 3\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{292}
$$

$$
\begin{aligned}
C^{xy}_{D_1} =&(\alpha_{x\xi}\alpha_{x\eta}\alpha^3_{z\zeta} - \alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{z\zeta} - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&+ \alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{293}
$$

$$
\begin{aligned}
C^{xy}_{D_2} =&-(\alpha_{x\xi}\alpha_{x\zeta}\alpha^3_{z\eta} - \alpha_{y\xi}\alpha_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{z\eta} + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&+ \alpha^2_{x\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{294}
$$

$$\begin{aligned}C^{xy}_{D_3} =&(\alpha_{z\xi}\alpha^2_{x\eta}\alpha^2_{y\zeta} + \alpha_{z\xi}\alpha^2_{x\eta}\alpha^2_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\\
&- 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- \alpha_{z\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta} - \alpha_{z\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{z\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&+ \alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/r_2,\end{aligned} \tag{295}$$

$$\begin{aligned}C^{xy}_{D_4} =&-(\alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} + \alpha^2_{x\eta}\alpha^3_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha^2_{y\eta}\alpha^3_{z\zeta}\\
&+ 2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/(\beta^+ r_2),\end{aligned} \tag{296}$$

$$\begin{aligned}C^{xy}_{D_5} =&(2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha^2_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\\
&+ \alpha^2_{x\zeta}\alpha^3_{z\eta} - \alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha^2_{y\zeta}\alpha^3_{z\eta})/(\beta^+ r_2),\end{aligned} \tag{297}$$

$$\begin{aligned}C^{xy}_{D_6} =&-(\alpha_{z\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{y\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} - \alpha_{z\xi}\alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\\
&+ \alpha_{y\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&+ \alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{z\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\\
&- \alpha_{z\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta})/r_2,\end{aligned} \tag{298}$$

$$\begin{aligned}C^{xy}_{D_7} =&(\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta}\\
&+ \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&+ \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + \alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\\
&+ \alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho\\
&- 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,\end{aligned} \tag{299}$$

Now we can write an expression for u^+_{xz} as,

$$\begin{aligned}u^+_{xz} =& C^{xz}_{xx}u^-_{xx} + C^{xz}_{xy}u^-_{xy} + C^{xz}_{xz}u^-_{xz} + C^{xz}_{yy}u^-_{yy} + C^{xz}_{yz}u^-_{yz} + C^{xz}_{D_1}D_1 + C^{xz}_{D_2}D_2 + C^{xz}_{D_3}D_3 + C^{xz}_{D_4}D_4\\
&+ C^{xz}_{D_5}D_5 + C^{xz}_{D_6}D_6 + C^{xz}_{D_7}D_7,\end{aligned} \tag{300}$$

where,

$$
\begin{aligned}
C_{xx}^{xz} =& - (\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^3 - \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3 + \alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 \\
& - \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 \\
& + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3 \\
& - \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta} + \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3 + \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta} \\
& - \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^3\rho + \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3\rho - \alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho \\
& + \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho + 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta} - 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2 \\
& + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
& + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 + \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} - \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
& + \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} \\
& + 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho + \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho \\
& - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2\rho + \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
& + \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3\rho - \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
& - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\rho \\
& + 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
& + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
& + \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
& + 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho - 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{301}
$$

$$
\begin{aligned}
C_{xy}^{xz} =&(\alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^3 + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2 \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3\rho + \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^3\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta} + 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&+ \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho \\
&- \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho + \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
&+ 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\rho - 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{302}
$$

$$
\begin{aligned}
C^{xz}_{xz} =&(\alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\\
&+ 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\\
&+ 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta}\\
&- 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta}\rho - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta}\rho\\
&+ \alpha^3_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\rho - \alpha^3_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\rho + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\\
&- 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\\
&- 3\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 3\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\\
&+ \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta}\rho + \alpha_{x\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho\\
&- \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\rho\\
&+ \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&+ 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho\\
&+ 3\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho - 3\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\rho - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho\\
&+ \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho\\
&+ 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho\\
&+ \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{303}
$$

$$
\begin{aligned}
C^{xz}_{yy} =& -(\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 - \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3 \\
& - \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta} + \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3 + \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta} \\
& - \alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho + \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
& + 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
& - \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} \\
& - 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
& + \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3\rho - \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
& - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
& + 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho + \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
& - \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
& - 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
& + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{304}
$$

$$
\begin{aligned}
C^{xz}_{yz} =& -(\alpha_{y\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3 - \alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^3 \\
& + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta} - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta} \\
& - \alpha_{y\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3\rho + \alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2 \\
& - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2 + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
& + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
& - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 + 3\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
& - 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^3\rho \\
& - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3\rho + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^2\rho \\
& - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
& + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
& - 3\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
& - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{305}
$$

$$\begin{aligned}
C^{xz}_{D_1} =& -(\alpha_{x\xi}\alpha_{x\eta}\alpha^3_{y\zeta} - \alpha_{z\xi}\alpha^3_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&+ \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - \alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta})/r_2,
\end{aligned} \tag{306}$$

$$\begin{aligned}
C^{xz}_{D_2} =& (\alpha_{x\xi}\alpha_{x\zeta}\alpha^3_{y\eta} - \alpha^3_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{z\eta} + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&+ \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + \alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{y\xi}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta})/r_2,
\end{aligned} \tag{307}$$

$$\begin{aligned}
C^{xz}_{D_3} =& (-\alpha_{y\xi}\alpha^2_{x\eta}\alpha^2_{y\zeta} - \alpha_{y\xi}\alpha^2_{x\eta}\alpha^2_{z\zeta} + 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} \\
&- 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta} + \alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{z\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} \\
&- 2\alpha_{z\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta})/r_2,
\end{aligned} \tag{308}$$

$$\begin{aligned}
C^{xz}_{D_4} =& (\alpha^2_{x\eta}\alpha^3_{y\zeta} + \alpha^2_{x\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta} \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta} + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} + 2\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^3_{y\zeta}\alpha^2_{z\eta})/(\beta^+ r_2),
\end{aligned} \tag{309}$$

$$\begin{aligned}
C^{xz}_{D_5} =& -(\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} - \alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&+ \alpha^2_{x\zeta}\alpha^3_{y\eta} + \alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta} - \alpha^3_{y\eta}\alpha^2_{z\zeta} + 2\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta})/(\beta^+ r_2),
\end{aligned} \tag{310}$$

$$\begin{aligned}
C^{xz}_{D_6} =& (\alpha_{z\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{z\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{y\xi}\alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} - \alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^3_{y\zeta}\alpha^2_{z\eta} + \alpha_{z\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} \\
&- \alpha_{z\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta})/r_2,
\end{aligned} \tag{311}$$

$$\begin{aligned}
C_{D_7}^{xz} =&(\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 - \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3 + \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3 \\
&- \alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3\rho + \alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 \\
&- \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 + \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&- 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
&+ \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - \alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^3\rho - \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho + \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&- 3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{312}$$

The expression for u_{yy}^+ is,

$$\begin{aligned}
u_{yy}^+ = C_{xx}^{yy}u_{xx}^- + C_{xy}^{yy}u_{xy}^- + C_{xz}^{yy}u_{xz}^- + C_{yy}^{yy}u_{yy}^- + C_{yz}^{yy}u_{yz}^- + C_{D_1}^{yy}D_1 + C_{D_2}^{yy}D_2 + C_{D_3}^{yy}D_3 \\
+ C_{D_4}^{yy}D_4 + C_{D_5}^{yy}D_5 + C_{D_6}^{yy}D_6 + C_{D_7}^{yy}D_7,
\end{aligned} \tag{313}$$

$$\begin{aligned}
C_{xx}^{yy} =&(2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3 - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 + 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2 \\
&- 2\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3\rho - 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{314}$$

$$
\begin{aligned}
C^{yy}_{xy} =&(2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{x\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\\
&+ 2\alpha_{x\xi}\alpha^3_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho + 2\alpha_{x\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho\\
&- 2\alpha_{x\xi}\alpha^3_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^3_{z\zeta}\rho + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho\\
&+ 4\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho\\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 4\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho\\
&+ 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho\\
&- 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{315}
$$

$$
\begin{aligned}
C^{yy}_{xz} =&-(2\alpha_{x\xi}\alpha^3_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta} - 2\alpha_{x\xi}\alpha^3_{x\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\\
&+ 2\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta}\\
&+ 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha^3_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\rho\\
&+ 2\alpha_{x\xi}\alpha^3_{x\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^3_{z\zeta}\rho - 2\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{z\eta}\rho\\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta}\\
&- 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho\\
&- 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&- 4\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha^2_{z\zeta}\rho\\
&+ 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{316}
$$

$$
\begin{aligned}
C_{yy}^{yy} =&(\alpha_{x\xi}\alpha_{y\eta}^{3}\alpha_{z\zeta}^{3} - \alpha_{x\xi}\alpha_{y\zeta}^{3}\alpha_{z\eta}^{3} + \alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{z\eta}^{3}\\
&- \alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{z\zeta}^{3} + \alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}^{3} - \alpha_{x\zeta}^{3}\alpha_{y\eta}^{3}\alpha_{z\xi}\\
&+ \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\eta}\alpha_{z\zeta}^{3} - \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\zeta}^{3}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\zeta}\alpha_{z\eta}^{3}\\
&+ \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{3}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^{2}\alpha_{z\zeta}^{3} - \alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{y\eta}^{2}\alpha_{z\eta}\\
&+ \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^{3}\alpha_{z\eta}^{2} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{3} + \alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\zeta}\\
&+ \alpha_{x\zeta}^{3}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^{2} - \alpha_{x\zeta}\alpha_{y\eta}^{3}\alpha_{z\xi}\alpha_{z\zeta}^{2} - \alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2}\\
&+ \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\zeta}\\
&+ 3\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta} - 3\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\\
&- \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} + \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\eta}^{2}\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta}\\
&- \alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^{2} + 3\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} + \alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^{2}\\
&+ 3\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta}\\
&+ \alpha_{x\eta}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{z\eta}\alpha_{z\zeta}^{2} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2} \qquad (317)\\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^{2}\alpha_{z\zeta}^{3}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{y\eta}^{2}\alpha_{z\eta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{3}\rho\\
&- 2\alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^{3}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^{2}\rho + 2\alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^{2} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^{2}\alpha_{z\zeta} - 2\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} - 2\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}\alpha_{z\zeta}\\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^{2}\alpha_{z\zeta} + 2\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^{2}\rho\\
&- 2\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^{2}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{z\eta}\alpha_{z\zeta}^{2}\rho\\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 4\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho\\
&+ 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&- 4\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^{2}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
$$

$$
\begin{aligned}
C_{yz}^{yy} =& - (2\alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 - 2\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3 - 2\alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta} + 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3 - 2\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\rho + 2\alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\rho - 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho + 2\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 4\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- 4\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{318}
$$

$$
\begin{aligned}
C_{D_1}^{yy} =& (\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta}^3 + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta}^3 - \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{z\eta} \\
&- \alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi} + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned}
\tag{319}
$$

$$
\begin{aligned}
C_{D_2}^{yy} =& - (\alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}^3 + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{z\zeta} - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} \\
&+ \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta})/r_2,
\end{aligned}
\tag{320}
$$

$$
\begin{aligned}
C_{D_3}^{yy} =& - (2\alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\zeta} + 2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{x\eta}^2\alpha_{x\zeta} - 2\alpha_{x\xi}\alpha_{y\zeta}\alpha_{x\eta}^2\alpha_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta} - 2\alpha_{y\eta}\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2 + 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{y\eta}\alpha_{z\xi}\alpha_{x\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{x\zeta}^2\alpha_{z\eta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&+ 2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{x\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2)/r_2,
\end{aligned}
\tag{321}
$$

$$C^{yy}_{D_4} = -(-2\alpha_{y\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\zeta} + 2\alpha_{y\zeta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\eta} + 2\alpha_{y\eta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\zeta} - 2\alpha_{y\zeta}\alpha_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{y\eta}\alpha_{x\eta}\alpha^3_{z\zeta} - 2\alpha_{y\eta}\alpha^3_{x\zeta}\alpha_{z\eta} + 2\alpha_{y\zeta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta})/(\beta^+ r_2), \tag{322}$$

$$C^{yy}_{D_5} = (-2\alpha_{y\zeta}\alpha^3_{x\eta}\alpha_{z\zeta} + 2\alpha_{y\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\eta} + 2\alpha_{y\eta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\zeta} - 2\alpha_{y\eta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\eta} - 2\alpha_{y\zeta}\alpha_{x\eta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{y\eta}\alpha_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{y\zeta}\alpha_{x\zeta}\alpha^3_{z\eta} - 2\alpha_{y\eta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/(\beta^+ r_2), \tag{323}$$

$$C^{yy}_{D_6} = (-\alpha_{y\xi}\alpha^3_{x\eta}\alpha^3_{z\zeta} + \alpha_{z\xi}\alpha_{y\zeta}\alpha^3_{x\eta}\alpha^2_{z\zeta} + 3\alpha_{y\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{y\eta}\alpha_{z\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha^2_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{x\eta}\alpha^3_{z\zeta} - 3\alpha_{y\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{z\xi}\alpha_{y\zeta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta} + 2\alpha_{y\eta}\alpha_{z\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{y\zeta}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} - \alpha_{y\eta}\alpha_{z\xi}\alpha^3_{x\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha^2_{x\zeta}\alpha^3_{z\eta} + \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta})/r_2, \tag{324}$$

$$C^{yy}_{D_7} = -(\alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} + \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta} - \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 3\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - 2\alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\rho + 2\alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2, \tag{325}$$

Now we can write an expression for u^+_{yz} as,

$$u^+_{yz} = C^{yz}_{xx}u^-_{xx} + C^{yz}_{xy}u^-_{xy} + C^{yz}_{xz}u^-_{xz} + C^{yz}_{yy}u^-_{yy} + C^{yy}_{yz}u^-_{yz} + C^{yz}_{D_1}D_1 + C^{yz}_{D_2}D_2 + C^{yz}_{D_3}D_3 + C^{yz}_{D_4}D_4 + C^{yz}_{D_5}D_5 + C^{yz}_{D_6}D_6 + C^{yz}_{D_7}D_7, \tag{326}$$

$$
\begin{aligned}
C_{xx}^{yz} =& - (\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3 + \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3 \\
&+ \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta} - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3 - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&- \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho + \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
&- \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\rho + \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&- 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&+ \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{327}
$$

$$
\begin{aligned}
C_{xy}^{yz} =& - (\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^3 - \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3 + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^3\rho + \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3\rho + 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta} \\
&- 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho + \alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\rho + 3\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{328}
$$

$$
\begin{aligned}
C^{yz}_{xz} =&(\alpha_{x\xi}\alpha^3_{x\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta} \\
&+ \alpha_{x\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} - \alpha_{x\xi}\alpha^3_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha^3_{x\eta}\alpha^3_{z\zeta}\rho + \alpha_{x\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} + 3\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&+ 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho \\
&- \alpha_{x\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\rho + \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho + \alpha_{x\xi}\alpha^3_{x\eta}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta}\rho - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\
&- 3\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 3\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
&+ 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{329}
$$

$$
\begin{aligned}
C_{yy}^{yz} =&(\alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^3 - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 \\
&+ \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2 - \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 \\
&- \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3 + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 + \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3 + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&- \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3\rho + \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^3\rho + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
&- \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3\rho - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta} + 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2 \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} + 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} \\
&+ \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho \\
&+ \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\rho - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\rho - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\rho \\
&- 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\rho - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
&+ \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho - 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho - \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{330}
$$

$$
\begin{aligned}
C_{yz}^{yz} =& (\alpha_{x\xi}\alpha_{y\eta}^{3}\alpha_{z\zeta}^{3} - \alpha_{x\xi}\alpha_{y\zeta}^{3}\alpha_{z\eta}^{3} \\
&+ \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\eta}\alpha_{z\zeta}^{3} - \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\zeta}^{3}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\zeta}\alpha_{z\eta}^{3} + \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{3}\alpha_{z\zeta} \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{y\eta}^{2}\alpha_{z\eta} - 2\alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\zeta} - 2\alpha_{x\zeta}^{3}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^{2} \\
&+ 2\alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2} + \alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{z\eta}^{3}\rho - \alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{z\zeta}^{3}\rho \\
&+ \alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}^{3}\rho - \alpha_{x\zeta}^{3}\alpha_{y\eta}^{3}\alpha_{z\xi}\rho + \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\zeta} + 2\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta} \\
&- \alpha_{x\xi}\alpha_{x\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} + \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\eta}^{2}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^{2} \\
&- 2\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^{2} + 3\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^{2}\alpha_{z\zeta}^{3}\rho - \alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{y\eta}^{2}\alpha_{z\eta}\rho + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^{3}\alpha_{z\eta}^{2}\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{3}\rho + \alpha_{x\eta}^{3}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\zeta}\rho + \alpha_{x\zeta}^{3}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^{2}\rho \\
&- \alpha_{x\zeta}\alpha_{y\eta}^{3}\alpha_{z\xi}\alpha_{z\zeta}^{2}\rho - \alpha_{x\eta}^{3}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^{2}\alpha_{z\eta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^{2} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{y\zeta}\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^{2}\alpha_{z\zeta} + 4\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\zeta}\rho + 3\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\rho \\
&- \alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^{2}\alpha_{z\eta}\rho - 3\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\rho - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta}\rho \\
&- \alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^{2}\rho + 3\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2}\rho + \alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^{2}\rho \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{z\eta}\alpha_{z\zeta}^{2}\rho \\
&- \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}^{2}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}^{2}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^{2}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^{2}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}^{2}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha_{y\eta}^{2}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{331}
$$

$$
\begin{aligned}
C_{D_1}^{yz} =& (\alpha_{y\xi}\alpha_{x\zeta}^{3}\alpha_{x\eta} - \alpha_{x\zeta}^{3}\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^{2} \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\zeta}^{2} - \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{x\eta}\alpha_{y\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^{2}\alpha_{y\zeta} \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\zeta}^{2} + \alpha_{x\xi}\alpha_{x\zeta}^{2}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^{2} \\
&+ \alpha_{x\eta}\alpha_{x\zeta}^{2}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{x\eta}\alpha_{z\zeta}^{2} + \alpha_{x\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^{2}\alpha_{z\eta})/r_2,
\end{aligned} \tag{332}
$$

$$
\begin{aligned}
C_{D_2}^{yz} =& -(\alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta} - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2 \\
& - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta} \\
& - \alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
& - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}^2 + \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta})/r_2,
\end{aligned} \tag{333}
$$

$$
\begin{aligned}
C_{D_3}^{yz} =& -(-2\alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta} + 2\alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2 \\
& - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\zeta}^2 + 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta} - 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta} \\
& + 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2 \\
& + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{z\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\xi}\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2)/r_2,
\end{aligned} \tag{334}
$$

$$
\begin{aligned}
C_{D_4}^{yz} =& -(\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2 - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\zeta}^2 \\
& - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\zeta}^3\alpha_{y\eta}^2 - \alpha_{x\zeta}^3\alpha_{z\eta}^2 \\
& + \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 - \alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2)/(\beta^+ r_2),
\end{aligned} \tag{335}
$$

$$
\begin{aligned}
C_{D_5}^{yz} =&(\alpha_{x\eta}^3\alpha_{y\zeta}^2 - \alpha_{x\eta}^3\alpha_{z\zeta}^2 - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2 - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta}^2 - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
& + \alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 + 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2)/(\beta^+ r_2),
\end{aligned} \tag{336}
$$

$$
\begin{aligned}
C_{D_6}^{yz} =& -(\alpha_{z\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2 - 2\alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& + \alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 - \alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta} + 2\alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta} \\
& + 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 \\
& - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - \alpha_{z\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta} \\
& + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2)/r_2,
\end{aligned} \tag{337}
$$

$$
\begin{aligned}
C_{D_7}^{yz} =&(\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3 + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 \\
&+ \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3 \\
&- \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^3\rho + \alpha_{x\zeta}^3\alpha_{z\xi}\alpha_{z\eta}^3\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}^3\rho \\
&- \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\rho + \alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^3\rho + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&- 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&+ \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{338}
$$

Now we can write an expression for u_{zz}^+ as,

$$
\begin{aligned}
u_{zz}^+ =& C_{xx}^{zz}u_{xx}^- + C_{xy}^{zz}u_{xy}^- + C_{xz}^{zz}u_{xz}^- + C_{yy}^{zz}u_{yy}^- + C_{yz}^{zz}u_{yz}^- + C_{D_1}^{zz}D_1 + C_{D_2}^{zz}D_2 + C_{D_3}^{zz}D_3 \\
&+ C_{D_4}^{zz}D_4 + C_{D_5}^{zz}D_5 + C_{D_6}^{zz}D_6 + C_{D_7}^{zz}D_7,
\end{aligned} \tag{339}
$$

$$
\begin{aligned}
C_{xx}^{zz} =& -(\alpha_{x\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3 - \alpha_{x\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3 \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{z\eta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{z\zeta}^3 + \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}^3 - \alpha_{x\zeta}^3\alpha_{y\eta}^3\alpha_{z\xi} \\
&+ \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\eta} \\
&- \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}^2\alpha_{z\zeta}^3 + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta} \\
&- \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2 + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3 - \alpha_{x\eta}^3\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} + \alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2 \\
&+ \alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^2 - \alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta} + 3\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta} \\
&+ \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta} - 3\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&+ 3\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 + 3\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} \\
&- 3\alpha_{x\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\eta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho - 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}^3\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_3,
\end{aligned}
\tag{340}
$$

$$
\begin{aligned}
C_{xy}^{zz} =&(2\alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{x\zeta} - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta} \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{x\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{x\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{x\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{x\zeta}\rho + 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{x\zeta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 4\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{x\zeta} + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{x\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{x\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{x\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{x\zeta}\rho - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{x\zeta}\rho)/r_3,
\end{aligned}
\tag{341}
$$

$$
\begin{aligned}
C_{xz}^{zz} =& - (2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta} - 2\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta} \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho + 2\alpha_{x\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^3\alpha_{z\eta}\rho + 2\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\xi}\alpha_{z\zeta}\rho \\
&+ 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 4\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&- 4\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho)/r_3,
\end{aligned}
\tag{342}
$$

$$
\begin{aligned}
C^{zz}_{yy} =& -(\alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} \\
& - \alpha^3_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} + \alpha^3_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta} - \alpha^3_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi} \\
& + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} \\
& + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^3_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} - \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} \\
& - \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + \alpha^3_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
& + \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta} + \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
& + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} \\
& + 3\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} - 3\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta} \\
& - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 3\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
& + 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} \\
& + 2\alpha_{y\xi}\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^3_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho \\
& - 2\alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\rho - 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 2\alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
& - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho \\
& - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
& - 4\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 4\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
& + 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_3,
\end{aligned}
\tag{343}
$$

$$\begin{aligned}
C^{zz}_{yz} = &-(2\alpha_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha^3_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha^3_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta} + 2\alpha^3_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha_{y\xi}\alpha^3_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha^3_{y\eta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha^3_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha^3_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho - 2\alpha^3_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 4\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
&+ 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho - 4\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 4\alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_3,
\end{aligned} \tag{344}$$

$$\begin{aligned}
C^{zz}_{D_1} = &-(\alpha_{x\xi}\alpha^3_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta} - \alpha_{y\xi}\alpha^3_{x\zeta}\alpha_{z\eta} \\
&- \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta})/r_3,
\end{aligned} \tag{345}$$

$$\begin{aligned}
C^{zz}_{D_2} = &(\alpha_{x\xi}\alpha^3_{y\eta}\alpha_{z\zeta} + \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi} - \alpha^3_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} - \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} \\
&+ \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} \\
&+ \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta})/r_3,
\end{aligned} \tag{346}$$

$$\begin{aligned}
C^{zz}_{D_3} = &(2\alpha_{z\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta} + 2\alpha_{y\xi}\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{x\zeta} - 2\alpha_{x\xi}\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{y\zeta} \\
&- 2\alpha_{z\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta} - 2\alpha_{y\xi}\alpha_{z\eta}\alpha_{x\eta}\alpha^2_{x\zeta} + 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} \\
&- 2\alpha_{y\xi}\alpha_{z\eta}\alpha_{x\eta}\alpha^2_{y\zeta} + 2\alpha_{x\xi}\alpha_{z\eta}\alpha^2_{x\zeta}\alpha_{y\eta} - 2\alpha_{z\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} \\
&+ 2\alpha_{y\xi}\alpha_{z\zeta}\alpha_{x\zeta}\alpha^2_{y\eta} - 2\alpha_{x\xi}\alpha_{z\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\xi}\alpha_{z\eta}\alpha_{y\eta}\alpha^2_{y\zeta})/r_3,
\end{aligned} \tag{347}$$

$$\begin{aligned}C^{zz}_{D_4} =&(-2\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta} + 2\alpha_{z\zeta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta} + 2\alpha_{z\eta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta} \\ &- 2\alpha_{z\zeta}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} + 2\alpha_{z\eta}\alpha_{x\eta}\alpha^3_{y\zeta} - 2\alpha_{z\eta}\alpha^3_{x\zeta}\alpha_{y\eta} + 2\alpha_{z\zeta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} \\ &- 2\alpha_{z\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta})/(\beta^+ r_3),\end{aligned} \tag{348}$$

$$\begin{aligned}C^{zz}_{D_5} =&-(-2\alpha_{z\zeta}\alpha^3_{x\eta}\alpha_{y\zeta} + 2\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta} + 2\alpha_{z\eta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta} - 2\alpha_{z\eta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta} \\ &- 2\alpha_{z\zeta}\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta} + 2\alpha_{z\eta}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} + 2\alpha_{z\zeta}\alpha_{x\zeta}\alpha^3_{y\eta} - 2\alpha_{z\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta})/(\beta^+ r_3),\end{aligned} \tag{349}$$

$$\begin{aligned}C^{zz}_{D_6} =&(\alpha_{z\xi}\alpha^3_{x\eta}\alpha^3_{y\zeta} - \alpha_{y\xi}\alpha_{z\zeta}\alpha^3_{x\eta}\alpha^2_{y\zeta} - 3\alpha_{z\xi}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta} \\ &+ 2\alpha_{y\xi}\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{z\eta}\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta} + \alpha_{x\xi}\alpha_{z\zeta}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta} \\ &- \alpha_{x\xi}\alpha_{z\eta}\alpha^2_{x\eta}\alpha^3_{y\zeta} + 3\alpha_{z\xi}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{z\zeta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta} \\ &- 2\alpha_{y\xi}\alpha_{z\eta}\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\xi}\alpha_{z\zeta}\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\xi}\alpha_{z\eta}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta} \\ &- \alpha_{z\xi}\alpha^3_{x\zeta}\alpha^3_{y\eta} + \alpha_{y\xi}\alpha_{z\eta}\alpha^3_{x\zeta}\alpha^2_{y\eta} + \alpha_{x\xi}\alpha_{z\zeta}\alpha^2_{x\zeta}\alpha^3_{y\eta} - \alpha_{x\xi}\alpha_{z\eta}\alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta})/r_3,\end{aligned} \tag{350}$$

$$\begin{aligned}C^{zz}_{D_7} =&(\alpha_{x\xi}\alpha^3_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^3_{y\zeta}\alpha^3_{z\eta} + \alpha_{y\xi}\alpha^3_{x\zeta}\alpha^3_{z\eta} \\ &- \alpha^3_{x\eta}\alpha_{y\xi}\alpha^3_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} \\ &- \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^3_{z\eta} \\ &+ \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta} + \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - \alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} \\ &- \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\eta}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ &- \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 3\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\ &+ 3\alpha_{x\xi}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 3\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ &- \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} \\ &+ 2\alpha_{x\eta}\alpha_{z\xi}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\eta}\rho - 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\ &+ 2\alpha^3_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ &+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &+ 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho \\ &- 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho - 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &- 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_3,\end{aligned} \tag{351}$$

where,

$$
\begin{aligned}
r_3 =&(\alpha_{z\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^3 - \alpha_{y\xi}\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\zeta} + \alpha_{z\xi}\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- \alpha_{y\xi}\alpha_{x\eta}^3\alpha_{z\zeta}^3 - 3\alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2 + 2\alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^2 + \alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{z\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 3\alpha_{y\xi}\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\zeta}^3 - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\eta} \\
&- \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 3\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta} - \alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{z\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&- 3\alpha_{y\xi}\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{z\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- \alpha_{y\xi}\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{z\zeta}^3 - 2\alpha_{z\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 \\
&+ \alpha_{z\xi}\alpha_{x\eta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 - \alpha_{y\xi}\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{z\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^3 + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\eta} \\
&- \alpha_{z\xi}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}^2 + \alpha_{y\xi}\alpha_{x\zeta}^3\alpha_{z\eta}^3 + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} \\
&+ \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^3 - \alpha_{z\xi}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 + 2\alpha_{z\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{z\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}^3 \\
&+ \alpha_{x\xi}\alpha_{y\eta}^3\alpha_{z\zeta}^3 - 3\alpha_{x\xi}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + 3\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}^3\alpha_{z\eta}^3),
\end{aligned}
\tag{352}
$$

And the expression for u_{zz}^- is,

$$u_{zz}^- = -u_{xx}^- - u_{yy}^- + D_7. \tag{353}$$

APPENDIX B

COORDINATE TRANSFORMATION OF JUMP RELATIONS FOR THREE-DIMENSIONAL PARABOLIC INTERFACE PROBLEMS

Here we explain how to obtain explicit expressions for u^+, u_x^+, u_y^+, u_z^+, u_{xx}^+, u_{xy}^+, u_{xz}^+, u_{yy}^+, u_{yz}^+ and u_{zz}^+ in terms of u^-, u_x^-, u_y^-, u_z^-, u_{xx}^-, u_{xy}^-, u_{xz}^-, u_{yy}^-, u_{yz}^- and u_{zz}^- as a part of finding the correction terms at the irregular grid points. To get that, we have to do the coordinate transformations of the jump relations (237)-(245) in chapter 5.

As the first nine jump relations are as the same as in the model for elliptic interface problem, coordinate transformations of the jump relations (237)-(244) in chapter 5 are given by (262)-(264) and (269)-(273) in Appendix A.

So, the expression for u_x^+, u_y^+, u_z^+ are as same as for three-dimensional elliptic interface problems and are given by the equations (265)- (267) in the Appendix A. However, expressions for u_{xx}^+, u_{xy}^+, u_{xz}^+, u_{yy}^+, u_{yz}^+ and u_{zz}^+ are different from the case for three-dimensional elliptic interface problems as the last jump relation for both cases are obtained from the PDE itself.

So, the coordinate transformation of the jump relation (245) in chapter 5 is given by,

$$\begin{aligned}&(u_{xx}^+ - \rho u_{xx}^-)\alpha_{x\xi}^2 + (u_{yy}^+ - \rho u_{yy}^-)\alpha_{y\xi}^2 + (u_{zz}^+ - \rho u_{zz}^-)\alpha_{z\xi}^2\\&+ 2\alpha_{x\xi}\alpha_{y\xi}(u_{xy}^+ - \rho u_{xy}^-) + 2\alpha_{x\xi}\alpha_{z\xi}(u_{xz}^+ - \rho u_{xz}^-)\\&+ 2\alpha_{y\xi}\alpha_{z\xi}(u_{yz}^+ - \rho u_{yz}^-) = D,\end{aligned} \tag{354}$$

We can solve the equations (269)-(273) in Appendix A together with the equation (354) to get expressions for u_{xx}^+, u_{xy}^+, u_{xz}^+, u_{yy}^+, u_{yz}^+, u_{zz}^+ in terms of u_{xx}^-, u_{xy}^-, u_{xz}^-, u_{yy}^-, u_{yz}^-, u_{zz}^-.

So the expression for u_{xx}^+ can be written as,

$$\begin{aligned}u_{xx}^+ = {}& C_{xx}^{xx}u_{xx}^- + C_{xy}^{xx}u_{xy}^- + C_{xz}^{xx}u_{xz}^- + C_{yy}^{xx}u_{yy}^- + C_{yz}^{xx}u_{yz}^-\\&+ C_{zz}^{xx}u_{zz}^- + C_{D_1}^{xx}D_1 + C_{D_2}^{xx}D_2 + C_{D_3}^{xx}D_3 + C_{D_4}^{xx}D_4 + C_{D_5}^{xx}D_5 + C_{D_8}^{xx}D_8,\end{aligned} \tag{355}$$

Notice that $D_8 = D_1 + D_2 - ([f] + w_t)/\beta^+$.

where,

$$
\begin{aligned}
C_{xx}^{xx} = (&\alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{z\zeta}^2 + \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}^2 + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{z\eta}^2 \\
&+ \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}^2 - \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 - \alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2 \\
&- \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2 + \alpha_{x\xi}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho + \alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho \\
&+ \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{y\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{356}
$$

$$
\begin{aligned}
C_{xy}^{xx} = -(&2\alpha_{x\eta}\alpha_{y\eta}^3\alpha_{z\zeta}^2 + 2\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta}^2 - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\eta}\alpha_{z\zeta}^2 \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2 - 2\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}\alpha_{y\eta}^3\alpha_{z\zeta}^2\rho - 2\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + 2\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{357}
$$

$$
\begin{aligned}
C^{xx}_{xz} = &-(2\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
&- 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&- 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\
&- 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
&+ 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&+ 4\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{358}
$$

$$
\begin{aligned}
C^{xx}_{yy} = &-(\alpha^4_{y\eta}\alpha^2_{z\zeta} + \alpha^4_{y\zeta}\alpha^2_{z\eta} - \alpha^2_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} \\
&- \alpha^2_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha^2_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha^2_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta} \\
&- \alpha^4_{y\eta}\alpha^2_{z\zeta}\rho - \alpha^4_{y\zeta}\alpha^2_{z\eta}\rho + \alpha^2_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho \\
&+ \alpha^2_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - \alpha^2_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - \alpha^2_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{y\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha^3_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{y\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha^3_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha^2_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2, ;
\end{aligned} \tag{359}
$$

$$
\begin{aligned}
C^{xx}_{yz} = &-(2\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + 2\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta} + 2\alpha^3_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha^3_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
&- 4\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 4\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho \\
&- 2\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta}\rho - 2\alpha^3_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 2\alpha^3_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha_{y\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 4\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
&+ 4\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 4\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{360}
$$

$$
\begin{aligned}
C^{xx}_{zz} = &-(\alpha^2_{y\eta}\alpha^4_{z\zeta} + \alpha^2_{y\zeta}\alpha^4_{z\eta} - \alpha^2_{y\eta}\alpha^2_{z\xi}\alpha^2_{z\zeta} \\
&- \alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha^2_{y\eta}\alpha^2_{z\eta}\alpha^2_{z\zeta} + \alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha^2_{z\zeta} \\
&- \alpha^2_{y\eta}\alpha^4_{z\zeta}\rho - \alpha^2_{y\zeta}\alpha^4_{z\eta}\rho + \alpha^2_{y\eta}\alpha^2_{z\xi}\alpha^2_{z\zeta}\rho \\
&+ \alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - \alpha^2_{y\eta}\alpha^2_{z\eta}\alpha^2_{z\zeta}\rho - \alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^3_{z\zeta} - 2\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\alpha_{z\zeta} + 2\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^3_{z\zeta}\rho + 2\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{361}
$$

$$C^{xx}_{D_1} = (\alpha^2_{y\xi}\alpha^2_{z\zeta} - 2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha^2_{z\xi}\alpha^2_{y\zeta})/r_2, \tag{362}$$

$$C^{xx}_{D_2} = (\alpha^2_{y\xi}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} + \alpha^2_{y\eta}\alpha^2_{z\xi})/r_2, \tag{363}$$

$$C^{xx}_{D_3} = (2\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta} + 2\alpha^2_{y\xi}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta})/r_2, \tag{364}$$

$$C^{xx}_{D_4} = -D_4(2\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta})/(\beta^+ r_2), \tag{365}$$

$$C^{xx}_{D_5} = (2\alpha_{y\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta})/(\beta^+ r_2), \tag{366}$$

$$C^{xx}_{D_8} = -(\alpha^2_{y\eta}\alpha^2_{z\zeta} - 2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha^2_{y\zeta}\alpha^2_{z\eta})/r_2, \tag{367}$$

where,

$$r_2 = (\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi})^2, \tag{368}$$

$$\begin{aligned} u^+_{xy} &= C^{xy}_{xx}u^-_{xx} + C^{xy}_{xy}u^-_{xy} + C^{xy}_{xz}u^-_{xz} + C^{xy}_{yy}u^-_{yy} + C^{xy}_{yz}u^-_{yz} \\ &\quad + C^{xy}_{zz}u^-_{zz} + C^{xy}_{D_1}D_1 + C^{xy}_{D_2}D_2 \\ &\quad + C^{xy}_{D_3}D_3 + C^{xy}_{D_4}D_4 + C^{xy}_{D_5}D_5 + C^{xy}_{D_8}D_8, \end{aligned} \tag{369}$$

where,

$$\begin{aligned} C^{xy}_{xx} = (&\alpha^3_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta} + \alpha^3_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\xi}\alpha^2_{z\zeta} + \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta} \\ &+ \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha^3_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho - \alpha^3_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho \\ &- \alpha^3_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} \\ &+ \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &+ \alpha^3_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha^3_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &+ \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\rho + \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{y\xi}\alpha^2_{z\zeta}\rho \\ &- \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho - \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho \\ &+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\ &- \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho \\ &- \alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &+ \alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2, \end{aligned} \tag{370}$$

$$C^{xy}_{xy} = (\alpha^2_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta} + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi} + 2\alpha^2_{x\eta}\alpha^2_{y\eta}\alpha^2_{z\zeta}$$
$$+ 2\alpha^2_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha^2_{x\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho + \alpha^2_{x\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho$$
$$+ \alpha^2_{x\eta}\alpha^2_{y\xi}\alpha^2_{z\zeta}\rho + \alpha^2_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho$$
$$- 2\alpha^2_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}$$
$$- 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}$$
$$- \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}$$
$$+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}$$
$$- 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}$$
$$+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \quad (371)$$
$$+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho$$
$$- \alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho$$
$$- 2\alpha_{x\eta}\alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho$$
$$+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho$$
$$+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho$$
$$+ 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho$$
$$+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho$$
$$+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,$$

$$C^{xy}_{xz} = -(\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha^2_{z\eta} + \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{z\zeta} - \alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\eta} - 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}$$
$$- \alpha^2_{x\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha^2_{x\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - 2\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}$$
$$- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^3_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}$$
$$+ \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}$$
$$+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^3_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho - \alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha^2_{z\eta}\rho - \alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{z\zeta}\rho$$
$$+ \alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\eta}\rho + 2\alpha^2_{x\eta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + \alpha^2_{x\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho$$
$$- 2\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}$$
$$- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta}\rho$$
$$- \alpha_{x\eta}\alpha_{x\zeta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho$$
$$+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,$$
$$(372)$$

$$
\begin{aligned}
C^{xy}_{yy} = (&\alpha_{x\eta}\alpha^3_{y\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha^3_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} \\
&- \alpha_{x\eta}\alpha^3_{y\eta}\alpha^2_{z\zeta}\rho - \alpha_{x\zeta}\alpha^3_{y\zeta}\alpha^2_{z\eta}\rho - \alpha_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho + \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
&- \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta}\rho - \alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\
&- \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned} \tag{373}
$$

$$
\begin{aligned}
C^{xy}_{yz} = -(&\alpha_{x\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
&- \alpha_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} \\
&+ 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta} - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho \\
&- \alpha_{x\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho - \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
&+ \alpha_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
&- 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho \\
&- \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{374}
$$

$$
\begin{aligned}
C^{xy}_{zz} = (&\alpha_{x\eta}\alpha_{y\eta}\alpha^4_{z\zeta} + \alpha_{x\zeta}\alpha_{y\zeta}\alpha^4_{z\eta} - \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha^2_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\zeta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha^2_{z\zeta} - \alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^3_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^3_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha^3_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\eta}\alpha^4_{z\zeta}\rho - \alpha_{x\zeta}\alpha_{y\zeta}\alpha^4_{z\eta}\rho + \alpha_{x\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^3_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha^3_{z\zeta}\rho \\
&+ \alpha_{x\zeta}\alpha_{y\eta}\alpha^3_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha^2_{z\zeta}\rho - \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\eta}\alpha^2_{z\zeta}\rho \\
&+ \alpha_{x\zeta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - \alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha^2_{z\zeta}\rho \\
&- \alpha_{x\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{375}
$$

$$
C^{xy}_{D_1} = -(\alpha_{x\xi}\alpha_{y\xi}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha^2_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\zeta})/r_2, \tag{376}
$$

$$
\begin{aligned}
C^{xy}_{D_2} = -(&\alpha_{x\xi}\alpha_{y\xi}\alpha^2_{z\eta} + \alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\xi} - \alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} \\
&- \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{z\eta})/r_2,
\end{aligned} \tag{377}
$$

$$
\begin{aligned}
C^{xy}_{D_3} = -(&\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha^2_{z\xi}\alpha_{y\zeta} + \alpha_{x\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta})/r_2,
\end{aligned} \tag{378}
$$

$$
\begin{aligned}
C^{xy}_{D_4} = -(&\alpha_{x\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha^2_{z\zeta} - \alpha_{x\xi}\alpha_{y\eta}\alpha^2_{z\zeta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta})/(\beta^+ r_2),
\end{aligned} \tag{379}
$$

$$
\begin{aligned}
C^{xy}_{D_5} = -(&\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha_{y\zeta}\alpha^2_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta})/(\beta^+ r_2),
\end{aligned} \tag{380}
$$

$$
C^{xy}_{D_8} = (\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta})/r_2, \tag{381}
$$

$$
\begin{aligned}
u^+_{xz} = &C^{xz}_{xx}u^-_{xx} + C^{xz}_{xy}u^-_{xy} + C^{xz}_{xz}u^-_{xz} + C^{xz}_{yy}u^-_{yy} + C^{xz}_{yz}u^-_{yz} + C^{xz}_{zz}u^-_{zz} + C^{xz}_{D_1}D_1 + C^{xz}_{D_2}D_2 \\
&+ C^{xz}_{D_3}D_3 + C^{xz}_{D_4}D_4 + C^{xz}_{D_5}D_5 + C^{xz}_{D_8}D_8,
\end{aligned} \tag{382}
$$

where,

$$
\begin{aligned}
C_{xx}^{xz} =&(\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\eta} + \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2 \\
&- \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi} + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} \\
&- \alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\zeta}\rho - \alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\rho + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\rho \\
&- \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho \\
&- \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho)/r_2;
\end{aligned}
\tag{383}
$$

$$
\begin{aligned}
C_{xy}^{xz} =&-(\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi} + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2 - \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} \\
&- 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} - \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta}\rho - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\rho \\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2\rho + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho \\
&- \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho)/r_2,
\end{aligned}
\tag{384}
$$

$$
\begin{aligned}
C_{xz}^{xz} =&(\alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{z\zeta}^2 + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2 \\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\xi}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho + \alpha_{x\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}^2\rho \\
&+ \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}^2\rho - 2\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} \\
&- \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{y\zeta}\rho - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
&- \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{385}
$$

$$
\begin{aligned}
C_{yy}^{xz} =&(\alpha_{x\eta}\alpha_{y\zeta}^4\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{y\eta}^4\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta} \\
&+ \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta} - \alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} + \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^3\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\eta}^3\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^3\alpha_{z\eta} \\
&- \alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\zeta}^4\alpha_{z\eta}\rho - \alpha_{x\zeta}\alpha_{y\eta}^4\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}^3\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\eta}^3\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^3\alpha_{z\eta}\rho \\
&+ \alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\eta}\alpha_{y\eta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\rho + \alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho \\
&- \alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{y\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{386}
$$

$$
\begin{aligned}
C^{xz}_{yz} =& -(\alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} + \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} \\
& - \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta} + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} \\
& - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^3_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\zeta}\alpha^3_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho \\
& + \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho \\
& + 2\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
& - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho \\
& - \alpha_{y\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\eta}\alpha^2_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
& + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned} \tag{387}
$$

$$
\begin{aligned}
C^{xz}_{zz} =& (\alpha_{x\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta} + \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^3_{z\zeta} - \alpha_{x\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} \\
& + \alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha^2_{y\zeta}\alpha^3_{z\eta}\rho \\
& - \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^3_{z\zeta}\rho - \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta} \\
& + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta} - \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} + \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^3_{z\eta}\rho \\
& + \alpha_{x\xi}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho - \alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - \alpha_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} \\
& - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha^2_{z\eta}\rho + \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho \\
& - 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned} \tag{388}
$$

$$
C^{xz}_{D_1} = -(\alpha_{x\xi}\alpha_{z\xi}\alpha^2_{y\zeta} + \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta})/r_2, \tag{389}
$$

$$
C^{xz}_{D_2} = -(\alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\xi} + \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\xi})/r_2, \tag{390}
$$

$$
\begin{aligned}
C^{xz}_{D_3} =& -(\alpha_{x\xi}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta} - \alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha^2_{y\xi}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
& - 2\alpha_{x\xi}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi})/r_2,
\end{aligned} \tag{391}
$$

$$
\begin{aligned}
C^{xz}_{D_4} =& -(\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta} - \alpha_{x\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
& - 2\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta})/(\beta^+ r_2),
\end{aligned} \tag{392}
$$

$$C^{xz}_{D_5} = -(\alpha_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi} - \alpha_{x\xi}\alpha^2_{y\eta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta})/(\beta^+ r_2), \quad (393)$$

$$C^{xz}_{D_8} = (\alpha_{x\eta}\alpha^2_{y\zeta}\alpha_{z\eta} + \alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta})/r_2, \quad (394)$$

$$u^+_{yy} = C^{yy}_{xx}u^-_{xx} + C^{yy}_{xy}u^-_{xy} + C^{yy}_{xz}u^-_{xz} + C^{yy}_{yy}u^-_{yy} + C^{yy}_{yz}u^-_{yz} + C^{yy}_{zz}u^-_{zz} + C^{yy}_{D_1}D_1 + C^{yy}_{D_2}D_2 + C^{yy}_{D_3}D_3 + C^{yy}_{D_4}D_4 + C^{yy}_{D_5}D_5 + C^{yy}_{D_8}D_8, \quad (395)$$

where,

$$\begin{aligned} C^{yy}_{xx} = &-(\alpha^4_{x\eta}\alpha^2_{z\zeta} + \alpha^4_{x\zeta}\alpha^2_{z\eta} - \alpha^2_{x\xi}\alpha^2_{x\eta}\alpha^2_{z\zeta} - \alpha^2_{x\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} \\ &+ \alpha^2_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta} + \alpha^2_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\zeta} - \alpha^4_{x\eta}\alpha^2_{z\zeta}\rho - \alpha^4_{x\zeta}\alpha^2_{z\eta}\rho + \alpha^2_{x\xi}\alpha^2_{x\eta}\alpha^2_{z\zeta}\rho + \alpha^2_{x\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta}\rho \\ &- \alpha^2_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\eta}\rho - \alpha^2_{x\eta}\alpha^2_{x\zeta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\eta}\alpha^3_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha^3_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &+ 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha^3_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha^3_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2, \end{aligned} \quad (396)$$

$$\begin{aligned} C^{yy}_{xy} = &-(2\alpha^3_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta} + 2\alpha^3_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta} + 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta} \\ &- 2\alpha^2_{x\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} + 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\zeta} \\ &- 2\alpha^3_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho - 2\alpha^3_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} \\ &- 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &+ 2\alpha^2_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\ &+ 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha^2_{z\eta}\rho \\ &+ 2\alpha^2_{x\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\ &+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha^2_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &- 2\alpha^2_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\ &- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2, \end{aligned} \quad (397)$$

$$\begin{aligned} C^{yy}_{xz} = &-(2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^3_{z\eta} + 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^3_{z\zeta} + 2\alpha^3_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta} \\ &+ 2\alpha^3_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{z\zeta} - 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha^2_{z\eta} - 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta} - 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta} \\ &- 2\alpha_{x\eta}\alpha^2_{x\zeta}\alpha^3_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^3_{z\zeta}\rho \\ &- 2\alpha^3_{x\eta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho - 2\alpha^3_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha^2_{x\eta}\alpha_{z\xi}\alpha^2_{z\zeta}\rho + 2\alpha_{x\xi}\alpha^2_{x\zeta}\alpha_{z\xi}\alpha^2_{z\eta}\rho \\ &+ 4\alpha_{x\eta}\alpha^2_{x\zeta}\alpha_{z\eta}\alpha^2_{z\zeta}\rho + 4\alpha^2_{x\eta}\alpha_{x\zeta}\alpha^2_{z\eta}\alpha_{z\zeta}\rho \\ &+ 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2, \end{aligned} \quad (398)$$

$$
\begin{aligned}
C_{yy}^{yy} =&(\alpha_{x\xi}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2 + \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}^2\\
&+ \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}^2 - \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}^2 - \alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\\
&+ \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{z\zeta}^2\rho + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2\rho\\
&+ \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\\
&- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{y\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2\rho\\
&- 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho\\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{399}
$$

$$
\begin{aligned}
C_{yz}^{yy} =&-(2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^3 + 2\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^3 - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{z\eta}\\
&+ 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^3\rho - 2\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^3\rho\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}^2\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2\rho\\
&+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}^2\alpha_{y\zeta}\alpha_{z\eta}\rho\\
&+ 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho\\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{400}
$$

$$
\begin{aligned}
C_{zz}^{yy} =&-(\alpha_{x\eta}^2\alpha_{z\zeta}^4 + \alpha_{x\zeta}^2\alpha_{z\eta}^4 - \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{z\xi}^2\alpha_{z\eta}^2\\
&+ \alpha_{x\eta}^2\alpha_{z\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}^2 - \alpha_{x\eta}^2\alpha_{z\zeta}^4\rho - \alpha_{x\zeta}^2\alpha_{z\eta}^4\rho + \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{z\zeta}^2\rho + \alpha_{x\zeta}^2\alpha_{z\xi}^2\alpha_{z\eta}^2\rho\\
&- \alpha_{x\eta}^2\alpha_{z\eta}^2\alpha_{z\zeta}^2\rho - \alpha_{x\zeta}^2\alpha_{z\eta}^2\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^3 - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}^3\alpha_{z\zeta}\\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}^3\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}^3\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}^2\alpha_{z\eta}\alpha_{z\zeta}\rho)/r_2,
\end{aligned}
\tag{401}
$$

$$
C_{D_1}^{yy} =(\alpha_{x\xi}^2\alpha_{z\zeta}^2 - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\zeta} + \alpha_{x\zeta}^2\alpha_{z\xi}^2)/r_2, \tag{402}
$$

$$
C_{D_2}^{yy} =(\alpha_{x\xi}^2\alpha_{z\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{z\xi}^2)/r_2, \tag{403}
$$

$$
C_{D_3}^{yy} =-(2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}^2 + 2\alpha_{x\xi}^2\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta})/r_2, \tag{404}
$$

$$C^{yy}_{D_4} = -\left(2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\zeta}\right)/(\beta^+ r_2), \tag{405}$$

$$C^{yy}_{D_5} = -\left(2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\right)/(\beta^+ r_2), \tag{406}$$

$$C^{yy}_{D_8} = -\left(\alpha_{x\eta}^2\alpha_{z\zeta}^2 - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\zeta}^2\alpha_{z\eta}^2\right)/r_2, \tag{407}$$

$$\begin{aligned} u^+_{yz} = {} & C^{yz}_{xx}u^-_{xx} + C^{yz}_{xy}u^-_{xy} + C^{yz}_{xz}u^-_{xz} + C^{yz}_{yy}u^-_{yy} + C^{yz}_{yz}u^-_{yz} + C^{yz}_{zz}u^-_{zz} + C^{yz}_{D_1}D_1 + C^{yz}_{D_2}D_2 \\ & + C^{yz}_{D_3}D_3 + C^{yz}_{D_4}D_4 + C^{yz}_{D_5}D_5 + C^{yz}_{D_8}D_8, \end{aligned} \tag{408}$$

where,

$$\begin{aligned} C^{yz}_{xx} = {} & (\alpha_{x\zeta}^4\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\eta}^4\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} - \alpha_{x\xi}^2\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} \\ & + \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\zeta} \\ & - \alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} \\ & - \alpha_{x\zeta}^4\alpha_{y\eta}\alpha_{z\eta}\rho - \alpha_{x\eta}^4\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} \\ & + \alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho + \alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\ & + \alpha_{x\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho + \alpha_{x\xi}^2\alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho - \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho \\ & - \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho - \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2, \end{aligned} \tag{409}$$

$$\begin{aligned} C^{yz}_{xy} = {} & -(\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2 + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi} - \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\ & - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta} - \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta} \\ & + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\zeta} - 2\alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\ & + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\ & + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\ & + 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\rho \\ & + \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\rho + \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho \\ & - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\ & - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} \\ & - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\ & - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho \\ & + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho)/r_2, \end{aligned} \tag{410}$$

$$
\begin{aligned}
C_{xz}^{yz} = &- (\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\eta}^2 + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\zeta}^2 \; - \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2 - \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2 + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\zeta}^2 + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\zeta}^2 \\
&- 2\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}^3\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\eta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\zeta}^2\rho + \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\zeta}^2\rho + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^2\rho \\
&+ \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\zeta}^2\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&- \alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - \alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned}
\tag{411}
$$

$$
\begin{aligned}
C_{yy}^{yz} = &(\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2 \\
&+ \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} \\
&- \alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\eta}\rho - \alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\zeta}\rho - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta} \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} \\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^3\alpha_{z\eta}\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}\rho + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}^2\rho - \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
&- \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho \\
&+ \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\rho)/r_2,
\end{aligned}
\tag{412}
$$

$$
\begin{aligned}
C_{yz}^{yz} =&(\alpha_{x\xi}^2\alpha_{y\eta}^2\alpha_{z\zeta}^2 + \alpha_{x\xi}^2\alpha_{y\zeta}^2\alpha_{z\eta}^2 + 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}^2 + 2\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta}^2 + \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\eta}^2\alpha_{z\xi}^2\alpha_{y\zeta}^2\rho + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{z\eta}^2\rho + \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\xi}^2\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\eta}^2\alpha_{y\zeta}^2\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2 \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2 - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta} - 2\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} \\
&- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2 - 2\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2 - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta} \\
&- 2\alpha_{x\xi}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} \\
&+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{z\zeta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}^2\alpha_{y\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}^2\rho \\
&- 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}^2\rho \\
&+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&+ \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_2,
\end{aligned} \tag{413}
$$

$$
\begin{aligned}
C_{zz}^{yz} =&(\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^3 + \alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^3 - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2 - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{z\zeta}^2 \\
&+ \alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta} + \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2 - \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}^3\rho - \alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}^3\rho \\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^3 - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^3 + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2 + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2 \\
&- \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2 + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}^3\rho + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}^3\rho \\
&+ \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\xi}\alpha_{z\eta}^2\rho + \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi}\alpha_{z\zeta}^2\rho - \alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho \\
&- \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho - \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}^2\rho \\
&- \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}^2\rho + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}^2\alpha_{z\zeta}\rho + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}^2\rho \\
&+ \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}\rho + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho \\
&- 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{z\eta}\alpha_{z\zeta}\rho)r_2,
\end{aligned} \tag{414}
$$

$$
C_{D_1}^{yz} = -(\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\xi} + \alpha_{x\xi}^2\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\zeta} - \alpha_{x\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta})/r_2, \tag{415}
$$

$$
C_{D_2}^{yz} = -(\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\xi} + \alpha_{x\xi}^2\alpha_{y\eta}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\eta} - \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi})/r_2, \tag{416}
$$

$$C_{D_3}^{yz} = -\,(\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{z\zeta} - \alpha_{x\xi}^2\alpha_{y\zeta}\alpha_{z\eta} - \alpha_{x\xi}^2\alpha_{y\eta}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha_{y\zeta} + \alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi})/r_2, \tag{417}$$

$$C_{D_4}^{yz} = -\,(\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\xi} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{z\eta} + \alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\zeta} + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta})/(\beta^+ r_2), \tag{418}$$

$$C_{D_5}^{yz} = -\,(\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}^2\alpha_{z\xi}\alpha_{y\zeta} - \alpha_{x\eta}^2\alpha_{y\xi}\alpha_{z\zeta} + \alpha_{x\xi}\alpha_{x\eta}\alpha_{y\zeta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\eta} + \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi})/(\beta^+ r_2), \tag{419}$$

$$C_{D_8}^{yz} = (\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} + \alpha_{x\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} - \alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta})/r_2, \tag{420}$$

$$u_{zz}^+ = C_{xx}^{zz}u_{xx}^- + C_{xy}^{zz}u_{xy}^- + C_{xz}^{zz}u_{xz}^- + C_{yy}^{zz}u_{yy}^- + C_{yz}^{zz}u_{yz}^- + C_{zz}^{zz}u_{zz}^- + C_{D_1}^{zz}D_1 + C_{D_2}^{zz}D_2 + C_{D_3}^{zz}D_3 + C_{D_4}^{zz}D_4 + C_{D_5}^{zz}D_5 + C_{D_8}^{zz}D_8, \tag{421}$$

where,

$$C_{xx}^{zz} = -\,(\alpha_{x\eta}^4\alpha_{y\zeta}^2 + \alpha_{x\zeta}^4\alpha_{y\eta}^2 - \alpha_{x\xi}^2\alpha_{x\eta}^2\alpha_{y\zeta}^2 - \alpha_{x\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2 + \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2 + \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\zeta}^2 - \alpha_{x\eta}^4\alpha_{y\zeta}^2\rho - \alpha_{x\zeta}^4\alpha_{y\eta}^2\rho + \alpha_{x\xi}^2\alpha_{x\eta}^2\alpha_{y\zeta}^2\rho + \alpha_{x\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2\rho - \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2\rho - \alpha_{x\eta}^2\alpha_{x\zeta}^2\alpha_{y\zeta}^2\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}^3\alpha_{y\eta}\alpha_{y\zeta}\rho + 2\alpha_{x\eta}^3\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\rho - 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\rho)/r_3, \tag{422}$$

$$C_{xy}^{zz} = -\,(2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^3 + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^3 + 2\alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}^2 + 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{y\zeta} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2 - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2 - 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2 - 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^3\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^3\rho - 2\alpha_{x\eta}^3\alpha_{y\eta}\alpha_{y\zeta}^2\rho - 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{y\zeta}\rho + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}^2\rho + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}^2\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\rho + 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\rho + 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} - 4\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\rho)/r_3, \tag{423}$$

$$
\begin{aligned}
C_{xz}^{zz} =& -(2\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\eta} + 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\zeta} - 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta} + 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta} \\
& - 2\alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\zeta} - 2\alpha_{x\eta}^3\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\zeta}^3\alpha_{y\eta}^2\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} \\
& + 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{z\eta}\rho + 2\alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho - 2\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\eta}^2\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}^2\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
& + 4\alpha_{x\eta}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_3,
\end{aligned}
\tag{424}
$$

$$
\begin{aligned}
C_{yy}^{zz} =& -(\alpha_{x\eta}^2\alpha_{y\zeta}^4 + \alpha_{x\zeta}^2\alpha_{y\eta}^4 - \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}^2 - \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2 + \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}^2 + \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}^2 \\
& - \alpha_{x\eta}^2\alpha_{y\zeta}^4\rho - \alpha_{x\zeta}^2\alpha_{y\eta}^4\rho + \alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}^2\rho + \alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}^2\rho - \alpha_{x\eta}^2\alpha_{y\eta}^2\alpha_{y\zeta}^2\rho - \alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}^2\rho \\
& - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^3 - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{y\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} \\
& + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^3\rho + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^3\alpha_{y\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\rho)/r_3,
\end{aligned}
\tag{425}
$$

$$
\begin{aligned}
C_{yz}^{zz} =& -(2\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\eta} + 2\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\zeta} - 2\alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta} \\
& + 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta} - 2\alpha_{x\zeta}^2\alpha_{y\eta}^3\alpha_{z\eta}\rho \\
& - 2\alpha_{x\eta}^2\alpha_{y\zeta}^3\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta} \\
& + 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta} - 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta} + 2\alpha_{y\xi}^2\alpha_{x\zeta}^2\alpha_{y\eta}\alpha_{z\eta}\rho \\
& - 2\alpha_{x\eta}^2\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\eta}\rho + 2\alpha_{x\eta}^2\alpha_{y\xi}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\zeta}^2\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\zeta}\rho \\
& + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}^2\alpha_{z\eta}\rho \\
& + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{y\xi}^2\alpha_{x\zeta}\alpha_{y\zeta}\alpha_{z\eta}\rho \\
& + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}^2\alpha_{y\zeta}\alpha_{z\eta}\rho + 4\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}^2\alpha_{z\zeta}\rho \\
& - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_3,
\end{aligned}
\tag{426}
$$

$$C^{zz}_{zz} =(\alpha^2_{x\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} + \alpha^2_{x\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} + \alpha^2_{x\eta}\alpha^2_{y\xi}\alpha^2_{z\zeta} + \alpha^2_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} - \alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta} - \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}$$
$$- \alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta} - \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta} + \alpha^2_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta}\rho$$
$$+ \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}\rho + \alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\eta}\rho + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\eta}\rho$$
$$+ \alpha^2_{x\eta}\alpha^2_{y\zeta}\alpha^2_{z\zeta}\rho + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\zeta}\rho - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta}$$
$$+ 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\eta}\alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta} - 2\alpha^2_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta}$$
$$+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta}\rho$$
$$- 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta}\rho - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta}\rho$$
$$- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\eta}\rho - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta}\alpha^2_{z\zeta}\rho$$
$$+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}\rho + 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho$$
$$+ 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}\rho + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}\rho)/r_3, \quad (427)$$

$$C^{zz}_{D_1} =(\alpha^2_{x\xi}\alpha^2_{y\zeta} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta} + \alpha^2_{y\xi}\alpha^2_{x\zeta})/r_3, \quad (428)$$

$$C^{zz}_{D_2} =(\alpha^2_{x\xi}\alpha^2_{y\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta} + \alpha^2_{x\eta}\alpha^2_{y\xi})/r_3, \quad (429)$$

$$C^{zz}_{D_3} =-(2\alpha_{x\eta}\alpha^2_{y\xi}\alpha_{x\zeta} + 2\alpha^2_{x\xi}\alpha_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta} - 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta})/r_3, \quad (430)$$

$$C^{zz}_{D_4} =-(2\alpha_{x\xi}\alpha_{x\eta}\alpha^2_{y\zeta} + 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta})/r_3, \quad (431)$$

$$C^{zz}_{D_5} =-(2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta} + 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{y\zeta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{y\zeta} - 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta})/r_3, \quad (432)$$

$$C^{zz}_{D_8} =-(\alpha^2_{x\eta}\alpha^2_{y\zeta} - 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{y\zeta} + \alpha^2_{x\zeta}\alpha^2_{y\eta})/r_3, \quad (433)$$

where,

$$r_3 =\alpha^2_{x\xi}\alpha^2_{y\eta}\alpha^2_{z\zeta} - 2\alpha^2_{x\xi}\alpha_{y\eta}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + \alpha^2_{x\xi}\alpha^2_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\eta}\alpha^2_{z\zeta}$$
$$+ 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\xi}\alpha_{y\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}$$
$$- 2\alpha_{x\xi}\alpha_{x\eta}\alpha_{z\xi}\alpha^2_{y\zeta}\alpha_{z\eta} + 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\eta}\alpha_{z\zeta}$$
$$- 2\alpha_{x\xi}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\zeta}\alpha^2_{z\eta} - 2\alpha_{x\xi}\alpha_{x\zeta}\alpha^2_{y\eta}\alpha_{z\xi}\alpha_{z\zeta}$$
$$+ 2\alpha_{x\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta} + \alpha^2_{x\eta}\alpha^2_{y\xi}\alpha^2_{z\zeta} - 2\alpha^2_{x\eta}\alpha_{y\xi}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\zeta}$$
$$+ \alpha^2_{x\eta}\alpha^2_{z\xi}\alpha^2_{y\zeta} - 2\alpha_{x\eta}\alpha^2_{y\xi}\alpha_{x\zeta}\alpha_{z\eta}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\zeta} + 2\alpha_{x\eta}\alpha_{y\xi}\alpha_{x\zeta}\alpha_{z\xi}\alpha_{y\zeta}\alpha_{z\eta}$$
$$- 2\alpha_{x\eta}\alpha_{x\zeta}\alpha_{y\eta}\alpha^2_{z\xi}\alpha_{y\zeta} + \alpha^2_{y\xi}\alpha^2_{x\zeta}\alpha^2_{z\eta} - 2\alpha_{y\xi}\alpha^2_{x\zeta}\alpha_{y\eta}\alpha_{z\xi}\alpha_{z\eta} + \alpha^2_{x\zeta}\alpha^2_{y\eta}\alpha^2_{z\xi}, \quad (434)$$

VITA

Kumudu Janani Gamage
Department of Computational and Applied Mathematics
Old Dominion University
Norfolk, Virginia 23529

Publications:

- Gamage, K. J. and Peng, Y., "*A Direct Method for Modeling and Simulations of Elliptic and Parabolic Interface Problems.*", College of Sciences Posters. 8. (2021)
- Gamage, K. J., and Fahrioglu, M., "*The current technological status of ground source heat pump systems and their potential use in Northern Cyprus*", International Journal of Wind and Renewable Energy (IJWRE) 3 (2014): 39-48.
- Gamage, K. J., Yousefzadeh, M., Uzgören, E. and Uzgören, Y. M., "*Optimization of a Ground Source Heat Pump System Using Monte-Carlo Simulation*", 7th International Ege Energy Symposium and Exhibition, Usak, Turkey, June 2014.
- Gamage, K. J., and Pathirana, W. P. M. R., "*A review of Green Remediation technologies and Feasibility study of integrating renewable energy into contaminated site in Northern Cyprus*", Renewable Energy Sources Symposium 2013, Turkish Republic of Northern Cyprus.

Presentations:

- Gamage, K. J., "*Modelling and Simulation of Interface Problems using Direct Immersed Interface Method.*", ODU Mathematics Awareness Conference, April 6, 2019.
- Gamage, K. J., "*Modeling and Simulation of Interface Problems using Immersed Interface Method*", ODU Mathematics Awareness Conference, April 21, 2018.

Typeset using LaTeX.