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STABLE AND CONVERGENT DIFFERENCE SCHEMES FOR WEAKLY SINGULAR CONVOLUTION INTEGRALS

WESLEY DAVIS AND RICHARD NOREN

We obtain new numerical schemes for weakly singular integrals of convolution type called Caputo fractional order integrals using Taylor and fractional Taylor series expansions and grouping terms in a novel manner. A fractional Taylor series expansion argument is utilized to provide fractional-order approximations for functions with minimal regularity. The resulting schemes allow for the approximation of functions in $C^\gamma[0,T]$, where $0<\gamma\le 5$. A mild invertibility criterion is provided for the implicit schemes. Consistency and stability are proven separately for the whole-number-order approximations and the fractional-order approximations. The rate of convergence in the time variable is shown to be $O(\tau^\gamma)$, $0<\gamma\le 5$ for $u\in C^\gamma[0,T]$, where τ is the size of the partition of the time mesh. Crucially, the assumption of the integral kernel K being decreasing is not required for the scheme to converge in second-order and below approximations. Optimal convergence results are then proven for both sets of approximations, where fractional-order approximations can obtain up to whole-number rate of convergence in certain scenarios. Finally, numerical examples are provided that illustrate our findings.

1. Introduction

We begin by recalling the Caputo fractional time-derivative [9; 10] of a given function $f(t)$ is

$$ {}_0^C D_t^\alpha f(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{df(s)}{ds}(t-s)^{-\alpha}\,ds,\qquad 0<\alpha<1, \tag{1}$$

which is a fractional derivative of order α. In [3], the Laplace transform was applied to the fractional order diffusion initial-boundary value problem

$$ {}_0^C D_t^\alpha u(x,t)=\frac{\partial^2}{\partial x^2}u(x,t)+g(x,t),\qquad x\in[0,1],\ t\in[0,T], \tag{2}$$

$$ u(x,0)=\phi(x),\qquad u(0,t)=u(1,t)=0, \tag{3}$$

to obtain the integral equation

$$ u(x,t)=\phi(x)+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(\frac{\partial^2}{\partial x^2}u(x,s)+g(x,s)\right)ds. \tag{4}$$

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Equation (4) was then studied numerically and convergent schemes were developed for this integral equation inspired by the works presented in [8; 11; 12; 15; 17; 20]. The regularity of (2) has been considered in [3; 11; 15]. Our discussion of the regularity of these schemes is motivated by the findings in [5; 4; 7]. We will derive and examine numerical schemes to discretize integrals of the form (4), motivated by the principles in [4; 6; 13; 14; 16]. Equations of the form (4) have numerous engineering and physics applications; see [15; 17; 20]. One of the major advantages of applying the Laplace transform to a fractional derivative term, as seen in [3], is the ability to relax the regularity assumption for fractional derivative discretizations in the time variable while preserving an optimal convergence rate. Namely, we now have the ability to relax the regularity assumption from requiring the objective function $u(t) \in C^2[0, T]$ under the well known L1-method (see [20]) to $u(t) \in C^\gamma[0, T]$, where $0 < \gamma \leq 2$. Further, we can strengthen this assumption to $u(t) \in C^\gamma[0, T]$, where $2 < \gamma \leq 5$ while obtaining optimal rate of convergence. This is achieved by a Taylor series expansion to obtain convergence results for whole number values of γ, and by using a fractional Taylor series expansion to approximate functions with a fractional order of regularity, see [19]. By requiring more regularity than $u(t) \in C^2[0, T]$ in the usual L1-method, we are able to obtain a higher order of convergence, as seen in Theorems 3.6 and 3.7. This method naturally generalizes to any convolution type-quadrature where the kernel function K is positive, nonincreasing, and satisfies $K \in L^1[0, T]$, as seen in Theorems 3.4 and 3.5. The space variable can be discretized by a stable finite difference operator presented in [3] and [20] to obtain a rate of convergence in the space variable of $O(h^4)$, where h denotes the size of the partition of the space variable interval. This ultimately yields a rate of convergence in both space and time for $u(x, t) \in C^\gamma([0, T]; C^6[0, 1])$ of $O(\tau^\gamma + h^4)$, where τ is the size of the partition of the time variable interval. We remark that a standard finite difference operator in the space variable can relax the regularity in space to $u(x, t) \in C^\gamma([0, T]; C^4[0, 1])$, where special consideration must be taken to ensure stability in the space variable.

The remainder of the paper is organized as follows. Section 2 will provide discretizations for fractional integrals of the form found in (4), and a general scheme is established for convolution integrals based on the integral kernel. We obtain general schemes of orders up to fifth order of accuracy in time. Section 3 establishes all of the necessary consistency, stability, and convergence results for each of these schemes, in addition to a discussion of the implementation of the schemes. We also prove optimal order of convergence of our stable schemes, and the instability of schemes of order 6 and above are presented as well. The main results are featured in Theorems 3.4–3.7. Section 4 presents numerical solutions of fractional integral equations demonstrating orders of convergence predicted by our theoretical results.

2. Discretized numerical schemes

In order to discretize the Caputo fractional integral (4), let $0 = t_0 < t_1 < \ldots < t_N = T$ be a uniform partition, define $\tau = \frac{T}{N} = t_k - t_{k-1}, k = 1, \ldots, N$ where N is the number of partitions of the time interval $[0, T]$ and let $s \in (0, T)$. Then,

$$f(s) = f(t_k) + (s - t_k) f'(t_k) + \frac{(s - t_k)^2}{2!} f''(t_k) + \frac{(s - t_k)^3}{3!} f'''(t_k) + \cdots . \tag{5}$$

From the above, similar Taylor expansions centered at any given t_k can be constructed for each of the previous points $t_{k-1}, t_{k-2}, \ldots, t_1, t_0 \in [0, t_k]$. We will use the notation $A = O(h)$ if A/h is bounded.

We obtain

$$
\begin{aligned}
f(t_k) &= f(t_k),\\
f(t_{k-1}) &= f(t_k) - \tau f'(t_k) + \frac{\tau^2}{2!} f''(t_k) - \frac{\tau^3}{3!} f'''(t_k) + O(\tau^4),\\
f(t_{k-2}) &= f(t_k) - 2\tau f'(t_k) + \frac{(2\tau)^2}{2!} f''(t_k) - \frac{(2\tau)^3}{3!} f'''(t_k) + O(\tau^4),\\
&\vdots\\
f(t_0) &= f(t_k) - k\tau f'(t_k) + \frac{(k\tau)^2}{2!} f''(t_k) - \frac{(k\tau)^3}{3!} f'''(t_k) + O(\tau^4).
\end{aligned} \tag{6}
$$

We will use the following equations to find the j-th order approximation of $f(s)$ for any point $s \in (0, T)$ and each $k = 0, \ldots, N$:

$$\sum_{i=0}^{j-1} c_i^k f(t_{k-i}) = f(s) + O((s-t_k)^j), \tag{7}$$

$$\sum_{i=0}^{j-1} c_i^k f(t_{k-i}) = \sum_{i=0}^{j-1} \frac{(s-t_k)^i}{i!} f^{(i)}(t_k) + O((s-t_k)^j). \tag{8}$$

We replace each $f(t_{k-i})$ by its Taylor expansion about the point t_k, neglect the higher order terms and solve the system resulting from equating coefficients of $f(t_k), f'(t_k), \ldots, f^{(j-1)}(t_k)$. For example, a second order approximation of $f(s)$ is obtained from

$$c_0^k f(t_k) + c_1^k (f(t_k) - \tau f'(t_k))i = f(t_k) + (s-t_k) f'(t_k),$$

by equating the coefficients of $f(t_k)$ and $f'(t_k)$ to obtain the system of equations

$$
\begin{aligned}
c_0^k + c_1^k &= 1,\\
-c_1^k \tau &= (s - t_k).
\end{aligned}
$$

Solving the above yields $c_1^k = (t_k - s)/\tau$, $c_0^k = 1 - (t_k - s)/\tau$. As an example, we may numerically approximate the integral as seen in [3] using c_0^k and c_1^k as solved for above:

$$\int_0^{t_n} \frac{(t_n - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{(t_n - s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds \approx \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{(t_n - s)^{\alpha-1}}{\Gamma(\alpha)} (c_0^k f(t_k) + c_1^k f(t_{k-1}))\, ds.$$

We remark that under this construction, we satisfy the condition $s \in [t_{k-1}, t_k]$. This directly implies that the coefficients c_0 and c_1 presented above are nonnegative. We now provide the values of the coefficients for each scheme up to fourth order accuracy. Higher order schemes can be derived using (7). In this way, the general method is outlined below. We remark that in general, $c_i = c_i(s)$ for each $i = 0, 1, \ldots, j-1$. For fixed $k = 1, 2, \ldots, N$:

First order accurate:

$$
\begin{aligned}
c_0^k &= 1,\\
f(s) &= f(t_k) + O(\tau).
\end{aligned}
$$

Second order accurate:

$$c_0^k = 1 - \frac{t_k - s}{\tau}, \quad c_1^k = \frac{t_k - s}{\tau}, \quad k \geq 1,$$
$$f(s) = c_0^k f(t_k) + c_1^k f(t_{k-1}) + O(\tau^2).$$

Third order accurate:

$$c_0^k = \frac{(\tau + s - t_k)(2\tau + s - t_k)}{2\tau^2}, \quad c_1^k = \frac{(t_k - s)(2\tau + s - t_k)}{\tau^2}, \quad c_2^k = \frac{(s - t_k)(\tau + s - t_k)}{2\tau^2}, \quad k \geq 2,$$
$$f(s) = c_0^k f(t_k) + c_1^k f(t_{k-1}) + c_2^k f(t_{k-2}) + O(\tau^3).$$

Fourth order accurate:

$$c_0^k = \frac{(\tau + s - t_k)(2\tau + s - t_k)(3\tau + s - t_k)}{6\tau^3}, \quad c_1^k = \frac{(t_k - s)(2\tau + s - t_k)(3\tau + s - t_k)}{2\tau^3},$$
$$c_2^k = \frac{(s - t_k)(\tau + s - t_k)(3\tau + s - t_k)}{2\tau^3}, \quad c_3^k = \frac{(t_k - s)(\tau + s - t_k)(2\tau + s - t_k)}{6\tau^3}, \quad k \geq 3,$$
$$f(s) = c_0^k f(t_k) + c_1^k f(t_{k-1}) + c_2^k f(t_{k-2}) + c_3^k f(t_{k-3}) + O(\tau^4).$$

As a generalization of the previous examples, after replacing each $f(t_{k-i})$ with its Taylor series, we equate the coefficients of $f(t_k)$, $f'(t_k)$, ..., $f^{(j-1)}(t_k)$ and neglect the higher order terms to obtain the following system of equations from (7)

$$V_\tau^T \vec{c_j^k} = \vec{y_j^k}, \tag{9}$$

where

$$V_\tau^T = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -\tau & -2\tau & \dots & -(j-1)\tau \\ 0 & (-\tau)^2 & (-2\tau)^2 & \dots & (-(j-1)\tau)^2 \\ & & \vdots & & \\ 0 & (-\tau)^{j-1} & (-2\tau)^{j-1} & \dots & (-(j-1)\tau)^{j-1} \end{bmatrix} \tag{10}$$

$$\vec{c_j^k} = \begin{bmatrix} c_0^k \\ c_1^k \\ c_2^k \\ \vdots \\ c_{j-1}^k \end{bmatrix}, \quad \vec{y_j^k} = \begin{bmatrix} 1 \\ (s - t_k) \\ (s - t_k)^2 \\ \vdots \\ (s - t_k)^{j-1} \end{bmatrix}. \tag{11}$$

Notice that V_τ^T is the transpose of the usual Vandermonde matrix [18] which has the determinant

$$\det(V_\tau^T) = \det(V_\tau) = \prod_{1 \leq i < n \leq j} (x_n - x_i) = \prod_{1 \leq i < n \leq j} (n - i)\tau \neq 0,$$

because, recall $x_j = jT/N$ and $\tau = T/N \neq 0$. This directly implies that the matrix V_τ^T is invertible under this condition. The following lemma is immediate from the above considerations.

Lemma 2.1. *Equation (9) has a unique solution, $\vec{c_n^k}$ for each $n \leq N$, $n \in \mathbb{N}$ and each $k = 1, \dots, N$.*

We now compute the unique solution ensured by the previous lemma. From [18], we can establish the generalized inverse of the Vandermonde matrix to solve (9).

Theorem 2.2. *Let $\tau > 0$. Then, (9) has a unique solution $\vec{c_n^k}$ for each $j = 1, 2, \ldots, N$, $N \in \mathbb{N}$ and $k \geq j$, with the solution*

$$\vec{c_i^k} = \begin{cases} \sum_{1\leq n\leq j}(s-t_k)^{n-1}(-1)^{j-i}\Big(\sum_{\substack{1\leq m_1<\ldots,m_{j-i}\leq j \\ m_1,\ldots,m_{j-1}\neq n}} x_{m_1}\ldots x_{m_{j-i}}/\prod_{1\leq i<n\leq j}(\tau(n-i))\Big) & 1 \leq i < j, \\ \sum_{1\leq n\leq j}(s-t_k)^{n-1}1/\prod_{1\leq i<n\leq j}(\tau(n-i)) & i = j, \end{cases} \tag{12}$$

where $x_n = n\tau$.

Proof. From Lemma 2.1, we may invert the matrix V_τ^T to obtain the solution

$$\vec{c_j^k} = (V_\tau^T)^{-1}\vec{y_j^k},$$

which, from [18], each entry of $(V_\tau^T)^{-1} = [v_{in}]$, $1 \leq i, j \leq n$ is calculated by

$$v_{in} = \begin{cases} (-1)^{j-i}\Big(\sum_{\substack{1\leq m_1<\ldots,m_{j-i}\leq j \\ m_1,\ldots,m_{j-1}\neq n}} x_{m_1}\ldots x_{m_{j-i}}/\prod_{1\leq i<n\leq j}(\tau(n-i))\Big) & 1 \leq i < j, \\ 1/\prod_{1\leq i<n\leq j}(\tau n - i) & i = j, \end{cases}$$

so we may solve component-wise to find each entry of $\vec{c_j}$, from

$$(V_\tau^T)^{-1}\vec{y_j^k} = \sum_{1\leq n\leq j} v_{in}y_n^k.$$

Thus,

$$\begin{aligned} \vec{c_i^k} &= (V_\tau^T)^{-1}\vec{y_j^k} \\ &= \sum_{1\leq n\leq j} v_{in}y_n^k \\ &= \begin{cases} \sum_{1\leq n\leq j}(s-t_k)^{n-1}(-1)^{j-i}\Big(\sum_{\substack{1\leq m_1<\ldots,m_{j-i}\leq j \\ m_1,\ldots,m_{j-1}\neq n}} x_{m_1}\ldots x_{m_{j-i}}/\prod_{1\leq i<n\leq j}(\tau(n-i))\Big) & 1\leq i<j, \\ \sum_{1\leq n\leq j}(s-t_k)^{n-1}1/\prod_{1\leq i<n\leq j}(\tau(n-i)) & i=j. \end{cases} \end{aligned} \tag{13}$$

□

Remark 2.3. By utilizing the fractional Taylor series expansion instead for $f(s)$ on $[0, T]$, as discussed in [19], we obtain similar results to those outlined in Theorem 2.2. This can further relax the regularity assumption to $f(s) \in C^\alpha[0, T]$ for $0 < \alpha < 1$.

Using the fractional Taylor series expansion, we define an α-order scheme by the following:

α **order accurate**:

$$c_0^k = 1,$$
$$f(s) = f(t_k) + O(\tau^\alpha).$$

We now examine the consistency, stability, and convergence of these schemes based on the generalized scheme

$$f(s) = \sum_{i=0}^{n-1} c_i^k f(t_{k-i}) + O((s-t_k)^n). \tag{14}$$

3. Numerical consistency, stability, and convergence

Numerical consistency. We motivate our discussion of stability and convergence by examining the results presented in [1]. The main results of this paper are established in Theorems 3.3–3.7. The quadrature studied in [1] is of the form

$$\int_0^T \phi(s)\,ds = \tau \sum_{j=0}^{N} w_j \phi(j\tau) + O(\tau^R), \tag{15}$$

where $R \in \mathbb{N}$. From [1], if $\phi \in C^R[0, T]$ then there exists of a sequence of constants, $\{c_l\}$, such that

$$\int_0^T \phi(s)\,ds - \tau \sum_{j=0}^{N} w_j \phi(j\tau) = \sum_{l=\rho+1}^{R} \tau^l (r^l c_l)\{\phi^{(l-1)}(T) - \phi^{(l-1)}(0)\} + O(\tau^R).$$

We will compare these results to the ones established in the previous section to prove stability and assert convergence. Our goal is to rewrite the integrand $\phi(s) = K(t_n - s) f(s)$ as a convolution integral, where we may relax the continuity assumptions on the kernel function K. We begin by recalling some basic definitions for quadrature methods. From (1.15) of [1], a quadrature method of the form (15) is said to be consistent if it satisfies

$$\sum_{j=0}^{N} w_j = N.$$

We will relate (15) and our findings in the previous section. We use the notation $\lceil \gamma \rceil = a$, where a is the smallest integer that satisfies $a \geq \gamma$.

Lemma 3.1. *Let $\gamma > 0$, $f \in C^\gamma[0, T]$, and $K \in L^1[0, T]$. Then, if w_j^k is given by(18) for $k \geq j$ and any t_n*

$$\int_0^{t_n} f(s) K(t_n - s)\,ds = \sum_{k=1}^{n} \sum_{j=0}^{\lceil \gamma \rceil - 1} w_j^k f(t_{k-j}) + O(\tau^\gamma). \tag{16}$$

Proof. By utilizing the Taylor expansion for $f(s)$ about the point t_k, we may readily obtain a similar quadrature rule by using Theorem 2.2 and the definition of each $c_j(s)$ defined in (12). By further remarking that for each $s \in [t_{k-1}, t_k]$, then we may write $O((t_k - s)^\gamma) = O(\tau^\gamma)$ to find

$$\begin{aligned}
\int_0^{t_n} f(s) K(t_n - s)\,ds &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(s) K(t_n - s)\,ds \\
&= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Bigg(\sum_{j=0}^{\lceil \gamma \rceil - 1} c_j^k(s) f(t_{k-j}) + O((s - t_k)^\gamma) \Bigg) K(t_n - s)\,ds \\
&= \sum_{k=1}^{n} \sum_{j=0}^{\lceil \gamma \rceil - 1} f(t_{k-j}) \int_{t_{k-1}}^{t_k} c_j^k(s) K(t_n - s)\,ds + O(\tau^\gamma).
\end{aligned} \tag{17}$$

By letting

$$w_j^k = \int_{t_{k-1}}^{t_k} c_j^k(s) K(t_n - s)\, ds, \tag{18}$$

we arrive at the conclusion. □

The following remark is a natural extension of the first lemma, which allows for direct comparison to prove stability using Theorem 3.7 in [1].

Remark 3.2. By expanding the series

$$\sum_{k=1}^{n} \sum_{j=0}^{\lceil\gamma\rceil-1} w_j^k f(t_{k-j}) + O(\tau^\gamma),$$

$\gamma \in \mathbb{Z}^+$, and by collecting all of the repeating terms for each $f(t_{k-j})$, provided $k \geq j$, we may condense the double summation into a single summation term

$$\sum_{k=1}^{n} \sum_{j=0}^{\lceil\gamma\rceil-1} w_j^k f(t_{k-j}) = \sum_{k=0}^{n} (w_0^k + w_1^{k+1} + \ldots + w_{\gamma-1}^{k+\gamma-1}) f(t_k), \tag{19}$$

where we define $w_0^0 = 0$ to satisfy the previous lemma. Further, by defining

$$\tilde{w}_k^\gamma = w_0^k + w_1^{k+1} + \cdots + w_{\gamma-1}^{k+\gamma-1}, \tag{20}$$

we arrive at a form identical to the generalized quadrature rule posed in [1], namely

$$\sum_{k=1}^{n} \sum_{j=0}^{\lceil\gamma\rceil-1} w_j^k f(t_{k-j}) + O(\tau^\gamma) = \sum_{k=0}^{n} \tilde{w}_k^\gamma f(t_k) + O(\tau^\gamma). \tag{21}$$

In particular, from the results in Theorems 3.4 and 3.6, these schemes are only stable and hence convergent for $0 < \gamma \leq 5$.

Theorem 3.3. *The approximation scheme (21) is consistent for any $0 < \gamma \leq 5$, where γ is the order of approximation.*

Proof. From the consistency requirement in [1], we must show that the scheme (21) satisfies, for any time step $\tau > 0$,

$$\begin{gathered} \int_0^{t_n} \phi(s)\, ds = \tau \sum_{j=0}^{n} w_{n-j} \phi(j\tau) + O(\tau^R), \\ \sum_{j=0}^{n} w_j = n, \end{gathered} \tag{22}$$

for any fixed γ. That is, we have from Remark 3.2, by combining (20) and (18),

$$\begin{aligned}\sum_{k=0}^{n} \tilde{w}_k^{\gamma} &= \sum_{k=0}^{n} w_0^k + w_1^{k+1} + \cdots + w_{\gamma-1}^{k+\gamma-1} \\ &= \sum_{k=1}^{n} \sum_{j=0}^{\lceil \gamma \rceil -1} w_j^k \\ &= \sum_{k=1}^{n} \sum_{j=0}^{\lceil \gamma \rceil -1} \int_{t_{k-1}}^{t_k} c_j^k(s) K(t_n - s)\, ds \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\sum_{j=0}^{\lceil \gamma \rceil -1} c_j^k(s) \right) K(t_n - s)\, ds.\end{aligned}$$

From (9), the first equation in the Vandermonde matrix requires

$$\sum_{j=0}^{\lceil \gamma \rceil -1} c_j^k(s) = 1,$$

hence,

$$\sum_{k=0}^{n} \tilde{w}_k^{\gamma} = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} K(t_n - s)\, ds = \int_0^{t_n} K(t_n - s)\, ds.$$

On the other hand, by relabelling the coefficients of (22) and by noting that $k\tau = t_k$,

$$\int_0^{t_n} f(s)K(t_n-s)ds = \tau \sum_{j=0}^{n} w_{n-j} f(j\tau)K(t_n-j\tau) + O(\tau^R) = \tau \sum_{k=0}^{n} w_{n-k} f(t_k)K(t_n-t_k) + O(\tau^R). \tag{23}$$

By equating (21) and (23), we have

$$\tau \sum_{k=0}^{n} w_{n-k} K(t_n - t_k) = \sum_{k=0}^{n} \tilde{w}_k^{\gamma} = \int_0^{t_n} K(t_n - s)\, ds.$$

Since each $w_{n-k} \geq 0$ under this construction, we select $\{w_k\}_{k=0}^{n}$ to satisfy $\sum_{k=0}^{n} w_k = n$. Thus, we have for the scheme (21)

$$\int_0^{t_n} f(s)K(t_n - s)\, ds = \tau \sum_{k=0}^{n} w_{n-k} f(t_k) K(t_n - t_k) + O(\tau^R)$$

$$\sum_{k=0}^{n} w_k = n,$$

hence the scheme (21) is consistent. For simplicity and for implementation, we take $w_k = 1$ for each k to trivially satisfy these conditions since $w_0^0 = w_0 = 0$. $\square$

Invertibility criteria. Given a Volterra integral equation of the second kind

$$u(t) = f(t) + \int_0^t K(t, s)u(s)\, ds,$$

the numerical approximation of order γ to the integral is

$$u(t_k) \approx f(t_k) + \int_0^{t_k} K(t, s)\bigg(\sum_{i=0}^{\lceil\gamma\rceil-1} c_i^k(s)u(t_{k-i})\bigg)\, ds = f(t_k) + \sum_{i=0}^{\lceil\gamma\rceil-1} u(t_{k-i}) \int_0^{t_k} K(t, s)c_i^k(s)\, ds,$$

which is solved for each $k = 1, 2, \dots, N$. As a result, we can rearrange the above to yield the approximate equation

$$\begin{aligned} u(t_k)\bigg(1 - \int_{t_{k-1}}^{t_k} K(t_k, s)\, c_0^k(s)\, ds\bigg) \\ = f(t_k) + \int_{t_{k-1}}^{t_k} K(t_k, s)\bigg(2\sum_{i=1}^{\lceil\gamma\rceil-1} c_i^k(s)u(t_{k-i})\bigg)\, ds + \sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} K(t_k, s)\bigg(\sum_{i=0}^{\lceil\gamma\rceil-1} c_i^k(s)u(t_{j-i})\bigg)\, ds, \end{aligned}$$

hence,

$$\begin{aligned} u(t_k) = \bigg(1 - \int_{t_{k-1}}^{t_k} K(t_k, s)\, c_0^k(s)\, ds\bigg)^{-1}\bigg[f(t_k) + \int_{t_{k-1}}^{t_k} K(t_k, s)\bigg(\sum_{i=1}^{\lceil\gamma\rceil-1} c_i^k(s)u(t_{k-i})\bigg)\, ds \\ + \sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j} K(t_k, s)\bigg(\sum_{i=0}^{\lceil\gamma\rceil-1} c_i^k(s)u(t_{j-i})\bigg)\, ds\bigg]. \end{aligned}$$

That is, for an implicit scheme, we must restrict

$$1 - \int_{t_{k-1}}^{t_k} K(t_k, s)\, c_0^k(s)\, ds \neq 0,$$

or equivalently,

$$\int_{t_{k-1}}^{t_k} K(t_k, s)\, c_0^k(s)\, ds \neq 1.$$

In practice, since γ is the predetermined order of approximation and K is given, we can select an appropriate order of approximation or an appropriate choice of parameters for K.

Numerical stability and convergence. As a remark, the consistency results hold for any $\gamma > 0$ using this argument, but the stability results do not hold in general for $\gamma > 5$. We must further satisfy stability requirements in order to prove the convergence of these schemes for any order $0 < \gamma \leq 5$. From [1], we have the following theorem asserting stability under arbitrary quadrature rules.

Theorem [1, Theorem 3.7]. *The stability polynomial*

$$\Sigma(\mu;\lambda\tau) = (1-\lambda\tau w_0 K(0))\mu^N - \lambda\tau w_1 K(\tau)\mu^{N-1} - \cdots - \lambda\tau w_N K(n\tau) \tag{24}$$

is Schur, if $|\lambda\tau|\sum_{k=0}^{N}|w_k K(k\tau)| < 1$. *Assuming each* $w_k \geq 0$ *and satisfy* $\sum_{k=0}^{N} w_k = N$, *the recurrence for*

$$y(t) = f(t) + \lambda\int_{t_n - T}^{t_n} K(t_n - s)y(s)\,ds$$

when $K(t) \equiv 1$ *for* $t \in [0, T]$ *is stable whenever* $|\lambda T| < 1$, *given* $\tau > 0$.

The Schur polynomial $\beta(\mu)$ in [1] is said to satisfy the requirement that the zeros of β lie inside the complex unit disc, namely $|\mu_n| < 1$ for all $n = 0, 1, \ldots, N$. This proof is achieved by the use of Rouche's theorem (see [2, Theorem 3.8]), which requires that for α and β analytic functions in μ inside and on the contour $\Gamma \subset \mathbb{C}$, we have $|\beta(\mu)| < |\alpha(\mu)|$ for each $\mu \in \Gamma$. This proof is completed by letting Γ be the unit disc such that $|\mu| = 1$, $\alpha(\mu) = \mu^N$, and $\beta(\mu) = -\lambda h(w_0 k(0)\mu^N + w_1 k(h)\mu^{N-1} + \cdots + w_N k(\tau))$. We remark that under these results, we must simply satisfy the requirement that each $\tilde{w}_k^{\gamma} \geq 0$ in (21) to satisfy a similar stability criterion for this generalized quadrature. This immediately leads to two stability results.

Theorem 3.4. *Let* K *be a positive function in* $L^1[0, T]$ *and let* $\tau > 0$. *Then, the approximation scheme* (21) *is stable for* $0 < \gamma \leq 2$, *where* γ *is the order of approximation.*

Proof. The case where $\gamma = 1$ is immediate since $c_0^k = 1$, hence $\tilde{w}_1^k \geq 0$. For $\gamma = 2$, recall that since $c_0^k(s) > 0$ on $[t_{k-1}, t_k]$, $c_1^{k+1}(s) > 0$ on $[t_k, t_{k+1}]$, and $K(s) > 0$, then

$$\begin{aligned}
\tilde{w}_2^k &= w_0^k + w_1^{k+1} \\
&= \int_{t_{k-1}}^{t_k} c_0^k(s)K(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)K(s)\,ds \\
&\geq \min_{s\in[t_1,T]}(K(s))\left(\int_{t_{k-1}}^{t_k} c_0^k(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)\,ds\right) \\
&= \min_{s\in[t_1,T]}(K(s))\left(\int_{t_k}^{t_{k+1}} c_0^{k+1}(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)\,ds\right) \\
&= \min_{s\in[t_1,T]}(K(s))\left(\int_{t_k}^{t_{k+1}} c_0^{k+1}(s) + c_1^{k+1}(s)\,ds\right) \\
&= \min_{s\in[t_1,T]}(K(s))\tau \geq 0.
\end{aligned}$$

Using similar analysis we are able to come to the same conclusion for $\gamma = \alpha$ and $\gamma = 1 + \alpha$, given $0 < \alpha < 1$. Therefore, when $\gamma \in [1, 2]$, the scheme (21) is stable. □

We require additional assumptions on the integral kernel K to ensure that the scheme is stable in the case where the order of approximation to (21) is any order $2 < \gamma \leq 5$.

Theorem 3.5. *Let K be a positive, nonincreasing function in $L^1[0,T]$ and let $\tau > 0$. The approximation scheme (21) is stable for any $2 < \gamma \leq 5$ order of accuracy.*

Proof. We begin by showing that $\tilde{w}_k^\gamma \geq 0$ for each $k = 1, 2, \dots, n$. That is, we use the relationship established in Remark 3.2. We will present the argument for the cases where $\gamma = 3, 4, 5$ and deduce the pattern from there. We remark that under the construction found in Theorem 2.2 that for $j = 2, 4, 6, \dots$ then $c_j^k(s) < 0$, provided $s \in [t_{k-1}, t_k]$. Therefore, when $\gamma = 3$, we have

$$\begin{aligned}
\tilde{w}_3^k &= w_0^k + w_1^{k+1} + w_2^{k+2} \\
&= \int_{t_{k-1}}^{t_k} c_0^k(s)K(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)K(s)\,ds + \int_{t_{k+1}}^{t_{k+2}} c_2^{k+2}(s)K(s)\,ds \\
&\geq K(t_{k+1})\left(\int_{t_{k-1}}^{t_k} c_0^k(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)\,ds + \int_{t_{k+1}}^{t_{k+2}} c_2^{k+2}(s)\,ds\right) \\
&= K(t_{k+1})\left(\int_{t_{k+1}}^{t_{k+2}} c_0^{k+2}(s)\,ds + \int_{t_{k+1}}^{t_{k+2}} c_1^{k+2}(s)\,ds + \int_{t_{k+1}}^{t_{k+2}} c_2^{k+2}(s)\,ds\right) \\
&= K(t_{k+1})\left(\int_{t_{k+1}}^{t_{k+2}} c_0^{k+2}(s) + c_1^{k+2}(s) + c_2^{k+2}(s)\,ds\right) \\
&= K(t_{k+1})\tau \\
&\geq 0.
\end{aligned}$$

Hence, when $\gamma = 3$, the scheme (21) is stable. When $\gamma = 4$, the argument is similar, but we must account for the extra positive term in w_3^{k+3}. That is, by recalling from (20) that $\tilde{w}_4^k = \tilde{w}_3^k + w_3^{k+3}$, where

$$\tilde{w}_3^k \geq K(t_{k+1})\left(\int_{t_{k-1}}^{t_k} c_0^k(s)\,ds + \int_{t_k}^{t_{k+1}} c_1^{k+1}(s)\,ds + \int_{t_{k+1}}^{t_{k+2}} c_2^{k+2}(s)\,ds\right).$$

Here,

$$\begin{aligned}
\tilde{w}_4^k &= w_0^k + w_1^{k+1} + w_2^{k+2} + w_3^{k+3} \\
&\geq K(t_{k+1})\left(\int_{t_k}^{t_{k+1}} c_0^{k+1}(s) + c_1^{k+1}(s) + c_2^{k+1}(s)\,ds\right) + \int_{t_{k+2}}^{t_{k+3}} c_3^{k+3}(s)K(t_{k+3})\,ds \\
&= K(t_{k+1})\left(\int_{t_{k+2}}^{t_{k+3}} 1 - c_2^{k+3}(s)\,ds\right) + \int_{t_{k+2}}^{t_{k+3}} c_2^{k+3}(s)K(t_{k+3})\,ds \\
&= \int_{t_{k+2}}^{t_{k+3}} K(t_{k+1}) + (K(t_{k+1}) - K(t_{k+3}))c_2^{k+3}(s)\,ds \\
&\geq 0.
\end{aligned}$$

Since K is nonincreasing, $K(t_{k+1}) \geq K(t_{k+3})$, and since $c_2^{k+3}(s) < 0$ where $s \in [t_{k+2}, t_{k+3}]$ by translating over to the correct interval, we then require $-1 \leq c_2^{k+3}(s)$, $s \in [t_{k+2}, t_{k+3}]$ to ensure that

$$\int_{t_{k+2}}^{t_{k+3}} K(t_{k+1}) + (K(t_{k+1}) - K(t_{k+3}))c_2^{k+3}(s)\,ds \geq \int_{t_{k+2}}^{t_{k+3}} K(t_{k+3})\,ds, \geq 0.$$

To satisfy the requirement, we find that the minimum attained on the interval $s \in [t_{k+2}, t_{k+3}]$ for the function $c_2^{k+3}(s)$ is found at $s = t_{k+3} + \frac{-4+\sqrt{7}}{3}\tau$ by the extreme value theorem and by evaluating the derivative of $c_2^{k+3}(s)$ on the interval $s \in [t_{k+2}, t_{k+3}]$. Hence, the minimum value for $c_2^{k+3}(s)$ is

$$c_2^{k+3}\big(t_{k+3} + \tfrac{-4+\sqrt{7}}{3}\tau\big) = \frac{\big(\frac{-4+\sqrt{7}}{3}\big)\big(1 + \frac{-4+\sqrt{7}}{3}\big)\big(3 + \frac{-4+\sqrt{7}}{3}\big)}{2\tau^2} = \frac{20 - 14\sqrt{7}}{54} \approx -0.31 \geq -1.$$

Therefore, when $\gamma = 4$, the scheme is stable. We now consider the case where $\gamma = 5$. In this case, we have a similar argument where $\gamma = 4$, but we add an additional negative term in $w_4^{k+4}(s) < 0$ for $s \in [t_{k+3}, t_{k+4}]$. Thus, by recalling that

$$\tilde{w}_5^k = \tilde{w}_3^k + w_3^{k+3} + w_4^{k+4},$$

we have

$$\begin{aligned}
\tilde{w}_5^k &= w_0^k + w_1^{k+1} + w_2^{k+2} + w_3^{k+3} + w_4^{k+4} \\
&\geq K(t_{k+1})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_0^{k+2}(s) + c_1^{k+2}(s) + c_3^{k+2}(s)\,ds\bigg) + K(t_{k+3})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_2^{k+3}(s) + c_4^{k+2}(s)\,ds\bigg) \\
&= K(t_{k+1})\bigg(\int_{t_{k+1}}^{t_{k+2}} 1 - c_2^{k+2}(s) - c_4^{k+2}(s)\,ds\bigg) + K(t_{k+3})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_2^{k+3}(s) + c_4^{k+2}(s)\,ds\bigg) \\
&= \int_{t_{k+1}}^{t_{k+2}} K(t_{k+1}) + (K(t_{k+1}) - K(t_{k+3}))(c_2^{k+2}(s) + c_4^{k+2}(s))\,ds.
\end{aligned}$$

We must restrict $-1 \leq c_2^{k+2}(s) + c_4^{k+2}(s) < 0$ to ensure the stability of the scheme. We remark that under the construction of the coefficients c_2^{k+2} and c_4^{k+2}, there is a common factor of $(s - t_{k+2})$ and $(s - t_{k+2} + \tau)$, hence $c_2^{k+2}(s) + c_4^{k+2}(s) = 0$ when $s = t_{k+2}$ and $s = t_{k+2} - \tau = t_{k+1}$. Since $c_2^{k+2},\ c_4^{k+2} < 0$ for $s \in (t_{k+1}, t_{k+2})$, then we may apply the extreme value theorem again to assert that $c_2^{k+2}(s) + c_4^{k+2}(s)$ attains a minimum value on $[t_{k+1}, t_{k+2}]$. Hence, the minimum of $c_2^{k+2}(s) + c_4^{k+2}(s)$ is attained at $s \approx t_{k+2} - 0.416\tau$, with a minimum value of

$$c_2^{k+2}(t_{k+2} - 0.416\tau) + c_4^{k+2}(t_{k+2} - 0.416\tau) \approx -0.603912 \geq -1.$$

Therefore, the scheme is stable when $\gamma = 5$.

We will now show that the above condition no longer holds when $\gamma = 6$. By repeating the same argument for when $\gamma = 6$, we have

$$\begin{aligned}
\tilde{w}_6^k &= w_0^k + w_1^{k+1} + w_2^{k+2} + w_3^{k+3} + w_4^{k+4} + w_5^{k+5} \\
&\geq K(t_{k+1})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_0^{k+2}(s) + c_1^{k+2}(s) + c_3^{k+2}(s) + c_5^{k+2}\,ds\bigg) + K(t_{k+3})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_2^{k+3}(s) + c_4^{k+2}(s)\,ds\bigg) \\
&= K(t_{k+1})\bigg(\int_{t_{k+1}}^{t_{k+2}} 1 - c_2^{k+2}(s) - c_4^{k+2}(s)\,ds\bigg) + K(t_{k+3})\bigg(\int_{t_{k+1}}^{t_{k+2}} c_2^{k+3}(s) + c_4^{k+2}(s)\,ds\bigg) \\
&= \int_{t_{k+1}}^{t_{k+2}} K(t_{k+1}) + (K(t_{k+1}) - K(t_{k+3}))(c_2^{k+2}(s) + c_4^{k+2}(s))\,ds,
\end{aligned}$$

where we again must satisfy $-1 \le c_2^{k+2}(s) + c_4^{k+2}(s) < 0$ to ensure the stability of the scheme. Using the same argument as before, we find that there exists a minimum for $s \in (t_{k+1}, t_{k+2})$, then we may apply the extreme value theorem again to assert that $c_2^{k+2}(s) + c_4^{k+2}(s)$ attains a minimum value on $[t_{k+1}, t_{k+2}]$. Using the definition of the coefficients c_2^{k+2} and c_4^{k+2} as defined by (12), we find that the minimum exists at the point $s = t_{k+2} - 0.38843\tau$ with the minimum value $c_2^{k+2}(s) + c_4^{k+2}(s) = -1.05315 \ngeq -1$. A similar analysis holds for each of the fractional order schemes and is therefore omitted. Hence, the condition is no longer satisfied and thus the scheme fails to be stable for when $\gamma = 6$, which completes the proof. □

With the consistency and stability results, we are now ready to present the convergence analysis. We first define the infinity norm by $\|\cdot\|_\infty = \max\{\cdot\}$.

Numerical convergence. We now consider an arbitrary stable scheme of the form (21) up to order γ where $0 < \gamma \le 5$. We present the convergence results for the usual Taylor series expansion first, followed by the fractional Taylor series expansion results.

Theorem 3.6. *Let $0 \le s \le t_n$ for any prescribed $t_n \in [0, T]$. Let $K \in L^1[0, T]$ be positive and nonincreasing on $[0, T]$ and let $\tau > 0$. Let $f(s) \in C^\gamma[0, T]$ satisfy the stable scheme (21) up to some order $\gamma = 1, 2, 3, 4, 5$, where γ is the order of approximation. Then, for some $C > 0$,*

$$\left\| \int_0^{t_n} f(s)K(t_n - s)\,ds - \sum_{k=0}^{n} \tilde{w}_k^\gamma f(t_k) \right\|_\infty \le C\tau^\gamma. \tag{25}$$

Proof. *We fix $\gamma \ge 1$ such that for some $C > 0$ by utilizing (16) and (21),*

$$\begin{aligned}
\left\| \int_0^{t_n} f(s)K(t_n - s)\,ds - \sum_{k=1}^{n} \tilde{w}_k^\gamma f(t_k) \right\|_\infty &\le \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \left(\frac{1}{\lceil\gamma\rceil!} \max_{0\le t\le t_n} |f^{(\gamma)}(t)|(t_k - s)^\gamma \right) K(t_n - s)\,ds \right\|_\infty \\
&\le \frac{1}{\lceil\gamma\rceil!} \max_{0\le t\le t_n} |f^{(\gamma)}(t)|\tau^\gamma \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |K(t_n - s)|\,ds \right\|_\infty \\
&= \frac{1}{\lceil\gamma\rceil!} \max_{0\le t\le t_n} |f^{(\gamma)}(t)|\tau^\gamma \left\| \int_0^{t_n} |K(t_n - s)|\,ds \right\|_\infty \\
&\le c\tau^\gamma,
\end{aligned}$$

where

$$c = \frac{1}{\lceil\gamma\rceil!} \left\| \int_0^{t_n} |K(t_n - s)|\,ds \right\|_\infty = \frac{1}{\lceil\gamma\rceil!} \|k\|_{l^1[0,t]} < \infty.$$ □

For the fractional order regularity assumption, we have the following convergence rate result.

Theorem 3.7. *Let $0 \le s \le t_n$ for any prescribed $t_n \in [0, t]$. Let $K \in l^1[0, t]$ be positive and nonincreasing on $[0, t]$ and let $\tau > 0$. let $f(s) \in C^\gamma[0, t]$ satisfy the stable scheme (21) for any $\gamma \in (0, 5) - \{1, 2, 3, 4\}$, where γ is the order of approximation. Let $\gamma = n + \alpha$, where $n = 0, 1, 2, 3, 4$ and $0 < \alpha < 1$. Then, for some $c > 0$,*

$$\left\| \int_0^{t_n} f(s)K(t_n - s)\,ds - \sum_{k=0}^{n} \tilde{w}_k^\gamma f(t_k) \right\|_\infty \le c\max(\tau^\gamma, \tau^{n+1}). \tag{26}$$

Proof. By fixing $\gamma = n + \alpha$ where $\gamma \in (0, 5) - \{1, 2, 3, 4\}$ and $0 < \alpha < 1$, we have for some $c_1 > 0$,

$$\begin{aligned}\left\| \int_0^{t_n} f(s)K(t_n - s)\,ds - \sum_{k=1}^{n} \tilde{w}_k^{\gamma} f(t_k) \right\|_\infty &= \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Big(f(s) - \sum_{j=0}^{\lceil\gamma\rceil-1} c_j^k(s) f(t_{k-j}) \Big) K(t_n - s)\,ds \right\|_\infty \\ &\le c \max(\tau^\gamma, \tau^{n+1}) \left\| \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} K(t_n - s)\,ds \right\|_\infty \\ &= c \max(\tau^\gamma, \tau^{n+1}) \left\| \int_0^{t_n} |K(t_n - s)|\,ds \right\|_\infty \\ &\le c \|K\|_{l^1[0,t]} \max(\tau^\gamma, \tau^{n+1}),\end{aligned}$$

where $\|K\|_{l^1[0,t]} < \infty$. □

We present an example demonstrating that the kernel K improves this estimate accordingly.

Example 3.8. Let $K(t) = t^{\alpha-1}$ for $0 < \alpha < 1$ and consider an order α approximation to $f(s)$ from the scheme (21). then, we define

$$\begin{aligned}|r_n| :&= \int_0^{t_n} (t_n - s)^{\alpha-1} |f(s) - f(t_k)|\,ds \\ &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} |(s - t_{k-1})^\alpha - \tau^\alpha + o({}_0^c d_t^{2\alpha} f)| (t_n - s)^{\alpha-1}\,ds \\ &\le \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \tau^\alpha (t_n - s)^{\alpha-1}\,ds \\ &\le \frac{\tau^\alpha}{\alpha} \max_{1\le k\le n} \tau^\alpha \\ &= c\tau^{2\alpha},\end{aligned}$$

which is attained under a uniform mesh size. however, if $2\alpha > 1$, we obtain the secondary estimate of $c\tau$, since then it is the maximum of that and $c\tau^{2\alpha}$.

4. Numerical examples

Our first example is a Volterra equation of the second kind with kernel $K(t) = t^{\alpha-1}$

$$u(t) = f(t) + \int_0^t u(s)(t - s)^{\alpha-1}\,ds \tag{27}$$

$$u(0) = 0, \quad \forall t \in [0, T], \tag{28}$$

where we consider the exact solution $u(t) = t^{6+\alpha} - t^{9/2}$. We define N to be the number of intervals in a uniform partition of the time domain $[0, T]$, $E_{3,\infty}(N)$ to be the maximum error attained over the total mesh for a third order accurate scheme, and

$$\text{rate}_3 = \log_2(E_{3,\infty}(N/2)/E_{3,\infty}(N)).$$

α	N	$E_{3,\infty}(N)$	rate_3
0.1	10	0.0010	*
	20	0.0002	2.6369
	40	2.4729e–5	2.7585
	80	3.4911e–6	2.8245
	160	4.7969e–7	2.8635
0.4	10	0.1148	*
	20	0.0144	2.992
	40	0.0020	2.8689
	80	0.0003	2.9013
	160	3.4531e–5	2.9364
0.5	10	0.0049	*
	20	0.0008	2.6731
	40	0.0001	2.8254
	80	1.4432e–5	2.9052
	160	1.8719e–6	2.9468
0.7	10	0.0022	*
	20	0.0003	2.7754
	40	4.3309e–5	2.8883
	80	5.6279e–6	2.944
	160	7.1747e–7	2.9716
0.9	10	0.0012	*
	20	0.0002	2.8067
	40	2.2852e–5	2.905
	80	2.9506e–6	2.9532
	160	3.7479e–7	2.9769

α	N	$E_{4,\infty}(N)$	rate_4
0.1	10	0.0003	*
	20	2.2041e–5	3.6373
	40	1.6168e–6	3.769
	80	1.1313e–7	3.8371
	160	7.6929e–9	3.8783
0.4	10	0.0362	*
	20	0.0028	3.7128
	40	0.0002	3.7667
	80	1.3999e–5	3.8564
	160	9.2778e–7	3.9154
0.5	10	0.0021	*
	20	0.0002	3.5777
	40	1.2673e–5	3.7898
	80	8.537e–7	3.8919
	160	5.5542e–8	3.9421
0.7	10	0.0008	*
	20	6.2265e–5	3.691
	40	4.3148e–6	3.8511
	80	2.8369e–7	3.9269
	160	1.822e–8	3.9607
0.9	10	0.0004	*
	20	3.417e–5	3.7087
	40	2.3549e–6	3.859
	80	1.5441e–7	3.9308
	160	9.8935e–9	3.9642

Table 1. Numerical error for $u(t) = t^{6+\alpha} - t^{9/2}$, $T = 1$ using a third order scheme on the left and numerical error for $u(t) = t^{6+\alpha} - t^{9/2}$, $T = 1$ using a fourth order scheme on the right.

Analogously, we will define $E_{4,\infty}(N)$, $E_{\alpha,\infty}(N)$, rate_4, and rate_α for the fourth-order accurate and α-order accurate schemes. We will take $\alpha = 0.1, 0.4, 0.5, 0.7, 0.9$ in this example. The numerical results are given in the left of Table 1. By applying the fourth order scheme to the first example, we have the results recorded in the right side of Table 1.

When we have $\alpha = 0.25$, we have spurious and large blowup for small values of N, but as $N \to \infty$, we still exhibit the appropriate order of convergence, and hence still preserve the stability condition. For example, using the fourth order scheme for the second example, with $\alpha = 0.25$, we have the for up to $N = 10240$, following rate of convergence listed in Table 2.

Another consequence of the α-order accurate scheme is that we can also numerically approximate $u(t)$ when the exact function is not known. Consider (27) where $f(t) = t^{2\alpha}$ and $u(t)$ is unknown. Since

α	N	rate_4
	10	*
	20	−9.1682
	40	−53.279
	80	−300.81
	160	127.59
0.25	320	11.223
	640	4.3639
	1280	3.8026
	2560	3.8409
	5120	3.8947
	10240	4.223

Table 2. Numerical rate for $u(t) = t^{6+\alpha} - t^{9/2}$, $T = 1$ using a fourth order scheme.

the exact solution is not known explicitly, we instead compute the error using the two mesh principle as outlined in [15] and the references therein. Given a uniform time mesh, we define u^n to be the numerical approximation to u at time $t = t_n$ for N total grid points, and z^n to be the numerical approximation to u at time $t = t_n$ for 2N total grid points. Then, the maximum error considered between the two meshes is computed by

$$E_{\alpha,\infty}(N) = \max_{1\le n\le N} |u^n - z^{2n}|.$$

We then define the rate of convergence in this case by

$$\text{rate}_\alpha = \log_2\left(\frac{E_{\alpha,\infty}(N/2)}{E_{\alpha,\infty}(N)}\right).$$

When $\alpha = 0.05, 0.25, 0.5, 0.75, 0.95$, the results given in the left of Table 3.

Our second example is the Volterra equation of the second kind that is motivated by the findings in [20] and [3]. This particular equation is obtained by applying the Laplace transform to (1.2) of [20] to obtain

$$u(x,t) = \phi(x) + \int_0^t \left(g(x,s) + \frac{\partial^2 u}{\partial x^2}(x,s) \right) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, ds, \tag{29}$$

$$u(x,0) = \phi(x), \quad u(t,0) = u(t,1) = 0 \tag{30}$$

on the interval $x \in [0, 1]$, $t \in [0, 1]$, which has the initial condition $\phi(x) = 0$ and the exact solution

$$u(x,t) = \sin(\pi x) t^{1-\alpha}.$$

We apply a fixed fourth order discrete Laplacian operator in space $\mathscr{H}_h$ as in [3] and in [20], combined with the α order scheme presented in Section 3 to analyze the problem. By fixing $M = 25$ space steps and using $\alpha = 0.05, 0.25, 0.5, 0.75, 0.75, 0.95$, we have the results show in the right of Table 3. Of particular interest is the cases where $\alpha \ge \frac{1}{2}$, which validate the findings in Example 3.8.

α	N	$E_{\alpha,\infty}(N)$	rate$_\alpha$
0.05	10	$*$	$*$
	20	5.0445e–5	$*$
	40	4.8623e–5	0.0531
	80	4.6863e–5	0.0532
	160	4.5163e–5	0.0533
0.25	10	$*$	$*$
	20	0.0025	$*$
	40	0.0019	0.3858
	80	0.0014	0.4054
	160	0.0011	0.4263
0.5	10	$*$	$*$
	20	0.0053	$*$
	40	0.0029	0.8886
	80	0.0015	0.9193
	160	0.0008	0.9422
0.75	10	$*$	$*$
	20	0.0100	$*$
	40	0.0052	0.9566
	80	0.0026	0.9753
	160	0.0013	0.9859
0.95	10	$*$	$*$
	20	0.0126	$*$
	40	0.0064	0.9797
	80	0.0032	0.9893
	160	0.0016	0.9944

α	N	$E_{\alpha,\infty}(N)$	rate$_\alpha$
0.05	10	0.0078	$*$
	20	0.0046	0.7578
	40	0.0026	0.8156
	80	0.0015	0.8490
	160	0.0008	0.8713
0.25	10	0.0226	$*$
	20	0.0132	0.7730
	40	0.0077	0.7767
	80	0.0045	0.7806
	160	0.0026	0.7856
0.5	10	0.0385	$*$
	20	0.0249	0.6265
	40	0.0158	0.6618
	80	0.0097	0.7017
	160	0.0058	0.7443
0.75	10	0.1757	$*$
	20	0.1197	0.5537
	40	0.0763	0.6497
	80	0.0456	0.7426
	160	0.0258	0.8219
0.95	10	0.4118	$*$
	20	0.2767	0.5735
	40	0.1683	0.7172
	80	0.0948	0.8284
	160	0.0507	0.9025

Table 3. Numerical error for (28), $f(t) = t^{2\alpha}$ using an α-order scheme, u unknown on the left and for $u(x,t) = \sin(\pi x)t^{1-\alpha}$, using an α-order scheme on the right.

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