UNOMAHA PROBLEM OF THE WEEK

A COLLECTION OF MATH PROBLEMS FOR THE ENTIRE YEAR!

by Brad Horner
and Jordan Sahs

UNOmaha Problem of the Week

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Dedicated to our past, current, and future students.
Thank you for making the act of teaching math
an absolute blessing and joy!

UNIVERSITY OF Nebraska Omaha

Contents

Preface

The University of Omaha math department's Problem of the Week was taken over in Fall 2019 from faculty by the authors. The structure: each semester (Fall and Spring), three problems are given per week for twelve weeks, with each problem worth ten points - mimicking the structure of arguably the most well-regarded university math competition around, the Putnam Competition, with prizes awarded to top-scorers at semester's end. The weekly competition was halted midway through Spring 2020 due to COVID-19, but relaunched again in Fall 2021, with massive changes.

Now there are three difficulty tiers to POW problems, roughly corresponding to easy/medium/hard difficulties, with each tier getting twelve problems per semester, and three problems (one of each tier) per week posted online and around campus. The tiers are named after the EPH classification of conic sections (which is connected to many other classifications in math), and in the present compilation they abide by the following color-coding:

- ELLIPTIC: Puzzles for the non-math types.
- PARABOLIC: Exercises for the math types.
- HYPERBOLIC: Challenges for the best at math.
- OLDER: Problems made before the tiers were.

In practice, when creating the problem sets, we begin with a large enough pool of problem drafts and separate out the ones which are most obviously elliptic or hyperbolic, and then the remaining ones fall into parabolic. The tiers don't necessarily reflect workload, though, only prerequisite mathematical background. Ideally, the solutions to elliptic problems, and any parts of solutions to parabolic and hyperbolic problems not covered in standard undergraduate courses, are meant to test participants' creativity. Beware, though, many solutions also include additional commentary which varies wildly in the reader's assumed mathematical maturity.

The first author's favorite things make frequent appearances in the problems, like the number twenty-four. (Any guesses why twenty-four? After reading this book, can you guess more favorite things?) A few problems (Finitessimal Accretion, Arts and Crafts, Joker's Wild) were suggested by a friend, Brad Tuttle. While variety is always sought after during the design phase, ultimately the problems we come up with are limited to the interest and familiarity of the authors.

The problems have all kinds of inspirations. Some recycle typical math contest fare; some bring attention to niche gems; some bring attention to already-iconic ideas within mathematical areas that could use some more attention anyway; some touch on random explorations of the authors; some are flimsy excuses to soapbox about cool stuff; some highlight recreational, "pop-math" from memes and mainstream media; some cover math with historical and cultural significance across time and space.

Finitessimal Accretion ①

Problem: Find the sum of the smallest element of every subset of

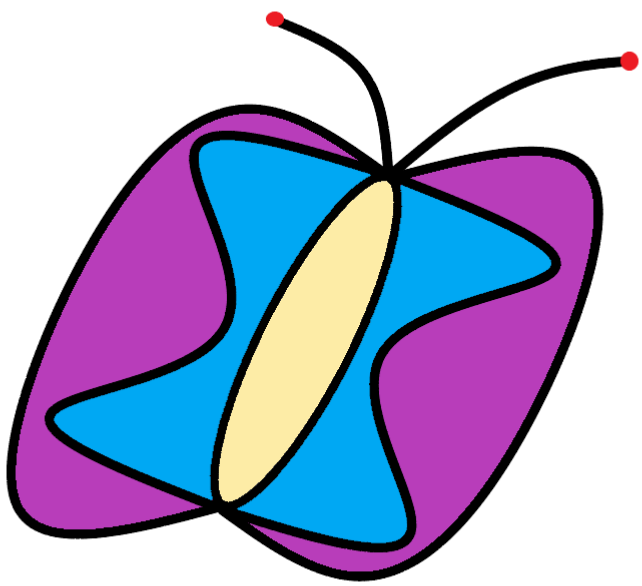
$$\left\{1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots,\ \frac{1}{2^{2021}}\right\}.$$

(Ignore the empty subset which has no elements. A number will generally be added multiple times, once for each subset it is the smallest member of.)

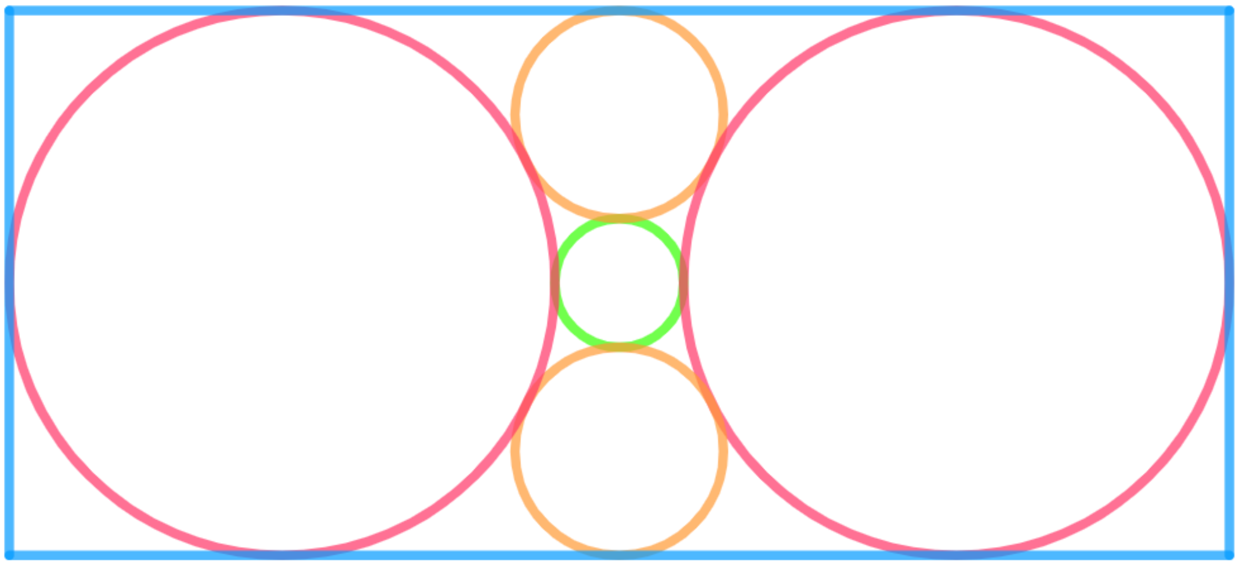
Rhopalocera ②

Problem: How many ways are there to trace this butterfly?

Assume you cannot lift your pen and must start from the left antenna.



Sākuru ③



Problem: Find the ratio of longer to shorter side of the rectangle above.

Transitive Property ④

Problem: An acquaintance shows you three fair 3-sided dice (colored cyan, lime, magenta) and asks if you want to play a game: each of you rolls one die, and whoever has the higher wins. You examine the strange shapes' faces:

Cyan: 1, 6, 8; Lime: 2, 4, 9; Magenta: 3, 5, 7.

The acquaintance offers to let you pick your die to roll first. Should you accept the offer? Why or why not? Explain your answer.

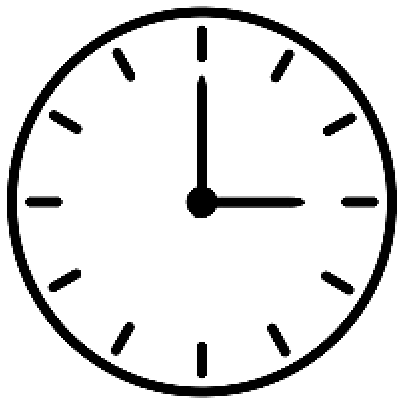
Prime Generation 5

Define the **potential** of a whole number n to be the proportion of integers x in the interval $[-n, n]$ for which $x^2 + x + n$ is a prime number.

Problem. Find and rank the top five highest potential numbers in $[1, 50]$.

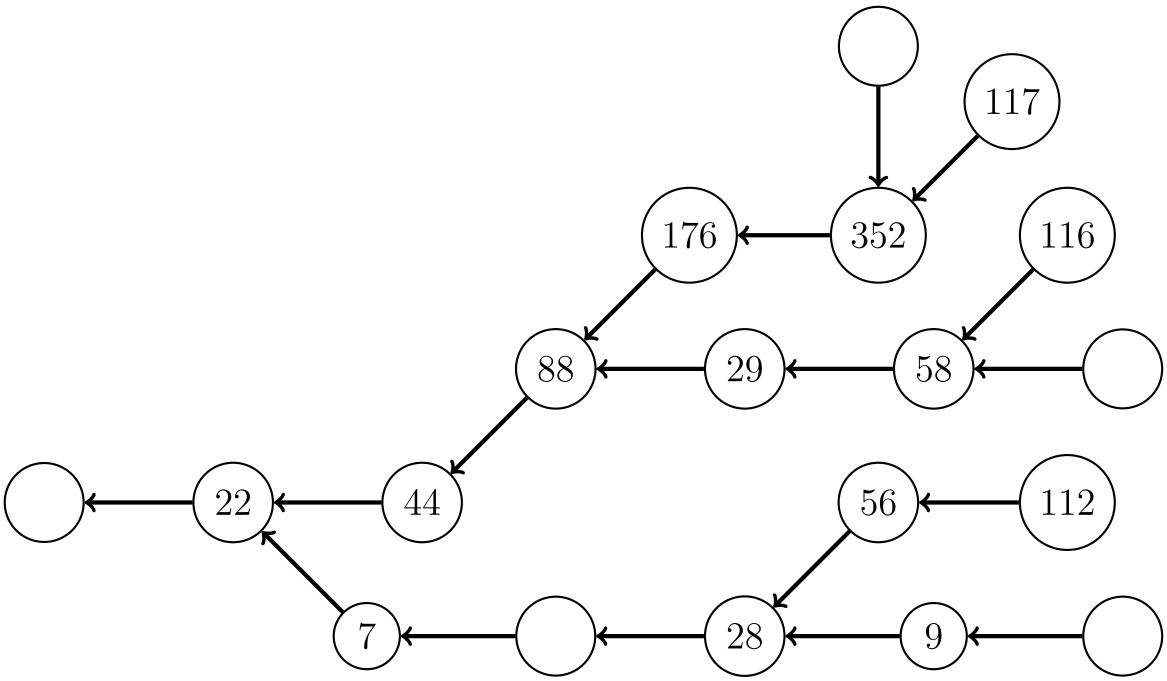
Hint: Try out a Computer Algebra System!

Synchronicity 6



Problem: To the nearest minute, what time is the last time before midnight that the minute hand is a right angle clockwise from the hour hand?

Not Yet Ready 7



Problem: Determine the five missing numbers above.

Cyclic Sieving 8

Problem. How many ways are there to triangulate a regular heptagon?

A triangulation of a polygon is a way of drawing nonintersecting line segments between vertices which partitions its interior into triangles. Assume that rotating a triangulation does not count as a different triangulation.

Unequal Booty 9

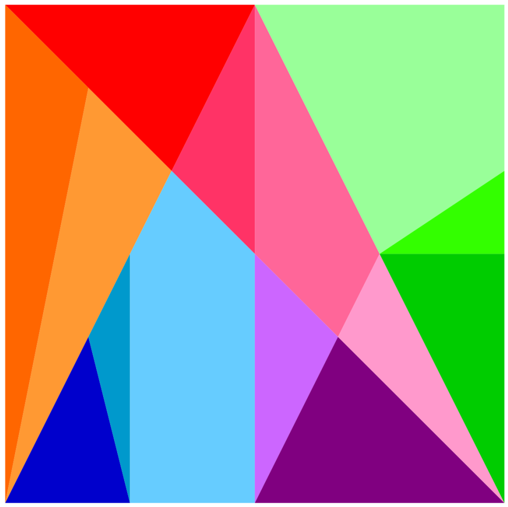
Four pirates and their monkey split a chest of gold coins.

First, the captain makes three equal piles, giving a single leftover coin to the monkey. He takes one of the piles for himself and pours the other two piles back in the treasure chest. The second-in-command does the exact same, followed by the second-to-last and then the swabbie. Finally, the last of the gold in the chest is split equally between the four pirates.

Problem: How few coins could have been in the chest?

Child's Play 10

Problem. Find a different way to arrange these tiles and get a square.

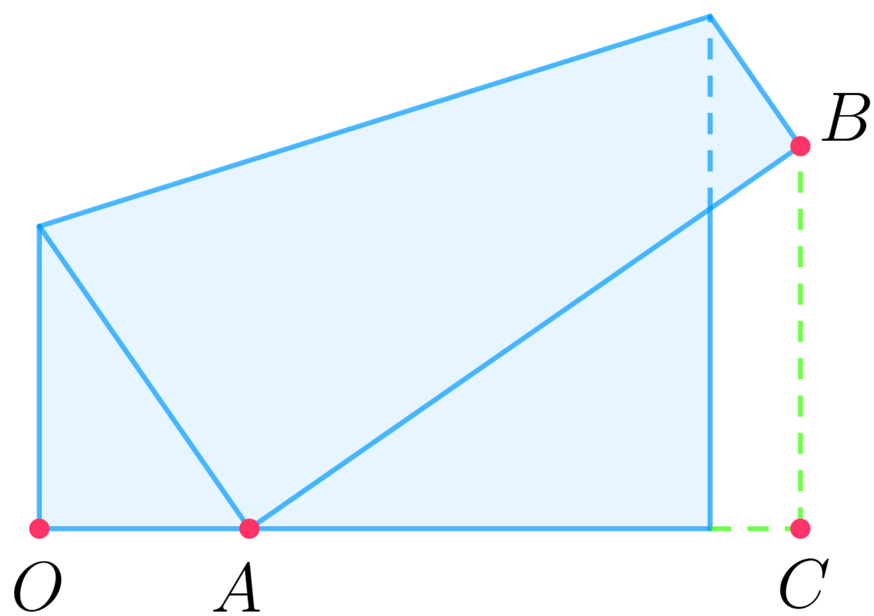


(Any rotation or reflection of an arrangement will not be considered a different one. More subtly, merely swapping identically-shaped pieces or groups of pieces, even of different colors, counts as an equivalent arrangement too.)

First Fold

11

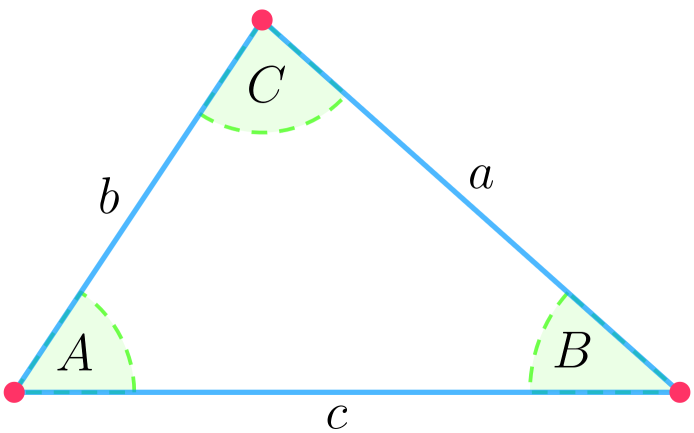
The upper left corner of a 1×1 square is folded down to its base:



Problem: Find the sides of triangle $\triangle ABC$ in terms of $x = \overline{OA}$.

Hint: Use the Pythagorean theorem and similar triangles.

Killer Triangle Problem 12



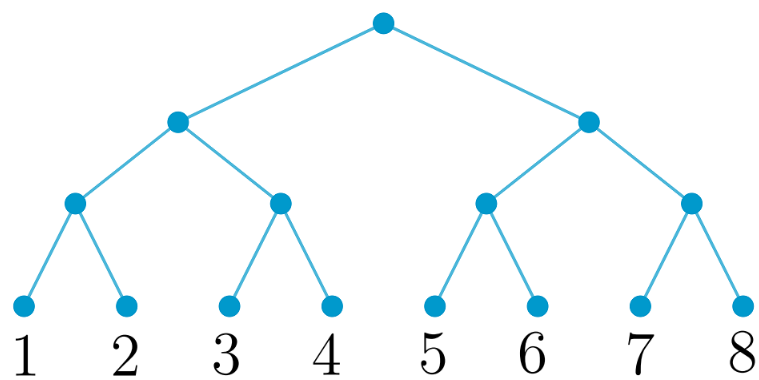
Problem: Prove any triangle $\triangle ABC$ satisfies the inequality

$$\frac{aA + bB + cC}{a + b + c} \leq 90^\circ.$$

Hint: A triangle's interior angles sum to 180°.

Trees and Wreaths 13

Every vertex not at the bottom of the binary tree below has two identical, left and right, subtrees below it. The bottom eight vertices are labelled 1-8.



Problem. Find how many permutations of 1 2 3 4 5 6 7 8 are attainable by swapping the left and right sides below any vertex (multiple swaps allowed)?

Arithmetic Jenga 14

A **composite** number is a nonzero number which is divisible by a whole number other than 1 and itself. (There are three kinds of noncomposite whole numbers: prime numbers $2, 3, 5, 7, \cdots$, the unit 1, and zero 0.)

Problem. Find all whole numbers n with the property that, for all positive divisors d (aka factors) of n the number $d - 1$ is noncomposite.

For a nonexample, consider 5. The number 5 does not have this property, because for the divisor $d = 5$ of 5, the number $d - 1 = 4$ is composite.

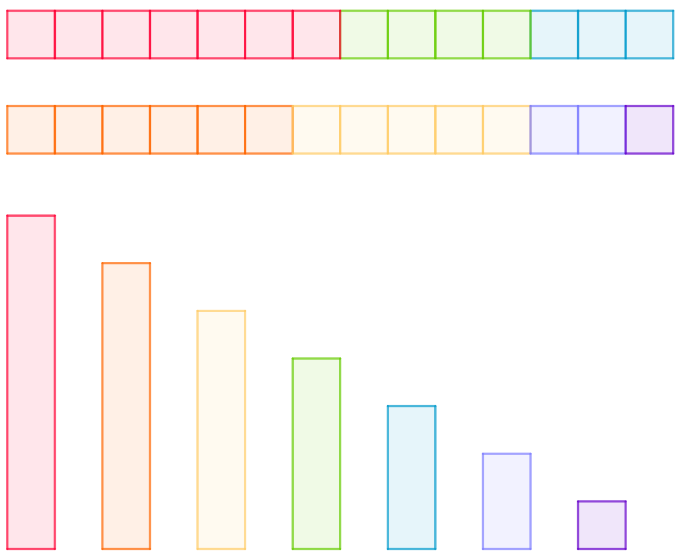
Cutting Sticks 15

Two sticks of (whole number) lengths $a, b \geq n$ have total length

$$a + b = 1 + 2 + \cdots + n.$$

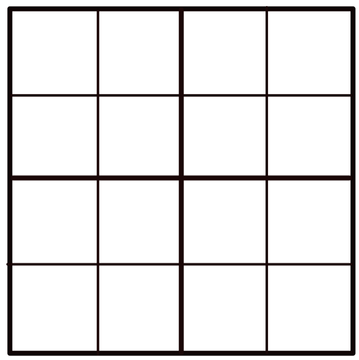
Problem. Show the two sticks may be cut into lengths $1, 2, \cdots, n$.

The pictures below illustrate the particular case $n = 7$ and $a, b = 14$.



Heat of Battle 16

In a one-player game of mini-battle ship, the computer places one 3-tile ship and one 2-tile ship on a 4×4 grid. (Each configuration is equally likely.)



Problem. Rank the tiles by their relative likelihood of being occupied.

Hint. Symmetry can reduce the computational load. But ultimately math isn't all clever tricks; sometimes it's just work.

Odd One Out 17

Suppose a sum of rational numbers is zero:

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0.$$

(Assume each fraction $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$ is nonzero and expressed in lowest terms.)

Problem. Explain why a prime power cannot evenly divide only one of the denominators; in other words, it must divide none or more than one of them.

A prime number has only two positive factors: 1 and itself. For instance $2, 3, 5, 7, 11, \cdots$ are prime but $1, 4, 6, 8, 9, 10, \cdots$ are not. A prime power is any prime number raised to a positive power. A prime factorization is a way of writing a number as a product of prime powers (and possibly ± 1).

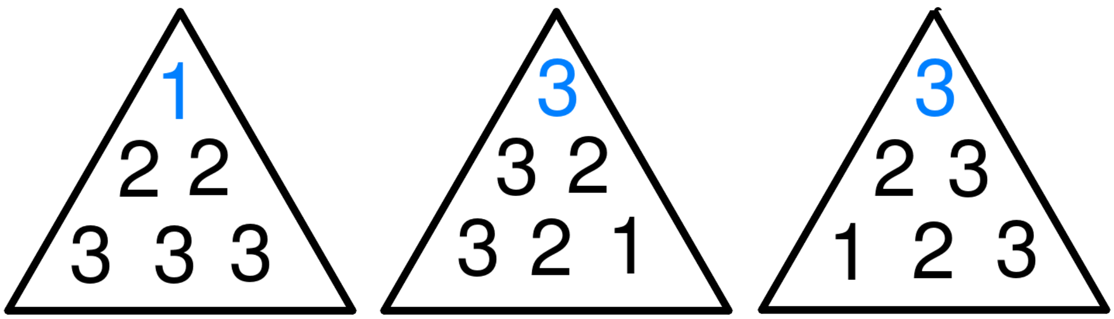
Solutions may assume the Fundamental Theorem of Arithmetic, which says all (nonzero) integers have unique prime factorizations.

Working Backwards 18

A friend proposes a game to decide who picks up the next tab. There are six marbles of three colors - one, two, and three of each color. You and your friend take turns removing any number of marbles of the same color (except zero; at least one marble must be removed per turn), and whoever removes the last marble wins. Your friend offers to let you go first.

Problem. Should you accept the offer? Explain.

Forty Two 19



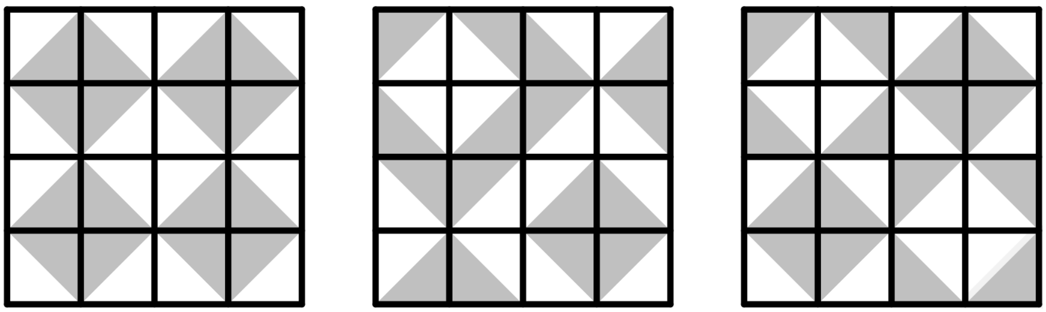
Problem. Explain how the equation below describes the picture above:

$$3(1^2 + 2^2 + 3^2) = (1 + 2 + 3)(1 + 2 \times 3)$$

This is the $n = 3$ instance of a more general identity. Providing the general identity, or even the $n = 4$ instance, is acceptable in lieu of an explanation.

Hint. Each triangle has six positions. Between the three triangles, the sum of all three numbers in a given position doesn't depend on the position.

Kaleidoscopic Diamonds 20



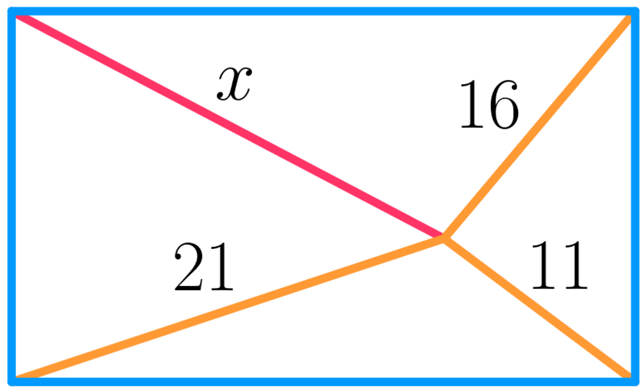
Problem. Pick a pair of diagrams above and show it is possible to turn one into the other using the following three kinds of moves:

- Swapping a pair of rows,
- Swapping a pair of columns,
- Swapping a pair of blocks.

(The blocks are the 2×2 quadrants in the four corners. In fact, it is possible to convert between any of the three pictures using these moves.)

Try online here: `http://finitegeometry.org/sc/16/kal/index.html`

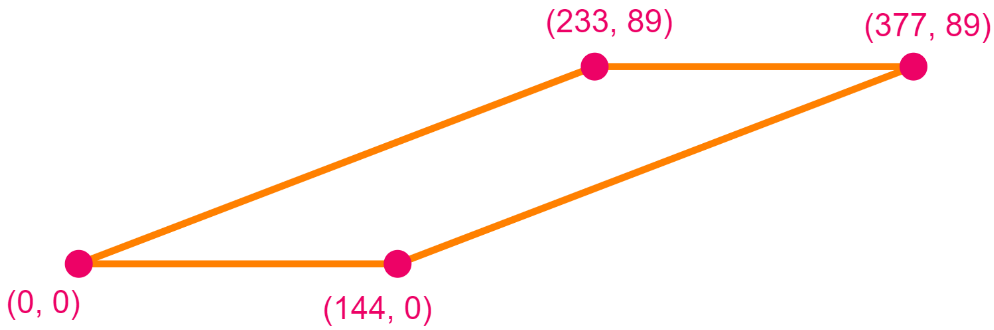
Vexing Vexillology 21



Problem. Solve for x.

Hint. Drop altitudes.

Fenced In 22

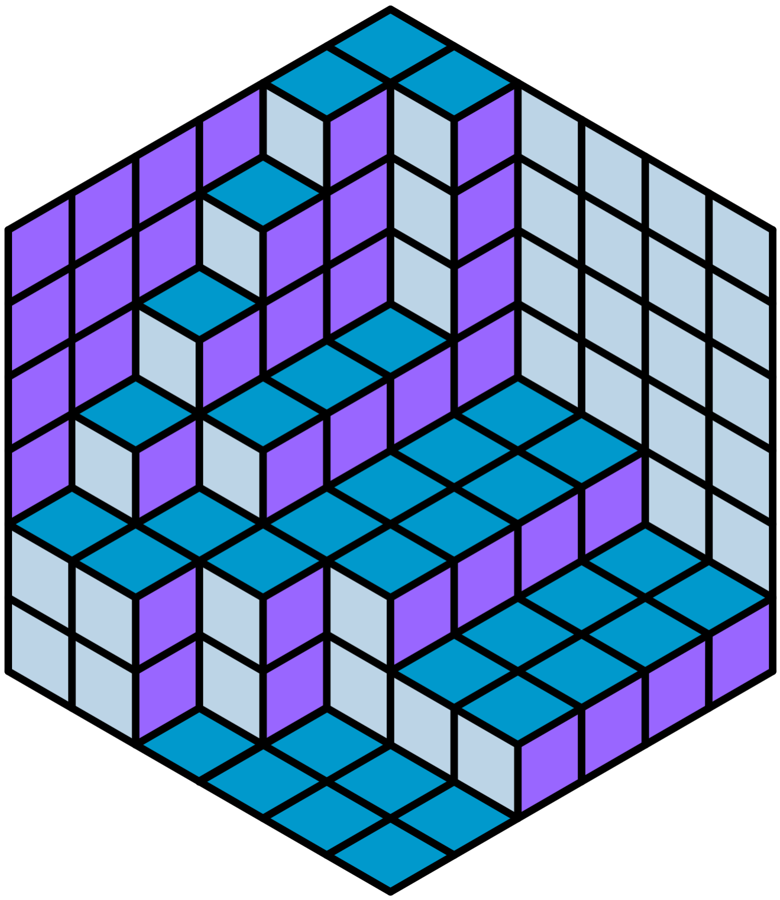


Problem. Determine the number of grid points (that is, points whose coordinates are integers) which lie strictly within the interior of the parallelogram.

Hint. What about the same question for rectangles or right triangles (with horizontal bases)? How are these related to this parallelogram?

Thinking Outside the Box 23

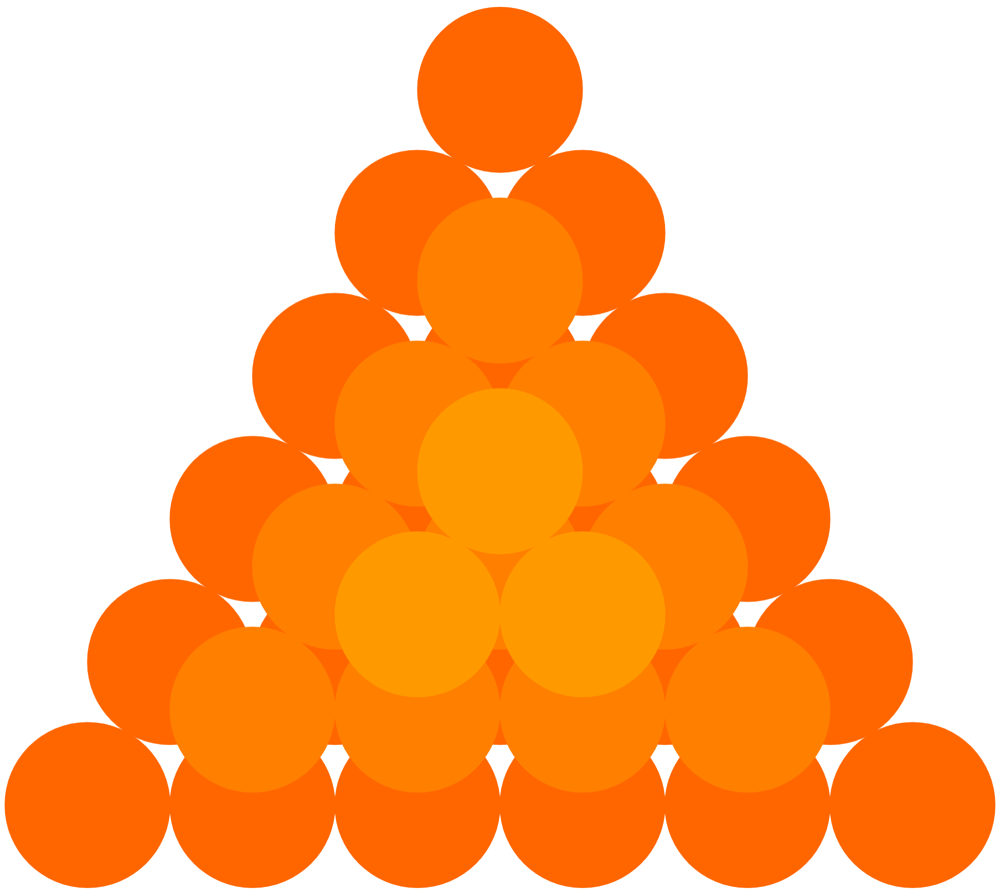
Any rhombic tiling of a hexagon may be interpreted as a pile of cubes:



Problem. Explain why, in any such rhombic tiling, there are an equal number of tiles of each of the three possible orientations.

Orange Stack 24

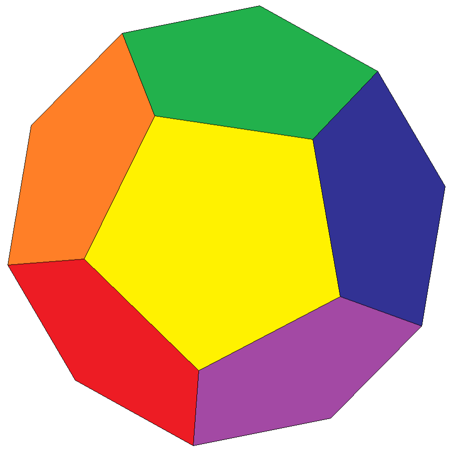
Three layers of oranges (identical spheres) are laid down. The top and bottom layers are not aligned (their centers do not overlap in the 2D projection).



A central orange in the middle layer has many neighboring spheres surrounding it; connecting the centers of touching oranges' centers forms a polyhedron.

Problem. How many vertices, edges and faces does this polyhedron have?

Icosian Palette ①

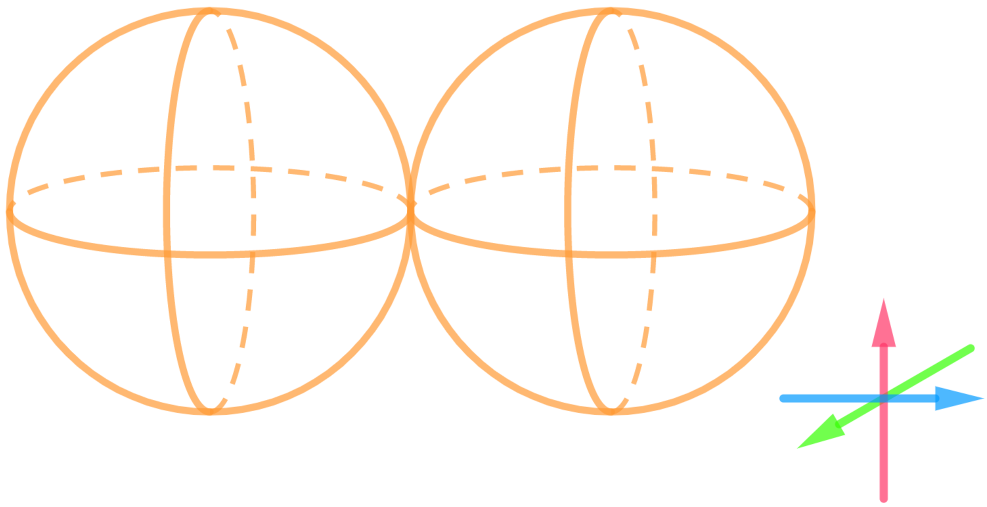


Problem: How many ways are there to paint the faces of a dodecahedron with six colors - say, the primary/secondary colors shown above - so that every face and its five neighbors exhibit all six colors? Assume that rotated colorings are considered equivalent, i.e. do not count as distinct colorings.

Rolling Spheres ②

Two spheres are in contact side-by-side, depicted below. While the left sphere remains fixed, the right sphere is rolled (without slipping) along the top-right fore octant of the left sphere - that is, from the left sphere's x-axis pole, to its y-axis pole, up to its z-axis pole, and then back to its x-axis pole.

Problem: What effect does this have on the right sphere's orientation? That is, describe what kind of rotation it has undergone overall.



Equational Sudoku ③

A number system is a set, whose elements are called "numbers," with two binary operations called addition $+$ and multiplication $\times$, satisfying identities

- **Commutativity**: $a + b = b + a$ and $a \times b = b \times a$
- **Associativity**: $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$
- **Distributivity**: $a \times (b + c) = (a \times b) + (a \times c)$

and the existence (and uniqueness) of certain elements,

- **Absorption**: An element called 0 satisfying the identity $0 \times a = 0$
- **Identities**: Elements 0 and 1 satisfying the identities $0 + a = a$ and $1 \times a = a$ (note this 0 is the same as the aforementioned element)
- **Inverses**: For any a there is an element "$-a$" satisfying $a + (-a) = 0$, and if a is nonzero there is an element "a^{-1}" satisfying $a \times a^{-1} = 1$

Problem: Consider a number system with exactly four distinct elements, say $\{0, 1, \star, ☾\}$. Complete the addition and multiplication tables:

$+$	0	1	$\star$	☾
0	0	1	$\star$	☾
1	1			
$\star$	$\star$			
☾	☾			

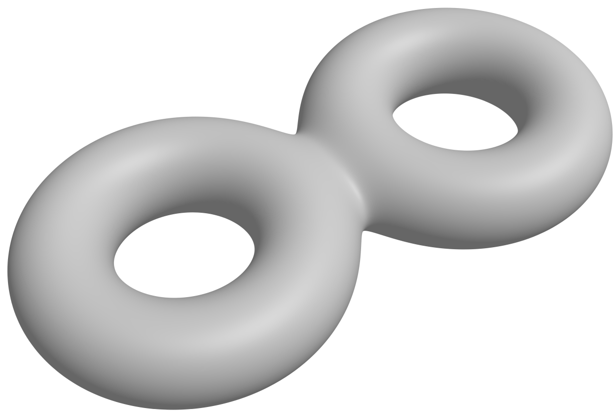
$\times$	0	1	$\star$	☾
0	0	0	0	0
1	0	1	$\star$	☾
$\star$	0	$\star$		
☾	0	☾		

Justify all answers with elementary algebra and logic.

Hint: Find $\star \times$ ☾ first, then $\star \times \star$ and ☾ $\times$ ☾ or $1 + \star$ and $1 +$ ☾, then the rest; make sure to use process of elimination and the fact $0, 1, \star,$ ☾ are distinct!

Totally Tubular ④

Two loops on a surface are called *equivalent* if one can be slid across the surface until it becomes the other. (Imagine rubber bands.) A loop is called *separating* if cutting the surface along the loop results in more than one piece.

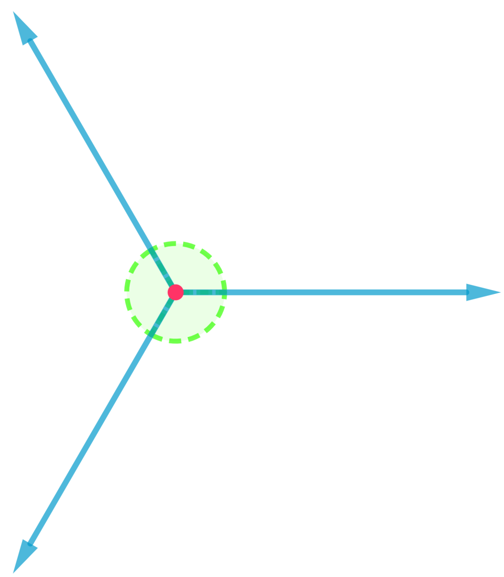


(McKay et al., 2015)

Problem: Draw six inequivalent, non-separating loops on a double torus.

Favorite Angle ⑤

Problem. Show any three 3D vectors at 120° with each other are coplanar.



Orloj Cog 6

Problem. Explain how the periodic sequence $1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 3, 2 \cdots$ (which bounces back and forth between 1 and 4) may be segmented, and the terms in each segment added together, to get the sequence $1, 2, 3, 4, 5, 6, 7, \cdots$.

For instance, 1 | 2 | 3 | 4 | 3 2 | 1 2 3 | 4 3 | 2 1 2 3 | 4 3 2 | 1 2 3 4 with its segments summed yields 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10. How to continue?

Use the periodicity of the $1, 2, 3, 4, \cdots$ sequence to conclude it suffices to check the pattern up to a certain point, then actually perform this check.

Like an Egyptian 7

Problem: How close can $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ be to 1 while still being less than it? (Assume a, b, c are distinct whole numbers.)

Pinching an Impulse ⑧

Suppose $f_1(x), f_2(x), f_3(x), \cdots$ and $f(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n(x)$ are all bounded functions defined on the interval $[0,1]$ with maximums $m_n \stackrel{\text{def}}{=} \max_{0\le x\le 1} f_n(x)$.

Problem: Give an example where $m_1, m_2, m_3, \cdots$ is unbounded.

Hint: Consider piecewise-linear functions known as *triangular* functions.

Categorical Imperative 9

A set S has two binary operations, $\circ$ and $\bullet$, satisfying the relation

$$(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d) \quad \text{for all } a, b, c, d.$$

Just as addition has 0 and multiplication has 1, these operations have their own identity elements, call them $\lozenge$ and $\blacklozenge$. That is, we have the relations

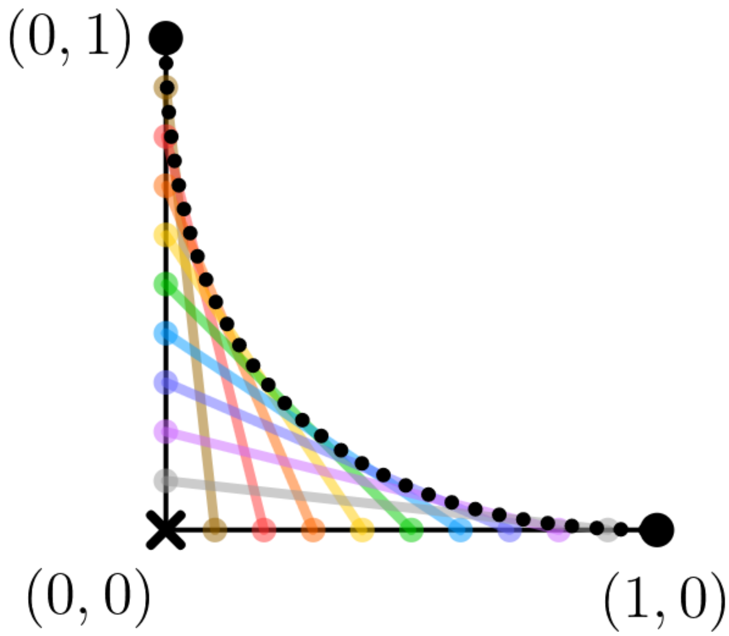
$$\begin{cases} \lozenge \circ s = s = s \circ \lozenge \\ \blacklozenge \bullet s = s = s \bullet \blacklozenge \end{cases} \quad \text{for all } s.$$

Problem. Prove $\lozenge = \blacklozenge$ are the same element of S, $\circ = \bullet$ (they are the same operation!), and indeed it is a commutative operation.

Hint. The identity has an unambiguous "two-dimensional" interpretation: the following expression may be evaluated row-wise or column-wise first.

$$\begin{array}{ccc} a & \circ & b \\ \bullet & & \bullet \\ c & \circ & d \end{array}$$

Arts and Crafts 10

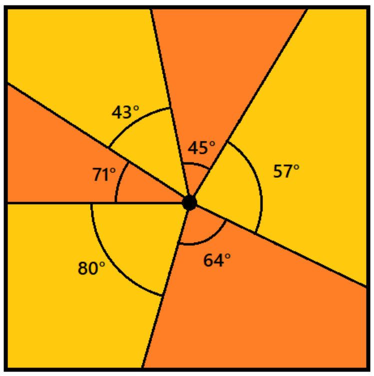


(cmglee, 2013)

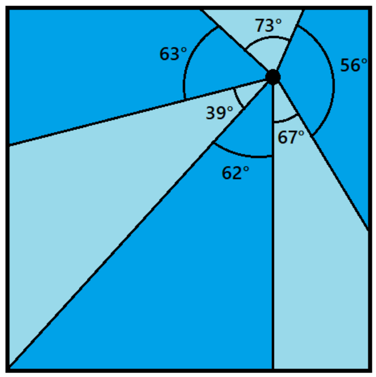
For all t in the interval $0 \leq t \leq 1$, a thread is pinned from the point $(0, t)$ on the y-axis to the corresponding point $(1-t, 0)$ on the x-axis.

Problem. What curve do the threads lie under?

Folding Point 11



A



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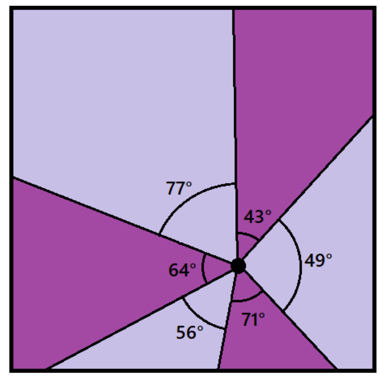


C

Problem. Which of the sheets above can be folded - alternating between into and out of the page - along the marked rays to get a figure that rests flat?

Hint. As the regions are alternately folded down and flattened, the angles all lay down flat too and can be seen zig-zagging back-and-forth.

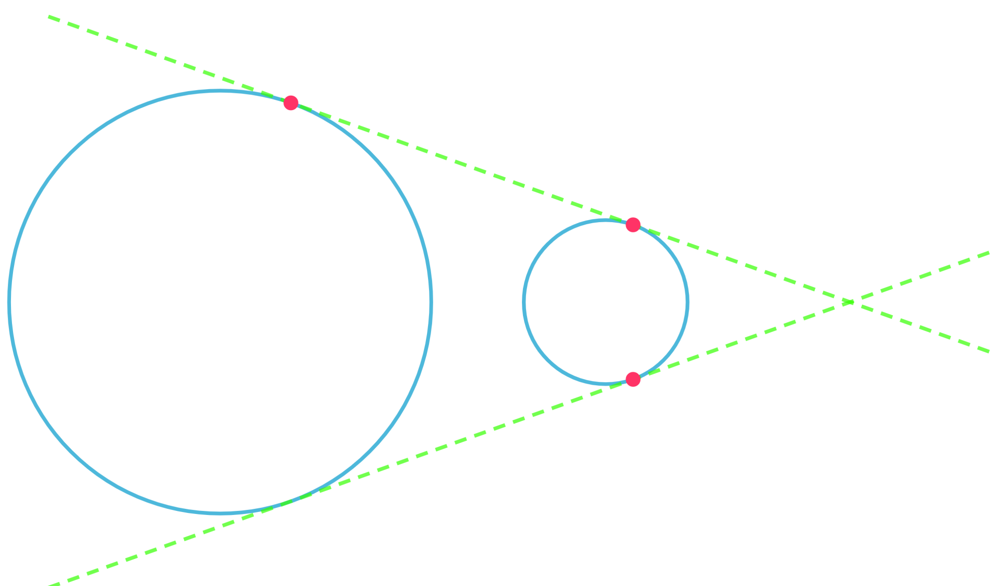
(Note the angles are not drawn accurately in the pictures. Even if they were, the numbers are close enough a hands-on experiment probably won't help.)

Tale of Two Tangents 12

Compass and straightedge construction is a way of creating geometric figures using a handful of specific moves. The modern set of moves:

- **Generic Points**: a random point may be drawn in any region or on any line or arc. (If you're looking to construct a specific point, you cannot assume the random point is the one you want - that's cheating!)
- **Intersections**: Any point of intersection (between two lines, two circles, or between a circle and a line) is considered constructed.
- **Lines**: A line may be drawn between any two constructed points. (This is considered using the "straightedge" tool.)
- **Circles**: A circle may be drawn with any constructed point as its center through another point. (This is considered using the "compass" tool.)

Problem. Two unknown circles have two tangent lines meeting them in four points. Show how to construct the circles given three of the four points.



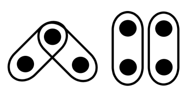
Check out the Euclidea app (`https://www.euclidea.xyz`) for practice.

Pair of Pairs 13

The binomial coefficient $\binom{n}{k}$ denotes the number of ways to draw k distinct elements from a set of size n. For example, $\binom{4}{2} = 6$ since there are six pairs

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{3,4\},\ \{2,4\}.$$

Problem. Explain why $\displaystyle\binom{\binom{n}{2}}{2} = 3\binom{n}{3} + 3\binom{n}{4}$.

Hint. 

Alfred's Ansatz 14

Problem. Find an intermediate-size factor of $2^{62}+1$ by hand.

Any factor other than 1 and 5 (which also means other than $2^{62}+1$ itself and the quotient $(2^{62}+1)/5$) is acceptable. Any submitted calculation that isn't obviously by hand may not count for credit. Your answer may be an arithmetic expression that is left unevaluated (not simplified).

Hint. Write $2^{62} = 4x^4$.

Twenty Four 15

The card game *24* is played with a standard deck as follows: players draw four cards, and the player who finds a way to represent 24 using the four face values (in the range 1-13, using J for 11, Q for 12, and K for 13), the four arithmetic operations ($+$, $-$, $\times$, $\div$), and parentheses wins the round.

Problem. How many solutions of the following form are there?

$$A\,/\,(B - C\,/\,D) = 24$$

Hint. Try your hand at some programming!

Anharmonic Asymmetry 16

A **probability distribution** on a finite set S is effectively a way of assigning a value between 0 and 1 (called a probability) to each element of the set (and these probabilities must sum to 1). If all the probabilities are the same, the distribution is called **uniform**. For instance, a single die roll has set of outcomes $S = \{1, 2, 3, 4, 5, 6\}$ with uniform probability distribution.

A **permutation** is a function on a set X for which there is an inverse function. Let S_3 denote the set of all six permutations of the set $X = \{1, 2, 3\}$.

Problem. Is there a *non*-uniform probability distribution on S_3 for which the probability a permutation sends $i \mapsto j$ is nonetheless equally likely for all nine pairs (i, j)? If so, give an example. If not, give a proof there isn't.

(In some contexts S_3 is called the anharmonic group for unrelated reasons.)

Twisted Clothesline 17

Suppose a function F of two variables has the double series expansion

$$F(x, y) = \sum_{m,n} f_{m,n}\, x^m y^n.$$

(The double series may cover both positive and negative values of m, n, but assume the coefficients decrease rapidly enough convergence is a non-issue.)

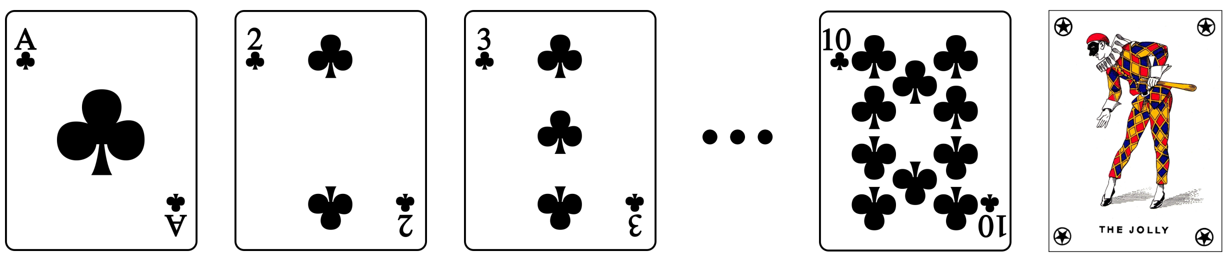
Problem. Express the following double series in terms of F:

$$\sum_{m,n} f_{m+n,2m+3n}\, x^m y^n \;= ?$$

Hint. Gotta reindex!

Joker's Wild 18

A game is played with a small deck of eleven cards: the numbers one (ace) through ten and a Joker. Each turn the player chooses to either draw a card or quit. If a number is drawn then it is added to the current score (which starts at zero), but if a Joker is drawn then the player's score drops to zero and the game ends. Assume the player aims for a target score of S, i.e. they will draw another card if their current score is below S, or quit otherwise.



Problem. What target S would maximize the expected value of their score?

Hint. The next card's expected value is a function of the current score.

Interesting Asymptotic 19

The constant e is often defined by the *interest* formula $e \stackrel{\text{def}}{=} \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$.

Problem. What values of a and b make $1 + \frac{a}{n} + \frac{b}{n^2}$ the best possible approximation to $\frac{1}{e}(1 + \frac{1}{n})^n$ as $n \to \infty$? We may define a for instance by

$$a = \lim_{n\to\infty} n \left[\frac{1}{e}\left(1 + \frac{1}{n}\right)^n - 1\right].$$

Hint. Consider the Newton-Mercator series for $\ln(1 + \frac{1}{n})$. (Look it up!)

Noncommutative Calculus 20

The exponential function is often given by the infinite power series formula

$$\exp X = \sum_{n=0}^{\infty} \frac{X^n}{n!} = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \frac{1}{24}X^4 + \cdots$$

It satisfies $\exp(X+Y) = \exp(X)\exp(Y)$ when X and Y commute (meaning $XY = YX$) but generally not otherwise, e.g. for matrices it is usually false.

But even for noncommuting X and Y there is an infinite series for which

$$\exp(tX)\exp(tY) = \exp\big(tZ_1 + t^2 Z_2 + t^3 Z_3 + \cdots\big)$$

where each Z_k is a degree k *noncommutative* polynomial of X and Y (and we assume t commutes with everything). In particular, for matrices the Z-series can be evaluated for scalar values t small enough relative to X and Y.

For example, $Z_1 = X + Y$ (unsurprisingly) and $Z_2 = \frac{1}{2}(XY - YX)$.

Problem. Find the noncommutative polynomial $Z_3(X, Y)$.

Versorial Validation 21

A **quaternion** is a pretend-sum $a + \mathbf{u}$ of a real number a and a 3D vector $\mathbf{u}$ (called the scalar part and vector part respectively) which does not simplify.

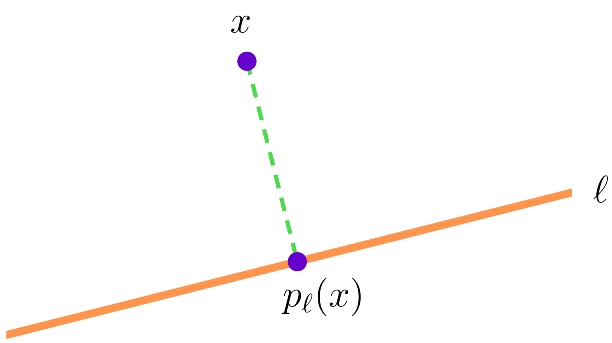
The sum of quaternions is $(a + \mathbf{u}) + (b + \mathbf{v}) = (a + b) + (\mathbf{u} + \mathbf{v})$. The product of two vectors has scalar and vector parts given by dot product and cross product: $\mathbf{uv} = (-\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \times \mathbf{v})$. To multiply two generic quaternions, i.e. $(a+\mathbf{u})(b+\mathbf{v})$, we would use the distributive property (or "FOIL").

The squared norm is $|a + \mathbf{u}|^2 := |a|^2 + \|\mathbf{u}\|^2$ where $\|\mathbf{u}\|$ is the vector norm.

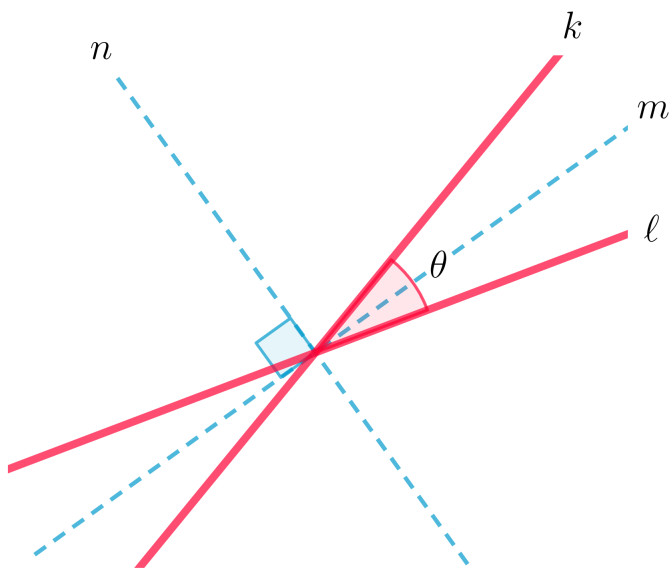
Problem. Show $|xy| = |x||y|$ (multiplicativity) for all quaternions x and y using identities for dot products, cross products and vector norms.

Projector Junction 22

For a line ℓ, the projector p_ℓ sends any point x to the nearest one $p_\ell(x)$ on ℓ:



The composition $p_k p_\ell$ of two projectors (as functions) is generally not a projector. However, the **symmetrization** $\frac{1}{2}(p_k p_\ell + p_\ell p_k)$ is a sum of orthogonal projectors, corresponding to the two bisectors m and n of k and ℓ:



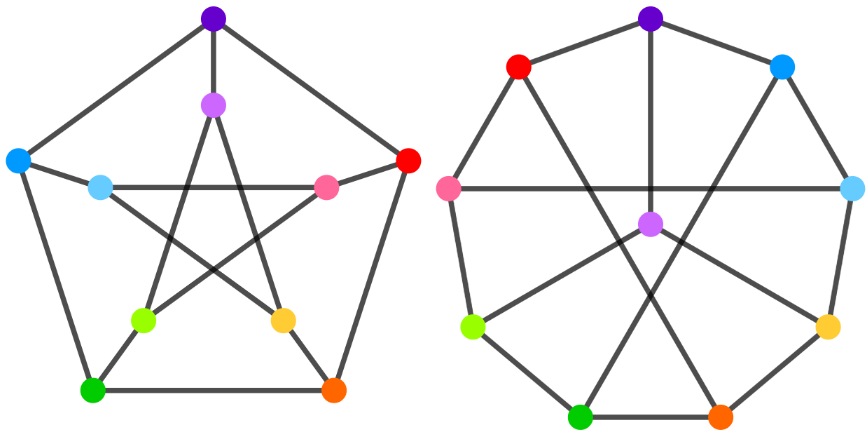
Problem. Express $a(\theta)$ and $b(\theta)$ in terms of trigonometric functions of θ:

$$\frac{1}{2}(p_k p_\ell + p_\ell p_k) = a(\theta) p_m + b(\theta) p_n$$

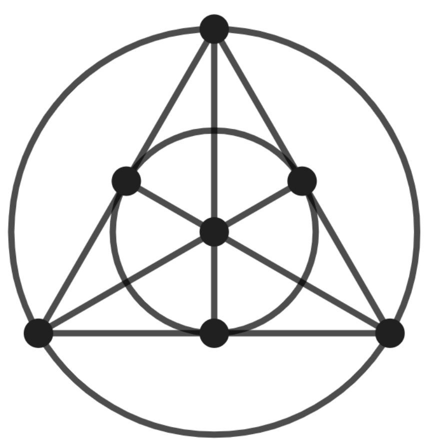
Perspective Shift 23

A graph (collection of vertices and edges) can be drawn in many ways. It is often not possible to illustrate all of a graph's symmetry in a single drawing.

To compensate, we can use multiple different drawings to illustrate different kinds of symmetry of the same graph. For instance, the Peterson graph can be drawn to illustrate either fivefold or threefold symmetry:



Problem. Redraw the graph below to illustrate *fourfold* symmetry:

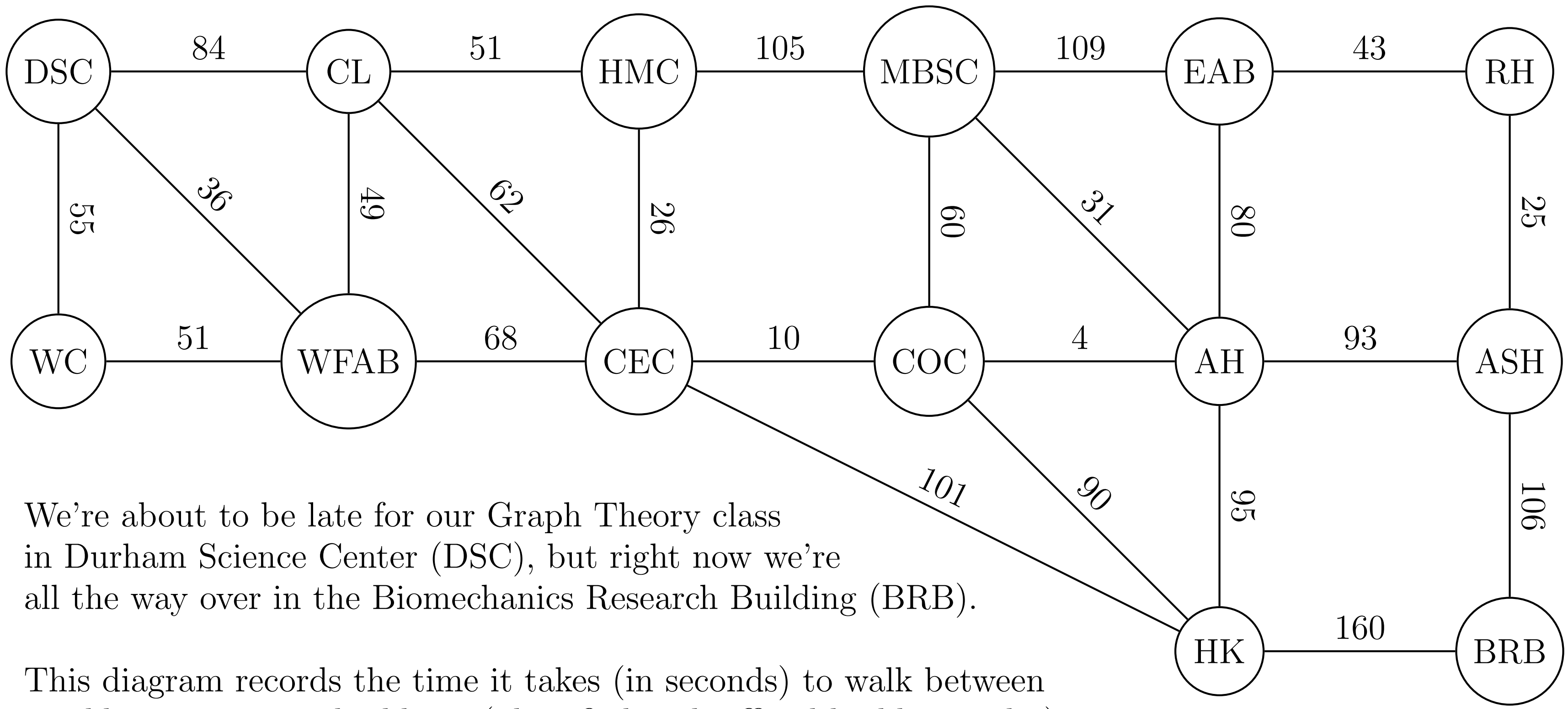


(Currently, of course, this picture illustrates threefold symmetry.)

Hint. Instead of drawing a simple two-dimensional figure, lift some pieces off the page (or out of the screen) and imagine a three-dimensional figure!

Campus Dash

24



We're about to be late for our Graph Theory class in Durham Science Center (DSC), but right now we're all the way over in the Biomechanics Research Building (BRB).

This diagram records the time it takes (in seconds) to walk between neighboring campus buildings (identified with official building codes).

Problem: What path from BRB to DSC is the quickest? What path is the slowest?

UNO North Campus Map: https://tinyurl.com/UNO-North-Campus-Parking-Map

Quadratic Pythagorean Triples ①

Call $(a(x), b(x), c(x))$ a *primitive quadratic Pythagorean triple* if

$$a(x)^2 + b(x)^2 = c(x)^2$$

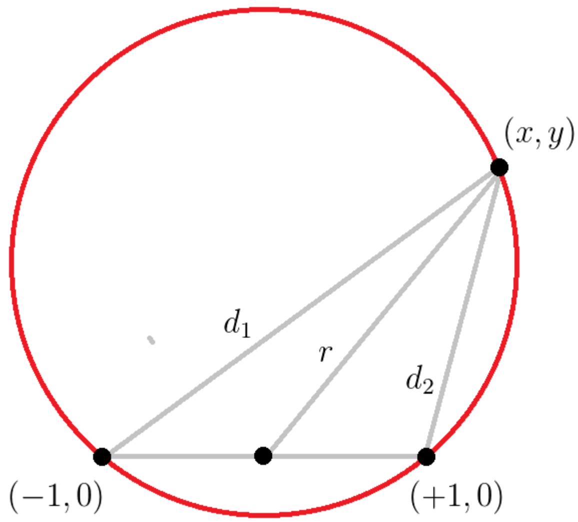
and $a(x), b(x), c(x)$ are quadratic polynomials with no common root.

Call $\Delta = B^2 - 4AC$ the *discriminant* of a quadratic $Ax^2 + Bx + C$.

Problem. Explain how the three discriminants $\Delta_1, \Delta_2, \Delta_3$ of a primitive quadratic Pythagorean triple (f_1, f_2, f_3) are related.

Hint. How are integer Pythagorean triples parametrized? (Look it up!) This parametrization works for polynomials just as it does for integers.

Bipolarity ②



Let σ be the angle between d_1 and d_2 above, and $\tau = \ln(d_1/d_2)$.

Problem. Find x and y in terms of σ and τ.

Hints. Express $d_1 d_2 \cos\sigma$ and $d_1 d_2 \sin\sigma$ in terms of x, y, r using trig, and ratios $f(\tau)/g(\sigma)$ between hyperbolic and standard trig functions.

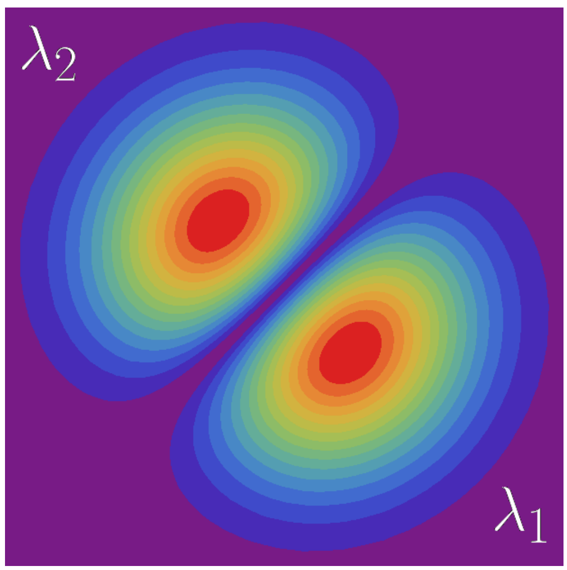
Ensemble Cast ③

H is a symmetric matrix-valued random variable with probability density

$$H = \begin{pmatrix} x+y & z \\ z & x-y \end{pmatrix}, \quad \rho_H(x,y,z) = \pi^{-3/2} e^{-(x^2+y^2+z^2)}.$$

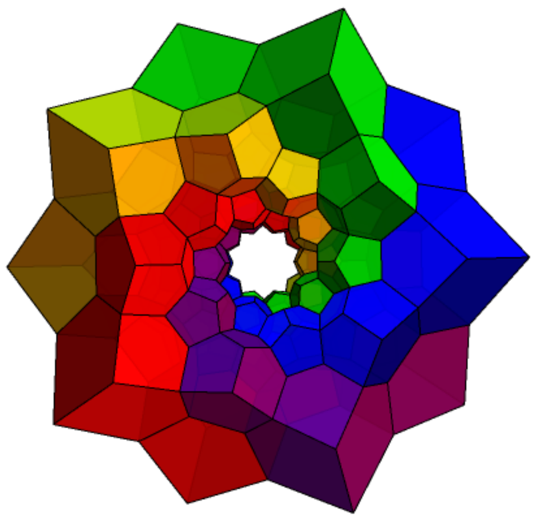
Problem. Find the joint probability density of its eigenvalues $\lambda_1 \leq \lambda_2$.

(In other words, integrate the density ρ_H over the space of all the symmetric matrices H with given eigenvalues λ_1, λ_2 in order to get a function of λ_1, λ_2.)



Good Fibrations ④

Three-dimensional space is filled in by gluing 120 dodecahedra face-to-face in a particularly symmetric fashion. A **bundle** is a way of partitioning these dodecahedra into a dozen rings, each consisting of ten dodecahedra. Every neighboring pair of dodecahedra can be extended to a unique ring, and every ring can be extended to a unique bundle.

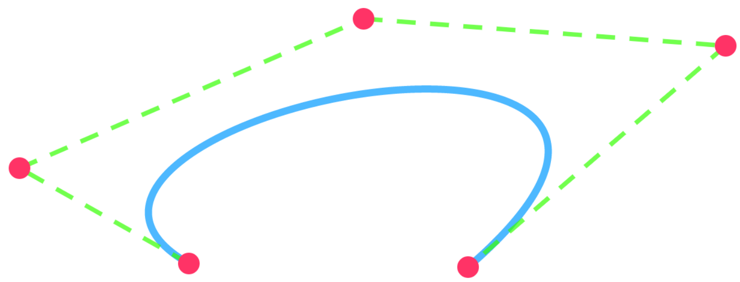


(apgoucher, 2021)

Five rings of one bundle are shown above. (Inevitably, one or more dodecahedra will be "inside out" and infinitely large, but this will not be an issue.)

Problem. How many bundles are possible?

Lazy Spline (5)



The degree n **Bèzier curve** which interpolates a polygonal path with vertices $\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_n$ (in order) is parametrized from 0 to 1 by the formula

$$\mathbf{B}(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \mathbf{v}_k.$$

The **energy** of a curve $\mathbf{x}(t)$ is the average square of its speed:

$$E = \int_0^1 \|\mathbf{x}'(t)\|^2 \, \mathrm{d}t.$$

Problem. Suppose a polygonal path connects five points, the first edge from $\mathbf{a}$ to $\mathbf{b}$ and the last edge from $\mathbf{c}$ to $\mathbf{d}$. What central vertex (in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$) minimizes the energy of the corresponding quartic Bèzier curve?

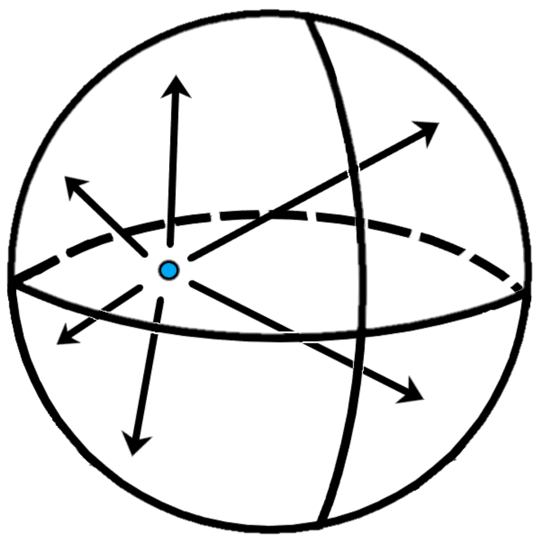
Hint. Remember, a scalar function E is extremized when $\nabla E = \mathbf{0}$.

Using software (e.g. Wolfram|Alpha) to aid calculations is recommended.

Blinding Sphere 6

The **light intensity** from a point source to another point in space is $1/r^2$, where r is the distance between the two points. This ensures that as light travels outward from a point in an expanding sphere the total light intensity is constant, because the sphere's area grows proportional to r^2.

Problem: If every point on a unit sphere is a light source, find the average intensity experienced by a point which is p units from the center.



Sample Energy 7

A first candidate for defining the "distance" between the distributions of random variables X and Y is the expected value of their absolute difference, $\mathbb{E}|X-Y|$. However, this would be nonzero if the variables had the same distribution yet were independent. It turns out, though, this can be fixed by subtracting from $\mathbb{E}|X-Y|$ the average of $\mathbb{E}|X-X'|$ and $\mathbb{E}|Y-Y'|$ (where X', Y' are distributed identically to but independent of X, Y respectively).

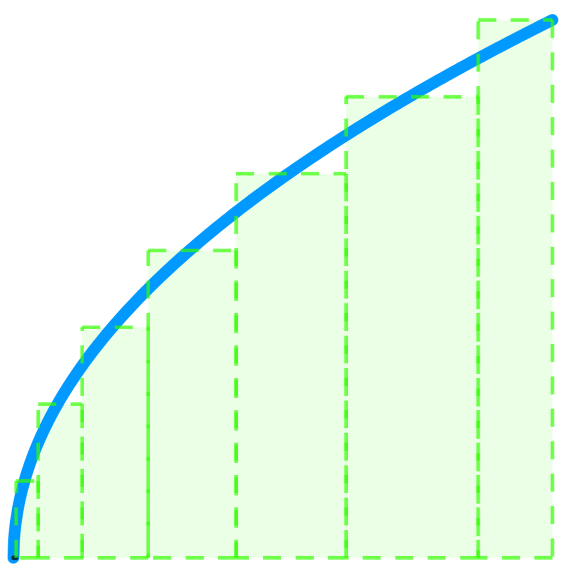
Thus the **energy distance** between X and Y is defined by the relation

$$\mathrm{d}(X,Y)^2 = \mathbb{E}|X-Y| - \tfrac{1}{2}\big(\mathbb{E}|X-X'| + \mathbb{E}|Y-Y'|\big).$$

Using the cumulative distribution functions, it is also given by the formula

$$\mathrm{d}(X,Y)^2 = \int_{-\infty}^{\infty} \big(F_X(t) - F_Y(t)\big)^2 \,\mathrm{d}t.$$

Problem. What three values $0 < u < v < w < 1$ minimize the energy distance between the discrete uniform distribution on the set $\{u, v, w\}$ and the continuous uniform distribution on the interval $[0,1]$?



Gyration Conjugation ⑧

Problem. Two of the sixty rotational symmetries of an icosahedron are chosen at random; what are the chances they satisfy the commutative property

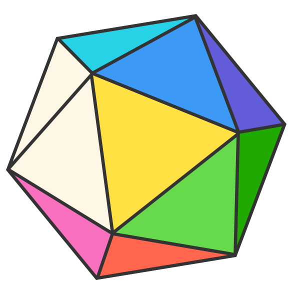
$$AB = BA,$$

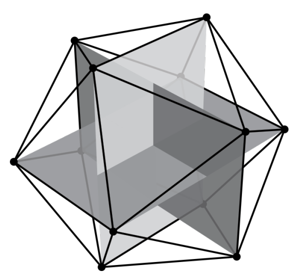
i.e. applying the rotations A and B in either order achieves the same effect? Here, the phrase "by symmetry" used in submissions will be accepted as an explanation whenever valid. Partial credit available for (i) explaining why there are sixty rotational symmetries, (ii) classifying them into types by axes and angles, and (iii) finding how many there are of each type you define.

A couple facts may or may not help in your calculations:

Conjugation Lemma. We call ABA^{-1} the "conjugation of B by A": this conjugate is a rotation by the same angle as B, but its (directed) axis of rotation is obtained by applying the rotation A to B's axis of rotation.

Rectangle Lemma. Every edge is contained in a unique inscribed compound of three perpendicular golden rectangles.





Halving Harmonics 9

Problem. Show, with explanation, there exists a rearrangement $a_1, a_2, a_3, \cdots$ of the natural numbers $1, 2, 3, \cdots$, a constant $c < 1$, and bound N for which

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) < c\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$$

for all $n \geq N$.

Note. Defining the nth harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, you may use the fact that the inequality $0 < H_n - \ln n < 1$ is true for all $n > 1$.

Factorial Frenzy 10

The **binomial theorem** tells us how to expand a power of a binomial:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

After expanding both sides of $(1+x)^{2m}(1+x)^{2n} = (1+x)^{2m+2n}$ with the binomial theorem, then multiplying and collecting like terms, and equating the coefficient of x^{m+n} on both sides, we get the following:

$$\sum_{k} \binom{2m}{m+k}\binom{2n}{n-k} = \binom{2m+2n}{m+n},$$

an instance of the **Vandermonde convolution identity**.

(Every x^{m+n} term that arises from expanding $(1+x)^{2m}(1+x)^{2n}$ originates from multiplying a x^{m+k} term from $(1+x)^{2m}$ with a x^{n-k} term from $(1+x)^{2n}$ for some k. Note k is allowed to range over both positive and negative integers - only finitely many summands are nonzero, though, since if the bottom number in a binomial coefficient is out of range then it evaluates to 0.)

Problem. Express the following sum in terms of binomial coefficients:

$$\sum_{k} (-1)^k \binom{2m}{m+k}\binom{2n}{n-k}$$

by examining the x^{2m} coefficient of $(1+x)^{m+n}(1-x)^{m+n} = (1-x^2)^{m+n}$.

Yoga of π 11

Problem. Transform one of the integrals below into the other,

$$\iint\limits_{x^2+y^2\leq 1} \mathrm{d}x\,\mathrm{d}y \quad \longleftrightarrow \quad \int\limits_{-1\leq u\leq 1} \frac{2\,\mathrm{d}u}{u^2+1},$$

using the following three techniques:

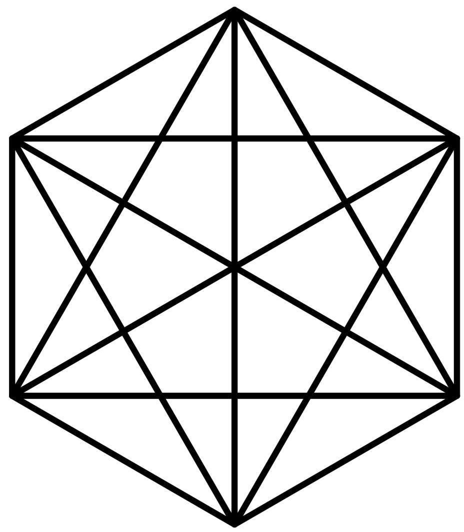
- **Linearity of Integrals** - any integral may be turned into a sum of integrals if the integrand is a sum or the domain is split up,
- **Fundamental Theorem of Calculus** to (un)evaluate integrals,
- **Change of Variables** (for 2D integrals, need Jacobian determinant),

and subject to the following rule:

- **Algebraicity**: at all times, integrands must be rational functions and domains must be characterized by polynomial inequalities.

In particular, transcendental functions such as exponentials, logarithms, and trigonometric functions are not allowed to appear, nor is the constant π.

Synthematics 12

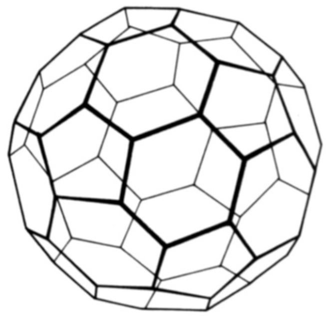


Six points are arranged as vertices of a hexagon, and all $\binom{6}{2} = 15$ possible line segments are drawn between them. A **synthematic** is created by coloring these line segments with 5 colors so that there are 3 of each color and no two segments share both a vertex and a color. Permuting the colors does not count as a different synthematic, but rotating or reflecting one can.

Problem. How many synthematics are possible? Explain.

Partial credit for how many triples of line segments share no vertices.

Buckminsterfullerene 13



Problem. Suppose a convex polyhedron has P pentagonal and H hexagonal faces (for instance, $P = 12$ and $H = 20$ for a soccer ball). Show how to find the number of **space diagonals** S it must have in terms of P and H.

(Any line segment joining two vertices is either an edge, a face diagonal, or a space diagonal through the interior, and these are mutually exclusive.)

*abc*s in the Margin 14

Some polynomial-related terminology:

- The **degree** $\deg f$ of a polynomial $f(t)$ is the highest power that appears in it, for instance a quadratic like $t^2 + 1$ has degree two.
- A polynomial that cannot be factored (into nonconstant polynomials of smaller degree), like $t^2 + 1$, is called **irreducible**. Every polynomial factors as a product of irreducible factors (ignoring scalar factors).
- Two polynomials are called **coprime** if they share no common (non-constant) factor. For example $t^2 + 1$ and $t^2 - 1$ are coprime.
- The **radical** $\operatorname{rad} f$ of a polynomial $f(t)$ is the product of its irreducible factors, for example $\operatorname{rad}\left[t(t-1)^2(t+1)^3\right] = t(t-1)(t+1)$.

The ***abc* theorem** says if $a(t), b(t), c(t)$ are coprime nonconstant polynomials and $a(t) + b(t) = c(t)$ then their product's radical has larger degree:

$$\deg a(t), \deg b(t), \deg c(t) < \deg \operatorname{rad}\left[a(t)b(t)c(t)\right].$$

Problem. Use the *abc* theorem to show that if $n > 2$ then there are no nonconstant coprime polynomials $a(t), b(t), c(t)$ for which $a(t)^n + b(t)^n = c(t)^n$.

Rational Corollary 15

Problem. Justify the following identity for 2×2 matrices A and B:

$$(I + A)\,(I - BA)^{-1}\,(I + B) \quad = \quad (I + B)\,(I - AB)^{-1}\,(I + A).$$

(You may assume A and B are close to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ if you think it might help.)

Heisenberg 16

Invertible matrices with matrix multiplication, and permutations with function composition, are both examples of noncommutative groups.

A **group** G is a set with an associative binary operation, a unique identity element e, and inverses. Elements x and y are said to **commute** if $xy = yx$.

- For whole numbers, the power g^n is $gg \cdots g$ (n times), with special cases $g^1 = g$ and $g^0 = e$. Negative powers g^{-n} may be defined by either $(g^{-1})^n$ or $(g^n)^{-1}$. Exponent rules like $(a^m)^n = a^{mn}$ and $a^m a^n = a^{m+n}$ apply.
- The socks-and-shoes rule says $(xy)^{-1} = y^{-1}x^{-1}$ for any pair of group elements x and y, which is illustrated in the fact that we put on socks before shoes but take them off in the reverse order.
- Equations are often manipulated by left-multiplying or right-multiplying by inverses, for example $x = y$ is equivalent to $y^{-1}x = e$ and $xy^{-1} = e$.
- Substitution is also often useful, for example if $g^3 = e$ then $g^2 = g^{-1}$, which means g^{-1} and g^2 are interchangeable in any expression.

Suppose a and b are elements of a group in which $g^3 = e$ for all elements g.

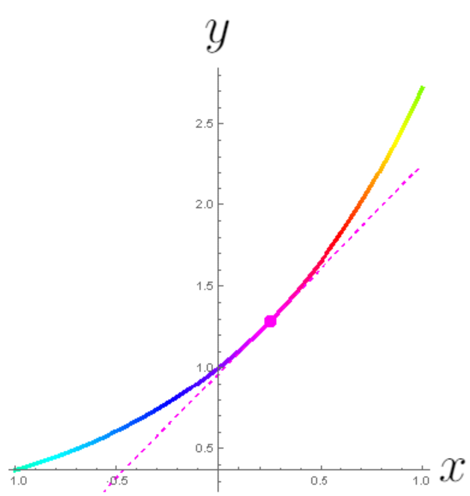
Problem. Do one of the following:

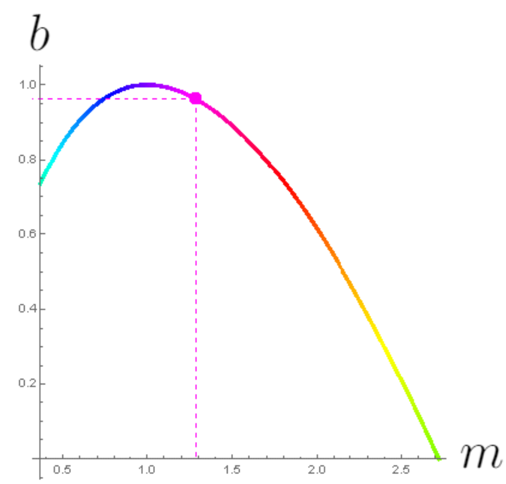
- Show the elements ab^{-1} and $b^{-1}a$ commute.
- Show $aba^{-1}b^{-1}$ commutes with one of a or b.

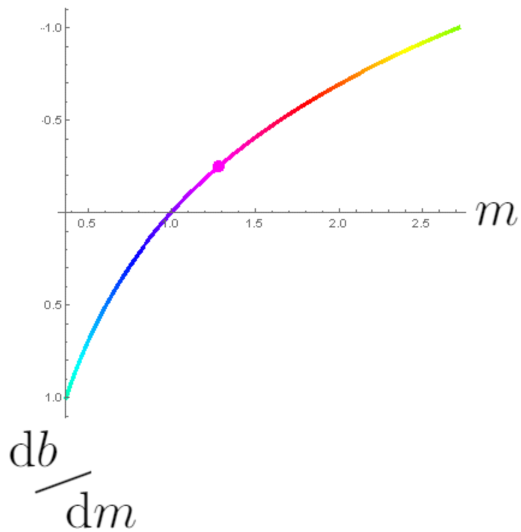
Slope-Intercept Coordinates 17

Suppose the graph $y = f(x)$ of a function f is smooth: at every point (x, y) there is a tangent line $y = mx+b$. Suppose further we plot the corresponding points (m, b) in a separate mb-plane, one for each tangent line of f's graph, and the result is the graph of a function $b = g(m)$ in this separate plane.

Problem. Show the derivatives f' and $-g'$ are inverse functions.







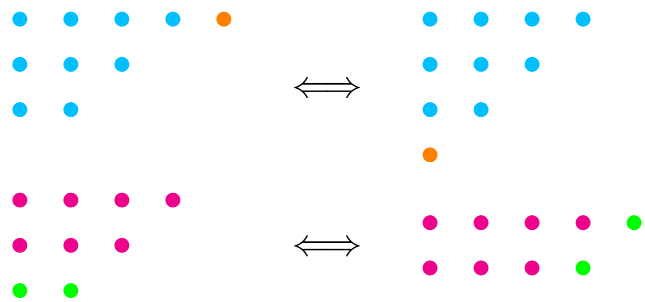
Top left is the graph of $y = f(x)$ for the exponential function. Top right is the graph of the function $b = g(m)$.

Below that is the graph of its derivative $\mathrm{d}b/\mathrm{d}m$, displayed upside-down to show it is the mirror reflection of $y = f(x)$ across the diagonal (emblematic of inverse functions).

Pentagonal Peculiarity 18

Consider diagrams which depict dots in a series of (left-aligned) rows, with each row having strictly less dots than the one above. The number of such diagrams with n dots and an even number of rows let's denote E, and the number of those with an odd number of rows let's denote O.

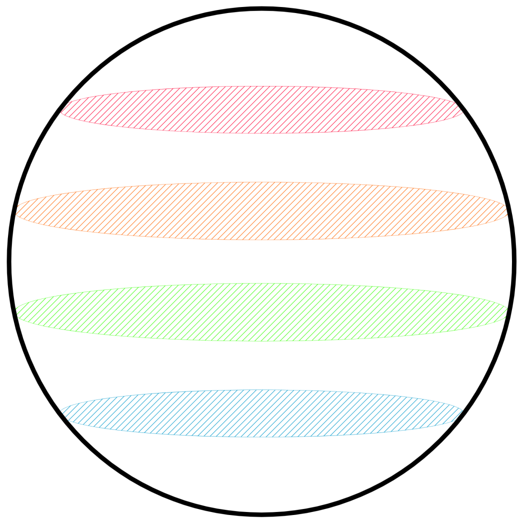
Problem. Explain why E and O differ by at most 1 (for any n).



Hint. Above is an illustration of how a diagram with an odd number of rows can be converted into one with an even number of rows, or vice versa: by pouring the right diagonal into the last row, or conversely scooping the last row into the right diagonal (depending on which has more dots). But this procedure doesn't always work though... Try lots of examples to see! (Note the term "pentagonal" is not a hint and does not refer to diagram shapes.)

Washers in Balance 19

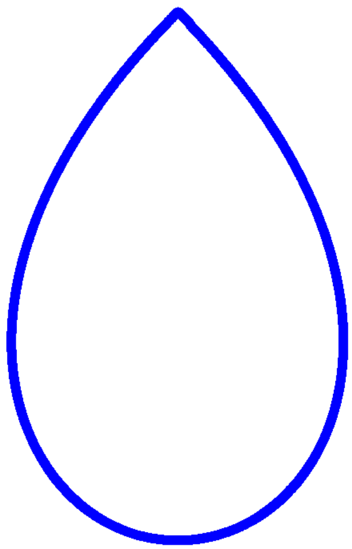
A sphere of volume Cn (where C is a constant to be determined) is sliced (along parallel horizontal planes) into n pieces of equal height.



Let P_n be the product of their volumes.

Problem. What value of C is necessary for $\lim\limits_{n\to\infty} P_n$ to exist (and $\neq 0$)?

Cusp of Crying 20



Problem. Find the angle at the top corner of the teardrop curve defined by the equation $r = e^{y-1}$ (written with the polar coordinate $r = \sqrt{x^2 + y^2}$).

Polarization 21

Suppose $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ is a real-valued function of four 3D vectors satisfying:

- *Multilinearity.* If any three of the arguments are held fixed then ϕ is a linear function of the fourth argument, for example

$$\phi(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \phi(\mathbf{a}_1, \mathbf{b}, \mathbf{c}, \mathbf{d}) + \phi(\mathbf{a}_2, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

- *Antisymmetry.* Swapping the first or second pair changes the sign:

$$\phi(\mathbf{b}, \mathbf{a}, \mathbf{c}, \mathbf{d}) = -\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \phi(\mathbf{a}, \mathbf{b}, \mathbf{d}, \mathbf{c})$$

- *Symmetry.* Swapping the first pair with the second doesn't change it:

$$\phi(\mathbf{c}, \mathbf{d}, \mathbf{a}, \mathbf{b}) = \phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$$

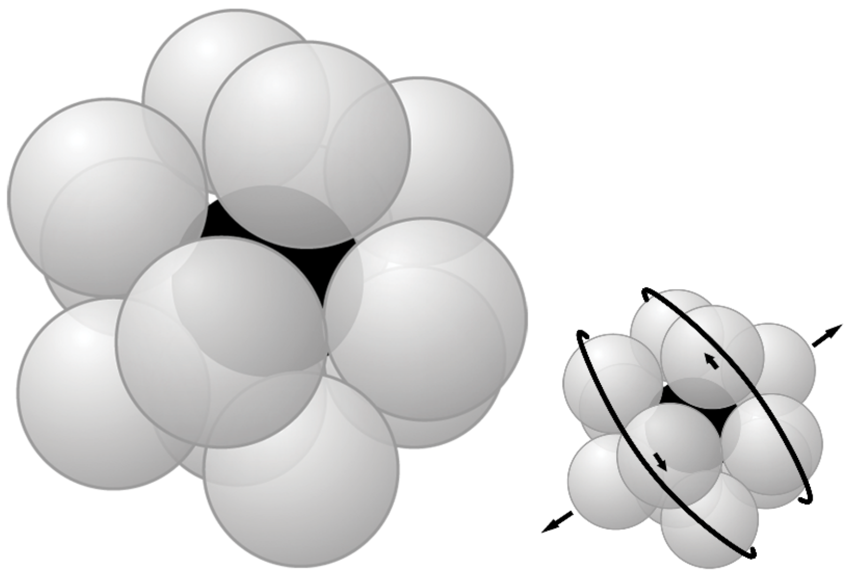
- *Vanishing.* If the first pair equals the second pair, the result is zero:

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{a}, \mathbf{b}) = 0$$

Problem. Show $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = 0$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$.

Sporadic Twists 22

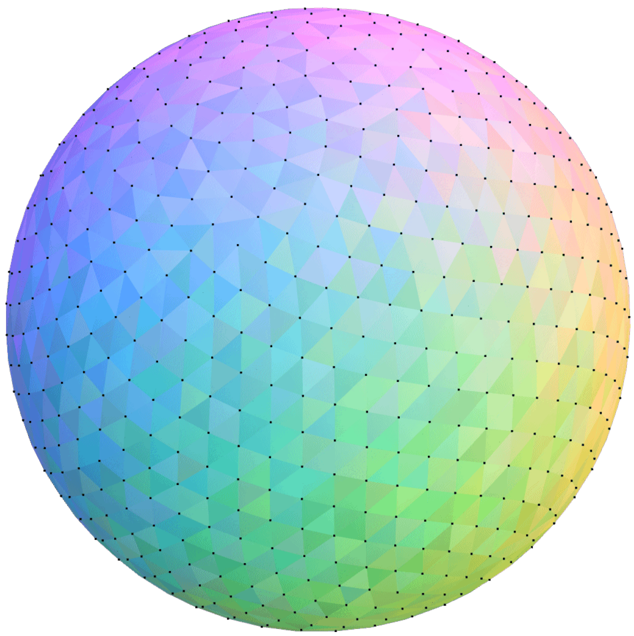
Twelve unit spheres are situated around a central unit ball with wiggle room between them. Each sphere is surrounded by a ring of five other spheres, followed by another ring of five beyond that, and then one final sphere on the opposite side. (In other words, arranged as an icosahedron.)



A **twist** consists of simultaneously rotating one such ring of five spheres one direction and the adjacent ring of five spheres the other direction (equally).

Problem. Show it is possible to move any pair of spheres to (the current positions of) any other pair of spheres with a sequence of twists.

Striking Gold 23



(Patel, 2018)

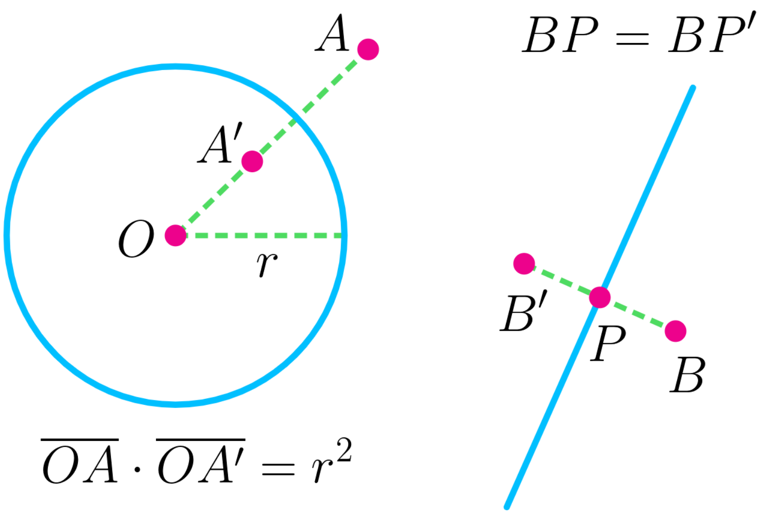
Problem. Find a spherical triangulation whose chromatic polynomial has a root x within 0.1 of the value 2.6. (Try out some small triangulations!)

The **chromatic polynomial** $P(x)$ of a graph (such as a triangulation) counts the number of ways to color the vertices of the graph so that no adjacent vertices are the same color, with x colors available to choose from. It is always a polynomial, so non-whole values of x can be plugged into it.

Look up how to calculate chromatic polynomials. Or, if you know a specific triangulation by name, you may look up its chromatic polynomial...

Celestial Shifting 24

Circle inversions swap the inside and outside of circles by flipping points across them: a point A and its image A' lie on a ray at reciprocal lengths (scaled with the circle's size) from the center. In this context, lines are considered infinitely large circles, and inversions across lines are just reflections.



Look up "inversive geometry" for more information on inversions.

Problem. Show that for any two pairs of points, there is a sequence of inversions which transforms the first pair of points into the second pair of points.

Hint. What happens if we compose inversions across two concentric circles, or across two lines (whether parallel or intersecting)?

Conical Conversions ①

Problem. Show the following are inverse functions:

$$S(t) = \int_0^t \frac{\mathrm{d}\tau}{\cosh \tau}, \qquad T(s) = \int_0^s \frac{\mathrm{d}\sigma}{\cos \sigma}.$$

Note hyperbolic cosine is given by $\cosh \tau = (e^\tau + e^{-\tau})/2$.

Homogenization 2

Problem. Suppose a function $f(x)$ is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Find a closed-form expression for

$$\sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{k+\ell+m+n} w^k x^\ell y^m z^n.$$

in terms of w, x, y, z and f. Assume w, x, y, z are all distinct.

Justify your answer. Ignore issues of convergence.

Hint: Consider a two-variable version first.

Dynamical Billiards ③

Problem. A point particle bounces around inside of the unit square. The particle has a constant speed of one unit per second, and begins in the corner $(0,0)$ at an angle of $\pi/12$ radians from the base. Where will the particle be in exactly one minute? Give an exact answer.

Bonus Credit: Explain how to do the same problem for an equilateral triangle with all sides one unit in length.

Cyberpunk 4

Problem. Express n in binary, where n is the number of trailing zeros in the binary representation of $1^1 \cdot 2^2 \cdot 3^3 \cdots 2049^{2049}$. No electronic calculations of any kind will count for credit, only hand calculations.

Hyperdiamond ⑤

In four dimensions, $\mathbb{R}^4$, pick 24 points a distance of 1 from $(0,0,0,0)$: the permutations of $(\pm 1, 0, 0, 0)$ and $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)$. "Draw" an edge between any two of these points which are 1 unit apart, as per the distance formula (generalized from two and three dimensions). It may help to draw five 3D cross-sections of this figure, corresponding to what the first coordinate is, and then somehow record which points are connected to which.

Problem. For increasing levels of credit:

(a) Give an example of a regular hexagon in this picture: list six of the vertices that are the correct distance of 1 and angle of $60°$ apart.

(b) Find the number of these regular hexagons, with explanation.

(c) Call a bundle any set of four of these regular hexagons which share no vertices or edges. Give an example of a bundle.

(*Bonus Credit.*) Find the number of bundles, with explanation.

Hints. Angles may be computed with the dot product just as they can in three dimensions. The most efficient calculations and readable explanations will use the phrase "by symmetry" multiple times! No symmetry argument will need justification, only correct usage.

Regularization 6

Problem. Find, with explanation, meaningful values for:

(a) $1 + 2 + 3 + 4 + \cdots$

(b) $1^2 + 2^2 + 3^2 + 4^2 + \cdots$

(c) $1^3 + 2^3 + 3^3 + 4^3 + \cdots$

Solutions ought to use ideas described in the appendix section.

Regularization: Addendum

Addition is not at first defined for infinitely many summands, it is eventually defined as a limit of partial sums. On this interpretation, there are series which don't have values (i.e. the limits don't exist), and these series are said to diverge. There many kinds of divergence, or even strange kinds of convergence. Consider the following examples:

- $1 + 2 + 3 + 4 + \cdots$: diverges to $+\infty$, as do the individual terms.
- $1-2+3-4+\cdots$: diverges, but not to $+\infty$ or $-\infty$; both the individual terms and the partial sums are unbounded but oscillate sign.
- $1 + 1 + 1 + 1 + \cdots$: diverges to $+\infty$, but the terms are bounded.
- $1-1+1-1+\cdots$: diverges, but the terms and partial sums are bounded, and if regrouped to $(1-1)+(1-1)+\cdots$ it converges.
- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges to $+\infty$, but slowly: the terms tend to 0 while nth partial sum is approximately $\ln n$.
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges conditionally to $\ln 2$; permuting infinitely many terms can cause it to converge to potentially *any* other real number (this follows from the Riemann rearrangement theorem).
- $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$ converges to $\frac{\pi^2}{6}$ (the solution to the Basel problem); it is an example of a p-series and is a particular value of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

If $\mathcal{S}$ denotes the set of all infinite sequences of real numbers and $\mathcal{C}$ the set of all sequences whose corresponding series converges, then $\mathcal{S}$ is a real vector space and $\mathcal{C}$ is a subspace of $\mathcal{S}$. The summation operator Σ which outputs the limit of partial sums of a sequence is a linear map $\mathcal{C} \to \mathbb{R}$.

An extension of Σ to a linear map $\mathcal{V} \to \mathbb{R}$ on a larger subspace $\mathcal{V}$ of $\mathcal{S}$ is called a **summability method**, it allows us to assign finite values to divergent series. The axiom of choice implies summability methods exist, but it doesn't guarantee they're anything but arbitrary number assignments. Fortunately, there are methods of interest.

Summability methods are a subset of a broader process of **regularization**, which resolves infinities out of calculations - not just sums, but also products, functional integrals (called Feynman path integrals in quantum theory), or other expressions to get meaningful results, even bridging the gap between theoretical calculations and physical observations, as in the Casimir effect.

The summability method we will describe let's call **baby-zeta** regularization. The grownup-zeta regularization would require exploring Dirichlet series, which we will substitute with Taylor series for simplicity. When grownup-zeta and baby-zeta regularization both work, they give the same result, but baby-zeta won't work as often and grownup-zeta won't behave as nicely.

Suppose $\sum_{n=1}^{\infty} a_n$ is an infinite series for which $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is true near $x = 1$ within a positive radius of convergence for some nice (i.e. analytic) function $f(x)$. Then $f(1)$, if it exists, is called the baby-zeta regularized value of the series. For example, consider the geometric series

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots$$

While this power series only converges for $|x| < 1$, the function on the left is defined for all values except $x = -1$, so setting $x = 1$ gives

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots$$

Heuristically, if we split it into $(1 + 0 + 1 + 0 + \cdots) - (0 + 1 + 0 + 1 + \cdots)$, then delete the 0s to get $(1+1+1+1+\cdots) - (1+1+1+1+\cdots)$ we should expect 0, which is a contradiction. This may resolved two different ways:

- Grown-up zeta does assign $1 + 1 + \cdots$ the value $-\frac{1}{2}$, but does not allow infinitely many 0s to be introduced or discarded.
- Baby zeta allows introducing/discarding 0s arbitrarily, but does not assign $1 + 1 + \cdots$ any value so there is no contradiction.

This unlocks new tools to define the zeta regularization besides simply analyzing $f(x)$ directly. Both grown-up and baby zeta regularization are linear:

$$\sum_{n=1}^{\infty}(a_n + b_n) = \left(\sum_{n=1}^{\infty} a_n\right) + \left(\sum_{n=1}^{\infty} b_n\right), \quad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

Introducing 0s between terms does not affect a baby zeta value, even infinitely many 0s. This means even if $f(x) = \sum_{n=1}^{\infty} a_n x^n$ does not have $f(1)$ defined, $\sum_{n=1}^{\infty} a_n$ may still potentially possess a zeta regularized value after all.

Integrating or differentiating the aforementioned geometric series gives

$$\ln\left(\frac{1}{1+x}\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

These are altered versions of the Mercator series and Newton-binomial series, respectively. Plugging $x = 1$ into the first gives the aforementioned series for $\ln 2$, although that one is already convergent. Might the second be of use?

Squared Cubic Roots 7

Problem. Find the polynomial whose roots are the squares of the roots of $T^3 + aT^2 + bT + c$ (counted with multiplicity if necessary) in terms of a, b, c.

Sphere Gears 8

Suppose two balls are rotating against each other without slipping. The line segment joining their centers makes acute angles θ_1 and θ_2 with their axes of rotation. Their angular velocities are ω_1 and ω_2.

(Angular velocities could be in e.g. revolutions or radians per time; units won't matter. Do not assume the balls have equal radii.)

Problem. Show the axes of rotation intersect at a point, and moreover if that point lies in the unique plane separating the two balls then

$$\omega_1 \cos\theta_1 = \omega_2 \cos\theta_2.$$

Circular Cocycle 9

Problem. Find, with explanation, a third order differential equation whose general solution on any open interval is $y = (ax + b)/(cx + d)$, where a, b, c, d are constants (and $-d/c$, if defined, is not in the interval).

Local Linear Fraction 10

Problem. Find, with explanation, the function of the form $(ax+b)/(cx+d)$ which best approximates an arbitrary twice-differentiable function $f(x)$ near a point $x = w$. For example, you may express a, b, c, d in terms of the values $f(w), f'(w), f''(w)$. Your function does not need to be written precisely in the given form, though it must be equivalent.

(For comparison, the best function of the form $ax + b$ is the first two terms $f(w) + f'(w)(x - w)$ of f's Taylor series around $x = w$.)

Lattice Chasing

11

Problem. What is the maximum number of distinct sets that can be created using unions, intersections, and three sets A, B, C? (So for instance, A, $A \cup B$, and $(A \cup B) \cap C$ are all counted.) Explain.

Involutive Units 12

Call a whole number n **tight** if it is a factor of $x^2 - 1$ for all integers x which are relatively prime to n (meaning, share no factor with n other than ± 1). For example, 2 is a factor of $x^2 - 1$ for all odd numbers x, so 2 is tight.

Problem. Find, with proof, all tight numbers.

Suggestion. (i) Consider the smallest prime which is not a factor of n, and (ii) prove the lemma $p_1 p_2 \cdots p_k > p_{k+1}^2$ (for sufficiently large k, where p_k is the kth prime) by induction using Bertrand's postulate.

(The postulate states there is always a prime between m and $2m$.)

Golden Architecture 13

In 2D, figures may be constructed using the abstract tools of *compass* and *straightedge*. The compass allows one to draw a circle around any known center through any known point, and the straightedge allows one to draw a line between any two known points. Intersection points which arise from lines and circles automatically become known. Points may also be chosen arbitrarily in space or on a line or circle, but one cannot assume anything else about such points when choosing them.

In 3D, figures shall be constructed using *astrolabe* and *flatedge*. The astrolabe allows us to construct a sphere around any known center through a given point, and the flatedge allows us construct the plane through any three noncollinear points, or the line between any two.

Problem. Explain how to construct a regular icosahedron.

Hint: the icosahedron inscribes three orthogonal golden rectangles.

Partial credit available for constructing (a) three perpendicular lines through a known point or (b) a golden rectangle with a known center.

Isoepiareal Ratio 14

The **isoperimetric quotient** of a simple closed loop in the plane is the ratio A/P^2, where A is the area enclosed and P the perimeter. It is maximized only when the loop is as symmetric as possible, i.e. a perfect circle. Among rectangles, squares have maximal quotient.

Consider the **isoepiareal ratio** V^2/S^3 for closed surfaces in three dimensions, where V is the volume enclosed and S the surface area.

Problem. Prove cubes have maximal isoepiareal ratio among cuboids.

(A cuboid is a right rectangular prism, i.e. ... a box.)

Superexponential 15

Define the k-times iterated logarithm $\ln^k x = \overbrace{\ln \cdots \ln}^{k} x$.

Problem. *Prove or disprove* the claim that there exist sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \left[\ln^k a_n - \ln^k b_n\right] = +\infty$ for all k.

Quadric Query 16

Define p_{ij} to be one of the six minors of a matrix M:

$$M = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}, \quad p_{ij} = \det \begin{bmatrix} x_i & x_j \\ y_i & y_j \end{bmatrix}.$$

Problem. Find a (nontrivial) polynomial equation valid for all M:

$$Q(p_{12}, p_{13}, p_{14}, p_{23}, p_{34}, p_{24}) = 0.$$

Hint. The expression Q involves each p_{ij} exactly once.

Blade Angle 17

Suppose two planes (in 3D space) intersect at an acute angle ϕ and are spanned by pairs of vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ respectively.

Problem. Show $\cos\phi$ may be expressed in terms of the ten possible dot products between the four vectors $\mathbf{a}, \mathbf{b}, \mathbf{b}, \mathbf{d}$.

Hint. Consider dot and cross product identities.

*abc*s in the Margin: Solution

Note that the degree is multiplicative: if $f(t)$ and $g(t)$ are polynomials, then

$$\deg\left[f(t)g(t)\right] = \deg f(t) + \deg g(t).$$

This means if $f(t)$ is a factor of $g(t)$ then $\deg f(t) \leq \deg g(t)$.

Assume for the sake of contradiction $a(t)^n + b(t)^n = c(t)^n$ for nonconstant coprime polynomials $a(t), b(t), c(t)$. (If they weren't coprime, we could divide the equation to reduce to the case where they are coprime or constant.) The powers $a(t)^n, b(t)^n, c(t)^n$ must then also be coprime.

Then the *abc* theorem applies to the polynomials $a(t)^n, b(t)^n, c(t)^n$:

$$\max\{\deg a(t)^n, \deg b(t)^n, \deg c(t)^n\} < \deg \operatorname{rad}\left[a(t)^n b(t)^n c(t)^n\right].$$

Since $f(t)^n$ has the same irreducible factors as $f(t)$, the radical on the right-hand side is unaffected by the power n. By multiplicativity, however, the left-hand side is affected, since $\deg f(t)^n = n \deg f(t)$ for each polynomial:

$$n \max\{\deg a(t), \deg b(t), \deg c(t)\} < \deg \operatorname{rad}\left[a(t)b(t)c(t)\right]$$

The assumption $n > 2$ implies

$$\begin{aligned}
& n \max\{\deg a(t), \deg b(t), \deg c(t)\} \\
\geq\ & 3 \max\{\deg a(t), \deg b(t), \deg c(t)\} \\
\geq\ & \deg a(t) + \deg b(t) + \deg c(t) \\
=\ & \deg\left[a(t)b(t)c(t)\right].
\end{aligned}$$

But putting this inequality together with the last one yields

$$\deg\left[a(t)b(t)c(t)\right] < \deg \operatorname{rad}\left[a(t)b(t)c(t)\right],$$

which is impossible because $\operatorname{rad}\left[a(t)b(t)c(t)\right]$ is a factor of $a(t)b(t)c(t)$.

The ***abc* conjecture** is actually about integers, not polynomials. It says, effectively, that for positive coprime integers (a, b, c) satisfying $a + b = c$, the value c rarely exceeds the radical $\operatorname{rad} abc$ by much. (The radical of an integer is the product of its prime factors, for example $\operatorname{rad} 24 = 6$.) More precisely, it says no matter how small $\varepsilon > 0$ is, there are only finitely many exceptions to the inequality $c < (\operatorname{rad} abc)^{1+\varepsilon}$. The version of *abc* for polynomials instead of integers is called the Mason-Stothers theorem and has a quick, (relatively) simple proof using Wronskians.

The *abc* conjecture has numerous implications in number theory, one being an alternate proof **Fermat's Last Theorem**, which says for $n > 2$ there are no nontrivial integer solutions (a, b, c) to $a^n + b^n = c^n$. This was written by Fermat (found by his son in the margin of his copy of *Arithmetic*, a 3rd century book by Diophantus about exactly these kinds of equations, now called Diophantine equations), famously adding "I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain."

While it's doubtful Fermat really had a proof, nonetheless, the mathematical community's subsequent quest for a proof is oft-touted as the birth of algebraic number theory. The first valid proof appeared three-and-a-half centuries later in the mid-90s by Andrew Wiles, by linking it to and then proving (a narrow version of) the Taniyama-Shimura conjecture, now called the **modularity theorem**, which asserts a rational correspondence between rational elliptic curves and classical modular curves.

This problem highlights similarities between integers and polynomials. Both admit factorizations into primes/irreducibles. Long division with quotients and remainders is possible for both. Relative size can be measured by absolute value or degrees. Even partial fraction decompositions are possible for rational numbers just as they are for rational functions. And as we've seen, both contexts have versions of the *abc* theorem, Fermat's Last Theorem, and many other theorems. When we use finite fields for polynomial coefficients this observation is called the **function field analogy**.

Alfred's Ansatz: Solution

Rewrite $2^{62}+1$ as $4x^4+1$ with $x=2^{15}$. The polynomial $4x^4+1$ has no real root, since x^4 is can't be negative, so it cannot have a linear factor, and thus instead must have only quadratic factors (if it factors at all). Indeed, it must be a product of two quadratics. Their leading terms are either both $2x^2$ or else one is x^2 and the other is $4x^2$. Let's look at the first case and if that doesn't pan out explore the second. Write out the factorization

$$4x^4+1=(2x^2+ax+b)(2x^2+cx+d).$$

When expanded out and like terms collected, the right-hand side becomes

$$4x^4+2(a+c)x^3+(2b+2d+ac)x^2+(ad+bc)x+bd.$$

The coefficient of x^3 must be 0, so $c=-a$. The constant coefficient must be 1, so b and d are either both 1 or both -1 (in particular, $b=d$). We can now substitute $-a$ for c and b for d so there are only two unknowns. The coefficient of x becomes $ab-ba$ which is automatically 0. The coefficient of x^2 is now $4b-a^2$, which must be 0, so $a^2=4b$. This forces b to be nonnegative, so $b=1$ and $a=\pm 2$. Putting this all together we get the factorizations

$$4x^4+1=(2x^2+2x+1)(2x^2-2x+1),$$

$$2^{62}+1=(2^{31}+2^{16}+1)(2^{31}-2^{16}+1).$$

Each of the two factors above would work.

In general, the polynomial x^n-1 (and by extension, x^n+1) can be factored into the irreducible *cyclotomic polynomials* $\Phi_n(x)$. These can be "factored" further in a different sort of way as $\Phi(x)=U(x)^2\pm\bigcirc V(x)^2$ for polynomials $U(x),V(x)$ and monomial $\bigcirc$ depending on n (due to Lucas, Gauss, Schinzel).

This allows us to factor $\Phi_n(x)$ as an integer for particular values of x, which Aurifeuille (pseudonym Alfred de Caston) did in the case of

$$\begin{aligned}2^{2(2n+1)}+1&=(2^{2n+1}+1)^2-(2^{n+1})^2\\&=(2^{2n+1}+2^{n+1}+1)(2^{2n+1}-2^{n+1}+1).\end{aligned}$$

for $n=14$ (our case being $n=15$).

Anharmonic Asymmetry: Solution

In **cycle notation**, $(a_1a_2\cdots a_k)$ represents the permutation which cycles the elements $a_1,\cdots,a_k$ of a set (in that order). For example, for a set $\{1,2,3,4\}$, the permutation (123) is the function ρ defined by the input/output pairs

$$\rho(1)=2,\quad \rho(2)=3,\quad \rho(3)=1,\quad \rho(4)=4.$$

(Any element x not listed in the cycle notation is fixed, i.e. $f(x)=x$.)

There are six permutations of $\{1,2,3\}$, listed below with probabilities:

$$\begin{array}{cccccc} (1) & (23) & (31) & (12) & (321) & (123) \\ Z & A & B & C & X & Y \end{array}$$

When drawing a permutation from S_3, the probability it sends $i\mapsto j$ is the sum of the probabilities associated with every permutation that sends $i\mapsto j$. This gives us a 3×3 table of probabilities:

$\nearrow$	1	2	3
1	$A+Z$	$C+Y$	$B+X$
2	$C+X$	$B+Z$	$A+Y$
3	$B+Y$	$A+X$	$C+Z$

The ij entry is the probability of $i\mapsto j$. Since there is a 100% chance 1 is sent to one of $1,2,3$ we may conclude the first row sums to 1; similarly there is a 100% chance one of $1,2,3$ is sent to 1 so the first column also sums to 1. The same is true for all rows and columns, so the table is a so-called doubly stochastic matrix. In particular, all its entries ought to be $1/3$.

Setting the entries equal to each other results in many of the variables being equal. For instance $A+X=A+Y=A+Z$ implies $X=Y=Z$ and $A+X=B+X=C+X$ implies $A=B=C$. Conversely, so long as A,B,C are equal and X,Y,Z are equal, the table's entries are all the same.

Thus, to get a nonuniform distribution on S_3 for which $i\mapsto j$ is equally likely for all pairs i,j it suffices to pick any solution of $A+X=\frac{1}{3}$ for which $A,X\geq 0$ and $A\neq X$, so such a distribution is possible.

Arithmetic Jenga: Solution

Suppose n has the desired property. This means, in particular, for every prime factor p of n, the number $p-1$ must be noncomposite. This is true for $p=2$, but for $p>2$ the number $p-1$ is even, so unless $p-1=2$ that would mean $p-1$ is composite, a contradiction. Therefore, the only primes that may appear in n's prime factorization are 2 and 3.

Write $n=2^a3^b$. Each of $2-1, 2^2-1, \cdots, 2^a-1$ must be noncomposite. We can check that $2-1, 2^2-1, 2^3-1$ are noncomposite but $2^4-1=3\cdot 5$ is composite, so $a\leq 3$. Similarly, $3-1$ is noncomposite but 3^2-1 is composite, so $b\leq 1$. The largest candidate is $n=2^3\cdot 3=24$. We can check:

d	24	12	8	6	4	3	2	1
$d-1$	23	11	7	5	3	2	1	0

By inspection we find that for every positive divisor d of 24, the number $d-1$ is noncomposite, so $n=24$ has the desired property. Similarly, every other number of the form $n=2^a3^b$ with $a\leq 3$ and $b\leq 1$ has the property as well, and these are precisely all the divisors of 24 listed above.

Arts and Crafts: Solution

For a given thread (associated to a value t), we may calculate its slope

$$m = \frac{y-t}{x-0} = \frac{0-t}{(1-t)-0}$$

Solving for y yields the formula

$$y \;=\; t - \frac{tx}{t-1} \;=\; t + x + \frac{x}{t-1}.$$

Using Algebra. We may see for which points (x, y) (in the unit square, $0 \leq x, y \leq 1$) there exists a solution t (in the interval $0 \leq t \leq 1$). Multiplying by $(t-1)$ and rearranging gives $t^2 + (x-y+1)t + y = 0$, with solution

$$t = \frac{(y-x+1) \pm \sqrt{(y-x+1)^2 - 4y}}{2}$$

Denote $b = y - x + 1$ and $\Delta = b^2 - 4y$. Since $0 \leq x, y \leq 1$, we know $b \geq 0$. Suppose $\Delta \geq 0$. Then, since $\Delta \leq b^2$, we know $\sqrt{\Delta} \leq b$, and therefore

$$0 \;\leq\; \frac{b-\sqrt{\Delta}}{2} \;\leq\; \frac{b}{2} \;\leq\; \frac{1-0+1}{2} \;=\; 1.$$

That is, as long as $0 \leq x, y \leq 1$ and $\Delta \geq 0$, there is a solution $t = (b - \sqrt{\Delta})/2$ in the interval $0 \leq t \leq 1$. In other words, the point (x, y) is on or below some thread. Thus, the curve is defined by the boundary of this inequality, $\Delta = 0$.

Square rooting $b^2 = 4y$ yields $y - x + 1 = 2\sqrt{y}$, then subtracting $2\sqrt{y}$ and adding x, we can factor as $(\sqrt{y} - 1)^2 = x$. Since $y \leq 1$, square rooting again yields $\sqrt{y} - 1 = -\sqrt{x}$. In conclusion, the curve is the so-called **superellipse**

$$\sqrt{x} + \sqrt{y} = 1.$$

Parabola. The curve opens up diagonally, so we ought to see what happens if we rotate our coordinate system by 45°. Let's introduce

$$u = \frac{x-y}{\sqrt{2}}, \quad v = \frac{x+y}{\sqrt{2}},$$

The equation $(y-x+1)^2 = 4y$ becomes $(1-\sqrt{2}u)^2 = 2\sqrt{2}(v-u)$, and then completing the square gets us $\sqrt{2}v = \frac{1}{2} + u^2$. Thus, the curve is a parabola.

With Calculus. Calculate the derivative of y with respect to t:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \frac{x}{(t-1)^2}.$$

The critical point (where the y coordinate is maximized, which is on the curve) occurs where $\mathrm{d}y/\mathrm{d}t = 0$, or in other words $(t-1)^2 = x$. Since we want $0 \le t \le 1$, this yields the solution $t = 1-\sqrt{x}$, and substituting back in and factoring yields $y = (1-\sqrt{x})^2$, or once again $\sqrt{x}+\sqrt{y} = 1$.

Bipolarity: Solution

Law of Cosines says $2^2 = d_1^2 + d_2^2 - 2d_1 d_2 \cos\sigma$.

Law of Sines says $\dfrac{\sin\sigma}{2} = \dfrac{(y/d_1)}{d_2} = \dfrac{(y/d_2)}{d_1}$.

Therefore $d_1 d_2 \cos\sigma = r^2 - 1$ and $d_1 d_2 \sin\sigma = 2y$. Plus, we have

$$\begin{cases} d_1^2 = (x+1)^2 + y^2 \\ d_2^2 = (x-1)^2 + y^2 \end{cases} \implies \begin{cases} d_1^2 + d_2^2 = 2(r^2+1) \\ d_1^2 - d_2^2 = 4x \end{cases}$$

Now compute $f(\tau)/g(\sigma)$ as follows:

$$\begin{aligned}
\frac{\cosh\tau}{\cos\sigma} &= \frac{\left(\dfrac{d_1}{d_2} + \dfrac{d_2}{d_1}\right)}{2\cos\sigma} &= \frac{d_1^2 + d_2^2}{2d_1 d_2 \cos\sigma} &= \frac{r^2+1}{r^2-1}, \\
\frac{\sinh\tau}{\cos\sigma} &= \frac{\left(\dfrac{d_1}{d_2} - \dfrac{d_2}{d_1}\right)}{2\cos\sigma} &= \frac{d_1^2 - d_2^2}{2d_1 d_2 \cos\sigma} &= \frac{2x}{r^2-1}, \\
\frac{\cosh\tau}{\sin\sigma} &= \frac{\left(\dfrac{d_1}{d_2} + \dfrac{d_2}{d_1}\right)}{2\sin\sigma} &= \frac{d_1^2 + d_2^2}{2d_1 d_2 \sin\sigma} &= \frac{r^2+1}{2y}, \\
\frac{\sinh\tau}{\sin\sigma} &= \frac{\left(\dfrac{d_1}{d_2} - \dfrac{d_2}{d_1}\right)}{2\sin\sigma} &= \frac{d_1^2 - d_2^2}{2d_1 d_2 \sin\sigma} &= \frac{x}{y}.
\end{aligned}$$

We also get $\tan\sigma = \dfrac{2y}{r^2-1}$ and $\tanh\tau = \dfrac{2x}{r^2+1}$ from these. Now

$$\begin{aligned}
\frac{\cosh\tau}{\sinh\tau} - \frac{\cos\sigma}{\sinh\tau} &= \frac{r^2+1}{2x} - \frac{r^2-1}{2x} &= \frac{1}{x} \\
\frac{\cosh\tau}{\sin\sigma} - \frac{\cos\sigma}{\sin\sigma} &= \frac{r^2+1}{2y} - \frac{r^2-1}{2y} &= \frac{1}{y}
\end{aligned}$$

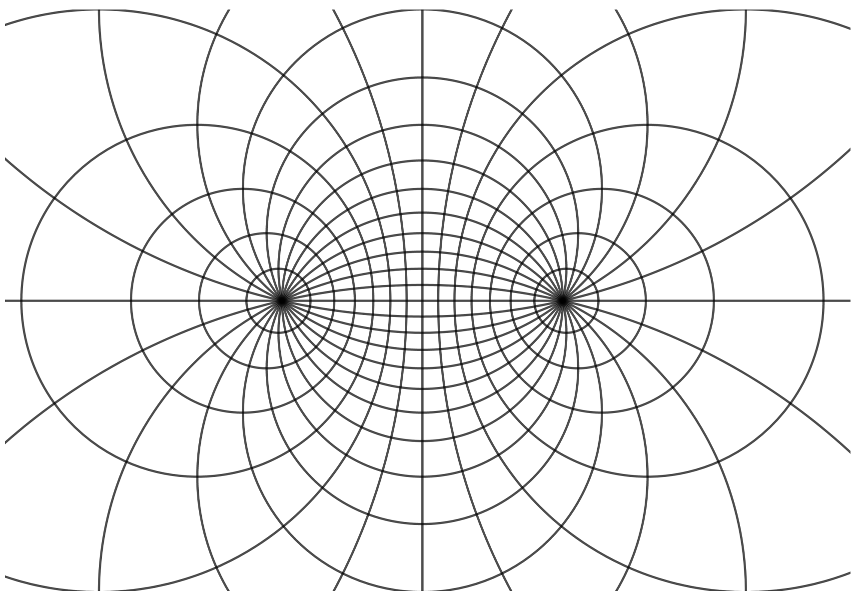
Therefore, $(x, y) = \left(\dfrac{\sinh\tau}{\cosh\tau - \cos\sigma}, \dfrac{\sin\sigma}{\cosh\tau - \cos\sigma} \right)$.

This gives rise to **bipolar coordinates** as follows.

First complete the square from the formulas for $\tan\sigma$ and $\tanh\tau$:

$$\begin{array}{c|c} \tan\sigma = \dfrac{2y}{r^2-1} & \tanh\tau = \dfrac{2x}{r^2+1} \\ x^2+y^2-2(\cot\sigma)y = 1 & x^2-2(\coth\tau)x+y^2 = -1 \\ x^2+(y-\cot\sigma)^2 = \csc^2\sigma & (x-\coth\tau)^2+y^2 = \operatorname{csch}^2\tau \end{array}$$

The circles of constant σ (whose centers are along the x-axis) and the circles of constant τ (whose centers are along the y-axis) are below:



This is what we get when we stereographically project longitude and latitude coordinates from the sphere to a plane from a point on the equator!

These are **circles of Apollonius**. The two kinds of circles are orthogonal families of circles, and every circle of constant σ is the locus of points with a constant ratio of distances to the two foci (aka poles), namely $d_1/d_2 = \exp\tau$.

Blade Angle: Solution

For comparison, recall the dot product satisfies $\mathbf{n}_1 \cdot \mathbf{n}_2 = \|\mathbf{n}_1\|\|\mathbf{n}_2\| \cos\theta$, where θ is the angle between two vectors, so $\cos\theta$ may be expressed as:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\sqrt{(\mathbf{n}_1 \cdot \mathbf{n}_1)(\mathbf{n}_2 \cdot \mathbf{n}_2)}}$$

Now suppose two planes Π_1 and Π_2 (spanned by pairs $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}, \mathbf{d}$ respectively) intersect in a line perpendicular to a third plane Π. The so-called **dihedral angle** between Π_1 and Π_2 is ϕ (chosen to not be obtuse).

In the 2D plane Π we can see the two normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ of the first two planes Π_1 and Π_2 intersecting at an angle of either ϕ or else its supplement, depending on which normal vectors are chosen. This choice will only affect the sign of $\cos\phi$, so we may as well use any choice of normal vectors and take the absolute value.

To this end, normalize the cross products $\mathbf{a} \times \mathbf{b}$, $\mathbf{c} \times \mathbf{d}$ for $\mathbf{n}_1$, $\mathbf{n}_2$ in the formula $\cos\phi = \mathbf{n}_1 \cdot \mathbf{n}_2$, and also use the Binet-Cauchy identity:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) \quad = \quad (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

and special case $\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\|\mathbf{b}\|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ (Lagrange's Identity).

This gives us the answer

$$\left| \frac{\mathbf{a} \times \mathbf{b}}{\|\mathbf{a} \times \mathbf{b}\|} \cdot \frac{\mathbf{c} \times \mathbf{d}}{\|\mathbf{c} \times \mathbf{d}\|} \right| = \frac{|(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})|}{\sqrt{\|\mathbf{a} \times \mathbf{b}\|^2 \|\mathbf{c} \times \mathbf{d}\|^2}}$$

$$= \quad \frac{|(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})|}{\sqrt{\big((\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2\big)\big((\mathbf{c} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{d}) - (\mathbf{c} \cdot \mathbf{d})^2\big)}}.$$

All ten possible dot products of $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ make an appearance.

Blinding Sphere: Solution

We wish to integrate the light intensity $\|\mathbf{r}-\mathbf{p}\|^{-2}$ as $\mathbf{r}$ varies over the surface of the unit sphere (centered at $\mathbf{0}$). Spherical coordinates are given by

$$\mathbf{r}(\phi,\theta) = \begin{bmatrix} \cos\theta\sin\phi \\ \sin\theta\sin\phi \\ \cos\phi \end{bmatrix}$$

This is the math convention, with ϕ the polar angle and θ the azimuthal angle; the physics convention is reversed. By symmetry, we can pick $\mathbf{p} = (0,0,p)$.

For the integral, our surface area element is given by

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sin\phi.$$

Chugging through the calculations we journey forth:

$$\begin{aligned}
\frac{1}{\operatorname{area}(S)} \oiint_S \frac{\mathrm{d}A}{\|\mathbf{r}-\mathbf{p}\|^2} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta}{(\cos\phi - p)^2 + \sin^2\phi} \\
= \frac{2\pi}{4\pi}\int_0^{\pi} \frac{-\mathrm{d}(\cos\phi)}{(\cos\phi-p)^2 + 1 - \cos^2\phi} &= \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}t}{(t-p)^2+1-t^2} \\
= \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}t}{(1+p^2)-(2p)t} &= \frac{1}{2}\left[-\frac{1}{2p}\ln\Big((1+p^2)-(2p)t\Big)\right]_{-1}^{1} \\
= -\frac{1}{4p}\ln\left(\frac{1+p^2-2p}{1+p^2+2p}\right) = \frac{1}{2p}\ln\left|\frac{1+p}{1-p}\right| &= \frac{1}{2p}\ln\left|\coth\left(\frac{1}{2}\ln p\right)\right|.
\end{aligned}$$

The last two expressions are both valid answers, whether $0 < p < 1$ or $p > 1$.

Buckminsterfullerene: Solution

Suppose a polyhedron has P pentagonal and H hexagonal faces. Let V, E, F denote the numbers of vertices, edges and faces it has.

Equation One. Euler's formula says that for convex polyhedra,

$$V - E + F = 2. \tag{1}$$

Equation Two. Every pentagonal face has 5 edges, and every hexagonal face has 6 edges, for a total of $5P + 6H$ edges. However, this double-counts the edges, since there are two faces on either side of each edge, so

$$E = \tfrac{1}{2}(5P + 6H). \tag{2}$$

Equation Three. Each face is either a pentagon or hexagon, so

$$F = P + H. \tag{3}$$

Note equations $(1), (2), (3)$ allow us to solve for V, E, F:

$$\begin{cases} F = P + H, \\ E = \frac{5}{2}P + 3H, \\ V = \frac{3}{2}P + 2H + 2. \end{cases}$$

(Substitute (2) and (3) into (1) to solve for V above.)

Equation Four. Finally, let S be how many space diagonals there are.

The number of line segments joining distinct vertices is the combination ${}_VC_2$, called "V choose 2," also denoted $\binom{V}{2}$. These segments come in three kinds: edges, face diagonals, and space diagonals. Counting by hand, there are 5 face diagonals per pentagon and 9 per hexagon, so $5P + 9H$ face diagonals.

So the fourth equation, from which we can solve for S by substituting, is

$$\binom{V}{2} = E + (5P + 9H) + S. \tag{4}$$

Campus Dash: Solution

Dijkstra's algorithm builds, piece by piece, the quickest paths from BRB to all other buildings. During each pass of the algorithm, each node will have: (i) a minimum-known time from BRB, and (ii) the name of the previous node in a quickest-known path from BRB to it (if applicable).

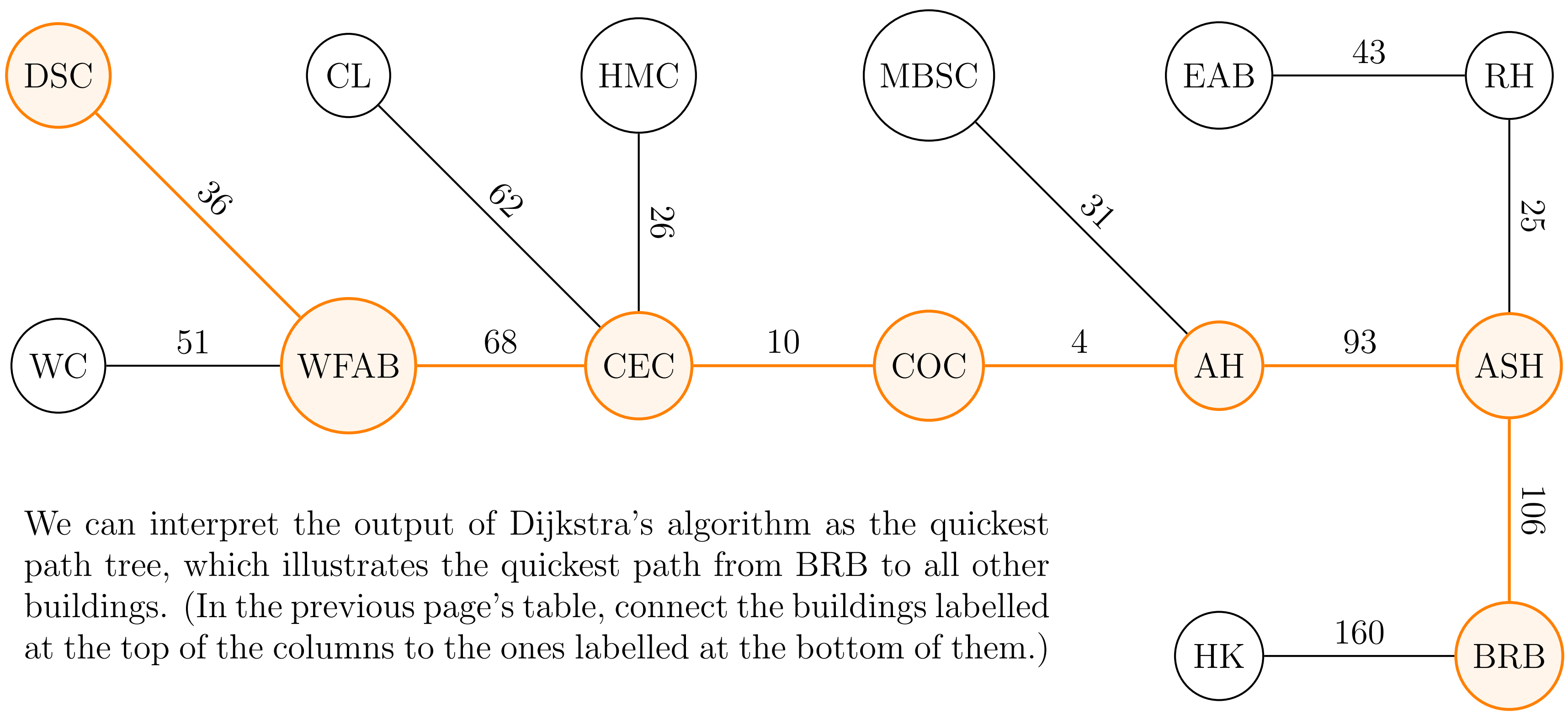
The first pass of the algorithm sets all minimum-known times to ∞, except BRB's minimum-known time is set to 0. Every pass after "visits" a new node N, examining all of its unvisited neighbors U - the pass compares (the current minimum-known time from BRB to U) to (the minimum time from BRB to N plus the time from N to U): if U's current minimum-known time is larger it is replaced and the previous node is updated to N.

Pass	MBSC	COC	EAB	AH	RH	ASH	HK
2	-	-	-	-	-	106 BRB	260 BRB
3	-	-	-	199 ASH	131 ASH		
4	-	-	174 RH				
5	283 EAB	-					
6	230 AH	203 AH					

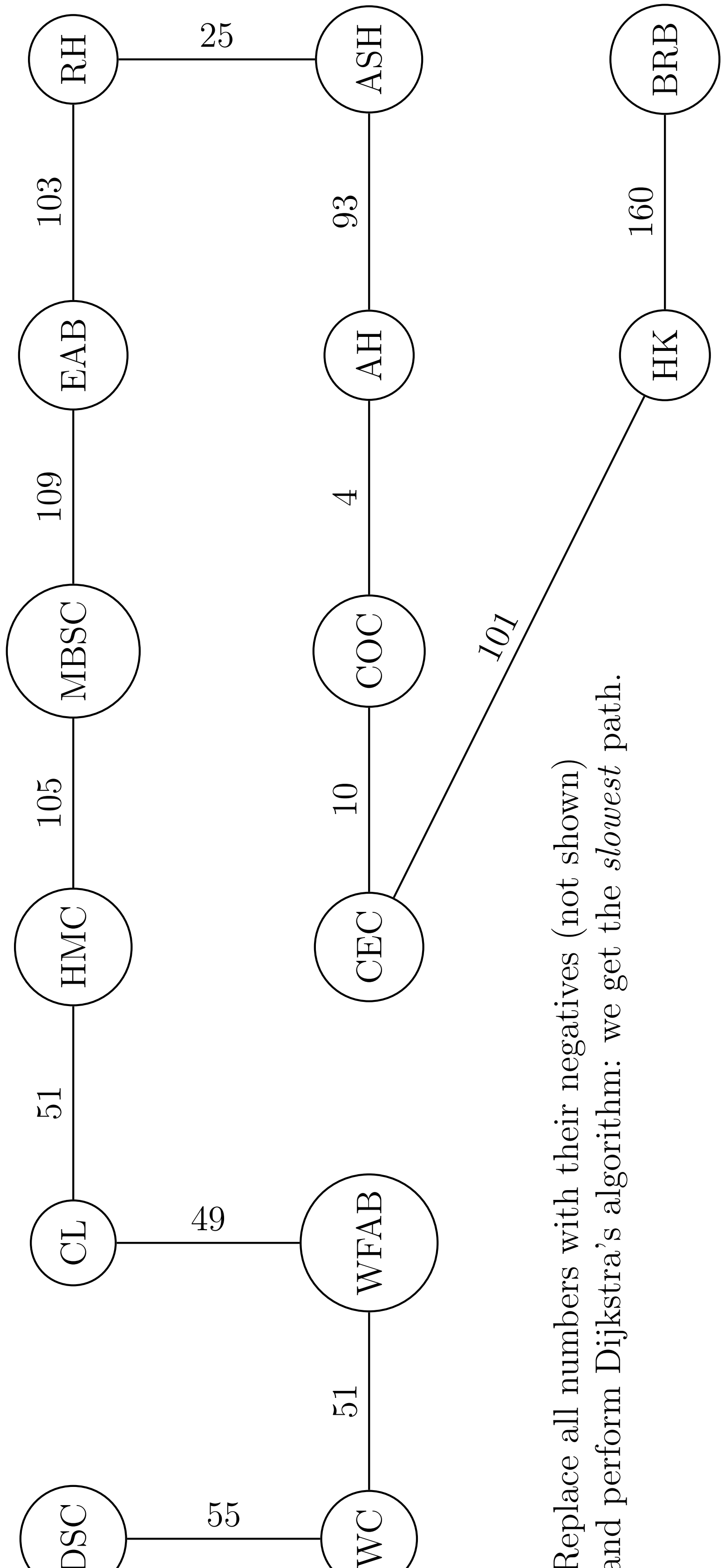
Above is how the first so many passes deal with the buildings closest to BRB. The dashes represent ∞, and whenever a minimum-time is found the rest of the column is left blank. The next passes where updates happen are below:

Pass	DSC	CL	WC	WFAB	HMC	CEC
7	-	-	-	-	-	213 COC
8	-	275 CEC	-	281 CEC	239 CEC	
12	359 CL		-			
13	317 WFAB		332 WFAB			

The quickest path from BRB to DSC, highlighted in orange below, takes 5 minutes and 17 seconds (plus the time it takes to walk through or around the buildings themselves, which we are ignoring).



We can interpret the output of Dijkstra's algorithm as the quickest path tree, which illustrates the quickest path from BRB to all other buildings. (In the previous page's table, connect the buildings labelled at the top of the columns to the ones labelled at the bottom of them.)



Replace all numbers with their negatives (not shown)
and perform Dijkstra's algorithm: we get the *slowest* path.

Categorical Imperative: Solution

First, we can set $a, d = \blacklozenge$ and $c, d = \lozenge$. In our 2D representation, we can either evaluate the expression row-wise to get $\blacklozenge \bullet \blacklozenge$ (which simplifies to $\blacklozenge$), or evaluate the expression column-wise to get $\lozenge \circ \lozenge$ (which simplifies to $\lozenge$):

$$
\begin{array}{ccccc}
\blacklozenge & \circ & \lozenge & & \blacklozenge \\
\bullet & & \bullet & \rightarrow & \bullet \\
\lozenge & \circ & \blacklozenge & & \blacklozenge \\
& \downarrow & & & \\
\lozenge & \circ & \lozenge & &
\end{array}
$$

To emphasize $\lozenge$ and $\blacklozenge$ are the same element, and an identity element for both operations $\circ$ and $\bullet$, we will simply call it $\textcolor{magenta}{\blacklozenge}$. Next, we can evaluate

$$
\begin{array}{ccc}
a & \circ & \textcolor{magenta}{\blacklozenge} \\
\bullet & & \bullet \\
\textcolor{magenta}{\blacklozenge} & \circ & b
\end{array}
$$

either row-wise to get $a \bullet b$ or column-wise to get $a \circ b$. Thus, $a \bullet b = a \circ b$ are the same operation! For this reason, we will now use $\textcolor{cyan}{\bullet}$ instead of $\circ$ or $\bullet$.

Finally, we can evaluate one last expression

$$
\begin{array}{ccc}
\textcolor{magenta}{\blacklozenge} & \textcolor{cyan}{\bullet} & a \\
\textcolor{cyan}{\bullet} & & \textcolor{cyan}{\bullet} \\
b & \textcolor{cyan}{\bullet} & \textcolor{magenta}{\blacklozenge}
\end{array}
$$

either row-wise to get $a \textcolor{cyan}{\bullet} b$ or column-wise to get $b \textcolor{cyan}{\bullet} a$. Thus, $a \textcolor{cyan}{\bullet} b = b \textcolor{cyan}{\bullet} a$, meaning the operation is commutative.

This line of reasoning, called the **Eckman-Hilton argument**, is a style of 2D diagrammatic proof which can be used, for example, to show higher homotopy groups are trivial. An approximately accurate version of this statement in plainer, albeit vaguer, language might read: conjoining together higher-dimensional holes in topological spaces is a commutative operation.

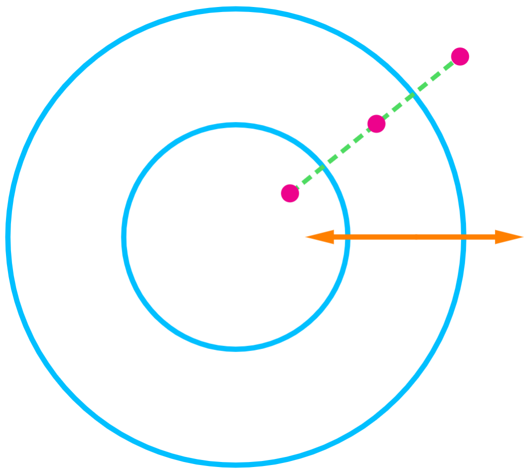
The **interchange law** $(a \circ b) \bullet (c \circ d) = (a \bullet c) \circ (b \bullet d)$ shows up naturally in the context of monoidal 2-categories. But what is a category?

A category is a collection of objects and composable arrows. The most well-known example is the category of sets: the objects are just sets, the arrows are functions. There are categories of topological spaces with continuous functions, lattices with monotone functions, groups with homomorphisms, and categories for numerous other kinds of mathematical objects.

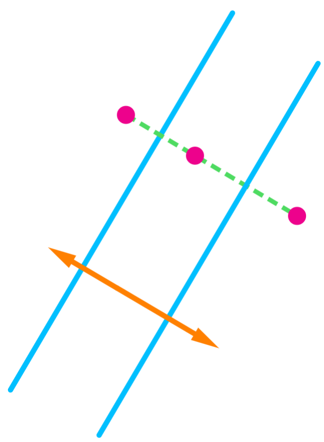
Categories are monoidal when they have an operation to combine objects together (which in turn combines arrows together). A 2-category has not only objects and composable arrows between objects, but composable arrows between those arrows! For instance, a path on a surface may be considered an arrow between endpoints, and then there are homotopies - ways of sliding paths across the surface to turn them into other paths.

Beyond this, there are n-categories, or even ∞-categories. One of the more popular math blogs, the n-Category Cafè, is named after them. Emily Riehl is a co-host of the cafè and has worked on the foundations of ∞-categories (which are hard to define).

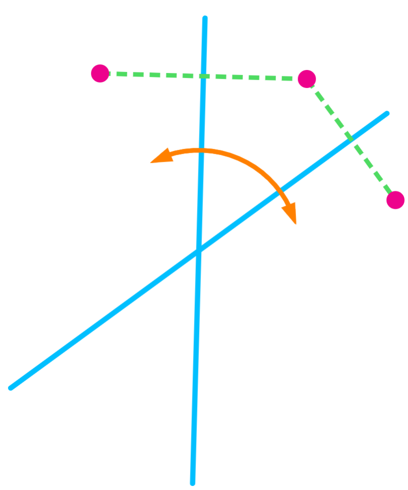
Celestial Shifting: Solution



Concentric Circles



Parallel Lines



Intersecting Lines

Following the hint, we summarize our investigation thusly:

Concentric circles. If the distance of a point from a radius r circle's center is denoted x, the effect of circle inversion is $x \mapsto r^2/x$. Thus, if we compose a circle inversion (radius r) followed by another circle inversion (radius R, concentric), the combined effect is $x \mapsto R^2/(r^2/x)$, or simply $x \mapsto (R/r)^2 x$.

In effect, we can stretch space from any point by any multiplicative factor (or shrink it towards any point, if we use a factor smaller than 1) if we compose inversions across concentric circles with the appropriate radii.

Parallel lines. Composing inversions across two parallel lines has the effect of sliding all points in a direction perpendicular to the lines by twice the distance between the lines.

Intersecting lines. A similar geometric argument shows composing inversions across two intersecting lines has the effect of rotating all points around the point of intersection by twice the angle between the lines.

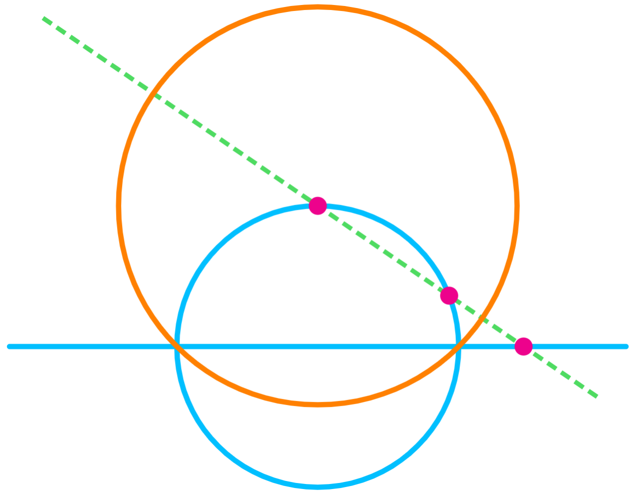
To turn any (distinct) pair of points A and B into any other pair C and D:

- First, apply an expansion or compression (from any central point), using two concentric circles, so the distance $\overline{AB}$ matches $\overline{CD}$.
- Second, apply a translation using two parallel lines, slide until $A = C$.
- Third, apply a rotation using two intersecting lines, to rotate around A (which is now also C) until $B = D$ too.

These three transformations (scale, translate, rotate) can be taken in pretty much any order to achieve the same effect, not just this particular order.

These dilations, translations, and rotations are special cases of **Möbius transformations**, which are complex-valued functions of the form $\frac{az+b}{cz+d}$.

A particular inversion of interest is **stereographic projection**:



Usually this projection is a one-to-one correspondence between points on a line and on a circle (not counting one point on the circle). It is set up by drawing lines through a point on the circle (called the pole) and recording the intersections with the circle and a line (which must be parallel to the circle's tangent line at the pole). This is always just the restriction of an inversion!

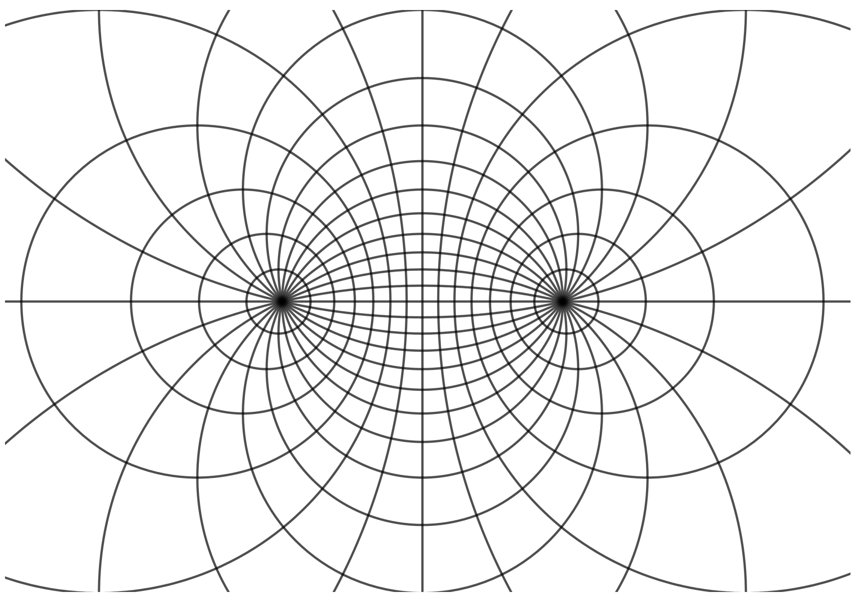
More generally, inversions preserve angles (they are **conformal**) and they also turn circles or lines into other circles or lines. Stereographically projecting the plane onto a sphere (itself a *spherical* inversion) gives us the perspective of the Riemann sphere - here, lines become circles on the sphere through the pole. It is from this perspective we see lines should be thought of as circles.

In astronomy this may be called the **celestial sphere**. The Möbius transformations (transported to the sphere from the plane) describe how the apparent night sky changes appearance if Earth were to travel through space at different speeds. For more serious changes to its appearance, speeds on the order of magnitude of the speed of light are necessary. The celestial sphere is the projectivized lightcone of special relativity, and Möbius transformations extend to linear Lorentz transformations of Minkowski spacetime.

The **Cartan-Dieudonnè theorem** for Euclidean space says every rotation is a composition of an even number of reflections across linear hyperplanes; here we've encountered a generalization of this theorem from Euclidean space to Minkowski spacetime: every Möbius transformation is a composition of an even number of inversions.

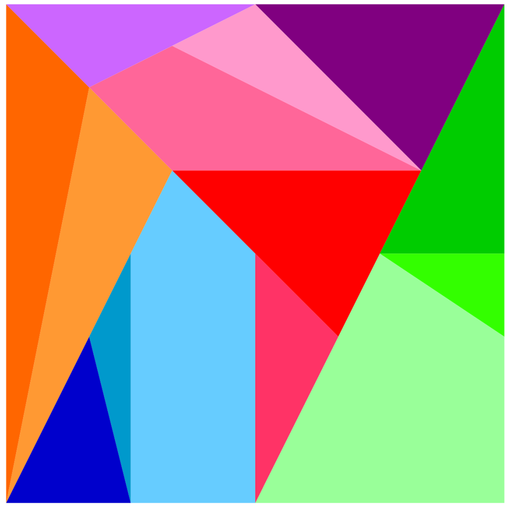
A group of transformations acting on a set of points is called **transitive** if it is possible to transform any point into any other point; we have shown the group of Möbius transformations is *doubly* transitive. In fact, it is *sharply* 3-transitive: given any three distinct points a, b, c and any other three distinct points u, v, w on the Riemann sphere, there is one and only one Möbius transformation f for which $f(a) = u, f(b) = v, f(c) = w$.

What about composing a pair of inversions across other combinations of generalized circles? (Generalized circle is an umbrella term for both circles and lines.) If the generalized circles intersect, the composition of inversions is a hyperbolic rotation around the two poles, whereas if they don't then the two poles become a repelling source and an attracting sink for moving points.



Notice all of the inversion compositions show a *doubling* effect: the distance, angle, scale factor, or appropriate bipolar coordinate "between" two lines or circles is doubled to give the effect of the composition of inversions. This is a manifestation of **spin**, a mysterious concept from physics once described by a mathematician as the "square root of geometry."

Child's Play: Solution



This is essentially a **tangram** - these are toy dissection puzzles, enjoyed by children, like jigsaw puzzles but with simple sold-colored polygon pieces and often with multiple different arrangements possible.

Indeed, this isn't just any tangram, it's the **Ostomachion** attributed to the mathematician Archimedes from Ancient Greece.

A natural question to ask is: *When is it possible to dissect one polygon and rearrange the pieces into another polygon?* Of course, the two polygons would need to have the same area. The Wallace–Bolyai–Gerwien theorem says this is not only necessary, it's actually sufficient too!

Hilbert posed the same question for 3D polyhedra. If two polyhedra can be dissected and rearranged into each other, they are called **scissors-congruent**.

Dehn answered the question by defining what we call the **Dehn invariant**, a kind of numerical signature a polyhedron has which does not change by dissection or rearrangement. In particular, the five Platonic solids have different Dehn invariants, so they are not scissors-congruent.

It turns out, two polyhedra are scissors-congruent if and only if they have both the same Dehn invariant *and* the same volume.

Conical Conversions: Solution

$$S(t) = \int_0^t \frac{\mathrm{d}\tau}{\cosh \tau}, \qquad T(s) = \int_0^s \frac{\mathrm{d}\sigma}{\cos \sigma}.$$

The function $S(t)$ is called the **Gudermannian function** and $T(s)$ the inverse Gudermannian function. Writing $S = S(T)$ or $T = T(S)$, they satisfy the following three equivalent identities:

$$\begin{array}{ll} (1) & \tan S = \sinh T \\ (2) & \sin S = \tanh T \\ (3) & \cos S = \operatorname{sech} T \end{array}$$

For example, (1) says $\tan S(t) = \sinh t$ and $\tan s = \sinh T(s)$.

One identity may be converted into another by applying transformations, using circular Pythagorean identities on the left and hyperbolic Pythagorean identities on the right. Applying $\sqrt{1-x^2}$ converts between (2) and (3); applying $x/\sqrt{1+x^2}$ converts from (1) to (2) and its inverse $x/\sqrt{1-x^2}$ from (2) to (1); applying $1/\sqrt{1+x^2}$ converts from (1) to (3) and its inverse $\sqrt{1-x^2}/x$ from (3) to (1). When converting from (3) it suffices to assume $S, T \geq 0$ since they are odd functions.

There is also a fourth equivalent half-angle identity

$$(4) \qquad \tan(S/2) = \tanh(T/2).$$

This follows from all (hence any) of (1),(2),(3) using either version of tan and tanh's half-angle formulas. For example,

$$\tan\frac{S}{2} = \frac{\sin S}{1+\cos S} = \frac{\tanh T}{1+\operatorname{sech} T} = \frac{\sinh T}{\cosh T + 1} = \tanh\frac{T}{2}.$$

Conversely, (4) may be converted to (1) by applying $2x/(1-x^2)$, to (2) by applying $2x/(1+x^2)$, and to (3) by applying $(1-x^2)/(1+x^2)$; therefore all of the identities (1),(2),(3),(4) are equivalent to each other.

To show $S(t)$ and $T(s)$ are inverse functions, it suffices to establish any of the four identities for $S(t)$ and t and any other one of the four for s and $T(s)$. For example, if evaluating $S(t)$ yields (1) and evaluating $T(s)$ yields (2), then (1) implies (2) so $S(t) = \sin^{-1}(\tanh t)$ and $T(s) = \tanh^{-1}(\sin s)$ and hence they are inverse functions.

Cofunction substitutions. Evaluate the definite integral $S(t)$ using the substitution $u = \sinh(\tau)$ (where $\mathrm{d}u = \cosh(\tau)\mathrm{d}\tau$) and the hyperbolic Pythagorean identity $\cosh^2 - \sinh^2 = 1$:

$$S(t) = \int_0^t \frac{\cosh(\tau)\mathrm{d}\tau}{1+\sinh^2(\tau)} = \int_0^{\sinh t} \frac{\mathrm{d}u}{1+u^2} = \tan^{-1}(\sinh(t)).$$

Evaluate $T(s)$ first by using the substitution $u = \sin(\sigma)$ (where $\mathrm{d}u = \cos(\sigma)\mathrm{d}u$) and the circular Pythagorean identity $\cos^2 + \sin^2 = 1$:

$$T(s) = \int_0^s \frac{\cos(\sigma)\mathrm{d}\sigma}{1-\sin^2(\sigma)} = \int_0^{\sin s} \frac{\mathrm{d}u}{1-u^2} = \tanh^{-1}(\sin(s)).$$

Without directly knowing or recognizing the derivative of $\tanh^{-1}$, it is also possible to use partial fraction decomposition:

$$\int_0^{\sin s} \frac{1}{2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)\mathrm{d}u = \frac{1}{2}\ln\left|\frac{1+\sin s}{1-\sin s}\right| = \tanh^{-1}(\sin(s)).$$

Exponential substitutions. Evaluate $S(t)$ using the substitution $u = e^\tau$ (where $\mathrm{d}u = e^\tau\mathrm{d}\tau$) and absorbing 2 into the integral by doubling the interval over which it is taken (since cosh is an even function):

$$S(t) = \int_0^t \frac{2\mathrm{d}\tau}{e^\tau + e^{-\tau}} = \int_{-t}^t \frac{e^\tau\mathrm{d}\tau}{e^{2\tau}+1} = \int_{e^{-t}}^{e^t} \frac{\mathrm{d}u}{u^2+1} = \tan^{-1}(e^t) - \tan^{-1}(e^{-t}).$$

Apply tangent with difference-angle identity to get

$$\tan S(t) = \frac{e^t - e^{-t}}{1+e^t e^{-t}} = \sinh(t).$$

We may instead have chosen to divide S by 2 in which case

$$\frac{S(t)}{2} = \int_0^t \frac{\mathrm{d}\tau}{e^\tau + e^{-\tau}} = \int_1^{e^t} \frac{\mathrm{d}u}{u^2+1} = \tan^{-1}(e^t) - \tan^{-1}(1).$$

Applying tangent (and multiplying by $e^{-t/2}/e^{-t/2}$) yields

$$\tan\frac{S(t)}{2} = \frac{e^t - 1}{1+e^t} = \frac{(e^{t/2} - e^{-t/2})/2}{(e^{t/2} + e^{-t/2})/2} = \tanh(t/2).$$

Phasor substitutions. With functions of complex variables and path integrals in the complex plane it is possible to evaluate $T(s)$ using the substitution $u = e^{i\sigma}$ (where $\mathrm{d}u = ie^{i\sigma}\mathrm{d}\sigma$) alongside the formula $\cos(\sigma) = (e^{i\sigma} + e^{-i\sigma})/2$. Absorbing 2 into the integral yields:

$$T(s) = \int_0^s \frac{2\mathrm{d}\sigma}{e^{i\sigma} + e^{-i\sigma}} = \int_{-s}^s \frac{e^{i\sigma}\mathrm{d}\sigma}{e^{2i\sigma} + 1} = \frac{1}{i}\int_{e^{-is}}^{e^{is}} \frac{\mathrm{d}u}{u^2 + 1}$$

$$= \frac{\tan^{-1}(e^{is}) - \tan^{-1}(e^{-is})}{i}.$$

Apply tangent with difference angle identity to get

$$\tan iT(s) = \frac{e^{is} - e^{-is}}{1 + e^{is}e^{-is}} = i\sin s,$$

which is equivalent to $\tanh T(s) = \sin s$. Dividing by 2 instead,

$$\frac{T(s)}{2} = \int_0^s \frac{d\sigma}{e^{i\sigma} + e^{-i\sigma}} = \frac{1}{i}\int_0^{e^{is}} \frac{\mathrm{d}u}{u^2 + 1} = \frac{\tan^{-1}(e^{is}) - \tan^{-1}(1)}{i}$$

Applying tangent (and multiplying by $e^{-is/2}/e^{-is/2}$) yields

$$\tan\frac{iT(s)}{2} = \frac{e^{is} - 1}{1 + e^{is}} = \frac{e^{is/2} - e^{is/2}}{e^{is/2} + e^{-is/2}} = \tan(s/2).$$

Differentiation. Another idea: we may show $(T \circ S)(t) = t$ by showing both sides have the same derivative and agree at the initial value $(T \circ S)(0) = 0$. Differentiating with the chain rule and solving for S indicates we need to show $S(t) = \pm\cos^{-1}(\operatorname{sech}(t))$. This, again, can be argued by showing both sides are equal at $t = 0$ and have the same derivative, though one needs to manage the continuity of the $\pm$ sign, and then the same can be done to show $(S \circ T)(s) = s$, or else argue S and T are one-to-one because they are monotonic because their integrands are always positive on S and T's domains.

Circular Cocycle: Solution

A function of the form $f(x) = \frac{ax+b}{cx+d}$ is called a linear fractional transformation, or a Möbius transformation. Its derivative has numerator $\delta = ad - bc$. Notice this is the determinant of $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$!

The first three derivatives of $y = (ax+b)/(cx+d)$ are

$$\begin{aligned} y' &= \frac{\delta}{(cx+d)^2}, \\ y'' &= \frac{-2c\delta}{(cx+d)^3}, \\ y''' &= \frac{6c^2\delta}{(cx+d)^4}. \end{aligned}$$

The three exponents are related by $2+4 = 3+3$, so find

$$\begin{aligned} y'y''' &= \frac{6c^2\delta^2}{(cx+d)^6}, \\ y''y'' &= \frac{4c^2\delta^2}{(cx+d)^6}. \end{aligned}$$

They differ only by their coefficients, so we may conclude

$$2y'y''' = 3\big(y''\big)^2.$$

Conversely, we should show this has no other solutions.

Given the above differential equation, we can first address when $y' \equiv 0$ or $y'' \equiv 0$ identically: in this case $y = ax + b$.

Otherwise, we may address an interval where $y'' \neq 0$, and hence (from the differential equation) $y' \neq 0$ too, and futhermore we may assume it is positive, $y' > 0$, by replacing y with $-y$ if necessary.

First, divide to put in logarithmic derivative form $\big(\ln|f|\big)' = f'/f$,

$$2\,\frac{y'''}{y''} = 3\,\frac{y''}{y'}$$

so that both sides integrate to become logarithms:

$$2\ln|y''| = 3\ln|y'| + C$$

Exponentiate, drop the absolute values by introducing a $\pm$ sign, then replace e^C with C (which absorbs the $\pm$ sign). We get

$$\big(y''\big)^2 = C\big(y'\big)^3.$$

Since $(y'')^2 > 0$ and $y' > 0$, it follows $C > 0$ and we may take square roots. Doing this, replace $\sqrt{C}$ with C (again absorbing an implicit $\pm$):

$$\big(y'\big)^{-3/2} y'' = C$$

Integrating (and dividing by -2, absorbing into constants) gives

$$\big(y'\big)^{-1/2} = Cx + D$$

Then isolate y' to get

$$y' = \frac{1}{(Cx+D)^2}$$

Integrating again gives

$$y = -\frac{1/C}{Cx+D} + B$$

which, when combined, is of the form $y = \frac{ax+b}{cx+d}$. On any interval where this is defined, y' and y'' cannot be 0, as assumed.

Note there are only three degrees of freedom to this form, despite there being four unknowns a, b, c, d. This is because multiplying all four by a value doesn't change the function (without loss of generality, we may assume $ad - bc = \pm 1$, which is done in some contexts).

Cusp of Crying: Solution

The cusp $(0,1)$ has two tangent lines. By symmetry, their slopes are $\pm m$ for some m. Their slopes will help us find the angle. Assume the curve is parametrized by $(x(t), y(t))$ leading up to the cusp.

Squaring $r = e^{y-1}$ yields $x^2 + y^2 = e^{2y-2}$. Differentiating and halving yields

$$xx' + yy' = y'e^{2y-2}$$

Collect like terms for y' on the right, replace e^{2y-2} with $x^2 + y^2$, then divide:

$$1 = \left(\frac{x^2 + y^2 - y}{x}\right)\frac{y'}{x'} = \left(x + y\,\frac{y-1}{x}\right)\frac{y'}{x'}.$$

The ratio $(y-1)/x$ is the slope of the secant line from $(0,1)$ to (x,y), and y'/x' is the slope of the tangent line at (x,y). Therefore, in the limit $(x,y) \to (0,1)$,

$$1 = \big(0 + 1\cdot m\big)\,m = m^2.$$

Thus, $m = \pm 1$, and the cusp is a right angle ($\angle = 90°$).

The exponential function has the globally convergent power series

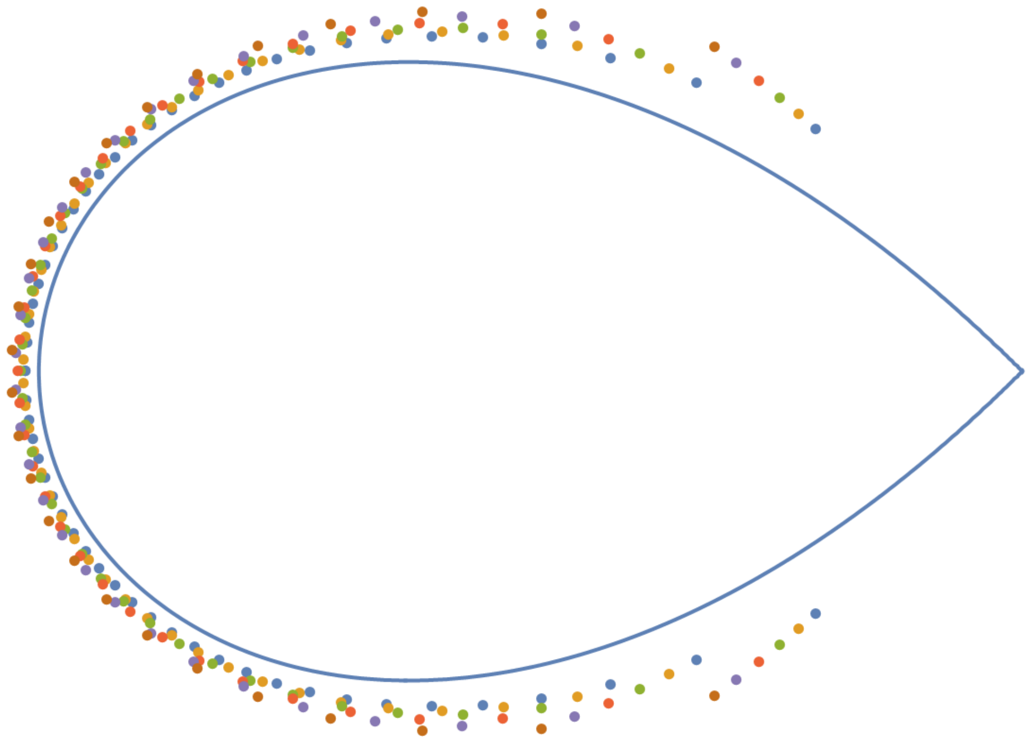
$$\exp z = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

The truncations of this series to the first so many terms,

$$\exp_n(z) := \sum_{k=0}^{n} \frac{z^k}{k!},$$

are polynomials and therefore have complex roots. But $\exp z$ itself has no complex roots! Thus it's no surprise the roots of $\exp_n(z)$, as $n \to \infty$, expand outward without bound. And yet, they still approach a certain shape.

The roots of $\exp_n$ divided by n tend towards this curve:



This is the **Szëgo curve** defined by $|ze^{1-z}| = 1$ in the complex plane. By rearranging this to $|z| = |e^{z-1}|$ and setting $z = x + yi$ we may rewrite it in Cartesian coordiantes as $\sqrt{x^2 + y^2} = e^{x-1}$; flipping this across the diagonal line $y = x$ leaves us with the teardrop curve of this problem.

Cutting Sticks: Solution

We present two algorithms for cutting the sticks.

Greedy Algorithm. Greedy algorithms try to get as close to an objective as possible at each step. A sufficiently greedy chess AI, for example, would fail to sacrifice a piece even if sacrificing would have meant winning the next turn, since it measures closeness to winning by material on the board and will thus choose to avoid losing pieces whenever possible.

We can try to maximize how much we cut from one of the sticks. Starting with the stick of length a, cut out the largest value from $1, 2 \cdots, n$ possible that hasn't already been cut from it, and continue doing so as long as possible.

Suppose this process terminated with some stick left over. If at any stage we didn't cut the maximum unused length from the set $\{1, \cdots, n\}$, then (because our original process had leftover stick) we could have increased the cut at that stage without problem, contradicting the assumption we chose lengths greedily. Thus, we must have cut lengths n, $n-1$, $n-2$, $\cdots$, and if our last cut wasn't 1 then the process didn't terminate because at the end we can now cut 1 out. But this means we cut the total net length $n+\cdots+2+1$ out of the first stick, so there can't be any leftover length, a contradiction.

The sum total of all the lengths from $1, 2, \cdots, n$ that weren't cut from the first stick must equal the length of the second stick. Therefore, these lengths can be cut out of the second stick in any order (e.g. greedily) to finish.

Notice how this algorithm does not require the hypothesis $a, b \geq n$. However, the condition is necessary for a generalization: if we have sticks of lengths $\ell_1, \cdots, \ell_k \geq n$ with sum total length $\ell_1+\cdots+\ell_k = 1+2+\cdots+n$ then we can cut them into lengths $1, 2, \cdots, n$. The greedy algorithm doesn't work for this generalization. Indeed, whether or not this assertion is true is an unsolved question - the **cutting sticks conjecture**!

The following recursive algorithm, on the other hand, does work for the generalization for all n but for only finitely many exceptions for any given number of sticks k (though the number of exceptions grows with k).

Recursive Algorithm. A recursive algorithm solves a problem by using itself on a subproblem at each step. For example, the recursive algorithm computing $1+2+\cdots+n$ would compute $1+2+\cdots+(n-1)$ then add n.

In the sticks problem, both lengths are $a, b \geq n$. If we cut out a length of n from the longer stick (suppose $a \geq b$), then either the remaining stick length $a-n$ is $\geq n-1$ or else $< n-1$. In the former case, we've reduced the problem for minimal length of n to the same problem for a minimal length $n-1$, so our recursive algorithm can use itself on this subproblem.

The cases where we can't use this algorithm are when the remaining stick length is $a-n < n-1$. In these cases, $b \leq a < 2n-1$ implies an upper bound on the total length $1+2+\cdots+n = a+b \leq 2a < 2(2n-1)$. It is well-known the so-called triangular number $1+2+\cdots+n$ is given by the formula $\frac{n(n+1)}{2}$, so our bound is $\frac{n(n+1)}{2} < 2(2n-1)$. Clearing denominators and distributing makes this $n^2+n < 8n-4$, or equivalently $n(n-7) < -4$, which is only possible for $n < 7$. Thus for $n = 3, 4, 5, 6$ we manually find and list a solution for the cases where $a < 2n-1$:

n	$\frac{n(n+1)}{2}$	$2n-1$	$a \to$	$\cdots$	$b \to$	$\cdots$
6	21	11				
5	15	9	8 $\to$	5+3	7$\to$	4+2+1
4	10	7	6 $\to$	3+2+1	4$\to$	4
			5 $\to$	4+1	5$\to$	3+2
3	6	5	4 $\to$	3+1	2$\to$	2
			3 $\to$	2+1	3$\to$	3

Cyberpunk: Solution

The number of trailing zeros in a number m, represented in binary, equals the number of times it is divisible by 2, or equivalently the power of 2 in its prime factorization; let $v_2(m)$ be the power of 2 in m's prime factorization. In number theory this is called the *2-adic valuation*. Much like a logarithm, it satisfies the product rule $v_2(ab) = v_2(a) + v_2(b)$. Therefore the valuation of $m = 1^1 2^2 3^3 \cdots 2049^{2049}$ is equal to the sum

$$1v_2(1) + 2v_2(2) + 3v_2(3) + \cdots + 2049v_2(2049)$$

We may tally the valuations $v_2(k)$ for $k = 1, \cdots, 16$ (for illustration) as in the below table on the left. To multiply k times $v_2(k)$ we may replace each dot with a k and insert plus signs, as on the right:

$$\begin{array}{rcccc}
1 & \circ & \circ & \circ & \circ \\
2 & \bullet & \circ & \circ & \circ \\
3 & \circ & \circ & \circ & \circ \\
4 & \bullet & \bullet & \circ & \circ \\
5 & \circ & \circ & \circ & \circ \\
6 & \bullet & \circ & \circ & \circ \\
7 & \circ & \circ & \circ & \circ \\
8 & \bullet & \bullet & \bullet & \circ \\
9 & \circ & \circ & \circ & \circ \\
10 & \bullet & \circ & \circ & \circ \\
11 & \circ & \circ & \circ & \circ \\
12 & \bullet & \bullet & \circ & \circ \\
13 & \circ & \circ & \circ & \circ \\
14 & \bullet & \circ & \circ & \circ \\
15 & \circ & \circ & \circ & \circ \\
16 & \bullet & \bullet & \bullet & \bullet
\end{array}
\quad \Longrightarrow \quad
\begin{array}{ccccccc}
 & & & & & & \\
2 & & & & & & \\
 & & & & & & \\
4 & + & 4 & & & & \\
 & & & & & & \\
6 & & & & & & \\
 & & & & & & \\
8 & + & 8 & + & 8 & & \\
 & & & & & & \\
10 & & & & & & \\
 & & & & & & \\
12 & + & 12 & & & & \\
 & & & & & & \\
14 & & & & & & \\
 & & & & & & \\
16 & + & 16 & + & 16 & + & 16
\end{array}$$

Evaluating $n = v_2(m)$, then, amounts to adding up all the numbers scattered above on the right. Instead of grouping the terms in rows, giving $2 + 4 \cdot 2 + 6 + 8 \cdot 3 + 10 + 12 \cdot 2 + 14 + 16 \cdot 4 + \cdots$, we will group the terms in columns because the column sums have a nice formula.

Grouping the summands according to columns on the last page,

$$
\begin{array}{lr}
 & (2+4+6+8+\cdots+2048) \\
 & (4+8+12+\cdots+2048) \\
 & (8+16+\cdots+2048) \\
 & (16+\cdots+2048) \\
 & \vdots \\
+ & (2048) \\
\hline
= & n
\end{array}
$$

From here we may factor out common factors:

$$
\begin{array}{lr}
 & 2(1+2+3+\cdots+1024) \\
 & 4(1+2+3+\cdots+\ \ 512) \\
 & 8(1+2+3+\cdots+\ \ 256) \\
 & 16(1+2+3+\cdots+\ \ 128) \\
 & \vdots \\
+ & 2048(1) \\
\hline
= & n
\end{array}
$$

Note 2048 is a power of 2, by hand calculation:

e	2^e
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048

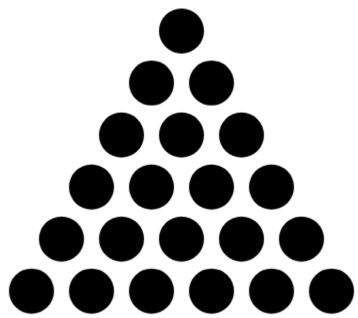
And so it is revealed that 2049 is more than a *Blade Runner* reference; it is closer to a perfect power of 2 than 2019 happens to be.

At this point we need to use the following:

Lemma. The nth *triangular number* is given by the formula

$$T_n \overset{\text{def}}{=} 1+2+3+\cdots+n \quad = \frac{n(n+1)}{2}.$$

They count how many balls are in a triangular stack (by row):



(*Proof 1.*) T_n counts how many subsets of $\{1,2,3,\cdots,n+1\}$ have the form $\{a,b\}$; make the following arrangement and count by rows:

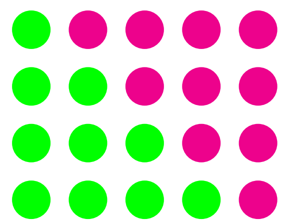
$$\begin{array}{c}
\{1,2\} \\
\{1,3\},\{2,3\} \\
\{1,4\},\{2,4\},\{3,4\} \\
\{1,5\},\{2,5\},\{3,5\},\{4,5\} \\
\{1,6\},\{2,6\},\{3,6\},\{4,6\},\{5,6\} \\
\{1,7\},\{2,7\},\{3,7\},\{4,7\},\{5,7\},\{6,7\} \\
\text{(and so on)}
\end{array}$$

On the other hand, if we pick a first then b, we have $n+1$ choices for a then n remaining choices for b, but then we must divide by 2 to undo our overcounting since $\{a,b\}=\{b,a\}$; this gives $n(n+1)/2$.

(*Proof 2.*) There is an oft-told story which says that when a young Carl Friedrich Gauss (considered one of the greatest mathematicians of all time) was a schoolboy, his teacher gave the students busywork by asking them to add the numbers 1 through 100, which Gauss solved immediately with the trick of adding the sum to itself in reverse order.

$$\begin{array}{rcccccccccc}
S & = & 1 & + & 2 & + & \cdots & + & 99 & + & 100 \\
+S & = & 100 & + & 99 & + & \cdots & + & 2 & + & 1 \\
\hline
2S & = & 101 & + & 101 & + & \cdots & + & 101 & + & 101
\end{array}$$

Summing gives $2S = 100(101)$. This generalizes to $2T_n = n(n+1)$, and may be visualized by combining two triangular stacks:



$T_4 + T_4 = 4 \times 5$

Applying the lemma to our aforementioned column sum for n,

$$n = 2\left(\tfrac{1024\cdot1025}{2}\right) + 4\left(\tfrac{512\cdot513}{2}\right) + 8\left(\tfrac{256\cdot257}{2}\right) + \cdots + 2048\left(\tfrac{1\cdot2}{2}\right)$$

$$= 1024(1025 + 513 + 257 + \cdots + 2)$$

For this we may employ yet another formula,

Lemma. The *geometric sum formula* for the kth partial sum of a geometric sequence with first term 1 and common ratio r is:

$$S = 1 + r + r^2 + \cdots + r^{k-1} = \frac{r^k - 1}{r - 1}.$$

(*Proof.*) Compare S with its multiple rS:

$$\begin{array}{rcccccccccc}
S & = & 1 & + & r & + & r^2 & + & \cdots & +r^{k-1} & & \\
rS & = & & & r & + & r^2 & + & \cdots & +r^{k-1} & + & r^k
\end{array}$$

Subtracting gives $rS - S = r^k - 1$. Applying with $r = 2$ and $k = 11$,

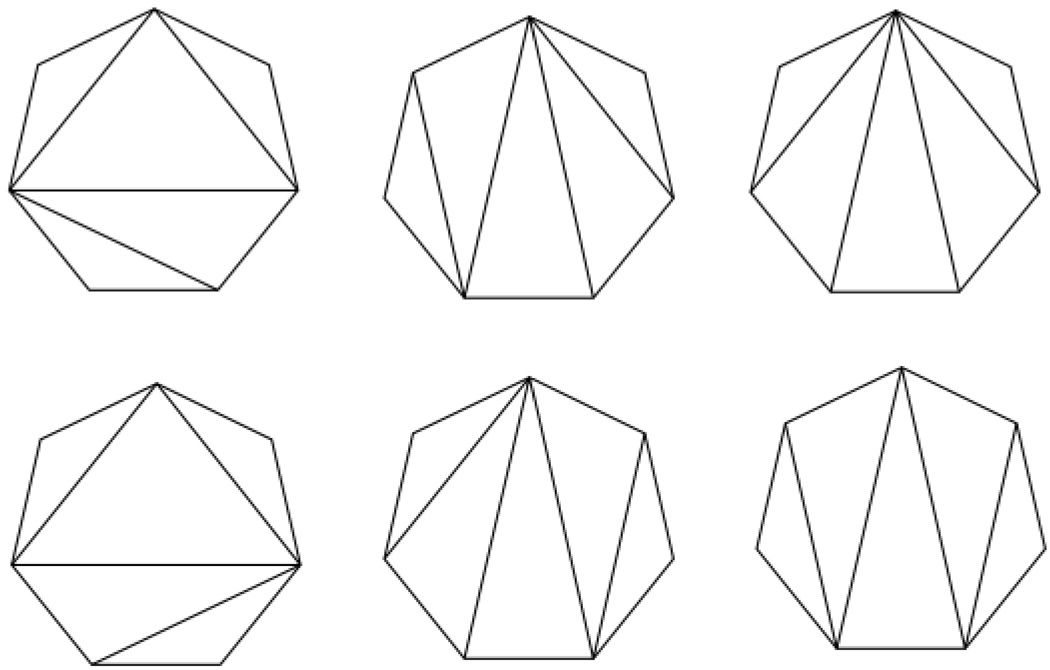
$$n = 2^{10}\big((2^{10}+1) + (2^9+1) + \cdots + (2^0+1)\big)$$

$$= 2^{10}\big((2^{10} + 2^9 + \cdots + 2^0) + (1 + 1 + \cdots + 1)\big)$$

$$= 2^{10}\left(\frac{2^{11}-1}{2-1} + 11\right) = 2^{10}(2^{11} + 2^3 + 2) = 2^{21} + 2^{13} + 2^{11}$$

expressed in binary is 1000000010100000000000_2.

Cyclic Sieving: Solution



Often, combinatorial formulas counting certain things with set-theoretic descriptions generalize to other formulas (their q-**analogs**) which count similar things with linear-algebra interpretations. (Arguably it would be more truthful to say projective-geometry interpretations.) The analogs use the traditional choice of variable q. Here, linear algebra is not done over the fields $\mathbb{R}$ or $\mathbb{C}$, but rather over a **finite field** with q scalars denoted $\mathbb{F}_q$.

The simplest example: n counts how many elements the set $\{1, \cdots, n\}$ has, while $[n] := q^{n-1}+\cdots+q+1$ counts how many 1D subspaces the n-dimensional vector space $\mathbb{F}_q^n$ has. More generally, the binomial coefficient $\binom{n}{k} = \frac{n(n-1)\cdots}{k(k-1)\cdots}$ (with k terms in the numerator and denominator) counts how many size-k subsets there are of $\{1, \cdots, n\}$, and its q-analog $\left[{n \atop k} \right] = \frac{[n][n-1]\cdots}{[k][k-1]\cdots}$ counts how many k-dimensional subspaces there are of the vector space $\mathbb{F}_q^n$.

Plugging $q = 1$ into q-analogs typically gives the original combinatorial formula. This suggests a "**field with one element**" is missing in field theory, however changing the definition of a "field" to allow only one element fails to reproduce the combinatorial formulas. Mathematicians have tried to remedy this by abstracting every possibly relevant definition until they can finally actually define $\mathbb{F}_1$, but this saga has yet to reach a conclusion.

The **cyclic sieving phenomenon** (CSP) occurs when a combinatorial formula counts certain things and then plugging a complex nth root of unity into the q-analog counts how many of those things have cyclic symmetry.

For instance, the formula $\binom{n}{k}$ counts how many ways there are to color k vertices of a polygon one color and the other $n-k$ vertices another (like a necklace with beads), and if d is a factor of n then plugging a (primitive) dth root of unity for q into the analog $\left[{n \atop k} \right]$ tells us how many of those colorings are unchanged by rotating the polygon by $1/d$th of a full turn.

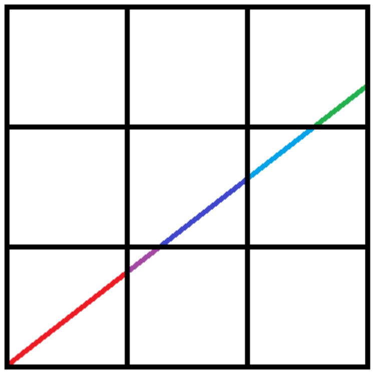
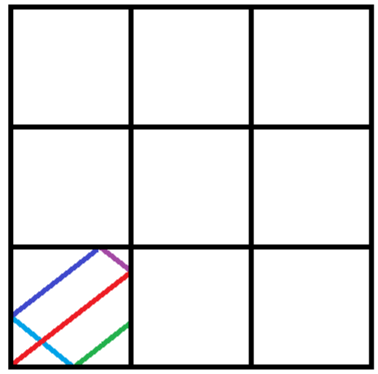
In our problem of polygon triangulations, however, we are not counting the "fixed points" of rotations (the configurations with cyclic symmetry) but rather the "orbits," or in other words we are grouping the triangulations according to rotations and then counting how many groups (orbits) we get. But we can count orbits using fixed points according to **Burnside's Lemma**.

(If we did the same for coloring vertices of a polygon, grouping the colorings according to rotations, **necklace polynomials** count the orbits.)

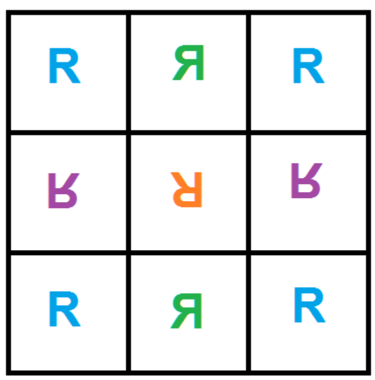
The total number of triangulations of a polygon with n vertices, where rotations are *not* counted as equivalent, is C_{n-2} where the **Catalan numbers** are given by the formula $C_n = \frac{1}{n+1}\binom{2n}{n}$. The q-analog $\frac{1}{[n+1]}\left[{2n \atop n} \right]$ exhibits CSP, so plugging a complex (primitive) dth root of unity for q in (where d is a factor of n) yields how many configurations are unchanged by $1/d$th of a full turn, which Burnside's Lemma tells us how to count orbits with.

Dynamical Billiards: Solution

Compare the trajectory of a point particle bouncing around the unit square with that of a point particle free to traverse the plane:

As the particle crosses the first edge, its path (purple) may be flipped across the edge to get the would-be trajectory had the particle instead bounced. Generalizing, the points in any square correspond to points in an adjacent square by flipping across the common edge. To illustrate, consider the effect of repeatedly flipping the letter 'R' as follows:



The pattern is seen to repeat every 2 units. Therefore, after computing where the point would be with no bouncing, we may subtract even numbers from each coordinate to obtain a point in the initial 2×2 square. Using half angle formulas, the coordinates are

$$x = 60\cos\frac{\pi}{12} = 60\sqrt{\frac{1+\cos\frac{\pi}{6}}{2}} = 30\sqrt{2+\sqrt{3}} \approx 57.95$$

$$y = 60\sin\frac{\pi}{12} = 60\sqrt{\frac{1-\cos\frac{\pi}{6}}{2}} = 30\sqrt{2-\sqrt{3}} \approx 15.53$$

Therefore, the corresponding point in the initial 2×2 square is

$$\left(30\sqrt{2+\sqrt{3}}-56,\ 30\sqrt{2-\sqrt{3}}-14\right) \approx (1.95, 1.53).$$

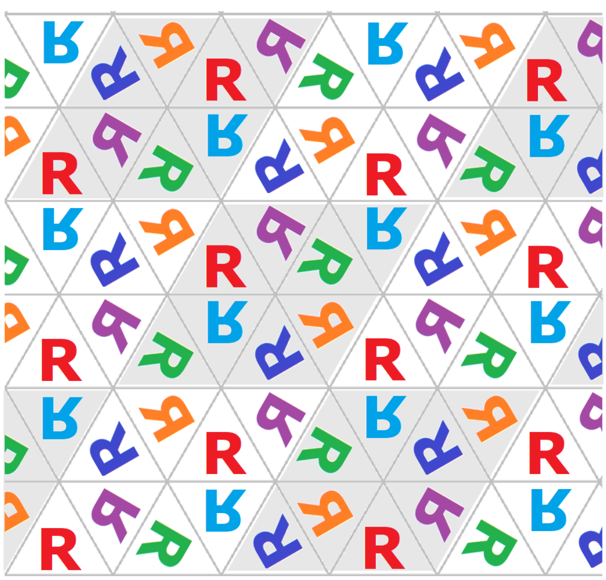
This point is catercorner to the original 1×1 square, so we must flip it across both the vertical line $x = 1$ and horizontal line $y = 1$. This means replacing x with $2 - x$ and y with $2 - y$. Thus, the answer is

$$\left(58-30\sqrt{2+\sqrt{3}},\ 16-30\sqrt{2-\sqrt{3}}\right).$$

Remark. Another way to evaluate $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$ is using the difference angle formulas, since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. In this case, we wind up with the curious "denesting" identities $2\sqrt{2 \pm \sqrt{3}} = \sqrt{6} \pm \sqrt{2}$.

Bonus: Triangular Billiards

With an equilateral triangle, there is again a repeating pattern of flipping triangles across edges, but now there are six possible orientations. The pattern now repeats in two (non-orthogonal) directions, say along vectors $\vec{u}$ and $\vec{v}$. Their magnitude is double the side length of the triangles. We may draw rhombi whose sides are these two vectors:



By copy-and-pasting this rhombus over and over again, edge to edge, we get a repeating pattern of rhombi. This repeating pattern is variously called a lattice, tiling, or tessellation. The prototypical tile that all the others are modeled on is called the fundamental polygon. These are studied in e.g. crystallography and hyperbolic geometry.

Any other parallelogram with sides $a\vec{u} + b\vec{v}$ and $c\vec{u} + d\vec{v}$ would also work as a fundamental polygon, so long as the integer matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has determinant 1 (for this it is necessary but not sufficient that $\gcd(a, b)$ and $\gcd(c, d)$ are both 1). In fact, other non-polygonal shapes may be used as fundamental regions for repeating patterns - for instance, replace the straight edges of a fundamental polygon with curves.

The triangles have base 1 and height $\sqrt{3}/2$, so the vectors are

$$\vec{u} = \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}.$$

Whatever point the particle would end up at by crossing edges, we may write it in terms of uv-coordinates:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

To find the components a and b, simply compute

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 1 & \sqrt{3} \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}.$$

For the square lattice we could subtract even numbers from either coordinate. For the triangular lattice, we may subtract $\vec{u}$ and $\vec{v}$, which corresponds to subtracting integers from the a and b components (to get their fractional parts) and then converting back to xy-coordinates.

Once this is done, the point will be in one of the handful of triangles overlapping the fundamental rhombus. Compare with the grid points (vertices of triangles) to figure out which triangle it is in and which lines to reflect over, then proceed to reflect it as appropriate until it is in the original triangle.

Ensemble Cast: Solution

The characteristic polynomial of the symmetric matrix H is

$$\det\begin{pmatrix} x+y-\lambda & z \\ z & x-y-\lambda \end{pmatrix} = (x-\lambda)^2 - y^2 - z^2.$$

The eigenvalues ($\lambda_1 \le \lambda_2$) of H are therefore given by

$$\begin{cases} \lambda_1 = x - \sqrt{y^2+z^2} \\ \lambda_2 = x + \sqrt{y^2+z^2} \end{cases}$$

Thus, the (x, y, z) corresponding to a given (λ_1, λ_2) satisfy

$$\begin{cases} x = \frac{1}{2}(\lambda_1+\lambda_2) \\ y^2+z^2 = \frac{1}{4}(\lambda_2-\lambda_1)^2 \end{cases}$$

This is a circle in xyz-space. The density ρ of H at every point on it is

$$\begin{aligned} \rho &= \pi^{-3/2}\exp\big(-(x^2+y^2+z^2)\big) \\ &= \pi^{-3/2}\exp\big(-\tfrac{1}{4}(\lambda_1+\lambda_2)^2 - \tfrac{1}{4}(\lambda_2-\lambda_1)^2\big) \\ &= \pi^{-3/2}\exp\big(-\tfrac{1}{2}(\lambda_1^2+\lambda_2^2)\big) \end{aligned}$$

Thus, the density ρ is constant on the circle associated to a given (λ_1, λ_2).

The circumference of the circle is $2\pi\sqrt{y^2+z^2} = \pi|\lambda_2 - \lambda_1|$. Since ρ is constant on the circle, our answer is simply ρ times this circumference:

$$\frac{1}{\sqrt{\pi}}\exp\left(-\frac{\lambda_1^2+\lambda_2^2}{2}\right)|\lambda_2-\lambda_1|$$

This is a Gaussian Orthogonal Ensemble (GOE). Using complex Hermitian matrices instead of real symmetric ones gives the **Gaussian Unitary Ensemble** (GUE), proposed by theoretical physicist Eugene Wigner as a way to model the spectral theory (energy levels) of heavy atomic nuclei.

Notice $\rho = 0$ on the line $\lambda_1 = \lambda_2$. As a result, (λ_1, λ_2) exhibits **repulsion**: the eigenvalues are not independent, they prefer to be apart from each other.

Equational Sudoku: Solution

First, we want to figure out which of $0, 1, \star, ☾$ that $\star \times ☾$ is. We will set it equal to elements and multiply by $\star^{-1}$ or $☾^{-1}$ to get contradictions:

- $\star \times ☾ = 0 \quad \Rightarrow \quad \star, ☾ = 0$
- $\star \times ☾ = \star \quad \Rightarrow \quad ☾ = 1$
- $\star \times ☾ = ☾ \quad \Rightarrow \quad \star = 1$

These are all contradictions because $0, 1, \star, ☾$ are all distinct. This leaves the only possibility $\star \times ☾ = 1$, which also means $☾ \times \star = 1$. In other words, $\star$ and $☾$ are each other's multiplicative inverses, $\star^{-1} = ☾$ and $☾^{-1} = \star$.

Second, we do the same for $\star \times \star$, multiplying by $\star^{-1} = ☾$:

- $\star \times \star = 0 \quad \Rightarrow \quad \star = 0$
- $\star \times \star = 1 \quad \Rightarrow \quad \star = ☾$
- $\star \times \star = \star \quad \Rightarrow \quad \star = 1$

This leaves only $\star \times \star = ☾$. Symmetrically, we must also have $☾ \times ☾ = \star$.

Third, we may do the same for $1 + \star$:

- $1 + \star = 1 \quad \Rightarrow \quad \star = 0 \quad$ (add -1)
- $1 + \star = \star \quad \Rightarrow \quad 1 = 0 \quad$ (add $-\star$)
- $1 + \star = 0 \quad \Rightarrow \quad \star = ☾$

From $1+\star = 0$ we may multiply by $☾$ to get $☾ + 1 = 0$. Setting $1+\star = ☾ + 1$, we may add -1 to get $\star = ☾$, a contradiction. This leaves only $1 + \star = ☾$, and symmetrically $1 + ☾ = \star$. Not much left to go!

Multiplying $1 + \star = ☾$ by $\star$ (or $1 + ☾ = \star$ by $☾$) gives $\star + ☾ = 1$.

Lastly, multiplying the element $\bigcirc := 1 + \star + ☾$ by either of $\star$ or ☾ leaves it unchanged - thus, we have $(1-\star)\bigcirc = (1 - ☾)\bigcirc = 0$, and multiplying by $(1-\star)^{-1}$ or $(1-☾)^{-1}$ (which is possible because $\star$, ☾, 1 are distinct) yields $\bigcirc = 0$. Replacing $\star + ☾$ in $\bigcirc = 0$ with 1, this equation is now $1 + 1 = 0$.

Our completed table now reads

+	0	1	⋆	☾
0	0	1	⋆	☾
1	1	0	☾	⋆
⋆	⋆	☾	0	1
☾	☾	⋆	1	0

×	0	1	⋆	☾
0	0	0	0	0
1	0	1	⋆	☾
⋆	0	⋆	☾	1
☾	0	☾	1	⋆

What this problem called a "number system" in math is known as a *field*, and when it has a finite number of elements it is a **finite field**.

For comparison, consider the **integers mod** n, denoted $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$. This effectively consists of $\{0, 1 \cdots, n-1\}$ with "clock arithmetic," where the addition and multiplication operations "wrap around," for instance $11+2 = 1$ in $\mathbb{Z}_{12}$ just as 2 hours after 11:00 is 1:00. If n is composite, then $\mathbb{Z}_n$ has nonzero elements without multiplicative inverses (anything not relatively prime to n), but if p is prime then $\mathbb{Z}_p$ is a finite field.

There is essentially only one finite field of size q for prime powers q, and none for other cardinalities, denoted $\mathbb{F}_q$ in math or $GF(q)$ in computer science ("Galois field"). For primes p, the finite field $\mathbb{F}_p$ is just $\mathbb{Z}_p$. But for higher prime powers q, we construct $\mathbb{F}_q$ by adding "imaginary" elements to $\mathbb{F}_p$ just as how we construct $\mathbb{C}$ from $\mathbb{R}$. For instance, we may construct $\mathbb{F}_4$ from the problem by adjoining a cube root of unity ☾ to $\mathbb{F}_2 = \{0, 1\}$.

Finite fields are indispensable to modern cryptography and error correction.

Factorial Frenzy: Solution

Expanding $(1+x)^{m+n}(1-x)^{m+n} = (1-x^2)^{m+n}$, the x^{2m} coefficient is

$$\sum_k \binom{m+n}{m+k}\binom{m+n}{m-k}(-1)^{m-k} = \binom{m+n}{m}(-1)^m.$$

To understand the left side, note each x^{2m} term arises from multiplying a $\binom{m+n}{m+k}x^{m+k}$ term from $(1+x)^{m+n}$ and a $\binom{m+n}{m-k}(-x)^{m-k}$ term from $(1-x)^{m+n}$ together for some k. The right side is just the coefficient of $\binom{m+n}{m}(-x^2)^m$.

Rewriting the binomial coefficients with factorials gives

$$\sum_k \frac{(m+n)!}{(m+k)!(n-k)!}\cdot\frac{(m+n)!}{(m-k)!(n+k)!}(-1)^{m-k} = \frac{(m+n)!}{m!n!}(-1)^m.$$

The $(-1)^m$ can be cancelled from both sides, and $(-1)^{-k}$ might as well be written $(-1)^k$. There is an abundance of $(m+n)!$s, so let's divide by both of them from the LHS numerator to end up with one in the RHS denominator. Furthermore, the $m \pm k$ and $n \pm k$ in the LHS denominator sum to $2m$ and $2n$ respectively, so let's multiply both sides by $(2m)!(2n)!$. We get

$$\sum_k \frac{(2m)!}{(m+k)!(m-k)!}\frac{(2n)!}{(n-k)!(n+k)!}(-1)^k = \frac{(2m)!(2n)!}{m!n!(m+n)!}.$$

The RHS can be "unsimplified" in preparation to write binomial coefficients by multiplying numerator and denominators by $m!$ and $n!$:

$$\frac{(2m)!(2n)!}{m!n!(m+n)!} = \frac{(2m)!}{m!m!}\cdot\frac{(2n)!}{n!n!}\cdot\frac{m!n!}{(m+n)!} = \binom{2m}{m}\binom{2n}{n}\Big/\binom{m+n}{m}$$

Thus we conclude the **von Szily convolution identity**:

$$\sum_k (-1)^k\binom{2m}{m+k}\binom{2n}{n-k} = \binom{2m}{m}\binom{2n}{n}\Big/\binom{m+n}{m}$$

Ira Gessel called $S(m,n) = \binom{2m}{m}\binom{2n}{n}/\binom{m+n}{m}$ the **super Catalan numbers**.

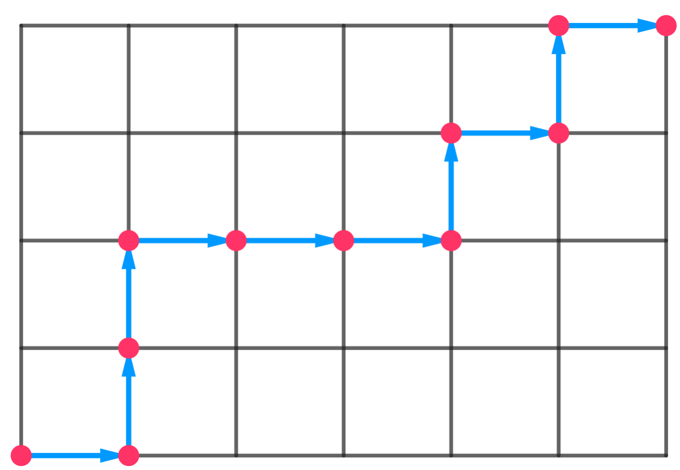
It is not obvious from the rational expression that these should even be whole numbers, let alone (surprisingly) what they ought to count!

A couple alternative ways to show they are whole numbers involve:

- **Legendre's formula**: the exponent of a prime p in the factorization of $n!$ is $(n-s)/(p-1)$, where s is the sum of n's digits in base p.
- **Kummer's theorem**: the exponent of a prime p in the factorization of $\binom{n}{k}$ is the number of carries when adding k and $(n-k)$ in base p.

It is possible to give $S(m,n)$ a combinatorial interpretation via von Szily.

A **lattice path** is a sequence of unit steps north and east from one grid point (usually the origin) to another in the Euclidean plane. The number of lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{m}$, because a lattice path corresponds to a choice of which m of the $m+n$ steps go right and which n go up.



Consider lattice paths $(0,0) \to (m+n, m+n)$. There are $\binom{2m+2n}{m+n}$ total. Any one of them must intersect the line $x+y=2m$ at some point $(m+k, m-k)$ which is k squares diagonally from (m,m) for some k. (Note this argument also works using $x+y=2n$ instead.) There are $\binom{2m}{m+k}$ paths $(0,0) \to (m+k, m-k)$ and $\binom{2n}{n-k}$ paths $(m+k, m-k) \to (m+n, m+n)$.

Thus, let E and O be the number of paths $(0,0) \to (m+n, m+n)$ which intersect $x+y=2m$ at a point an even or odd number of squares from (m,m). Then super Catalan numbers are the difference $S(m,n) = E - O$.

Favorite Angle: Solution

Suppose $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are (WLOG) unit vectors at 120° angles to each other.

Remember the dot product is bilinear and $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$. That means each of the three dot products $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{b} \cdot \mathbf{c}$, $\mathbf{c} \cdot \mathbf{a}$ is equal to $-\frac{1}{2}$.

The most "nose-to-the-ground" solution solves for $\mathbf{c}$ as a linear combination of $\mathbf{a}, \mathbf{b}$ using this information. We can pick a unit normal vector $\mathbf{n}$ to the plane spanned by $\{\mathbf{a}, \mathbf{b}\}$, then $\{\mathbf{a}, \mathbf{b}, \mathbf{n}\}$ is a basis and $\mathbf{c} = u\mathbf{a} + v\mathbf{b} + w\mathbf{n}$ for some coefficients u, v, w. The equations $\mathbf{a} \cdot \mathbf{c} = -\frac{1}{2}$ and $\mathbf{b} \cdot \mathbf{c} = -\frac{1}{2}$ become

$$\begin{cases} u - \frac{1}{2}v = -\frac{1}{2} \\ -\frac{1}{2}u + v = -\frac{1}{2} \end{cases} \implies \begin{cases} u = -1 \\ v = -1 \end{cases}$$

and then $1 = \|\mathbf{c}\|^2 = u^2 - uv + v^2 + w^2$ becomes $1 = 1 + w^2$ which forces $w = 0$, thus $\mathbf{c} = -\mathbf{a} - \mathbf{b}$ and we conclude $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are linearly dependent.

This implies the identity $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, which in turn suggests there might be a simple solution involving symmetry. Indeed, we can just distribute

$$\begin{aligned} \|\mathbf{a} + \mathbf{b} + \mathbf{c}\|^2 &= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= 3(1) + 6(-\tfrac{1}{2}) \ = \ 0. \end{aligned}$$

and immediately conclude from this that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$.

A similar trick can be used to show the vertices of regular tetrahedron, centered at the origin, are at $\cos^{-1}(-\frac{1}{3})$ angles to each other. Note this proof showing three vectors at 120° must be coplanar actually works in any number of dimensions, not just 3D. (The previous solution can also be made to work in n dimensions, by decomposing $\mathbf{c} = \mathbf{c}_\parallel + \mathbf{c}_\perp$ into parallel and perpendicular components WRT $\mathrm{span}\{\mathbf{a}, \mathbf{b}\}$, then $\mathbf{c}_\perp$ takes the role of $\mathbf{n}$.)

The volume of the parallelepiped generated by $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is the triple product

$$\mathrm{vol} = \det(\mathbf{a}\ \mathbf{b}\ \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

If we define the matrix $V = (\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ (assuming the three vectors are column vectors), we can use the so-called **Gramian determinant**

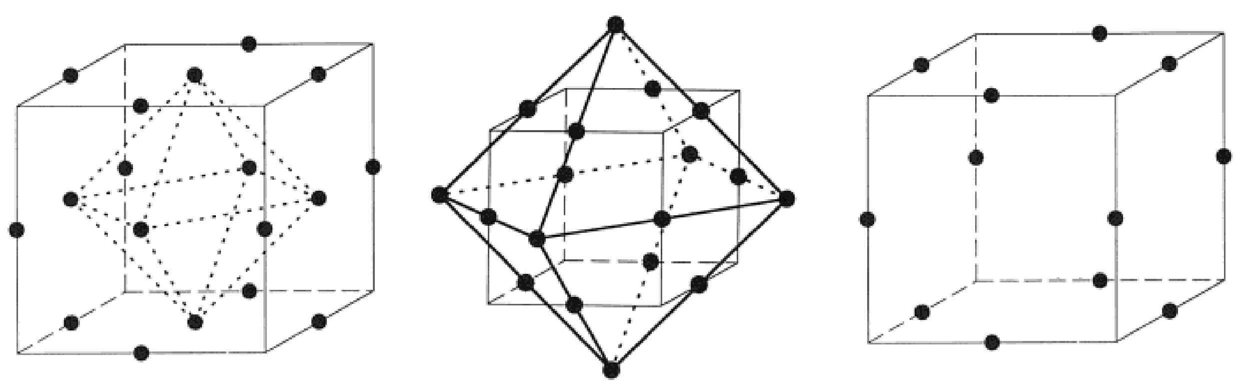
$$(\det V)^2 = (\det V)(\det V) = (\det V^T)(\det V) = \det(V^T V)$$
$$= \det \begin{pmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c} \end{pmatrix} = \det \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} = 0.$$

But $\det V = 0$ implies the columns of $V = (\mathbf{a}\ \mathbf{b}\ \mathbf{c})$ are linearly dependent!

(In fact, $\det(V^T V) = \mathrm{vol}^2$ applies for the parallelepiped generated by any number of vectors in any number of dimensions. The expression $\det V$ by itself doesn't make sense unless V is a square matrix.)

Note the last two solutions begin with the serendipitous decision of squaring, after which the algebra works out. This is a common trick in some circles, a special case of two mutually inverse tricks, polarization and symmetrization: loosely speaking, these convert between bilinear and quadratic gadgets.

A **root system** Φ is a particularly symmetric set of vectors (called roots); any line through a root contains only one other, its antipode; reflecting one root across the plane perpendicular to a second root gives a third root; projecting one root onto another produces an integer or half-integer multiple.



(Ruen, 2010)

By adding complex numbers into the mix, root systems can be used to classify (infinitesimal versions of) *smooth* symmetries. To compare smooth vs. discrete symmetry, consider the symmetry of a sphere vs. of a polyhedron.

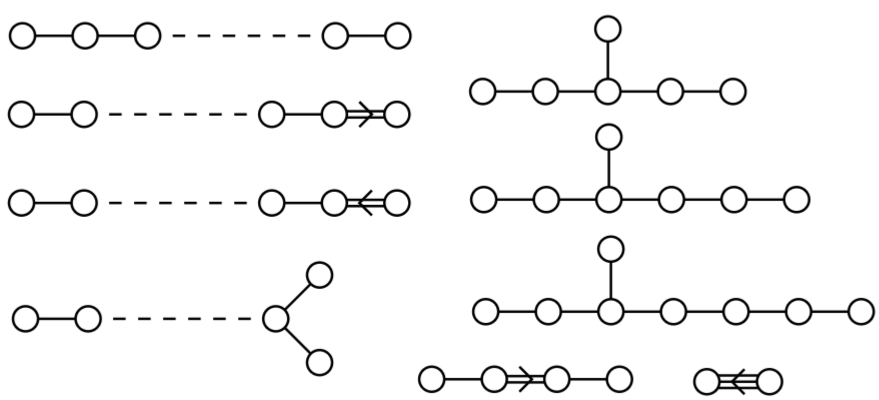
The angles between roots are quite restricted - they are among the special angles studied in school trigonometry, the 30° and 45° families. The ratio between lengths of non-orthogonal roots can only be one of $\sqrt{1}, \sqrt{2}, \sqrt{3}$.

This is also related to the **crystallographic restriction theorem**, which validates the empirical observation that crystals in nature have twofold, threefold, fourfold, or sixfold rotational symmetry and no other kind.

Picking a plane separates a root system Φ into "positive" and "negative" halves, $\Phi = \Phi^+ \cup \Phi^-$. A basis $\Delta \subseteq \Phi^+$ of "simple" roots can be chosen so that all positive roots are sums of simple ones. All but at most one pair of simple roots are either orthogonal or at 120° angles to each other.

From Δ we can construct a **Dynkin diagram**: a graph, consisting of one node for each simple root and a simple edge for each pair at 120° angles (the only possible exceptions are directed double or triple edges for the supplements of 45° or 30° respectively, pointing from larger to shorter root).

It is possible to reconstruct root systems from their Dynkin diagrams. And more generally, Dynkin diagrams are used to classify many kinds of interrelated mathematical objects involving symmetry and geometry.

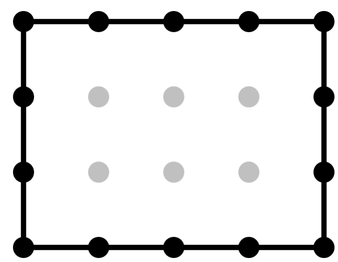


(Nonenmacher, 2008)

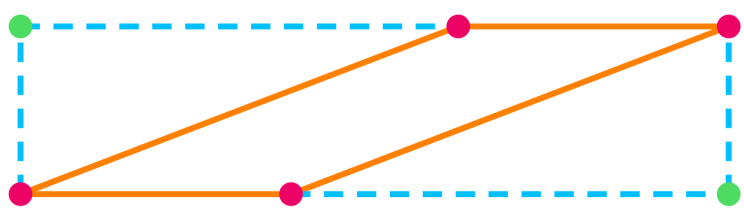
The Dynkin diagrams for "irreducible" root systems are classified.

Fenced In: Solution

For an upright rectangle with length L and width W, each lengthwise edge has $L+1$ grid points, each widthwise edge has $W+1$ grid points, and the interior has $(L-1)(W-1)$ grid points. See below for a small example:



We can extend the orange-magenta parallelogram to a larger rectangle by adding on blue-green right triangles to its upper left and lower right corners:



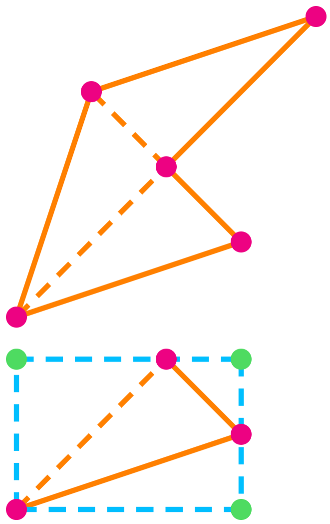
The larger rectangle has dimensions 89×377. If the two blue-green triangles were combined into a smaller triangle, it would have dimensions 89×233.

Are there any points on the orange diagonals? The slope $m = 89/233$ of the left diagonal is already in lowest terms, so cannot be expressed as a fraction b/a with smaller numbers, and so there is no point (a, b) on it between its endpoints. Any point on the right diagonal would correspond to one on the left diagonal, so there are no points on the right diagonal either.

The number of grid points within the parallelogram is therefore the difference between the numbers within the larger and smaller rectangles:

$$(88)(376) - (88)(232) \;=\; 88 \cdot 144 \;=\; \mathbf{12672}.$$

What about counting interior grid points of an arbitrary polygon?



Pick's Theorem says the area A of a polygon with integer-coordinate vertices can be expressed in terms of the number i of interior grid points and the number b of grid points on the boundary:

$$A = i + \tfrac{1}{2}b - 1. \tag{PT}$$

We can divide any polygon into triangles and add their PT equations up to get PT for the polygon. PT is true for rectangles because $A = LW$, $i = (L-1)(W-1)$, and $b = 2L+2W$, and right triangles (by halving rectangles), so it's true for triangles in general (by fitting them in bounding boxes alongside right triangles).

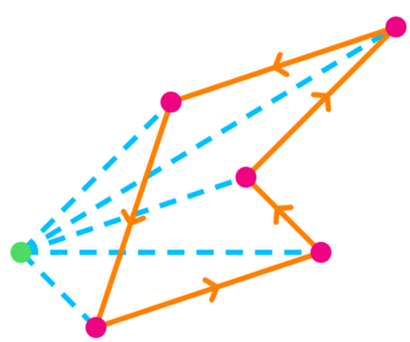
Meanwhile, the **Shoelace Formula** says the area of a polygon with vertices (x_1, y_2), (x_2, y_2), $\cdots$, (x_n, y_n) (write $(x_{n+1}, y_{n+1}) = (x_1, y_1)$, too) is given by

$$A \;=\; \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \end{vmatrix} \;\overset{\text{def}}{=}\; \sum_{k=1}^{n} \frac{1}{2} \underbrace{\begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix}}_{x_k y_{k+1} - x_{k+1} y_k}.$$

The formula is so-named because in the $2 \times n$ array above if we draw lines to pair up the xs and ys that get multiplied, it looks like we're lacing them up.

Each vertex may be interpreted as a vector, and then any edge of the polygon, alongside the pair of vectors to its endpoints, forms a triangle.

If we orient the edges around the polygon in a loop, then we can say the triangles have positive or negative area as a appropriate, and then the sum of their signed areas is the area of the polygon!



(Note if a triangle has two edges meeting at the origin, interpreted as column vectors $\mathbf{a}$ and $\mathbf{b}$, its signed area is half of the determinant $\det(\mathbf{a}\ \mathbf{b}) = \left|\begin{smallmatrix} a_1 & b_1 \\ a_2 & b_2 \end{smallmatrix}\right|$.)

Using $x_k y_{k+1} - x_{k+1} y_k = x_k(y_{k+1} - y_k) - (x_{k+1} - x_k) y_k$ we may rewrite the summands of the shoelace formula. In the limit the formula becomes

$$A = \frac{1}{2} \sum \left(x \Delta y - y \Delta x\right) \longrightarrow \frac{1}{2} \oint x \, \mathrm{d}y - y \, \mathrm{d}x.$$

In vector calculus this contour integral is a special case of **Green's theorem**, which is itself a special case of the **curl theorem**. This is the theoretical basis for the real-life **planimeter** tool used to calculate areas.

Finitessimal Accretion: Solution

Any set with n elements has 2^n subsets. This follows from the fundamental counting principle: to construct a subset, we need to make a binary decision to either include or not include each of the n elements.

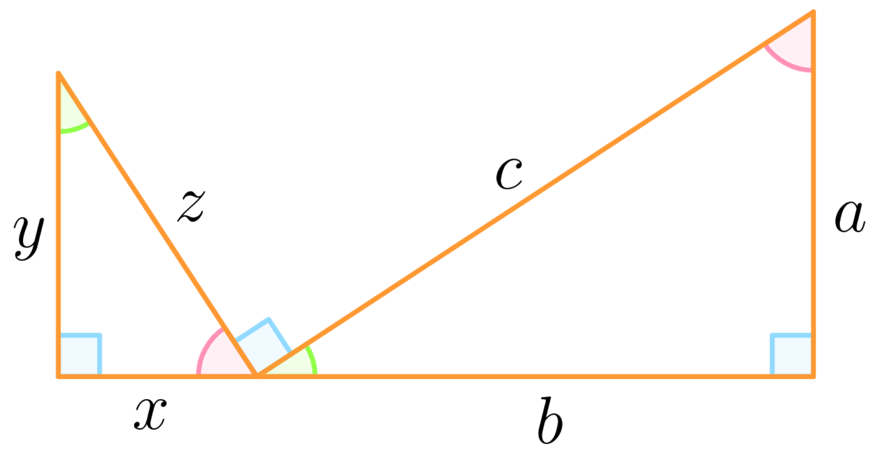
Let m_k be how many subsets of $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \cdots, \frac{1}{2^{2021}}\}$ have $\frac{1}{2^k}$ as its smallest element. Then m_k is how many times $\frac{1}{2^k}$ is counted in the sum.

Any subset X whose smallest element is $\frac{1}{2^k}$ is just a subset of $\{1, \frac{1}{2}, \cdots, \frac{1}{2^{k-1}}\}$ with $\frac{1}{2^k}$ added. There are k elements in $\{1, \frac{1}{2}, \cdots, \frac{1}{2^{k-1}}\}$, so there are 2^k subsets, i.e. $m_k = 2^k$. Summing this over all k we get an answer

$$1\Big(1\Big) + 2\left(\frac{1}{2}\right) + \cdots + 2^{2021}\left(\frac{1}{2^{2021}}\right) \;=\; 2022.$$

First Fold: Solution

Highlight and label the lower two triangles as follows:



The original top-left corner of the square is a right angle, which forces the two resulting triangles to be similar. The left side of the square is folded into $y + z = 1$ and the top side of the square becomes $c+? = 1$.

The Pythagorean theorem $x^2 + y^2 = z^2$ says $x^2 = z^2 - y^2 = (z - y)(z + y)$, which is just $z - y$. This establishes the linear system

$$\begin{cases} z + y = 1 \\ z - y = x^2 \end{cases} \implies \begin{cases} z = \frac{1}{2}(1 + x^2) \\ y = \frac{1}{2}(1 - x^2) \end{cases}$$

This follows from adding or subtracting, then halving, the equations.

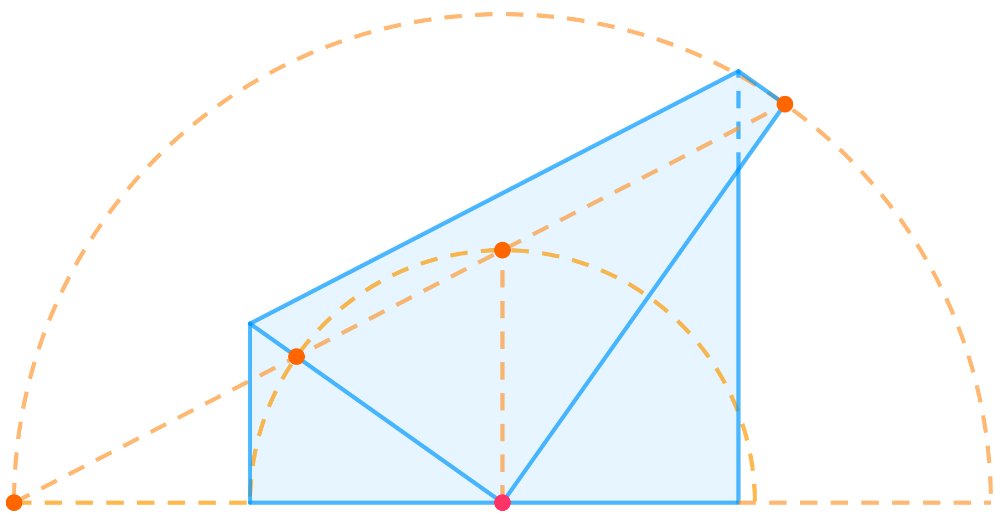
The similarity of the triangles means the proportions (ratios between corresponding sides) match. This is sometimes written $[a : b : c] = [x : y : z]$.

Given $c = 1$, equating ratios $[a : c] = [x : z]$ and $[b : c] = [y : z]$ yields

$$a = \frac{x}{z} = \frac{2x}{1 + x^2}, \qquad b = \frac{y}{z} = \frac{1 - x^2}{1 + x^2}.$$

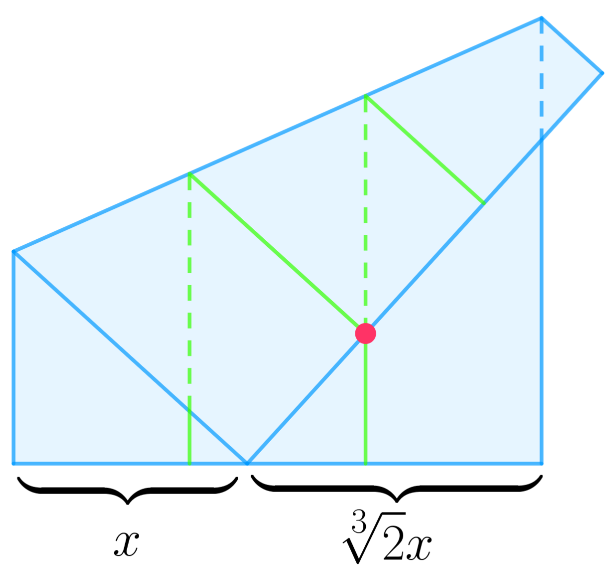
In fact, these are the formulas for stereographic projection!

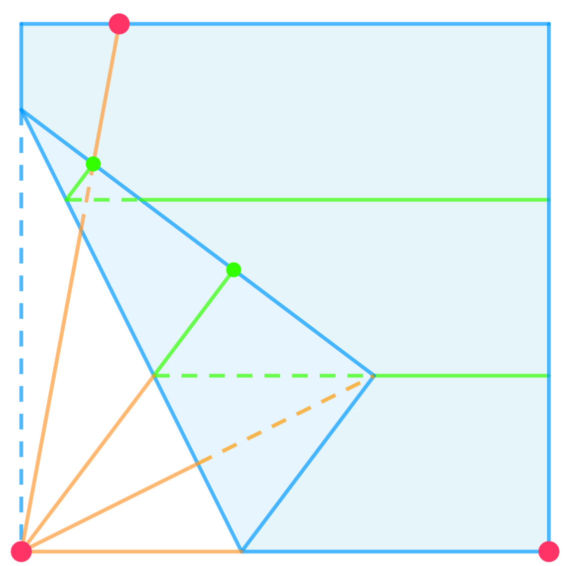
The picture of the fold with stereographic projection looks like:



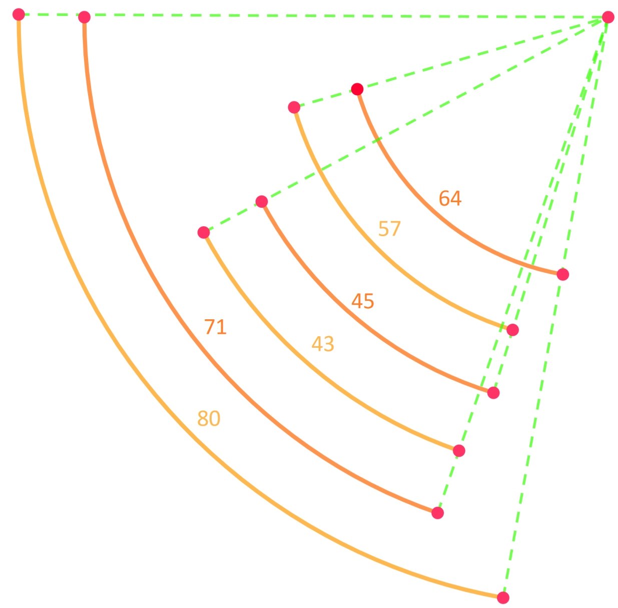
This is called a **Haga fold** in mathematical paper folding.

By drawing lines to split the square into thirds, Haga folds can be used to solve the problems of doubling the cube and trisecting the angle, which are intractable with compass-and-straightedge constructions.





Folding Point: Solution



After folding the first image onto the 80° region, the angles laid flat look like this. In order for the picture to make sense, though, the clockwise and counterclockwise angular dispalcements must exactly cancel each other out. In other words, the sum of the angles in the clockwise direction must match the sum of the angles in the counterclockwise direction. Calculate

$$71^\circ + 45^\circ + 64^\circ = 80^\circ + 43^\circ + 57^\circ$$
$$62^\circ + 63^\circ + 56^\circ \neq 39^\circ + 73^\circ + 67^\circ$$
$$64^\circ + 43^\circ + 71^\circ \neq 77^\circ + 49^\circ + 56^\circ$$

Therefore the first picture is the correct answer.

This exemplifies **Kawasaki's theorem**, which says the sum of every other angle must equal the sum of the other alternating set of angles for the paper to be flat-foldable. (But Kawasaki himself called it Husimi's theorem.) This also requires **Maekawa's theorem**: the number of creases must be even.

Origami is a branch of math!

Forty Two: Solution

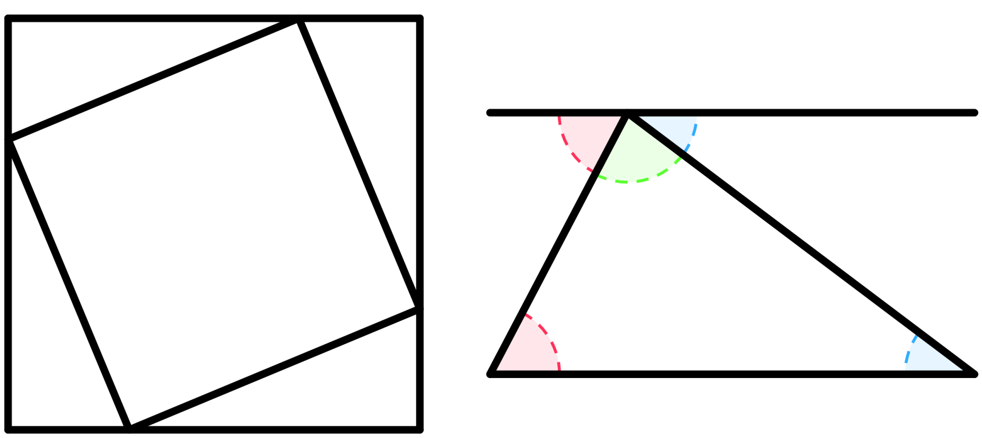
Each triangle has one 1, two 2s and three 3s. Thus, if all the numbers in a triangle are added together, the result is $1^2 + 2^2 + 3^2$. There are three triangles, so adding all the numbers up leaves us with $3(1^2 + 2^2 + 3^2)$.

On the other hand, instead of summing the numbers within a triangle first, we could instead add them *across* triangles. In the top position, we get $1 + 2 \times 3$. Moving from one position to another either adds 1, does nothing, or subtracts 1 - thus, the sum of the three numbers in a given position is constant! The first row has 1 position, the second row 2 positions, and the third row 3 positions, so the sum of all the numbers is $1+2+3$ times $1+2\times 3$.

The general version of this has three triangles with n rows, yielding

$$3(1^2 + 2^2 + \cdots + n^2) = (1 + 2 + \cdots + n)(1 + 2n)$$

This is an example of a so-called **proof without words**: a picture which, if studied closely, can reveal a complete explanation of an interesting mathematical fact. Ideally, these proofs are supposed to be “self-evident” from the pictures, however realistically some thought, guidance, or mathematical background is still often necessary for real understanding.



Can you see $a^2 + b^2 = c^2$ or $A + B + C = 180°$ above? Deeper facts may also have proof without words, e.g. Dandelin spheres or Monge’s Theorem.

Golden Architecture: Solution

2D Constructions. There are certain compass and straightedge constructions that can also be generalized to 3D and will be useful for our solution. We consider these as a warm-up.

One may construct the line through a point P which is perpendicular to the line L between points P and Q as follows. First, draw a circle C around P through Q, call the intersection between C and L the point Q'. Draw a circle D through Q' around Q and another circle D' through Q around Q', and let R be one of the intersection points between D and D'. Finally, draw the line between P and R.

Similarly, one can construct the midpoint of a line segment between points P and Q as follows. Draw a circle C around P through Q and another circle D around Q through P. Call the intersection points between circles C and D the points R and S. Draw the line L between R and S, and then the intersection M between the line L and the line segment from P to Q will be the midpoint of P and Q.

Given a known plane in 3D, any 2D construction may be performed in that plane using astrolabe and flatedge by intersecting with the plane at each step. For example, given two points C and X in the plane, construct the sphere around C through X, the intersect that sphere with the plane to get the circle around C through X within that plane.

Constructing Three Perpendicular Lines.

Given a point C, pick any two other points W and X and form the plane P between the three points C, W, X with the flatedge tool. Within the plane P, draw the line L from C to X and construct the perpendicular line M through C. Draw the sphere S around C through X and intersect with the line M to get another point Y. Say the intersection of S with line L consists of points X, X' and the intersection of S with line M consists of points Y, Y'. We want two points Z, Z' on an axis N perpendicular to L and M.

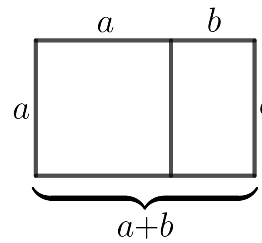
Form spheres around X through X' and vice versa and intersect to get a circle C in the YZ-plane perpendicular to L. Similarly, form spheres and Y

through Y' and vice-versa and intersect to get a circle D in the XZ-plane perpendicular to M. The circles C and D intersect desired points Z, Z' through which we may form the line N. Note in this picture Z, Z' are further from the center C than X, X', Y, Y'.

The three lines L, M, N are three perpendicular lines through C.

Constructing Golden Rectangles.

A rectangle is called *golden* if the smaller rectangle alongside an inscribed square is similar to (same proportions as) the whole rectangle:



The equation $(a+b)/a = a/b$ becomes $1 + 1/x = x$ if we define $x = a/b$, and solving the subsequent quadratic equation $x^2 - x - 1 = 0$ yields the golden ratio $\varphi = (1+\sqrt{5})/2$. Thus, a rectangle is golden if the proportion a/b between its sides is the golden ratio.

To construct a golden rectangle with a given center C in a plane, it suffices to construct four equal-size golden rectangles around it. Thus, given perpendicular lines L and M intersecting at a corner C in a plane, it suffices to be able to construct a golden rectangle with a given line segment $\overline{CX}$ along line L. By fiat declare this to be unit length.

Draw a circle around C through X and intersect with line M to get two points Y, Y'. Construct a line L' perpendicular to L through X, and another line M' perpendicular to M through Y. Call C' the intersection of lines L' and M'. Thus $\square CXC'Y$ is a unit square.

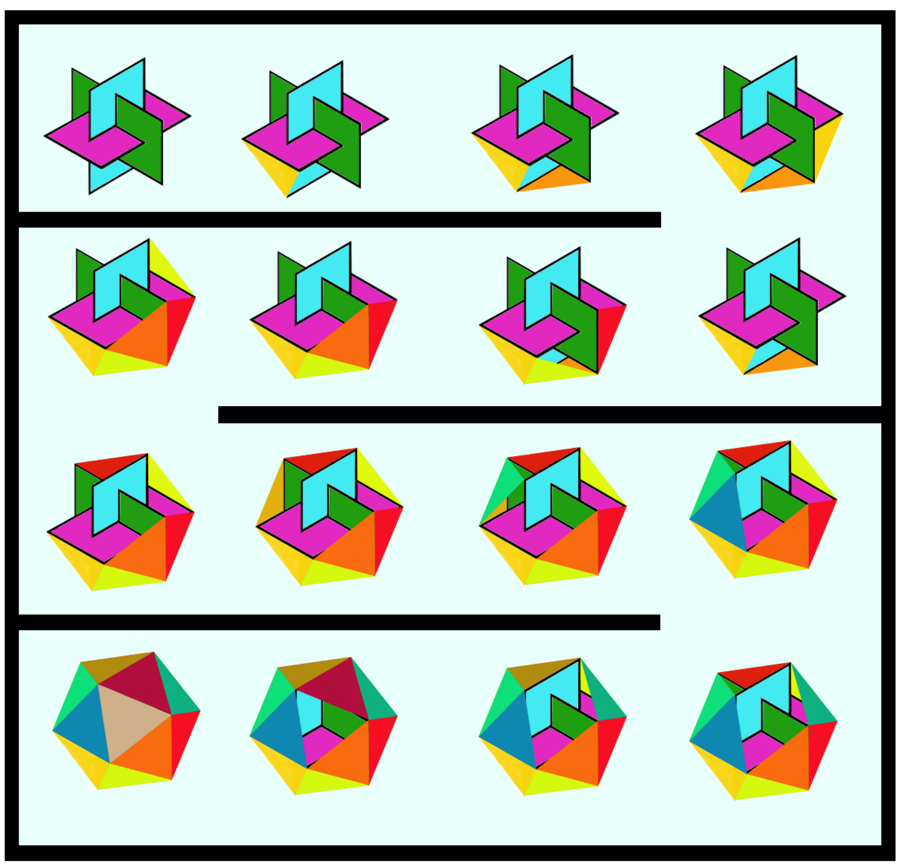
Next, draw a circle around Y' through C' and intersect with M to get a point W on the other side of Y' from C. Draw a circle around C through W and again intersect with M to get another point W'. Note W and W' are both a distance of $\sqrt{5}$ from C, by the Pythagorean theorem applied to the right triangle $\triangle C'YY'$.

Finally, construct the midpoint G of the line segment $\overline{CW'}$. Construct a line M'' perpendicular to M at G. The lines L' and M'' intersect at a point, say H. Then $CXGH$ is a golden rectangle.

Constructing Icosaheda.

Finally, after constructing three perpendicular lines intersecting at a point C, pick a point X on one of them, form a sphere around C through X and intersect with the three lines to get pairs X, X' and Y, Y' and Z, Z' on the three axes. And may use these line segments to construct three golden rectangles in the three corresponding planes.

By drawing line segments between neighboring corners of the golden rectangles we obtain a regular icosahedron.



Good Fibrations: Solution

Every neighboring pair of dodecahedra extends to a unique ring of ten dodecahedra, and every ring extends to a unique bundle of a dozen rings:

$$\text{pair} \to \text{ring} \to \text{bundle}$$

This is not a one-to-one correspondence, though: each bundle arises from any of its twelve rings, and similarly each ring arises from any of its ten neighboring pairs of dodecahedra.

To construct a neighboring pair of dodecahedra within a given ring, we can first pick one of the ten dodecahedra of the ring, then either of its two neighbors, but notice this overcounts by a factor of two since we can pick the two dodecahedra of a pair in two different orders - which is picked 1st vs 2nd.

Thus, there are $12 \times 10 = 120$ neighboring pairs per bundle.

To construct a neighboring pair in general, we can pick any of the 120 dodecahedra in the picture, then pick any of its 12 neighbors (a dodecahedron has twelve faces), and divide by 2 for the same reason as before.

Thus, there are $120 \times 12 \,/\, 2 = 720$ neighboring pairs in total.

Since there are 720 pairs total, and 120 pairs per bundle (and no pair shared between bundles), there must be $720/120 = 6$ bundles.

This counting argument also works in the game SET.

In SET, each of the cards has a picture with four features (color, shape, number, shading), each with three possible variations, for a total of $3^4 = 81$:

- color: red, purple, green
- shape: oval, squiggly, diamond
- number: one, two, three
- shading: blank, solid, hatching

A "SET" is three cards in which each feature either has the same variation on each card or all three variations. We can write down the equation

$$(\text{SETs}) \cdot (\text{pairs per SET}) = (\text{pairs}) \cdot (\text{SETS per pair})$$

There are $\binom{3}{2} = 3$ pairs of cards per SET, and there is 1 SET per pair (in any SET, the features of the third card are determined by those of the first two). And the total number of pairs is $\binom{3^4}{2}$, so the number of SETs is

$$\binom{3^4}{2} \Big/ \binom{3}{2} = 1080.$$

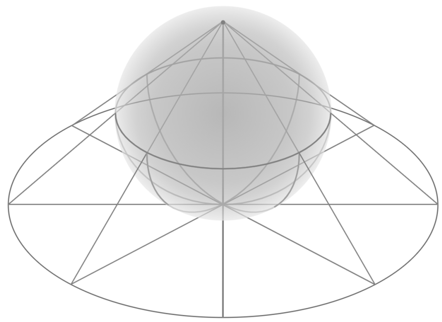
For dodecahedral bundles we used the same reasoning, with adjacent pairs of dodecahedra instead of pairs of cards and bundles of rings instead of SETs!

SET is an example of a **Steiner system**. A system $S(t, k, n)$ is a collection of k-subsets (called blocks) of an n-set for which every t-subset is contained within exactly one block. By our counting argument, there are $\binom{n-\ell}{t-\ell}/\binom{k-\ell}{t-\ell}$ blocks containing any ℓ-subset. SET is a $S(2, 3, 3^4)$ and $\ell = 0$ counts SETs.

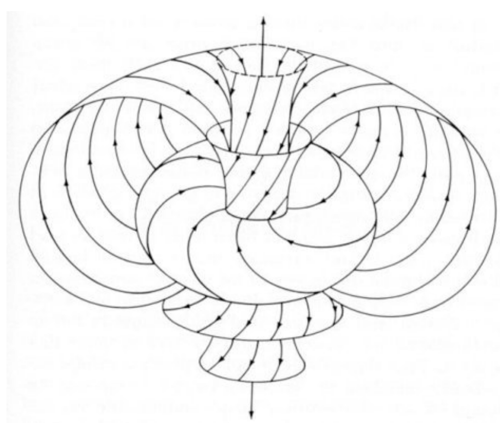
There are infinitely-many lines (not necessarily through the origin) in Euclidean space. If we consider 4D space, and instead of using real numbers for coordinates use the integers mod 3, then the vectors and lines respectively correspond to cards and SETs from the game SET!

This problem's title and bundle picture are taken from a post of the same name on the blog "Complex Projective 4-Space" by A.P. Goucher.
The dodecahedral bundles are discrete versions of the **Hopf fibration**.

Visualizing the fibration requires stereographic projection. Usually, we project a circle onto a line, or a sphere onto a plane, but for this, we need to project the "three-sphere" sitting in 4D down to 3D Euclidean space.



Just as a Möbius band is a bunch of line segments arranged in a circle, or a Klein bottle is a bunch of circles arranged in a circle, the three-sphere is, somewhat miraculously, a bunch of circles arranged in the shape of a (2D) sphere! When stereographically projected, that means all of 3D space is filled in with circles, with one "infinitely large" circle (i.e. a line).



The circles can be bunched together into wreaths (solid Dupin cyclides, to be exact), then those wreathes turned into rings of dodecahedra.

In 4D space, these dodecahedra are the cellular panels of the "120-cell" polytope. The centers of the dodecehdra form the dual polytope, the "600-cell," which is also the group of unit-length **icosians** in the quaternions. Because of how quaternions model 3D rotations, every antipodal pair of icosians corresponds to one of the 60 rotational symmetries of an icosahedron!

Gyration Conjugation: Solution

An icosehdron has 12 vertices, 30 edges, and 20 triangular faces. If we mark a distinguished vertex then any rotational symmetry may be specified by first saying which vertex the marked one is sent to and then choosing one of the five rotated orientations around that final vertex, for a total of $12 \times 5 = 60$ rotations. A similar argument works for edges (30×2) or faces (20×3). This is an instance of the **orbit-stabilizer theorem** in group theory.

By symmetry, there are three types of axis a rotational symmetry of the icosehedron can have: one through a vertex or through the midpoint of either an edge or a face. Every vertex corresponds to five rotations, every edge to two, and every face to three. In the first case, for example, we have fifths of a full turn, or multiples of $360°/5 = 72°$. (Angles are measured counterclockwise according to the right-hand rule.)

A reflex-angle rotation around a ray is the same as a convex-angle rotation around the opposite ray (e.g. 270° around the South Pole is the same as 90° around the North Pole), so WLOG we may consider only convex angles. Then, not counting 0°, every vertex has two angles, every pair of opposite edges has one angle, and every face has one angle. In summary:

order	angle	count	$\times$	commute	$=$	total
1	0°	1	$\times$	60	$=$	60
2	180°	15	$\times$	4	$=$	60
3	120°	20	$\times$	3	$=$	60
5	72°	12	$\times$	5	$=$	60
5	144°	12	$\times$	5	$=$	60

As $AB = BA$ is equivalent to $ABA^{-1} = B$, we can count the (A, B) that commute by counting for each type of rotation A the rotations B unchanged by conjugation, tabulated above. In general, B simply must have the same axis of rotation, unless A is 180° around an edge midpoint and then B can also have a perpendicular axis (through another inscribed rectangle). Then

$$P = \frac{5 \times 60}{60 \times 60} = \frac{5}{60} = \frac{1}{12}.$$

Halving Harmonics: Solution

Fill in the list $a_1, a_2, a_3, \cdots$ with powers-of-two for the non-square indices, and all other numbers in increasing order for square indices:

$$\sum_{k=1}^{n} \frac{1}{a_k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{3} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{5} + \cdots$$

In the first n terms there are $\sqrt{n}$ (rounded) harmonic terms, and the others are geometric terms. The harmonic terms can be increased by decreasing the denominators to simply be $1, 2, 3, \cdots, \sqrt{n}$, so the harmonic part is bounded above by $H_{\sqrt{n}} < \ln \sqrt{n} + 1$. The geometric part is bounded above by the infinite geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$.

Therefore, for any choice $0 < \varepsilon < \frac{1}{2}$ (setting $c = \frac{1}{2} + \varepsilon$), we have

$$\sum_{k=1}^{n} \frac{1}{a_k} < \left[\ln \sqrt{n} + 1\right] + 1 < \left(\frac{1}{2} + \varepsilon\right) \ln n < c \sum_{k=1}^{n} \frac{1}{k}$$

for all $n > N$ so long as $1 + 1 < \varepsilon \ln N$.

Heat of Battle: Solution

The idea is to overlay all ship configurations together to get a **heat map** which for each cell counts how many configurations have a ship occupying it.

Wherever a 3-tile ship is, we can rotate or flip the board so that it is either vertical in the top left corner, or just to the right of that. In both cases, determine how many 2-tile ship placements cover the other cells (the yellow values $1, 2, 3, 4$ below). Each configuration, the 2-tile ship contributes $+2$ to the yellow total, so the number of configurations with the 3-tile ships is the yellow totals divided by 2 (the yellow-orange values $18, 15$).

18	2	3	2
18	3	4	3
18	3	4	3
1	3	3	2

&

1	15	2	2
2	15	3	3
2	15	3	3
2	2	3	2

For both arrays above, there are a total of four rotated versions of the array, and then four more arrays that can be obtained by flipping those rotated versions. Instead of writing down sixteen arrays and then adding sixteen numbers for each of the sixteen cells, we can see what rotating and flipping does to individual cells. For example, any one corner cell when rotated and flipped lands at each of the four corner cells twice.

Considering this for all cells (or even just for three appropriate cells in a corner 2×2 block), we fill in the grid with sky, blue, and purple multipliers shown in the next figure. After adding the yellow and yellow-orange grids above together, we must apply the multipliers and then sum all the values in cells of a given multiplier color to get our final heatmap.

For example, after adding the grids, the corners clockwise from the top left are $18 + 1 = 19$, $2 + 2 = 4$, $2 + 2 = 4$, and $1 + 2 = 3$; doubling these values gives 38, 8, 8, and 6; adding these values gives $38 + 8 + 8 + 6 = 60$.

x2	x1	x1	x2
x1	x2	x2	x1
x1	x2	x2	x1
x2	x1	x1	x2

=

60	85	85	60
85	100	100	85
85	100	100	85
60	85	85	60

23% 32% 38%

Each cell counts how many configurations have a ship occupying it, but to get probabilities we need to know how many configurations there are in total; each configuration contributes +5 tiles to the total sum of orange-red values above, so if we add all these orange-red values and divide by 5 we get a total of $4(60 + 2 \cdot 85 + 100)/5 = 264$ configurations. Dividing the orange-red values by 264 gives the probabilities listed below the grid above.

Computing the heatmap for a larger battleship grid using more pieces of more shapes would require a computer.

The video game *The Legend of Zelda: The Wind Waker* includes an 8×8 battleship minigame, "Sploosh Kaboom," which was once the bane of speedrunners because the game generates the ship configuration randomly. (Speedrunning is a competetive activity where very capable players beat video games, or certain levels in them, under certain conditions as fast as possible.)

A breakthrough occured after an app was created which not only uses a heatmap to show where the most probable hits are, but *updates* the heatmap after each hit or miss by throwing out all configurations inconsistent with the hit or miss. This is effectively an example of a **Bayesian search**.

Bayesian statistics has been applied successfully to real life searches where traditional searches failed (with cost-benefit predictions to boot): many lost ships and flights have been found from the ocean depths, including even a lost hydrogen bomb. The U.S. Coast Guard and U.S. Air Force adopted it for use in search and rescue operations after its success.

Heisenberg: Solution

(I). We show ab^{-1} and $b^{-1}a$ commute:

$$\begin{array}{rccc} & (ab^{-1})(b^{-1}a) & = & (b^{-1}a)(ab^{-1}) \\ \iff & ab^{-2}a & = & b^{-1}a^2b^{-1} \\ \iff & aba & = & b^{-1}a^{-1}b^{-1} \\ \iff & aba & = & (bab)^{-1} \\ \iff & (aba)(bab) & = & e \\ \iff & (ab)(ab)(ab) & = & e \\ \iff & (ab)^3 & = & e \end{array}$$

(II). And a commutes with $aba^{-1}b^{-1}$ because (much harder):

$$\begin{array}{rrcll} & a(aba^{-1}b^{-1}) & = & (aba^{-1}b^{-1})a & (1) \\ \iff & a(aba^{-1}b^{-1})a^{-1}(aba^{-1}b^{-1})^{-1} & = & e & (2) \\ \iff & a(aba^{-1}b^{-1})a^{-1}(bab^{-1}a^{-1}) & = & e & (3) \\ \iff & a^{-1}ba^{-1}b^{-1}a^{-1}bab^{-1}a^{-1} & = & e & (4) \\ \iff & a^{-1}b(bab)bab^{-1}a^{-1} & = & e & (5) \\ \iff & a^{-1}b^{-1}ab^{-1}ab^{-1}a^{-1} & = & e & (6) \\ \iff & a^{-1}b^{-1}ab^{-1}ab^{-1} & = & a & (7) \\ \iff & ab^{-1}ab^{-1}ab^{-1} & = & e & (8) \\ \iff & (ab^{-1})^3 & = & e & (9) \end{array}$$

(1) $\Rightarrow$ (2) rewrites $xy = yx$ as $xyx^{-1}y^{-1} = e$, where $x = a$ and $y = aba^{-1}b^{-1}$; (2) $\Rightarrow$ (3) uses the socks-and-shoes rule; (3) $\Rightarrow$ (4) rewrites a^2 as a^{-1} in the front; (4) $\Rightarrow$ (5) rewrites $a^{-1}b^{-1}a^{-1} = (aba)^{-1}$ as bab (compare with the middle of the last derivation); (5) $\Rightarrow$ (6) rewrites b^2 as b^{-1}; (6) $\Rightarrow$ (7) $\Rightarrow$ (8) right-multiplies by a and left-multiplies by a^{-1} (replacing a^{-2} with a).

Suppose G is freely generated by a and b, or in other words all group elements are products of powers of a and b, and it is not possible to express a or b in terms of each other (in particular e, a, a^2, b, b^2 are all distinct).

In the first derivation **(I)**, the observation $(ab)^3 = e$ is equivalent to aba and bab being inverses is prescient. Another consequence, to be used momentarily:

$$\begin{array}{rrcccl} & aba & = & (ab^{-1})(b^{-1}a) & = & pq \\ \iff & bab & = & (pq)^{-1} & = & q^{-1}p^{-1}, \end{array}$$

denoting $p = ab^{-1}$ and $q = b^{-1}a$ for convenience.

Interpret the equation $(ab^{-1})b = a = b(b^{-1}a)$ as a **sliding rule**: a recipe for how to slide one group element past another (with compromises along the way). In particular, the rule $pb = bq$ says we can slide p past b from left to right as long as we turn the p into q, or conversely we can slide q past b from right to left as long as we turn the q into a p. But then how do we slide p and q past b the other directions? Using $b^{-1} = b^2$ we can determine

$$\begin{aligned} qb &= (b^{-1}a)b \\ &= b(bab) \\ &= bq^{-1}p^{-1} \end{aligned} \qquad \text{and} \qquad \begin{aligned} bp &= b(ab^{-1}) \\ &= (bab)b \\ &= q^{-1}p^{-1}b \end{aligned}$$

In conclusion, if we have an expression which is a bunch of ps and qs on one side of b, these sliding rules let us convert it into an expression with a (probably different) bunch of ps and qs on the other side of b. Since p and q also commute, we can conclude all group elements can be put into a "standard form" like $b^u p^v q^w$ with $-1 \le u, v, w \le 1$ (or $0 \le u, v, w \le 2$, same difference); in particular, this means the order (cardinality) is $|G| = 3^3 = 27$.

In **(II)** we work with the **commutator** $[a, b] := aba^{-1}b^{-1}$ of two elements a and b, so-called because it "measures" the extent to which a and b fail to commute. (This intuition extends further to describe the structure of a group; see *central series* and *nilpotence class*.) In particular, two elements commute ($xy = yx$) if and only if the commutator is trivial ($[x, y] = e$).

Our derivation in **(II)** only showed a commutes with $[a, b]$, or in other words $[a, [a, b]] = e$, but it is possible to show this implies $[b, [a, b]] = e$ too.

Note $xy = xy$ implies $y^{-1}x = xy^{-1}$ and $yx^{-1} = x^{-1}y$ (multiply on the left or right by x^{-1} or y^{-1} appropriately); also, it implies e.g. $x^2y = xxy = xyx = yxx = yx^2$; similar reasoning shows any power of x commutes with any power of y (positive or negative). Socks-and-shoes implies $[x, y]^{-1} = [y, x]$. By symmetry, we could have done the derivation in **(II)** with the letters a and b swapped, which gives $[b, [b, a]] = e$, which thus implies $[b, [a, b]] = e$.

Since $c := [a, b]$ commutes with G's generators a and b, it is **central**: it commutes with all group elements. We can interpret $ab = cba$ or $ba = abc^{-1}$ as another sliding rule for how to move a and b past each other, from which we may conclude all group elements are expressible in a standard form like $a^u b^v c^w$ with $-1 \leq u, v, w \leq 1$ (or $0 \leq u, v, w \leq 2$, if so inclined).

We can also express these ideas in the esoteric language of group theory.

For **(I)**, consider the subgroup $H = \langle ab^{-1}, b^{-1}a \rangle$ of G. To show it's normal, it suffices to check conjugating H's generators by G's generators doesn't leave H: both $a(b^{-1}a)a^{-1}$ and $b(b^{-1}a)b^{-1}$ simplify to ab^{-1}, and both $a(ab^{-1})a^{-1}$ and $b(ab^{-1})b^{-1}$ simplify to bab, which we found earlier is $(b^{-1}a)^{-1}(ab^{-1})^{-1}$.

We can say "$a \equiv b \bmod H$" because $b^{-1}a$ and ab^{-1} are in H. Thus in the quotient group G/H, all bs turn into as and so all elements can be represented by a power a^u with $0 \leq u \leq 2$. Moreover, ab^{-1} and $b^{-1}a$ commute and have order 3 in H, so H is elementary abelian of order $3^2 = 9$. From this we can conclude the order of G is $|G| = [G : H]|H| = 3 \cdot 3^2 = 27$.

Or for **(II)**, consider the subgroup $K = \langle [a, b] \rangle$. It is cyclic of order 3. As $[a, b]$ is central, so is K, so in particular it is normal. We can say "$ab \equiv ba \bmod K$" because $(ab)(ba)^{-1}$ is in K. Thus in the quotient group G/K all elements are expressible as $a^u b^v$ with $0 \leq u, v \leq 2$, or in other words G/K is elementary abelian of order 3^2. Once again, $|G| = [G : K]|K| = 3^2 \cdot 3 = 27$.

Our group G has an explicit matrix representation, by writing

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this matrix group, numbers are interpreted mod 3, meaning all matrix entries are in $\mathbb{F}_3 = \{0, 1, 2\}$ where addition and multiplication "wrap around" (for comparison, clock arithmetic is mod 12), which means e.g. $-1 \equiv 2$ represent the same scalar. Here, $c = aba^{-1}b^{-1}$, and the group of matrices generated by a and b using matrix multiplication are the unitriangular ones:

$$G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \middle| \, x, y, z \text{ in } \mathbb{F}_3 \right\}.$$

This is the **Heisenberg group** $H_3(\mathbb{F}_3)$. The continuous version $H_3(\mathbb{R})$ (which uses real numbers instead of integers mod 3) has infinitesimal generators analogous to a and b which represent position and momentum operators in quantum mechanics (also present in the Heisenberg uncertainty principle).

The **Burnside group** $B(k, n)$ is the "free"-est group of exponent n generated by k generators. That means all group elements are products of powers of generators $a_1, \cdots, a_k$ and the only relations that exist between the generators are those that can be derived from the assumption that $g^n = e$ for all group elements g. Our group is $H_3(\mathbb{F}_3) = B(2, 3)$. In general, if $n = 3$ the Burnside group has order $|B(k, 3)| = 3^{\binom{k}{1}+\binom{k}{2}+\binom{k}{3}}$. While it is known whether or not $B(k, n)$ is finite for many small values of (k, n), no general rule is known.

The complexity of the derivation for **(II)** is not at all an outlier in computational group theory. The **word problem** for groups asks if there is an algorithm that, when given a group (presented by a set of generators and relations between them) can decide when two "words" represent the same element. It turns out to be **undecidable**: there is no such algorithm.

Homogenization: Solution

For the two-variable version, we may combine like terms:

$$a_0 + a_1(x + y) + a_2(x^2 + xy + y^2) + a_3(x^3 + x^2y + xy^2 + y^3) + \cdots$$

(Arbitrary rearrangement and grouping is legal since we do not need to worry about convergence issues.) This means

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m+n}x^m y^n = \sum_{s=0}^{\infty} a_s(x^s + \cdots + y^s) = \sum_{s=0}^{\infty} a_s \left(\frac{x^{s+1} - y^{s+1}}{x - y}\right)$$

$$= \frac{x\left(\sum_{s=0}^{\infty} a_s x^s\right) - y\left(\sum_{s=0}^{\infty} a_s y^s\right)}{x - y} = \frac{xf(x) - yf(y)}{x - y}.$$

For the four-variable version, notice that the sum of all monomials $w^k x^\ell y^m z^n$ such that $k + \ell = r$ and $m + n = s$, where r and s are fixed, can be factored as (the sum of all monomials $w^k x^\ell$ with $k + \ell = r$) times (the sum of all monomials $y^m z^n$ with $m + n = s$). This is because we may choose the pairs (k, ℓ) and (m, n) independently of each other when deciding on which monomial $w^k x^\ell y^m z^n$ to write down. Therefore

$$\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{k+\ell+m+n} w^k x^\ell y^m z^n$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r+s}(w^r + \cdots + x^r)(y^s + \cdots + z^s) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r+s}\left(\frac{w^{r+1} - x^{r+1}}{w - x}\right)\left(\frac{y^{s+1} - z^{s+1}}{y - z}\right)$$

$$= \frac{wy\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} w^r y^s\right) - wz\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} w^r z^s\right) - xy\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x^r y^s\right) + xz\left(\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} x^r z^s\right)}{(w - x)(y - z)}$$

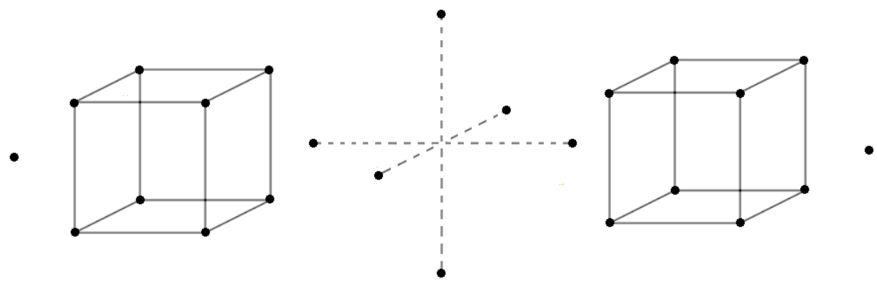
$$= \frac{wy\left(\frac{wf(w) - yf(y)}{w - y}\right) - wz\left(\frac{wf(w) - zf(z)}{w - z}\right) - xy\left(\frac{xf(x) - yf(y)}{x - y}\right) + xz\left(\frac{xf(x) - zf(z)}{x - z}\right)}{(w - x)(y - z)}$$

$$= \frac{w^3 f(w)}{(w - x)(w - y)(w - z)} + \frac{x^3 f(x)}{(x - w)(x - y)(x - z)} + \frac{y^3 f(y)}{(y - w)(y - x)(y - z)} + \frac{z^3 f(z)}{(z - w)(z - y)(z - x)}.$$

Hyperdiamond: Solution

How are the points (w, x, y, z) arranged?

The $w = \pm 1$ cross-sections each have a single point. The $w = 0$ cross-section has the vertices of a regular octahedron (the six unit vectors along the 3D coordinate axes), but no edges within this slice. The $w = \pm\frac{1}{2}$ cross-sections each contain a unit cube's worth of vertices and edges.



The points with $w = \pm 1$ are connected to all vertices of the cube in the corresponding $w = \pm\frac{1}{2}$ slice. Each cube vertex is connected to the corresponding vertex of the other cube. Each cube face corresponds to an octahedral vertex, and any cube vertex is connected to an octahedral vertex corresponding to an adjacent cube face.

(a) and (b):

First, let's find how many regular hexagons are incident to $(1, 0, 0, 0)$, by first considering which cross-section an adjacent vertex may lie.

The distances from $(1, 0, 0, 0)$ to other points are as follows:

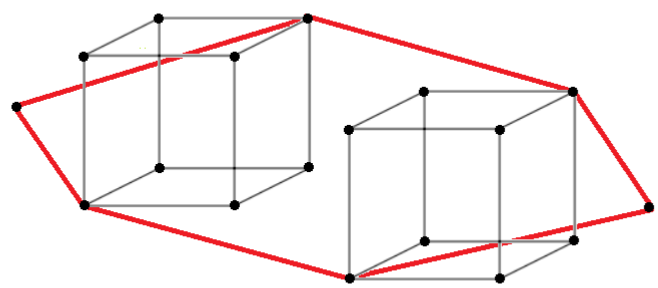
- $\sqrt{1}$ for those in the $w = +\frac{1}{2}$ slice,
- $\sqrt{2}$ for those in the $w = 0$ slice,
- $\sqrt{3}$ for those in the $w = -\frac{1}{2}$ slice, and
- $\sqrt{4}$ for those in the $w = -1$ slice.

The second point in a regular hexagon must be 1 unit away, which forces it to be any of the eight points in the $w = \frac{1}{2}$ slice.

The third point must be 1 unit away from the second and $\sqrt{3}$ units from the first, forcing it to be the cube vertex in the $w = -\frac{1}{2}$ slice corresponding to the one chosen second in the $w = \frac{1}{2}$ slice. So if we picked $\frac{1}{2}(1, 1, 1, 1)$ for the second point, then the third must be $\frac{1}{2}(-1, 1, 1, 1)$.

For the next three points, use the fact the regular hexagon is symmetric across the origin. This means it must contain the opposites of the three points we already picked. This leaves us with one hexagon:

$$H_1 = \{(\pm 1, 0, 0, 0), \tfrac{1}{2}(\pm 1, \underbrace{\pm 1, \pm 1, \pm 1}_{\text{same sign}})\}.$$



Notice the entire hexagon was determined by the choice of second point. There was some redundancy in this choice, however, since the hexagon ultimately ended up having a pair of vertices antipodal within the $w = \frac{1}{2}$ slice's cube, $\frac{1}{2}(1, 1, 1, 1)$ and $\frac{1}{2}(1, -1, -1, -1)$.

Therefore, the number of hexagons incident to $(1, 0, 0, 0)$ is the number of antipodal pairs of vertices of the cube, which is 4. By symmetry, every point of the 24-cell is incident to 4 hexagons. Therefore,

$$(\#\text{hexagons})(\#\text{points per hexagon}) = (\#\text{points})(\#\text{hexagons per point})$$

implies the number of hexagons is $24 \times 4 \,/\, 6 = 16$. Alternatively, we can note that the second choice of vertex determining the rest of the hexagon is equivalent to saying a single edge (between the first and second vertex) fully determines the hexagon. Therefore,

$$(\#\text{hexagons})(\#\text{edges per hexagon}) = \#\text{edges}$$

implies the number of hexagons is $96 \,/\, 6 = 16$. The edge count

$$2(8 + 12 + 24) + 8 = 96$$

follows from counting edges between the five w slices:

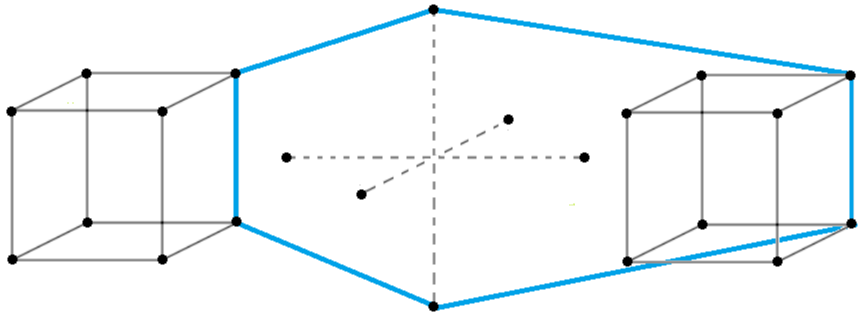
- 8 edges from $w = 1$ to the $w = \frac{1}{2}$,
- 12 edges within the $w = \frac{1}{2}$ slice,
- 24 edges between the $w = \frac{1}{2}$ slice and the $w = 0$ slice:
  - 8×3 by picking a vertex in $w = \frac{1}{2}$ then $w = 0$, or
  - 6×4 by picking a vertex in $w = 0$ then $w = \frac{1}{2}$,
- by symmetry, ditto for the negative side of $w = 0$,
- 8 edges between the $w = \pm\frac{1}{2}$ slices.

In summary, every edge is contained in exactly one hexagon, so there are 16 hexagons and they form a partition of the 96 edges.

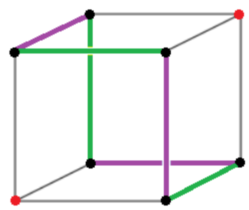
(c) and (d):

We will construct a bundle by first picking a hexagon H_1, then a disjoint hexagon H_2, and so on. At each stage, we note how many hexagons we have to choose from. By symmetry, we expect the number of hexagons available at any stage is independent of which hexagons were chosen before. By the fundamental counting principle, we multiply the numbers to count the ordered bundles (H_1, H_2, H_3, H_4).

Even before this, though, there are 16 hexagons total and 4 through $w = \pm 1$; where are the other 12? Pick any of the 12 edges in the $w = \frac{1}{2}$ slice, pick the corresponding edge in the $w = -\frac{1}{2}$ slice, then connect them to the endpoints of the parallel axis in the $w = 0$ slice:



We've already picked H_1 out of 16 options. Our next hexagon H_2 cannot use either of the vertices in $w = \pm 1$ so must entirely be contained in the middle three slices. It must also avoid the cube-antipodal points already used in the $w = \pm\frac{1}{2}$ slices, as well as all of the edges connected to them. This leaves the following six edges available:



Any of the six edges may be chosen, but the next three hexagons must be vertex-disjoint, so we must choose all three edges of one or the other color and determine three hexagons H_2, H_3, H_4 from them.

Picking the three green edges, we have the following bundle:

$$\begin{aligned} H_1 &= \{(\pm 1, 0, 0, 0), \tfrac{1}{2}(\pm 1, \overbrace{\pm 1, \pm 1, \pm 1}^{\text{same sign}})\}, \\ H_2 &= \{(0, \pm 1, 0, 0), \tfrac{1}{2}(\pm 1, \pm 1, -1, +1)\}, \\ H_3 &= \{(0, 0, \pm 1, 0), \tfrac{1}{2}(\pm 1, +1, \pm 1, -1)\}, \\ H_4 &= \{(0, 0, 0, \pm 1), \tfrac{1}{2}(\pm 1, -1, +1, \pm 1)\}. \end{aligned}$$

There are 4! permutations of (H_1, H_2, H_3, H_4), and when constructing the bundle we had 16 options for H_1 followed by $3! = 6$ permutations of 2 color choices for the remaining three hexagons H_2, H_3, H_4, thus

$$\#\text{bundles} = \frac{16 \times 6 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 8.$$

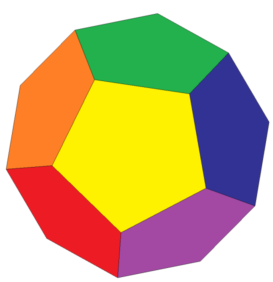
Viewing (w, x, y, z) as a quaternion $w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the 24-cell Q_{24} is a finite group under multiplication which contains the usual quaternion group $Q_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. The elements of orders $1, 2, 3, 4, 6$ correspond to vertices a distance of $\sqrt{0}, \sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}$ away from 1 in the slices $w = 1, -1, -\frac{1}{2}, 0, \frac{1}{2}$.

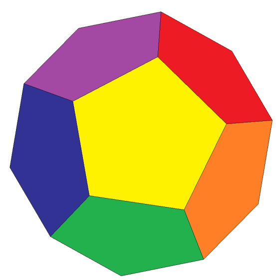
The order six subgroups of Q_{24}, which are cyclic, are the four hexagons through ± 1, and are all conjugate. The 16 hexagons are the cosets of these subgroups, and the 8 bundles are the left and right coset spaces of the four hexagon subgroups. (Note every left coset is a right coset of a conjugate subgroup, but left coset spaces are not right coset spaces.)

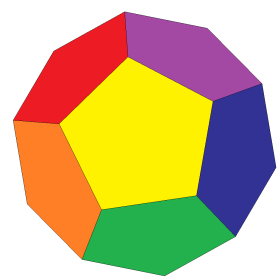
Icosian Palette: Solution

All six colors must be present on the dodecahedron. Once one half (one face and its five neighbors) has been painted, the other half is determined: every face must be painted the same color as the face on the opposite side. This is shown with front and back sides of an example below.

To see why this is true, suppose we start by painting one face our favorite color. The first ring of five neighboring faces must exhibit the five other colors. Call this the "front" side. The second ring of five faces beyond that, on the back side, cannot be colored the first color, or else one of the five faces in the first ring would be adjacent to two faces of the same color. This backside ring of five faces must exhibit five different colors, since they are all adjacent to the final face they surround and which is opposite the very first one. As the backside ring exhibits the same five colors as the frontside ring, that leaves only the first color for the final face opposite the very first one.


(front from front pov)


(back from front pov)


(back from back pov)

(The second picture above is what is seen if the front half is deleted, the third is what is seen if the solid is rotated 180° in 3D around the North-South axis.)

Thus, any painting of the dodecahedron is determined by the ring of colors around the face with our favorite color. There are 5! ways to put five colors to five faces, but we must take into account these are equivalent under rotation and reflection (from looking at the ring on the opposite side). There are five rotations and five reflections of a pentagon, ten total, so our answer is

$$\frac{5!}{10} = 12.$$

The title of the problem is based on the *Icosian Game*, a toy puzzle invented by mathematician William Rowan Hamilton in the late 1850s. The toy challenges the player to trace a circuit along a dodecahedron's edges which connects all the vertices, hitting each exactly once. Such a path is today called a Hamiltonian circuit. A general version of the problem was posed a couple years earlier by the graph theorist Kirkman.

(Royal Irish Academy Library, 2016)

Hamilton himself considered the game as a way to apply what he called the icosian calculus, a way to describe the symmetries of the dodecahedron (which are the same as those of an icosahedron, because they are dual polytopes) in the language of group theory, aptly calling the symmetries "noncommutative roots of unity." His friend John T. Graves suggested turning the puzzle into a commercial venture (which apparently failed).

Isoepiareal Ratio: Solution

Let $Q = V^2/S^3$ by the isoepiareal ratio. Suppose a cuboid with maximal Q has dimensions x, y, z so its volume and surface area are

$$V = xyz, \qquad S = 2xy + 2yz + 2zx.$$

Solution 1. If Q is maximized, then so is $\ln Q$, which is given by

$$\ln Q = 2\ln(xyz) - 3\ln(xy + yz + zx) - 3\ln 2.$$

If two of x, y, z are fixed while the third is changed, the ratio is decreased, so $\ln Q$ has a local maximum and its first derivative (with respect to the nonconstant variable) must vanish. Thus,

$$\begin{aligned}
\frac{\partial \ln Q}{\partial x} &= \frac{2}{x} - \frac{3(y+z)}{xy+yz+zx} &= 0 \\
\frac{\partial \ln Q}{\partial y} &= \frac{2}{y} - \frac{3(x+z)}{xy+yz+zx} &= 0 \\
\frac{\partial \ln Q}{\partial z} &= \frac{2}{z} - \frac{3(x+y)}{xy+yz+zx} &= 0
\end{aligned}$$

Solve for $\frac{2}{3}(xy + yz + zx)$ in each, equate the results:

$$x(y + z) = y(x + z) = z(x + y).$$

Subtracting pairs of expressions from this equation gives

$$0 = (x - y)z = (y - z)x = (x - z)y.$$

Since $x, y, z > 0$, the differences (e.g. $x - y$) are zero, so $x = y = z$.

This means the cuboid must be a cube.

Solution 2. Let a, b, c be the products xy, yz, zx. Then $V^2 = abc$ and $S = 2(a + b + c)$. With surface area S held constant, the AM-GM inequality implies the volume is bounded above by

$$\sqrt[3]{abc} \le \frac{a+b+c}{3} \implies V^{2/3} \le \frac{S}{6} \implies \frac{V^2}{S^3} \le \frac{1}{6^3}$$

with equality (hence when the ratio is maximized), crucially, if and only if $a = b = c$, which in turn is equivalent to $x = y = z$.

Interesting Asymptotic: Solution

By repeatedly differentiating $\ln(1+x)$ we can reasonably guess, and then prove, a formula for its nth derivative, and then determine the coefficients of its Taylor-Maclaurin power series. Alternatively, we can find the definite integral of the geometric series for $1/(1+t)$ from 0 to x.

Either way, we arrive at the so-called Newton-Mercator series:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Thus we may rewrite $\frac{1}{e}(1+\frac{1}{n})^n$ as powers of e and then

$$\begin{aligned}
\frac{1}{e}\left(1+\frac{1}{n}\right)^n &= \exp\left[-1 + n\ln\left(1+\frac{1}{n}\right)\right] \\
&= \exp\left[-1 + n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right] \\
&= \exp\left(-\frac{1}{2n} + \frac{1}{3n^2} - \cdots\right) \\
&= 1 + \left(-\frac{1}{2n} + \frac{1}{3n^2} - \cdots\right) + \frac{1}{2!}\left(-\frac{1}{2n} + \cdots\right)^2 + \cdots \\
&= 1 - \frac{1}{2n} + \frac{1}{3n^2} + \frac{1}{8n^2} + \cdots
\end{aligned}$$

And therefore we conclude $a = -\frac{1}{2}$, $b = \frac{1}{3} + \frac{1}{8} = \frac{11}{24}$, or in other words

$$\frac{1}{e}\left(1+\frac{1}{n}\right)^n \approx 1 - \frac{1}{2n} + \frac{11}{24n^2}.$$

Involutive Units: Solution

Solution 1. Suppose n is tight and the $(j+1)$st prime p_{j+1} is the largest prime not dividing n, so n must be a factor of $p_{j+1}^2 - 1$, and indeed $n < p_{j+1}^2$.

On the other hand, n is divisible by the first j primes $p_1, p_2, \cdots, p_j$ so it is divisible by their product, hence their product satisfies $p_1 p_2 \cdots p_j \leq n$.

Putting this together we conclude $p_1 p_2 \cdots p_j < p_{j+1}^2$.

Check when this comparison first fails:

$$\begin{array}{lcl} 2 \cdot 3 & < & 5^2 \\ 2 \cdot 3 \cdot 5 & < & 7^2 \\ 2 \cdot 3 \cdot 5 \cdot 7 & > & 11^2 \end{array}$$

It will follow that $p_1 p_2 \cdots p_k > p_{k+1}^2$ for all $p_{k+1} \geq 11$, since if it holds for one prime p_{k+1} on the right, then for the next prime p_{k+2} we have

$$p_1 p_2 \cdots p_k p_{k+1} > p_1 p_2 \cdots p_k \cdot 4 > p_{k+1}^2 \cdot 4 > p_{k+2}^2.$$

This uses Bertrand's postulate, which implies $p_{k+2} < 2p_{k+1}$.

If the largest prime not dividing n is $p_{k+1} = 7$, then n is a factor $7^2 - 1 = 48$. This would imply 5 is not a factor of n. Therefore n is a factor of one of the numbers $5^2 - 1 = 24$ or $3^2 - 1 = 8$ or $2^2 - 1 = 3$.

It turns out the tight numbers are precisely the factors of 24:

$$n = 1, 2, 3, 4, 6, 8, 12, 24.$$

To check that one of these numbers n is tight, we can't directly check it is a factor of all $x^2 - 1$ for *all* relatively prime values x because there are infinitely many such values of x! But, it suffices to check n is a factor of $x^2 - 1$ for *just* relatively prime values $x < n$. This is because if y is any value relatively prime to n and x is its remainder upon division by n then $y^2 - 1 = (x^2 - 1) + (2k + n)n$ has n as a factor if and only $x^2 - 1$ does.

Manually check each listed number is right.

Solution 2. The condition that n is a factor of $x^2 - 1$ for values x coprime to n may be restated as $x^2 \equiv 1 \bmod n$ for all units x mod n.

In other words, all nontrivial elements of the unit group $U(n) \overset{\text{def}}{=} (\mathbb{Z}/n\mathbb{Z})^\times$ have order two, i.e. they are all **involutions** and $U(n)$ has exponent two.

The Chinese Remainder Theorem indicates $U(n)$ is a direct product of $U(p^v)$ for all prime powers p^v in n's prime factorization. This group, for odd primes, is cyclic of order $\phi(p^v) = p^{v-1}(p-1)$, hence if $p - 1 > 2$ then $U(n)$ contains elements of order not 2. Thus, if n is tight it cannot be divisible by any prime $p > 3$ and can only be divisible by 3 at most once. For $p = 2$, we have $U(2^w) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{w-2}}$, which contains an element of order 4 if $w - 2 > 1$, so if n is right it can only be divisible by 2 at most 3 times.

In conclusion, $n = 2^v 3^w$ with $v \in \{0, 1, 2, 3\}$ and $w \in \{0, 1\}$.

Joker's Wild: Solution

Suppose a player's current score is X. Let $c_1, \cdots, c_{k-1}, J$ be the cards they haven't drawn yet. The expected value of the next hand is then

$$E = \frac{c_1 + \cdots + c_{k-1} - X}{k}$$

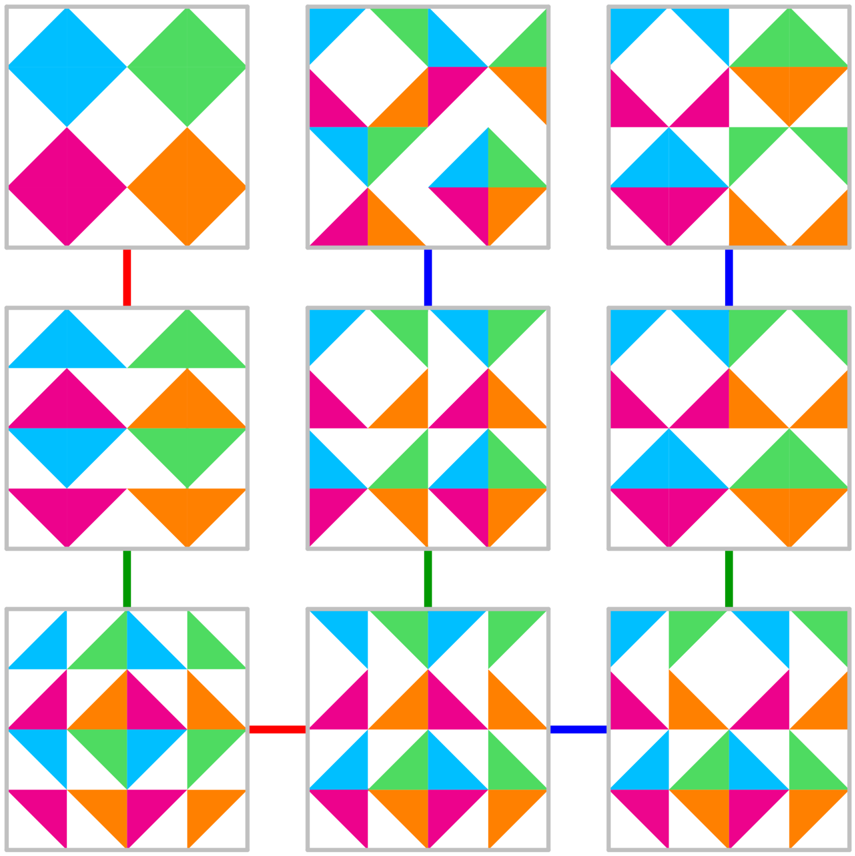
If the Joker is drawn next then the value of the next hand is $-X$, since the score drops to 0. Note that X is the sum of all the cards that have been drawn so far, so $(c_1 + \cdots + c_{k-1}) + X = 1 + 2 + \cdots + 10 = 55$. Therefore,

$$E = \frac{(55 - X) - X}{k}$$

The expected value of the next hand is positive precisely when $55 - 2X > 0$, or $X < 27.5$. Thus we ought to set the target score at $S = 28$.

Kaleidoscopic Diamonds: Solution

There are tons of ways to go between the top three diagrams (indeed, infinitely many if we allow backtracking and going in circles between diagrams).



Above is a fairly compact web showing how they can all be connected, where:

- red lines mean row swap,
- green lines mean column swap,
- blue lines mean block swap.

The tiles are also color-coded so that all of the sixteen tiles have a unique combination of color and orientation. Can you see which swaps take place?

The full problem, created and hosted online by Steven H. Cullinane, a self-described finite geometry enthusiast, includes many more diagrams. There is also a harder version challenging the player to turn a pair of diagrams into another pair if the moves apply to both diagrams simultaneously!

Finite geometry studies what happens when the axioms and operations of geometry apply to finite sets of points - indeed, the coordinates and equations of finite geometry replace the real number system with finite fields (number systems which have the usual four arithmetic operations $+, -, \times, \div$ but only finitely many numbers, like the integers mod p where p is prime).

Let G be the group of permutations of the sixteen positions generated from swapping rows, columns, or blocks. Cullinane published a "Diamond theorem" which says any permutation of G applied to the first diamond figure always results in another figure with symmetry involving ordinary rotation, reflection and/or color-swapping the white and gray of tiles. The number of diagrams attainable from the first one is $24 \cdot 35 = 840$.

G has a a subgroup H (with 1/16th of the permutations of G) which has two other interesting incarnations: (a) the group A_8 of all even permutations of $\{1, \cdots, 8\}$, those that arise from an even number of swaps, and (b) the group $\mathrm{GL}_4\mathbb{F}_2$ of all invertible 4×4 matrices with bit entries (0 or 1) and bitwise arithmetic (where $+$ is logical `XOR` and $\times$ is logical `AND`). The full group $G \cong \mathrm{Aff}_4\mathbb{F}_2$ is equivalent to all affine transformations of $\mathbb{F}_2^4$ (four-dimensional space with bit coordinates), and has $8! \cdot 8 = 322\,560$ permutations.

Killer Triangle Solution

In fact, we can show a stronger inequality:

$$60^\circ \leq \frac{aA + bB + cC}{a \ + \ b \ + \ c} \leq 90^\circ.$$

The upper bound is realized exactly when $\triangle ABC$ is degenerate (one-dimensional), and the lower bound is realized exactly when $\triangle ABC$ is equilateral.

First, we may replace 90° with $(A + B + C)/2$ and cross-multiply:

$$2aA + 2bB + 2cC \leq (a + b + c)(A + B + C).$$

Expanding the right side (viewing $a + b + c$ as a single coefficient), then subtracting the terms from left to right and combining like terms yields

$$0 \leq (b + c - a)A + (c + a - b)B + (a + b - c)C$$

The triangle inequality says each of these coefficients is positive (assuming it is nondegenerate). In other words, the shortest path between two points (in Euclidean space) is a straight line, which means the distance c on the triangle is shorter than the zig-zag distance $a+b$ the other way around the triangle.

As all of our operations were reversible, the truth of this last inequality implies the truth of the first one. That is, 90° is indeed the upper bound.

We can proceed in a similar fashion for the lower bound: replace 60° with $(A + B + C)/3$, and then cross-multiply to get the inequality

$$(a + b + c)(A + B + C) \leq 3aA + 3bB + 3cC$$

Subtracting the left side from the right side, the right side then admits an elementary but nonetheless extremely non-obvious factorization:

$$0 \leq (a - b)(A - B) + (b - c)(B - C) + (c - a)(C - A)$$

The triangle's angles and sides have the same order rankings (for example, if $A \geq B \geq C$ then $a \geq b \geq c$). Thus, every pair of factors above has the same sign (e.g. $a - b$ and $A - B$), and so all three products above are nonnegative. Indeed, the only way all of them are zero is if the triangle is equilateral.

As before, all the operations performed were reversible, so the truth of this last inequality implies the truth of the $60°$ lower bound.

This is an example of a **Coffin Problem**. These were examination problems given to Jewish candidates at Moscow State University during the 70s and 80s which had solutions that were wildly easier to understand than they were to discover (especially in a test setting), thus giving the mathematics department a means of discrimination with plausible deniability.

This particular problem is attributed to Podol'skii, Aliseichik, 1989 in the article *Entrance Examinations to the Mekh-mat* by A. Shen. Ilan Vardi has written a set of solutions to twenty of the problems listed in Shen's article:

`http://www.lix.polytechnique.fr/Labo/Ilan.Vardi/mekh-mat.html`

Tanya Khovanova also keeps an online collection of coffin problems:

`http://www.tanyakhovanova.com/coffins.html`

A similar-looking problem is attributed to Dranishnikov, Savchenko, 1984:

$$\frac{a+b-2c}{\sin C/2} + \frac{b+c-2a}{\sin A/2} + \frac{a+c-2b}{\sin B/2} \geq 0$$

which follows from rearranging the sum of three nonnegative terms

$$\sum_{\text{cyc}} \, (a-b)\left(\frac{1}{\sin B/2} - \frac{1}{\sin A/2}\right).$$

The notation $\sum_{\text{cyc}}$ (which is conventional, since expressions with these kinds of symmetry show up in certain contexts a lot) means to cycle through the letters alphabetically (wrapping back around as appropriate). For instance, the $60°$ lower bound earlier followed from rearranging $\sum_{\text{cyc}}(a-b)(A-B)$.

Lazy Spline: Solution

Denote the central vertex $\mathbf{v}$. The curve is given by the formula

$$\mathbf{x}(t) = t^4\,\mathbf{a} + 4t^3(1-t)\,\mathbf{b} + 6t^2(1-t)^2\,\mathbf{v} + 4t(1-t)^3\,\mathbf{c} + (1-t)^4\,\mathbf{d}.$$

with velocity vector $\mathbf{x}'(t) = \mathbf{p}(t) + q(t)\mathbf{v}$, where

$$\mathbf{p}(t) = 4t^3\,\mathbf{a} + 4(3-4t)t^2\,\mathbf{b} + 4(1-4t)(1-t)^2\,\mathbf{c} - 4(1-t)^3\,\mathbf{d}$$

and $q(t) = 12t(2t^2 - 3t + 1)$, computed by software (because we're lazy, but certainly possible to compute by hand). Then the energy functional is

$$E = \int_0^1 \|\mathbf{p}(t) + q(t)\mathbf{v}\|^2\,\mathrm{d}t$$

which, using $\|\mathbf{r}\|^2 = \mathbf{r}\cdot\mathbf{r}$ and FOILing out becomes

$$E = \int_0^1 \|\mathbf{p}(t)\|^2 + 2\mathbf{p}(t)\cdot q(t)\mathbf{v} + q(t)^2\|\mathbf{v}\|^2\,\mathrm{d}t.$$

This is extremized when $\nabla E = \mathbf{0}$, where E is a scalar function of $\mathbf{v}$. And

$$\nabla E = \int_0^1 2\mathbf{p}(t)q(t) + 2q(t)^2\mathbf{v}\,\mathrm{d}t.$$

Setting $\nabla E = \mathbf{0}$ and solving for $\mathbf{v}$ yields

$$\mathbf{v} = -\Big(\int_0^1 \mathbf{p}(t)q(t)\,\mathrm{d}t\Big)\Big/\Big(\int_0^1 q(t)^2\,\mathrm{d}t\Big).$$

Again using software we compute all integrals and coefficients and get

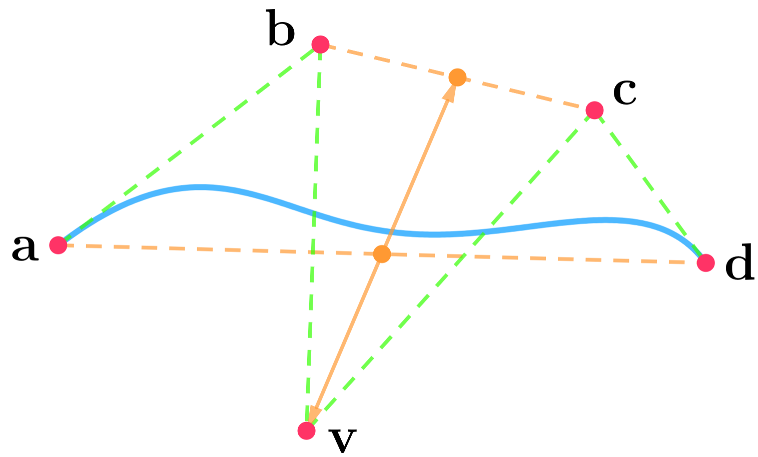
$$-\frac{(-24/35)\,\mathbf{a} + (12/35)\,\mathbf{b} + (12/35)\,\mathbf{c} - (24/35)\,\mathbf{d}}{24/35}$$

which simplifies to $\mathbf{v} = \mathbf{a} - \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{c} + \mathbf{d}$.

The solution $\mathbf{v}$ has the following interpretation: it is the midpoint of BC reflected across the midpoint of AD. This is evident in the formula

$$\mathbf{v} = \tfrac{1}{2}(\mathbf{a}+\mathbf{d}) - \left(\tfrac{1}{2}(\mathbf{b}+\mathbf{c}) - \tfrac{1}{2}(\mathbf{a}+\mathbf{d})\right)$$

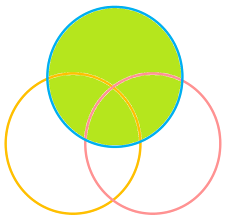
Essentially, while $\mathbf{b}$ and $\mathbf{c}$ draw the curve outwards, the solution $\mathbf{v}$ is located so as to pull the curve equally and oppositely back inward:



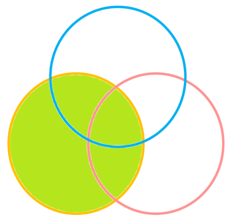
Quadratic and cubic Bèzier curves are used in **vector graphics**.

While common filetypes like PNG and JPG store a rectangular array of pixels, other filetypes like SVG store equations that describe curves and gradients; the latter kind are *scalable* - they look smooth no matter how zoomed-in.

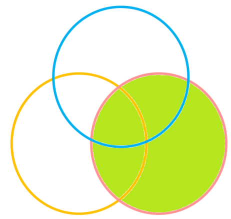
Lattice Chasing: Solution



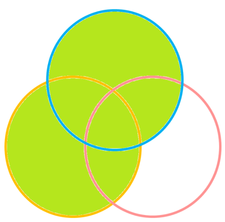
A



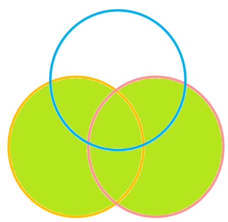
B



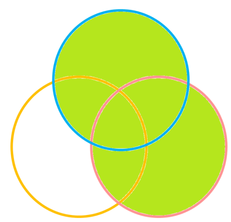
C



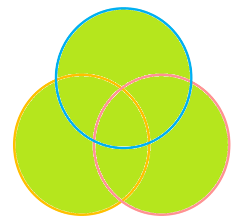
$A \cup B$



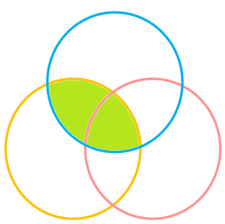
$B \cup C$



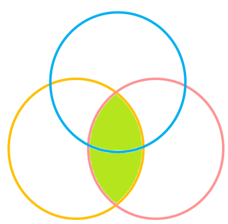
$C \cup A$



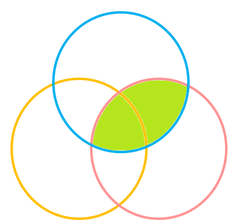
$A \cup B \cup C$



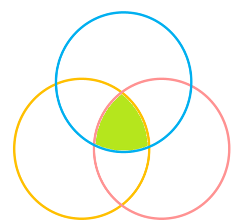
$A \cap B$



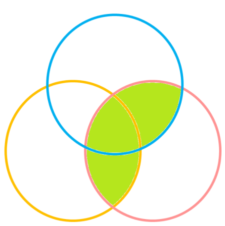
$B \cap C$



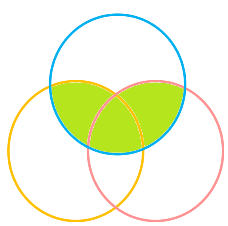
$C \cap A$



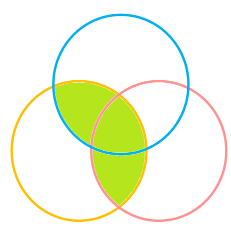
$A \cap B \cap C$



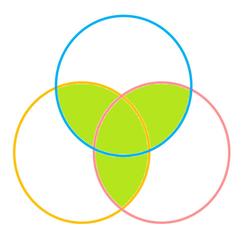
$(A \cup B) \cap C$



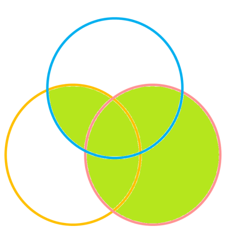
$(B \cup C) \cap A$



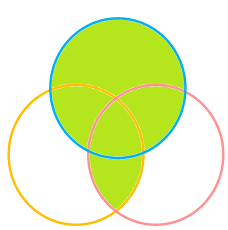
$(C \cup A) \cap B$



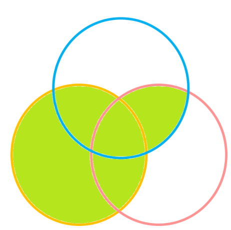
X



$(A \cap B) \cup C$



$(B \cap C) \cup A$

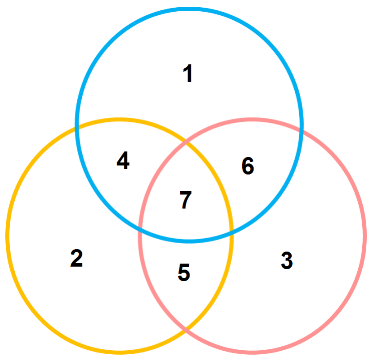


$(C \cap A) \cup B$

Note the last set in the lower right corner has two representations:

$$X = (A \cap B) \cup (B \cap C) \cup (C \cap A) = (A \cup B) \cap (B \cup C) \cap (C \cup A)$$

All of these diagrams can be guaranteed distinct when each of the seven possible regions in the circles is nonempty. For instance, let A, B, C literally be the circles depicted in the Euclidean plane. Or label each of the regions 1-7 and construct the corresponding sets A, B, C:



$A = \{1, 4, 6, 7\}$

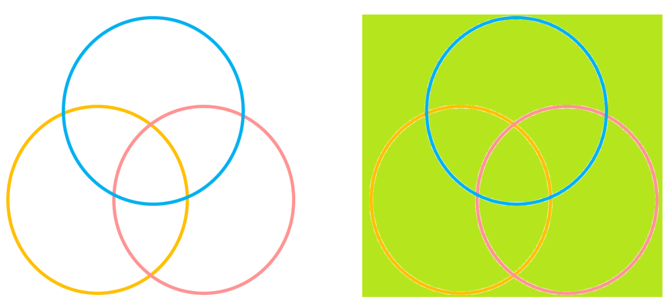
$B = \{2, 4, 5, 7\}$

$C = \{3, 5, 6, 7\}$

It can be manually checked that unioning or intersecting any two of the listed sets from the previous page yields another one of the listed sets. Therefore, the maximum number of distinct sets we can get is 18 sets.

If counting empty union and empty intersection, there are 20 sets.

The empty union is the union of no sets, which must be empty because the empty set is the only set that doesn't affect anything else when unioning. Similarly, the empty intersection is the intersection of no sets, which must be the entire universe of elements under consideration. In a Venn diagram, this would be all three circles filled in, plus the rest of a rectangle outside of it.



Like an Egyptian: Solution

Without loss of generality, $a < b < c$. Then $a > 1$ and $b > 2$. Furthermore,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1,$$

which means for $c > 6$ we can maximize the sum using $a = 2$ and $b = 3$:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42} < 1.$$

We maximize the above sum by maximizing the summands individually, which in turn we do by minimizing the denominators.

If $c = 6$, we know $(a, b) \neq (2, 3)$ which means we can similarly bound

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{6} \leq \frac{1}{2} + \frac{1}{b} + \frac{1}{6} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{11}{12} < \frac{41}{42}.$$

We can manually check all four possibilities for $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ with $c < 6$:

$$\begin{aligned}
\frac{1}{3} + \frac{1}{4} + \frac{1}{5} &= \frac{47}{60} < \frac{41}{42}, \\
\frac{1}{2} + \frac{1}{4} + \frac{1}{5} &= \frac{19}{20} < \frac{41}{42}, \\
\frac{1}{2} + \frac{1}{3} + \frac{1}{5} &= \frac{31}{30} > 1, \\
\frac{1}{2} + \frac{1}{3} + \frac{1}{4} &= \frac{13}{12} > 1.
\end{aligned}$$

Thus we conclude $\frac{1}{2} + \frac{1}{3} + \frac{1}{7}$ is closest, being $\frac{1}{42}$ less than 1.

Local Linear Fraction: Solution

The first three terms in $f(x)$s Taylor series around $x = w$:

$$f(x) \approx f(w) + f'(w)(x-w) + \tfrac{1}{2}f''(w)(x-w)^2$$

If $f'(w) = 0$, the only Mobius transformation with derivative 0 somewhere are constant functions, so the best approximation is the constant $f(w)$.

Otherwise, we may subtract $f(w)$ and divide by $f'(w)(x-w)$ to get

$$\frac{1}{f'(w)}\frac{f(x)-f(w)}{x-w} \approx 1 + \frac{f''(w)}{2f'(w)}(x-w)$$

The right side is the first two terms of a geometric series,

$$1 + \frac{f''(w)}{2f'(w)}(x-w) \approx \frac{1}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

Putting this together gives

$$\frac{1}{f'(w)}\frac{f(x)-f(w)}{x-w} \approx \frac{1}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

which becomes

$$f(x) \approx f(w) + \frac{f'(w)(x-w)}{1 - \frac{f''(w)}{2f'(w)}(x-w)}$$

which is expressible as $\dfrac{ax+b}{cx+d}$. Notice if $f'(w) = 0$, this becomes $f(x) \approx f(w)$, and if $f''(w) = 0$ it becomes the linear approximation.

Another solution method is to rewrite the Möbius transformation $g(x)$ to only depend on three parameters, write the system of equations

$$g(w) = f(w), \quad g'(w) = f'(w), \quad g''(w) = f''(w),$$

and then solve for the three parameters.

Noncommutative Calculus: Solution

We can work "mod t^4," meaning ignore any powers of t higher than t^3.

The left-hand side $\exp(tX)\exp(tY)$ is

$$\Big(1 + tX + \tfrac{1}{2}t^2X^2 + \tfrac{1}{6}t^3X^3 + \cdots\Big)\Big(1 + tY + \tfrac{1}{2}t^2Y^2 + \tfrac{1}{6}t^3Y^3 + \cdots\Big) =$$

$$1 + t(X+Y) + \tfrac{1}{2}t^2\big(X^2 + 2XY + Y^2\big) + \tfrac{1}{6}t^3\big(X^3 + 3X^2Y + 3XY^2 + Y^3\big) + \cdots$$

On the other hand, the right-hand side $\exp(tZ_1 + t^2Z_2 + t^3Z_3 + \cdots)$ is

$$1 + (tZ_1 + t^2Z_2 + t^3Z_3 + \cdots) + \tfrac{1}{2}(tZ_1 + t^2Z_2 + \cdots)^2 + \tfrac{1}{6}(tZ_1 + \cdots)^3 + \cdots$$

$$= 1 + tZ_1 + \tfrac{1}{2}t^2(Z_1^2 + 2Z_2) + \tfrac{1}{6}t^3(Z_1^3 + 3Z_1Z_2 + 3Z_2Z_1 + 6Z_3) + \cdots$$

Equating coefficients of t gives $Z_1 = X + Y$, and equating coefficients of $\frac{1}{2}t^2$ gives $Z_1^2 + 2Z_2 = X^2 + 2XY + Y^2$: substituting Z_1 into the latter we can solve $Z_2 = \frac{1}{2}(XY - YX)$. Equating coefficients of $\frac{1}{6}t^3$ and substituting gives

$$(X+Y)^3 + \tfrac{3}{2}(X+Y)(XY-YX) + \tfrac{3}{2}(XY-YX)(X+Y) + 6Z_3$$

$$= X^3 + 3X^2Y + 3XY^2 + Y^3.$$

Distributing, subtracting, cancelling, and dividing by 6 gives

$$Z_3 \;=\; \tfrac{1}{12}\big(X^2Y - 2XYX + YX^2 \;+\; Y^2X - 2YXY + XY^2\big).$$

The noncommutative polynomials $Z_k(X,Y)$ may be expressed much more compactly using the **commutator** operation $[X,Y] := XY - YX$:

$$\begin{aligned}
Z_1 &= X + Y,\\
Z_2 &= \tfrac{1}{2}[X,Y],\\
Z_3 &= \tfrac{1}{12}[X,[X,Y]] + \tfrac{1}{12}[Y,[Y,X]],\\
Z_4 &= \tfrac{1}{24}[Y,[X,[Y,X]]],\\
&\ \ \vdots
\end{aligned}$$

A formula of Dynkin says that in $Z_n(X, Y)$ the coefficient of

$$[\underbrace{X, [X, \cdots [X,}_{r_1} \underbrace{[Y, [Y, \cdots [Y,}_{s_1} \cdots \underbrace{[X, [X, \cdots [X,}_{r_n} \underbrace{[Y, [Y, \cdots Y]}_{s_n} \cdots\cdots]$$

is $\frac{(-1)^{n-1}}{n}$ times the reciprocal of $(r_1+\cdots+r_n+s_1+\cdots+s_n)r_1!\cdots r_n!s_1!\cdots s_n!$

The full solution $Z(X, Y) = \sum_{n=0}^{\infty} Z_n(X, Y)$ to $\exp(X)\exp(Y) = \exp Z$ (so, when $t = 1$) is known as the **Baker-Campbell-Hausdorff formula**.

Beginning with Klein's Erlangen Program at the turn of the 20th century, mathematicians began studying the geometry of homogeneous spaces from the perspective of symmetry groups. ("Homogeneous," here, means no point in space is more special than any other point in space.) The symmetry of a sphere, for example, is the matrix group SO(3) of 3D rotation matrices.

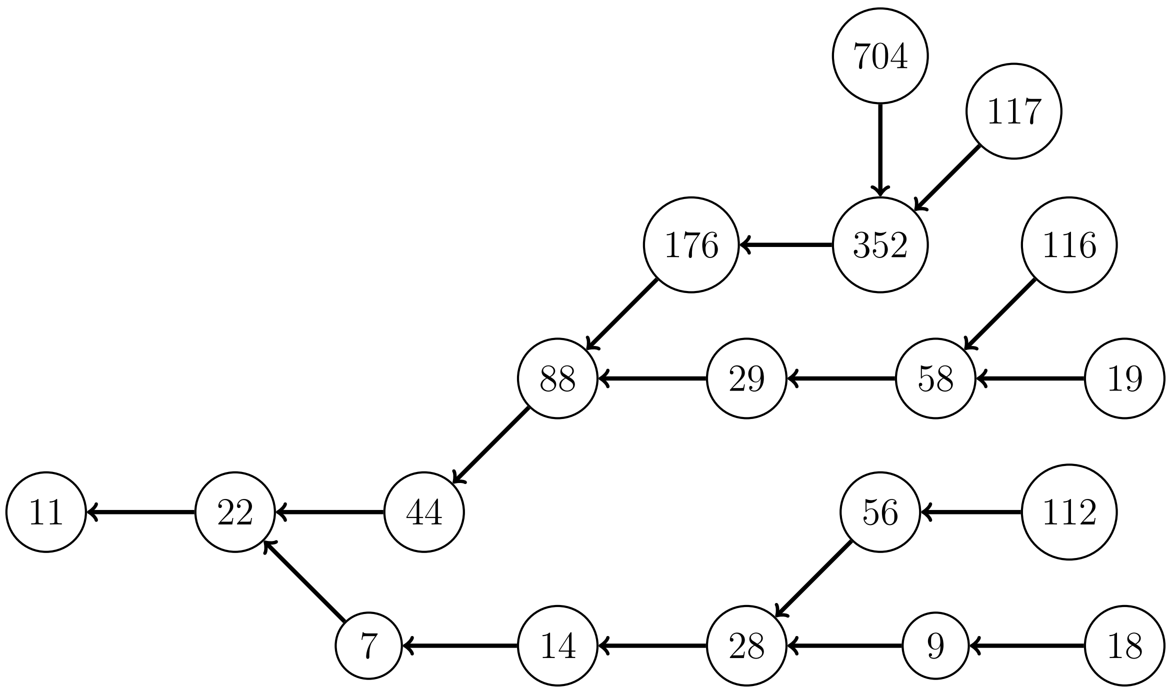
Born from this was an interest in the action of continuous symmetry groups, called **Lie groups**, particularly when it came to solving differential equations describing motion and dynamics. Lie's idea was to think about so-called *infinitessimal* symmetries. In other words, if we parametrize a family of symmetries to create an animation (imagine, for instance, an animation of a sphere rotating around an axis), we may differentiate the parametrization at time $t = 0$. The infinitessimal symmetries form a **Lie algebra**.

The BCH formula implies it is possible to reconstruct the composition operation of the Lie group (locally, at least) from the bracket operation of the Lie algebra. This is one of many results of a larger Lie-group−Lie-algebra correspondence. More generally, the composition, conjugation, and commutator operations of a Lie group correspond respectively (via differentiation) to the addition, adjoint, and bracket operations in the corresponding Lie algebra.

Not Yet Ready: Solution

Notice the progression $22 \leftarrow 44 \leftarrow 88$. It suggests a process of simply dividing by 2. Indeed, this explains what happens to even numbers on the graph.

The arrows from odd numbers are $352 \leftarrow 117$, $88 \leftarrow 29$, $28 \leftarrow 9$, $22 \leftarrow 7$; in these cases, by inspection, the ratio between neighboring numbers is closer to 3 than it is to 2. Suspiciously close, in fact. Indeed, the larger numbers are all exactly 1 more than 3 times the smaller odd numbers! So we have:



Our graph is none other than a graph of the function

$$T(x) = \begin{cases} x/2 & x \text{ even} \\ 3x+1 & x \text{ odd} \end{cases}$$

The **Collatz conjecture** proposed in 1937 says this function applied over and over again to any whole number must eventually reach 1. Paul Erdös, one of the most prolific and renowned mathematicians of the 20th century, said of the conjecture “Mathematics may not be ready for such problems.”

Odd One Out: Solution

Suppose for the sake of contradiction a prime power p^v evenly divided only one of the denominators. It doesn't matter which one, so just say it was x.

This means p can only be a factor at most $v - 1$ times in the other denominators, so if we multiply the equation by p^{v-1} and move the other two fractions to the other side we get a new equation like the following:

$$\frac{A}{X} = \frac{B}{Y} + \frac{C}{Z}$$

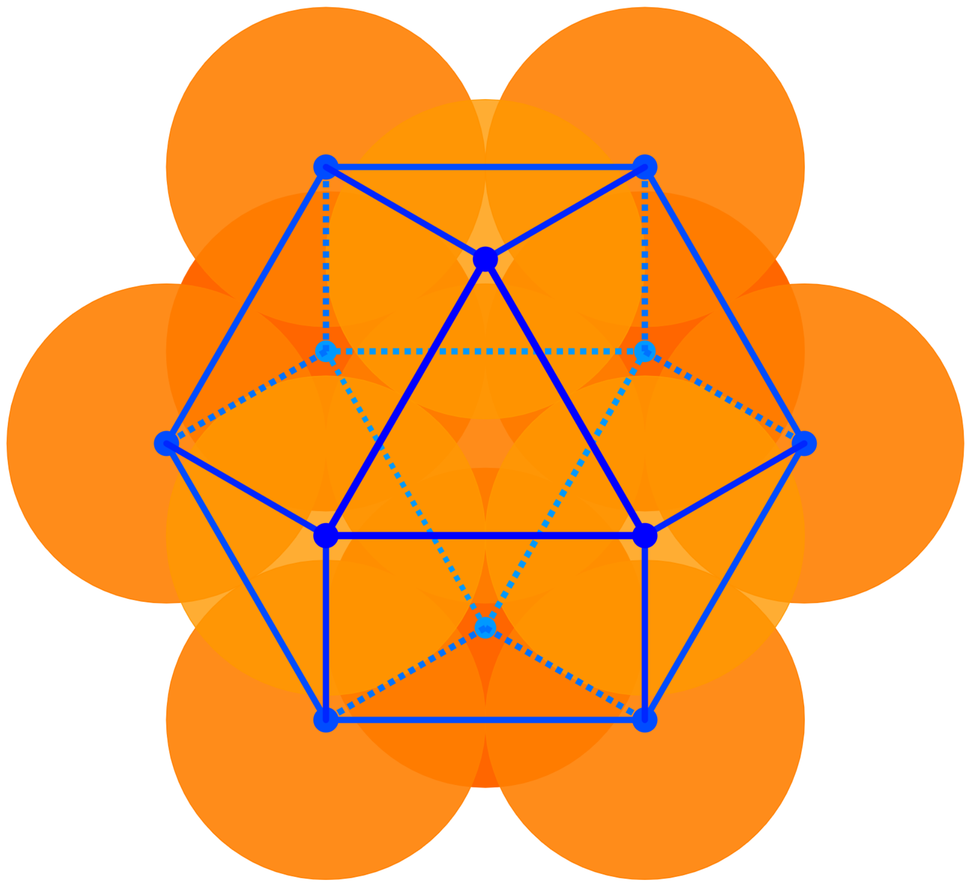
We only multiplied by p^{v-1}, so the denominator X is still divisible by p. On the other hand, the new denominators Y and Z no longer have p as a factor (if they ever did to begin with...), which means when we add the fractions on the right, the new denominator (after simplifying) will be a factor of YZ, and so the denominator on the right will not be divisible by p:

$$\frac{A}{X} = \frac{D}{W}$$

In summary, when the fractions on both sides are in lowest terms, the left denominator is divisible by p but the right one isn't. This is a contradiction!

Orange Stack: Solution

There are three layers and a central sphere is chosen from the middle one; it has three neighboring spheres in both the top and bottom layers, and six neighboring spheres in the middle layer, for a total of twelve neighbors.



Counting, we find 12 vertices, 24 edges, and 14 faces. Specifically, 8 triangular and 6 square faces - the quadrilateral faces may not look like squares in the 2D projection because in 3D they are sloped. This is a **cuboctahedron**.

This illustrates the **kissing number** (the most unit spheres which fit around a central one) in 3D is 12. In 2D the kissing number is 6, corresponding to a hexagon, which also extends to a circle packing. (This is why honeycombs use hexagons!) In 4D the kissing number is 24, corresponding to a 24-cell.

Observe $1+2+3+4+3+2=15$. Up to this and beyond we have:

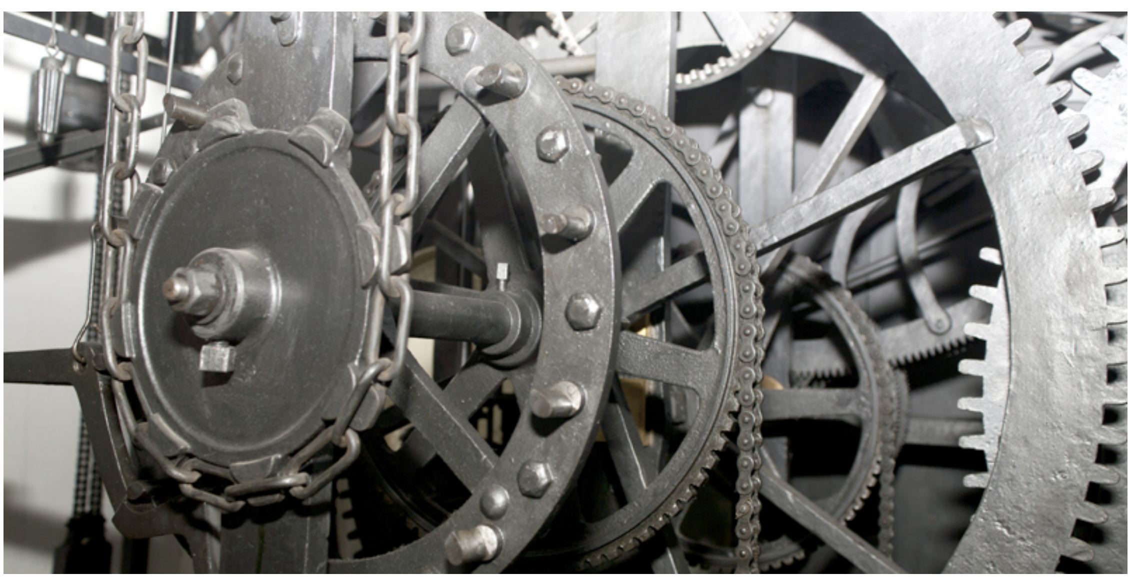
1	1							
2	2							
3	3							
4	4							
5	3	2						
6	1	2	3					
7	4	3						
8	2	1	2	3				
9	4	3	2					
10	1	2	3	4				
11	3	2	1	2	3			
12	4	3	2	1	2			
13	3	4	3	2	1			
14	2	3	4	3	2			
15	1	2	3	4	3	2		
16	1	2	3	4	3	2	1	
17	2	3	4	3	2	1	2	
18	3	4	3	2	1	2	3	
19	4	3	2	1	2	3	4	
20	3	2	1	2	3	4	3	2

Moreover, notice after segmenting the sequence up to $n=15$, the sequence is back at its first term of 1. Thus, to segment terms of the sequence to sum to any n beyond 15, we may first include the next 6 terms of the sequence (which is the period, so any 6 consecutive terms sum to 15) and then include whatever comes next that was used to segment the earlier value of $n-15$.

Any sequence with the same property (that it can be segmented to get the sequence of whole numbers) is called a **Šindel sequence**. If the sequence is periodic with period p, and the sum of its first p terms is s, then we need only check the first $(s-1)/2$ terms to conclude it is a Šindel sequence.

(Prague Astronomical Clock - Orloj 2022)

The **Orloj** is a medieval astronomical clock in Prague. Started in 1410, it is the oldest running clock in the world. Its inner workings contain a gear with teeth spaced at intervals $1, 2, 3, \cdots, 24$ and an auxillary gear with teeth spaced at $1, 2, 3, 4, 3, 2$. This allows it to strike k times on the kth hour.



(FG Forrest)

Pair of Pairs: Solution

The number $\binom{n}{2}$ counts how many unordered pairs $\{a, b\}$ there are with a, b distinct numbers drawn from $\{1, \cdots, n\}$. Furthermore, the expression

$$\binom{\binom{n}{2}}{2}$$

counts how many pairs of pairs $\{\{a, b\}, \{c, d\}\}$ there are with a, b, c, d from $\{1, \cdots, n\}$, with $\{a, b\} \neq \{c, d\}$. There are two cases: the sets $\{a, b\}$ and $\{c, d\}$ either share one number in common, or none.

In the first case, there are $\binom{n}{3}$ ways to pick three distinct numbers $\{a, b, c\}$ from $\{1, \cdots, n\}$, and then 3 ways to construct a pair of pairs out of them:

$$\{\{a, b\}, \{a, c\}\}, \qquad \{\{a, b\}, \{b, c\}\}, \qquad \{\{a, c\}, \{b, c\}\},$$

each corresponding to a choice of one of a, b, c (assume $a < b < c$ so these subsets are distinguishable) to be repeated. In the second case, there are $\binom{n}{4}$ ways to pick four distinct numbers $\{a, b, c, d\}$ (again assume $a < b < c < d$), and then there are four ways to partition these four into two pairs of pairs:

$$\{\{a, b\}, \{c, d\}\}, \qquad \{\{a, c\}, \{b, d\}\}, \qquad \{\{a, d\}, \{b, c\}\}.$$

Putting it all together, we can write this as an equation

$$\binom{\binom{n}{2}}{2} = 3\binom{n}{3} + 3\binom{n}{4}.$$

Alternatively, we could have used the formula $\binom{n}{k} = \frac{n}{k}\frac{n-1}{k-1}\frac{n-2}{k-2}\cdots$ (with k fractions being multiplied) to verify the identity algebraically:

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} - 1\right)}{2} = 3\,\frac{n(n-1)(n-2)}{6} + 3\,\frac{n(n-1)(n-2)(n-3)}{24}.$$

Pentagonal Peculiarity: Solution

Let r be the number of dots in the bottom row and d on the right diagonal.

► If $r > d$, we can pour the right diagonal into a new row. This increases the number of rows by one. The new row will have strictly less dots than the one above, so all rows have distinct numbers of dots, *unless* a dot from the *original* last row gets poured into the *new* last row! This will happen if the last row and right diagonal share a corner dot, in which case the new row will fail to have fewer dots than the row above precisely if $r = d + 1$.

In this case, the number of dots is

$$\begin{aligned} n &= r + (r+1) + \cdots + (r + r - 2) \\ &= (r-1)r + \frac{(r-2)(r-1)}{2} \\ &= \frac{(3r-2)(r-1)}{2} = \frac{(3d+1)d}{2}. \end{aligned}$$

► If $r \leq d$, we can scoop the last row into the right diagonal. The rows will still have distinct numbers of dots. This decreases the number of rows by one, *unless* we scoop a dot back into the last row! This will happen if $r = d$ and again a corner dot is shared by the last row and right diagonal.

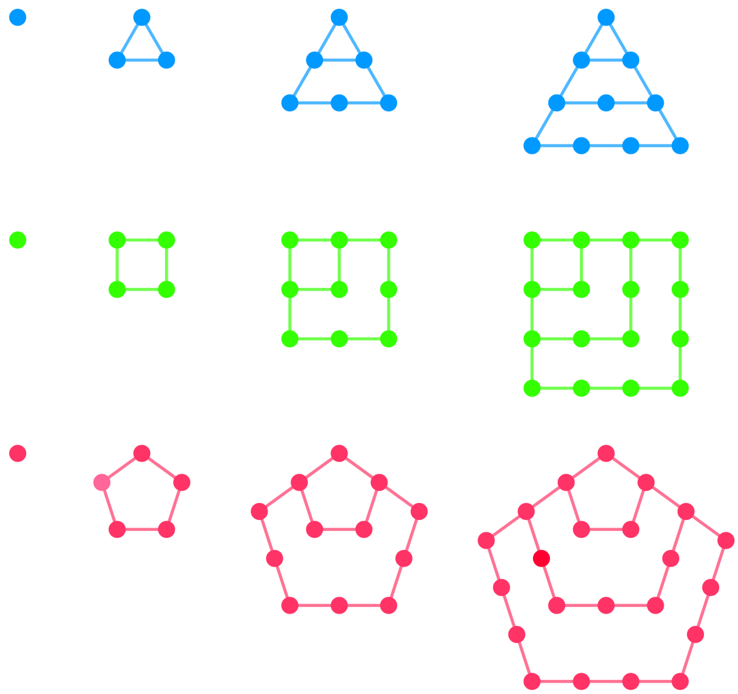
In this case the number of dots is

$$\begin{aligned} n &= r + (r+1) + \cdots + (r + r - 1) \\ &= r^2 + \frac{r(r-1)}{2} = \frac{(3r-1)r}{2}. \end{aligned}$$

Defining the kth **pentagonal number** $g(k) = (3k-1)k/2$, the second case has $n = g(r)$ and the first case has $n = g(-d)$. Note g is a one-to-one function since $g(0) = 0$ and $g(k) < g(-k) < g(k+1)$ for all $k > 0$. Thus, either one exception or the other can occur, depending on n, but not both.

In conclusion, the pouring-and-scooping procedure pairs the even-row diagrams with the odd-row diagrams, with at most one exception, so $|E - O| = 1$.

The triangular numbers, square numbers, and pentagonal numbers are so-named because they count dots in series of expanding geometric figures:



This has applications to expanding a certain infinite product into a series,

$$\prod_{m=1}^{\infty}(1-q^m) = \sum_{n=0}^{\infty} \square\, q^n$$

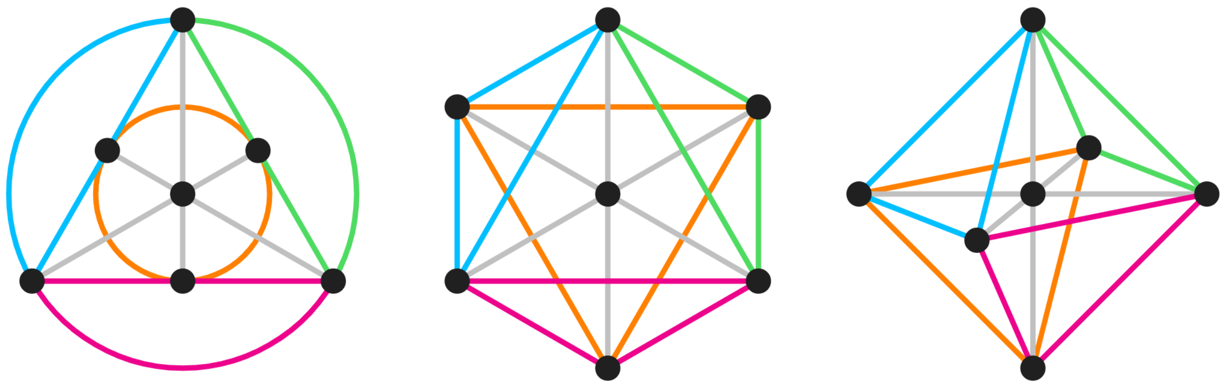
When we expand out the product, infinitely, the resulting terms are of the form $(-1)^r q^{m_1+\cdots+m_r}$ for distinct exponents $m_1, \cdots, m_r$. These terms correspond to diagrams with n dots - specifically, with m_1 in the first row, m_2 in the second row, and so on. Each diagram contributes ± 1 to the coefficient $\square$ depending on whether the number of rows is even or odd.

Thus, $\square = E - O$. We've seen this is 0 except when $n = g(k)$ is a generalized pentagonal number. In both kinds of exceptions we examined, the number of rows was k (either $k = d$ when $r = d + 1$ or $k = r$ when $r = d$). Therefore

$$\prod_{m=1}^{\infty}(1-q^m) \;=\; \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2}.$$

This is the **Pentagonal Number Theorem**.

Perspective Shift: Solution



First, stretch out the triangle edges' midpoints, and tighten the circular arcs until they're taut, to get a hexagon. Then, pull those same three vertices out of the page to get a "triangular antiprism," i.e. an octahedron!

Around all three axes there is fourfold rotational symmetry. In fact, we can color opposite faces the same color using four colors, and then there is exactly one rotational symmetry for each of the $4! = 24$ permutations of these colors!

The graph of this problem, minus its outer circle, is a famous depiction of the **Fano plane** $\mathbb{F}_2\mathbb{P}^2$ studied in finite geometry.

Finite geometry studies finite sets with combinatorial structures satisfying axioms from geometry, often modeled with equations involving finite fields (number systems with only finitely many numbers). In this context, the axioms come from projective geometry, a branch of geometry historically influenced by the development of perspective drawing in Renaissance art.

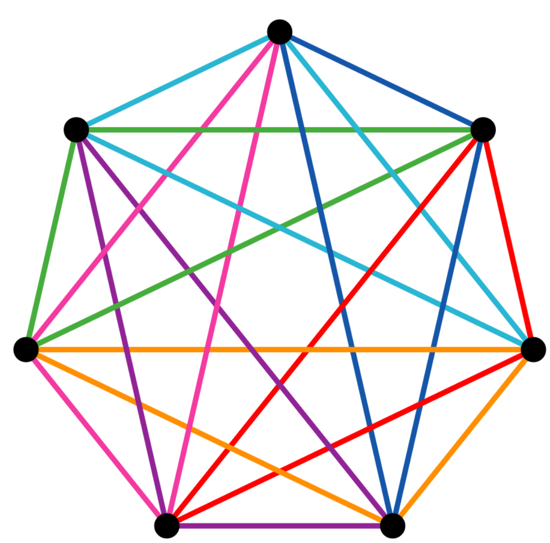
The real projective plane $\mathbb{RP}^2$ is the 2D space whose points represent all possible 1D subspaces of 3D Euclidean space $\mathbb{R}^3$. Every 1D subspace of $\mathbb{R}^3$ can be represented by a pair of unit vectors, so $\mathbb{RP}^2$ is like the sphere S^2 but where antipodal points $\pm\mathbf{v}$ of S^2 count as the same point of $\mathbb{RP}^2$. The real projective plane, like the Klein bottle, is impossible to embed in $\mathbb{R}^3$ without self-intersection, although immersions are possible (notably Boy's surface).

The Fano plane $\mathbb{F}_2\mathbb{P}^2$ is defined the same way, but uses the finite field $\mathbb{F}_2$ instead of the real numbers $\mathbb{R}$. The term field means there is addition and multiplication satisfying commutativity, distributivity and associativity, and $\mathbb{F}_2$ in particular has only two numbers, an additive identity called 0 and a multiplicative identity called 1. Arithmetic is what you'd expect, except $1+1 = 0$.

This means $\mathbb{F}_2^3$ has $2^3 = 8$ vectors (a, b, c), and every 1D subspace contains exactly one nonzero vector so there are seven elements of the Fano plane $\mathbb{F}_2\mathbb{P}^2$, corresponding to the seven vertices of the graphical depiction. Every 2D subspace of $\mathbb{F}_2^3$ (called a line of $\mathbb{F}_2\mathbb{P}^2$) contains exactly three 1D subspaces, which suggests drawing every possible edge between seven vertices (i.e., a complete graph K_7) and coloring the 3-cycles which are projective lines. The usual triangular Fano plane depiction is missing edges, unfortunately.
The fact that the Fano plane's symmetry group $\mathrm{PGL}_3\mathbb{F}_2$ has

$$(2^3 - 1)(2^3 - 2)(2^3 - 2^2) \;=\; 168 \;=\; 24 \cdot 7$$

elements means there ought to be a picture which captures its sevenfold symmetry (which is not apparent in the octahedral picture). This picture can be obtained by considering the exceptional isomorphism $\mathrm{PGL}_3\mathbb{F}_2 \cong \mathrm{PSL}_2\mathbb{F}_7$ and the metacyclic subgroup of $\mathrm{PSL}_2\mathbb{F}_7$ (whose Möbius transformations acting on the projective line $\mathbb{F}_7\mathbb{P}^1$ are affine transformations fixing $1/0 = \infty$). It can also be a helpful mnemonic for octonion multiplication! Octonions are an eight-dimensional nonassociative number system generalizing quaternions.

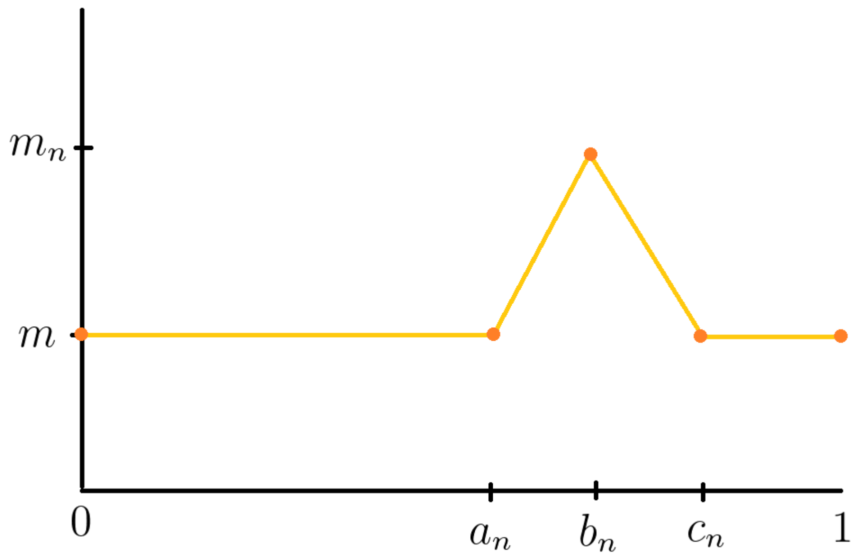


Pinching an Impulse: Solution

Pick any constant m, unbounded sequence $m_1, m_2, m_3, \cdots$ and three sequences (i) $a_1, a_2, a_3, \cdots$; (ii) $b_1, b_2, b_2, \cdots$; and (iii) $c_1, c_2, c_3, \cdots$ satisfying $0 < a_n < b_n < c_n < 1$ and $\lim_{n\to\infty} a_n = 1$. For example we could pick

$$m = 0, \quad m_n = n, \quad a_n = 1 - \frac{1}{n+1}, \quad b_n = 1 - \frac{1}{n+2}, \quad c_n = 1 - \frac{1}{n+3}.$$

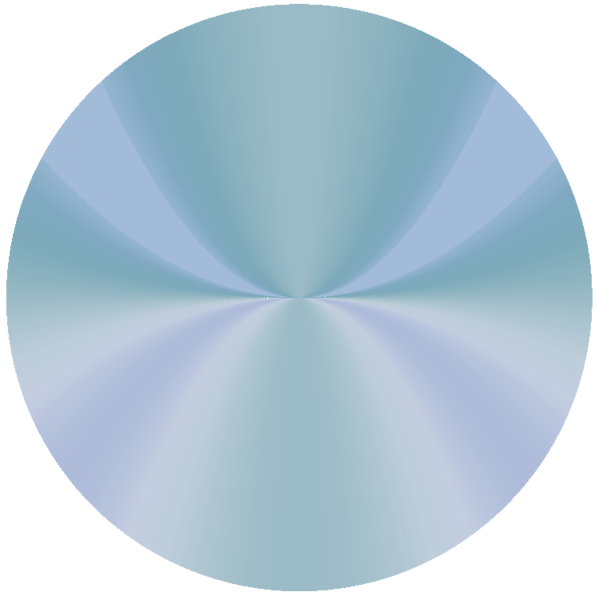
Then define $f_n(x)$ to be the piecewise-linear function whose graph consists of the line segments joining the points $(0, m)$, (a_n, m), (b_n, m_n), (c_n, m), $(1, m)$:



Since $f_n(1) = m$ for all n, so too does $f(1) = m$. For any $x < 1$ in the domain, eventually $a_n > x$ (or in other words, the triangle passes to the right of x), after which point $f_n(x) = m$, and thus $f(x) = m$ there too. Thus, f is the constant function identically equal to m, which is bounded.

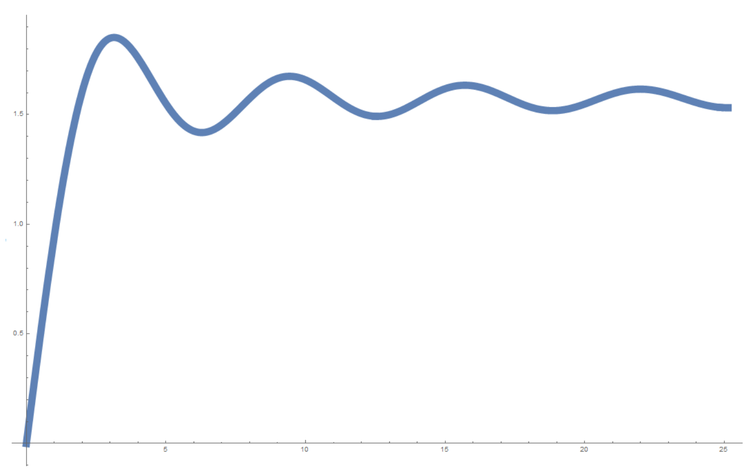
This phenomenon shows up in some places.

Consider the continuity of $f(x, y) = 2x^2y/(x^4 + y^2)$ at $(0, 0)$. Approaching the origin along any line, the limit is 0. But approaching the origin along one of the parabolas $y = \pm x^2$, the limit is ± 1. This, and riffs on it, are common counterexamples given in introductory calculus classes.



Restricting the function f to a circle of radius r centered at the origin, the graph shows ridges on either side which, as $r \to 0$, get squeezed towards the East and West poles of the circle, never really reaching them.

Another illustration of this is seen in **Gibbs phenomenon**. Almost any "nice" function is expressible as an infinite sum of trigonometric functions, called a Fourier series. The partial sums of a Fourier series converge, yet there are "ringing artifacts" that are squeezed towards jump discontinuities.



Polarization: Solution

First we extend the vanishing condition. Substitute $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ into the vanishing condition, and then distribute (aka "FOIL") with multlinearity for

$$\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) + \phi(\mathbf{a}_2, \mathbf{b}, \mathbf{a}_1, \mathbf{b}) = 0$$

The terms $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_1, \mathbf{b})$ and $\phi(\mathbf{a}_2, \mathbf{b}, \mathbf{a}_2, \mathbf{b})$ are zero so do not appear. By the symmetry condition, the two remaining terms are equal, so $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) = 0$.

Similarly, substituting $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ into $\phi(\mathbf{a}_1, \mathbf{b}, \mathbf{a}_2, \mathbf{b}) = 0$ gives

$$\phi(\mathbf{a}_1, \mathbf{b}_2, \mathbf{a}_2, \mathbf{b}_1) = -\phi(\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2)$$

Thus, swapping the second and fourth arguments changes the sign. If we had instead substituted $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ first and $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ second we would have found swapping the first and third arguments also changes the sign.

In conclusion, ϕ is fully **antisymmetric**: swapping any two of its arguments changes its sign. This also forces ϕ to be **alternating**: if any two of its arguments are equal, ϕ vanishes (equals 0). This is because if two arguments are equal, then swapping them changes the sign but also does nothing, and the only scalar value ϕ satisfying $\phi = -\phi$ is $\phi = 0$.

Finally, for $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$, we can use basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to write

$$\begin{cases} \mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 \\ \mathbf{c} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3 \\ \mathbf{d} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2 + d_3\mathbf{e}_3 \end{cases}$$

which means $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \displaystyle\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{\ell=1}^{3} a_i b_j c_k d_\ell \; \phi(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_\ell)$.

Two of i, j, k, ℓ must be equal by the pigeonhole principle, which means all of the summands above are 0, forcing $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = 0$ for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$.

This solution shows how the two-vector **Lagrange identity**

$$\|\mathbf{a} \times \mathbf{b}\|^2 = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

in three dimensions implies the four-vector **Binet-Cauchy identity**

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$= \det \begin{pmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{d} \\ \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{d} \end{pmatrix} = \det \begin{pmatrix} | & | \\ \mathbf{a} & \mathbf{b} \\ | & | \end{pmatrix}^T \begin{pmatrix} | & | \\ \mathbf{c} & \mathbf{d} \\ | & | \end{pmatrix}$$

of which the Lagrange identity is a special case: set $\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ to be the difference between the left and right sides of Binet-Cauchy, then show $\phi \equiv 0$.

The situation is different in higher dimensions - in four dimensions, for instance, there is a nonzero alternating form satisfying all four properties:

$$\phi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) = \det \begin{pmatrix} | & | & | & | \\ \mathbf{a} & \mathbf{b} & \mathbf{c} & \mathbf{d} \\ | & | & | & | \end{pmatrix}.$$

Even more generally, in n dimensions the set of all multilinear alternating forms of k variables forms an $\binom{n}{k}$-dimensional vector space called the **exterior power** $\Lambda^k \mathbb{R}^n$ (or technically its dual, depending on definitions).

Besides the pigeonhole principle, this solution uses **polarization**, a technique for converting between homogeneous multivariable polynomials of degree d and multilinear forms of d variables. The simplest nontrivial case is converting between quadratic and bilinear forms, as seen in any of the many **polarization identities** relating squared norms and inner products:

$$\|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a}\|^2 + 2(\mathbf{a} \cdot \mathbf{b}) + \|\mathbf{b}\|^2.$$

The relation $\|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}$ tells us how to write norms in terms of dot products and leads to this identity by substituting $\mathbf{v} = \mathbf{a} + \mathbf{b}$, and conversely this identity tells us how to rewrite dot products in terms of norms.

Another equivalent polarization identity does the same trick,

$$\mathbf{a} \cdot \mathbf{b} = \tfrac{1}{4}\left(\|\mathbf{a}+\mathbf{b}\|^2 - \|\mathbf{a}-\mathbf{b}\|^2\right),$$

and is the antisymmetrized sibling of the **parallelogram law**

$$2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right) = \|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2.$$

Exercise 3.7 of *The Cauchy-Schwarz Master Class* challenges the reader to upgrade the n-dimensional version of the two-vector Lagrange identity to the n-dimensional version of the four-vector Binet-Cauchy identity,

$$(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) = \sum_{k<\ell} \begin{vmatrix} a_k & b_k \\ a_\ell & b_\ell \end{vmatrix} \begin{vmatrix} c_k & d_k \\ c_\ell & d_\ell \end{vmatrix}.$$

(The text is a dedicated compendium of applications and offshoots of the **Cauchy-Schwarz inequality** $|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$, which itself follows from polarizing the positivity condition $\|\mathbf{a}-\mathbf{b}\|^2 \geq 0$.)

Surprisingly, the text's hint to use polarization seems erroneous, since the difference between the left and right sides of Binet-Cauchy satisfy the four properties given in the problem (which are the algebraic features of the form that allow for polarization) but we saw for $n \geq 4$ there are nonzero forms.

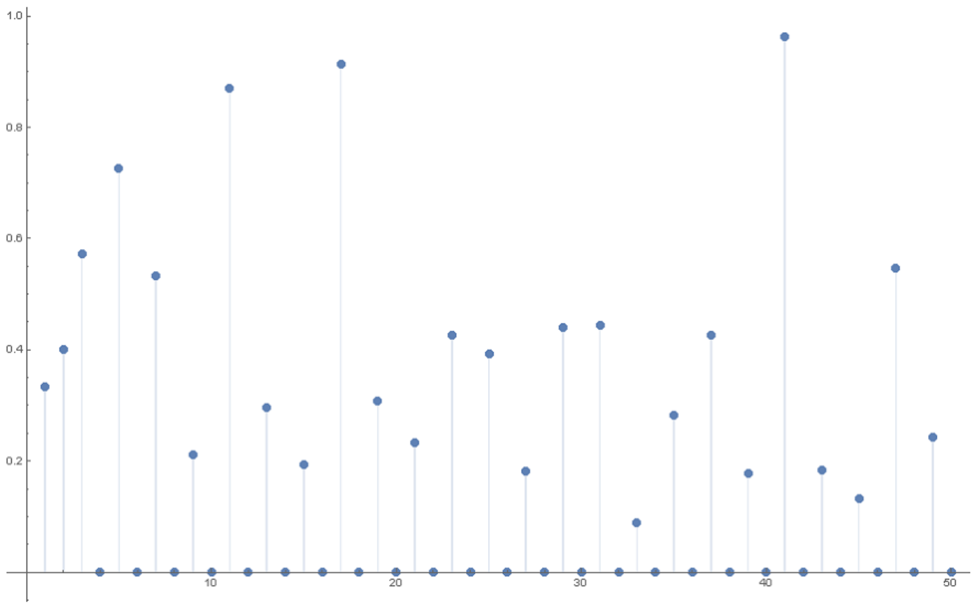
Prime Generation: Solution

Using the following two lines of input in Mathematica,

```
potential[n_]:=Sum[Boole[PrimeQ[x^2+x+n]],{x,-n,n}]/(2*n+1)

DiscretePlot[potential[n],{n,1,50}]
```

we receive the following output:



By inspection, the five highest-potential numbers are $n = 3, 5, 11, 17, 41$.

The corresponding discriminants $\Delta = 1 - 4n$ of the quadratics $x^2 + x + n$ are (minus) the largest five of the so-called **Heegner** numbers:

$$-\Delta = 1, 2, 3, 7, 11, 19, 43, 67, 163$$

These have significance in algebraic number theory.

An algebraic number is one which is the root of an integer-coefficient polynomial (in contrast to transcendental numbers), and an **algebraic integer** is one which is the root of a *monic* integer coefficient polynomial. All rational numbers are algebraic numbers, but the integers are the only rational numbers which are algebraic integers. More generally, any algebraic number is an algebraic integer divided by a whole number.

The Heegner numbers $-\Delta$ are the squarefree positive integers for which the algebraic integers generated from the imaginary surds $\sqrt{\Delta}$ enjoy **unique factorization** into irreducible elements. To illustrate, 5 is not a Heegner number because $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ are two inequivalent factorizations.

On April 1st, 1975, mathematical columnist Martin Gardner said $\exp(\pi\sqrt{163})$ was a whole number. In fact, it's not a whole number, but incredibly close:

$$\exp(\pi\sqrt{163}) \;\approx\; 262537412640768743.99999999999925\ldots$$

This is explained in modern number theory. More specifically, modular forms and elliptic forms. More specifically still, the j-invariant and "complex multiplication" (which is not what you think it is).

The **Bateman-Horn conjecture** predicts how often a family of polynomials $f_1(n), \cdots, f_m(n)$ are simultaneously prime. It says the number of $n \le x$ for which each polynomials evaluate to a prime has the asymptotic estimate

$$\sim \frac{C}{D} \int_a^x \frac{\mathrm{d}t}{(\ln t)^m}$$

where a doesn't matter, $D = (\deg f_1) \cdots (\deg f_m)$, and the constant C is

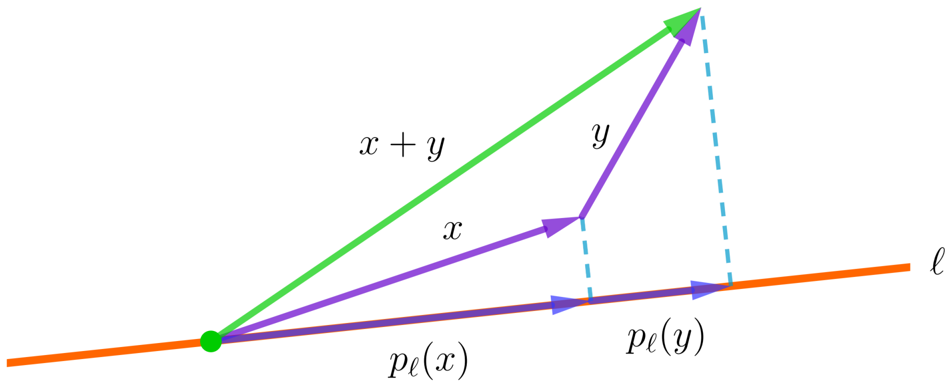
$$C = \prod_p \frac{1 - N(p)/p}{(1-1/p)^m},$$

the infinite product taken over all primes p and $N(p)$ counting values mod p for which one of the values $f_1(n), \cdots, f_m(n)$ is 0 mod p.

This vastly generalizes the Twin Prime Conjecture (which says there are infinitely many pairs of primes 2 apart) and the Prime Number Theorem (which says the number of primes $\le x$ is asymptotically $\int_0^x \mathrm{d}t/\ln t$).

Projector Junction: Solution

Picking an origin on the line ℓ, the projector p_ℓ is a linear map:



Every vector may be decomposed into a sum $v = v_\parallel + v_\perp$ of a parallel and perpendicular component with respect to ℓ, and the projector extracts the parallel component $v_\parallel$ (the perpendicular component $v_\perp$, meanwhile, is called the *rejection* instead of the projection). In other words, if $\ell^\perp$ is the line perpendicular to ℓ through the chosen origin, then p_ℓ is characterized by

$$p_\ell(v) = \begin{cases} v & \text{if } v \text{ is on } \ell \\ 0 & \text{if } v \text{ is on } \ell^\perp \end{cases}$$

Since p_ℓ is linear, it may be represented as a matrix. In fact, if we pick either of the two unit vectors $\pm u$ in ℓ (represented as column vectors), then the projector is $p_\ell = uu^T$. To see this, note the u^T in uu^T ensures the kernel (aka nullspace) is $\ell^\perp$ and the u in uu^T ensures the image (aka columnspace or range) is ℓ. More specifically, uu^T has the same characterization as p_ℓ since $(uu^T)v = (u \cdot v)u$ and $u \cdot v$ is 0 if v is on $\ell^\perp$ or is λ if $v = \lambda u$ is on ℓ.

For the diagram with lines k and ℓ, it will be convenient to choose their intersection as the origin and use m and n for our x and y axes. Then the lines are at an angle $\phi = \theta/2$ to m, so we can choose unit vectors $\left[\begin{smallmatrix} \cos\phi \\ \pm\sin\phi \end{smallmatrix}\right]$ on them. The corresponding projectors are then

$$p_k = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix}, \text{ and}$$

$$p_\ell = \begin{bmatrix} \cos\phi \\ -\sin\phi \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \end{bmatrix} = \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix}.$$

The products $p_k p_\ell$ and $p_\ell p_k$ may be calculated as

$$\begin{aligned}
p_k p_\ell &= \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \\
&= (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ \sin\phi\cos\phi & -\sin^2\phi \end{bmatrix}, \\
p_\ell p_k &= \begin{bmatrix} \cos^2\phi & -\cos\phi\sin\phi \\ -\sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \\
&= (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ -\sin\phi\cos\phi & -\sin^2\phi \end{bmatrix}.
\end{aligned}$$

So, by $\theta = 2\phi$ and the double angle formulas for cos, the symmetrization is

$$\begin{aligned}
\frac{1}{2}(p_k p_\ell + p_\ell p_k) &= (\cos^2\phi - \sin^2\phi) \begin{bmatrix} \cos^2\phi & 0 \\ 0 & -\sin^2\phi \end{bmatrix} \\
&= \cos\theta \begin{bmatrix} \frac{1}{2}(\cos\theta + 1) & 0 \\ 0 & \frac{1}{2}(\cos\theta - 1) \end{bmatrix}
\end{aligned}$$

Since $p_m = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $p_n = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, this finally means

$$a(\theta) = \tfrac{1}{2}\cos\theta(\cos\theta + 1), \qquad b(\theta) = \tfrac{1}{2}\cos\theta(\cos\theta - 1).$$

This calculation is relevant in a certain kind of algebraic structure: Jordan algebras, which were an early attempt to formalize quantum observables.

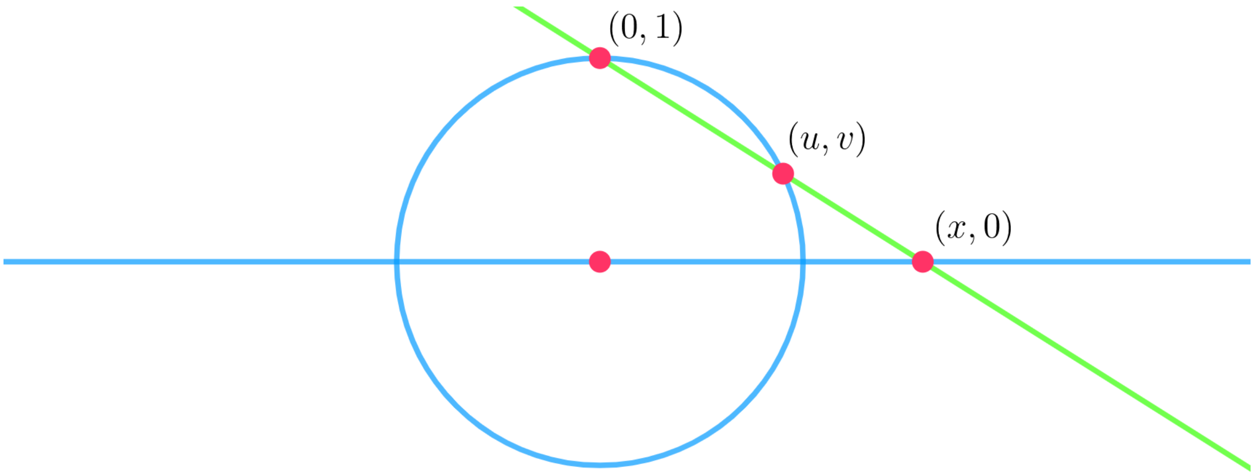
A **Jordan algebra** is a power-associative algebra (where the associative property may fail, but powers like $x^3 = (xx)x = x(xx)$ are still well-defined) where left-multiplication and right-multiplication by powers commute, i.e. $x^m(yx^n) = (x^m y)x^n$ for all elements x, y in the algebra. (The Jordan identity $(xy)x^2 = x(yx^2)$ is a special case, and implies all the other cases.)

The **formally real** Jordan algebras are those where no sum of nonzero squares is zero (just like the real numbers). As with many other algebraic structures, we can define homomorphisms, ideals, direct sums and the like.

Simple Jordan algebras are those with no proper nonzero ideals, or equivalently no proper nonzero homomorphic images. Every formally real Jordan algebra is a direct sum of simple ones. All simple formally real Jordan algebras are of Clifford type or matrix type. The latter are $n \times n$ self-adjoint matrices over real numbers, complex numbers, or quaternions (or octonions for $n \leq 3$), but instead of using the usual matrix multiplication they use the (normalized) **anticommutator** $\{A, B\} := \frac{1}{2}(AB + BA)$.

The **spectral theorem** implies any real symmetric (i.e. self-adjoint) matrix is a linear combination of orthogonal projectors (which are not only orthogonal geometrically, but algebraically as well: $p_k p_\ell = 0$ if k and ℓ are perpendicular lines). This problem shows how to find this decomposition for the anticommutator of two projectors.

Quadratic Pythagorean Triples: Solution



Stereographic projection establishes a one-to-one correspondence between points on a circle and points on a line through it. Below are the formulas for the x-axis and the unit circle (which can be determined by characterizing the line through $(0,1)$ and (u,v) on the unit circle and $(x,0)$ on the x-axis, using point-slope form and different pairs of points for the slope):

$$(u,v) \mapsto \left(\frac{u}{1-v}, 0\right), \qquad \left(\frac{2x}{x^2+1}, \frac{x^2-1}{x^2+1}\right) \leftarrow\!\shortmid (x,0).$$

Since the formulas in both directions send rationals to rationals, this establishes a one-to-one correspondence between rational numbers and rational points on the unit circle. The pole $(0,1)$ corresponds to ∞ in the "extended" real number line $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, which one can imagine "wraps around."

Any Pythagorean triple (a,b,c) can be turned into a rational point $(\frac{a}{c}, \frac{b}{c})$ on the unit circle, and conversely any rational point $(\frac{x}{y}, \frac{w}{z})$ can be written using the lowest common denominator $(\frac{a}{c}, \frac{b}{c})$, which can be turned into a Pythagorean triple (a,b,c). Note (a,b,c) is primitive if and only if $(\frac{a}{c}, \frac{b}{c})$ has no smaller common denominator.

Stereographic projection sends the rational m/n to a rational point, which can in turn be turned into a primitive Pythagorean triple as follows:

$$\frac{m}{n} \mapsto \left(\frac{2mn}{m^2+n^2}, \frac{m^2-n^2}{m^2+n^2}\right) \mapsto (m^2-n^2, 2mn, m^2+n^2).$$

Note the formulas also work to turn rational functions $m(x)/n(x)$ into polynomial Pythagorean triples $(f_1(x), f_2(x), f_3(x))$ of the form

$$\begin{aligned} f_1(x) &= m(x)^2 - n(x)^2, \\ f_2(x) &= 2m(x)n(x), \\ f_3(x) &= m(x)^2 + n(x)^2. \end{aligned}$$

Again the statement about primitive triples and lowest terms holds, since sharing no common root is equivalent to sharing no common factor. If the second polynomial $f_2(x) = 2m(x)n(x)$ is quadratic, then either

- one of $m(x), n(x)$ is quadratic, the other constant; or
- both are linear, so that their product is quadratic.

If (f_1, f_2, f_3) is a quadratic triple then the former is impossible, since it would imply f_1 and f_3 have degree four (not quadratic), so we must have

$$\begin{aligned} m(x) &= Ax + B, \\ n(x) &= Cx + D. \end{aligned}$$

and therefore (f_1, f_2, f_3) has the form

$$\begin{aligned} f_1(x) &= (A^2 - C^2)x^2 + 2(AB - CD)x + (B^2 - D^2), \\ f_2(x) &= (2AC)x^2 + 2(AD + BC)x + (2BD), \\ f_3(x) &= (A^2 + C^2)x^2 + 2(AB + CD)x + (B^2 + D^2). \end{aligned}$$

The discriminants can then be calculated as

$$\begin{aligned} \Delta_1 &= [2(AB - CD)]^2 - 4(A^2 - C^2)(B^2 - D^2) \\ &= +4(AD - BC)^2, \\ \Delta_2 &= [2(AD + BC)]^2 - 4(2AC)(2BD) \\ &= +4(AD - BC)^2, \\ \Delta_3 &= [2(AC + BD)^2] - 4(A^2 + B^2)(C^2 + D^2) \\ &= -4(AD - BC)^2. \end{aligned}$$

Thus, by inspection, $\Delta_1 = \Delta_2 = -\Delta_3$. (Curiously, $AD - BC = \det[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$.)

Quadric Query: Solution

A nontrivial linear equation like the following is not possible:

$$Ap_{12} + Bp_{13} + Cp_{14} + Dp_{23} + Ep_{34} + Fp_{24} = 0.$$

Such an equation cannot exist because each term $x_i y_j$ appears no more than once, hence these terms cannot cancel in pairs. Thus, if an equation exists it must contain a product $p_{ij}p_{k\ell}$. When i, k, k, ℓ are all four numbers $1, 2, 3, 4$ we get the following three expressions:

$$\begin{aligned}
p_{12}p_{34} &= (x_1y_2 - x_2y_1)(x_3y_4 - x_4y_3) \\
&= x_1y_2x_3y_4 - x_1y_2y_3x_4 - y_1x_2x_3y_4 + y_1x_2y_3x_4 \\
p_{13}p_{42} &= (x_1y_3 - x_3y_1)(x_4y_2 - x_2y_4) \\
&= x_1y_2y_3x_4 - x_1x_2y_3y_4 - y_1y_2x_3x_4 + y_1x_2x_3y_4 \\
p_{14}p_{23} &= (x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) \\
&= x_1x_2y_3y_4 - x_1y_2x_3y_4 - y_1x_2y_3x_4 + y_1y_2x_3x_4
\end{aligned}$$

The terms above have been written so the p-products' indices are even permutations of $1, 2, 3, 4$ and the x, y-products indices are simply $1, 2, 3, 4$ in order.

There are six possible "words" of two xs and two ys:

$$xxyy, \quad xyxy, \quad xyyx, \quad yxxy, \quad yxyx, \quad yyxx.$$

Each such term appears twice in the three p-products, once with a plus sign and once with a minus sign, hence their sum is zero:

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

(Or, equivalently, $p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0$.)

Note this polynomial in the ps is half the determinant

$$\det \begin{bmatrix} x_1 & y_1 & x_1 & y_1 \\ x_2 & y_2 & x_2 & y_2 \\ x_3 & y_3 & x_3 & y_3 \\ x_4 & y_4 & x_4 & y_4 \end{bmatrix} = 2(p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23}).$$

This follows from expansion-by-minors: first pick one of the six minors p_{ij} in the first two columns, then a corresponding minor $p_{k\ell}$ in the last two columns; each product $p_{ij}p_{k\ell}$ is obtained in two ways. The determinant must vanish (i.e. equal zero) since the columns of the matrix are linearly dependent.

$\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ are known as **Plücker coordinates**.

Notice the p_{ij}s do not change under column operations (adding multiples of $\mathbf{x}$ to $\mathbf{y}$ or vice-versa), hence $\mathbf{p}$ depends only on $\operatorname{span}\{\mathbf{x}, \mathbf{y}\}$.

The set of 2D subspaces of four-dimensional Euclidean space corresponds in Plücker coordinates to a subset of $\mathbb{R}^6$ defined by a system of algebraic equations. In affine space this is known as the (oriented) **Grassmanian** $\widetilde{G}(4, 2)$, or in projective space it is known as the **Klein quadric**.

Rational Corollary: Solution

Define the rational function

$$f(A,B) = (I+A)(I-BA)^{-1}(I+B)$$

We want to show f is symmetric, i.e. $f(A,B) = f(B,A)$. First, note

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

This formula can be derived by solving a general linear system ($XY = I$) with elimination. It is generalized by Cramer's rule for matrix inverses, which says $X^{-1} = (\det X)^{-1}\operatorname{adj} X$, where $\operatorname{adj} X$ is the adjugate matrix. From here, we can plug two generic 2×2 matrices (thus, a total of eight unknowns) into the rational function f and *laboriously* calculate the result.

Or... we can use a Computer Algebra System like Mathematica:

```
A := {{a11, a12}, {a21, a22}}

B := {{b11, b12}, {b21, b22}}

I2 := IdentityMatrix[2]

(I2 + A) (I2 - B*A)^(-1) (I2 + B)
```

These four lines of input yield the following output:

$$\begin{bmatrix} \frac{(1+a_{11})(1+b_{11})}{(1-a_{11}b_{11})} & -1 \\ -1 & \frac{(1+a_{22})(1+b_{22})}{(1-a_{22}b_{22})} \end{bmatrix}$$

By inspection we see that swapping as and bs does not change the result.

For scalars, there is a geometric sum formula, which provides the power series expansion $(1-x)^{-1} = 1 + x + x^2 + \cdots$ which converges when $|x| < 1$.

The same power series expansion works for $(I-X)^{-1}$ when X is close enough to the zero matrix. "Closeness" is measured by the **Frobenius norm** (aka **Hilbert-Schmidt norm**) given by $\|X\|^2 = \operatorname{tr}(X^T X) = \sum_{i,j} |x_{ij}|^2$, which is the Euclidean norm with respect to the standard basis of matrices.

If $\|X\| < 1$ then $(I-X)^{-1} = I + X + X^2 + \cdots$. The norm is "submultiplicative," meaning $\|AB\| \leq \|A\|\|B\|$. Thus, when $\|A\|, \|B\| < 1$, we get:

$$f(A,B) = (I+A)(I-BA)^{-1}(I+B)$$

$$= (I+A)\big(I + BA + (BA)^2 + \cdots\big)(I+B)$$

$$\begin{array}{ccccccccc} = & I & + & (BA) & + & (BA)^2 & + & \cdots \\ + & A & + & A(BA) & + & A(BA)^2 & + & \cdots \\ + & B & + & (BA)B & + & (BA)^2B & + & \cdots \\ + & AB & + & A(BA)B & + & A(BA)^2B & + & \cdots \end{array}$$

$$\begin{array}{ccccccccc} = & I & + & BA & + & BABA & + & \cdots \\ + & A & + & ABA & + & ABABA & + & \cdots \\ + & B & + & BAB & + & BABAB & + & \cdots \\ + & AB & + & ABAB & + & ABABAB & + & \cdots \end{array}$$

This exhibits all possible "words" made from the letters A and B with no repetitions. Which row a word is located in depends on the first and last letter of the word. Swapping A or B does not change this description, so $f(B,A) = f(A,B)$, for sufficiently "small" matrices, at least.

A theorem due to Krob essentially says any rational identity (like this one) that holds true in any ring (like the ring of matrices) is an algebraic consequence of the geometric sum formula. See *"How would you solve this tantalizing Halmos problem?"* on MathOverflow for more information.

Regularization: Solution

The divergent sums we want meaningful values for are

$$\begin{array}{ccccccccccc} A & = & 1 & + & 2 & + & 3 & + & 4 & + & \cdots \\ B & = & 1 & + & 2^2 & + & 3^2 & + & 4^2 & + & \cdots \\ C & = & 1 & + & 2^3 & + & 3^3 & + & 4^3 & + & \cdots \end{array}$$

Solution 1. Differentiating the geometric series

$$\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \cdots$$

and then multiplying by x gives

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

Doing this twice more yields two more series expansions

$$\begin{aligned} \frac{x(1-x)}{(1+x)^3} \qquad &= x - 2^2x^2 + 3^2x^3 - 4^2x^4 + \cdots \\ \frac{1-4x+x^2}{(1+x)^4} \qquad &= x - 2^3x^2 + 3^3x^3 - 4^3x^4 + \cdots \end{aligned}$$

Setting $x = 1$ in the last three equations yields

$$\begin{aligned} \frac{1}{4} \qquad &= 1 - 2 + 3 - 4 + \cdots \\ 0 \qquad &= 1 - 2^2 + 3^2 - 4^2 + \cdots \\ -\frac{1}{8} \qquad &= 1 - 2^3 + 3^3 - 4^3 + \cdots \end{aligned}$$

To obtain the regularized values of the non-alternating versions of these sums, we may use zero-padding and linearity. In particular,

$$a_1 + a_2 + a_3 + a_4 + \cdots = 0 + a_1 + 0 + a_2 + 0 + a_3 + 0 + \cdots$$

Then we may rewrite

$$\begin{aligned}
\frac{1}{4} &= 1-2+3-4+\cdots \\
&= \begin{array}{r}(1+2+3+4+\cdots) \\ -2(0+2+0+4+\cdots)\end{array} \\
&= \begin{array}{r}(1+2+3+4+\cdots) \\ -4(0+1+0+2+\cdots)\end{array} \\
&= A-4A \quad = \quad -3A
\end{aligned}$$

which implies $A=-\frac{1}{12}$ and then

$$\begin{aligned}
0 &= 1-2^2+3^2-4^2+\cdots \\
&= \begin{array}{r}(1+2^2+3^2+4^2+\cdots) \\ -2(0\ +2^2+0\ +4^2+\cdots)\end{array} \\
&= \begin{array}{r}(1+2^2+3^2+4^2+\cdots) \\ -8(0\ +1^2+0\ +2^2+\cdots)\end{array} \\
&= B-8B \quad = \quad -7B
\end{aligned}$$

which implies $B=0$ and then

$$\begin{aligned}
-\frac{1}{8} &= 1-2^3+3^3-4^3+\cdots \\
&= \begin{array}{r}(1+2^3+3^3+4^3+\cdots) \\ -2(0\ +2^3+0\ +4^3+\cdots)\end{array} \\
&= \begin{array}{r}(1+2^3+3^3+4^3+\cdots) \\ -16(0\ +1^3+0\ +2^3+\cdots)\end{array} \\
&= C-16C \quad = \quad -15C
\end{aligned}$$

which implies $C=\frac{1}{120}$.

Solution 2. We may find A, B, C and their alternating versions, which we'll call X, Y, Z, without differentiating the geometric series formula.

$$\begin{aligned}
W &= 1 - 1 + 1 - 1 + \cdots \\
X &= 1 - 2 + 3 - 4 + \cdots \\
Y &= 1 - 2^2 + 3^2 - 4^2 + \cdots \\
Z &= 1 - 2^3 + 3^3 - 4^3 + \cdots
\end{aligned}$$

The geometric series formula already gives $W = \frac{1}{2}$, but also

$$2W \;=\; \begin{array}{r} (1 - 1 + 1 - 1 + \dots) \\ +\,(0 + 1 - 1 + 1 - \cdots) \end{array} \;=\; 1$$

which implies $W = \frac{1}{2}$ as well. Similarly,

$$2X \;=\; \begin{array}{r} (1 - 2 + 3 - 4 + \dots) \\ +\,(0 + 1 - 2 + 3 - \cdots) \end{array} \;=\; W$$

and $2X = \frac{1}{2}$ implies $X = \frac{1}{4}$.

The next one requires splitting up into two previous alternating sums:

$$\begin{aligned}
2Y \;&=\; \begin{array}{r} (1 - 2^2 + 3^2 - 4^2 + \dots) \\ +\,(0 + 1^2 - 2^2 + 3^2 - \cdots) \end{array} \\
&=\; 1 - 3 + 5 - 7 + \cdots \\
&=\; \begin{array}{r} (1 - 1 + 1 - 1 + \dots) \\ -2(0 + 1 - 2 + 3 - \cdots) \end{array} \\
&=\; W - 2X
\end{aligned}$$

and $2Y = \frac{1}{2} - 2(\frac{1}{4})$ implies $Y = 0$.

And the last one requires splitting into three previous sums.

$$\begin{aligned}
2Z &= \begin{array}{r}(1-2^3+3^3-4^3+5^3-\dots)\\ +(0+1^3-2^3+3^3-4^3+\cdots)\end{array}\\
&= 1\,-\,7\,+\,19\,-\,37\,+\,61\,-\cdots\\
&= \begin{array}{r}(1-\;1+\;1-\;1+\;1-\dots)\\ -6(0+1-3+6-10+\cdots)\end{array}
\end{aligned}$$

Notice 1, $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$ are the triangular numbers, which satisfy $1 + 2 + 3 + \cdots + n = \frac{1}{2}(n^2 + n)$. Continuing,

$$\begin{aligned}
&= \begin{array}{r}(1-\;1+\;1-\;1+\;1-\dots)\\ -3(0+1-2+3-\;4\;+\cdots)\\ -3(0+1-4+9-16+\cdots)\end{array}\\
&= W-3X-3Y
\end{aligned}$$

and $2Z = \frac{1}{2} - 3(\frac{1}{4}) - 3(0)$ implies $Z = -\frac{1}{8}$.

We could have also said $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ to similar effect.

Then A, B, C can be gotten from X, Y, Z as in Solution 1.

The Riemann zeta function, defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$ (its abscissa of convergence), also exists for complex numbers s with real parts less than or equal to 1, except at $s = 1$ itself. This is like the situation for $(1-x)^{-1} = \sum_{n=1}^{\infty} x^n$, where the series converges for $|x| < 1$ but the function exists for all x except $x = 1$. The process of extending the domain of a function in the complex plane is called **analytic continuation**, common for this kind of regularization.

In general, the values $\zeta(-n) = 1 + 2^n + 3^n + \cdots$ have the formula $\zeta(-n) = B_{n+1}/(n+1)$, where the **Bernoulli numbers** B_n appear in the exponential generating function $x/(e^x - 1) = \sum_{n=0}^{\infty} (B_n/n!)x^n$, as well as coefficients in all of the so-called Faulhaber polynomials $P_s(n)$ defined by $P_s(n) = \sum_{k=1}^{n} k^s = 1 + 2^s + 3^s + \cdots + n^s$, e.g. $P_1(n) = \frac{1}{2}(n^2 + n)$.

Rhopalocera: Solution

This is an application of the **Fundamental Counting Principle**, which says that if some outcome depends on a set of choices, and the number of options for each choice is constant, then the total number of possible outcomes is the product of how many options each choice has.

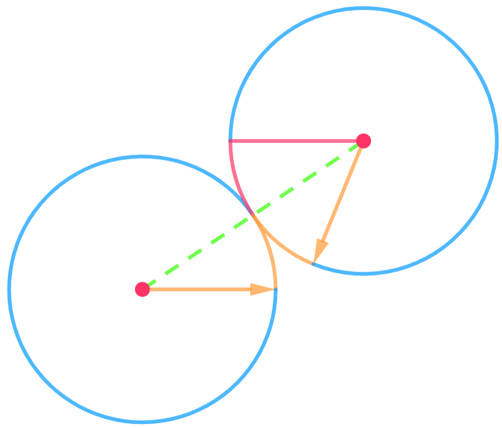
For example, suppose you want to buy a meal that includes one of 2 sides, 3 entrees and 4 drinks: then the total number of possible meals is $2 \times 3 \times 4 = 24$.

The left antenna must be the first segment traced, and the right antenna must be the last segment traced, so we're left to consider the other segments.

There are six possible paths to trace from the head to tail. After this, there are five possible paths to trace back up to the head. And then four possible paths to trace back down. And so on. So the total number of possible ways to trace the butterfly is $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Rolling Spheres: Solution

First, consider the rolling coin paradox. Holding one coin in place on a flat surface, suppose we roll an identical coin around it (without slippage). If the moving coin goes once around the stationary coin, how many times did the moving coin rotate around its own center? The surprising answer is: twice!



In general, as the moving coin revolves by an angle of θ around the stationary coin's center, it rotates by the double angle 2θ around its own center. Cool!

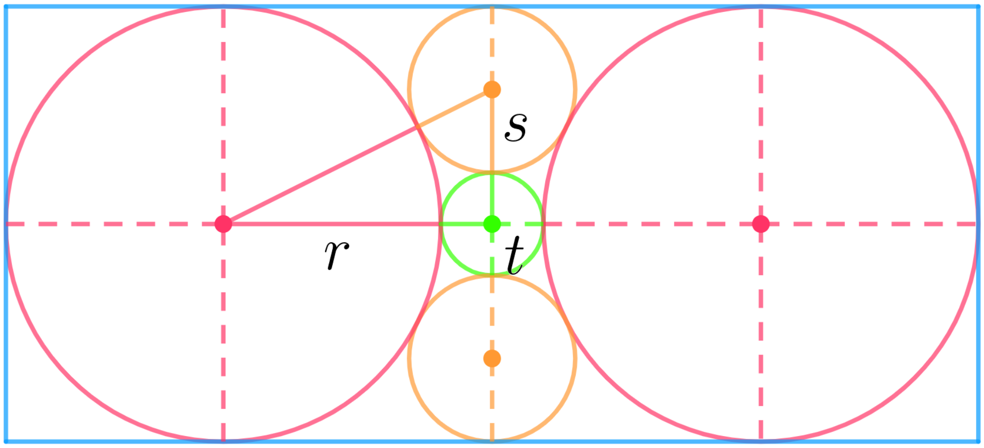
Since the spherical arcs being traversed correspond to right angles from the fixed sphere's center, the rolling sphere undergoes three 180° rotations around its center. First, rolling from the fixed sphere's x-axis pole to its y-axis pole causes the rolling sphere to undergo a 180° rotation around its z-axis. Then it rotates 180° around its x-axis, then around its y-axis.

Pay attention to what happens to a ray, initially pointing in the direction of an axis, when it undergoes these rotations. Whether the ray points in the x-, y-, or z-axes - it always ends up pointing in the same direction again! Thus, the rolling sphere has undergone a net *zero* overall rotation. Indeed,

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

expressed with rotation matrices. (Indeed, doing the same with quaternions, $\mathbf{jik} = 1$, shows the "orientation entanglement" is also ultimately unchanged.)

Sākuru: Solution



The rectangle's length is $4r + 2t$ and its height is $2r$ or also $4s + 2t$. Then the ratio of length to height is $(4r + 2t)/(2r) = 2 + (t/r)$. We can rescale all lengths without changing proportions, so without loss of generality $r = 1$.

Equating the two expressions for height and halving, we can say $1 = 2s + t$.

The right triangle has base $1 + t$, altitude $s + t$ and hypotenuse $1 + s$. Then the Pythagorean theorem says $(1 + t)^2 + (s + t)^2 = (1 + s)^2$. Expanding the squares and cancelling $1 + s^2$ from both sides yields $2t + 2t^2 + 2st = 2s$. We can replace $2s$ with $1 - t$ and this becomes $2t + 2t^2 + (1 - t)t = 1 - t$, or

$$t^2 + 4t - 1 = 0.$$

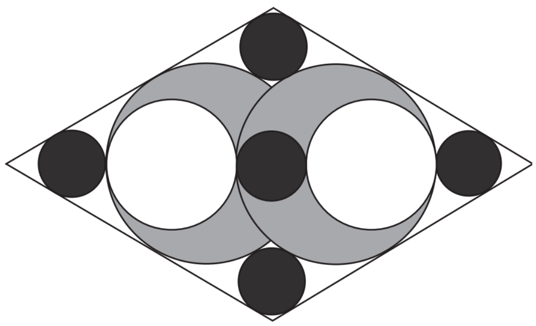
The quadratic formula tells us the positive root of this is

$$t = \frac{-4 + \sqrt{16 + 4}}{2} = -2 + \sqrt{5}.$$

Therefore, the ratio is $2 + t = 2 + (-2 + \sqrt{5}) = \boldsymbol{\sqrt{5}}$.

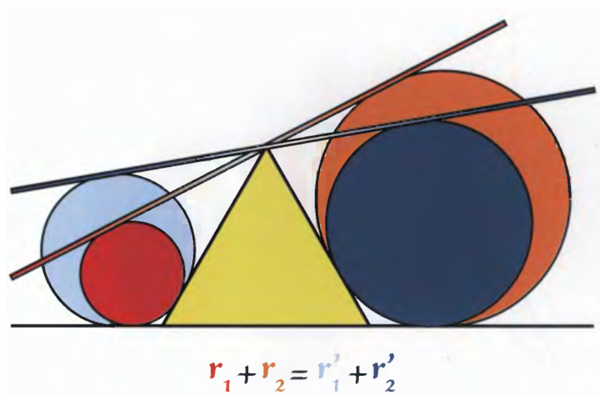
This is a **Sangaku**: "Japanese geometrical problems or theorems on wooden tablets which were placed as offerings at Shinto shrines or Buddhist temples during the Edo period by members of all social classes." (Wikipedia)

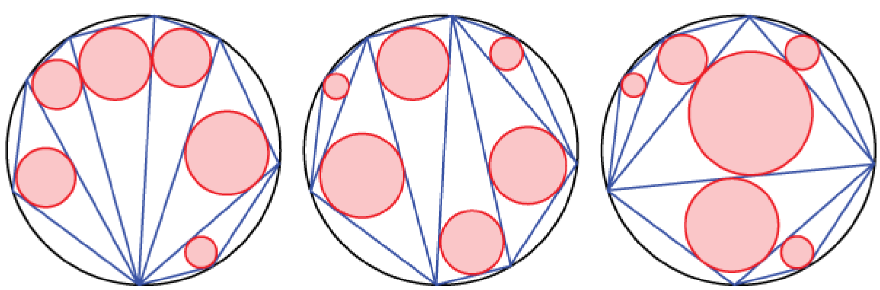
For this problem, J. Marshall Unger, in *A Collection of 30 Sangaku Problems*, cites Fukagawa, Hidetoshi, and Dan Pedoe in *Japanese temple geometry problems*, themselves apparently citing a tablet from the Iwate Prefecture, 1820.



The black circles are half the width of the white ones. “Proposed by Yamasaki Tsugujirou ... the second problem from the right on the Meiseirinji tablet, color plate 8” from *Sacred Mathematics: Japanese Temple Geometry* (Hidetoshi, Rothman)

The sum of the red and orange circle’s radii matches the sum of the light and dark blue circle’s radii. The skew lines on top the equilateral triangle are arbitrary. Taken from the compilation *Sangaku - le mystère des énigmes géométriques japonaises* (Huvent)





The **Japanese Theorem** says the sum of radii of the inscribed circles of any triangulation of a cyclic polygon is independent of the triangulation.

The quadrilateral case is a sangaku of Ryōkwan Maruyama, whose original is lost but is recorded in Kagen Fujita’s *Zoku-Sinpeki-Sanpō* (1807).

Sample Energy: Solution

Let X be the continuous uniform distribution on $[0, 1]$, with

$$F_X(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq 1 \\ 1 & 1 \leq t \end{cases}$$

Let Y be the discrete uniform distribution on $\{u, v, w\}$, with

$$F_Y(t) = \begin{cases} 0 & t < u \\ 1/3 & u \leq t < v \\ 2/3 & v \leq t < w \\ 1 & w \leq t \end{cases}$$

The integrand of the squared energy distance $\int_{-\infty}^{\infty}(F_X(t) - F_Y(t))^2 \, \mathrm{d}t$ is 0 outside of the interval $[0, 1]$, so we can restrict the domain of integration, then split $[0, 1]$ into four intervals and thus the integral into four:

$$\int_0^u t^2 \, \mathrm{d}t + \int_u^v (t - \tfrac{1}{3})^2 \, \mathrm{d}t + \int_v^w (t - \tfrac{2}{3})^2 \, \mathrm{d}t + \int_w^1 (t - 1)^2 \, \mathrm{d}t$$

Instead of evaluating the integrals right away, let's expand the quadratics, then collect the t^2 terms from the four integrals into just one:

$$\int_0^1 t^2 \, \mathrm{d}t + \int_u^v -\tfrac{2}{3}t + \tfrac{1}{9} \, \mathrm{d}t + \int_v^w -\tfrac{4}{3}t + \tfrac{4}{9} \, \mathrm{d}t + \int_w^1 -2t + 1 \, \mathrm{d}t$$

Evaluating the integrals we then get a number of terms:

$$\begin{aligned} = \quad \tfrac{1}{3} \quad &- \quad \tfrac{1}{3}(v^2 - u^2) \quad - \quad \tfrac{2}{3}(w^2 - v^2) \quad - \quad \tfrac{3}{3}(1 - w^2) \\ &+ \quad \tfrac{1}{9}(v - u) \quad + \quad \tfrac{4}{9}(w - v) \quad + \quad \tfrac{9}{9}(1 - w) \end{aligned}$$

Combining like terms simplifies this to

$$\left(\tfrac{1}{3}u^2 + \tfrac{1}{3}v^2 + \tfrac{1}{3}w^2\right) - \left(\tfrac{1}{9}u + \tfrac{3}{9}v + \tfrac{5}{9}w\right) + \left(\tfrac{1}{3} - 1 + 1\right)$$

Way may complete the square to turn this into

$$\tfrac{1}{3}\Big[(u - \tfrac{1}{6})^2 - \tfrac{1}{6^2} + (v - \tfrac{1}{2})^2 - \tfrac{1}{2^2} + (w - \tfrac{5}{6})^2 - \tfrac{5^2}{6^2}\Big] + \tfrac{1}{3}$$

which simplifies to

$$\tfrac{1}{3}\Big[(u-\tfrac{1}{6})^2+(v-\tfrac{1}{2})^2+(w-\tfrac{5}{6})^2\Big]+\tfrac{1}{108}.$$

Thus, when $\{u,v,w\}=\{\frac{1}{6},\frac{1}{2},\frac{5}{6}\}$ the energy distance is minimized to $\frac{1}{\sqrt{108}}$.

In general the squared energy distance between the discrete uniform distribution on $\{u_1,\cdots,u_n\}$ and the continuous uniform distribution on $[0,1]$ is

$$E=\frac{1}{12n^2}+\frac{1}{n}\sum_{k=1}^{n}\left(u_k-\frac{2k-1}{2n}\right)^2.$$

In general, the squared energy distance $E=\mathrm{d}(X,U)^2$ between a random variable X and a discrete uniform random variable on $U=\{u_1,\cdots,u_n\}$ is

$$E=\sum_{k=0}^{n}\int_{u_n}^{u_{n+1}}\left(F_X(t)-\frac{k}{n}\right)^2\mathrm{d}t$$

where $u_1<\cdots<u_n$ and $u_0:=0,u_{n+1}:=1$. The partial derivatives are

$$\begin{aligned}\frac{\partial E}{\partial u_k}\quad &-\Big(F_X(u_k)-\frac{k}{n}\Big)^2+\Big(F_X(u_k)-\frac{k-1}{n}\Big)^2\\ &=\frac{1}{n}\Big(2F_X(u_k)-\frac{2k-1}{n}\Big)\end{aligned}$$

Setting $\nabla E=0$ and solving, we find the minimum $E_{\min}$ is attained when $U=\{F_X^{-1}(\frac{2k-1}{2n})\mid 1\leq k\leq n\}$ (intuitively, the narrower U is, the less it would approximate a continuous variable X, suggesting we needn't check boundary cases). Integrating ∇E from U to $V=\{v_1,\cdots,v_n\}$ yields

$$E=E_{\min}+\frac{2}{n}\int_{F_X^{-1}(\frac{2k-1}{2n})}^{v_k}(v_k-u)f_X(u)\,\mathrm{d}u.$$

Slope-Intercept Coordinates: Solution

Substitute $b = y - mx$ into $b = g(m)$ and differentiate with respect to x, get

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \left(\frac{\mathrm{d}m}{\mathrm{d}x}x + m\frac{\mathrm{d}x}{\mathrm{d}x}\right) = g'(m)\frac{\mathrm{d}m}{\mathrm{d}x}$$

by the product and chain rules. Replace $\mathrm{d}y/\mathrm{d}x$ with m and $\mathrm{d}x/\mathrm{d}x$ with 1, the terms m and $-m$ will cancel, then we may divide by $-\mathrm{d}m/\mathrm{d}x$ and replace the m inside $g'(m)$ with $f'(x)$ to get $x = -g'(f'(x))$. Because g' and f' are one-to-one functions, this is sufficient to show $-g'$ and f' are inverse.

In the language of differentials, $y = mx + b$ with the chain rule yields

$$\mathrm{d}y = (\mathrm{d}m)x + m(\mathrm{d}x) + \mathrm{d}b$$

Using $\mathrm{d}y = m\mathrm{d}x$ we may rearrange this to

$$0 \;=\; \mathrm{d}y - m(\mathrm{d}x) \;=\; \mathrm{d}b + x(\mathrm{d}m)$$

or in other words

$$\begin{cases} y = f(x) \\ b = g(m) \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}x} = m \\[2ex] \dfrac{\mathrm{d}b}{\mathrm{d}m} = -x \end{cases}$$

We may rewrite $\mathrm{d}b/\mathrm{d}m = g'(m)$ as $x = -g'(f'(x))$, just as above.

Functions are **Legendre transformations** of each other when their derivatives are opposite inverses; their graphs are each other's **dual curves**.

The Legendre transformation allows physicists to convert between the so-called Lagrangian and Hamiltonian formulations of classical mechanics, and convert between thermodynamic variables or even create new ones (pressure, volume, temperature, entropy, enthalpy, various energies, and more).

Sphere Gears: Solution

While a point is rotated around an axis, it traces out a circle which is contained in a plane perpendicular to the axis. In particular, its velocity vector must be orthogonal to the axis at all times.

The key idea is that for two balls to be rotating against each other without slipping, the velocity vector at their point of contact must be the same with respect to both rotations. Thus the axes must both be contained in the plane perpendicular to this velocity vector. Since the angles the axes make with the line segment joining the balls' centers are acute, the axes cannot be parallel, so they must intersect at a point.

To find the speed of a particular point under a rotation, we may find the circumference of the circle it traverses and multiply by the angular velocity ω in revolutions per unit time. (This is the usual distance-rate-time formula, where distance = revolutions $\times$ circumference.)

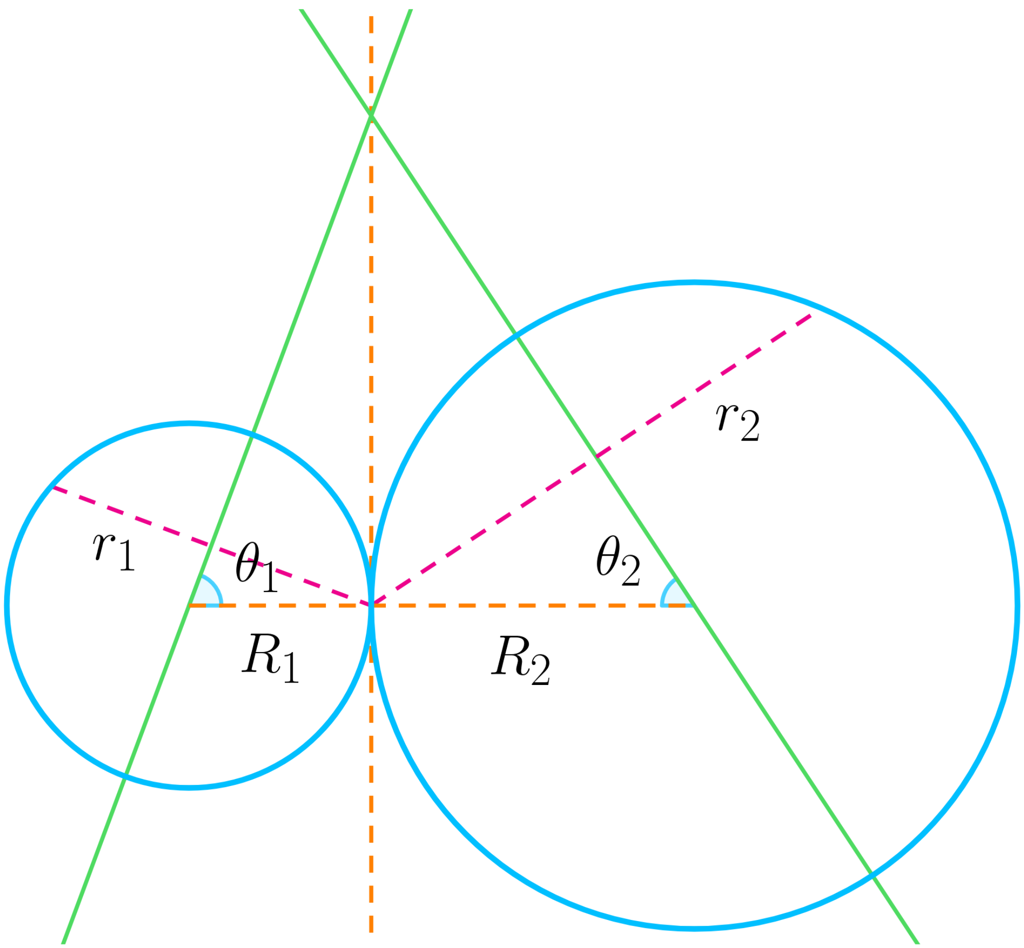
Suppose balls have radii, acute angles (between axes of rotation and the line segment joining their centers) and angular velocities R_1, θ_1, ω_1 and R_2, θ_2, ω_2 respectively. Let r_1 and r_2 be the smaller radii of the circles which are traced out by the point of contact under rotation.

By drawing a triangle between a ball's center, the smaller circle's center, and the point of contact between the balls, we see the smaller radius is $r_i = R_i \sin\theta_i$ for $i = 1, 2$. Setting the speeds v_1 and v_2 equal to each other then gives the equation $2\pi\omega_1 R_1 \sin\theta_1 = 2\pi\omega_2 R_2 \sin\theta_2$.

Then, if the axes' intersection is in the plane between the balls, the line segment joining it to the point of contact must be perpendicular to the other line segment joining the balls' centers. Thus, it is an altitude of the triangle formed by the axes and the line segment joining the centers, which by trigonometry is $R_1 \tan\theta_1 = R_2 \tan\theta_2$.

Dividing the last two equations yields $\omega_1 \cos\theta_1 = \omega_2 \cos\theta_2$.

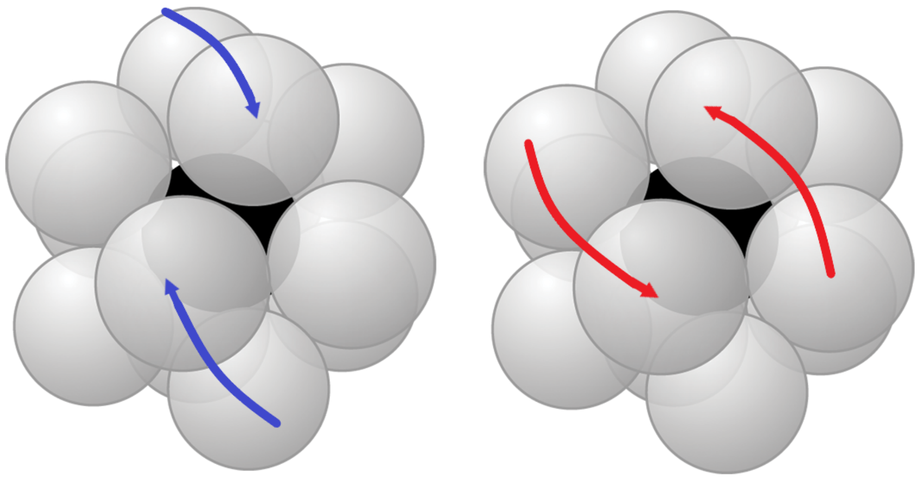
Cross-section in the plane perpendicular to the velocity vectors:



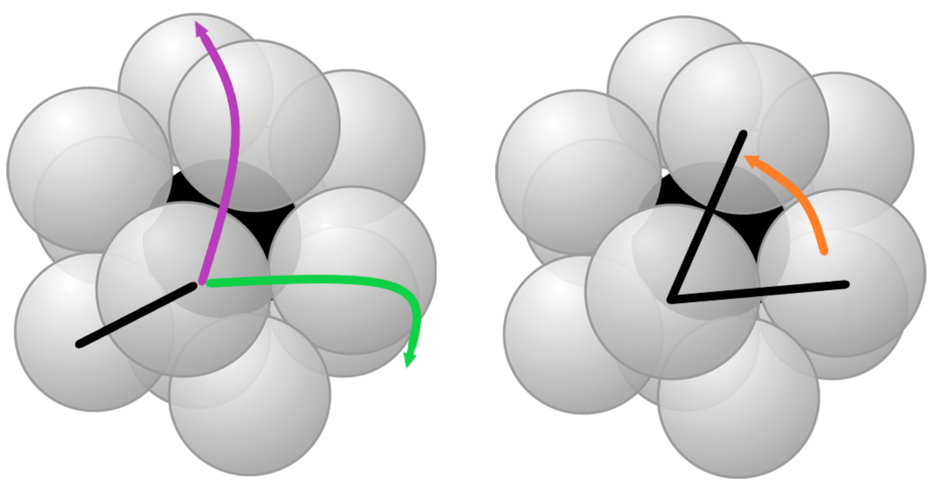
- The velocity vectors point into or out of the page (depending on which sphere we're talking about) at the point of contact between the spheres.
- R_1 and R_2 are the radii of the spheres.
- θ_1 and θ_2 are the angles the rotation axes make with the line segment joining the spheres' centers.
- r_1 and r_2 are the radii of the smaller circles traced out on the surface of the spheres by the point of contact between the spheres under rotation.

Sporadic Twists: Solution

Any two spheres are either (i) adjacent, (ii) share a common adjacent sphere, or (iii) are antipodal (on opposite sides); in other words, spheres are a distance of 1, 2, or 3 spheres apart from each other. Below we illustrate how if spheres are 2 or 3 spheres apart they can be twisted to become adjacent.



Any adjacent pair of spheres is part of two rings. Every other adjacent pair of spheres is either on one of these two rings or shares a sphere with one of them. Thus, given two adjacent pairs, it is possible to slide the first until it is either the second or at least shares a common sphere with the second. In the latter case, a twist can swivel one pair to become the other.



Let A_1, A_2 and B_1, B_2 denote two pairs of spheres. A sequence of twists turn them into adjacent pairs X_1, X_2 and Y_1, Y_2. Thus, there is a sequence of twists that turns $A_1, A_2 \to X_1, X_2$, one which twists $X_1, X_2 \to Y_1, Y_2$ since they're both adjacent, and we can reverse the sequence turns $B_1, B_2 \to Y_1, Y_2$ to get a sequence of twists that turn $Y_1, Y_2 \to B_1, B_2$.

Putting it all together, we get $A_1, A_2 \to X_1, X_2 \to Y_1, Y_2 \to B_1, B_2$.

The group of permutations generated by twists is the **Mathieu group** M_{12}, the subscript indicating it acts on a set of 12 objects. It is the second smallest among a family of five Mathieu groups $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$.

M_{12} is actually **sharply 5-transitive**: given 5 spheres A_1, A_2, A_3, A_4, A_5 and 5 other spheres B_1, B_2, B_3, B_4, B_5 there is a unique permutation of M_{12} which sends $A_i \to B_i$ for $i = 1, 2, 3, 4, 5$. (Our problem merely showed it is 2-transitive!) Besides a full group of *all* permutations on a set (or half that, the group of all even permutations), the Mathieu groups are the *only* permutation groups which act higher than 3-transitively on sets.

In fact, all 2-transitive groups and higher are classified, but the proofs seem to rely on the Classification of Finite Simple Groups, also called the Enormous Theorem. (The Mathieu groups are all simple groups.) The proof of the CFSG is tens of thousands of pages in hundreds of papers by about a hundred mathematicians. It is widely considered the greatest mathematical achievement of the 20th century. A second-generation proof, simplifying the arguments and culling the unnecessary tangents, is in the works.

A **simple** permutation group has no action induced on another (not singleton) set with fewer permutations. For instance, the symmetry group G of a cube can be interpreted as permutations of the 4 space diagonals, but then it has an induced action, involving strictly fewer permutations, on the set of 3 axes, so G is not simple. Contrast with a (regular) icosahedron's symmetry group, which is simple. The Jordan-Hölder theorem describes how all finite groups are built from simple groups.

Squared Cubic Roots: Solution

Solution 1. Define $f(T) = T^3 + aT^2 + bT + c$. By the fundamental theorem of algebra, it can be factored as $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$ for three (not necessarily distinct) roots α, β, γ. Expanding yields:

$$f(T) = T^3 - (\alpha + \beta + \gamma)T^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)T - (\alpha\beta\gamma).$$

Vieta's formulas state that, for monic (i.e. leading coefficient 1) polynomials of *any* degree, each coefficient is equal to $\pm$ a corresponding **elementary symmetric polynomial** of the roots α, β, γ.

In this case, we have:

$$\begin{aligned} \alpha + \beta + \gamma &= -a \\ \alpha\beta + \beta\gamma + \gamma\alpha &= b \\ \alpha\beta\gamma &= -c \end{aligned}$$

On the other hand, define $g(T) = (T - \alpha^2)(T - \beta^2)(T - \gamma^2)$, and assume it expands as $g(T) = T^3 + AT^2 + BT + C$, then Vieta's formulas say

$$\begin{aligned} \alpha^2 + \beta^2 + \gamma^2 &= -A \\ (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 &= B \\ (\alpha\beta\gamma)^2 &= -C \end{aligned}$$

The easiest to find is $C = -(\alpha\beta\gamma)^2 = -c^2$.

Next, notice $(-a)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ (after regrouping and combining like terms) which is $-A + 2b$, and so $A = 2b - a^2$.

Finally, $b^2 = (\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2 + 2(\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2)$, by the same token. The latter part may be factored as $2\alpha\beta\gamma(\alpha + \beta + \gamma)$, so this equation states $b^2 = B + 2(-c)(-a)$, and thus $B = b^2 - 2ac$.

Putting it all together, we conclude

$$g(T) = T^3 + (2b - a^2)T^2 + (b^2 - 2ac)T - c^2.$$

Solution 2. The formula $A^2 - B^2 = (A-B)(A+B)$, which says a difference of squares factors as a product of conjugates, may be used:

$$g(T) = (T-\alpha^2)(T-\beta^2)(T-\gamma^2)$$
$$= (\sqrt{T}-\alpha)(\sqrt{T}+\alpha)\cdot(\sqrt{T}-\beta)(\sqrt{T}+\beta)\cdot(\sqrt{T}-\gamma)(\sqrt{T}+\gamma)$$
$$= (\sqrt{T}-\alpha)(\sqrt{T}-\beta)(\sqrt{T}-\gamma)\cdot(\sqrt{T}+\alpha)(\sqrt{T}+\beta)(\sqrt{T}+\gamma),$$

valid for $T \geq 0$, or even for $T < 0$ if we adopt the convention $\sqrt{-x} = ix$ whenever $-x$ is negative. The first three factors are $f(\sqrt{T})$, however the last three terms have + signs. To remedy this, multiply by $(-1)^4$ and distribute the (-1)s out like so:

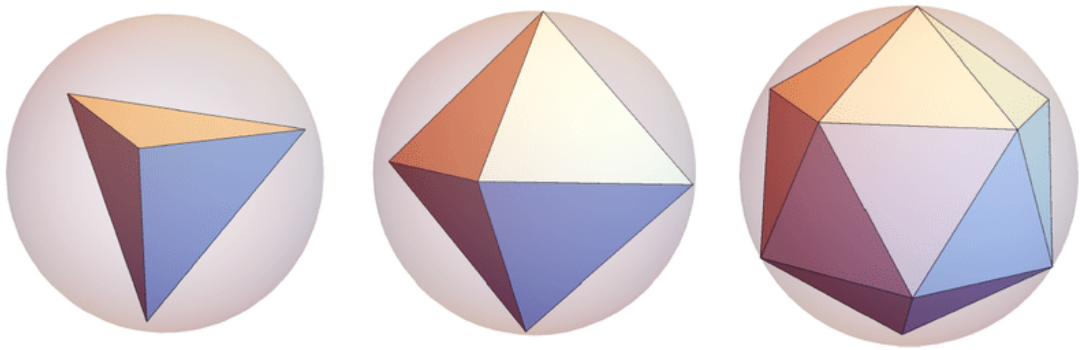
$$(\sqrt{T}+\alpha)(\sqrt{T}+\beta)(\sqrt{T}+\gamma) = -(-\sqrt{T}-\alpha)(-\sqrt{T}-\beta)(-\sqrt{T}-\gamma).$$

Thus, we have $g(T) = -f(\sqrt{T})f(-\sqrt{T})$. Multiplying this out,

$$g(T) = (T^{3/2} + aT + bT^{1/2} + c)(T^{3/2} - aT + bT^{1/2} - c)$$
$$= T^3 + (2b - a^2)T^2 + (b^2 - 2ac)T - c^2$$

The fractional powers cancel out in the end. (Interpret $T^{3/2}$ and $T^{1/2}$ as placeholders for $T\sqrt{T}$ and $\sqrt{T}$ for negative numbers if necessary.)

Striking Gold: Solution



(Tavakoli et al, 2020)

One of the simplest triangulations involves inflating a tetrahedron until its edges become arcs on the sphere. According to MathWorld, its chromatic polynomial is $x(x-1)(x-2)(x-3)$, whose roots are not close enough.

Again by MathWorld, the octahedral graph has chromatic polynomial

$$x(x-1)(x-2)(x^3-9x^2+29x-32).$$

Plugging the last factor into Wolfram|Alpha, we find a root $x \approx 2.5466$.

The icosahedral graph has chromatic polynomial $x(x-1)(x-2)(x-3)$ times

$$x^8 - 24x^7 + 260x^6 - 1670x^5 + 6999x^4 - 19698x^3 + 3640x^2 - 40240x + 20170.$$

Plugging the last factor into W|A again, we find a root $x \approx 2.6182$.

A theorem due to W. Tutte says spherical triangulations' chromatic polynomials tend to have a real root near $\varphi + 1$, where $\varphi = \frac{1+\sqrt{5}}{2} \approx 1.618$ is the golden ratio. More precisely, if G is a planar graph with V vertices then

$$|P_G(\varphi+1)| \leq \varphi^{5-V}.$$

Superexponential: Solution

The **tetration** operation $a \uparrow\uparrow b$ can initially be interpreted as repeated exponentiation, similar to how exponentiation can be interpreted as repeated multiplication and multiplication as repeated addition. (This kindergarten interpretation fails beyond counting numbers, of course.)

Also known as a "power tower," it is defined by the formula:

$$a \uparrow\uparrow b := \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{b}.$$

E.g. $2 \uparrow\uparrow 3 := 2 \wedge 2 \wedge 2 = 16$. It satisfies the recurrence

$$a \uparrow\uparrow b = a \wedge (a \uparrow\uparrow (b-1)).$$

A pattern emerges applying iterated logarithms to tetrations, e.g.

$$\ln^3(a \uparrow\uparrow 5) = \ln\ln\ln a^{a^{a^{a^{a}}}} = a^a \ln(a \ln(a \ln a))$$

Notice $a^a = a \uparrow\uparrow 2$ and $\ln(a \ln(a \ln a)) = \ln^3(a \uparrow\uparrow 3)$. More generally,

$$\ln^c(a \uparrow\uparrow b) = (a \uparrow\uparrow (b-c)) \ln^c(a \uparrow\uparrow c)$$

when $c \le b$ (and $a \uparrow\uparrow 0 = 1$) by induction. So define

$$a_n := n \uparrow\uparrow (n+1), \qquad b_n := n \uparrow\uparrow n.$$

Then for $n > k$ (with k fixed) the difference $\ln^k a_n - \ln^k b_n$ is

$$\big[n \uparrow\uparrow (n+1-k) - n \uparrow\uparrow (n-k)\big] \ln^k(n \uparrow\uparrow k).$$

Both parts of the above product diverge as $n \to \infty$. The first may be written as $n^m - m$ where $m = n \uparrow\uparrow (n-k)$. Since $n^m \ge nm$ when $n > 1, m \ge 1$, which can be proved by induction (for fixed n), the bracketed expression is $\ge (n-1)m = (n-1)(n \uparrow\uparrow (n-k))$. So the bracketed expression diverges. For the non-bracketed expression,

$$\ln^k(n \uparrow\uparrow k) = \ln n + \ln\ln n + \ln\ln\ln n + \cdots + \ln^k n$$

by induction, which diverges since it is a sum of k divergent terms.

Synchronicity: Solution

At 11:00pm the minute hand is $360°/12 = 30°$ clockwise from the hour hand, and at 11:15pm the minute hand is a little more than $90°$ clockwise from the hour hand, so our solution is somewhere between these two times.

Every minute that passes, the minute hand rotates $360°/60 = 6°$ forward, and the hour hand rotates $6°/12 = 0.5°$ forward. If m is the number of minutes after 11:00pm for our solution, then $30 + 6m - 0.5m = 90$ is the angle between the hands, and solving yields $m = 60/5.5 \approx 11$, or 11:11pm.

A common superstition has it that 11:11 is connected to coincidence.

Mathematics is not without its share of numerology and mysticism. Take for instance this exchange (from *The Man Who Knew Infinity*, p312):

> Once, in the taxi from London, Hardy noticed its number, 1729. He must have thought about it a little because he entered the room where Ramanujan lay in bed and, with scarcely a hello, blurted out his disappointment with it. It was, he declared, "rather a dull number," adding that he hoped that wasn't a bad omen. "No, Hardy," said Ramanujan. "It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."

Indeed, $1729 = 1^3 + 12^3 = 9^3 + 10^3$. There is a 1729-dimensional Fourier transform used in a so-called galactic algorithm for computing the product of two integers. It is the fastest-known algorithm to multiply numbers in the long run, but the "long run" means its optimal efficiency doesn't kick in until numbers start having around `2^1729^12` bits.

Nature's favorite number might just be 24. Because of its balance between high divisibility and smallness, it is also a favorite choice (along with its maximal divisor 12) of humans for systems that involve frequent division (such as timekeeping; there are 24 hours per day and 12 months per year). There are much deeper mathematical coincidences involving 24, though.

Some considerations show strings wiggling in 1D have ground state energy

$$\frac{1}{2}\big(1+2+3+4+\cdots\big) \;=\; -\frac{1}{24},$$

appropriately "regularized." In string theory, "strings" are actually 2D tubes, and them vibrating in $(24+2)$-dimensional spacetime leads to a ground state energy of $24(-\frac{1}{24}) = -1$, which turns out to explain why 26 is the only consistent number of dimensions for the theory.

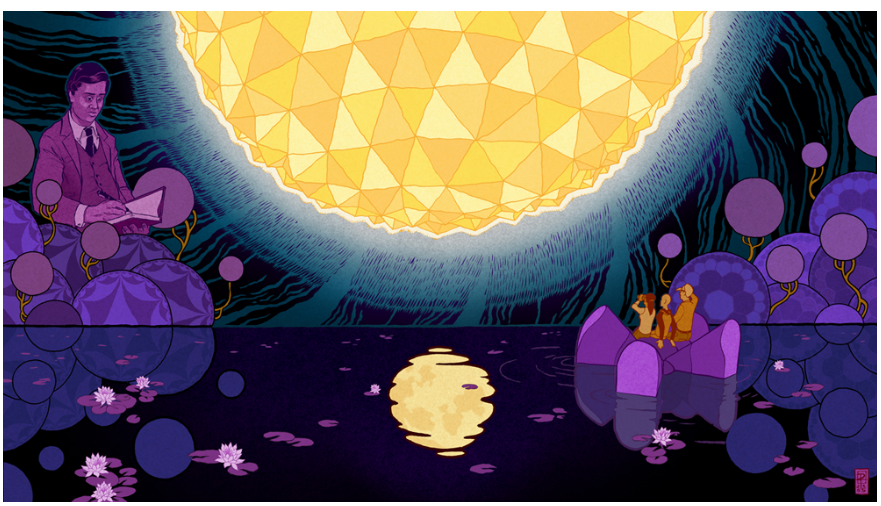
The only time the sum of the first n squares is itself a square is

$$1^2+2^2+3^2+4^2+\cdots+23^2+24^2 \;=\; 70^2.$$

For this reason, the null vector $(0,1,2,\cdots,24,70)$ in $(24+2)$-dimensional spacetime can be used to construct the Leech lattice Λ_{24}, a unique 24-dimensional crystal pattern. The symmetry group of the string theory built on Λ_{24} is the Monster group M, the largest sporadic finite simple group. The smallest nontrivial irreducible representation of M has dimension 196884. Seemingly completely unrelatedly, the first nontrivial coefficient of the Fourier expansion of the j-invariant from the theory of modular forms is 196883. When McKay pointed out to Conway that

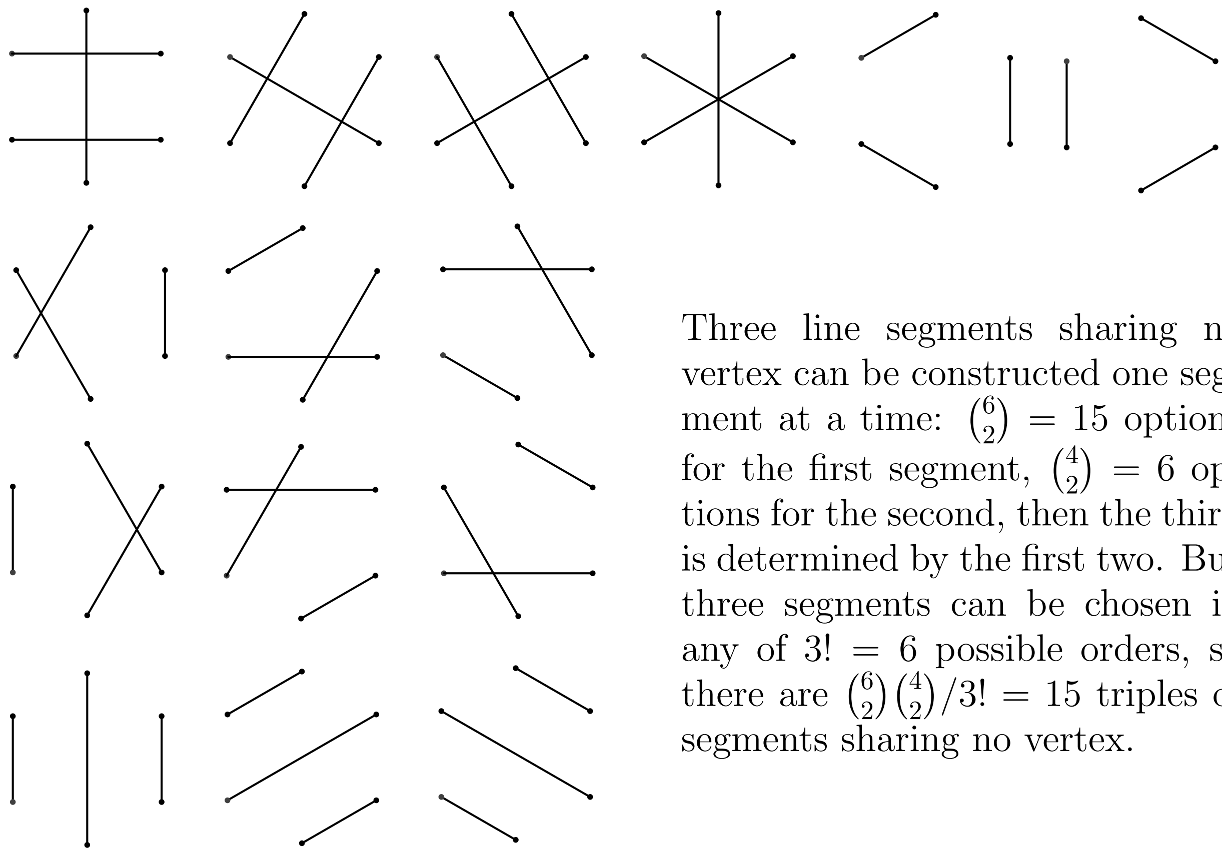
$$196884 = 196883 + 1,$$

Conway called it moonshine (i.e. "crazy"). This coincidence, and the area of math created in its wake to explain it, is known as Monstrous Moonshine.

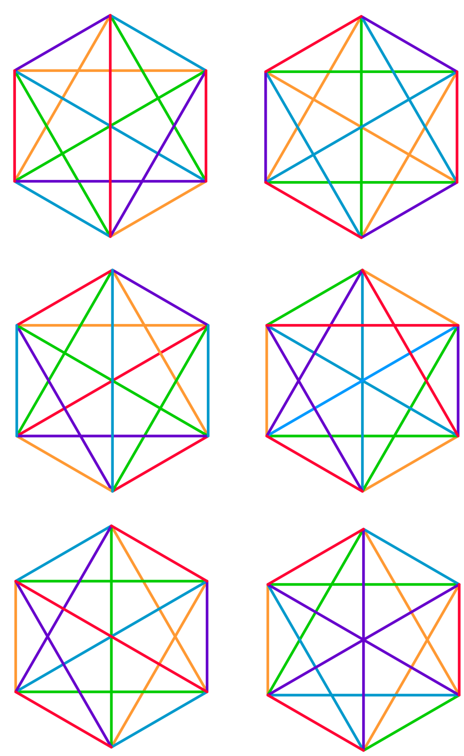


by Peter Diamond for Quanta Magazine

Synthematics: Solution



Three line segments sharing no vertex can be constructed one segment at a time: $\binom{6}{2} = 15$ options for the first segment, $\binom{4}{2} = 6$ options for the second, then the third is determined by the first two. But three segments can be chosen in any of $3! = 6$ possible orders, so there are $\binom{6}{2}\binom{4}{2}/3! = 15$ triples of segments sharing no vertex.



Imagine constructing the top right synthematic to the right. There are 15 options for the red triple, then 8 disjoint from the red to choose the purple. Only 4 options left, the top left triples above, but choosing the snowflake prevents us from choosing any more, so the last three triples are determined. The shape is irrelevant to the counting process - by symmetry, the number of options for each choice does not depend on previous choices. The triples can be chosen 5! orders, so the answer is

$$\frac{15 \cdot 8 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 6.$$

The vertices may be labelled one through six. The line segments, or **duads**, then, are subsets $\{a, b\}$. The triples are partitions of $\{1, 2, 3, 4, 5, 6\}$ of the shape $\{\{a, b\}, \{c, d\}, \{e, f\}\}$, called **synthemes** by Sylvester. The number of synthemes can be expressed in terms of multinomial coefficients as $\frac{1}{3!}\binom{6}{2,2,2}$. The colorful hexagons, called **synthematic totals**, are partitions of set of all 15 line segments $E = \{\{a, b\} \mid 1 \leq a < b \leq 6\}$ into 5 synthemes.

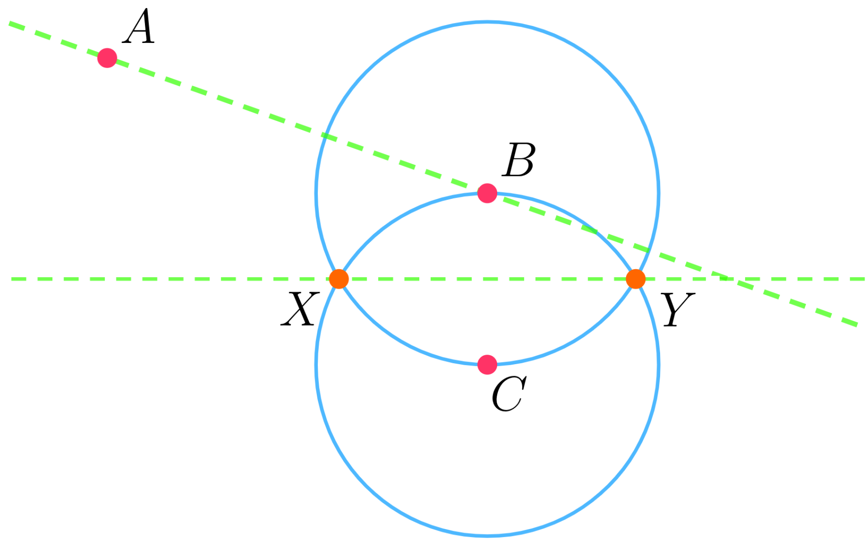
A permutation of the six vertices induces a permutation of the six synthematic totals. Swapping two vertices winds up swapping three pairs of totals (and vice-versa), and cycling three vertices winds up cycling two sets of three totals (and vice-versa). In the language of group theory, this exhibits the unique nontrivial outer automorphism of the symmetric group .

The hexagon with all its line segments forms a complete graph K_6. This can be considered a **hemi-icosehedron** (a kind of projective polyhedron), since an icosahedron has 6 opposite pairs of vertices and 15 opposite pairs of edges. Synthemes correspond to inscribed compounds of orthogonal golden rectangles. The symmetry group of the octahedron, A_5, is an **exotic copy** of the usual alternating subgroup $A_5 \subset S_6$.

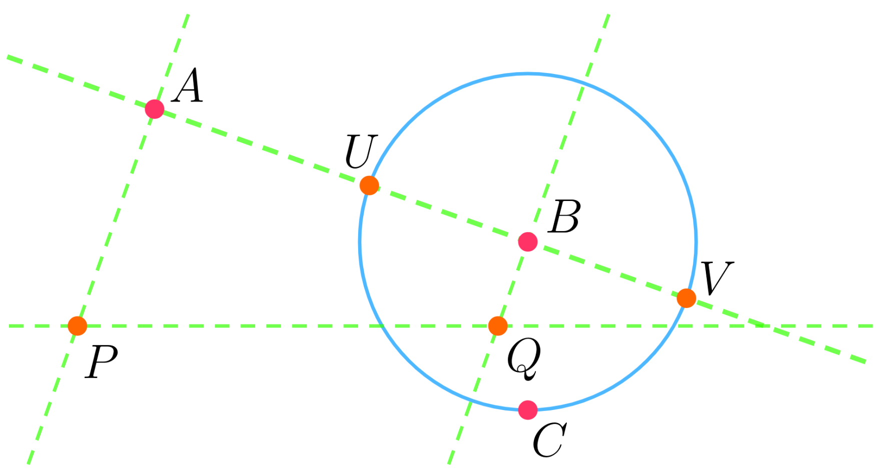
The fact $\operatorname{Out} S_6 \cong \mathbb{Z}_2$ has order two indicates there is *duality.* Indeed, constructing the set of synthematic totals out of a six element set can be considered (the restriction of a) **combinatorial species** which has order two under composition (up to natural isomorphism). Moreover, the duads and synthemes are the vertices and edges of the self-dual **Cremona-Richmond configuration** (whose vertices may represent duads and synthematics).

Tale of Two Tangents: Solution

Label the three points given A, B, C. Draw a circle around B through C and a circle around C through B, then label their points of intersection X, Y. Draw a line through X and Y. This is the *perpendicular bisector* of B, C.

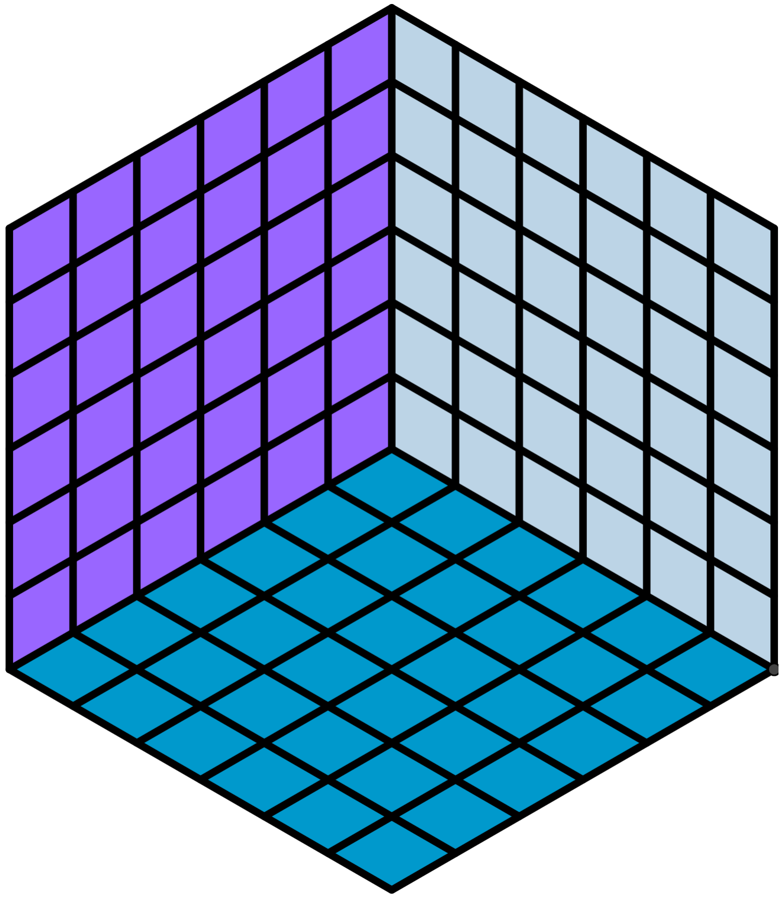


The circle around B intersects the line through A, B twice, call these points U, V. We know how to construct perpendicular bisectors now, so construct one through B. Do the same process with A as well.



These bisectors intersect the other line at points P, Q. Our final answer is the circle centered at P through A and the circle centered at Q through B.

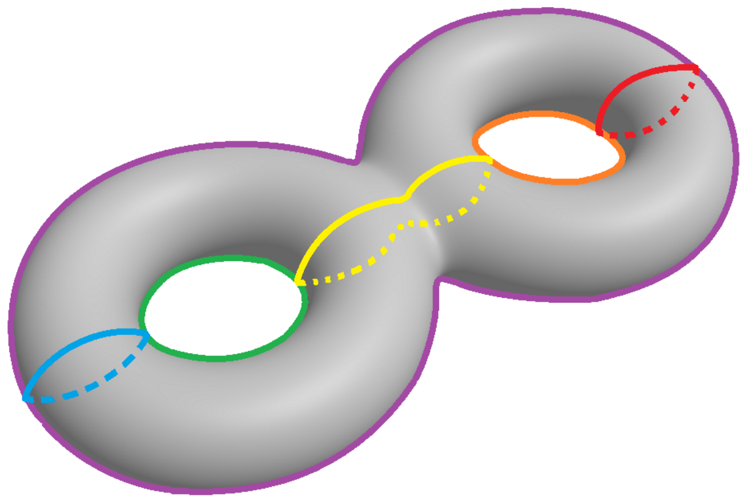
Thinking Outside the Box: Solution



If we tilt our head until we're looking straight at the floor, or the left wall, or the right wall, then all the squares we see are the same color!

Totally Tubular: Solution

Below are six inequivalent nonseparating loops in six different colors.



Slicing the double donut vertically between the left and right halves would separate it into two donuts; slicing it vertically the other way would yield the primary (blue, yellow, and red) colored loops; slicing it horizontally would yield the secondary (purple, green, and orange) colored loops.

Any loop on the surface can be tightened until it becomes a **geodesic** (a curve that minimizes surface-distance between any two sufficiently close points on it; in flat space, it is well-known the geodesics are just straight lines).

(Oberwolfach Photo Collection 2014)

Maryam Mirzakhani won the Fields Medal in 2014 for her work on the dynamics and geometry of Riemann surfaces. In particular, she showed the number of geodesics of length $\leq L$ can be estimated similar to the prime numbers $\leq x$.

A corollary says nonseparating loops on a double torus outnumber separating ones six to one.

Transitive Property: Solution

We may outline all (equally likely) outcomes:

●	2	4	9
1	●	●	●
6	●	●	●
8	●	●	●

●	3	5	7
2	●	●	●
4	●	●	●
9	●	●	●

●	1	6	8
3	●	●	●
5	●	●	●
7	●	●	●

The dots represent winners. The upper left dot represents the die which is more likely to win between the two dice in a given table. Therefore,

$$\text{Prob}(L > C) = \text{Prob}(M > L) = \text{Prob}(C > M) = \tfrac{5}{9}.$$

In other words, lime beats cyan, magenta beats lime, and cyan beats magenta.

Thus, whatever die you pick, your acquaintance may pick the superior die and have the better odds, so you should not accept the offer.

Twenty Four: Solution

A simple Python script is below. (Try it on your favorite online Python IDE!)

```
hands = []

for A in range(1, 14):
   for B in range(1, 14):
      for C in range(1, 14):
         for D in range(1, 14):
            if (A * D == (B * D - C) * 24):
               hands.append([A, B, C, D])
               print(hands[-1])

print("Total:  " + str(len(hands)))
```

Note denominators are cleared for the equation $AD = (BD - C) \cdot 24$ so the program doesn't need to handle fractions. This returns the following output:

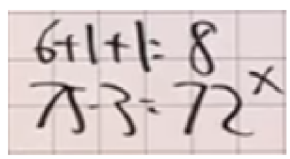
```
[2, 1, 11, 12]
[3, 1, 7, 8]
[4, 1, 5, 6]
[4, 1, 10, 12]
[4, 2, 11, 6]
[6, 1, 3, 4]
[6, 1, 6, 8]
[6, 1, 9, 12]
[6, 2, 7, 4]
[6, 3, 11, 4]
[8, 1, 2, 3]
[8, 1, 4, 6]
[8, 1, 6, 9]
[8, 1, 8, 12]
[8, 2, 5, 3]
[8, 2, 10, 6]
[8, 3, 8, 3]
[8, 4, 11, 3]
[9, 1, 5, 8]
[9, 2, 13, 8]
[10, 1, 7, 12]
[12, 1, 1, 2]
[12, 1, 2, 4]
[12, 1, 3, 6]
[12, 1, 4, 8]
[12, 1, 5, 10]
[12, 1, 6, 12]
[12, 2, 3, 2]
[12, 2, 6, 4]
[12, 2, 9, 6]
[12, 2, 12, 8]
[12, 3, 5, 2]
[12, 3, 10, 4]
[12, 4, 7, 2]
[12, 5, 9, 2]
[12, 6, 11, 2]
[12, 7, 13, 2]
Total:  37
```

According to statistics from the website 4nums.com, the three quadruples (out of the 1362 total solvable quadruples) which take humans the longest time to solve are $(2, 3, 5, 12)$, $(1, 3, 4, 6)$, and $(1, 4, 5, 6)$, all three of which have solutions exclusively of the form $A/(B - C/D) = 24$.

24 is similar to the numbers round of ***Countdown***, a long-running British game show. There are 24 numbers to choose from: four large $(25, 50, 75, 100)$ and twenty small (two of each of the numbers 1-10). Contestants, of course, only get to choose the sizes of the six numbers they receive (small or large), not the actual numbers themselves. A 3-digit number is randomly generated, and contestants win points based on how close they can get to it in 30 seconds using any of the 6 numbers and the four arithmetic operations $(+, -, \times, \div)$.

Replacing Carol Vorderman in 2008, Rachel Riley is more than just a Vanna White-style hostess for the show: she not only checks the contestants' answers, but routinely finishes the rounds off by providing the best or better solutions purely from mental math.

A crossover *8 Out of 10 Cats Does Countdown* began in 2012, which was the same show but played (less seriously) by comedians and celebrity guests. In episode 5 of series 14, contestants failed to get close to 576. The contestants were incredulous when Rachel revealed an exact solution::



After a friendly ribbing for having no friends ("the numbers are my friends!"), she pointed out a better solution is $576 = 24^2 = ((75/3 + 1) \times ((1 + 3) \times 6)$.

Twisted Clothesline: Solution

For each of the seven ($1+2+4=7$) vertices above the last row, we have a binary choice of whether or not to swivel the subtree below it. Thus, the number of swivellable permutations of the last row of vertices is $2^7 = 128$.

It may not be obvious why this exhausts all possibilities; what about the order in which we swivel subtrees, or the option to swivel multiple times at a vertex (just not in a row), why don't these lead to more permutations?

Consider the general problem of counting the number W_n of permutations possible if our tree has n levels (our situation is $n = 3$). The permutations are of two kinds: those where the left half of the numbers remain on the left and the right half of numbers remain on the right, or those where the opposite is true. There are an equal number of each, because swivelling below the top vertex converts permutations of the one kind to the other and then back.

The number of permutations of the first kind is $W_{n-1} \times W_{n-1}$, because there are W_{n-1} permutations possible for the first half of numbers and W_{n-1} also for the second half. Therefore, we have the recursion

$$W_n = 2W_{n-1}^2.$$

The first couple values are $W_0 = 1$ and $W_1 = 2$ by inspection, which leads to $W_2 = 2^3 = 8$ and $W_3 = 2^7 = 128$. The general formula we then guess is

$$W_n = 2^{1+2+\cdots+2^{n-1}} = 2^{2^n-1}.$$

We can verify this is true by induction: this satisfies $W_0 = 1$ and $W_n = 2W_{n-1}^2$.

This means swivelling in different orders or multiple times at vertices achieves no more permutations than if we always swivel the vertices in the same order at most once each. Indeed, each permutation can be interpreted as a function of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ which means we have a **permutation group**: a set of invertible functions on a set which is closed under composition and inverses.

To understand the structure of this permutation group, it's necessary to understand **wreath products**. One way to understand the wreath products (and direct products) of permutation groups is with the *product action*. Suppose G is a permutation group acting on the set $\{1, \cdots, m\}$ and H is a permutation group acting on the set $\{1, \cdots, n\}$. Make a table with m rows and n columns, filling the entries with the numbers 1 through $m \times n$:

	1	2	3	$\cdots$	n
1	1	$\cdot$	$\cdot$	$\cdots$	$\cdot$
2	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
3	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
m	$\cdot$	$\cdot$	$\cdot$	$\cdots$	mn

The direct product $H^m = \overbrace{H \times \cdots \times H}^{m}$ (in some contexts called a direct sum, for which we will use the shorthand $mH = H \oplus \cdots \oplus H$) is a permutation group acting on $\{1, \cdots, mn\}$, the set of labels in the table. The elements of H^m are attained by using permutations from H on each row individually.

The wreath product $H \wr G$ is a larger permutation group, containing H^m as a subgroup, acting on $\{1, \cdots, mn\}$. The functions of $H \wr G$ are attained by using row-permutations from H^m followed by using a permutation from G to shuffle the rows amongst each other. This means the order (cardinality) of the wreath product is $|H \wr G| = |H|^m |G|$.

The permutation group which cycles the elements $\{1, 2, \cdots, p\}$ around in a circle we can denote $\mathbb{Z}_p$. The permutation group of order 2^7 we found acting on $\{1, \cdots, 8\}$ is actually a **wreath power** $\mathbb{Z}_2^{\wr 3} = \mathbb{Z}_2 \wr \mathbb{Z}_2 \wr \mathbb{Z}_2$ (we don't need to distinguish between $\mathbb{Z}_2 \wr (\mathbb{Z}_2 \wr \mathbb{Z}_2)$ and $(\mathbb{Z}_2 \wr \mathbb{Z}_2) \wr \mathbb{Z}_2$ because the wreath product, as an operation on permutation groups, is associative).

Wreath powers yield **Sylow subgroups** of symmetric groups. The first Sylow theorem asserts that if a finite group G has order n and p^k is the largest power of a prime p dividing n, then G has a subgroup of order p^k, called a Sylow subgroup (this is a partial converse to Lagrange's theorem, which says the order of any subgroup H is a divisor of n; the full converse is false in general).

In particular, our $\mathbb{Z}_2^{\wr 3}$ is a Sylow subgroup of S_{2^3}. In general, a Sylow subgroup P of S_n can be constructed as a direct sum of wreath powers analogous to representing n in base-p. Specifically, if $n = \sum n_k p^k$ (with digits n_k taken from $\{0, 1, \cdots, p-1\}$) is n's base-p representation, then $P = \bigoplus n_k \mathbb{Z}_p^{\wr k}$. This can be verified a Sylow subgroup with Legendre's formula from number theory.

Trees and Wreaths

We begin by substituting (k, ℓ) for the indices (so $f_{m+n,2m+3n}$ becomes $f_{k,\ell}$), and then solving for (m, n) by elimination (to rewrite $x^m y^n$):

$$\begin{cases} k = m + n \\ \ell = 2m + 3n \end{cases} \implies \begin{cases} m = 3k - \ell \\ n = \ell - 2k \end{cases}$$

Then the double series becomes

$$\sum_{m,n} f_{m+n,2m+3n}\, x^m y^n = \sum_{k,\ell} f_{k,\ell}\, x^{3k-\ell} y^{\ell-2k}$$

$$= \sum_{k,\ell} f_{k,\ell}\, (x^3 y^{-2})^k (x^{-1} y)^\ell = F(x^3/y^2, y/x).$$

Unequal Booty: Solution

Call the number of coins in the chest A, the number of coins the captain gets first B, the number of coins the second pirate gets first C, the number of coins the third gets first D, the number the swabbie gets first E, and the number of coins each pirate gets in the final handout F, so that

$$\begin{aligned} A &= 1 + 3B \\ 2B &= 1 + 3C, \\ 2C &= 1 + 3D, \\ 2D &= 1 + 3E, \\ 2E &= 1 + 3F. \end{aligned}$$

A series of substitutions allows us to write

$$\begin{aligned} A \quad &= 1 + 3B \\ &= 1 + \tfrac{3}{2}(1 + 3C) \\ &= 1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(1 + 3D)) \\ &= 1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(1 + 3E))) \\ &= 1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(1 + \tfrac{3}{2}(3F)))). \end{aligned}$$

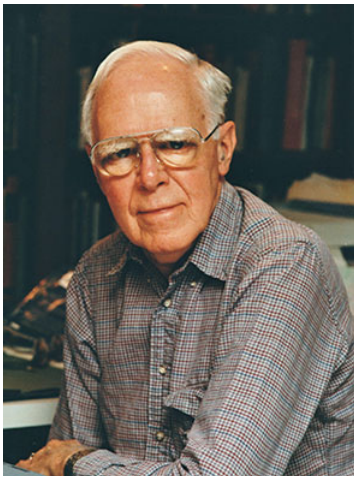
Multiplying by $2^4 = 16$ rids fractions, then find quotient/remainder of $\div 16$:

$$\begin{aligned} 16A \quad &= 16 + 3(8 + 3(4 + 3(2 + 3(6F)))) \\ &= 16 + 24 + 46 + 54 + 243F \\ &= 130 + 243F \\ &= 16(8 + 15F) + (2 + 3F) \end{aligned}$$

Subtracting and factoring allows us to write

$$16(A - 8 - 15F) = 2 + 3F.$$

The larger F is, the larger the number of coins in the chest is. We seek the smallest value of F that makes $2 + 3F$ divisible by 16. Or, equivalently, the smallest multiple of 16 that is 2 greater than a multiple of 3. This last interpretation lends itself to a quick answer: 32 is the smallest such multiple of 16, yielding $F = 10$, and subsequently $E = 15$, $D = 23$, $C = 35$, $B = 53$ and the smallest number of coins in the chest is $A = 160$.



This is an adaptation of the *monkey and the coconuts* problem, the favorite problem of probably the most famous mathematical columnist of all, Martin Gardner, writer for the Mathematical Games column of the *Scientific American* magazine for a quarter century, and publisher of over a hundred books. Besides popularizing recreational mathematics, he was also an expert on Lewis Carroll, and founded the now Committee for Skeptical Inquiry (CSI) to combat pseudoscience.

Versorial Validation: Solution

Setting $p = a + \mathbf{u}$ and $q = b + \mathbf{v}$, we begin by expanding $|pq|^2$:

$$\begin{aligned} |(a+\mathbf{u})(b+\mathbf{v})|^2 &= |(ab - \mathbf{u}\cdot\mathbf{v}) + (a\mathbf{v} + b\mathbf{u} + \mathbf{u}\times\mathbf{v})|^2 \\ &= (ab - \mathbf{u}\cdot\mathbf{v})^2 + \|a\mathbf{v} + b\mathbf{u} + \mathbf{u}\times\mathbf{v}\|^2. \end{aligned}$$

We can FOIL $(ab - \mathbf{u}\cdot\mathbf{v})^2$, and additionally using the relation $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w}$ we can distribute the vector norm above and combine like terms to get

$$\begin{array}{ccccccc} = & a^2b^2 & - & 2ab(\mathbf{u}\cdot\mathbf{v}) & + & (\mathbf{u}\cdot\mathbf{v})^2 \\ + & a^2\|\mathbf{v}\|^2 & + & b^2\|\mathbf{u}\|^2 & + & \|\mathbf{u}\times\mathbf{v}\|^2 \\ + & 2ab(\mathbf{u}\cdot\mathbf{v}) & + & 2a\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) & + & 2b\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}). \end{array}$$

The $\pm 2ab(\mathbf{u}\cdot\mathbf{v})$ terms cancel. The cross product $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, which makes the dot products $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v})$ and $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v})$ zero, so the magenta terms vanish. For the orange terms, we can use the facts

$$\begin{cases} \mathbf{u}\cdot\mathbf{v} & = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta, \\ \|\mathbf{u}\times\mathbf{v}\| & = \|\mathbf{u}\|\|\mathbf{v}\|\sin\theta. \end{cases}$$

Using $\cos^2\theta + \sin^2\theta = 1$, the orange terms combine to $\|\mathbf{u}\|^2\|\mathbf{v}\|^2$. So we have

$$= a^2b^2 + a^2\|\mathbf{v}\|^2 + b^2\|\mathbf{u}\|^2 + \|\mathbf{u}\|^2\|\mathbf{v}\|^2$$

$$= (a^2 + \|\mathbf{u}\|^2)(b^2 + \|\mathbf{v}\|^2) = |a+\mathbf{u}|^2|b+\mathbf{v}|^2.$$

In conclusion, we have shown $|pq|^2 = |p|^2|q|^2$.

Hamilton spent about a decade searching for a 3D number system that would model rotations similar to how complex numbers model 2D rotations:

> Every morning in the early part of October 1843, on my coming down to breakfast, your brother William Edwin and yourself used to ask me: "Well, Papa, can you multiply triples?" Whereto I was always obliged to reply, with a sad shake of the head, "No, I can only add and subtract them."

Orthogonality (the assumption that 1 and i point in orthogonal directions) and multiplicativity are the key properties that let phasors (unit-norm complex numbers of the form $e^{i\theta}$) act as rotations of the complex plane. Hamilton sought a similar system with triples $a + bi + cj$, but to no avail, until one day he realized making i and j anticommute ($ij = -ji$) and ij jut out into a fourth dimension made all the algebra work out:

> An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw, immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k; namely,
>
> $$i^2 = j^2 = k^2 = ijk = -1.$$

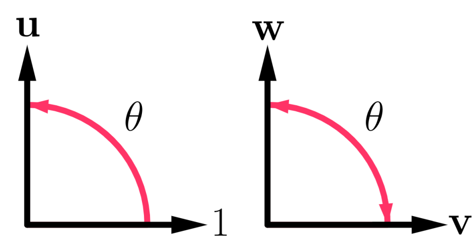
(It is a quick exercise to verify this very symmetric equation is equivalent to the usual relations $i^2 = j^2 = k^2 = -1$ and $k = ij = -ji$.) In hindsight, it makes sense three imaginaries are necessary: there are three perpendicular planes of rotation possible in 3D, unlike only one plane of rotation in 2D.

Hamilton may have spent the rest of his life evangelizing quaternions, but they eventually fell out of favor - imagining four dimensions is a hard ask - but Gibbs came along later and cut out the real and imaginary parts of the product of two pure imaginary quaternions and gave us what we now call the dot product and cross product, now standard curriculum today. This story is but a subplot in a larger 'war' waged over various kinds of algebras - other notable names include Gibbs and Heaviside on the side of vectors, and Clifford and Grassman with a multivector generalization of quaternions.

The quaternions are denoted $\mathbb{H}$ in honor of Hamilton.

The real and imaginary parts of quaternions are also called the scalar and vector parts. A couple characterizations: two quaternions commute ($xy = yx$) if and only if their vector parts are parallel, and they anticommute ($yx = -xy$) if and only if they are perpendicular vectors.

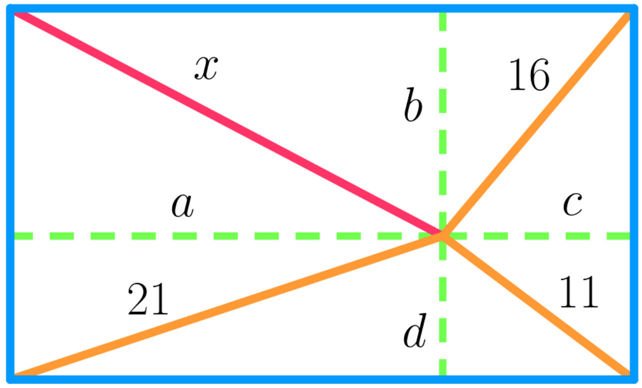
In $\mathbb{H}$, the only square roots of $+1$ form a 'zero-sphere' $S^0 = \{\pm 1\}$, the only square roots of -1 are 3D unit vectors forming the two-sphere S^2, and the versors (unit quaternions) form a hypersphere S^3. All nonzero quaternions have a polar form $r\exp(\theta\mathbf{u})$, where r is the norm, $\mathbf{u}$ is a unit vector, θ is a convex angle $0 \leq \theta \leq \pi$, and Euler's formula applies to $\exp(\theta\mathbf{u})$.



Any unit vector $\mathbf{u}$ can be extended to an orthonormal basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for 3D space (oriented according to the right-hand rule), which extends to a basis $\{1, \mathbf{u}, \mathbf{v}, \mathbf{w}\}$ for $\mathbb{H}$. If $p = \exp(\theta\mathbf{u})$ then the left-multiplication function $L_p(x) = px$ acts as a rotation by θ in a pair of 2D subspaces, the $1\mathbf{u}$-plane and the $\mathbf{vw}$-plane. The right-multiplication map $R_p(x) = xp$ is the same, but rotates the opposite direction in the $\mathbf{vw}$-plane.

The composition $L_p \circ R_{p^{-1}}$ is conjugation $x \mapsto pxp^{-1}$. These left and right multiplications cancel out in the $1\mathbf{u}$-plane, so when restricted to 3D vectors the effect is rotation around the $\mathbf{u}$-axis by the double angle 2θ. This is how quaternions model 3D rotations. Indeed, they also model 4D rotations: any rotation of four-dimensional space is equivalent to the "bimultiplication" $L_p \circ R_{q^{-1}}$ for some pair of versors p and q.

Vexing Vexillology: Solution



The Pythagorean theorem yields the system of equations

$$\begin{cases} a^2 + b^2 = x^2 \\ b^2 + c^2 = 16^2 \\ c^2 + d^2 = 11^2 \\ d^2 + a^2 = 21^2 \end{cases}$$

which yields

$$a^2 + b^2 + c^2 + d^2 \;=\; x^2 + 11^2 \;=\; 16^2 + 21^2$$

and therefore

$$x = \sqrt{16^2 + 21^2 - 11^2} = 24.$$

This is an example of the **British Flag Theorem**.

Washers in Balance: Solution

Say the slice volumes are $V_1, \cdots, V_n$ and the radius is r (all depending on n).

Let A_i be the area of a cross-section of the corresponding slice of a unit sphere. Scaling by r^2, we can say $r^2 A_i$ is the area of a cross-section of the volume V_i. Its height is $\frac{2r}{n}$, so we may approximate $V_i \approx (\frac{2r}{n})(r^2 A_i)$.

The volume equation $\frac{4}{3}\pi r^3 = Cn$ implies $\frac{2r^3}{n} = \frac{3C}{2\pi}$ which means $V_i \approx (\frac{3C}{2\pi})A_i$.

The logarithm $\ln P_n = \ln V_1 + \cdots + \ln V_n$ we may estimate by

$$\begin{aligned} &\approx \quad n \ln\left(\tfrac{3C}{2\pi}\right) + \ln A_1 + \cdots + \ln A_n \\ &= \quad n\left[\ln\left(\frac{3C}{2\pi}\right) + \frac{1}{n}\sum_{i=1}^{n} \ln A_i\right] \end{aligned}$$

For the limit to exist, the expression inside brackets must tend to 0 as $n \to \infty$.

The summation is akin to a Riemann sum. But for what integral? We need to parametrize the cross-sectional areas using the interval $[0, 1]$. The cross-sections are circles with radii $\sqrt{1-z^2}$ and areas $A = \pi(\sqrt{1-z^2})^2$, and we can parametrize $-1 \le z \le 1$ from $0 \le t \le 1$ using $z(t) = 2t - 1$. Thus, the cross-sectional area is $A(t) = \pi\big(1 - (2t-1)^2\big) = 4\pi t(1-t)$, and then

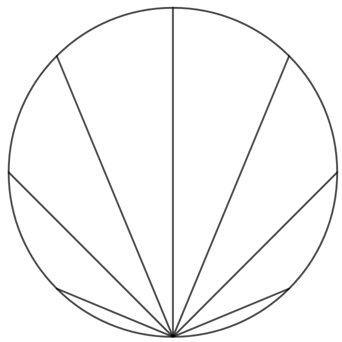
$$\begin{aligned} \ln\left(\frac{3C}{2\pi}\right) &= -\int_0^1 \ln\big[4\pi t(1-t)\big]\,\mathrm{d}t. \\ &= -\ln(4\pi) - \int_0^1 \ln t\,\mathrm{d}t - \int_0^1 \ln(1-t)\,\mathrm{d}t \end{aligned}$$

We may add $\ln(4\pi)$ over to the left, then combine the integrals on the right since they are equal (by symmetry - use the substitution $s = 1 - t$):

$$\ln(6C) = -2\int_0^1 \ln t\,\mathrm{d}t = -2\Big[t(\ln t - 1)\Big]_0^1 = 2$$

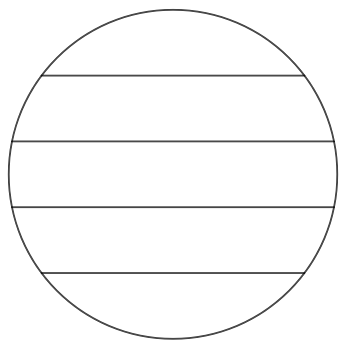
(since $\lim\limits_{t\to 0^+} t\ln t = 0$), which leads to $C = \dfrac{e^2}{6}$.

Similar results are possible slicing up a circle by chords (equal angle apart), into parallel chords (equal distance apart), or a sphere into concentric shells (of equal thickness), provided by Dan (user 398708) on Math.StackExchange:



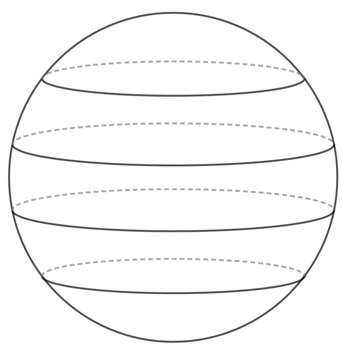
$$A = 2n$$

$$P \to \left[2\cosh\left(\tfrac{\pi}{2\sqrt{3}}\right)\right]^2$$



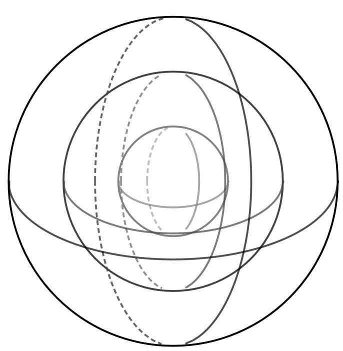
$$A = \left(\tfrac{\pi e}{8}\right)n$$

$$P \to 2\cos\left(\tfrac{\pi}{2\sqrt{3}}\right)$$



$$V = \left(\tfrac{e^2}{6}\right)n$$

$$P \to 2$$



$$V = \left(\tfrac{e^2}{3}\right)n$$

$$P \to 2\cosh\left(\tfrac{\pi}{2\sqrt{3}}\right)$$

Working Backwards: Solution

We can represent the state of the game with three numbers (a, b, c), which means there are a, b, and c marbles of each color left. Say (a, b, c) is a winning position if perfect play starting from that position has a guaranteed win, and a losing position otherwise. (Order doesn't matter to whether the game state is a winning or losing position, so we might as well assume $a \leq b \leq c$ so that we don't have to write as many triples.)

A position is winning if (and only if) either you can win the game on that turn or else it is possible to make a move which leaves your friend with a losing position. Conversely, a position is losing if (and only if) no matter what move you make you leave your friend with a winning position.

The game states where you can win on your turn are those where all marbles have only one color left, i.e. when two of the numbers a, b, c are zero:

Win: (0,0,1), (0,0,2), (0,0,3).

The losing position with fewest marbles is $(0, 1, 1)$, since no matter which marble you take you leave your friend with the winning game position $(0, 0, 1)$; this means any state that can reach $(0, 1, 1)$ on the next turn is winning:

Win: (0,1,2), (0,1,3),
(1,1,1), (1,1,2), (1,1,3).

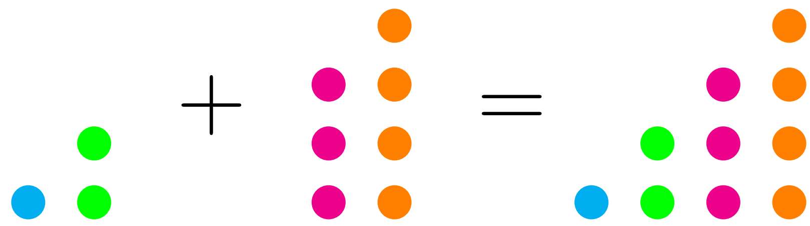
This means (0,2,2) must be a losing position, because it can only leave your friend with one of the winning positions (0,1,2) or (0,0,2), which also means

Win: (1,2,2), (0,2,3).

Finally, (1,2,3) is a losing position because it too can only leave your friend one of the winning positions (0,2,3), (1,1,3), (1,2,2), (1,1,2), (1,1,1), (0,1,2).

In conclusion, if you believe your friend is banking on this fact, or else can recognize and capitalize on it immediately, you should **decline** her offer.

This is the game of **Nim**. The game can start with any number of "heaps," each heap having any number of items, and you take turns with your opponent removing any number of items from one heap per turn. Games can be added:



In game theory terms, the combined game is the **disjunctive sum** of the smaller games. The sum $A+B$ of two games A and B has players playing the games A and B "in parallel," meaning each turn a player performs a move in either game A or game B (until one of them is concluded, after which the players finish the remaining game), and the sum game is finished when both A and B are, with the final outcome the same as that of the final game.

Compare with simultaneous exhibitions where a high-ranking player plays multiple other players (in chess or Go, for example) at the same time. The disjunctive sum is like this, but played 1v1 instead of against multiple other players, and only the last-concluded game's outcome counts.

Maybe we can determine if the sum $A + B$ is a winning or losing position based on the positions of Nim games A and B? Consider the possibilities:

- $L + L = L$. If A and B are both losing positions, your opponent can play perfectly in both games and win both, hence the sum.
- $L+W = W$. If (say) A is a winning position and B is a losing position, then you can play a correct move in A to leave your opponent with two games in losing position, which we just said is a losing position.
- $W + W$ is indeterminate, as our previous work reveals - for example $(1) + (1) = (1, 2)$ is a losing position but $(1) + (2) = (1, 2)$ is winning.

Perhaps we can study some special cases and generalize? The simple game $(1, 1, \cdots, 1)$ with n one-item heaps is winning or losing based on n's parity, i.e. it is a winning position for odd n and a losing position for even n.

If we analyze the two-heap game (a, b) for small values of a and b we will find it is only a losing position when $a = b$. What makes equal-size pairs of heaps so special? For the game (a, a), you cannot take all of one heap because then your opponent can win next turn. But then whatever number of items you take from one heap, your opponent can mirror your move by taking an equal number of items from the other heap! This generalizes: given any game A, the game $A + A$ must be a losing position, since whatever move you do in one of the A games, your oppononent can copy it in the other A game.

If we analyze the three-heap game (a, b, c) for small values of a, b, c, no obvious pattern emerges. (We will find out in a bit that the three-heap games are the key to solving the full game with any number of heaps.)

Consider the effect of adding one heap to a game. For any game A, there is *at most* one number a for which $A + (a)$ is in a losing position. Indeed, if there were two such numbers $a < b$, then faced with $A+(b)$ you could remove items from (b) to leave your opponent with the losing position $A+(a)$, which means $A + (b)$ was a winning position for you, a contradiction. The value a (which will turn out to always exist) is called the **nim sum** of A.

To anyone familiar with computer science, this may feel like a **parity bit**, or more generally a **checksum**. Computers deal in bits, or `1`s and `0`s (corresponding to the presence or absence of electrical signals in circuits, or to magnetic poles in physical media, or to `True` and `False` respectively in general). Computers often append a parity bit to the end of a code (string of bits) which is the `XOR` (exclusive or) of previous bits. In practice this means the parity bit has the same parity as the number of `1`s in the code (**parity** means whether a number is even or odd).

A checksum is a larger block of data that functions the same way. Parity bits and checksums are examples of error-detection systems. More sophisticated coding schemes can be used to not only detect but also correct reasonable levels of errors in signals travelling over noisy channels on-the-fly. Modern computers and the internet as we know it wouldn't be possible without error correction fixing flipped bits caused by interference.

Just as a parity bit is the unique bit added to the end of a code to make the total `XOR` equal `0`, the nim-sum of a game is the unique heap size (possibly zero) to add to make it a losing position, hence the analogy.

Let's use the notation Σ_A for the nim sum of a game A. By definition, $A + (\Sigma_A)$ and $B + (\Sigma_B)$ are losing positions, as is $A + B + (\Sigma_{A+B})$. That means the sum of all three of them is also a losing position:

$$A + (\Sigma_A) + B + (\Sigma_B) + A + B + (\Sigma_{A+B}).$$

We said earlier $L + L = L$ and $L + W = W$, which means subtracting any losing game (if possible) from a sum does not change winning or losing position. As $A+B+A+B$ is a losing position, we can subtract it to conclude

$$(\Sigma_A) + (\Sigma_B) + (\Sigma_{A+B})$$

is also a losing position, which means not only is Σ_{A+B} the nim-sum of A and B, it is also the nim-sum of $(\Sigma_A) + (\Sigma_B)$! This means we can replace the last two heaps of a game $(a_1, \cdots, a_{n-1}, a_n)$ with their nim-sum as a single heap and the overall nim-sum is unaffected. Thus, finding the nim-sum of any game is reduced to finding the nim-sum of two-heap games!

Let's pick notation that indicates we're considering a binary operation: let $a \oplus b$ denote the nim sum of $(a) + (b)$. We've found $1 \oplus 2 = 3$ in this problem, for instance. It takes quite a bit of legwork (working out case after case by hand) before we can observe any patterns in a nim-sum table:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

Every entry in the table is the nim-sum of its row's first number and its column's top number.

The diagonal is filled with 0s (as we said earlier, $A + A$ is always a losing position, which implies $a \oplus a = 0$). Also notice there are many runs of consecutive numbers within rows and columns of different sizes.

There also seems to be interlacing and reversing effects - for example, in the row beginning with 1, the numbers after 0 are just the same numbers above them but every consecutive pair is swapped. The effect is more pronounced if we subdivide the 8×8 array into a 2×2 array of 4×4 arrays: each 4×4 array is identical to the catercorner 4×4 array. The same is true if we subdivide the 4×4 arrays into 2×2 arrays, and this pattern continues if we zoom out to look at 8×8 arrays within a 16×16 or beyond.

This fractal-like structure (self-similarity at different scales) is reminiscent of numbers represented in **binary**. In our decimal (i.e. base-10) number system (chosen because we have ten fingers), each digit represents higher powers of ten, so for instance $100 = 10^2$ and $1000 = 10^3$. In binary, the digits represent powers of two, so `111` in binary is $4 + 2 + 1 = 7$ in decimal for example.

binary	decimal
`000`	0
`001`	1
`010`	2
`011`	3
`100`	4
`101`	5
`110`	6
`111`	7

`000`	`001`	`010`	`011`	`100`	`101`	`110`	`111`
`001`	`000`	`011`	`010`	`101`	`100`	`111`	`110`
`010`	`011`	`000`	`001`	`110`	`111`	`100`	`101`
`011`	`010`	`001`	`000`	`111`	`110`	`101`	`100`
`100`	`101`	`110`	`111`	`000`	`001`	`010`	`011`
`101`	`100`	`111`	`110`	`001`	`000`	`011`	`010`
`110`	`111`	`100`	`101`	`010`	`011`	`000`	`001`
`111`	`110`	`101`	`100`	`011`	`010`	`001`	`000`

In the left table of binary representations, notice the units digit alternates between `0` and `1`, the twos digit alternates between two `0`s and two `1`s, the fours digit alternates between four `0`s and four `1`s, and this pattern continues.

On a hunch, let's rewrite the nim-sum table in binary. It may not be obvious, but this is also the table for bitwise `XOR`! That is, a binary digit of $a \oplus b$ is `0` if the corresponding binary digits of a and b match, and is `1` otherwise.

This completely solves the game of Nim. For example, consider the game $(1, 2, 3, 4)$. The nim-sum is the bitwise `XOR` of the numbers $1, 2, 3, 4$:

$$\begin{array}{rr} & \texttt{0001} \\ & \texttt{0010} \\ & \texttt{0011} \\ \oplus & \texttt{0100} \\ \hline & \texttt{0100} \end{array}$$

The nim-sum is $1 \oplus 2 \oplus 3 \oplus 4 = 4$. Since the nim-sum is not zero, we conclude $(1, 2, 3, 4)$ is a winning position. In fact, this gives us the strategy for perfect play: always leave your opponent with a nim-sum of zero for the start of their turn.

The earliest European references to Nim are from the 1500s, but it's possible the game originated in China, and some of its many variations have been played since ancient times. The first published mathematical solution was in 1901, for comparison, so a lot of time went by before the solution was finally found. Don't get hung up if you didn't see it; almost nobody does!

Yoga of π: Solution

The first trick is that we can rewrite a positive constant α as $\alpha = \int_{0\le v\le\alpha} \mathrm{d}v$, or equivalently $1/\alpha = \int_{0\le\alpha v\le 1} \mathrm{d}v$, which we can use to rewrite the integrand as its own integral ($\alpha = u^2+1$, ignoring the 2), creating a double integral:

$$\int\limits_{-1\le u\le 1} \frac{2\,\mathrm{d}u}{u^2+1} \quad = \iint\limits_{\substack{-1\le u\le 1\\ 0\le(u^2+1)v\le 1}} 2\,\mathrm{d}u\mathrm{d}v$$

Observe the resemblance of $(u^2+1)v = u^2v+v$ to x^2+y^2; both are bounded by the same inequality. This suggests setting $u^2v = x^2$ and $v = y^2$, meaning

$$\begin{cases} x = u\sqrt{v} \\ y = \sqrt{v} \end{cases} \quad\Longleftrightarrow\quad \begin{cases} u = x/y \\ v = y^2 \end{cases}$$

With this change of variables, the double integral becomes

$$\iint\limits_{\substack{-1\le u\le 1\\ 0\le(u^2+1)v\le 1}} 2\,\mathrm{d}u\mathrm{d}v \;= \iint\limits_{\substack{-1\le x/y\le 1\\ x^2+y^2\le 1}} 2\,\frac{\partial(u,v)}{\partial(x,y)}\mathrm{d}x\mathrm{d}y$$

We may calculate the Jacobian determinant to be

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1/y & -x/y^2 \\ 0 & 2y \end{vmatrix} = 2$$

and therefore the double integral becomes

$$\iint\limits_{\substack{-y\le x\le y\\ x^2+y^2\le 1}} 4\,\mathrm{d}x\mathrm{d}y = \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y.$$

The domain Q_2 is the top quarter sector of the unit disk. We can use the substitution $(x,y)\mapsto(y,x)$ to turn two of the Q_2 domains into Q_1, the right quarter sector. Then, we can use the substitution $(x,y)\mapsto(-x,-y)$ to turn one Q_1 and one Q_2 into the other two quarters Q_3 and Q_4. Also note these substitutions do not change the integrands or differentials.

Thus the four integrals become

$$\iint\limits_{Q_1} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_2} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_3} \mathrm{d}x\mathrm{d}y + \iint\limits_{Q_4} \mathrm{d}x\mathrm{d}y = \iint\limits_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y.$$

And so we have followed the rules to conclude

$$\int\limits_{-1\leq u\leq 1} \frac{2\,\mathrm{d}u}{u^2+1} \quad\longrightarrow\quad \iint\limits_{x^2+y^2\leq 1} \mathrm{d}x\mathrm{d}y.$$

In school we encounter an increasing progression of larger number systems:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

$\mathbb{N} = \{1, 2, 3, \cdots\}$ is the set of natural numbers (whether or not 0 is included depends on author and context), $\mathbb{Z}$ is the set of integers, $\mathbb{Q}$ is the set of rational numbers (quotients), $\mathbb{R}$ is the set of real numbers, and $\mathbb{C}$ is the set of complex numbers. A different extension of $\mathbb{Q}$ besides $\mathbb{R}$ is the set $\overline{\mathbb{Q}}$ of all algebraic numbers (roots of integer-coefficient polynomials). Other than $\mathbb{N}$ these are all rings, meaning they are closed under the arithmetic operations of addition, subtraction, and multiplication.

A larger class of numbers is the **ring of periods**. A period is any number expressible as a (possibly multivariable) integral of a rational function (with rational coefficients) over a domain defined by polynomial inequalities; while the set of periods includes the set of algebraic numbers, "most" periods are transcendental, and yet the set of periods is still countable.

Examples of periods include π, the natural logarithm of algebraic numbers, and special values of broad families of zeta and hypergeometric functions and elliptic integrals. This problem not only illustrates π is a period, but:

Conjecture. If a period is expressible as two different integrals, it is possible to convert one integral to the other using just the rules outlined in this problem. (Bonus conjecture: there is an algorithm that can do this.)

Disappointingly, $e \approx 2.718$ is not expected to be a period. This suggests extending the ring of periods to "exponential periods," which would include for example the Gaussian integral $\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-x^2}\, \mathrm{d}x$. There is an analogous extension of field theory to "differential field theory" which allows us to actually prove functions like x^x and e^{x^2} have no elementary antiderivatives.

It is also not expected that the Euler-Mascheroni constant, defined by the limit $\gamma \stackrel{\text{def}}{=} \lim\limits_{n\to\infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n) \approx 0.577$, is a period. This is unlikely to be proven any time soon, though; it has not even been proven that γ is irrational! In the context of regularization and asymptotic analysis γ can be considered an honorary zeta value $\zeta(1)$ (which technically doesn't exist).

Following this trend, Kontsevich and Zagier in their survey article "Periods" opine that "all classical constants are periods in an appropriate sense."

Appendix

AM-GM

(Isoequiareal Quotient)

Bertrand's Postulate

(Involutive Units)

Bèzier Curves

(Arts and Crafts) (Lazy Spline)

Calculus

(Interesting Asymptotic) (Noncommutative Calculus) (Blinding Sphere) (Halving Harmonics) (Yoga of π) (Slope-Intercept Coordinates) (Washers in Balance) (Cusp of Crying)

Chromatic Polynomials

(Striking Gold)

Compass & Straightedge

(Tale of Two Tangents) (Golden Architecture)

Computer Aid

(Prime Generation) (Twenty Four)

Factoring Polynomials

Fundamental Counting Principle

Fundamental Theorem of Arithmetic

Generating Functions

Geometric Sums

Group Theory

Graph Theory

Homogenization

Hopf Bundle

Hyperbolic Functions

Induction

Möbius Transformations

Pick's Theorem

Platonic Solids

Probability Theory

Polytopes

Triangle Inequality

(Killer Triangle Problem)

Vector Algebra

(Favorite Angle) (Versorial Validation) (Projector Junction) (Polarization) (Blade Angle)

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