8-1-2018

# Client–Server and Cost Effective Sets in Graphs

Mustapha Chellali
*Université Saad Dahlab de Blida*

Teresa W. Haynes
*East Tennessee State University*, haynes@etsu.edu

Stephen T. Hedetniemi
*Clemson University*



### Citation Information
Chellali, Mustapha; Haynes, Teresa W.; and Hedetniemi, Stephen T.. 2018. Client–Server and Cost Effective Sets in Graphs. *AKCE International Journal of Graphs and Combinatorics*. Vol.15(2). 211-218. https://doi.org/10.1016/j.akcej.2017.09.001 ISSN: 0972-8600


Taylor & Francis
Taylor & Francis Group

AKCE International Journal of Graphs and Combinatorics

ISSN: 0972-8600 (Print) 2543-3474 (Online) Journal homepage: https://www.tandfonline.com/loi/uakc20


# Client–server and cost effective sets in graphs


Mustapha Chellali, Teresa W. Haynes & Stephen T. Hedetniemi

Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 15 (2018) 211–218
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

# Client–server and cost effective sets in graphs

Mustapha Chellali[a], Teresa W. Haynes[b,*], Stephen T. Hedetniemi[c]

[a] *LAMDA-RO Laboratory, Department of Mathematics University of Blida, B.P. 270, Blida, Algeria*
[b] *Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614, USA*
[c] *Professor Emeritus, School of Computing Clemson University Clemson, SC 29634, USA*

Received 10 February 2016; received in revised form 30 August 2017; accepted 2 September 2017
Available online 14 December 2017

**Abstract**

For any integer $k \geq 0$, a set of vertices $S$ of a graph $G = (V, E)$ is *k-cost-effective* if for every $v \in S$, $|N(v) \cap (V \setminus S)| \geq |N(v) \cap S| + k$. In this paper we study the minimum cardinality of a maximal $k$-cost-effective set and the maximum cardinality of a $k$-cost-effective set. We obtain Gallai-type results involving the $k$-cost-effective and global $k$-offensive alliance parameters, and we provide bounds on the maximum $k$-cost-effective number. Finally, we consider $k$-cost-effective sets that are also dominating. We show that computing the $k$-cost-effective domination number is NP-complete for bipartite graphs. Moreover, we note that not all trees have a $k$-cost-effective dominating set and give a constructive characterization of those that do.


*Keywords:* Cost effective sets; $k$-cost-effective sets; Domination; $k$-cost effective domination; Global $k$-offensive alliances

## 1. Introduction

In a graph model of a computer network, specific vertices act as servers, in that they provide to neighboring vertices various computing facilities, such as data bases and specialized software. Ideally, for economical reasons, each server serves as many vertices (non-servers, or clients) as possible. Thus, in general, we would seek to establish a set of servers, each of which is serving a maximal number of clients.

In this paper, we introduce a generalization of cost effective sets suggested by Hedetniemi, Hedetniemi, Kennedy, and McRae [1]. As an introduction and motivation, in Section 2, we set up several different client–server models and objectives. The remainder of the paper will focus on the generalization of cost effective sets, namely, $k$-cost-effective sets (defined in Section 2).

We consider finite, undirected, and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. We shall use the following terminology. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of $v$, denoted by $\deg_G(v)$, is the cardinality

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
*E-mail addresses:* m_chellali@yahoo.com (M. Chellali), haynes@etsu.edu (T.W. Haynes), hedet@clemson.edu (S.T. Hedetniemi).

http://dx.doi.org/10.1016/j.akcej.2017.09.001

of its open neighborhood. We denote by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ the *maximum degree* and the *minimum degree* of a vertex in $V(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A tree $T$ is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively $p$ and $q$ leaves attached at each support vertex is denoted by $S_{p,q}$. The *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood of a set* $S$ is the set $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$. For any $S \subseteq V$, we denote the subgraph of $G$ induced by $S$ as $G[S]$.

A set $S \subset V$ is called *independent* if no two vertices in $S$ are adjacent. The *independent domination number* $i(G)$ is the minimum cardinality of a maximal independent set, and the *independence number* $\beta(G)$ equals the maximum cardinality of an independent set in $G$. A set $S \subset V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in $G$. In what follows, for any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-*set*.

A *Gallai Theorem* is a result of the form: $\alpha(G) + \lambda(G) = n$, where $G$ is a graph of order $n = |V|$ and $\alpha(G)$ and $\lambda(G)$ are two, non-negative integer-valued parameters of a graph. An example of a Gallai Theorem follows.

A set $S \subset V$ is called *enclaveless* if no vertex $u \in S$ satisfies $N[u] \subseteq S$, that is, every vertex $u \in S$ has at least one neighbor in $V \setminus S$. The *enclaveless number* $\Psi(G)$ equals the maximum cardinality of an enclaveless set in $G$. The following result is found in [2].

**Proposition 1.** *For any graph $G$ of order $n$, $\gamma(G) + \Psi(G) = n$.*

In Section 3, we obtain Gallai theorems involving $k$-cost-effective numbers, and in Section 4, we provide bounds on the maximum $k$-cost-effective number. Finally, in Section 5, we consider $k$-cost-effective sets that are also dominating. We show that computing the $k$-cost-effective domination number is NP-complete for bipartite graphs. Moreover, we give a constructive characterization of the trees having a $k$-cost-effective dominating set, since not all trees have such sets.

We conclude this section by mentioning a generalization of independence and domination that will be useful in our results. In [3,4], Fink and Jacobson introduced the concepts of $p$-domination and $p$-dependence. Let $p$ be a positive integer. A subset $S$ of $V$ is a *$p$-dominating set* of $G$ if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq p$. A *$p$-dependent set* is a subset $D$ of $V$ such that the maximum degree in the subgraph $G[D]$ induced by the vertices of $D$ is at most $p-1$. The *$p$-domination number* $\gamma_p(G)$ is the minimum cardinality of a $p$-dominating set of $G$, and the *$p$-dependence number* $\beta_p(G)$ is the maximum cardinality of a $p$-dependent set of $G$. Notice that a 1-dominating set (respectively, a 1-dependent set) is a dominating set (respectively, an independent set), and so $\gamma(G) = \gamma_1(G)$ and $\beta_1(G) = \beta(G)$. For more information on $k$-domination and $k$-dependence, see the survey by Chellali et al. [5].

## 2. Client–server models

In this section, we describe several client–server models, concluding with the definition of $k$-cost-effectiveness, the focus of this paper.

**Differential sets**

In this model, we seek a set of servers that collectively serve a maximum number of clients minus servers. The following definitions were introduced by Haynes et al. in 2006 [6]. The *differential of a set* $S \subset V$ is defined as $\partial(S) = |N(S)| - |S|$, while the *differential of a graph* $G$ is defined as $\partial(G) = max\{\partial(S) : S \subset V\}$. Thus, it is apparent that a set $S$ of servers that maximizes the number of clients served minus the number of servers served is just a set that defines the differential of a graph.

**Client number**

In the next model, we seek to find a set of servers that collectively serves a maximum number of clients. Suppose we define the *client number* $CN(G)$ to equal the maximum number of clients that can be served by a set $S$ of servers, that is, $CN(G) = max\{|N(S) \cap (V \setminus S)| : S \subset V\}$. As it turns out, it is easy to show the following; we leave the proof to the interested reader.

**Proposition 2.** *For any graph $G$ of order $n$, $\gamma(G) + CN(G) = n$.*

**Corollary 3.** *For any graph $G$, $\Psi(G) = CN(G)$.*

**Client–server sets**

For this model, we require that each server must serve at least $k$ clients, independent of the number of edges between servers. With this in mind we can introduce the following new definitions. A set $S \subset V$ is called a *k-client–server set (or kcs-set)* if for every vertex $u \in S$, $|N(u) \cap (V \setminus S)| \geq k$. The *k-client–server number* $CS_k(G)$ equals the maximum cardinality of a $k$cs-set in $G$, while the *lower k-client–server number* $cs_k(G)$ equals the minimum cardinality of a maximal $k$cs-set in $G$. Although newly defined here, we will not study $k$-client–server sets in this paper.

**Client–server colorings**

Another client–server model can be defined as follows. With a $k$ -client–server set $S$ a server vertex must serve at least $k$ clients in $V \setminus S$. But they can serve the same clients as other servers. Thus, there can be redundancy or inefficiency in serving clients. A stronger requirement is that each server can be assigned to serve at least $k$ clients that no other server is assigned to serve. In this way, we can define a *k-client–server coloring* to be a partition $\pi = \{V_1, V_2, \ldots, V_m, V_{m+1}\}$ of $V(G)$ having the property that (i) for $1, \leq i \leq m$, each set $V_i$ contains a vertex $v_i$, called a *server*, that is adjacent to every other vertex in $V_i$, and (ii) for every $1 \leq i \leq m$, $|V_i| \geq k+1$. Neither requirements (i) or (ii) need hold for the set $V_{m+1}$, as these vertices are neither servers nor clients of a server. We are not aware that these kinds of colorings have been studied before.

**Cost effective sets**

For cost effective sets, each server must serve more clients than servers. This idea was first observed by Hedetniemi et al. in [1] and studied further in [7–9]. A vertex (server) $u \in V$ is called *cost effective* if $|N(u) \cap (V \setminus S)| \geq |N(u) \cap S|$. A set $S \subset V$ is called *cost effective* if every vertex $u \in S$ is cost effective. The *cost effective number* $CE(G)$ equals the maximum cardinality of a cost effective set in $G$. The *lower cost effective number* $ce(G)$ equals the minimum cardinality of a maximal cost effective set in $G$.

If the inequality is strict, that is, if $|N(u) \cap (V \setminus S)| > |N(u) \cap S|$, then $v$ is said to be *very cost effective*. A set $S$ is *very cost effective* if every vertex of $S$ is very cost effective. The *very cost effective number* $VCE(G)$ equals the maximum cardinality of a very cost effective set in $G$, and the *lower very cost effective number* $vce(G)$ equals the minimum cardinality of a maximal very cost effective set in $G$.

**$k$-cost-effective sets**

Finally, we define a model that is the focus of the rest of this paper. It is a natural generalization of the concept of cost effective sets as follows. Let $k \geq 0$ be an integer. A vertex $v$ in a set $S \subset V$ is said to be *k-cost-effective* or *kce* if $|N(v) \cap (V \setminus S)| \geq |N(v) \cap S| + k$. A set $S$ is *k-cost-effective* if every vertex in $S$ is $k$-cost-effective. The *lower k-cost-effective number* of a graph $G$, denoted $ce_k(G)$, equals the minimum cardinality of a maximal $k$ce set in $G$, and the *k-cost-effective number* of $G$, $CE_k(G)$, equals the maximum cardinality of a $k$ce set in $G$.

From the above definition, it is clear that a 0ce set is a cost effective set and 1ce set is a very cost effective set. Consequently, for every graph $G$, we have $ce_0(G) = ce(G)$, $ce_1 = vce(G)$, $CE_0(G) = CE(G)$, and $CE_1(G) = VCE(G)$. Next we state some useful observations. Since every vertex of any $k$ce set has degree at least $k$, the following observation is immediate.

**Observation 4.** *No vertex of degree less than $k$ is a member of any kce set of $G$.*

**Observation 5.** *For every positive integer $k$, if $G$ is a graph with maximum degree at most $k-1$, then $ce_k(G) = CE_k(G) = 0$.*

Because of Observation 5, we will only consider $k$-cost-effective sets for graphs $G$ when $k \leq \Delta(G)$. For the particular case $k = \Delta$, it is clear that every $k$ce set is independent and every vertex belonging to any $k$ce set has degree $\Delta$. Let $V_\Delta$ be the set of vertices of maximum degree. The following observation is straightforward.

**Observation 6.** *For every graph $G$, $ce_\Delta(G) = i(G[V_\Delta])$ and $CE_\Delta(G) = \beta(G[V_\Delta])$.*

Since every $(k+1)$ce set is also a $k$ce set, we have:

**Observation 7.** *For every graph $G$,*

$$CE_0(G) \geq CE_1(G) \geq \cdots \geq CE_{\Delta-1}(G) \geq CE_\Delta(G) > 0,$$

$$0 < ce_0(G) \leq ce_1(G) \leq \cdots \leq ce_{\Delta-1}(G) \leq ce_\Delta(G).$$

**Observation 8.** *Every independent set in a graph without isolated vertices is a $\delta$ce set.*

According to Observations 7 and 8, we obtain the following.

**Corollary 9.** *For every graph $G$ without isolated vertices, $CE_k(G) \geq \beta(G)$ for every integer $k$, $1 \leq k \leq \delta$.*

**Proposition 10.** *For every graph $G$ and integer $k$ with $1 \leq k \leq \delta$, $CE_k(G) + \gamma_k(G) \leq n$.*

**Proof.** Let $D$ be a $k$ce set of $G$ of cardinality $CE_k(G)$. Since $k \leq \delta$, every vertex of $D$ has a neighbor in $V \setminus D$. Also, since $|N(v) \cap (V \setminus D)| \geq |N(v) \cap D| + k$ for every $v \in D$, $V \setminus D$ is a $k$-dominating set of $G$. Hence, $\gamma_k(G) \leq |V \setminus D| = n - CE_k(G)$. ■

As defined in [10], let $G = H \circ K_k$, for some integer $k$, be the graph formed by a copy of a graph $H$ and $|V(H)|$ copies of a clique $K_k$, where the $i$th vertex of $H$ is adjacent to every vertex in the ith copy of $K_k$. Let $H_i$ denote the subgraph induced by the $i$th vertex of $H$ and its copy of $K_k$. Clearly, $\delta(G) \leq k$, $n = (k+1)\,|V(H)|$, and $\gamma_k(G) = k\,|V(H)| = \frac{k}{k+1}n$ (see Theorem 4, [11]). Moreover, every $CE_k$-set of $G$ contains at most one vertex of each $H_i$, implying that $CE_k(G) \leq |V(H)|$. The equality is obtained from the $kce$ set containing exactly one vertex of each copy of $K_k$. It is worth mentioning that the same graph also provides the sharpness for Corollary 9 since $\beta(G) = CE_k(G) = |V(H)|$.

On the other hand, one can easily see that both inequalities in Corollary 9 and Proposition 10 can be strict for complete graphs of large order.

## 3. $k$-cost-effective sets and alliances

In [12], Kristiansen, Hedetniemi, and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive $k$-alliances, given by Shafique and Dutton [13,14]. Let $k$ be an integer such that $0 \leq k \leq \Delta$. A set $S$ of vertices of a graph $G$ is called a *global $k$-offensive alliance* if for every $v \in V \setminus S$, $|N(v) \cap S| \geq |N(v) \setminus S| + k$, that is, every vertex of $V \setminus S$ has at least $k$ more neighbors in $S$ than it has in $V \setminus S$. The *global $k$-offensive alliance number* $\gamma_o^k(G)$ is the minimum cardinality of a global $k$-offensive alliance in $G$. Note that every vertex of degree less than $k$ belongs to every global $k$-offensive alliance of $G$. A global 1-offensive alliance is a global offensive alliance and a global 2-offensive alliance is a global strong offensive alliance, as defined in [12–14]. A global $k$-offensive alliance $S$ is said to be *minimal* if for every $x \in S$, $S - \{x\}$ is not a global $k$-offensive alliance of $G$. We define the *upper global $k$-offensive alliance number* $\Gamma_o^k(G)$ as the maximum cardinality of a minimal global $k$-offensive alliance of $G$. As far as we know, this parameter has not been previously defined.

Our aim in this section is to establish Gallai theorems involving the cost $k$-cost-effective and global $k$-offensive alliance parameters.

**Theorem 11.** *For every graph $G$ without isolated vertices and for every non-negative integer $k$,*

(a) $CE_k(G) + \gamma_o^k(G) = n$,
(b) $ce_k(G) + \Gamma_o^k(G) = n$.

**Proof.** Clearly, if $k \geq \Delta + 1$, then $ce_k(G) = CE_k(G) = 0$ and $\gamma_o^k(G) = \Gamma_o^k(G) = n$, and so the results are valid. Hence, we assume that $k \leq \Delta$.

Let $D$ be a maximal $k$ce set of $G$. Since $G$ has no isolated vertices and $|N(v) \cap (V \setminus D)| \geq |N(v) \cap D| + k$ for every $v \in D$, it follows that $V \setminus D$ is a global $k$-offensive alliance of $G$. Thus, $\gamma_o^k(G) \leq |V \setminus D| \leq n - |D|$. Since $D$ is any maximal $k$ce set of $G$, we can choose it to be maximum and have that $\gamma_o^k(G) \leq n - CE_k(G)$.

We next show that $V \setminus D$ is a minimal global $k$-offensive alliance. If this is not the case, then there exists a vertex $y \in V \setminus D$ such that $(V \setminus D) \setminus \{y\} = Y$ is a global $k$-offensive alliance of $G$. It follows that for every vertex $x \notin Y$, $|N(x) \cap Y| \geq |N(x) \cap (V \setminus Y)| + k$. But then $D \cup \{y\}$ is $k$-cost-effective, contradicting the maximality of $D$. Hence, $V \setminus D$ is a minimal global $k$-offensive alliance of $G$, implying that $\Gamma_o^k(G) \geq |V \setminus D| = n - |D| \geq n - ce_k(G)$.

Next we let $S$ be a minimal global $k$-offensive alliance of $G$. Recall that every vertex of degree less than $k$ belongs to $S$. Since $S$ is a global $k$-offensive alliance, $|N(v) \cap S| \geq |N(v) \setminus S| + k$ for every $v \in V \setminus S$. Hence, $V \setminus S$ a $k$ce set. Letting $S$ be a minimum global $k$-offensive alliance of $G$, we have that $CE_k(G) \geq |V \setminus S| = n - \gamma_o^k(G)$. Thus, $CE_k(G) = n - \gamma_o^k(G)$.

Finally, we show that $V \setminus S$ is a maximal $k$ce set. Suppose to the contrary, that $V \setminus S$ is not a maximal $k$ce set. Then there is a vertex $x \in S$ such that $(V \setminus S) \cup \{x\} = X$ is a $k$ce set, that is, for every $v \in X$, $|N(v) \cap (V \setminus X)| \geq |N(v) \cap X| + k$. It follows that $S \setminus \{x\}$ is a global $k$-offensive alliance of $G$, contradicting the minimality of $S$. Hence, $V \setminus S$ is a maximal $k$ce set of $G$, implying that $ce_k(G) \leq |V \setminus S| = n - |S| \leq n - \Gamma_o^k(G)$. Thus, $ce_k(G) = n - \Gamma_o^k(G)$. ■

As an immediate consequence, we obtain the following.

**Corollary 12.** *For every graph G without isolated vertices,*

(a) $CE(G) + \gamma_o(G) = n$.
(b) $VCE(G) + \gamma_o^1(G) = n$.
(c) $ce(G) + \Gamma_o(G) = n$.
(d) $vce(G) + \Gamma_o^1(G) = n$.

We note that since determining the number $\gamma_o^1(G)$ for an arbitrary graph is NP-complete [15], it follows from the previous corollary that computing $VCE(G)$ is also NP-complete.

## 4. Bounds on the $k$-cost-effective number

Let $A = \{x \in V \mid \deg_G(x) \leq k - 1\}$, and let us denote by $\delta^* = \min\{\deg_G(x) \mid x \in V \setminus A\}$. Clearly, $\delta^* = \delta$ if $A = \emptyset$, and $\delta^* \geq \delta + 1$, otherwise. Next we present an upper bound for the $k$-cost-effective number $CE(G)$ of a graph $G$ in terms of the order, maximum degree and minimum degree $\delta^*$ of $G$.

**Proposition 13.** *If G is a graph without isolated vertices, then*

$$CE_k(G) \leq \frac{\Delta n - |A| (\Delta - k + 1)}{\Delta + \lceil (\delta^* + k) /2 \rceil}.$$

**Proof.** Let $D$ be a $CE_k$-set of $G$. Since $\delta \geq 1$, if $k \in \{0, 1\}$, then $A = \emptyset$. Also, $D$ dominates $V \setminus A$. Let $F$ be the set of edges with one end in $D$ and the other in $V \setminus D$. Note that a vertex of $A$ has at most $k - 1$ neighbors in $D$. Hence,

$$\begin{aligned}(k-1)|A| + \Delta |V - (D \cup A)| \geq |F| &\geq k|D| + \sum_{v \in D} \left\lceil \left(\deg_G(v) - k\right) /2 \right\rceil \\ &\geq k|D| + \left\lceil \left(\delta^* - k\right) /2 \right\rceil |D|.\end{aligned}$$

This leads to

$$|D| \leq \frac{\Delta n - |A| (\Delta - k + 1)}{\Delta + \lceil (\delta^* + k) /2 \rceil}. \quad ■$$

**Corollary 14.** *If G is a graph without isolated vertices, then*

$$ce(G) \leq CE(G) \leq \frac{\Delta n}{\Delta + \lceil \delta/2 \rceil}.$$

**Corollary 15.** *If G is a connected regular graph, then*

$$ce(G) \leq CE(G) \leq \begin{cases} 2n/3 & \text{if } \Delta \text{ is even,} \\ 2\Delta n/(3\Delta + 1) & \text{if } \Delta \text{ is odd.} \end{cases}$$

Note that the previous corollary gives a partial answer to open questions in [1], namely, is $ce(G) \leq 2n/3$ and is $ce(G) \leq 3n/5$? Clearly, the first question is true for every nontrivial regular graph, and the second one is true for cubic graphs $G$. In fact, the bound on $ce(G)$ can be improved for $d$-regular graphs $G$ when $d$ is odd.

**Corollary 16.** *If G is a connected, d-regular graph and d is odd, then* $ce(G) < 2dn/(3d + 1)$.

**Proof.** Let $G$ be a connected, $d$-regular graph. By Corollary 15, $ce(G) \leq 2dn/(3d+1)$. Suppose that $ce(G) = 2dn/(3d+1)$, and let $S$ be a $ce(G)$-set. Then every vertex in $S$ has at least $(d+1)/2$ neighbors in $V \setminus S$. Hence, $((d+1)/2)|S| = ((d+1)/2)(2dn/(3d+1)) = (d^2n + dn)/(3d+1) \leq d|V \setminus S| = d(n - (2dn/(3d+1))) = (d^2n + dn)/(3d+1)$. It follows that every vertex in $V \setminus S$ has exactly $d$ neighbors in $S$, that is, $V \setminus S$ is an independent set. Also, every vertex in $S$ has exactly $(d+1)/2$ neighbors in $V \setminus S$ and $(d-1)/2$ neighbors in $S$. But then no superset of $V \setminus S$ is a cost effective set, that is, $V \setminus S$ is a maximal cost effective set with cardinality $n - (2dn/(3d+1)) < 2dn/(3d+1) = ce(G)$, a contradiction. ■

**Corollary 17.** *If $G$ is a connected, cubic graph, then $ce(G) < 3n/5$.*

**Corollary 18.** *If $G$ is a graph without isolated vertices, then*

$$vce(G) \leq VCE(G) \leq \frac{\Delta n}{\Delta + \lceil(\delta+1)/2\rceil}.$$

**Corollary 19.** *If $G$ is a connected regular graph, then*

$$vce(G) \leq VCE(G) \leq \begin{cases} 2\Delta n/(3\Delta+2) & \textit{if } \Delta \textit{ is even,} \\ 2\Delta n/(3\Delta+1) & \textit{if } \Delta \textit{ is odd.} \end{cases}$$

Next we show that $d$-regular graphs $G$ have equal $\beta_{\lceil(d-k+1)/2\rceil}(G)$ and $CE_k(G)$.

**Theorem 20.** *If $G$ is a $d$-regular graph, then for every integer $k \leq d$, we have $\beta_{\lceil(d-k+1)/2\rceil}(G) = CE_k(G)$.*

**Proof.** Clearly, if $d = 0$, then $\beta_{\lceil(d-k+1)/2\rceil}(G) = CE_k(G) = n$. Hence, we assume that $d \geq 1$. First let $D$ be a $k$ce set of $G$ of cardinality $CE_k(G)$. Then every vertex of $D$ has at most $\lfloor(d-k)/2\rfloor$ neighbors in $D$. Since $\lfloor(d-k)/2\rfloor = \lceil(d-k+1)/2\rceil - 1$, we deduce that $D$ is a $\lceil(d-k+1)/2\rceil$-dependent set of $G$, implying that $\beta_{\lceil(d-k+1)/2\rceil}(G) \geq |D| = CE_k(G)$.

Now let $S$ be a maximum $\lceil(d-k+1)/2\rceil$-dependent set of $G$. Since $G[S]$ has maximum degree $\lceil(d-k+1)/2\rceil - 1 = \lfloor(d-k)/2\rfloor$ and $G$ is $d$-regular, it follows that every vertex $x$ of $S$ satisfies $|N(x) \cap (V \setminus S)| \geq |N(x) \cap S| + k$. Hence, $S$ is a $k$ce set and so, $CE_k(G) \geq |S| = \beta_{\lceil(d-k+1)/2\rceil}(G)$. Therefore, $\beta_{\lceil(d-k+1)/2\rceil}(G) = CE_k(G)$. ■

**Corollary 21.** *If $G$ is a $d$-regular graph, then $\beta(G) = CE_d(G)$.*

**Corollary 22.** *If $G$ is a cubic graph, then $\beta_2(G) = CE(G)$.*

We conclude this section by giving a lower bound on the global offensive alliance number for every graph without isolated vertices. Indeed, according to Corollaries 12-(b) and 18, we obtain the following.

**Corollary 23.** *For every graph without isolated vertices,*

$$\gamma_o^1(G) \geq \frac{n}{1 + \frac{\Delta}{\lceil(\delta+1)/2\rceil}}.$$

## 5. $k$-cost-effective domination

In this section, we consider sets that are both $k$-cost-effective and dominating. A dominating set $S$ is $k$-cost-effective, or a $k$ce dominating set, if every vertex in $S$ is $k$-cost-effective. The *$k$-cost-effective domination number* $\gamma_{kce}(G)$ is the minimum cardinality of a $k$ce dominating set of $G$. It is worth mentioning, according to Observation 4, that not all graphs have $k$ce dominating sets. For example, the corona of a nontrivial complete graph $K_p \circ K_1$ do not admit $k$ce dominating sets for every $k \geq 2$. The content of this section is divided into two subsections. In the first one we establish a complexity result on $k$-cost-effective domination for each positive integer $k$, while the second subsection is devoted to the class of trees.

### *5.1. Complexity result*

Our aim in this subsection is to determine the complexity of the following decision problem, to which we shall refer as $k$-COST-EFFECTIVE DOMINATING SET.

$k$-COST-EFFECTIVE DOMINATING SET

**Instance**: Graph $G = (V, E)$, positive integer $p \leq |V|$.

**Question**: Does $G$ have a $k$-cost-effective dominating set of size at most $p$?

We show that this problem is NP-complete by reducing the well-known NP-complete problem, Exact-3-Cover (X3C), to $k$-COST-EFFECTIVE DOMINATING SET.

EXACT 3-COVER (X3C)

**Instance**: A finite set $X$ with $|X| = 3q$ and a collection $C$ of 3-element subsets of $X$.

**Question**: Is there a subcollection $C'$ of $C$ such that every element of $X$ appears in exactly one element of $C'$?

**Theorem 24.** *$k$-COST-EFFECTIVE DOMINATING SET is NP-Complete for bipartite graphs for each positive integer $k$.*

**Proof.** $k$-COST-EFFECTIVE DOMINATING SET is a member of $\mathcal{NP}$, since we can check in polynomial time that any set of vertices is a $k$ce dominating set of $G$. Now let us show how to transform any instance of X3C into an instance $G$ of $k$-COST-EFFECTIVE DOMINATING SET so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \dots, x_{3q}\}$ and $C = \{C_1, C_2, \dots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we create a vertex $y_i$. Let $Y = \{y_1, y_2, \dots, y_{3q}\}$. For each $C_j \in C$, we build a graph $H_j$ obtained from a double star $S_{k+1,k}$ with support vertices $v_j$ and $u_j$, and leaf neighbors $\{v_{j,1}, v_{j,2}, \dots, v_{j,k+1}\}$ and $\{u_{j,1}, u_{j,2}, \dots, u_{j,k}\}$, respectively, by adding a new vertex $z_j$ attached to each leaf neighbor of $u_j$. Let $Z = \{z_1, z_2, \dots, z_t\}$. Now to obtain a graph $G$, we add edges $u_j y_i$ if $x_i \in C_j$. Clearly, $G$ is a bipartite graph. Let $H$ be the subgraph of $G$ induced by all $V(H_j)$ and set $p = 2t + q$. Observe that $\{v_j, z_j\}$ is a minimum $k$ce dominating set of $H_j$.

Suppose that the instance $X, C$ of X3C has a solution $C'$. We construct a $k$ce dominating set $D$ of $G$ of size at most $p$ as follows. For every $j \in \{1, \dots, t\}$, if $C_j \in C'$, then put $v_j$, $z_j$ and $u_j$ in $D$, and if $C_j \notin C'$, then put $v_j$ and $z_j$ in $D$. Note that since $C'$ exists, its cardinality is precisely $q$, and so the number of $u_j$'s is $q$, having disjoint neighborhoods in $\{y_1, y_2, \dots, y_{3q}\}$. Now using the fact that $C'$ is a solution for X3C, it is straightforward to see that $D$ is a $k$ce dominating set of $G$ with size $q + 2t = p$.

Conversely, suppose that $G$ has a $k$ce dominating set $D$ of size at most $p$. Clearly, $\left|D \cap V(H_j)\right| \geq 2$ and $v_j \in D$ for every $j \in \{1, 2, \dots, t\}$. Now if $k \geq 2$, then $z_j \in D$, and if $k = 1$, then $D$ contains either $z_j$ or $u_{j,1}$. Hence, without loss of generality, we can assume that $z_j \in D$. Consequently, $|D \cap V(H)| \geq 2t$, implying that $\left|D \cap \{y_1, y_2, \dots, y_{3q}\}\right| \leq q$. Since $|Y| = 3q$, $U = D \cap \{u_1, u_2, \dots, u_t\} \neq \emptyset$. Let $|U| = a$. Clearly, $a \leq q$. Also, since each $u_j$ has exactly three neighbors in $\{y_1, y_2, \dots, y_{3q}\}$, it follows that $q - a \geq 3q - 3a$ and so $a \geq q$. Therefore, $a = q$, and hence, no vertex of $Y$ belongs to $D$. Consequently, $C' = \{C_j : u_j \in D\}$ is an exact cover for $C$. ■

### *5.2. k-cost-effective domination in trees*

As mentioned above, not all graphs have $k$ce dominating sets. For the case of trees, for $k \geq 2$, a double star $S_{p,q}$ has a $k$ce dominating set if and only if $p \geq q \geq k + 1$. From there, an interesting question arises regarding the characterization of trees having $k$ce dominating sets.

In what follows, we give a constructive characterization of the trees having $k$ce dominating sets, for $k \geq 2$. Since any non-trivial tree $T$ is a bipartite graph, it has a unique bipartition $(X, Y, E)$. For $k \geq 2$, we say that $T$ is a *ck-tree* if every vertex in one of the partite sets has degree at least $k$. Clearly, such a partite set of $T$ is a $k$ce dominating set. Let $T = (X, Y, E)$ be a ck-tree and $X$ its $k$ce dominating set.

For the purpose of the characterization, we define the family $\mathcal{T}_k$ to include all trees $T$ that can be constructed from $r$ ($r \geq 1$) ck-trees $T_i = (X_i, Y_i, E_i)$ with $k$ce sets $X_1, \dots, X_r$, by adding $r - 1$ edges, where each new edge joins two vertices belonging to different sets, either $Y_i$ and $Y_j$ or $X_i$ and $X_j$ such that $T$ is connected with the condition that the total number of new edges incident with any vertex $x$ of $X_p$ for each $p$ is at most $\deg_{T_p}(x) - k$.

**Theorem 25.** *For any integer $k \geq 2$, a tree $T$ has a kce dominating set if and only if $T \in \mathcal{T}_k$.*

**Proof.** Assume that $T \in \mathcal{T}_k$. Then $T$ is obtained from $r$ ($r \geq 1$) ck-trees $T_i = (X_i, Y_i, E_i)$ with $ckd$-sets $X_1, \ldots, X_r$, by adding $r-1$ edges, where each new edge joins two vertices belonging to different sets, either $Y_i$ and $Y_j$ or $X_i$ and $X_j$ such that $T$ is connected with the condition that the number of new edges incident with any vertex $x$ of $X_p$ for every $p$ is at most $\deg_{T_p}(x) - k$. Clearly, $X = \bigcup_i^r X_i$ is a $k$ce dominating set of $T$.

Conversely, assume that $T$ has a $k$ce dominating set, say $S$. Let $F$ be the set of edges such that every edge of $F$ joins two vertices that are both in $S$ or both in $V \setminus S$. Consider the forest $H$ obtained by removing all edges of $F$. Obviously, each of $S \cap V(H)$ and $(V \setminus S) \cap V(H)$ is independent, and so each component $T_i$ of $H$ has partite sets $X_i = S \cap V(T_i)$ and $Y_i = (V \setminus S) \cap V(T_i)$. In this case, it is clear that each $X_i$ is a $k$ce dominating set of $T_i$, which implies that every vertex of $X_i$ has degree at least $k$, where $k \geq 2$. Hence, each component of $H$ is a ck-tree. Moreover, every edge of $F$ links two vertices that belong to different sets either $Y_i$ and $Y_j$ or $X_i$ and $X_j$. It should be noted that since $S$ is a $k$ce dominating set of $T$, the number of edges of $F$ incident with any vertex $x \in X_i$ for some $i$ equals $\deg_T(x) - \deg_{T_i}(x)$, which should not exceed $\deg_{T_i}(x) - k$, for otherwise, $x$ is not a $k$-cost-effective vertex in $S$. According to the previous facts, we conclude that $T \in \mathcal{T}_k$. ■

## 6. Open problems

1. Under what conditions is a $CE_k(G)$-set obtained by choosing a maximum independent set? This happens, for example, with $k$-regular graphs, where $\beta(G) = CE_k(G)$.
2. In Corollary 15 we show that the conjecture that $cd(G) \leq 2n/3$ holds for connected regular graphs $G$ of even degree. Find other classes of graphs where this bound holds.
3. In Corollary 15 we show that the conjecture that $cd(G) \leq 3n/5$ holds for connected cubic graphs $G$. Find other classes of graphs where this bound holds.
4. An $m$-by-$n$ grid graph is a graph of the form $G_{m,n} = P_m \Box P_n$, which is the Cartesian product of a path $P_m$ and a path $P_n$. It is easy to see that any maximum independent set in an $m$-by-$n$ grid graph is a 2ce dominating set, and it is easy to see that no grid graph has a 4ce dominating set. This raises the question: which grids, if any, have 3ce dominating sets? For example, it is easy to see that no 2-by-$n$ grid graph has a 3ce dominating set.

## Acknowledgment

The second author research was supported in part by the University of Johannesburg.

## References

[1] S.M. Hedetniemi, S.T. Hedetniemi, K.E. Kennedy, A.A. McRae, Self-stabilizing algorithms for unfriendly partitions into two disjoint dominating sets, Parallel Process. Lett. 23 (1) (2003) 11.
[2] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc, New York, 1995.
[3] J.F. Fink, M.S. Jacobson, $n$-domination in graphs, in: Y. Alavi, A.J. Schwenk (Eds.), Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 283–300.
[4] J.F. Fink, M.S. Jacobson, On $n$-domination, $n$-dependence and forbidden subgraphs, in: Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, pp. 301–312.
[5] M. Chellali, O. Favaron, A. Hansberg, L. Volkmann, $k$-domination and $k$-independence in graphs: A survey, Graphs Combin. 28 (2012) 1–55.
[6] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, J.R. Lewis, P.J. Slater, Differentials in graphs, Util. Math. 69 (2006) 43–54.
[7] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, T.L. McCoy, I. Vasylieva, Cost effective domination in graphs, Congr. Numer. 211 (2012) 197–209.
[8] T.W. Haynes, S.T. Hedetniemi, T.L. McCoy, T. Rodriguez, Bounds on cost effective domination numbers, Quaest. Math. 39 (6) (2016) 773–783.
[9] T.W. Haynes, S.T. Hedetniemi, I. Vasylieva, Very cost effective bipartitions in graphs, AKCE Int. J. Graphs Combin. 12 (2015) 155–160.
[10] R.W. Frucht, F. Harary, On the corona of two graphs, Aequationes Math. 4 (1970) 322–325.
[11] O. Favaron, A. Hansberg, L. Volkmann, On $k$-domination and minimum degree in graphs, J. Graph Theory 57 (2008) 33–40.
[12] S.M. Hedetniemi, S.T. Hedetniemi, P. Kristiansen, Alliances in graphs, J. Combin. Math. Combin. Comput. 48 (2004) 157–177.
[13] K.H. Shafique, R.D. Dutton, Maximum alliance-free and minimum alliance-cover sets, Congr. Numer. 162 (2003) 139–146.
[14] K.H. Shafique, R.D. Dutton, A tight bound on the cardinalities of maximum alliance-free and minimum alliance-cover sets, J. Combin. Math. Combin. Comput. 56 (2006) 139–145.
[15] A. Cami, H. Balakrishnan, N. Deo, R. Dutton, On the complexity of some global alliance problems, J. Combin. Math. Combin. Comput. 58 (2006) 23–31.