5-7-2021

Laplace's Equation in Fractional-Dimension Spaces

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Laplace’s Equation in Fractional-Dimension Spaces

by

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A senior thesis submitted to the faculty of
Loyola Marymount University
in partial fulfilment of the requirements for the degree of

Bachelor of Science

Department of Physics
Loyola Marymount
University

May 2021

THESIS COMMITTEE APPROVAL

of a thesis submitted by

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This thesis has been read by each member of the following thesis committee and by majority vote has been found to be satisfactory.

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ABSTRACT

The correct way to model gravity is a question in physics whose answer continues to elude our understanding. One major difficulty is the dark matter problem, which exists due to the mass discrepancy between predicted and measured values in our universe. One possible solution to this problem is Modified Newtonian Dynamics (MOND). MOND is an alternative gravity model that modifies Newtonian Dynamics with the hope to avoid the necessity of dark matter.

Dr. Varieschi has done work connecting MOND to Newtonian Fractional-Dimension Gravity—the application of fractional calculus and fractional mechanics to classical gravitation laws. In this formulation, we can consider dimension (D) to be somewhere between 1 and 3. Laplace's equation has already been found in the spherical coordinate system for this model, but the cylindrical case has not been explored. My work will answer two questions: "What is Laplace's equation in cylindrical coordinates for varying fractional dimensions?" and "How can this result be applied to model galactic systems?"

First, I conducted a thorough review of Laplace's equation in spherical coordinates for both the three-dimensional and fractional-dimensional cases. I then compared these two cases and analyzed the results of that comparison. Then, I utilized Mathematica to determine Laplace's equation in cylindrical coordinates. Finally, I applied the equation I found to galactic models, concluding that this formulation might be a promising start towards modeling gravity correctly.

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank Dr. Varieschi for his patience and guidance as my mentor. Without him, this thesis would have never been completed. I also want to acknowledge the LMU Physics department and thank them for my four years here. To my parents, thank you so much for everything you do and for your continued support of me during my journey through life. I would also like to thank the curators of Japanese Jazz music playlists on YouTube. Without those videos to help me focus, who knows how long this thesis would have taken to finish. Last and certainly not least, I want to thank all my friends in LMU Physics, who made this academic journey one that was filled with laughs, support, and love. You all have a very special place in my heart.

CHAPTER 1: INTRODUCTION

1.1 Motivation

One of the most pressing problems in physics is finding the correct way to model gravity. The biggest stumbling block comes in the form of the dark matter problem which arises from the mass discrepancy between predicted and measured values in our universe. One possible solution to this problem is an alternative gravity model called Modified Newtonian Dynamics (MOND). Essentially, MOND makes modifications to Newtonian dynamics in terms of inertia or gravity in order to describe celestial bodies' motion in the gravitational field of a galaxy without invoking dark matter. Recent work has shown that connecting MOND to Newtonian Fractional-Dimension Gravity—the application of fractional calculus and fractional mechanics to classical gravitation laws—may yield interesting results. Specifically, the possibility of considering dimension (D) to be somewhere between 1 and 3—fractional—instead of simply 3. Varieschi has made use of this MOND/Newtonian Fractional-Dimension Gravity hybrid to find Laplace's equation for the spherical fractional dimensions case. However, galaxies are essentially disks, and disks are simply very thin cylinders. Thus, Laplace's equation in cylindrical coordinates should serve as a better model on the galactic level. My work will focus on finding this cylindrical coordinate equation. Before we get to that discussion, however, a review of Laplace's equation and a better explanation of MOND and Newtonian Fractional Gravity are necessary.

1.2 Laplace's Equation in Cartesian Coordinates

Laplace's Equation is named for Pierre-Simon Laplace, who first studied its properties in the late eighteenth century. It has become ubiquitous in the world of physics; the equation has been evaluated and utilized in countless ways since its inception. One of these ways is in the field of galactic dynamics, where Laplace's equation is foundational. One important application of Laplace's equation in galactic dynamics is to find the disk potentials of Bessel functions, which I will discuss later. Thus, a brief review of the formulation of Laplace's equation will be useful. I will follow the process laid out by Griffiths, who succinctly discusses Laplace's equation through the lens of electrostatics [1].

We can start with a familiar equation for potential of a volume charge:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\boldsymbol{r}')d\tau' \tag{1.2.1}$$

Where V is potential, ϵ_0 is the permittivity of free space, $\mathscr{r}$ is the distance from the charge to $\mathbf{r}$, and ρ is the volume charge density. Due to the difficulty of working with this integral in an analytical manner, it is useful to rewrite the equation in differential form. This can be done with ease using Poisson's equation. Thus, we are left with:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \tag{1.2.2}$$

If we are looking for the potential in a region of space where the volume charge density is zero, Poisson's equation simply reduces to Laplace's equation.

$$\nabla^2 V = 0 \tag{1.2.3}$$

We can rewrite this equation in Cartesian coordinates instead using the definition of the del operator:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1.2.4}$$

Obviously, electric potential is not always the function being considered. Thus, a more generic form with F in place of V is more useful.

1.3 Laplace's Equation in 1-D and 2-D Cartesian Coordinates

Equipped with a brief description of how to arrive at Laplace's equation, we can now discuss the solutions of Laplace's equations in one dimension and two dimensions. In one dimension, potential depends only on one variable. Let us choose x for this example. Clearly, in this case, Laplace's equation can now be written:

$$\frac{\partial^2 F}{\partial x^2} = 0 \tag{1.3.1}$$

This is a very straightforward differential equation. Its general solution is the classic:

$$F(x) = mx + b \tag{1.3.2}$$

Which is the equation of a straight line. Now, for the two-dimensional case. In two dimensions, V depends on two variables, giving us Laplace's equation as follows:

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0 \tag{1.3.3}$$

Because this is a partial differential equation, there is no simple solution that we can give. There is something important to note in both one and two dimensions; Laplace's equation ensures V has no local minima or maxima.

1.4 Laplace's Equation in 3-D Spherical/Cylindrical Coordinates

As mentioned prior, Laplace's equation is far more useful for the purposes of modeling gravity when it is in spherical or cylindrical coordinates. For the spherical case, Laplace's equation is written as follows:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}=0 \qquad (1.4.1)$$

Where r is the radial distance from the origin, φ is the azimuthal angle (the angle around from the x axis to the y axis), and θ is the polar angle (the angle down from the z axis to the xy-plane). See Figure 1 for a visual representation [1].

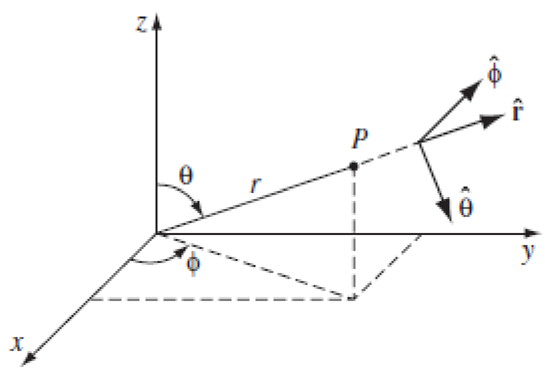

Figure 1: A visual representation of the spherical coordinates system.

For the cylindrical case, we can consult Binney and Tremaine's *Galactic Dynamics* [2]:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial F}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 F}{\partial\varphi^2}+\frac{\partial^2 F}{\partial z^2}=0 \qquad (1.4.2)$$

Where R is the radius of the cylinder, z is the height of the cylinder, and φ is the polar angle. With these definitions under our belt, we can discuss MOND, Newtonian Fractional-

Dimension Gravity, and how those two concepts help find a new form of Laplace's Equation.

CHAPTER 2: MOND, NFDG, AND SPHERICAL LAPLACE

2.1 MOND

Modified Newtonian Dynamics (MOND) is a leading alternative gravity model. It was first developed in 1982 by Milgrom as a potential answer to the mass-discrepancy problem [8]. Most importantly, it attempts to account for the mass discrepancy in the Universe without invoking dark matter. The MOND review in this thesis will be brief. For a fuller discussion, see [16] or [17]. MOND modifies Newtonian dynamics in terms of inertia or gravity. Thus, essential to the MOND model is the acceleration constant a_0. Following the discussion in [3], the estimated value of this constant is [13]:

$$a_0 \equiv 1.20 \pm 0.02\ (\text{random}) \pm 0.24\ (\text{syst}) \times 10^{-10}\ \text{ms}^{-2} \tag{2.1.1}$$

Below that acceleration scale, MOND corrections are used. There are two ways to express the modification to Newtonian dynamics [9]:

$$\mathbf{F} = m\mu\left(\frac{a}{a_0}\right)\mathbf{a} \tag{2.1.2}$$

$$\mu\left(\frac{g}{a_0}\right)\mathbf{g} = \mathbf{g}_N \tag{2.1.3}$$

The first equation is clearly a modification to Newton's second law; **F** is some arbitrary static force and *m* is the gravitational mass of the test particle. If the static force is the force of gravity, $\mathbf{F} = m\mathbf{g}_N$, where $\mathbf{g}_N = -\nabla\phi_N$ and ϕ_N is the Newtonian gravitational potential derived from the standard form of Poisson's equation. Equation 2.1.2 can be applied to any type of force, modifying the law of inertia since acceleration $\boldsymbol{a}$ is replaced by $\mu\left(\frac{g}{a_0}\right)\boldsymbol{a}$. On the other hand, Equation 2.1.3 only impacts the gravitational field **g** and does not affect Newton's Second Law **F** = m**a**. The equations differ in the face that Equation 2.1.2 modifies Newton's Laws of motion while Equation 2.1.3 modifies Newton's law of universal gravitation. In both equations, the driving force behind the modifications is the interpolation function $\mu(x)$. The interpolation function is defined as:

$$\mu(x) \equiv \mu\left(\frac{a}{a_0}\right) \tag{2.1.4}$$

for laws of motion modifications and is defined as:

$$\mu(x) \equiv \mu\left(\frac{g}{a_0}\right) \tag{2.1.5}$$

for Newtonian gravitational modifications. Continuing to reference the analysis from [3], we see MOND posits that:

$$\mu(x) \approx \begin{cases} 1 \text{ for } x \gg 1 \\ x \text{ for } x \ll 1 \end{cases} \tag{2.1.6}$$

The $x \gg 1$ case is known as the Newtonian regime while the $x \ll 1$ case is known as the deep-MOND regime. Clearly, MOND only has an impact when considering values of *x* that are much smaller than one; $\mu(x)$ becomes irrelevant if *x* is much greater than one. Milgrom initially used very basic forms for the interpolation function (called the "standard" form) like $\mu_2(x) = \frac{x}{\sqrt{1+x^2}}$ or $\mu(x) = 1 - e^{-x}$ [14,15]. However, as time has passed, different interpolation functions have seen much more popularity and use. One such function is the “simple” interpolation function $\mu_1(x) = \frac{x}{(1+x)}$. The general family of functions $\mu_n(x) = x(1 + x^n)^{-1/n}$ is another frequently seen function in the literature. Furthermore, Equation 1.1.2 is commonly inverted to become [10]:

$$\boldsymbol{g} = \mu^{-1}(x)\boldsymbol{g}_N \equiv v(y)\boldsymbol{g}_N \tag{2.1.7}$$

where y = g_N/a_0. A multitude of $v(y)$ functions have been formulated in the years since Milgrom’s initial work, but Varieschi focused on two main families:

$$v_n(y) = \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4y^{-n}}\right)^{\frac{1}{n}} \tag{2.1.8}$$

$$\hat{v}_n(y) = \left[1 - \exp\left(-y^{\frac{n}{2}}\right)\right]^{-\frac{1}{n}} \tag{2.1.9}$$

Equation 1.1.7 is the inverse of $\mu_n(x)$ previously described. Equation 1.1.8 corresponds to interpolation functions that are comparable to the aforementioned “simple” function on galactic scales ($\sim a_0$). This family of functions has no impact in inner solar system levels ($\sim 10^8 a_0$). The preferred interpolation function of Varieschi’s work is:

$$\hat{v}_1(y) = \left[1 - \exp\left(-y^{\frac{1}{2}}\right)\right]^{-1} \tag{2.1.10}$$

MOND has three main outcomes that can be written as three laws of rotationally-supported galaxies, which are discussed in [4]. First, rotation curves attain an approximately constant velocity that continues to persist indefinitely. Second, the observed baryonic mass goes as the fourth power of the amplitude of the flat rotation curve ($M_{bar} \sim V_f^4$). Third, there is a one-to-on correspondence between the radial force and the observed distribution of baryonic matter (the mass discrepancy-acceleration relation $M_{tot}/M_{bar} \cong V_{obs}^2/V_{bar}^2$).

The first outcome proves correct the initial MOND prediction regardless of interpolation function; at large galactic radii (the deep-MOND regime) $g_N \approx \frac{GM}{r^2}$ and $a = g = \frac{V^2}{r}$ can be replaced to obtain $V^4(r) \equiv V_f^4 \approx GMa_0$. This reduces to $V_f \approx \sqrt[4]{GMa_0}$ in which M is total mass of the galaxy being considered. This work allowed for the precise fitting of galactic rotation curves of multiple shapes without the need for dark matter, solidifying MOND as an alternative to the dark matter hypothesis.

The third outcome resulted in something very valuable: relating the radial acceleration traced by rotation curves (g_{obs}) to the radial acceleration predicted by the observed distribution of baryonic matter (g_{bar}). The discovered relation (known as the Radial Acceleration Relation, or RAR) is as follows:

$$g_{obs} = \frac{g_{bar}}{1 - e^{-\sqrt{\frac{g_{bar}}{g_\dagger}}}} \tag{2.1.11}$$

In this equation, $g_\dagger$ is an empirical acceleration parameter that corresponds to the MOND acceleration scale previously mentioned (a_0).

2.2 NFDG

Newtonian Fractional-Dimension Gravity (NFDG) is the next piece of the puzzle. This section will mainly be summarizing the work undertaken by Dr. Varieschi; for additional information regarding equations and formulations, see [3] and [5]. Varieschi first introduced NFDG through an extension of Gauss's law for gravitation to a lower dimensional space-time $D + 1$ where $D \leq 3$ can be non-integer (or fractional). He employs a scale length l_0 in order to achieve dimensional correctness whenever $D \neq 3$. This makes it much more effective to use dimensionless coordinates in each of his formulas. Some examples include radial distance $w_r \equiv r/l_0$, the more general dimensionless coordinate $\mathbf{w} \equiv \mathrm{x}/l_0$ for the field point, and the similarly dimensionless $\mathbf{w}' \equiv \mathrm{x}'/l_0$ for the source point. Part of Varieschi's work also included finding a rescaled mass density $\tilde{\rho}(\mathbf{w}') = \rho(\mathbf{w}'l_0)l_0^3 = \rho(\mathbf{x}')l_0^3$ with $\rho(\mathbf{x}')$ representing the standard mass density in kg m^{-3}. Another equation, $d\widetilde{m}_{(D)} = \tilde{\rho}(\mathbf{w}')d^D\mathbf{w}'$, represented the infinitesimal mass in a D-dimensional space. One of the most important equations discussed by Varieschi was the NFDG gravitational potential,

$$\tilde{\phi}(\mathbf{w}) = -\frac{2\pi^{1-\frac{D}{2}}\Gamma\left(\frac{D}{2}\right)G}{(D-2)l_0}\int_{V_D}\frac{\tilde{\rho}(\mathbf{w}')}{|\mathbf{w}-\mathbf{w}'|^{D-2}}d^D\mathbf{w}'\,;\ D \neq 2 \qquad (2.2.1)$$

For the case where $D = 2$, the equation is much different:

$$\tilde{\phi}(\mathbf{w}) = \frac{2G}{l_0}\int_{V_2}\tilde{\rho}(\mathbf{w}')\ln|\mathbf{w}-\mathbf{w}'|d^2; D = 2 \qquad (2.2.2)$$

With these equations, we can connect $\tilde{\phi}(\mathbf{w})$ and $\mathbf{g}(\mathbf{w})$ through $\mathbf{g}(\mathbf{w}) = -\nabla_D\tilde{\phi}(\mathbf{w})/l_0$ where the D-dimensional gradient ∇_D is equivalent to the standard gradient, but the derivatives are computed with respect to the **w** coordinates (the rescaled coordinates). A simple but important check can be performed by plugging in $D = 3$ to the above equations; this causes them all to reduce to the standard Newtonian case. Without this equivalency, the equations would clearly be incorrect.

Part of Varieschi's efforts included recovering fundamental MOND predictions through NFDG, which he showed was possible if the Deep-MOND limit was considered equivalent to reducing dimension to $D \approx 2$. With a focus on spherically symmetric mass distributions, we see that the observed gravitational field can be expressed as:

$$\mathbf{g}_{obs}(w_r) = -\frac{4\pi G}{l_0 w_r^{D(w_r)-1}} \int_0^{w_r} \bar{\rho}\,(w_r')\omega_r^{D(w_r)-1} dw_r' \widehat{\mathbf{w}}_r \tag{2.2.3}$$

for $1 \leq D \leq 3$. The gravitational field in Equation 2.2.3 is classified as "observed" ($\mathbf{g}_{obs}$) while the "baryonic" gravitational field $\mathbf{g}_{bar}$ applies in cases of fixed dimension $D = 3$. The baryonic gravitational field is:

$$\mathbf{g}_{bar}(w_r) = -\frac{4\pi G}{l_0^2 w_r^2} \int_0^{w_r} \tilde{\rho}\,(w_r') {w_r'}^2 dw_r' \widehat{\mathbf{w}}_r \tag{2.2.4}$$

Thus, the ratio of $\mathbf{g}_{obs}$ and $\mathbf{g}_{bar}$ shows the connection between MOND and RAR, obtained in NFDG, is:

$$\left(\frac{g_{obs}}{g_{bar}}\right)_{NFDG} (w_r) = w_r^{3-D(w_r)} \frac{\int_0^{w_r} \tilde{\rho}(w_r') {w_r'}^{D(w_r)-1} dw_r'}{\int_0^{w_r} \tilde{\rho}(w_r')\, {w_r'}^2 dw_r'} \tag{2.2.5}$$

This ratio was compared to the similar ratio from the RAR in (2.1.10) for analysis of multiple different forms of spherically-symmetric mass distributions [3]. As anticipated, the variable dimension $D(w_r)$ approached $D \approx 3$ in regions where Newtonian gravity held true but decreased continually toward $D \approx 2$ in regions of the Deep-MOND limit. Further discussion of the intricacies of NFDG can be found in [3] and [5].

2.3 Laplace's Equation in Spherical Variable Dimension

Equipped with MOND and NFDG, now the final order of business is Laplace's equation in variable dimension in the spherical coordinate system. Giusti introduced a fractional

Poisson equation for a point mass M through his work [6] that Varieschi compares to the equation found through his formulation. He concludes that the equations share the same overall form of the fractional potential while the detailed expressions are different. Both models are MOND-like theories because they reproduce the asymptotic behavior of MOND while dropping all non-linearities. This occurs because both models are based on differential operators that are linear [3]. Keeping this information in mind, Varieschi works through some calculations before arriving at Laplace's equation in spherical coordinates for a D-dimensional space:

$$\nabla_D^2\phi = \frac{1}{r^{D-1}}\frac{\partial}{\partial r}\left(r^{D-1}\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin^{D-2}\theta}\frac{\partial}{\partial\theta}\left(\sin^{D-2}\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta\sin^{D-3}\varphi}\frac{\partial}{\partial\varphi}\left(\sin^{D-3}\varphi\frac{\partial\phi}{\partial\varphi}\right) = 0 \tag{2.3.1}$$

This equation is the foundation of the work I undertook to find the cylindrical coordinate Laplace's equation for variable dimension. The equation is separable, as shown in both [11] and [12], allowing us to create radial and angular equations:

$$\left[\frac{1}{r^{D-1}}\frac{d}{dr}\left(r^{D-1}\frac{d}{dr}\right) + \frac{k_1}{r^2}\right]R(r) = 0 \tag{2.3.2}$$

$$\left[\frac{1}{\sin^{D-2}\theta}\frac{d}{d\theta}\left(\sin^{D-2}\theta\frac{d}{d\theta}\right) - k_1 - \frac{k_2}{\sin^2\theta}\right]\Theta(\theta) = 0 \tag{2.3.3}$$

$$\left[\frac{1}{\sin^{D-3}\varphi}\frac{d}{d\varphi}\left(\sin^{D-3}\varphi\frac{d}{d\varphi}\right) + k_2\right]\Phi(\varphi) = 0 \tag{2.3.4}$$

In these formulas, the k values are separation constants; $k_2 = m(m + D - 3)$, $m = 0,1,2,\dots$ and $k_1 = -l(l + D - 2)$, $l = 0,1,2,\dots$ with $m \le l$. For sake of simplicity, I will only discuss cases of axial symmetry. Thus, the radial equation results in two independent solutions: $R(r) = r^l$ and $R(r) = \frac{1}{r^{l+D-2}}$. The angular solution can be expressed in terms of Gegenbauer polynomials as follows:

$$\Theta(\theta) = C_l^{\left(\frac{D}{2}-1\right)}(\cos\theta) \tag{2.3.5}$$

Gegenbauer polynomials ($C_l^{(\lambda)}(x)$) [7] form a set of orthogonal polynomials that are defined from (-1,1) with the requirements that $\lambda > -\frac{1}{2}, \lambda \neq 0$. Gegenbauer polynomials are also known as ultraspherical functions. The weight function of Gegenbauer polynomials is $w(x) = (1 - x^2)^{\lambda - \frac{1}{2}}$ and the normalization factor is $h_l = \frac{2^{1-2\lambda}\pi\Gamma(l+2\lambda)}{(l+\lambda)\left(\Gamma(\lambda)\right)^2 l!}$. For a more complete discussion of Gegenbauer polynomials, see [3]. Applying the definition of this set of polynomials to the physical solutions discussed prior leads to the ortho-normality condition. We can achieve this by setting $x \equiv \cos\theta$ and $\lambda \equiv \frac{D}{2} - 1$ while constraining the λ limit to $D > 1, D \neq 2$. Thus, the overall ortho-normality condition can be expressed as:

$$\int_0^\pi C_l^{\left(\frac{D}{2}-1\right)}(\cos\theta)\, C_{l'}^{\left(\frac{D}{2}-1\right)}(\cos\theta) \sin^{\mathrm{D}-2}\theta\, d\theta = h_l \delta_{ll'} \tag{2.3.6}$$

The normalization factor in this case is:

$$h_l = \frac{2^{3-D}\pi\Gamma(l + D - 2)}{\left(l + \frac{D}{2} - 1\right)\left[\Gamma\left(\frac{D}{2} - 1\right)\right]^2 l!} \tag{2.3.7}$$

The first few Gegenbauer polynomials in $\cos\theta$ are:

$$C_0^{\left(\frac{D}{2}-1\right)}(\cos\theta) = 1;\ C_1^{\left(\frac{D}{2}-1\right)}(\cos\theta) = (D-2)\cos\theta\,;$$
$$C_2^{\left(\frac{D}{2}-1\right)}(\cos\theta) = \left(\frac{D}{2}-1\right)(D\cos^2\theta - 1) \tag{2.3.8}$$

As always, it is important to check that setting $D = 3$ will return typical Newtonian results. In this case, setting $D = 3$ will reduce our Gegenbauer polynomials to the standard Legendre polynomials $P_l(\cos\theta)$. Another reduction of note is if gravitational potential depends on both angular variables θ and φ. If that condition is true and $D = 3$, the Gegenbauer polynomials will become the standard spherical harmonics $Y_{l,m}(\theta,\varphi)$ [3].

Once again, considering only cases with axial symmetry, we see that the general solution for boundary condition $\phi(R,\theta) = \phi_0(\theta)$ is:

$$\phi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+D-2}}\right) C_l^{\left(\frac{D}{2}-1\right)}(\cos\theta) \tag{2.3.9}$$

$$A_l = \frac{1}{R^l h_l}\int_0^{\pi}\phi_0(\theta) C_l^{\left(\frac{D}{2}-1\right)}(\cos\theta)\sin^{\mathrm{D}-2}\theta d\theta$$

$$B_l = \frac{R^{l+D-2}}{h_l}\int_0^{\pi}\phi_0(\theta) C_l^{\left(\frac{D}{2}-1\right)}(\cos\theta)\sin^{\mathrm{D}-2}\theta d\theta$$

Thus concludes the review of MOND and NFDG. Through past work, Laplace’s equation for spherical coordinates in varying dimension spaces was discovered. The next step is to find this equation in cylindrical coordinates. With luck, this form of the equation will be far more useful for the purposes of modeling galaxies because galaxies themselves are disks, and disks are very thin cylinders.

CHAPTER 3: CYLINDRICAL LAPLACE'S EQUATION

3.1 First Steps

To begin the work towards finding Laplace's equation in cylindrical coordinates, I will outline some basic information. The notation I will use for the directions in this cylindrical coordinate system is (R, φ, z). Each of the directions has its own fractional dimension as well: $0 < \alpha_R \leq 1$, $0 < \alpha_\varphi \leq 1$, and $0 < \alpha_z \leq 1$. It follows that the overall space dimension is equivalent to $D \equiv \alpha_R + \alpha_\varphi + \alpha_z$; thus, the domain of dimension becomes $0 < D \leq 3$. The generalized form of the cylindrical Laplacian will take the form $\Phi = \Phi(R, \varphi, z)$ where Φ represents gravitational potential. Based on the results of the spherical case and knowledge of Laplace's equation, it is simple to create a lengthier version of this generalized form:

$$\nabla_D^2 \Phi = \frac{1}{R^{D-2}} \frac{\partial}{\partial r}\left(R^{D-2} \frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2 \sin^{D-3} \varphi} \frac{\partial}{\partial \varphi}\left(\sin^{D-3} \varphi \frac{\partial \Phi}{\partial \varphi}\right) + \frac{1}{z^{\alpha_z - 1}} \frac{\partial}{\partial z}\left(\mathrm{z}^{\alpha_z - 1} \frac{\partial \Phi}{\partial z}\right) \tag{3.1.1}$$

The first two terms can be grouped as the generalized Laplacian for plane-polar coordinates (R, φ) and the third term can be viewed as the generalized Laplacian for one cartesian coordinate (z). After some rewriting with these groupings in mind, we see:

$$\nabla_D^2 \Phi = \left[\frac{\partial^2 \varphi}{\partial R^2} + \frac{D-2}{R} \frac{\partial \Phi}{\partial R}\right] + \frac{1}{R^2}\left[\frac{\partial^2 \varphi}{\partial \varphi^2} + \frac{D-3}{\tan \varphi} \frac{\partial \Phi}{\partial \varphi}\right] + \left[\frac{\partial^2 \varphi}{\partial \varphi^2} + \frac{(\alpha_z - 1)}{z} \frac{\partial \Phi}{\partial z}\right] \tag{3.1.2}$$

However, since the intention of this work is to study thin-disk galaxies, I will set $\alpha_z = 1$. Thin-disk galaxies contain almost the entirety of their matter in the $z = 0$ plane. Setting $\alpha_z = 1$ also indicates that there is no fractional dimension in the vertical-z direction. I can simplify Laplace's equation if thin disk galaxies are the main target of study. The simplification maintains the same first two terms as (3.1.1) but the third term is far more straightforward.

$$\nabla_D^2 \phi = \frac{1}{R^{D-2}} \frac{\partial}{\partial r}\left(R^{D-2} \frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2 \sin^{D-3} \varphi} \frac{\partial}{\partial \varphi}\left(\sin^{D-3} \varphi \frac{\partial \Phi}{\partial \varphi}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (3.1.3)$$

By distributing R's in (3.1.3) we can get to an even simpler equation:

$$\nabla_D^2 \phi = \left[\frac{\partial^2 \varphi}{\partial R^2} + \frac{(D-2)}{R} \frac{\partial \Phi}{\partial R}\right] + \frac{1}{R^2}\left[\frac{\partial^2 \varphi}{\partial \varphi^2} + \frac{(D-3)}{\tan \varphi} \frac{\partial \Phi}{\partial \varphi}\right] + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad (3.1.4)$$

As always, it is important to check that upon setting $D = 3$, we obtain the standard Laplace's equation. After confirming that this is in fact true, we can move forward to separation of variables.

3.2 Separation of Variables

Using the cylindrical symmetry of gravitational potential, I can write the gravitational potential as:

$$\Phi = \Phi(R, z) = J(R) F(\varphi) Z(z) \qquad (3.2.1)$$

Notice there is no dependence on φ in the $\Phi(R, z)$ portion of the equation, only on R and z. We can insert the entirety of Equation (3.2.1) into Equation (3.1.4) and use separation of variables on the new equation. These calculations follow the form laid out by Binney and Tremaine in [2]. After replacing Φ in Equation (3.1.4) with Equation (3.2.1) and separating, we arrive at:

$$\frac{1}{J(R)}\left[\frac{d^2J}{dR^2}+\frac{(D-2)}{R}\frac{dJ}{dR}\right]+\frac{1}{R^2F(\varphi)}\left[\frac{dF^2}{d\varphi^2}+\frac{(D-3)}{\tan\varphi}\frac{dF}{d\varphi}\right]+\frac{1}{Z(z)}\frac{d^2z}{dz^2}=0 \quad (3.2.2)$$

Setting the third term equal to k^2 and the first two terms equal to $-k^2$ with k being a constant greater than 0 brings us quickly to a solution for the $Z(z)$ equation:

$$Z(z)=ae^{kz}+be^{-kz} \quad (3.2.3)$$

This is a common differential equation; thus, we know the answer is:

$$Z(z)=e^{-k|z|} \quad (3.2.4)$$

It is necessary that the solution to this equation goes to zero as z approaches positive or negative infinity. To test this, we replace z with infinity or negative infinity and can see that $Z(z)$ will go to zero. Regardless of the value of k, the right-hand side of the equation will become $e^{-\infty}$, which equals zero.

3.3 Angular Solution

Now that the $Z(z)$ section has been solved, we can move on to the $J(R)$ and $F(\varphi)$ parts of Equation (3.2.1). These parts can be rewritten as:

$$\frac{R^2}{J(R)}\left[\frac{d^2J}{dR^2}+\frac{(D-2)}{R}\frac{dJ}{dR}\right]+k^2R^2+\frac{1}{F(\varphi)}\left[\frac{d^2F}{d\varphi^2}+\frac{(D-3)}{\tan\varphi}\frac{dF}{d\varphi}\right]=0 \quad (3.3.1)$$

Because the entire expression is equal to zero, we can set the first two terms equal to $+m(m+2\lambda)$ and set the third term equal to $-m(m+2\lambda)$. The angular part of the equation $\left(F(\varphi)\right)$ can be solved using the aforementioned Gegenbauer polynomials, keeping in mind the $m(m+2\lambda)$ value. Replacing the first two terms of Equation (3.2.5) with $m(m+2\lambda)$ and multiplying by $F(\varphi)$ throughout results in:

$$\frac{d^2F}{d\varphi^2} + \frac{(D-3)}{\tan\varphi}\frac{dF}{d\varphi} + m(m+2\lambda)F(\varphi) = 0 \tag{3.3.2}$$

Through some clever application of Gegenbauer polynomials, we eventually arrive at:

$$\frac{d^2F}{d\varphi^2} + \frac{2\lambda}{\tan\varphi}\frac{dF}{d\varphi} + m(m+2\lambda)F(\varphi) = 0 \tag{3.3.3}$$

If $2\lambda = D-3$ and thus $\lambda = \frac{(D-3)}{2}$ and we choose $x = \cos\varphi$, the angular solution is:

$$F(\varphi) = C_m^{(\lambda)}(x) = C_m^{\left(\frac{D-3}{2}\right)}(\cos\varphi) \tag{3.3.4}$$

See Equation (2.3.8) from above for the value of the Gegenbauer polynomials and check [7] for more information on the general theory of Gegenbauer polynomials.

3.4 Radial Solution

With both the $Z(z)$ solution and the angular solution determined, finding the radial solution is all that remains. After replacing the third term with $-m(m+2\lambda)$ and rearranging, Equation (3.2.5) becomes:

$$\frac{R^2}{J(R)}\left[\frac{d^2J}{dR^2} + \frac{(D-2)}{R}\frac{dJ}{dR}\right] + k^2R^2 = m(m+2\lambda) = m(m+D-3) \tag{3.4.1}$$

Recall that $2\lambda = D-3$ from above. If we multiply both sides of the equation by $\frac{J(R)}{R^2}$ and subtract $m(m+D-3)$, it becomes:

$$\frac{d^2J}{dR^2} + \frac{(D-2)}{R}\frac{dJ}{dR} + J(R)\left[k^2 - \frac{m(m+D-3)}{R^2}\right] = 0 \quad (3.4.2)$$

Next, we can perform u-substitution by setting $u = kR$, which makes $du = kdR$. After some calculations through Mathematica, this process eventually leads to:

$$\frac{d^2J}{du^2} + \frac{(D-2)}{u}\frac{dJ}{du} + J(u)\left[1 - \frac{m(m+D-2)}{u^2}\right] = 0 \quad (3.4.3)$$

A few additional calculations in Mathematica later, we arrive at:

$$J(u) = u^{\frac{3-D}{2}} J_{\frac{D-3}{2}+m}(u) \quad (3.4.4)$$

Finally, replacing the u with its original value of kR gives us the final radial solution:

$$J(R) = (kR)^{\frac{3-D}{2}} J_{\frac{D-3}{2}+m}(kR) \quad (3.4.5)$$

In Equation (3.4.5), the $J_{\frac{D-3}{2}+m}(kR)$ component is one kind of Bessel function. More on Bessel functions can be found on page 103 of [2]. Part of the calculations resulted in a second kind of Bessel function $Y_{\frac{D-3}{2}+m}(kR)$, but that one typically diverges at $u = 0$ and $R = 0$, so we chose to include only the J_m solutions.

3.5 Final NFDG Solution

With each of the three solutions determined, we can simply combine Equation (3.2.4), Equation (3.3.4), and Equation (3.4.5) to find the full solution of the separation of variables method. The result is a solution to Laplace's equation in cylindrical coordinates for varying dimension D:

$$\Phi(R,z) = J(R)F(\varphi)Z(z) = (kR)^{\frac{3-D}{2}} J_{\frac{D-3}{2}+m}(kR) C_m^{\left(\frac{D-3}{2}\right)}(cos\varphi) e^{-k|z|} \tag{3.5.1}$$

As always, checking the equation by choosing $D = 3$ to ensure it reduces to the typical Newtonian equation is important. Choosing $D = 3$ for Equation (3.5.1) results in:

$$\Phi(R,z) = J_m(kR) e^{im\varphi - k|z|} \tag{3.5.2}$$

This result is what we expect for the typical $D = 3$ case, as outlined on page 104 of Binney and Tremaine's *Galactic Dynamics* [2].

CHAPTER 4: APPLICATIONS AND DISCUSSION

4.1 Equations for Thin Disk Galaxies

To analyze thin disk galaxies, it is important to first derive a few more equations. Starting with Equation (3.5.1), note that it is the solution of $\nabla^2_{(\mathrm{D})}\Phi = 0$ everywhere except on the $z = 0$ plane of a thin disk galaxy (where the mass is located). For this section, I will generalize the procedure laid out by Binney and Tremaine in [2]. Figure 2 shows the mass within a disk cross-section [2].

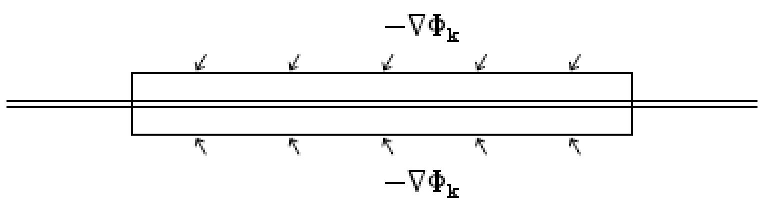

Figure 2: The disk mass within the box shown in cross-section equals $-(4\pi G)^{-1}$ times the integral of the normal component of $\boldsymbol{\nabla}\Phi_{km}$ over the surface of the box. The horizontal component of $\boldsymbol{\nabla}\Phi_{km}$ is due to the gravitational attraction from the rest of the galaxy.

The box is also known as a Gaussian Box. For this Gaussian Box, I applied the Fractional Gauss Theorem outlined by Varieschi [3]:

$$\Phi_g^{(D)} = \oint_{S_d} \mathbf{g} \cdot \mathrm{d}\mathbf{a} = -\frac{4\pi G}{l_0^2} \oint_{V_d} \tilde{\rho}(\mathbf{w})\, d^D\mathbf{w} = -\frac{4\pi G}{l_0^2} \widetilde{\mathrm{M}}_{(\mathrm{D})} \qquad (4.1.1)$$

Through extensive calculations in Mathematica, we eventually were able to arrive at an equation for axisymmetric thin disk potential. A few considerations before that equation, however, are as follows. This analysis takes place in the $z = 0$ plane, and setting $z = 0$ makes the $Z(z)$ portion of Equation (3.5.1) go to one: $e^{-k|z|} = e^0 = 1$. Due to cylindrical symmetry, only the $m = 0$ term is in series and

$$C_0^{\left(\frac{D-3}{2}\right)} \cos\varphi = 1 \tag{4.1.2}$$

Thus, our final expression for thin disk potential in the $z = 0$ plane is:

$$\Phi(R, z=0) = \int_0^\infty \mathrm{d}k\, S_0(k)(kR)^{\frac{3-D}{2}} J_{\frac{D-3}{2}}(kR) =$$
$$(-2\pi G)\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{\mathrm{D}}{2}-1\right)}\right] R^{\frac{3-D}{2}} \int_0^\infty \mathrm{d}k\, J_{\frac{D-3}{2}}(kR) \int_0^\infty \mathrm{d}R' R'^{\frac{(D-1)}{2}} \sum\nolimits_0 (R')\, J_{\frac{D-3}{2}}(kR') \tag{4.1.3}$$

With a little more Mathematica wrangling, it was possible to find a Main Thin-Disk Formula to apply to thin disk galaxies. This equation takes the form of $\mathbf{g}(\mathrm{R}, \mathrm{D})$, or, as we called it previously, $\mathbf{g}_{obs}$:

$$\mathbf{g}_{obs} = -\frac{\partial\Phi}{\partial R}\hat{R} =$$
$$\left\{-(2\pi G)\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{\mathrm{D}}{2}-1\right)}\right] R^{\frac{3-D}{2}} \int_0^\infty \mathrm{d}k\, k\, J_{\frac{D-1}{2}}(kR) \int_0^\infty \mathrm{d}R' R'^{\frac{(D-1)}{2}} \sum (R')\, J_{\frac{D-3}{2}}(kR')\right\}\hat{R} \tag{4.1.4}$$

This $\mathbf{g}_{obs}$ value refers to the observed field; specifically, when $D \neq 3$. In contrast, $\mathbf{g}_{bar}$ refers to the baryonic field, otherwise known as the field when $D = 3$. In this case, $\mathbf{g}_{bar}$ is very similar to $\mathbf{g}_{obs}$, with some differences:

$$\mathbf{g}_{bar} = -\frac{\partial\Phi}{\partial R}\hat{R} = \left\{-(2\pi G)\int_0^\infty \mathrm{d}k\, k\, J_1(kR) \int_0^\infty \mathrm{d}R' R' \sum (R')\, J_0(kR')\right\}\hat{R} \tag{4.1.5}$$

However, it would be preferable to write these equations in terms of the rescaled dimensionless coordinates and mass densities. I first mentioned these dimensionless coordinates in Section 2.2, but I will briefly review them here: $\mathrm{w}_R \equiv R/l_0$, $\mathrm{w}'_R \equiv R'/l_0$, $\tilde{k} \equiv k l_0$, $\sum(\mathrm{w}'_R) = \sum(R')\, l_0^2$. Remember, with MOND acceleration, $a_0 = g_\dagger \equiv \frac{GM}{l_0^2} = 1.2 \text{ x } 10^{-10}\ \mathrm{m/s^2}$, $l_0 = \sqrt{\frac{GM}{g_\dagger}}$ where M is galaxy mass, and $G = 6.674 \text{ x } 10^{-11}\, \frac{\mathrm{m^3}}{\mathrm{kg\ s^2}}$. Thus, we can rewrite $\mathbf{g}_{obs}$ and $\mathbf{g}_{bar}$ with these coordinates in mind:

$$\mathbf{g}_{obs} = \left\{-\left(\frac{2\pi G}{l_0^2}\right)\left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{D-1}{2}\right)}{\Gamma\left(\frac{\mathrm{D}}{2}-1\right)}\right]\mathrm{w}_R^{\frac{3-D}{2}}\int_0^\infty \mathrm{d}\tilde{k}\,\tilde{k}\,J_{\frac{D-1}{2}}\!\left(\tilde{k}\mathrm{w}_R\right)\int_0^\infty \mathrm{d}\mathrm{w}'_R\,\mathrm{w}'^{\frac{(D-1)}{2}}_R\sum(\mathrm{w}'_R)\,J_{\frac{D-3}{2}}\!\left(\tilde{k}\mathrm{w}'_R\right)\right\}\hat{R} \quad (4.1.6)$$

$$\mathbf{g}_{bar} = \left\{-\left(\frac{2\pi G}{l_0^2}\right)\int_0^\infty \mathrm{d}\tilde{k}\,\tilde{k}\,J_1\!\left(\tilde{k}\mathrm{w}_R\right)\int_0^\infty \mathrm{d}\mathrm{w}'_R\,\mathrm{w}'_R\sum(\mathrm{w}'_R)\,J_0\!\left(\tilde{k}\mathrm{w}'_R\right)\right\}\hat{R} \qquad (4.1.7)$$

4.2 Circular Velocity, NGC 6503, and Mass Distribution

To apply these equations to think disk galaxies, the last things we need are circular velocity and a chosen mass distribution. Finding circular velocity is as simple as applying the elementary formula for centripetal acceleration, $a_c = v_c^2/R$. Using rescaled coordinates, starting with:

$$|\boldsymbol{g}(\mathrm{w}_R)| = \frac{V_c^2(\mathrm{w}_R)}{\mathrm{w}_R l_0} \qquad (4.2.1)$$

allows us to rearrange to:

$$V_c(\mathrm{w}_R) = \sqrt{|\boldsymbol{g}(\mathrm{w}_R)|\mathrm{w}_R l_0} \qquad (4.2.2)$$

With this information, we can move forward to discuss the galaxy chosen for the application of these equations. We decided on NGC 6503, a field dwarf spiral galaxy that spans 30,000 light years and is 17 million light years away from Earth. It is located in a constellation called Draco. This galaxy is useful for our purposes because it is very thin, and our work has relied on thin-disk approximations. Thus, the thinner the galaxy we choose, the better our equations should work. Finally, mass distribution. Because of time restraints, we decided upon an exponential approximation of the mass distribution. This approximation is:

$$\sum(\mathrm{w}_R') = \frac{M}{2\pi \mathrm{w}_d^2} e^{-\frac{\mathrm{w}_R'}{\mathrm{w}_d}} \tag{4.2.3}$$

In this case, $l_0 = \sqrt{GM/g_\dagger} = 9.788 \text{ x } 10^9$ m, and $\mathrm{w}_d = R_d/l_0 = 0.681$.

4.3 NGC 6503 Application – Acceleration

The first application of equations through Mathematica resulted in a graph of acceleration versus radial distance, shown in Figure 3. The dotted line near the bottom of the graph is the typical Newtonian acceleration in the $D = 3$ case. The thinner dotted line shows the threshold where the MOND acceleration scale takes place. In this simulation, we used between twenty and fifty points for each curve. Each curve represents a different D value, but anything past $D = 1.8$ resulted in curves too small to be relevant. By $D = 2$, the curves were essentially zero. Beyond $D = 1.8$, all curves were below the Newtonian one, meaning they could not account for the “dark matter” effect that we’re trying to explain.

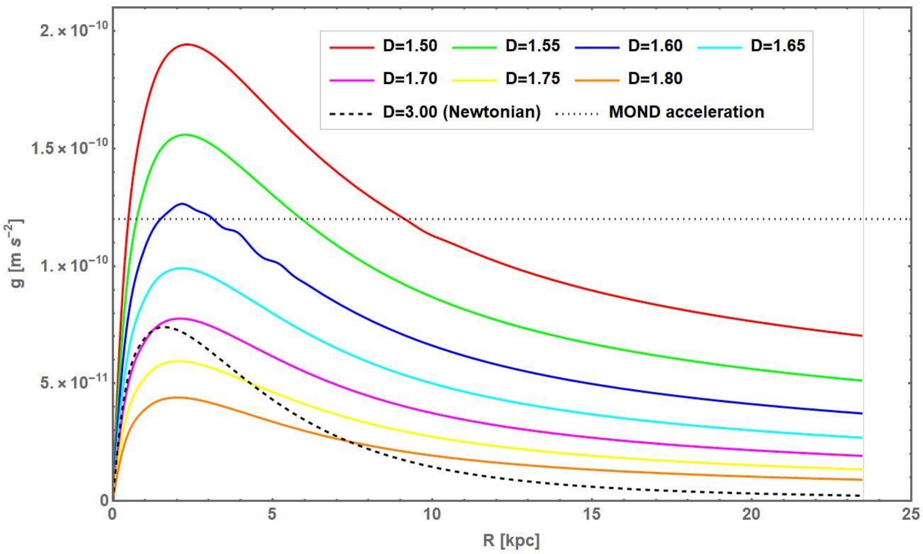

Figure 3: Application of equations to NGC 6503 to find the relationship between acceleration (m/s^2) and radial distance (kpc) for different values of D.

4.4 NGC 6503 Application – Circular Velocity

The second application of equations through Mathematica resulted in a graph of circular velocity versus radial distance, shown in Figure 4. This graph gives a bit more interesting information to work with. The large dots in the middle of the graph represent the SPARC data, which are observed values by astronomers for the NGC 6503 galaxy. The band in the middle is the flat velocity band, which is simply a visual aid that shows where most of the points are contained. The thin dotted line is the Newtonian case. The problem is very clear; the Newtonian case varies greatly from the actual experimentally determined values. Our current models do not line up with reality. So, we plotted a bunch of different values for D to try and see what best fit. At the furthers away from the center, we see that $D = 1.70$ fits nicely. Much of the middle of the graph looks like a dimension value between 1.70 and 1.65 would be the best. Near the beginning, we see that $D = 1.50$ best matches the experimental results. As anticipated, it seems that a continually varying dimension for D will give us the best result, because we can match the SPARC values perfectly if we vary dimension precisely throughout.

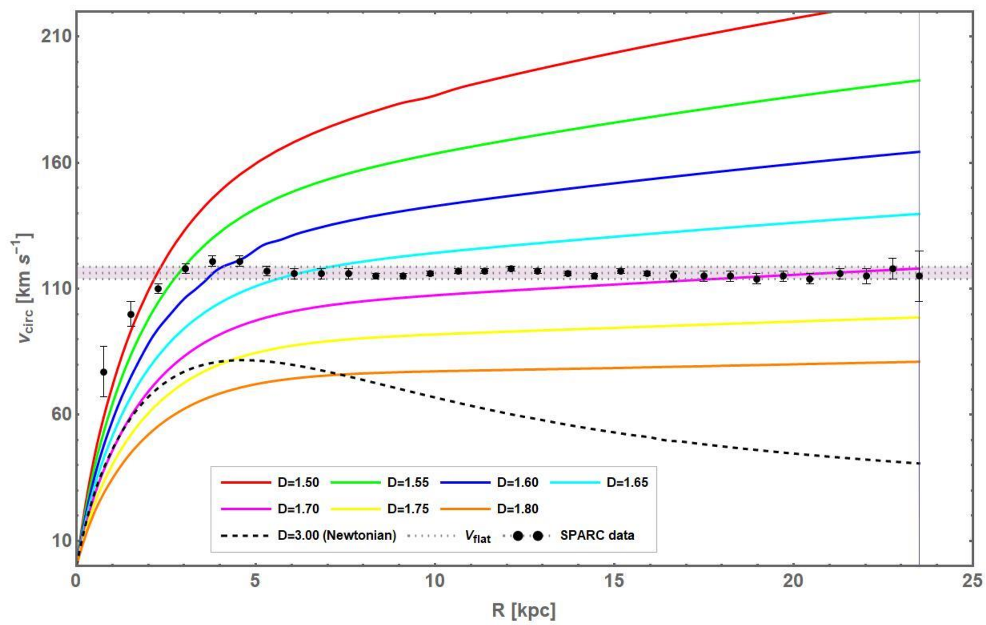

Figure 4: Application of equations to NGC 6503 to find the relationship between circular velocity (m/s) and radial distance (kpc) for different values of D.

CHAPTER 5: CONCLUSIONS AND FUTURE WORK

5.1 Conclusions

This thesis achieved the outcomes I laid out at the beginning of my work. I summarized the formulation of Laplace's equation from one dimension to three dimensions. I discussed the important background of MOND and NFDG, setting the stage for my own work. With a little help from Mathematica and a lot of help from Dr. Varieschi, I managed to present a final equation for Laplace's equation in cylindrical coordinates. After finding this equation, we were successful in determining a main thin disk equation for the purposes of our model.

These equations were applied to the NGC 6503 galaxy, resulting in some graphs that point towards a promising future for the model. With some tweaking, a varying dimension model appears to be capable of matching SPARC data values, an incredible improvement from the Newtonian predictions. With more work, this model has the potential to describe galactic dynamics without the need to include dark matter in the discussion. Since the motivation of the project from the beginning was to model galactic dynamics without dark matter, this thesis was a success.

5.2 Future Work

Clearly, the models we have created are not perfect. One possible reason for this is the assumption of a thin disk galaxy with no thickness because all galaxies will have some thickness, even if they are very thin. We also did not use the full mass distribution, but only an exponential approximation. This inevitably will lead to less accurate results. With circular velocity, we did not plot a graph of varying dimension D, we only plotted certain values of D. This was due both to time limitations and software limitations. Mathematica struggled to plot varying dimension on the graph, often getting stuck for hours on end without making any progress. For future work, considering the full mass distribution, considering thickness, and finding a way to plot the varying dimension D would all hopefully be fruitful in improving the model. These improvements might even make a full model for galactic dynamics that does not invoke dark matter a true possibility.

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