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Jakob Bernoulli Finds Exact Sums of Infinite Series (Capstone version)

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1 On the Convergence and Summation of Infinite Series

Consider this infinite series:

$$\frac{1}{2} + \frac{8}{10} + \frac{27}{50} + \frac{64}{250} + \cdots .$$

Task 1

(a) Prove that the series converges. Be sure to identify which convergence test you applied and how it justified this claim.

(b) Can you find the exact sum of the series? If so, explain how you determined it. If you find you can't, what does your answer in (a) tell you that helps out with this question?

The questions posed in the above Task should clarify for the reader that *knowing that a series converges is independent from discovering its sum.* Consequently, the mastery of tests for convergence of infinite series, while important for determining whether a given series *has* a sum, does not help the mathematician to learn what the series sums to exactly.

As it happens, in standard presentations of the theory of infinite series, we often stop short of finding the *exact* values of most of the convergent series we encounter. The focus of the theory of series is placed squarely on developing tests for determining convergence, and there is a long list of such criteria that we are asked to practice and apply (the comparison test, the ratio test, the root test, the integral test, etc.). To be sure, there are justifiable reasons for being concerned with the convergence of series—there is no hope to determine the sum of a series if we can show that it diverges!—but if we demonstrate that a series does converge, we are generally no closer to determining its sum. Indeed, the student introduced to series comes across few examples of these objects whose sums are precisely determinable. Notable exceptions to this include geometric series,

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such as $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$, and telescoping series like

$$\begin{aligned}\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots \\ &= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{5}\right)+\cdots \\ &= 1,\end{aligned}$$

whose sums are straightforward to discover.

This project will introduce you to work of Jakob Bernoulli (1655–1705), who, in a time before the full development of the theory of convergent series, discovered methods for determining the sums of a wide variety of infinite series beyond the few types listed above. Since producing the sum of a series is a guarantee of its convergence, what Bernoulli offered us is a straightforward and accessible method for evaluating sums of these convergent series, providing a very satisfying result—a series which converges because we know its sum, as compared with a series which we know converges although its sum is largely a mystery to us.

2 Jakob Bernoulli's *Tractatus de Seriebus Infinitis*

Jakob Bernoulli was born in Basel, Switzerland, into a well-to-do Protestant family, the eldest son of a town Councillor and Master of the local artist's guild. Jakob's father sent him off to university in Basel to prepare him for a career in theology, but while at school, the young Bernoulli became enthralled by the exciting developments being made in the seventeenth century in astronomy and mathematics. He eventually became a leading member of an active community of mathematicians across Europe working on experimental science and analytic mathematics, and he published frequently in the newly established academic journals of the day. He took a position as Professor of mathematics in Basel in 1683, and trained his younger brother Johann (1667–1748) in mathematics, famously becoming his professional rival in later years. Both Bernoulli brothers mastered the new analytic calculus that was being developed by Gottfried Leibniz (1646–1716) on the Continent and Isaac Newton (1643–1727) in England, and the Bernoulli brothers were instrumental in helping to develop its principles.

Over a period of almost twenty years, from the late 1680s to the early 1700s, Jakob Bernoulli wrote five treatises on the theory of infinite series. These works were collected, combined and published in the decade after Jakob's death, as *Tractatus de Seriebus Infinitis* [*A Treatise on Infinite Series*] (1713) [Bernoulli, 1713]. The work was bundled together in the same publication with an even more influential treatise named *Ars Conjectandi* [*The Art of Conjecturing*], which represented the most comprehensive work to date in the theory of probability.

In the pages below, we will read a portion of the first part of this treatise on series, which Jakob Bernoulli wrote in 1689.[1] It represented the first systematic treatment of series on their own terms. In Proposition XIV of the *Tractatus*, Bernoulli set forward a method for determining the sums of certain kinds of infinite series by means of a strategic rewriting of their terms.

Before we take up Bernoulli's work in detail, let's review first some of the terminology that he employed, some of which has evolved over the intervening years, and the central ideas regarding

[1] All translations of selections from Bernoulli's *Tractatus* were prepared by the first author of this project.

series. Sums of numbers in certain patterns have been investigated since ancient times; indeed, Greek mathematicians like Archimedes of Syracuse (ca. 287–212 BCE) famously used sums of this kind to solve problems in geometry like the *quadrature of the circle and the parabola*.[2] These patterns were identified and studied in the centuries before even Archimedes did his work by mathematicians in the school attributed to Pythagoras of Elea (6th century, BCE).

Today, we call any ordered list of numbers $a_1, a_2, a_3, \ldots$ a *sequence*. For mathematicians in the ancient world, a different term, *progression*, was often favored. And three types of progression were the primary objects of study:

- an *arithmetic* (pronounced "ar-ith-**met**-ik") progression is a sequence of numbers $a_1, a_2, a_3, \ldots$ such that the difference between any consecutive terms is a fixed constant. That is, for all $n \geq 1$, $a_{n+1} - a_n = c$ for some fixed c, often called the *common difference* of the progression.
  - The *natural numbers*, $1, 2, 3, 4, 5, \ldots$, provide a clear example of an arithmetic sequence. Here, $c = 1$.
  - The sequence $1, 3, 5, 7, \ldots$ of odd positive integers is also arithmetic. Here, $c = 2$.
  - The sequence $1, 4, 9, 16, 25 \ldots$ of squares is not arithmetic. Notice that $16 - 9 = 7$ while $4 - 1 = 3$, and $7 \neq 3$, so the sequence has no common difference.
- A *geometric* progression is a sequence of numbers $a_1, a_2, a_3, \ldots$ such that the ratio between any two consecutive terms is a fixed nonzero constant. That is, for all $n \geq 1$, $\frac{a_{n+1}}{a_n} = r$ for some fixed $r > 0$. This value r is often referred to as the *common ratio* of the sequence.
  - The sequence $1, 2, 4, 8, 16, \ldots$, is an example of a geometric progression. Here, $r = 2$.
  - The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$, is also geometric. Here, $r = \frac{1}{3}$.
- A *harmonic* progression is one whose reciprocals form an arithmetic progression.[3] That is, for all $n \geq 1$, $\frac{1}{a_{n+1}} - \frac{1}{a_n} = c$ for some constant value c. The quintessential example of a harmonic progression is the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; a second example is afforded by the reciprocals of the positive even numbers: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$.[4]

This is by no means an exhaustive list of all the kinds of progressions known to the ancients, and to Jakob Bernoulli in 1689, when he began his systematic study of these objects. In the pages below, you will see how Bernoulli described more of these types of number sequences.

An expression of the form $a_1 + a_2 + a_3 + \cdots$ which adds together numbers in an infinite sequence is what we call an *infinite series*. Of course, we learned from an early age that adding together finitely many numbers always produces a sum, but what would it mean to say that an infinite series had a (finite) sum? When an infinite series sums to a finite value, we say that the series *converges*, and when this is impossible, we say it *diverges*. Indeed, the truly fascinating thing about infinite series is that many are such that we *can* logically and definitively assign them finite sums!

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[2] *Quadrature* is an old term derived from Latin used to denote the determination of areas.

[3] An early application of mathematics to music theory was known to disciples of Pythagoras in the 5th century BCE. They understood that plucked strings of the same material but of lengths in the ratio of $1 : \frac{2}{3} : \frac{1}{2}$ made a pleasing harmonic sound together. Since the reciprocals of these numbers—1, $\frac{3}{2}$ and 2, respectively—were in arithmetic progression, they called any sequence of numbers with reciprocals in arithmetic progression a harmonic progression.

[4] The reader should be alerted that while arithmetic and geometric progressions will be found in Bernoulli's work below, we will not see a harmonic progression (although he did consider harmonic progressions elsewhere in his *Tractatus*); we include this definition here for the sake of providing a full picture of the landscape in which Bernoulli worked.

Task 2 Consult your favorite reference work or calculus textbook for the answer to this question; what does it really mean to say that we can "assign an infinite series a finite sum"? In other words, what is the formal definition of a convergent series?[5]

3 On Progressions and Figurate Numbers

In the early pages of his *Tractatus*, after stating some simple axioms to govern the behavior of the quantities which will be involved in the series he studied,[6] Bernoulli discussed what it meant for a sequence of numbers to continue without end to infinity. He then turned his attention to the first type of infinite series.

VIII. *To find the sum S of any geometric progression A, B, C, D, E.* ...

Corollary If a decreasing geometric progression continues to infinitely many terms, then the final term vanishes ...and the sum of all [terms] equals $\frac{A^2}{A-B}$; whence, moreover, it is made clear by this stipulation that infinitely many terms can produce a finite sum.

Task 3 Bernoulli uses the term "progression" here, a common synonym for *sequence*. What conditions must be satisfied by the (positive) numbers $A, B, C, D, E, \ldots,$ if this sequence is to be a *geometric* progression? (Hint: Let $r = \frac{B}{A}$.)

Task 4 We know that the exact value of a geometric series

$$a + ar + ar^2 + ar^3 + \cdots$$

is given by

$$\frac{a}{1-r},$$

provided the ratio r satisfies the inequality $|r| < 1$.

(a) Use the formula above to determine the exact value of the geometric series

$$1 + \frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \cdots + \frac{1}{7^n} + \cdots$$

(b) Determine the exact value of this geometric series:

$$5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots .$$

[5]This standard definition is largely credited to Augustin-Louis Cauchy (1789–1857) and Niels Henrik Abel (1802–1829); other different formulations came later in the nineteenth and twentieth centuries. but none of these notions was established until more than 100 years after Bernoulli wrote the treatise we will consider here!

[6]Bernoulli called these quantities "magnitudes," evoking the terminology found in the classical *Elements* of Euclid (ca. 300 BCE), a paradigm of the kind of systematic and rigorous theoretical approach that Bernoulli attempted to reproduce in his treatise on series.

(c) Explain why the geometric series

$$1+\frac{10}{9}+\frac{10^2}{9^2}+\frac{10^3}{9^3}+\cdots$$

does not converge.

(d) In the text above, Bernoulli gave the formula $\frac{A^2}{A-B}$ for "the sum S of any geometric progression". Show that his formula agrees with the one given at the beginning of this Task.

Task 5 What do you think Bernoulli meant in the excerpt above by "the final term" of a sequence that "continues to infinitely many terms"? And why did he say that it "vanishes"?

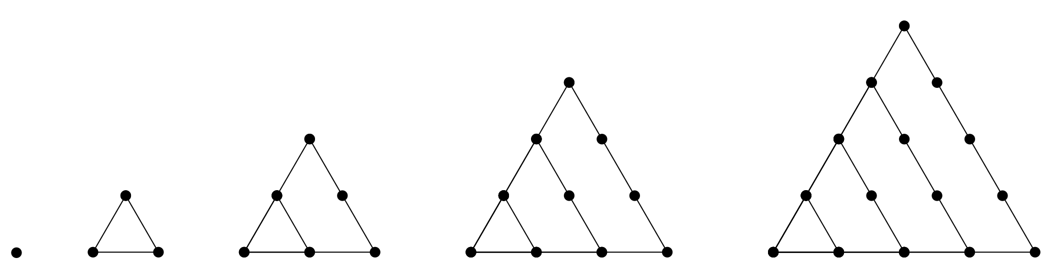
Before we look further into Bernoulli's work on infinite series, we next note his consideration of other "progressions" of numbers, of interest to him and his contemporaries because of their historical importance to mathematicians, dating back to the ancient Greeks. These sequences of positive integers, called *figurate numbers*, are the subject of the next portion of text we will study.

XIV. *To find the sum of an infinite series of fractions whose denominators grow howsoever by a geometric progression, and whose numerators proceed in like manner as the natural numbers* $1, 2, 3, 4,$ *etc., or triangular numbers* $1, 3, 6, 10,$ *etc., or pyramidal numbers* $1, 4, 10, 20,$ *etc., or in like manner as the squares* $1, 4, 9, 16,$ *etc., or cubes* $1, 8, 27, 64,$ *etc., or their equimultiples.*

By longstanding tradition dating back to the Greeks, mathematicians viewed the sequences that Bernoulli presented here in terms of very simple figures of increasing dimensions. As a starting point, the *natural numbers* $1, 2, 3, 4, \ldots$ counted the dots in an (indefinitely) extendable pattern of figures consisting of one-dimensional lines of points:

Other simple geometric patterns lead to different sequences of numbers. The tasks below will introduce us to the classical sequences that Bernoulli mentioned in the passage above, so they merit careful consideration.

Task 6 Consider the following triangular arrays of dots.



Counting the dots in each of these clusters, we obtain the sequence

$$1, 3, 6, 10, 15, \ldots.$$

(a) Copy down the first five terms of this sequence in a horizontal row. Then, just underneath and between consecutive terms in this row, write a new row which contains their "first differences": $3 - 1, 6 - 3, 10 - 6$, and so on. What is the well-known sequence that is represented by these first differences? (Bernoulli referenced this sequence in his comments above.)

(b) Assuming that this pattern of first differences continues indefinitely, what must be the next three numbers in this sequence after 15?

(c) The arrays pictured above suggest another simple pattern for the numbers in this sequence of *triangular numbers*:

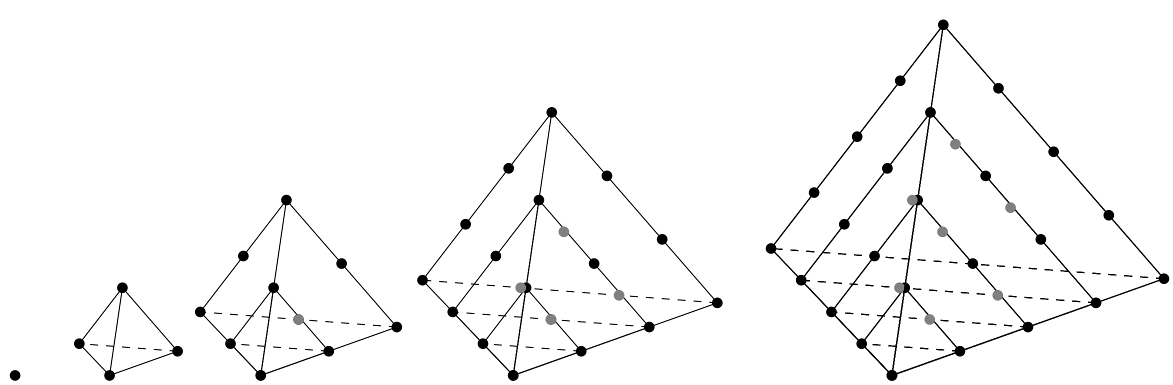
$$\begin{aligned} T_1 &= 1 = 1 \\ T_2 &= 3 = 1 + 2 \\ T_3 &= 6 = 1 + 2 + 3 \\ T_4 &= 10 = 1 + 2 + 3 + 4 \\ T_5 &= 15 = 1 + 2 + 3 + 4 + 5 \\ &\vdots \end{aligned}$$

In line with this pattern, the tenth triangular number can be identified as

$$T_{10} = 55 = 1 + 2 + 3 + \cdots + 10.$$

Write a similar expression for the nth triangular number T_n.

Task 7 Bernoulli next mentioned the sequence he called the *pyramidal numbers*, by which he meant the sequence that counts the following arrays of dots, one that raises the geometric dimension yet again:



Counting the dots in each of these clusters, we obtain the sequence

$$1, 4, 10, 20, 35, \ldots.$$

(a) Copy down the first five terms of this sequence of pyramidal numbers in a horizontal row. Then, just underneath and between consecutive terms in this row, write a new row which contains their "first differences": $4 - 1, 10 - 4, 20 - 10$, and so on. What sequence do these first differences represent?

(b) Assuming that the pattern of first differences continues indefinitely, what must be the next three pyramidal numbers in the sequence after 35?

Task 8 Following the pattern of Tasks 6 and 7, we now consider the construction of a new sequence using the pyramidal numbers as first differences.

(a) Start with the number 1, and add to this the pyramidal number 4.

$$1 + 4 = 5$$

Then, add to this 5 the next pyramidal number, which is 10:

$$5 + 10 = 15$$

We now have the start of a new sequence, $1, 5, 15, \ldots$ built up by starting with 1 and using the pyramidal numbers as first differences. Bernoulli and his contemporaries called this the sequence of *pyramido-pyramidal* numbers.

(b) Assuming this pattern of first differences continues (using the pyramidal numbers as the first differences), what must be the next three pyramido-pyramidal numbers after 15?

(c) Note that the nth pyramidal number is the sum of the first n triangular numbers; this can be visualized by stacking up the first n triangular numbers to build the nth pyramidal number. Describe the challenges of attempting a similar procedure to visualize the nth pyramido-pyramidal number in terms of pyramidal numbers.

The observant reader may notice that the sequences we have identified above:

the natural numbers	1, 2, 3, 4, 5, ...;
the triangular numbers	1, 3, 6, 10, 15, ...;
the pyramidal numbers	1, 4, 10, 20, 35, ...; and
the pyramido-pyramidal numbers	1, 5, 15, 35, 70, ...;

can all be found as consecutive diagonals within *Pascal's triangle*:

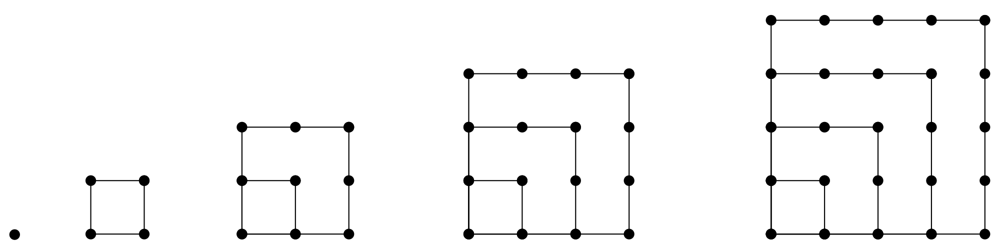
$$
\begin{array}{c}
1 \\
\mathit{1} \quad 1 \\
\mathbf{1} \quad \mathit{2} \quad 1 \\
\mathit{1} \quad \mathbf{3} \quad \mathit{3} \quad 1 \\
\mathbf{1} \quad \mathit{4} \quad \mathbf{6} \quad \mathit{4} \quad 1 \\
1 \quad \mathbf{5} \quad \mathit{10} \quad \mathbf{10} \quad \mathit{5} \quad 1 \\
1 \quad 6 \quad \mathbf{15} \quad \mathit{20} \quad \mathbf{15} \quad \mathit{6} \quad 1 \\
1 \quad 7 \quad 21 \quad \mathbf{35} \quad \mathit{35} \quad \mathbf{21} \quad \mathit{7} \quad 1 \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots
\end{array}
$$

This well-known array, in which each entry is the sum of the two just above it, has many applications, but perhaps the best known is its use in providing the values of the coefficients in the expansion of powers of a binomial sum like $x + y$.[7] For instance, the entries in row 5 of the array (where the top of the triangle is row 0) are found in the expansion

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

In the passage we read above from Proposition XIV of Bernoulli's *Tractatus*, in addition to the figurate number sequences, he included the sequences of squares and cubes. These were also understood as sequences of geometrical arrays of dots.

Task 9 Consider the following square arrays of dots.



Counting the dots in each of these clusters, we obtain the sequence

$$1, 4, 9, 16, 25, \ldots.$$

(a) Copy down the first five terms of this sequence in a horizontal row. Then, just underneath and between consecutive terms in this row, write a new row which contains their "first differences": $4 - 1, 9 - 4, 16 - 9$, and so on. What is the well-known sequence that is represented by these first differences?

(b) Assuming that this pattern of first differences continues indefinitely, what must be the next three square numbers in the sequence after 25?

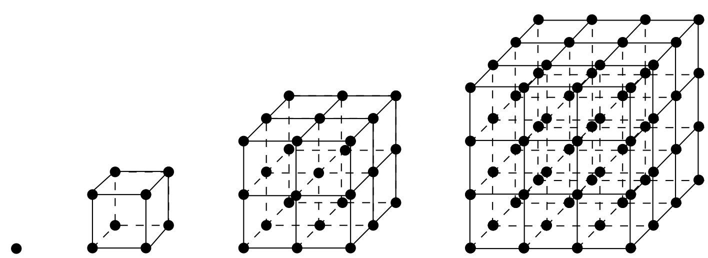
[7] To say any more about this here would take us too far afield of our work with infinite series in this project. For more on Pascal's triangle and its many amazing mathematical properties, see the project "Figurate Numbers and Sums of Consecutive Powers: Fermat, Pascal, Bernoulli" by David Pengelley [Pengelley, 2013].

(c) Another way to envision the pattern found in part (a) is to realize it in this array of equations:

$$\begin{aligned} 1 &= 1 \\ 4 &= 1 + a \\ 9 &= 1 + a + b \\ 16 &= 1 + a + b + c \\ 25 &= 1 + a + b + c + d \end{aligned}$$

Replace the letters a, b, c, d above with the natural numbers that realize the pattern. Then complete this sentence to describe the general pattern: *the square number n^2 is the sum of* ____________.

We note that Bernoulli also mentioned the *cubes* among the various sequences of figurate numbers[8] in the passage above. Bernoulli commented that these are the numbers $1, 8, 27, 64, \ldots$, that is, the numbers $1^3, 2^3, 3^3, 4^3, \ldots$, or more compactly, n^3 for $n = 1, 2, 3, 4, \ldots$. These numbers count a 3-dimensional cubical array of dots, with n dots along each edge of the cube, pictured here.



With this brief introduction to figurate numbers in hand, we now consider how Bernoulli worked to sum certain kinds of series, which according to him, consisted of "fractions whose denominators grow howsoever by a geometric progression, and whose numerators" are represented as figurate numbers.

4 Bernoulli's Summation Techniques

In Proposition XIV of his treatise Bernoulli posed the problem of summing certain types of series. Bernoulli was able to determine their exact sums by a method wherein he "split" the series into a collection of simpler series, a technique that then allowed him to re-express the original series by another which was much easier to sum exactly.

[8]For a more extensive study of figurate numbers, the interested reader may wish to consult Jerry Lodder's Primary Source Project "Construction of the Figurate Numbers", available at the TRIUMPHS website, `https://digitalcommons.ursinus.edu/`.

4.1 "If the numerators grow as the natural numbers"

XIV. *To find the sum of an infinite series of fractions whose denominators grow howsoever by a geometric progression, ...*

1. *If the numerators grow as the natural numbers:*

The sum may be found by resolving the proposed series A into infinitely many series B, C, D, E, etc., each of which proceeds geometrically, whose sums are found, and (if you exclude the first) constitute ...a new geometric progression F, whose sum is found in the same way as the others by Corollary VIII. Showing the work in detail:

$$A = \frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots = B + C + D + E + \cdots$$

$$\left.\begin{array}{l} B = \frac{a}{b} + \frac{a}{bd} + \frac{a}{bd^2} + \frac{a}{bd^3} + \cdots = \frac{ad}{bd-b} \\ C = . + \frac{c}{bd} + \frac{c}{bd^2} + \frac{c}{bd^3} + \cdots = \frac{c}{bd-b} \\ D = \ldots\ldots \frac{c}{bd^2} + \frac{c}{bd^3} + \cdots = \frac{c}{bd^2-bd} \\ E = \ldots\ldots\ldots\ldots \frac{c}{bd^3} + \cdots = \frac{c}{bd^3-bd^2} \\ \ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} F = \frac{cd}{b(d-1)^2}, \text{ which when added to the first term } \frac{ad}{bd-b} \text{ produces the total of the proposed series } A$$

$$\frac{ad}{b(d-1)} + \frac{cd}{b(d-1)^2} = \text{the sum.}$$

Let's begin to make sense of what Bernoulli did here by considering a specific example of the series A.

Task 10

(a) Using $a = 0$, $b = 5$, $c = 1$, and $d = 3$, write the first four nonzero terms of A.

(b) What is the general term of series A using the values for a, b, c, and d specified above? What convergence test would you use to determine whether this series A converges? Use the test to determine whether A converges or diverges.

(c) Given that $a = 0$, what does the sum B (which appears in Bernoulli's table above) equal?

Task 11

(a) In words, describe what Bernoulli did to build the individual terms of the series $C, D, E, \ldots$ using the original series A.

(b) Write the first four terms of C using the values a, b, c, and d that were used above. What kind of series is C? Find the exact value of C using a known formula for this type of series. Does your answer for C match what you get by using Bernoulli's general formula for C which appears in the table above?

(c) Write the first four terms of D using the values a, b, c, and d that were used above. What kind of series is D? Find the exact value of D using a known formula for this type of series. Does your answer for D match what you get by using Bernoulli's general formula for D which appears in the table above?

(d) Write out the first four terms of E using the values a, b, c, and d that were used above. How do the terms of this series differ from those in series C and D above? How are they similar? What is the value of the sum of series E?

(e) Do you see a pattern in the values of the series C, D, and E? What do you think the value of the next sum would be?

(f) Given the pattern that you just observed, what kind of series is $C+D+E+\ldots$? Given this, compute the sum that Bernoulli labeled $F = C + D + E + \ldots$.

(g) Now let's put everything together. We have determined the values of B and $F = C+D+E+\ldots$ for the specific values $a = 0$, $b = 5$, $c = 1$, and $d = 3$. Given that $A = B + F$, determine the value of A in this particular case.

Task 12 Now repeat Tasks 10 and 11 using these values: $a = 5$, $b = 7$, $c = 1$, and $d = 3$.

Let's reflect on what Bernoulli accomplished here. He artfully pulled apart the terms of the series A to create a list of other geometric series, thereby decomposing the original into component parts, each of which is a series whose sum was straightforward to evaluate. It just so happened that the resulting set of sums formed another series whose sum was straightforward to find.[9] *Et voilá*! He now had a formula that itself applies to an infinite number of different series (as we vary the values of the parameters a, b, c, d):

$$A = \frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots = \frac{ad}{b(d-1)} + \frac{cd}{b(d-1)^2}. \tag{1}$$

In the case where $a = 0$, this reduces to the even simpler, but still useful, formula

$$A_0 = \frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots = \frac{cd}{b(d-1)^2}. \tag{2}$$

4.2 "If the numerators are as the triangular numbers"

Bernoulli was not satisfied with only one successful summation. He recognized that this splitting and recombining of terms could apply to sum other series.

2. *If the numerators are as the triangular numbers:*

The given series G is resolvable into another H, whose numerators are as in the preceding hypothesis, as such:

[9] In Bernoulli's day, it was not yet recognized that problems might arise when series were manipulated by blithely reordering or rearranging their terms. In later centuries mathematicians would be led to investigate which properties of series allowed for such rearrangements without affecting their convergence or the values of their sums. They were ultimately led to the notion of *absolute convergence* (which we will not investigate here), now a standard topic in mathematical analysis. Luckily for Bernoulli, the series he worked with here were already absolutely convergent, and thus his rearrangements never led him astray!

$$
G = \frac{c}{b}+\frac{3c}{bd}+\frac{6c}{bd^2}+\frac{10c}{bd^3}+\cdots
$$

$$
\left.\begin{array}{rl}
\frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+\cdots = & \frac{cd}{bd-b} \\
+\frac{2c}{bd}+\frac{2c}{bd^2}+\frac{2c}{bd^3}+\cdots = & \frac{2c}{bd-b} \\
+\frac{3c}{bd^2}+\frac{3c}{bd^3}+\cdots = & \frac{3c}{bd^2-bd} \\
+\frac{4c}{bd^3}+\cdots = & \frac{4c}{bd^3-bd^2} \\
+\cdots = & \cdots
\end{array}\right\}
$$

$H = \frac{cd^3}{b(d-1)^3}$, since this series is to the preceding $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots = \frac{cd}{b(d-1)^2}$ as d^2 is to $d-1$.

Bernoulli seemed less interested here in paragraph 2 in explaining himself as carefully as he did in paragraph 1 above. Still, the method used to sum the series G is essentially of the same type as the one employed earlier: a splitting of the individual terms to lay out a sequence of geometric series whose sums themselves formed a new series which itself could be summed because it possessed a familiar structure. Let us now see how this was done.

Task 13 Here is the splitting that Bernoulli performed in paragraph 2 on the initial series G:

$$
\begin{array}{rl}
G = & \frac{c}{b}+\frac{3c}{bd}+\frac{6c}{bd^2}+\frac{10c}{bd^3}+\cdots \\
\hline
G_1 = & \frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+\cdots = \frac{cd}{bd-b} \\
G_2 = & \quad +\frac{2c}{bd}+\frac{2c}{bd^2}+\frac{2c}{bd^3}+\cdots = \frac{2c}{bd-b} \\
G_3 = & \qquad +\frac{3c}{bd^2}+\frac{3c}{bd^3}+\cdots = \frac{3c}{bd^2-bd} \\
G_4 = & \qquad\quad +\frac{4c}{bd^3}+\cdots = \frac{4c}{bd^3-bd^2} \\
\vdots = & \qquad \vdots \qquad\qquad = \vdots
\end{array}
$$

We have labeled the individual component series $G_1, G_2, \ldots$, noting that the splitting required infinitely many new series, each beginning one term further forward in the array than the one above it.

(a) Bernoulli could have split series G into component parts in a number of different ways. What pattern did he exploit to create the different subscripted series G_1 and G_2 and G_3 and ...? For instance, why was the term $\frac{6c}{bd^2}$ split into the three terms $\frac{c}{bd^2}, \frac{2c}{bd^2}, \frac{3c}{bd^2}$? Similarly, why was the term $\frac{10c}{bd^3}$ split into the four terms $\frac{c}{bd^3}, \frac{2c}{bd^3}, \frac{3c}{bd^3}, \frac{4c}{bd^3}$? As you formulate your answer, consider the work you did in Task 6, where we laid out the sequence of triangular numbers.

(b) Are each of the series $G_1, G_2, \ldots$ geometric? How can you tell?

(c) Verify the sums that Bernoulli gave for the series G_1, G_2, G_3.

(d) Recall that in paragraph 2, Bernoulli defined the series H to be

$$
H = G_1 + G_2 + G_3 + G_4 + \cdots = \frac{cd}{bd-b} + \frac{2c}{bd-b} + \frac{3c}{bd^2-bd} + \frac{4c}{bd^3-bd^2} + \cdots
$$

and concluded that paragraph by remarking that "this series is to the preceding $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots$ as d^2 is to $d-1$." Compare the series identified here by

Bernoulli, $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots$, which is also equal to the series we called A_0 in equation (2) above, with this series for H; show that the comparison leads to the proportion

$$\frac{H}{A_0} = \frac{d^2}{d-1}.$$

(e) Now use the known closed formula (2) for the sum of A_0 to conclude (as did Bernoulli) that

$$G = H = \frac{cd^3}{b(d-1)^3}. \tag{3}$$

Task 14 Identify the values of the parameters that Bernoulli called b, c and d in his expression for the series G which lead to the series

$$\frac{1}{5} + \frac{3}{10} + \frac{6}{20} + \frac{10}{40} + \cdots .$$

Then use Bernoulli's results in paragraph 2 as formulated in Task 13(e) to find the exact value of this series.

4.3 "If the numerators are as the pyramidal numbers"

3. *If the numerators are as the pyramidal numbers:*

The series may be resolved into another whose numerators grow as the triangular numbers, which has a ratio to the preceding series as d to $d-1$; whence its sum is found to be $= \frac{cd^4}{b(d-1)^4}$. More generally, if the numerators of the given series are as the figurate numbers of any degree, their sum will have [a ratio] to the sum of a similar series with the previous degree as d is to $d-1$; whence the sum of all remaining terms is quite easily found.

The reader has no doubt noticed that in paragraph 3, Bernoulli relied on carrying out a pattern of analysis which he had established in paragraphs 1 and 2, and which he then expected his readers to follow here, providing little in the way of details himself. The object of concern in paragraph 3 is therefore a series similar in form to series

$$G = \frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bd^2} + \frac{10c}{bd^3} + \cdots$$

from paragraph 2, but modified so that "the numerators are as the pyramidal numbers." (Recall what this sequence is; see Task 7.) In other words, Bernoulli devoted paragraph 3 to determining the sum of the series

$$\frac{c}{b} + \frac{4c}{bd} + \frac{10c}{bd^2} + \frac{20c}{bd^3} + \cdots .$$

To carry forward Bernoulli's own alphabetical conventions, we will label this series I. Let us turn our attention now to figuring out how Bernoulli was able to determine that "its sum is found to be $= \frac{cd^4}{b(d-1)^4}$."

Task 15 We mimic Bernoulli's analysis of the series G to sum the series I by decomposing it into an infinite sequence of geometric series.

(a) Keeping in mind that the pyramidal numbers are generated as cumulative sums of the triangular numbers, that is,

$$1 = 1, \quad 4 = 1 + 3, \quad 10 = 1 + 3 + 6, \quad \dots,$$

write out the array given below and fill in the missing (underlined) terms in the decomposition of I into the series I_1, I_2, I_3, I_4. (Note that the first term of the series I_n has the nth triangular number in its numerator.)

$$\begin{array}{rcl}
I = & \frac{c}{b} + \frac{4c}{bd} + \frac{10c}{bd^2} + \frac{20c}{bd^3} + \cdots & \\
\hline
I_1 = & \frac{c}{b} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \cdots & = \square \\
I_2 = & + \frac{3c}{bd} + \underline{\quad} + \underline{\quad} + \cdots & = \square \\
I_3 = & + \frac{6c}{bd^2} + \underline{\quad} + \cdots & = \square \\
I_4 = & + \frac{10c}{bd^3} + \cdots & = \square \\
\vdots = & \vdots & = \vdots
\end{array}$$

Then fill in the (boxed) expressions on the right by determining the exact values of each of the (geometric) sums I_1, I_2, I_3, and I_4.

(b) Now let J be the series whose terms are the sums of the series $I_1, I_2, I_3, I_4, \dots$, that is,

$$J = I_1 + I_2 + I_3 + I_4 + \cdots$$

Using your work in (a) above, write out J as an infinite series of terms, and give the details to verify that

$$J = \frac{d}{d-1}\left(\frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bd^2} + \frac{10c}{bd^3} + \cdots\right).$$

(c) In light of your work in (b) above and in Task 13, explain what Bernoulli meant when he said that series I "may be resolved into another whose numerators grow as the triangular numbers, which has a ratio to the preceding series as d to $d-1$." In particular, to what did he refer as "the preceding series"?

(d) Finally, verify Bernoulli's claim that "its sum is found to be $= \frac{cd^4}{b(d-1)^4}$." Of course, you must also identify which series he was referring to here!

Task 16 Using Bernoulli's results in his paragraph 3, find the exact value of the series

$$\frac{1}{5} + \frac{4}{10} + \frac{10}{20} + \frac{20}{40} + \dots$$

As we have seen, Bernoulli successfully discovered formulas for the sums of a number of "infinite series of fractions whose denominators grow howsoever by a geometric progression" and whose numerators "grow as the natural numbers" $1, 2, 3, \ldots$ (par. 1); "as the triangular numbers" $1, 3, 6, \ldots$ (par. 2); and "as the pyramidal numbers" $1, 4, 10, \ldots$ (par. 3). But at the end of paragraph 3, he made a final sweeping claim that indicated his awareness that this pattern of discovery could be indefinitely extended:

> More generally, if the numerators of the given series are as the figurate numbers of any degree, their sum will have [a ratio] to the sum of the similar series with the previous degree as d is to $d-1$; whence the sum of all remaining terms is quite easily found.

Bernoulli did not go on to explore the next series in this pattern, in which "the numerators of the given series are as the figurate numbers of any degree", namely, the series whose numerators contain the pyramido-pyramidal numbers. This may be because visualizing the pyramido-pyramidal numbers required four-dimensional representations, a serious challenge for a geometer (recall Task 8(c)). However, there is nothing barring us from exploring this series.

Task 17

(a) When we set $a = 0$ in the series A from paragraph 1, and multiply through by d, we get the first series A^* below. Copy down this series. Then, as shown here, write out the similar series G from paragraph 2 underneath A^*, followed by series I from paragraph 3.

$$A^* = \frac{c}{b} + \frac{2c}{bd} + \frac{3c}{bd^2} + \frac{4c}{bd^3} + \cdots$$
$$G = \frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bd^2} + \frac{10c}{bd^3} + \cdots$$
$$I = \frac{c}{b} + \frac{4c}{bd} + \frac{10c}{bd^2} + \frac{20c}{bd^3} + \cdots$$

What should the next series in this sequence look like? Write it down also, label it K, and identify the *name* of the sequence of numbers that appears in the numerators of its terms. You can recognize it from work we did back in Section 3.

(b) From the work we've done above, determine the values of the sums of A^*, G, and I in terms of b, c, and d. Using these facts, infer what the sum of the series K "must" be.

(c) Confirm your guess for the formula for K by performing an analysis similar to the one in Task 15. Now isn't that satisfying?

4.4 "If the numerators are as the squares"

Bernoulli next turned his attention to working with series of similar form whose numerators contain the powers of the natural numbers. In particular, he chose to investigate the series whose numerators included the squares and cubes, both of which are easily visualized in two or three dimensions, respectively (see Task 9 and the comment that follows it).

4. *If the numerators are as the squares:*

Series L is resolved into another M, whose numerators are arithmetic progressions, and are therefore as in the first case:

$$L = \tfrac{c}{b}+\tfrac{4c}{bd}+\tfrac{9c}{bd^2}+\tfrac{16c}{bd^3}+\cdots$$

$$\left.\begin{array}{rl}
\frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+\cdots = & \frac{cd}{bd-b} \\
+\frac{3c}{bd}+\frac{3c}{bd^2}+\frac{3c}{bd^3}+\cdots = & \frac{3c}{bd-b} \\
+\frac{5c}{bd^2}+\frac{5c}{bd^3}+\cdots = & \frac{5c}{bd^2-bd} \\
+\frac{7c}{bd^3}+\cdots = & \frac{7c}{bd^3-bd^2} \\
+\cdots = & \cdots
\end{array}\right\} \quad \begin{array}{l} M = \frac{cd^2}{b(d-1)^2}+\frac{2cd^2}{b(d-1)^3} \\ \quad = \frac{cd^3+cd^2}{b(d-1)^3} \end{array}$$

Task 18 (a) Bernoullli's array of series in paragraph 4 is reproduced below.

$$L = \tfrac{c}{b}+\tfrac{4c}{bd}+\tfrac{9c}{bd^2}+\tfrac{16c}{bd^3}+\_\_\_+\cdots$$

$$\begin{array}{rl}
L_1 = & \frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+___+\cdots = \square \\
L_2 = & +\frac{3c}{bd}+\frac{3c}{bd^2}+\frac{3c}{bd^3}+___+\cdots = \square \\
L_3 = & +\frac{5c}{bd^2}+\frac{5c}{bd^3}+___+\cdots = \square \\
L_4 = & +\frac{7c}{bd^3}+___+\cdots = \square \\
L_5 = & +___+\cdots = \square \\
\vdots = & \vdots \qquad = \vdots
\end{array}$$

Copy down this array and fill in the blanks to identify the fifth term in each of the series of the array. Task 8 may help you to see why Bernoulli decomposed L into these given series.

(b) Now determine formulas for the sums of the geometric series L_1, L_2, L_3, L_4, L_5, providing appropriate expressions to fill in the boxes above. Do these formulas coincide with the expressions given by Bernoulli in his array in paragraph 4?

(c) In paragraph 4, Bernoulli defined the series

$$\begin{aligned} M = & \; L_1 + L_2 + L_3 + L_4 + \cdots \\ = & \; \tfrac{cd}{bd-b} + \tfrac{3c}{bd-b} + \tfrac{5c}{bd^2-bd} + \tfrac{7c}{bd^3-bd^2} + \cdots \end{aligned}$$

He then performed a second decomposition in order to determine a formula for M:

$$\begin{aligned} M = & \; \tfrac{cd}{bd-b} + \tfrac{c}{bd-b} + \tfrac{c}{bd^2-bd} + \tfrac{c}{bd^3-bd^2} + \cdots \\ & + \tfrac{\Box c}{bd-b} + \tfrac{\Box c}{bd^2-bd} + \tfrac{\Box c}{bd^3-bd^2} + \cdots \end{aligned}$$

Copy this last equation, filling in the missing numbers in each of the boxes to complete Bernoulli's decomposition of M into two series, which we label M_1 (terms without a $\Box$) and M_2 (terms with a $\Box$).

(d) Verify that the series M_1 is geometric and give an expression for its sum. Then verify that M_2 is a multiple of the series A^* (see Task 17), a series "as in the first case", i.e., as determined in paragraph 1; use this fact to find a simple expression for M_2. Combine these to get the two-term formula for M that Bernoulli obtained in paragraph 4, then simplify algebraically to get the one-term formula he gave as the final form for M.

4.5 "If the numerators are as the cubes"

5. *If the numerators are as the cubes:*

The series may be resolved into another whose numerators are six times the triangular numbers plus a unit; whence its sum is found like that in the second case, $\frac{cd^2}{b(d-1)^2} + \frac{6cd^3}{b(d-1)^4} = \frac{cd^4+4cd^3+cd^2}{b(d-1)^4}$.

Bernoulli was extremely terse in his presentation of this last result, since the analysis that led him to it was entirely similar to the work he performed in paragraph 4. We can follow along similarly, modeling our steps with what we did in Task 18.

Task 19 (a) Bernoulli was clearly interested in the series

$$N = \frac{c}{b} + \frac{8c}{bd} + \frac{27c}{bd^2} + \frac{64c}{bd^3} + \cdots ,$$

which the reader should note is identical to L (from paragraph 4), except that the coefficients which are the perfect squares in L are now the perfect cubes in N. Our analysis will mimic that of Task 18, beginning with the array below.

$$
\begin{array}{l}
N = \frac{c}{b} + \frac{8c}{bd} + \frac{27c}{bd^2} + \frac{64c}{bd^3} + \_\_\_ + \cdots \\
\hline
N_1 = \frac{c}{b} + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \cdots = \bigcirc \\
N_2 = \quad + \frac{7c}{bd} + \_\_\_ + \_\_\_ + \_\_\_ + \_\_\_ + \cdots = \bigcirc \\
N_3 = \qquad\quad + \frac{19c}{bd^2} + \_\_\_ + \_\_\_ + \_\_\_ + \cdots = \bigcirc \\
N_4 = \qquad\qquad\quad + \frac{37c}{bd^3} + \_\_\_ + \_\_\_ + \cdots = \bigcirc \\
N_5 = \qquad\qquad\qquad\quad + \frac{61c}{bd^4} + \cdots = \bigcirc \\
\vdots \; = \qquad \vdots \qquad\qquad\qquad\quad = \vdots
\end{array}
$$

Copy down this array and fill in the blanks to identify the missing terms in each of the series of the array. Then fill in the circled expressions on the right by determining the exact values of each of the (geometric) sums N_1, N_2, N_3, N_4, and N_5.

(b) Next, we consider the "recomposed" series

$$
\begin{aligned}
O = {} & N_1 + N_2 + N_3 + N_4 + N_5 + \cdots \\
= {} & \frac{\square cd}{b(d-1)} + \frac{\square c}{b(d-1)} + \frac{\square c}{bd(d-1)} + \frac{\square c}{bd^2(d-1)} + \frac{\square c}{bd^3(d-1)} + \cdots
\end{aligned}
$$

(We use boxes to represent the coefficients of O here so as not to spoil the reader's satisfaction of working them out for themselves!) In paragraph 5, Bernoulli mentioned that the newly "resolved" series we are now calling O has coefficients which "are six times the triangular numbers plus a unit." In light of the values of the boxed coefficients, what did Bernoulli mean by this?

(c) Mimicking the analysis from paragraph 4, we decompose O once more into two series, exploiting what Bernoulli realized about the form of its coefficients:

$$
\begin{aligned}
O = {} & \frac{cd}{b(d-1)} + \frac{c}{b(d-1)} + \frac{c}{bd(d-1)} + \frac{c}{bd^2(d-1)} + \cdots \\
& + 6\left(\frac{\triangle c}{b(d-1)} + \frac{\triangle c}{bd(d-1)} + \frac{\triangle c}{bd^2(d-1)} + \cdots \right)
\end{aligned}
$$

Copy down this last equation, filling in the missing numbers which appear above as little triangles, to complete the decomposition of O into two other series O_1 (terms outside the parentheses) and O_2 (terms within the parentheses).

(d) Verify that the series O_1 is geometric and give an expression for its sum. Then verify that O_2 is a multiple of the series G (see paragraph 2, Task 13 and Task 17); use this fact to find a simple expression for the sum of O_2. Combine these to obtain the two-term formula for O that Bernoulli obtained at the end of paragraph 5. Finally, collect these two terms into one to get the final expression he found for O.

4.6 Taking Stock

Bernoulli summarized his results in the next brief section of text by presenting some examples of his results.

In this vein, let examples be provided by the following series, whose numerators are the

Natural nos.	$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots=2$
Triangular nos.	$\frac{1}{2}+\frac{3}{4}+\frac{6}{8}+\frac{10}{16}+\frac{15}{32}+\cdots=4$
Pyramidal nos.	$\frac{1}{2}+\frac{4}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\cdots=8$
Square nos.	$\frac{1}{2}+\frac{4}{4}+\frac{9}{8}+\frac{16}{16}+\frac{25}{32}+\cdots=6$
Cubic nos.	$\frac{1}{2}+\frac{8}{4}+\frac{27}{8}+\frac{64}{16}+\frac{125}{32}+\cdots=26$

Task 20 (a) Identify the first three series given here by Bernoulli (labeled as examples with natural numbers, triangular numbers and pyramidal numbers in the numerators of their terms) with the appropriate series from Task 17. Give the values of the parameters b, c, and d in each of these three series. Then, using these values and Bernoulli's results, verify the sums that Bernoulli offered above.

(b) Identify the last two series given here by Bernoulli (labeled as examples with square numbers and cubic numbers in the numerators of their terms) with the appropriate series in paragraphs 4 and 5. Give the values of the parameters b, c, and d in each of these series. Using these values and Bernoulli's results, verify the two sums for these series given by Bernoulli.

Task 21 Using the various results of Bernoulli that we have considered above, find exact values of the sums of the following series:

(a) $\dfrac{2}{1}+\dfrac{6}{7}+\dfrac{12}{49}+\dfrac{20}{343}+\dots$

(b) $\dfrac{1}{2}+\dfrac{8}{10}+\dfrac{27}{50}+\dfrac{64}{250}+\dots$ [Back where we began in Task 1: thanks, M. Bernoulli!]

(c) $\dfrac{2}{5}+\dfrac{16}{25}+\dfrac{72}{125}+\dfrac{256}{625}+\dots$ [Hint: take $d=\frac{5}{2}$.]

(d) $\dfrac{3}{2}+1+\dfrac{5}{12}+\dfrac{5}{36}+\dots$ [Hint: take $d=6$.]

5 The Obvious Pattern is the Right One!

In this section, our goal is to extend a pattern that Bernoulli mentioned in the final excerpt we encountered in Section 4.6. The first three examples in his list were the following:

Natural nos. $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots=2$
Triangular nos. $\frac{1}{2}+\frac{3}{4}+\frac{6}{8}+\frac{10}{16}+\frac{15}{32}+\cdots=4$
Pyramidal nos. $\frac{1}{2}+\frac{4}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\cdots=8$

Task 22 Recall from Section 3 that the figurate number sequences given by the natural numbers, triangular numbers and pyramidal numbers establish the pattern in that analysis which continues with the sequence of pyramido-pyramidal numbers (Task 8). Use this information to write down the next series in the pattern of series results quoted here by Bernoulli. What do you expect is the sum of this series?

Might it be possible to naturally extend this list so that the sums in question continue the pattern of summing to powers of 2? The answer to this question is a resounding YES!

To do this, we begin by setting some notation for the d-dimensional *pyramidal numbers* in question. Let $f_{d,n}$ be the nth d-dimensional number in question; that is,

$$\begin{array}{llllll} f_{1,1}=1 & f_{1,2}=2 & f_{1,3}=3 & f_{1,4}=4 & \cdots & \text{(natural numbers)} \\ f_{2,1}=1 & f_{2,2}=3 & f_{2,3}=6 & f_{2,4}=10 & \cdots & \text{(triangular numbers)} \\ f_{3,1}=1 & f_{3,2}=4 & f_{3,3}=10 & f_{3,4}=20 & \cdots & \text{(pyramidal numbers)} \\ f_{4,1}=1 & f_{4,2}=5 & f_{4,3}=15 & f_{4,4}=35 & \cdots & \text{(pyramido-pyramidal numbers)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \end{array}$$

The key property we need about these sequences, and which we have identified back in Section 3, is that the nth term in each of these sequences is the sum of the first n terms in the sequence one dimension lower. For instance, $f_{3,4}=20$ is the sum of the first four entries in row $d=2$; i.e., $20=1+3+6+10$. More generally then, the $f_{d,n}$ satisfy the following important identity[10]

$$f_{d+1,n}=f_{d,1}+f_{d,2}+f_{d,3}+\cdots+f_{d,n}. \tag{4}$$

The new notation we have built now allows us to recast the results that Bernoulli quoted in the excerpt above:

$$\begin{aligned} \frac{f_{1,1}}{2}+\frac{f_{1,2}}{4}+\frac{f_{1,3}}{8}+\frac{f_{1,4}}{16}+\frac{f_{1,5}}{32}+\cdots&=2, \\ \frac{f_{2,1}}{2}+\frac{f_{2,2}}{4}+\frac{f_{2,3}}{8}+\frac{f_{2,4}}{16}+\frac{f_{2,5}}{32}+\cdots&=4, \\ \frac{f_{3,1}}{2}+\frac{f_{3,2}}{4}+\frac{f_{3,3}}{8}+\frac{f_{3,4}}{16}+\frac{f_{3,5}}{32}+\cdots&=8. \end{aligned}$$

[10]The perceptive reader will notice that the numbers in the $f_{d,n}$ array form a subset of the familiar Pascal's triangle. Moreover, the result in equation (4) is sometimes called the "hockey stick" theorem, evoked by the positions of the numbers that sum to give the later entries.

This then leads us to the following general result.

Theorem: For all $d \geq 1$,

$$\frac{f_{d,1}}{2} + \frac{f_{d,2}}{4} + \frac{f_{d,3}}{8} + \frac{f_{d,4}}{16} + \frac{f_{d,5}}{32} + \cdots = 2^d.$$

Proof: The result is clear for $d = 1, 2, 3$ based on the work that Bernoulli (and the readers of this project) has done above.

To complete the argument, we need to be able to verify the next infinitely many similar formulas! However, all we really need do is to show how the truth of the d-dimensional case implies the truth of the $(d+1)$-dimensional case. (In essence, this is a proof by mathematical induction!) In order to do this, use Bernoulli's series "splitting" approach to verify the formula works at dimension $d+1$.

Consider the following array, modeled on the work of Bernoulli in the source texts we have already studied:

$$\begin{array}{cccccccccccc}
\multicolumn{12}{c}{\frac{f_{d+1,1}}{2}+\frac{f_{d+1,2}}{4}+\frac{f_{d+1,3}}{8}+\frac{f_{d+1,4}}{16}+\frac{f_{d+1,5}}{32}+\cdots} \\
\hline
\frac{f_{d,1}}{2} & + & \frac{f_{d,1}}{4} & + & \frac{f_{d,1}}{8} & + & \frac{f_{d,1}}{16} & + & \frac{f_{d,1}}{32} & + & \cdots \\
 & + & \frac{f_{d,2}}{4} & + & \frac{f_{d,2}}{8} & + & \frac{f_{d,2}}{16} & + & \frac{f_{d,2}}{32} & + & \cdots \\
 & & & + & \frac{f_{d,3}}{8} & + & \frac{f_{d,3}}{16} & + & \frac{f_{d,3}}{32} & + & \cdots \\
 & & & & & + & \frac{f_{d,4}}{16} & + & \frac{f_{d,4}}{32} & + & \cdots \\
 & & & & & & & + & \frac{f_{d,5}}{32} & + & \cdots \\
 & & & \vdots & & & & & & &
\end{array}$$

Task 23

(a) What kind of series is the series listed in row one of the array above,

$$\frac{f_{d,1}}{2} + \frac{f_{d,1}}{4} + \frac{f_{d,1}}{8} + \frac{f_{d,1}}{16} + \frac{f_{d,1}}{32} + \cdots \quad ?$$

Use this fact to determine its sum, noting that the answer will depend on $f_{d,1}$.

(b) What can you say about the type of the series that appear in rows two through five of the array above? Based on this recognition, determine each of their sums.

(c) The $(d+1)$-dimensional series at the top of the array,

$$\frac{f_{d+1,1}}{2} + \frac{f_{d+1,2}}{4} + \frac{f_{d+1,3}}{8} + \frac{f_{d+1,4}}{16} + \frac{f_{d+1,5}}{32} + \cdots,$$

can now be rewritten in a new way as a series whose terms are the sums of the series we have considered in parts (a) and (b) above. Do this.

(d) Verify now that

$$\frac{f_{d+1,1}}{2} + \frac{f_{d+1,2}}{4} + \frac{f_{d+1,3}}{8} + \frac{f_{d+1,4}}{16} + \frac{f_{d+1,5}}{32} + \cdots = 2\left(\frac{f_{d,1}}{2} + \frac{f_{d,2}}{4} + \frac{f_{d,3}}{8} + \frac{f_{d,4}}{16} + \frac{f_{d,5}}{32} + \ldots\right)$$

This equation states, in essence, the induction step of the argument we are making to formulate our proof of the theorem above.

(e) Thus, if we assume that we know the value of the d-dimensional sum in parentheses in part (d), argue that

$$\frac{f_{d+1,1}}{2} + \frac{f_{d+1,2}}{4} + \frac{f_{d+1,3}}{8} + \frac{f_{d+1,4}}{16} + \frac{f_{d+1,5}}{32} + \cdots = 2^{d+1}.$$

This completes the proof.

6 A Modern Approach to Bernoulli's Methods: When the Numerators are Powers of n

Let's revisit the series that Bernoulli considered in his paragraphs 1, 4, and 5. There, Bernoulli presented examples of the methods he employed in these paragraphs to find the sums of the following series:

Natural nos. $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots = 2$

...

Square nos. $\frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \cdots = 6$

Cubic nos. $\frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \frac{64}{16} + \frac{125}{32} + \cdots = 26$

We verified the last two sums that Bernoulli reported here in Task 20(b) above. Note that all three of these series have the form

$$\frac{1^k}{d} + \frac{2^k}{d^2} + \frac{3^k}{d^3} + \frac{4^k}{d^4} + \cdots = \sum_{n=1}^{\infty} \frac{n^k}{d^n}$$

when $d = 2$ and $k = 1$ for the first series, $k = 2$ for the second, and $k = 3$ for the third.

In what follows, we wish to consider how one might utilize more modern tools to arrive at these results of Bernoulli (and many more besides). Interestingly enough, these tools, which involve an understanding of the geometric series, differentiation of power series, and evaluation of power series at certain values of the variable, are often taught in many calculus courses, so we trust that you, dear reader, will be able to discover the same results with a little bit of help.

To that end, recall, as was mentioned in Section 2 above, that the exact value of

$$a + ax + ax^2 + ax^3 + \cdots$$

is given by

$$\frac{a}{1-x}$$

where a is the initial term of the series and the *ratio* x of consecutive terms satisfies the inequality $|x| < 1$. When $a = 1$, this yields the classic result

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\,, \tag{5}$$

which holds whenever $|x| < 1$. We can then use well-known rules of differentiation to produce many new results quickly.

For example, we can differentiate both sides of (5) with respect to x to yield

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\,.$$

(Note that the left-hand side of the above simply requires the quotient rule,[11] while the right-hand side is simply generated by applying the power rule to each of the terms in the series.) We then multiply both sides of this equation by x to yield

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots\,. \tag{6}$$

Since the steps completed above do not alter the *interval of convergence* of the power series in (6), we know that we can now evaluate (6) at any value of x which satisfies $|x| < 1$ and obtain the sum of an infinite convergent series.

Task 24 Write out the first five terms of the series in (6) obtained by setting $x = \frac{1}{2}$. Then set $x = \frac{1}{2}$ on the left-hand side of equation (6); what does this give us for the sum of this series? How does this relate to what Bernoulli stated in the excerpt at the beginning of this section?

Notice that there's nothing special about the value $x = \frac{1}{2}$ that was used above; we could have used $x = -\frac{1}{3}$ or $x = \frac{2}{7}$ or any value x, so long as $|x| < 1$. This simple observation means that we can now find the exact value of an infinite family of convergent infinite series.

Of course, we could apply the same procedure we just employed to (6): if we differentiate both sides of (6) with respect to x, and simplify the left-hand side after the quotient rule has been applied, we arrive at

$$\frac{1+x}{(1-x)^3} = 1 + 4x + 9x^2 + 16x^3 + \cdots\,,$$

and multiplying both sides of this equation by x yields

$$\frac{x(1+x)}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + \cdots\,. \tag{7}$$

Task 25 Write out the first five terms of the series in (7) obtained by setting $x = \frac{1}{2}$. Then set $x = \frac{1}{2}$ on the left-hand side of equation (7); what does this give us for the sum of this

[11] Or, the power rule and chain rule.

series? How does it relate to what Bernoulli stated in the excerpt at the beginning of this section?

Task 26 (a) Using equation (6) above, determine the **exact** value of the series

$$\frac{1}{4} + \frac{2}{16} + \frac{3}{64} + \frac{4}{256} + \cdots$$

(b) Using equation (7) above, determine the **exact** value of the series

$$\frac{1}{5} + \frac{4}{25} + \frac{9}{125} + \frac{16}{625} + \cdots$$

Task 27 In this task, we will employ a similar method to the one used above to sum another of Bernoulli's series.

(a) Differentiate both sides of equation (7) with respect to x, simplifying the resulting equation as much as possible.

(b) Multiply both sides of this new equation by x. Confirm that you obtain the formula

$$\frac{x(1+4x+x^2)}{(1-x)^4} = x + 8x^2 + 27x^3 + 64x^4 + \cdots . \tag{8}$$

(c) Use this last equation (part (b)) to find an exact value of the series

$$\frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \frac{64}{16} + \cdots$$

Does your result agree with one of Bernoulli's results mentioned earlier?

Let's briefly take stock of what we have accomplished in producing the formulas (6), (7) and (8). These series are all of a strikingly similar form:

$$F_k(x) = 0^k + 1^k x + 2^k x^2 + 3^k x^3 + 4^k x^4 + \cdots = \sum_{n=0}^{\infty} n^k x^n, \quad k \geq 1. \tag{9}$$

Indeed, equations (6), (7) and (8) give closed formulas for these series, which we have used to produce Bernoulli's sums:

$$\begin{aligned} F_1(x) &= \sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \\ F_2(x) &= \sum_{n=0}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \\ F_3(x) &= \sum_{n=0}^{\infty} n^3 x^n = \frac{x(1+4x+x^2)}{(1-x)^4} \end{aligned} \tag{10}$$

$\boxed{\textbf{Task 28}}$ Identify at least three characteristics you observe about the expressions that appear on the right sides of the equations for $F_k(x)$ in (10). (Resist the temptation to read on until you have completed this task!)

Did you notice that this sequence of formulas follows the pattern

$$F_k(x) = \sum_{n=0}^{\infty} n^k x^n = \frac{x \cdot E_{k-1}(x)}{(1-x)^{k+1}}, \tag{11}$$

where $E_{k-1}(x)$ is a polynomial of degree $k-1$? Did you also observe that the coefficients of the numerator polynomials $E_{k-1}(x)$ are all positive integers? In fact, these results were identified and conjectured by the great Leonhard Euler (1707–1783), which he described in his *Institutiones calculi differentialis* (1755, Ch. 9, par. 173, pp. 485–486). These polynomials are now called *Eulerian polynomials* in his honor. From (10) we know that the first few polynomials in this list are

$$\begin{aligned} E_0(x) &= 1, \\ E_1(x) &= 1 + x, \\ E_2(x) &= 1 + 4x + x^2, \end{aligned} \tag{12}$$

and we would like to figure out how to determine the rest of them, since they provide us the necessary tool for summing all series of the form

$$F_k\left(\frac{1}{d}\right) = \sum_{n=0}^{\infty} \frac{n^k}{d^n} = \frac{1^k}{d} + \frac{2^k}{d^2} + \frac{3^k}{d^3} + \frac{4^k}{d^4} + \cdots. \tag{13}$$

$\boxed{\textbf{Task 29}}$ Repeat the same procedure you used in Task 27, beginning by taking the derivative of the formula for $F_3(x)$ in (10), to find the series representation and closed-form formula for $F_4(x)$ that represents the next in the sequence of equations in (10). Deduce from this what the cubic polynomial $E_3(x)$ is.

The most important feature of the process we used to generate the formulas in (10) and the result in Task 29 is the differentiation step that led us to find (6) from (5), (7) from (6), (8) from (7), etc. We found there that

$$F_{k+1}(x) = x \cdot F_k'(x) \tag{14}$$

holds for $k = 1, 2,$ and 3, and clearly this pattern should continue for all nonnegative integer values of k.

$\boxed{\textbf{Task 30}}$ Combine formulas (11) and (14); show that this implies

$$E_k(x) = x(1-x)E_{k-1}'(x) + (1+kx)E_{k-1}(x), \quad k \geq 1. \tag{15}$$

The relationship in (15) will now allow us to work out in detail the coefficients of each of the polynomials $E_k(x)$. To achieve this goal, let us write out the terms of the polynomial $E_k(x)$ as follows:

$$E_k(x) = e_{k,0} + e_{k,1}x + \cdots + e_{k,k-1}x^{k-1} + e_{k,k}x^k. \tag{16}$$

Observe that we use a pair of subscripts here for each coefficient; the first of the subscripts identifies the index of the polynomial in which it lies and the second identifies the exponent of the term to which it belongs. Consistent with (12), we must set $e_{0,0} = 1$ to obtain $E_0(x) = 1$. It will then be convenient for us to define values of $e_{k,l}$ to equal 0 whenever l is larger than k or smaller than 0. Our goal then, accomplished in the task below, will be to construct a *recursive* formula that will allow us to compute the coefficients $e_{k,l}$ of $E_k(x)$ in terms of previously computed coefficients for Eulerian polynomials for smaller values of k.

Task 31

(a) Use equation (16) to write out expansions of $E_{k-1}(x)$ and then also $E'_{k-1}(x)$. Keep at least the two highest degree terms and the two lowest degree terms in these (and all subsequent) expansions; use ellipses ($\cdots$) to stand for the remaining terms, as in (16).

(b) Multiply the expansion of $E'_{k-1}(x)$ by $x(1-x) = x - x^2$ and collect like terms in x to produce the polynomial that represents the first term on the right of equation (15).

(c) Multiply the expression for $E_{k-1}(x)$ you found in part (a) by $1 + kx$ to produce the polynomial that represents the second term on the right of equation (15).

(d) Add the two polynomial expansions you found in parts (b) and (c) and collect like terms in x to obtain a polynomial expression for the right side of equation (15).

(e) This means that the coefficient of the term x^l on the left side of equation (16) must equal the coefficient of x^l on the right (for every choice of l). You will notice that the constant coefficient and the leading coefficient of $E_k(x)$ will have to be considered carefully as they behave differently from the interior coefficients. Verify from these comparisons that the following identity expresses all these relations simultaneously:

$$e_{k,l} = (k - [l-1])e_{k-1,l-1} + (l+1)e_{k-1,l}\ , \quad \text{valid for } k \geq 1 \text{ and all } l, \tag{17}$$

where again we take $e_{i,j}$ to be 0 whenever l is larger than k or smaller than 0.

The coefficients of the Eulerian polynomials are thus completely determined by the identities given in (12) together with the recurrence we have found in (17). These coefficients $e_{k,l}$ of these polynomials are called the *Eulerian numbers*. They are easily visualized in an array which we call the *Eulerian triangle*. (It may remind readers of the much more familiar Pascal triangle.) We list the (nonzero) values of $e_{k,l}$ row by row as the larger numbers in the table below, starting with row 0, and within each row by increasing values of l. Because of (12), we know that the Eulerian triangle begins:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 1^{(1)} & & {}^{(1)}1 & \\
1^{(1)} & & {}^{(2)}4^{(2)} & & {}^{(1)}1
\end{array}
$$

In rows $k > 0$ of this array, above and to the left and right of the number $e_{k,l}$ (for $l = 1, 2, \ldots, k-1$) we write the appropriate multiplier as indicated in (17) as small numbers in parentheses: we place the value of $k-(l-1)$ to the left of the Eulerian number and the value of l to the right. This makes the calculation of the entry $e_{k,l}$ in position l of row k particularly easy to work out; it is the sum of the number in the row above it and to the left multiplied by the auxiliary superscript in parentheses at its left with the number in the row above it and to the right multiplied by the auxiliary superscript at its right. At the beginning and end of each row, the number 1 will always appear, as only a single 1 above it, with multiplier 1, contributes to that value of $e_{k,0}$ or $e_{k,k}$.

Task 32

(a) Use (17) to determine the Eulerian numbers in row $k = 3$ of the Eulerian triangle. Include the appropriate superscripts to simplify the calculations.

(b) Identify the corresponding Eulerian polynomial $E_3(x)$ and compare with your answer in Task 29.

(c) Finally, use equation (11) to represent $F_4(x)$ as a rational function of x, and then use equation (13) to determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{n^4}{2^n}.$$

Task 33

(a) Use (17) to determine the Eulerian numbers in rows $k = 4$ through $k = 6$ of the Eulerian triangle, extending the work of Task 32(a). Continue to include the appropriate auxiliary superscripts in this array.

(b) Use equation (11) to represent $F_7(x)$ as a rational function of x, and then use equation (13) to determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{n^7}{2^n}.$$

Knowledge of how to compute the Eulerian numbers—now without messy derivative computations!—allows us, through equations (11) and (13), to determine the values of series of the form

$$\sum_{n=0}^{\infty} \frac{n^k}{d^n}$$

for any choice of integers $k \geq 1$ and real number $d > 1$. What is more, when d is rational, equation (11) makes it clear that the sum is rational as well!

7 Conclusion

There is much more in Bernoulli's *Tractatus* about the summation of infinite series than we can discuss here. For instance, in the very next proposition following the one we studied in this project, he considered this problem:

XV. To find the sum of the infinite series of fractions whose numerators make up a series of equal numbers, and whose denominators are either the triangular numbers or equimultiples of these.

The simplest example of such a series is the sum of the reciprocals of the triangular numbers,

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots ,$$

a series which you will almost certainly find presented in any standard textbook introduction to the theory of infinite series. Rather than give away the answer here, we urge all our readers to look up the very simple "telescoping" method presented in such books for summing this series.

Later in the *Tractatus*, Bernoulli considered what turned out be the deceptively thorny problem of summing the series of the reciprocals of the squares,

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots .$$

He himself joined a long line of his contemporaries who tried without success to resolve the exact sum of this series.

…it is more difficult than one might expect to seek out this sum, even though we learn that it is finite by [comparison with] other [series], as [this one] is clearly smaller: If someone should discover this [sum], communicate with us, for which we would be greatly appreciative, for it has eluded our diligence up to now.

This problem awaited the mathematical skills of the great Leonhard Euler (1707–1783), a fellow Swiss countryman of Bernoulli, but who was born shortly after Jakob Bernoulli's death.[12] Indeed, it was the resolution of this problem, in which Euler determined that

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6},$$

that essentially launched his career as one of the most prolific and accomplished mathematicians in history.[13]

[12] Jakob Bernoulli died at age 50 in 1705, of tuberculosis, a common (and unfortunately, often fatal) ailment of the times.

[13] For more about this series, see the project *Euler's Calculation of the Sum of the Reciprocals of the Squares* by Kenneth M. Monks, available for download at `https://blogs.ursinus.edu/triumphs/`.

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Notes to Instructors

PSP Content: Topics and Goals

This project is designed to provide students an immersive experience in determining the sums of infinite series by following the work of Jakob Bernoulli (1655–1705) in his *Tractatus de Seriebus Infinitis*. Students typically encounter the theory of infinite series in their second semester course in calculus, in which the focus is the determination of the *convergence* of series. While this topic is vital to the study and use of series in mathematical analysis, it turns out that, with the notable exceptions of geometric series and telescoping series, whose sums are easy to determine, students generally conclude this course realizing that very few of the infinite series which they can determine to be convergent can also be exactly summed. There are a few clear motivations for this approach to infinite series, characterized by a focus on the convergence tests and approximation methods. One such motivation is the ease with which many can now calculate the sum of a very large number of terms of a series thanks to advances in technology (calculators and computers) and the advent of computer algebra systems. Another motivation is that the idea of approximation is "natural" or important among calculus students who are intending to complete degrees in a variety of STEM fields, including a multitude of students in engineering programs.

The transition from a study of convergence tests for infinite series to an introduction to power series and their representations of the classical analytic functions, can be rather jarring for many students. Having spent considerable time and energy on the mechanics of determining whether a series converges, there are few opportunities for them to realize exact values for the series which they find do converge. One should be drawn to the question of finding the sum of a series once its convergence is determined, but this question is so little considered. Of course, students must become familiar with the ideas of series convergence to appreciate the more sophisticated notion of power series and the associated intervals on which they converge. Even so, we believe that this transition (from series of numbers to power series) can be strengthened pedagogically by providing ways for students to see first hand, once they appreciate the difference between convergence and divergence, how to find the exact sums of certain convergent infinite series before they encounter power series, and Jakob Bernoulli's work offers an ideal opportunity to do this.

Bernoulli's work in his *Tractatus* demonstrates an accessible approach to determining the exact sums of certain kinds of infinite series, specifically those of the form

$$\sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} \binom{n+k-1}{k} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} n^k,$$

where $b \neq 0$ and c are arbitrary, $|d| > 1$ and $k = 1, 2$ or 3. Section 5 invites the student to explore cases that correspond to integer values of $k > 3$ in the first series above. In Section 6, students are led through a modern approach to determination of the sums of the second type of series. Completing this PSP should provide the student with a wealth of examples of series whose sums can be found *exactly*.

This project is designed as a capstone experience for undergraduate students in their third or fourth year, particularly for pre-service teachers who may one day be teaching calculus. It is also suitable as an enrichment experience for any student who has completed the traditional introduction to the theory of infinite series, or for students in a history of mathematics course as an opportunity to consider the work of a prominent seventeenth century mathematician at the cusp of the many innovations that heralded the "invention" of calculus in this generation.

Another shorter version of this project is also available that substantially abbreviates the treatment of figurate numbers in Section 3 and omits Subsections 4.4–4.5 on Benoulli's treatment of the cases where the numerators are square and cubes. That version also replaces Sections 5 and 6 with a single section that presents a more elementary and streamlined treatment of modern techniques for summation of series. The shorter version is designed for students of calculus in high schools or in the first year or two of college-level mathematics, who have recently been introduced to summation of infinite series. For more about this other version of the PSP, see the section Connections to other Primary Source Projects below.

Student Prerequisites

Students taking on this project should be familiar with some of the topics found in a typical first year calculus course including summing geometric series, the notion of convergence versus divergence of a series, and the more common convergence tests (for instance, Task 1 expects the student to employ a ratio test to check the convergence of a series). While a comprehensive working knowledge of all the standard convergence tests is not required or used in this project, the more students are aware of the contingent nature of convergence, the more satisfying the impact should be of coming to know methods for computing the sums of series which do converge. In Sections 5 and 6, students are asked to differentiate the terms of a power series and some rational expressions; we trust that the typical second-semester calculus student will be comfortable with this. (We recognize that asking students to differentiate power series could lead to problems with convergence of the resulting series. However, we believe that discussing the validity of such operations with beginning calculus students is counterproductive and better left for another time and place!)

The authors of this project have endeavored to minimize the use of summation notation in this PSP so as to remove as many obstacles as possible for students for whom this symbolism is new and challenging. We would rather that students focus on grappling with Bernoulli's clever methods for manipulating the terms of his series to discover their sums. Of course, this does not prevent instructors from using—and asking their students to employ—this standard notation for infinite series.

PSP Design and Task Commentary

The project begins with a simple task designed to alert students that knowing that a series converges tells one nothing about the sum of that series. This establishes a focus for students' work with Bernoulli on the summation (rather than the convergence) of certain infinite series. The types of series that Bernoulli summed in his treatise are organized by him through the classical language of arithmetic and geometric progressions, and figurate numbers: natural numbers, triangular numbers, pyramidal numbers, etc. These objects are defined and explored in Section 3 of the project after a brief historical account that provides the context for what students will be reading. Instructors who wish to explore figurate numbers more deeply with their students can do no better than turn to the PSP *Construction of the Figurate Numbers* by Jerry Lodder (available for download at the TRIUMPHS website printed at the end of this project).

Section 4 is the heart of the project. It assists students as they work through Bernoulli's Proposition XIV, using variations on a technique that splits the given infinite series to be summed into an infinite number of simpler series (all geometric), each of whose sums then produce a reorganized representation of the given series whose sum was handled in an earlier paragraph of the treatise! The series for which Bernoulli found exact sums in this Proposition are those of the form

$$\sum_{n=1}^{\infty} \frac{p_n}{q_n},$$

where q_n is a geometric progression (with common ratio having absolute value greater than 1 to ensure convergence) and

- in subsection 4.1 (and Tasks 10-12), p_n is an arithmetic progression;
- in subsection 4.2 (and Tasks 13 and 14), p_n is proportional to the sequence of triangular numbers;
- in subsection 4.3 (and Tasks 15 and 16), p_n is proportional to the sequence of pyramidal numbers (with a mention—see Task 17—of how to proceed for higher-order pyramidal numbers).
- in subsection 4.4 (and Task 18), p_n is proportional to the sequence of squares; and
- in subsection 4.5 (and Task 19), p_n is proportional to the sequence of cubes.

In Section 5, students extend Bernoulli's techniques to handle series in which p_n is proportional to higher-order pyramidal numbers $\binom{n+k}{k}$. Task 23 is central to this work. Finally, in Section 6, a much more modern approach involving the generation of Eulerian polynomials is presented (Tasks 28-33) to allow students to determine the sums of series in which p_n is proportional to n^k. We recommend special care be taken with Task 31 as it requires that students be particularly meticulous in computing various polynomial expansions and equating like terms in order to derive a needed recursive formula.

Suggestions for Classroom Implementation

We expect that students will be doing preparatory work before each class day, that they will work (at least some of the time) in small groups with each other in class, and will complete homework and prepare formal write-ups of their classroom work after class. See the implementation schedule below for details on our specific suggestions about this.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Possible Modifications of the PSP

Expect about two weeks of class time to work through this entire project if implemented in a traditional classroom (see the Sample Implementation Schedule below for detailed suggestions for how to do this). But there are other ways to navigate the project for a variety of uses.

For instructors with more serious time restrictions, it should be possible to omit Sections 5 and 6 altogether and save a full week of implementation time. This allows students to follow the work of Bernoulli as presented in his *Tractatus*, but omits the very elegant and tidy generalizations that are found the last two sections of the project. Others may choose to omit only Section 6. Know that the technical demands on students increase through Sections 4, 5, and 6, so more assistance will likely be needed as students progress through the project.

Additionally, this project may profitably be used for independent study by a single student, or a small group of students, allowing for a good deal of freedom in working through the material.

Sections 1–3 may be assigned without much guidance to good students, especially those who may already be familiar with the idea of figurate numbers (Section 3).

Sample Implementation Schedule (based on a 50-minute class period)

Instructors planning to implement this project in a course that meets twice a week in 75-minute periods will need to adapt the schedule given here appropriately. Of course, the actual number of class periods spent on each section naturally depends on the instructor's goals; what follows is merely our suggestion.

Day 1

– Ask students to read Sections 1 and 2 of the project, to jot down any questions they may have as a result of their reading, and to write up Tasks 1 and 2 in advance of the first meeting.

– Address any questions at the opening of the period. Have a student read aloud the opening of Section 3, including the first brief excerpt of Bernoulli's writing. Set students to work in small groups on Tasks 3–5 (whole class discussions may be needed for students to agree on how to answer the questions in Task 3 and 5). The remaining discussion of Section 3 concerns the recursive structure of the figurate numbers. Tasks 6–9 should provide students with enjoyable interactions as they discover how to predict counts of larger and larger terms in these geometrical sequences. The class period might be brought to a close with a common reading of the statement of Proposition XIV and the beginning of paragraph 1 in the primary text that begins Subsection 4.1.

– Have students write up clean solutions for any of the Tasks they have completed by the end of this day's class that you believe will serve them well as a written record of their thinking.

Day 2

– In preparation for Day 2, ask students to read ahead through the end of Subsection 4.1. It should become clear at a reasonably close reading of this what the technical demands will be to make sense of Bernoulli's statements. The instructor should be ready to query the students about what Bernoulli has attempted to do to sum the series in the opening text of Subsection 4.1 to get them engaged in the day's real work.

– Start the period by fielding questions from the students' earlier work, then ask them what Bernoulli has done in this portion of text in Subsection 4.1. Set them to work in small groups on the tasks in this subsection. Repeat with the material in Subsections 4.2 and 4.3. Tasks 11, 13, and 15 are the key tasks, so it is important that classroom time be given to working through their details, which may require discussion across the classroom. We recommend ensuring not that students necessarily complete all the pieces of these (and the other) tasks during the class period as much as that they "get the main idea" about how Bernoulli has worked out the sum of the series. Strive to push them towards making their way through Subsection 4.3 by the end of the period; the goal of the day should be to make sense of Bernoulli's methods here.

- Again, decide what you believe is useful for students to formally write up as homework from this day's material. Task 17 summarizes the work of the first three subsections of Section 4 and should be assigned now to help set up what will be needed in Section 4.6.

Day 3

- Familiarity with the mechanical details from Subsections 4.1–4.3 should serve students well as they work through the rest of Section 4, where the same methods are used, but with minor alterations. So no further preparations should be needed for this day's class.
- The goal for the day is to complete Section 4 of the project. Bernoulli's paragraphs 4 and 5 complete the presentation of his Proposition XIV. The key tasks for today's work will be Task 18 in Subsection 4.4, Task 19 in Subsection 4.5, and a summary task in Subsection 4.6, Task 20.
- Instructors should assign for homework from this part of the project what they will find useful for student assessment.

Day 4

- Section 5 extends the discussion that concludes Bernoulli's Proposition XIV. Ask students to prepare for this class by reading the analysis presented at the beginning of Section 5, up to Task 22. Students who encounter any difficulties should make notes of what the issues are and bring them to class to resolve with discussion at the beginning of the period.
- After resolving the students' questions, complete Tasks 22 and 23. Instructors should realize that the work of Section 5 is more mathematically demanding than what students faced in earlier sections of the project because it is more theoretical. Students are now being asked to prove a general statement about an infinite family of sums.
- While there are only two tasks in Section 5, both are of sufficient weight that it would serve students well to write up their work for both of these. This exercise can also assist them in thinking through the sophistication of these new theoretical tools.

Day 5

- Section 6 includes a modern approach to extending the results quoted by Bernoulli in the lines of summary formulas given at the opening of this part of the project, and offers perhaps the steepest technical challenges. To set the stage for this work, which is laid out over these last two class days, ask students to read through Section 6 up to and including the statement of Task 26. Students should try their hands at answering Tasks 24, 25 and 26 in preparation for this day's class.
- To begin, have students share their progress with Tasks 24–26. This leads very naturally into work (in small groups) on Task 27, which has been organized in summary in equation (10). Tasks 28 and 29 are engineered to get students to think about the patterns evidenced in equation (10); this is vital to help them identify the connection between the forms of the rational expressions for $F_k(x)$ in equation (11) and the Euler polynomials $E_{k-1}(x)$ (see also equation (12)) that appear in their numerators. This connection is the key to working out a method for writing down those polynomials, an achievement that will allow

for the determination of all sums of the form

$$\sum_{n=0}^{\infty} \frac{n^k}{d^n}$$

(for integers $k \geq 1$ and real numbers $d > 1$). The recursive formula expressed in equation (14) summarizes the pattern that students have been working with up to now; this will translate into the recursive formula (15) for the Eulerian polynomials. Task 30 challenges students to derive (15) themselves. We suspect that the bulk of time in class will be spent by students working out this nontrivial relationship.

– Have students formally write up their work on Task 30 (and get a good night's rest in preparation for Day 6).

Day 6

– Ask students to carefully review their work from Day 5, and especially on Task 30, before they come to class this day. The goal of the day is to distill from the recursion in (15) a method for generating the triangle of Eulerian numbers, the main tool for summing series that generalize what Bernoulli provided in his work.

– Do Task 31! It offers carefully scaffolded steps for deriving the recursion in equation (17) from equation (15). The relationships between the double subscripts in these expressions may challenge many students, so we suggest being alert to offer guidance as needed here. The technical work of this task will pay off in Task 32, when students find the neat integer-valued sum of the series

$$\sum_{n=0}^{\infty} \frac{n^4}{2^n}.$$

– Instructors may want to have students write up their work from Task 31 for homework. However, it may be sufficient to assign only Task 33 as proof that students have understood this approach to summing Bernoulli's series.

Connections to other Primary Source Projects

The following TRIUMPHS PSPs are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name is listed (together with the general content focus, if this is not explicitly given in the project title). Each of these can be completed in 1–2 class days, with the exception of the four projects followed by an asterisk (*) which require 3, 4, 3, and 6 days respectively for full implementation. It should be noted that four projects in the list are devoted specifically to the topic of infinite series. Indeed, the first project below is a much shorter version of the present PSP designed specifically for students of calculus who have recently been introduced to summation of infinite series. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- *Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus version)**, Daniel E. Otero and James A. Sellers (infinite series)
- *The Derivatives of the Sine and Cosine Functions*, Dominic Klyve
- *L'Hôpital's Rule*, Daniel E. Otero

- *Fermat's Method for Finding Maxima and Minima*, Kenneth M Monks
- *Beyond Riemann Sums: Fermat's Method of Integration*, Dominic Klyve
- *How to Calculate π: Buffon's Needle (Calculus Version)*, Dominic Klyve (integration by parts)
- *How to Calculate π: Machin's Inverse Tangents*, Dominic Klyve (infinite series)
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, Janet Heine Barnett
- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, Janet Heine Barnett
- *Investigations Into d'Alembert's Definition of Limit (Calculus Version)*, Dave Ruch (definition of limit)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, Kenneth M Monks (infinite series)
- *Fourier's Proof of the Irrationality of e*, Kenneth M Monks (infinite series)
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, Kenneth M Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem**, Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent**, Kenneth M Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens**, Jerry Lodder

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.