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Understanding the Potential of Anticipation, Teaching, and Response to Struggle in the Learning of Mathematics

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UNDERSTANDING THE POTENTIAL OF ANTICIPATION, TEACHING, AND RESPONSE TO STRUGGLE IN THE LEARNING OF MATHEMATICS

by

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ABSTRACT

UNDERSTANDING THE POTENTIAL OF ANTICIPATION, TEACHING, AND RESPONSE TO STRUGGLE IN THE LEARNING OF MATHEMATICS

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This qualitative semi-structured interview study investigated how the opportunity to learn with productive struggle emerges in a teacher's beliefs, anticipation, planning, teaching, and response to struggle in in learning mathematics. In this study, the experiences of student struggle in the teaching and learning of fractions was investigated through the experiences of three fifth grade and one sixth grade teacher. The purpose was to understand how the phenomenon of productive struggle in learning mathematics was impacted by teachers anticipation, planning, teaching, and response to struggle.

The central research question asked: What role does productive struggle play in the design and implementation of mathematics lessons? Three interviews provided a progressive focus on the phenomenon as each participant's experience with productive struggle was illuminated. Attendant questions guided the interviews: How do teachers perceive their role and the role of students as it relates to learning with productive struggle? How do teachers prepare for anticipated struggle when planning for mathematics instruction? How do teachers respond to

evidence of struggle in student learning? and Do the responses have the potential to invoke a productive struggle for students?

The findings provide compelling evidence that the phenomenon of productive struggle in learning mathematics is directly impacted by a teacher's beliefs about the role of struggle in learning mathematics. Seven findings emerged *(1) Teachers describing a student-centered and constructivist learning environment were more likely to indicate an opportunity for learning that fostered productive struggle, (2) Teachers describing a teacher-centered and transmissionist learning environment were more likely to indicate a diminished opportunity for learning with productive struggle, (3) Teacher descriptions of students' mathematical understanding indicated a relationship to their teaching philosophy, (4) Teachers beliefs provide a strong indication of an opportunity or lack of opportunity for their students to learn with productive struggle, (5) Teachers who believe that struggle is a benefit to student learning create this opportunity, while teachers who believe that struggle is a barrier remove or diminish this opportunity, (6) Teacher inability to recognize productive struggle among students in their classrooms impacted their responses to evidence of student struggle, and (7) Teachers removed the opportunity for learning with productive struggle when students demonstrated a prolonged struggle following probing*. Expanding upon important research on productive struggle, the findings of this study suggest that understanding the relationship between our beliefs, practices, and ability to identify productive struggle has a direct impact on students' opportunity to learn with productive struggle.

Dedication

To John, Valerie, Abigail, and Michael

Without your love and support, this journey would not have been possible.

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CHAPTER 1

INTRODUCTION

Teaching and learning are inextricably linked, involving iterative interactions between the teacher and students, as well as engagement with content. The National Research Council (2001) recognized the importance of this interaction in mathematics, acknowledging that "although much is known about characteristics of effective instruction, research on teaching has often been restricted to describing isolated fragments of teaching and learning rather than examining continued interactions among the teacher, the students, and the content" (p. 9). The consequence of these interactions has broad implications, impacting students' development of mathematical understanding of a given concept, as well impacting a students' ability to apply this knowledge in novel applications throughout their learning journey.

The field of mathematics education has explored opportunities to learn mathematics, focusing on recommendations that build competence, resulting in mathematical proficiency for all students (National Council of Teachers of Mathematics [NCTM], 1980; NCTM, 2014; National Research Council [NRC], 2001). Mathematical proficiency in students has been defined to include conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001) which must be fostered in the classroom "as teachers and learners interact over the curriculum" (Ball & Forzani, 2011, p. 17). With a focus on building these strands of mathematical proficiency, planned learning opportunities must provide students with "experience in the application of mathematics" (NCTM, 1980, p. 3) in a manner that "engage[s] students in solving and discussing tasks that promote mathematical reasoning and problem solving" (NCTM, 2014, p. 17).

Background of the Problem

The recognition that mathematics needed to encompass "a more balanced approach" was in response to an unsuccessful "back-to-basics movement" emphasizing rote mechanical skills in the 1970s (Kilpatrick & Stanic, 1995, p. 5; Schoenfeld, 1992). The National Council of Teachers of Mathematics ([NCTM], 1980) redefined "basic skills" to "include problem solving, understanding, and applications to realistic situations" (p. 5).

This catalyst for reform efforts in mathematics instruction, coupled with the need to understand how teaching and students' engagement impact learning, led to focused efforts to illuminate learning opportunities that met this new definition of "basic skill." NRC (2001) defined a students' opportunity to learn as "circumstances that allow students to engage in and spend time on academic tasks such as working on problems, exploring situations and gathering data, listening to explanations, reading texts, or conjecturing and justifying" (p. 334). These opportunities are considered important predictors of student achievement (NRC, 2001).

Resulting investigations into teaching and learning of mathematics have provided strategies to facilitate learning, with emphases on (a) problem solving (Carpenter et al., 1989; Fennema et al., 1993; Stein & Lane, 1996), (b) tasks (Hiebert & Wearne, 1993; Stein et al., 1996; Stein & Lane, 1996), (c) questioning (Boaler, 1998; Hiebert & Wearne, 1993; Hiebert et al., 1996; Wood, 1998), and (d) discourse (Lampert, 1986; Silver & Smith, 1996; Yackel et al., 1991). Barring explicit consideration of synergistic effects, the individual impacts of these strategies are promising. The noted investigations into teaching and learning in a reform era of instruction, while not comprehensive, have provided recommendations on instructional practices that ensure students the opportunity to learn mathematics with understanding.

An additional consideration in learning engagement is a student's opportunity to struggle with mathematical concepts while building understanding. NCTM (2014) asserted that "effective mathematics teaching uses students' struggles as valuable opportunities to deepen their understanding of mathematics. Students come to realize that they are capable of doing well in mathematics with effort and perseverance in reasoning, sense-making, and problem-solving" (p. 50). The opportunity for students to experience struggle and disequilibrium in learning is important at all levels, though elementary mathematics is of greatest interest to the researcher. Investigations at the elementary level indicate that students develop a deeper understanding and ability to solve problems when instructional practices foster opportunity for reasoning and sense-making as opposed to memorization and procedural practice, alluding to the presence of struggle in learning (Fennema et al., 1993; Hiebert & Carpenter, 1992).

The Problem

The term *struggle* often elicits a personal response that is conceptualized through an individual's experience with great effort in accomplishing a task. In mathematics, the struggle encountered during a time of disequilibrium in learning, while necessary, is often perceived as negative. A teacher's response to student struggle can have positive implications that facilitate learning or undermine the efforts of students as they struggle to develop understanding (Doyle, 1988; Stein et al., 2000). Stein, Grover, and Henningsen (1996) describe the impacts of this, as

> teachers find it difficult to stand by and watch students struggle…step in prematurely to relieve them of their uncertainty and (sometimes) emotional distress at not being able to make headway…do[ing] too much for the students, taking away students' opportunities to discover and make progress on their own. (p. 480)

This struggle is an important feature in student learning and the development of conceptual understanding, including their opportunity to build connections between mathematical facts, ideas, and procedures (Hiebert & Grouws, 2007).

In the field of education, the understanding that struggle is a necessary component for constructing knowledge is not novel. Dewey (1910) shared that the process of deep learning begins with “some perplexity, confusion, or doubt” (p. 12), a learning structure supported by what Vygotsky (1978) identified as the zone of proximal development (ZPD). Vygotsky (1978) defined ZPD as the distance between a child’s ability to solve problems independently and when provided guidance and support from an adult or peer. Implicit in this learning structure is the opportunity for students to grapple with new learning. Relatedly, Piaget’s (1960) theory of learning captures that essence of disequilibrium, or struggle, experienced by students when new information is introduced without an immediately apparent connection. When disequilibrium occurs, the learner either assimilates this information, reinterpreting it to fit in with old ideas, or accommodates this new information, restructuring old ideas to include new understandings (Piaget, 1976).

Reframing struggle in learning mathematics with the potential for this to be positive and productive requires the acknowledgement of the purpose of this struggle, demonstrated in teaching and learner engagement that affords this opportunity (Hiebert & Grouws, 2007). The learning fostered in this place of disequilibrium is where understanding begins (Vygotsky, 1978) and is supported through teaching practices that privilege productive struggle, the learning phenomenon that occurs when students must “expend effort to make sense of something that is not immediately apparent…solving problems that are within reach and grappling with key ideas that are comprehendible but not yet well formed” (Hiebert & Grouws, 2007, p. 386). Ultimately, this struggle develops the ability for students to use their understanding in procedural constructs, as they demonstrate an ability to choose and apply "procedures flexibly, accurately, efficiently, and appropriately" in solving mathematical problems (NRC, 2001, p. 116).

NCTM (2000) recognized the importance of struggle in learning, recommending that students be provided appropriately challenging tasks, fostering their self-efficacy and belief in "their ability to tackle difficult problems, eager[ness] to figure things out on their own, flexib[ility] in exploring mathematical ideas and trying alternative solution paths, and willingness to persevere" (p. 21). Moving students from conceptual understanding of mathematical concepts to fluency in connecting mathematical knowledge in procedural constructs, one must then consider how we build mathematical knowledge. When students are not afforded the opportunity to learn mathematics with understanding, their chance for future success is greatly inhibited. In this, an essential consideration is that mathematics instruction must include the opportunity for students "to engage in productive struggle as they grapple with mathematical ideas and relationships" (NCTM, 2014).

Students respond to the learning culture that we foster, recognizing and valuing what we build as the teaching and learning expectations. Consequently, a teacher's acknowledgement and understanding of the role of productive struggle in learning, with a commitment to providing students the opportunity to grapple with learning while finding appropriate strategies, influences the "ultimate shape and form of the task" and learning therein. This requires that mathematics classrooms be structured in a manner that provides students with opportunities to understand and value the role of struggle in the learning process, to see it as productive (Carter, 2008). Students are more likely to benefit from the opportunity to struggle with mathematical ideas and feel success if they can understand that the struggle is a part of the learning process (NCTM, 1980, 2000, 2014). Engaging students in productive struggle as a learning opportunity must be considered with intentionality.

Understanding this phenomenon requires investigation of the teaching and learning of mathematics holistically, considering the role of anticipation, planning, teaching, and teacher response to student struggle that supports this learning dynamic. Warshauer (2011) investigated the impacts of productive struggle on learning in middle school mathematics classrooms, with the incorporation of tasks that "require[d] non-algorithmic and complex thinking, hav[ing] a greater likelihood of causing struggle among the students" (p. 30). As a result of this investigation, Warshauer (2011) identified four ways that the tasks elicited visible struggle in the students, while also identifying teacher responses to these struggles that either afforded or constrained the opportunity for learning with productive struggle. These findings provide important guidance on teacher responses to student struggle, to ensure that learning opportunities are not diminished, in manners that maintain cognitive demand.

While the phenomenon of productive struggle in learning mathematics has been investigated with positive implications in sixth and seventh grade (Warshauer, 2014), understanding the implications of productive struggle in elementary mathematics is an important consideration as well. Expanding upon the efforts of Warshauer (2011) to understand the implications of productive struggle on learning at the intermediate elementary level is necessary, as students at this level must develop a deep knowledge and proficient skills associated with rudimentary operations of elementary arithmetic, as this competence "provides an important foundation for further education in mathematics and in fields that use mathematics" (NCTM, 2000, p. 2). Investigating how elementary mathematics teachers anticipate, plan, teach, and support students as they encounter struggle while learning mathematics, can provide valuable and holistic insights into the opportunity afforded or constrained for learning with productive struggle, during learning at this pivotal time.

Investigations into the teaching and learning of elementary mathematics have provided important guidance to enhance students' learning and understanding of mathematics, including the use of tasks that elicit problem solving and learner engagement; questioning techniques to funnel or focus student learning efforts; and the use discourse to illuminate student thinking and solution pathways. While noteworthy, these investigations fail to demonstrate the potential of synergistic effects of anticipation, planning, teaching, and response to student struggle in learning mathematics. Understanding the synergistic effects of teaching and learning with productive struggle, provides a consideration worthy of exploration in these practices.

Purpose of the Study

The purpose of this study is to understand the potential impacts of anticipating, planning, teaching, and responding to struggle in learning. While the benefits of productive struggle in learning mathematics are well documented, this cannot be left to chance. Understanding the synergistic impacts of anticipating struggle, planning strategies to support this struggle in a manner that fosters productive struggle, and implementing these strategies as struggles arise during learning, holds the potential to further inform our practices related to teaching and learning mathematics.

Anticipation is an integral consideration in planning mathematics instruction. Educators must consider what students know, understand, and are able to do related to the mathematics topic and should consider "prerequisite skills, big ideas, and [potential] problem-solving strategies" (Hudson et al., 2006, p. 26) to support engagement opportunities that foster productive struggle. Additionally, the anticipation of student misconceptions and potential struggles are important considerations in our instructional design of mathematics. NCTM (2014) emphasizes the importance of anticipating potential struggles and misconceptions in learning, in order that "teachers [can] plan ways to support students productively without removing the opportunities for students to develop a deeper understanding of the mathematics" (p. 49). A student's learning of mathematics is influenced by the opportunity to build understanding through the activities and engagement designed by the teacher. In this planning process, careful anticipation of "how students might mathematically interpret a problem, [and] the array of strategies – both correct incorrect – that they might use to tackle it" (Smith & Stein, 2011, p. 8) will afford greater opportunity for the teacher to respond in a manner that supports students' understanding of a concept and response to struggle in a more productive way.

While the teaching of mathematics with productive struggle often relies on the use of rich math tasks, students will also demonstrate struggle when building understanding of foundational concepts within a typical lesson. Student learning hinges on engagement with mathematical concepts in manners that support the opportunity to build conceptual understanding and skill, while eliciting flexible solution strategies and critical thinking (Franke et al., 2007) during these learning opportunities. The opportunity to investigate how a teacher anticipates student needs, plans for those needs, reacts to student struggle, and what they believe about the role of struggle in learning mathematics will provide valuable insights into the synergistic impacts, providing the opportunity to understand how these responses might afford or constrain the opportunity to learn with productive struggle. A qualitative semi-structured interview study was conducted to illuminate how teachers afford or constrain the opportunity for students to struggle productively in learning fractions. This method was selected as "interviews are best suited for understanding people's perceptions and experiences" (Blandford, 2013, 6.4 Semi-Structured Interviews, para.

2), while making the accounts of practices accessible (Flick, 2009). Fractions have been chosen as an instructional topic to focus this study "because of their inherent role in more advanced mathematics, the strong predictive relation between earlier knowledge of them and later mathematics achievement, and the difficulty many children and adults have in learning about them" (Siegler et.al., 2013, p.13).

Research Questions

The dynamic nature of teaching and learning, with iterative patterns of interaction, required the investigation of this phenomenon in a holistic manner. Interviewing teachers to learn about their personal experience as mathematics educators and the processes they engage in related to instructional design, teaching, and response to student struggle, provided the opportunity to understand the impacts of each and the potential they hold to afford or constrain students' opportunity to learn with productive struggle. As a result of these considerations, the following questions guided this work:

Research question:

What role does productive struggle play in the design and implementation of mathematics lessons?

Attendant questions:

How do teachers perceive their role and the role of students as it relates to learning with productive struggle?

How do teachers prepare for anticipated struggle when planning for mathematics instruction?

How do teachers respond to evidence of struggle in student learning? Do the response(s) have the potential to invoke a productive struggle for students?

Each of these questions informed the semi-structured interviews, data collection, and data analysis in each individual participant "retain[ing] the integrity of each case in its entirety [before] compar[ing] or synthesiz[ing] any within-case patterns across cases" (Yin, 2018, p. 268).

Overview of Methodology

This research is a qualitative semi-structured interview study that used multiple participants to study the role of struggle in the teaching and learning of fractions. Three semi-structured interviews provided the opportunity to document each teacher's engagement with the phenomenon, as they shared their experiences in anticipating, planning, and teaching. Additionally, five stimulus prompts in the second interview provided a context to illuminate how student struggles in learning fractions elicited specific teacher responses, resulting in identified themes and patterns of responses, that build understanding and insight into this phenomenon. Conducting this interview study with multiple participants provides greater opportunity to investigate the research questions more thoroughly, with an opportunity to understand individual experiences, provide greater precision in defining the findings, and increase the trustworthiness across participants (Miles & Huberman, 1994).

Significance of Study

Productive struggle in learning mathematics is recognized as an important aspect of student learning and engagement to build students' conceptual understanding. Similarly, anticipation of potential student misconceptions and struggles, as well as planning and teaching with these in mind, are practices utilized by teachers to guide and engage students in learning mathematics with understanding. Understanding the synergy these practices afford teaching and learning, with a focus on the impact and opportunity students are afforded to struggle productively, could provide meaningful guidance on teaching practices that ensure productive learner engagement in order to understand mathematics. It is the belief of the researcher that by investigating the holistic impacts on teaching and learning when struggle is a focus, the potential exists for this work to supplement current efforts to understand how we can support productive struggle in learning.

Definition of Terms

While terms in this work will be familiar to some in the field of mathematics, the application of these terms can vary. Definitions of the following terms will impact the collection, interpretation, and dissemination of the findings in this work:

Productive struggle – struggle that a student engages in as they continue to grapple with a developmentally appropriate mathematical idea while making meaningful progress, and without ownership being eliminated in order to construct conceptual understanding (Hiebert & Grouws, 2007)

Affordance – an action or inaction on behalf of a student or teacher during learning that provides the opportunity for student ownership of a problem situation

Constraint – the opposite of learning opportunity, an action or inaction on behalf of the student or teacher, that creates a barrier to learning

CHAPTER 2

REVIEW OF THE LITERATURE

The purpose of this study is to understand the potential impacts of anticipating, planning, teaching, and responding to struggle in learning. Understanding the synergistic impacts of anticipating struggle, planning strategies to support this struggle in a manner that fosters productive struggle, and implementing these strategies as struggles arise during learning, holds the potential to further inform our practices related to teaching and learning mathematics.

The literature reviewed will provide a historical perspective on mathematics teaching and learning that led to reform efforts in mathematics teaching today. Within these reform efforts, insights into the importance of productive struggle in learning mathematics, teaching philosophy, advances made in teaching methodologies that foster students' learning of mathematics with understanding, student engagement, and teaching practices that hold the potential to ensure the opportunity for students to learn mathematics with productive struggle are explored. While the importance of teaching and learning mathematics with an opportunity for struggle is recognized by many, the purpose and potential of productive struggle in learning mathematics does not hold a shared significance for all who teach mathematics, nor a common vision of how to ensure that the potential is realized. With the belief that all people should acquire mathematical literacy, the synergistic implications of teaching practices that support this opportunity are considered.

Reform Efforts

Within the last 100 years, the focus of discussion in mathematics education has shifted from who *needs* to learn mathematics, to *how* mathematics should be taught in order that all students have the opportunity to learn mathematics (Kilpatrick & Stanic, 1995). The history of mathematics in our country has greatly impacted our journey, influencing where we are today.

Initial beliefs that we should only teach those who were believed to be capable or interested in mathematics resulted in a diminished emphasis on the subject, often "throw[ing] out mathematics altogether or mak[ing] it an elective" (NCTM, 1921, pp. 1-2). It was not until the late 1950s that the launch of Sputnik by the Soviet Union spurred a response by the U.S. government who "responded by dramatically increasing its support…to improve school mathematics" (p. 4). While seemingly insignificant at first, our beliefs which prevented some from learning mathematics, provides an important perspective on our perceptions related to struggle in learning mathematics.

Ensuring that all students would have the opportunity to learn mathematics was an important decision, motivated by the recognition that learning mathematics was a necessity of purpose. Unfortunately, a directive to provide mathematics instruction for all failed to recognize that all students can and should learn mathematics. The "new math" which followed did not last very long as the perception that students were not mastering the basic skills or prepared to grapple with the mathematics in a manner that focused on thinking and problem solving (Schoenfeld, 1992), was further complicated by "complaints that the new math was too abstract, impractical, and confusing" (Stanic & Kilpatrick, 1992, p. 413). The struggle related to teaching and learning mathematics this way led to a change once more in how mathematics was taught and the "back-to-basics" movement of the 1970s, with an emphasis on rote memorization and computational fluency, ensued.

This historical perspective provides insight into the present state of mathematics education, as the emphasis on memorization and computational fluency as focal points in instructional practice did not result in "developing the functionally *competent* student that all desire" (NCTM, 1980, p. 5). The understanding that "what is learned relative to a topic, how long it is retained, how readily it is applied – all…depend on the learning process the students pass through and how effectively they are engaged in that process" (p. 11) moved the mathematics community to a re-envisioned approach to instruction, with an emphasis on problem solving. Interweaving the mathematical knowledge, skills, and concepts recommended by NCTM (1980), in the context of problem solving, still informs our teaching practices today as we aspire to realize the vision of mathematical literacy for all.

Learning Mathematics

Learning of mathematics is a process in which students develop knowledge, skills, understanding, and the ability to make connections between concepts. An important yet often overlooked consideration in this process is students' development of a positive disposition for learning and engaging with mathematics. Moving students from conceptual understanding to fluency in connecting and applying mathematical knowledge in procedural constructs, one must then consider how we build mathematical knowledge through learning experiences. When students are not afforded the opportunity to learn mathematics with understanding, their chance for future success is greatly inhibited (NCTM, 2014). To ensure the opportunity for students to learn with understanding, an essential consideration is that students "engage in productive struggle as they grapple with mathematical ideas and relationships" (NCTM, 2014, p. 48). Inherent in the process of learning with struggle is the opportunity afforded to students when struggle is expected and productive, structured in a manner that requires students to apply prior knowledge, skills, and understanding, as they make connections to new learning. The navigation of disequilibrium during learning by the teacher and students, as students struggle to learn or understand mathematical concepts, is an important consideration to explore.

In a mathematical perspective, Hiebert and Grouws (2007) defined struggle “to mean that students expend effort to make sense of mathematics, to figure something out that is not immediately apparent” (p. 387). Hiebert and Grouws’s explanation of struggle in student learning and understanding of mathematics is analogous to the work of Vygotsky and Piaget, where students are “solving problems that are within reach and grappling with key mathematical ideas that are comprehensible but not well informed” (p. 387). Vygotsky (1978) captured the pedagogical implications of this understanding, relating that

> pedagogy must be oriented not to the yesterday, but to the tomorrow of the child's development. Only then can it call to life in the process of education those processes of development which now lie in the zone of proximal development. (p. 92)

Understanding of the aforementioned learning theories should compel teachers to engage students in learning opportunities that elicit productive struggle, with students understanding and expecting struggle as a necessary aspect of learning and building understanding. Unfortunately, the reality of mathematics learning is often devoid of such opportunity for students to struggle toward understanding and discovery, as Albers and Anderson (1985) reflected, “too often like walking on a path that is carefully laid out through the woods; it never comes up against any cliffs or thickets; it is all nice and easy” (p. 42). While the removal of struggle depicted in this description of learning mathematics is often perceived as a more effective and efficient approach, teachers’ removal of struggle through guided teaching or diminished cognitive demand (Henningsen & Stein, 1997; Stein et al., 1996) becomes a barrier to student learning opportunity.

Engagement with Mathematics

With a goal of mathematical proficiency for all learners, the recognition that learning mathematics is not an isolated engagement of a student with their mathematics, but rather the complete interaction of teacher, student, and learning task, as students engage in the sense-making process is essential (e.g., Cohen et.al, 2002; Kilpatrick et. al, 2001; Lampert, 2001). Conceptualizing how to ensure student learning and understanding in this dynamic is paramount. As a result, the instructional strategies a teacher chooses, as they navigate the questions and thought processes of students who struggle in their learning is consequential (Leinhardt & Steele, 2005).

The opportunity for students to learn with productive struggle is predicated on engagement with mathematics that affords the opportunity for thinking, reasoning, sense-making and problem-solving (NCTM, 2014). While struggle is a natural occurrence when students engage in learning and building understanding of mathematics, it should be devoid of "needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems…[leading to] feelings of despair that some students can experience when little of the material makes sense" (p. 387). Thus, the dynamic between a teacher and student during times of struggle must be understood, because "different kinds of teaching facilitate different kinds of learning [with] a direct consequence of opportunity to learn" (Hiebert & Grouws, 2007, p. 380). The opportunity for a student's struggle to be productive is greater when potential struggles are anticipated by the teacher, struggle is situated to provide the opportunity for productive struggle during learning, the teacher provides productive responses to the student struggles that emerge, and the teacher values learning mathematics in a manner that affords productive struggle.

Opportunity for students to learn with productive struggle and build understanding is constructivist in nature. In this structure of learning, students experience "an active process, in which each student builds his or her own mathematical knowledge, coupled with feedback from peers, teachers and other adults, and themselves" (NCTM, 2014, p. 9). Piaget's learning theory recognized that students construct their knowledge through assimilation, incorporating new ideas with pre-existing knowledge and understanding, or accommodation, where new knowledge is acquired through a reorganization of pre-existing knowledge and understanding, to allow for new learning to take hold (Blake & Pope, 2008). As students learn in a constructive manner, they are engaged in the opportunity to "connect new learning with prior knowledge and informal reasoning and, in the process, address preconceptions and misconceptions" (NCTM, 2014, p. 9). Simultaneously, Vygotsky's sociocultural learning theory necessitates the building of knowledge through social means, encompassing interactions between students and their teacher, between students in a classroom, and individual student engagement with the mathematical concept being explored. By "construct[ing] knowledge socially, through discourse, activity, and interaction related to meaningful problems" (NCTM, 2014, p. 9), both Piaget's and Vygotsky's theories of knowledge construction support meaningful mathematical learning opportunities. As a teacher uses strategies like affording time and opportunity for students to engage in discourse and questioning, while scaffolding learning and fostering connections as students grapple with mathematics concepts, student learning is fostered (Blake & Pope, 2008).

When struggle is externalized by students, the opportunity for teacher response to the student struggle in a productive manner emerges. Warshauer (2011, 2014) identified categories of student struggles, noting student difficulty in determining meaningful solution strategies, procedures, or concepts to guide them to an entry point into a problem; how to progress through a solution pathway to an answer or model their thinking and reasoning; how to explain their thinking, reasoning, solution strategy or models; or how to reevaluate their thinking when an answer does not match the response criteria. When one of the noted struggles occurs, the student may externalize their struggle as they express difficulty in starting a task, confusion in completion of a task, or ask questions when an answer does not fit the parameters of the required response (Borasi, 1996; Inagaki et al., 1998).

Background on Productive Struggle

The term *struggle* often elicits a personal response, conceptualized through an individual's experience with great effort in accomplishing a task. In learning mathematics, the preferred conceptualization of struggle comes from work by Hiebert and Grouws (2007) "to mean that students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). Struggle in this manner should be productive, not "needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems" (p. 387). Supporting student engagement in learning and building understanding, in a manner that affords the opportunity to grapple with key mathematical ideas that while not yet well informed, are within reach, is inherently productive (Hiebert & Grouws, 2007).

Teaching and learning perspective grounded in the construct of cognitive dissonance promotes intellectual growth, affording the type of learning opportunity where productive struggle is fostered. Piaget described learning as a process through which we experience, or are exposed to, something new and must adapt to this knowledge through either assimilation or accommodation, to find understanding (Piaget, 1960). Similar perspective is found in work by developmental psychologist Lev Vygoysky, who believed that learning occurs in the Zone of Proximal Development, with new learning informed by previous understanding, yet with consideration of future learning (Vygotsky, 1978). Acknowledgement that learning involves a degree of struggle, and that when this struggle is supported with appropriate scaffolding it can be productive, mirrors work from other researchers into the benefits and necessity of struggle in learning (e.g., Dewey, 1926; Stein et al., 1996).

Struggle in Learning Mathematics

Learning mathematics requires the ability of a student to access and apply previous understanding to novel tasks. However, the application of previous knowledge to novel tasks is not an automatic process for learners and often requires some level of struggle to make connections and build understanding. As a construct, *struggle* often elicits a personal interpretation, predicated on an individual's experience when great effort was required in accomplishing a task. While the purpose of struggle in learning is acknowledged by many, implementation is complicated by educators of mathematics whose own construct of struggle is a negative one. In mathematics, struggle is often perceived as an indication of a deficit, resulting in students' disengagement with the content and a teacher lowering the cognitive demand of a task through the removal of the challenging aspects (Henningsen & Stein, 1997; Stein et al., 1996). The implications for teaching approaches that remove struggle completely, in an effort to support students when they struggle, are enduring.

Student struggles often occur during learning when students are expected to solve a problem using previous knowledge in novel applications (NCTM, 2017). The relationship between a student's success and their ability to determine the appropriate strategies needed for success, the application of these strategies to tasks based on previous experience, and the attribution of difficulties to ineffective strategies rather than ability have previously been investigated (e.g., Kitsantas et al., 2004; Palmer & Wehmeyer, 2003; Cleary & Zimmerman, 2004). It was determined that teacher modeling of thinking, reasoning, and problem-solving, prior to the expectation of independent success, is imperative. In doing so, students are provided an approach to solving problems, supporting their ability to engage in the solution process with a more critical lens, and ultimately effective strategies to find success.

NCTM (2014) has provided explicit guidance that effective teaching provide opportunity for students to engage in productive struggle in their learning. Effective teaching then requires intentional consideration of the relationship between teaching, engagement, and the expected learning, reflected in Hiebert and Grouws (2007) assertion that "different kinds of teaching facilitate different kinds of learning [with] a direct consequence of opportunity to learn" (p. 380). Thus, the opportunity for productive struggle in learning cannot be expected to happen organically, but rather is the consequence of a teacher's understanding that "*how* skills are taught is as critical as *which* ones are taught" (emphasis in original Kilpatrick & Stanic,1995, p. 10), with intentional planning that fosters learning with productive struggle.

Ultimately, when students encounter a struggle, barrier, or mistake, they must be expected to apply their previous understanding and skills as they think through the problem. Mathematical learning opportunity that builds competence with the goal of proficiency requires teaching that "supports students in struggling productively as they…delv[e] more deeply into understanding the mathematical structure of problems and relationships among math ideas, instead of simply seeking mathematical solutions" (NCTM, 2014, p. 48). Acknowledging that mathematics teaching and student engagement have a significant impact on student learning outcomes has resulted in efforts to better understand how we can provide such learning opportunities for students (Dewey, 1938; Franke et al., 2007; Hiebert & Grouws, 2007; NCTM, 2014; NRC, 2001; Stein & Lane, 1996). The relationship between teaching practice and learning outcomes suggests a potentially synergistic relationship between a teacher's anticipation of struggle, planning opportunity for and potential responses to struggle, their response to students who struggle in learning mathematics, and the potential to ensure productive struggle in learning.

Productive Struggle in Learning Mathematics

While struggle in learning mathematics is often perceived as an obstacle, struggle should be capitalized upon as an opportunity for learning, when the introduction of new ideas and understanding elicits this response (Piaget, 1960; Vygotsky, 1978). In learning environments that provide an expectation of learning with struggle, the opportunity for productive struggle in learning provides "an argument for a *delay of structure* in learning and problem-solving situations, be it in the form of feedback and explanations, coherence in texts, or direct instruction" (Kapur, 2009, p. 524). Such learning structure allows students to grapple with prior knowledge and experience, while determining a productive solution pathway, with teacher guidance and support, in a delayed manner. In the intermediate elementary grades, learning and understanding becomes more apparent through the application of foundational skills to novel learning and tasks, when students are provided an opportunity to struggle productively. The responsibility for creating and fostering a learning environment where struggle is expected, while also being supported productively, becomes the imperative of the teacher.

Studies on the elementary years of mathematics demonstrate the long-term implications of foundational knowledge and "have suggested that students develop deeper levels of understanding and increased capacity to solve appropriately complex mathematical problems when they participate in instruction focused on reasoning and sense making rather than on memorization and the mechanical use of procedures" (Stein & Lane, 1996, p. 51). These findings mirror those of more recent investigations founded on productive struggle in learning and understanding mathematical concepts. Hiebert et al. (1996) investigated the development of arithmetic in a second-grade classroom using problem-solving as an approach to learning. In instructional design focused on problem-solving, mathematics is "problematic…allowing students to wonder why things are to inquire, to search for solutions, and to resolve incongruities" (p. 12). With this model of instruction, students were encouraged to develop solution pathways to tasks otherwise considered computational in nature. The productive struggle which ensued was a result of open-ended questioning, as well as student produced connections between the algorithmic solution model and a pictorial one. While struggle was not a focus here, an instructional design which fosters engagement in problem-solving has the potential for struggle to emerge.

Kapur (2012) investigated the notion of productive failure in learning mathematics. Similar to productive struggle, students engage in learning in a manner that provides opportunity to grapple with a solution pathway, prior to being provided formal instruction. Contrary to a traditional instructional model, where students are taught concepts and procedures in advance of solving problems that require them to activate prior knowledge and solve problems, students are expected to grapple with the unknown and discover meaning within the mathematical tasks. Kapur found that students engaging with the mathematics through productive failure generated significantly more solution strategies, engaged in deeper comparison between solutions, attended to critical features of the problems, and were better prepared to learn in the subsequent instruction phase (Kapur, 2012). Findings from Kapur's work are similar to those of Ball (1993), Lampert (2001), and Schoenfeld (1998) where student engagement in rich mathematics tasks that elicit struggle provide greater opportunity for students' development of conceptual understanding.

In more recent work, Warshauer investigated the types of struggles encountered by students engaged in problem-solving tasks in middle school mathematics. Categorizing observed student struggles, she determined that students' inability to get started, carry out a process, uncertainty in explaining and sense-making, or expressing misconceptions or errors were predominate struggles. Identification of student struggles allowed for analysis and categorization of teacher responses to student struggles, including telling, directed guidance, probing guidance, and affordance (Warshauer, 2014). Building on previous work related to productive struggle, Warshauer has provided opportunity for the mathematics community to consider and respond to the relationship between student struggle, specific teacher response(s), and the resulting student learning. Her work reinforces work of many in the exploration of struggles in learning mathematics.

Teaching Philosophy

Today we find two approaches to instruction that tend to dominate mathematics – a direct instruction approach and a problem-solving approach. A direct instruction approach aligns most closely with a traditional or transmissionist teaching philosophy. A problem-solving approach aligns with a constructivist teaching philosophy. Hiebert and Carpenter (1992) described some teachers as endorsing a top-down approach to teaching and learning mathematics; what one might find in a traditionalist's or transmissionist's classroom setting. Common to this teaching and learning dynamic is the belief that effective teaching requires a focus on students learning procedures, memorizing rules and facts, and practicing skills in a repeated fashion that have been modeled by the teacher first (Schoen & LaVenia, 2019). Conversely, Hiebert and Carpenter (1992) described other teachers who endorsed a bottom-up approach to teaching and learning; what one might find in a constructivist classroom setting. In this teaching and learning dynamic, students are encouraged to solve problems using strategies that require them to share their strategies, reasoning, and sense-making in a variety of ways (NCTM, 2020), as they construct understanding. In either methodology, the potential for students to experience difficulties in learning and building understanding of mathematical concepts exists.

With the imperative that students in the elementary level develop a solid foundation of knowledge and skills associated with basic operations of elementary arithmetic, it is important to investigate how teachers anticipate, plan, teach and respond when students struggle to learn these concepts and apply previous understanding to novel contexts. To begin the process of moving students from conceptual understanding to fluency in connecting their knowledge in procedural constructs, one must then consider how we build this knowledge. When students are provided meaningful opportunity to engage with challenging mathematics, which results in potential struggle, learning opportunity must be considered with intentionality. Such practice is supported in findings by Hiebert and Grouws (2007), which states that

> learners construct their own interpretations and memories of activities rather than simply absorbing them from the environment. When learners struggle (within reason), they must work more actively and effortfully to make sense of the situation, which in turn, leads them to construct interpretations more connected to what they already know and / or to reexamine and restructure what they already know…yield[ing] content and skills learned more deeply. (p. 389)

Learning environments that require students' active engagement in learning mathematics in this manner, with the expectation and acknowledgement of struggle as an important part of the learning process, hold the potential for students learn with productive struggle. Cleary's (2009) work recognized that without students' recognition of the role of struggle in learning mathematics, their focus on and interpretation of the struggles experienced may be external or uncontrollable factors. Acknowledging these impacts affirms that students need to recognize and value the role of struggle in learning mathematics; a dynamic and expectation set by the teacher.

> [A] general premise that classroom environments and instructional practices impact students' desire to engage in learning behaviors and activities…show[s] that classrooms that promote the use of diverse learning activities and seek to cultivate student autonomy, control, and interest in the learning process tend to have a positive effect on students' enjoyment, interest, and regulatory behaviors in school. (p. 155)

The responsibility for creating and fostering a learning environment where struggle is expected, while also being supported productively, becomes the imperative of the teacher.

Productive Struggle: Defining a Model to Investigate Teaching, Learning, and Understanding Synergistically

Productive struggle is not a new concept, but a term that defines an important aspect of learning mathematics. Warshauer (2011) investigated this "necessary component of learning mathematics with understanding" as she researched and expanded upon work by Hiebert and Grouws (2007). Recognizing struggle in learning as a need has provided a focus for the study reported here. The study reported herein expanded upon Warshauer's (2011) identification of student struggle and teacher response, exploring the impacts of a teacher's anticipation of struggle in learning, planning learning opportunities for and potential responses to struggle, teaching with the expectation and opportunity for productive struggle, and responding to elementary students struggles in fraction multiplication productively.

Anticipation

Teaching mathematics begins with the recognition of what students know, understand, and are able to do as it relates to engaging and learning a concept in a productive manner. While inherent in this consideration is prior learning opportunity, it also requires the consideration and anticipation of potential struggle in learning that could occur because of gaps or misconceptions, as well as how to respond to student struggles productively. Smith and Stein (2011) recognized the importance of anticipation, noting that

> Anticipating students' responses involves developing considered expectations about how students might mathematically interpret a problem, the array of strategies – both correct and incorrect – that they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn. (p. 8)

Anticipating learning needs and opportunities in a proactive manner is an essential consideration in planning efforts, providing the opportunity for more purposeful responses to students' struggle that allow for thoughtful consideration of "*how* to support students without taking away the challenge *before* they completely disengage in frustration (Stein et al., 2007, p. 352). Though anticipation of every potential student response to learning is an unrealistic expectation, without consideration of possible struggles, teachers must respond in a reactive manner, potentially resulting in a diminished task or funneling approach that inhibits or removes the opportunity for students' ownership of learning (AMTE, 2017; Chazen & Ball, 1999; NCTM, 1991; NCTM, 2014; NRC, 2001). The study reported herein noted the importance of anticipation of struggle in learning.

Planning

Opportunity to learn is not only predicated on the planning of engagement with mathematics in our classrooms, but how student engagement evolves in a productive manner, with a focus on students' building understanding of the concepts being learned. The planning process requires decisions that impact students' opportunity for learning, with a consideration of learning goals, tasks and activities, student needs, and teaching practices to ensure learning (Kilpatrick et al., 2001). While reform-oriented mathematics includes these planning activities, additional considerations of materials, classroom discussion, expectations for engagement, anticipation of potential problems, and consideration of how to navigate student problems are noted (e.g., Hiebert & Wearne, 1993; Kilpatrick, 2003; Stein et al., 2000; Superfine, 2008).

Learning opportunity must then consider both the *what,* or curriculum informed by standards, as well as the *how* students must engage to build these connections. NCTM (2020) recognized this connection, asserting that "children [must be] active doers and sense makers of mathematics, who author and generate mathematical strategies and share their mathematical insights, not just become passive recipients of information" (p. 70) for this opportunity to exist. Student engagement with mathematics with that ensures learning with productive struggle then requires intentional planning with strategies that support the opportunity to learn with productive struggle, focusing on the *what* both the teacher and students are doing during learning engagement, as well as *how* it is implemented.

Historically, teacher planning was prescriptive in nature, emphasizing what the teacher would be doing over student engagement in learning (Warren, 2000). While Cazden (1988) coined the model of instruction called IRE, meaning Initiate – Respond – Evaluate, this model is considered more traditional or transmissionist in nature and has been found to be less effective in promoting student learning and understanding. Today, many reform-oriented classrooms follow a different planning model that incorporates more of the vision described by NCTM. Lesson planning with a focus on student learning and engagement may incorporate an indirect model where students: Engage, Explore, Explain, Extend and Evaluate mathematics concepts (Polly, 2017), or a four-phase problem-solving model, where teachers present the problem for the day, students solve the problem independently and then compare and discuss solution strategies, and the teacher sums up the lesson at the end, discussing the various solution strategies and their value (Fuji, 2016).

Teaching

The act of teaching mathematics, as well as the impacts of specific strategies to enhance learning, have been investigated with important findings and recommendations. While the areas considered in the study reported here are not comprehensive, a focus on specific teaching practices that support learning with understanding are explored. The specific teaching practices include problem-solving, task implementation, questioning, and discourse due to their significance in providing opportunity to learn with productive struggle.

Problem Solving

Mathematics has long been associated with thinking and reasoning; ostensibly connected. Stanic (1986) notes that even Plato believed "those who are by nature good at calculation are, as one might say, naturally sharp at every other study, and…those who are slow at it, if they are exercised and educated in this study, nevertheless improve and become sharper than they were" (p. 178). It is doubtful that Plato's statement intended to convey that such thoughtfulness was associated with memorization and rote skill, but rather with problem solving.

NCTM (1980) recognized the importance of broadening mathematics instruction to not only encompass, but rather focus on problem solving in our instructional efforts, as these skills are "essential to the day-to-day living of every citizen" (p. 2). While the recommendations put forth almost four decades ago by NCTM mirror those of today, we have not fully realized their potential. Problem solving, serving as a foundation for change in instructional focus, was present in the NCTM (1980) agenda which recommended that “problem solving be the focus of school mathematics in the 1980s” (p. 1) enduring today in our five interrelated strands of mathematical proficiency, as well as our standards for mathematical practice (NCTM, 2014). The development of skill is ever present as well, in both the second recommendation in the NCTM (1980) agenda that “basic skills in mathematics be defined to encompass more than computational fluency” (p. 1) with the expectation for learner engagement in mathematics today focused on “acquiring conceptual knowledge as well as procedural knowledge, so [students] can meaningfully organize their knowledge, acquire new knowledge, and transfer and apply knowledge to new situations” (NCTM, 2014, p. 9).

The instructional approach initially proposed by NCTM (1980), with the mathematics learning organized around problem solving, fosters the opportunity for students to deepen their understanding of mathematics as they utilize their computational skills in application. Refinement of the vision of mathematics instruction shared by NCTM and refined over the past four decades, approaches to learning mathematics have been guided by the expectations that students: make sense of the problem at-hand; make meaningful conjectures about the mathematics and strategies necessary to solve the problem; enact solution strategies, using the appropriate tools and models; and revise their initial strategies when a solution is inaccurate (NCTM, 2014).

Carpenter et al. (1989) investigated the effects of teaching which focused on problem solving that validated student thinking and reasoning, in a cognitively guided manner of instruction, finding positive impacts on student learning. As these classrooms emphasized “develop[ing] understanding by stressing relationships between skills and problem solving…[with] instruction buil[t] on students existing knowledge” teachers were better equipped to adapt instruction to meet student learning needs, with high expectations for student engagement (p. 525). Fostering these learning opportunities has focused efforts related to the teaching methodologies that support and ensure this dynamic learning structure becomes a reality, while affording students the opportunity to learn with productive struggle.

Tasks

Instructional approaches in teaching mathematics can vary greatly. Practices ranging from a focus on practicing skills in support of developing procedural fluency, to ones that incorporate problem solving tasks that support meaningful connections and the development of conceptual understanding, are common (Carpenter & Lehrer, 1999). Evidence suggests that "instructional environments characterized by an increased emphasis on thinking, reasoning, and problem solving" will foster a learning environment where "students develop deeper, more sophisticated understandings of mathematics" (Stein & Lane, 1996). Thus, a learning environment that provides the opportunity for students to engage in problem-solving tasks is consequential. With a goal of developing student understanding of mathematical concepts, how a teacher engages students in tasks to support their learning and concept development is of great consequence (Doyle, 1988).

Underlying the challenge of teaching with an emphasis on problem solving is the task with which a student engages to build understanding of a concept. While the recognition that "effective teaching of mathematics engages students in problem solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies" (NCTM, 2014, p. 17), creating this reality remains a challenge when standard curricular resources may not provide this level of learning opportunity (Hiebert et al., 1997; Schoenfeld, 1998; Stein et al., 2009).

Hiebert and Wearne (1993) investigated the impacts of addition and subtraction instruction, with an emphasis on construction of relationships, studying the impacts on student learning of strategies, procedures, skill development, and comprehension. Supported by task engagement, the instructional emphasis where "understanding of place value and grouping-by-10 ideas…encourage[d] students to develop procedures based on their understanding of place value," (p. 398) sat in contrast to instruction that engaged students in a traditional, algorithmic understanding of these concepts. The findings provided compelling evidence for the engagement of students in the former learning model, as it supports development of students' ability to develop "patterns of relationships among quantities in addition and subtraction word problems that are not typically addressed in instruction…[as] successful problem solvers of such problems have knowledge of these patterns, but the knowledge is tacit" (Putnam et al., 1990, p. 74).

The examples shared explore one domain of mathematics, though additional studies exist in other domains of mathematics, demonstrating positive relationships between task engagement and student learning (e.g., Ball, 1990; Chazen & Ball, 1999; Cobb et al., 1991; Rittle-Johnson et al., 2001; Lampert, 1986). When teaching provides opportunity for student understanding of mathematical concepts, engages students in solving complex problems, and involves instruction that focuses on reasoning and sense-making, understanding deepens (Carpenter et al., 1989; Fennema et al., 1993; Hiebert & Carpenter, 1992). Task selection and engagement, however, are only two considerations in building student understanding in a manner consistent with recommendations from NCTM (1980, 2000, 2014), with teacher strategies to support student engagement an essential consideration in providing opportunity for students to struggle productively in learning.

While many opportunities exist to study the phenomenon of teacher's anticipation, planning, teaching and response to struggle in the learning of mathematics, the mathematical topic of fractions was chosen to focus the study reported here. Children often experience difficulties solving rational number tasks, due to their abstract and sophisticated nature, and the expectation that students can draw from whole number concepts previously developed (Kieran, 1980). The recognition that students often find difficulty in moving from whole number concepts to fraction number concepts has resulted in efforts to better understand how we can teach this topic in a manner that builds understanding (e.g., Ball, 1993; Behr & Post, 1988; Empson & Levi, 2011; Hannula, 2003; Pearn & Stephens, 2007; Siegler et al., 2010; Siegler et al., 2013).

The expectation for teaching and learning of mathematics "that centers on the development of mathematical understanding and mathematical power – the capacity to make sense with and about mathematics" (Ball, 2009, pp. 157-158) is especially compelling in the area of fractions. Recognition that the opportunity for struggle in learning fractions is a motivating factor in the study herein, however, the long-term implications are of greater consequence (e.g., Fennell & Karp, 2017). Fractional understanding is "educationally important because of [its] inherent role in more advanced mathematics, the strong predictive relation between earlier knowledge of [fractions] and later mathematics achievement, and the difficulty many children and adults have in learning about them" (Siegler et al., 2013). Thus, studying the synergistic impacts of a teacher's anticipation of potential struggle, planned opportunity for and potential responses to these struggles, and teacher responses to struggles that emerged in learning fractions, has the potential to add to the field of knowledge on teaching fractions in a manner that ensures the opportunity for students to learn fractions with productive struggle.

Questioning

Problem solving as a basis for learner engagement, when enacted thoughtfully and intentionally, is a powerful model to ensure student learning and understanding. Acknowledging that "students can acquire conceptual understandings of mathematics *if* teaching attends explicitly to concepts – connections among mathematical facts, procedures, and ideas" must also acknowledge the recommended teaching strategies that support these efforts effectively (Hiebert & Grouws, 2007, p. 383). In classrooms that place an emphasis on problem solving and sense-making, the use of questioning as an instructional strategy that supports students effectively is one consideration.

The impacts of questioning have been investigated since the 1980s, following the NCTM (1980) agenda that called for changes to mathematics instruction, with a focus on problem solving. Perry et al. (1993) first investigated differences in the use of questioning between Japan, Taiwan, and the United States in first-grade mathematics classrooms, to understand the use of questioning in these settings, and the impact on student learning and achievement. Inherent in this effort was the examination of differences in the "degree to which teachers attempt to engage students in higher order thinking, through the kinds of questioning they ask students" (p. 31). Previous work related to learning had indicated that

> asking more conceptually challenging questions leads to better student achievement is that these questions engage children in integrative thought, which leads to better learning than either answering questions that deserve rote responses or passively taking in material. (p. 33)

While the results found little difference between each country when considering the use of questions that focused on rote computation or recall, larger discrepancies existed in teaching examples. Teachers in Asian countries demonstrated a tendency to engage students in developing understanding and connections between a problem, their experiences, and real-life applications. Contrary to this, examples from teachers in the United States demonstrated students engaged in solving mathematics questions with an arbitrary context (Perry et al., 1993). Relatedly, Asian students engaged with more problem-solving strategies and problems to build conceptual understanding, supported by higher level questioning strategies, than students in the United States (Perry, et al., 1993). While Perry et al. (1993) caution readers that the apparent findings of causal relationship are not definitive, it does recognize that students in these higher performing Asian countries "are challenged more by the questions teachers ask them than are U.S. students" (p. 39).

Efforts to better understand the impacts of questioning, in a similar regard, have led to categorization of the types of questions we ask of students and the potential benefits or drawbacks to each. Wood (1998) investigated the pattern of teacher questioning and the way these questions "serve[d] to constrain or enhance children's opportunities to actively construct mathematical meaning" (p. 170). Wood elaborates on Bauersfeld's (1980) work on questioning, which delineated the strategy of questioning into two patterns: funneling or focusing. In a funneling method, the teacher is guiding the student thinking in a specific direction, resulting in "the student trying to figure out the response the teacher wants" (p. 172) in lieu of making sense of the task themselves. This method of questioning results in students relinquishing the learning and instead focusing on the correct response the teacher is seeking (p. 171). The use of funneling questions has been found less effective in building student understanding, as it typically involves lower-level cognition, such as "gathering of information" (NCTM, 2014, p. 36).

Contrary to funneling is a focusing pattern of questioning, in which a teacher creates the opportunity for a student to engage in building understanding, while reflecting on their thinking and the thinking of their peers (Wood, 1998, p. 172). The learning potential in focusing questions is present in recommendations for purposeful questioning in effective instruction today, with "questions that encourage students to explain and reflect on their thinking as an essential component of meaningful mathematical discourse" (NCTM, 2014, p. 35). In focusing interactions, the teacher uses questioning that will illuminate student ideas, explore thinking and solution pathways in a manner that makes them visible to others, and supports the opportunity for all to deepen their understanding through meaningful mathematical connections (Wood & Hackett, 2017).

Investigations into the role and impact of questioning on the learning of mathematics has led to the categorization of questions by type and purpose (e.g., Boaler & Brodie, 2004; Chapin & O'Connor, 2007; Franke et al., 2009; Herbel-Eisenmann & Breyfogle, 2005). As a result of these efforts, recommendations for the use of intentional questioning to support learning have provided meaningful guidance on the type and purpose of questions to ensure student learning and engagement (e.g., Hiebert et al., 1996; Huinker & Bill, 2017; NCTM, 2000, 2014; Smith et al., 2018) with specific frameworks to guide teachers as they implement this practice (e.g., Fennema et al., 1997; Hufferd-Ackles et al., 2004). The positive implications from work in the area of questioning as an instructional strategy has informed current recommendations by NCTM (2014) that effective teaching include the use of purposeful questioning as a strategy to support students in advancing their reasoning, making sense of the mathematics, and making meaningful connections.

While the use of questioning to engage students in higher-level thinking, deepening their understanding and supporting meaningful mathematical connections, struggle in learning is still possible. While the studies mentioned do not explicitly address the use of questioning as an opportunity to support students in struggling productively in their learning, Smith et al. (2018) note how a teacher supported students as they struggled to make sense of a task by "ask[ing] questions to help determine what they understood about the problem situation and then [making] suggestions that would likely help them get a foothold on the problem" (p. 41). The impacts of questioning were recognized in the study reported here, considering the impacts on learning with productive struggle.

Discourse

An additional aspect of reform recommended by NCTM (1980), while not stated explicitly, discussed the environment in which "problem solving can flourish" (p. 4). In such an environment, students are encouraged to provide explanations for varied solution strategies, communicating their ideas with precise mathematical language (NCTM, 1980), alluding to the presence of discourse in these classrooms. Recognizing and valuing student ideas and contributions in their learning became an emphasis in work that followed and explored the role of discourse in student learning.

With an emphasis on problem solving as a means of developing mathematical proficiency, teaching practices and student engagement were redefined. Lampert (1990) noted how that "*knowing*, *revising*, *thinking*, *explaining*, *problem* and *answer* took on new meanings," (p. 38) with a shared responsibility between teachers and learners, in turn fostering this learning opportunity. Approaching instruction with an emphasis on problem-solving requires students to make conjectures, explain their reasoning, abstract mathematical properties, validate their findings, and revise their thinking based upon reflection and feedback from others (pp. 32-33). Development of this learning culture takes effort on behalf of the teacher and learners, as “it requires courage and modesty to expose one’s exploratory thinking to others in hopes that by engaging in the exchange of ideas in classroom discourse, one might end up with better ideas in the end” (p. 54).

Ball (1993) continued to explore the learning opportunity afforded through classroom discourse, with an emphasis on creating a classroom dynamic where the teacher and students “strive to be a learning *community* and also be a *learning* community” (p. 388) which values the insights and knowledge from each student, provides opportunity for students learn by grappling with difficult ideas, and with the opportunity to revise their thinking when necessary. While Ball recognized the dilemmas within such a classroom structure, she also recognized the potential in shifting our practices and the positive impacts on student learning. Franke et al. (2007) further illuminated challenges in bringing this concept to reality, with a focus on learning and the manner in which teachers go beyond talking, using strategies to “scaffold, monitor, and facilitate discourse around mathematical ideas in ways that support student learning” (p. 232).

Subsequent work details key functions for discourse: eliciting student thinking, supporting student-to-student exchanges, and guiding and extending the math in support of student learning (e.g., Boerst et al., 2011; Franke et al., 2015; Hufferd-Ackles et al., 2004; Staples & King, 2017; Stein et al., 2003). These overarching functions are further supported by recommendations to establish classroom norms for engagement, where student contributions are validated and valued (Cobb et al., 1993, p. 95). The incorporation of practices to effectively integrate student responses in discussion, including anticipating, monitoring, selecting, sequencing, and making connections between student responses (Stein et al., 2003, p. 2); and effective talk moves to foster students' sustained discourse efforts, including: revoicing, repeating, reasoning, adding on, and waiting (Chapin et al., 2009, pp. 12-17) are essential considerations in this learning dynamic. NCTM (2014) recognizes the importance and effectiveness of classroom discourse, incorporating this in their eight recommended teaching practices.

Previous work which provides in-depth analysis of classroom discourse, guiding the interpretation and understanding of the use of discourse described by participants. Franke et al. (2015) indicated that "one way students can productively struggle with the mathematics is through their communication with others – both through explaining one's own thought processes…and discussing other students' reasoning processes" (p. 127). The efforts of the study reported here intend to build on previous knowledge, describing patterns of discourse shared that afford or constrain opportunity for productive struggle in learning fractions.

Responding

Teacher response to student struggle offers insight into perceptions and beliefs related to struggle in learning mathematics. Bremholm and Skott (2019) found that mathematics teachers stated values related to teaching and student engagement, their professional values, held greater emphasis in practice than their subject-related values. Understanding teachers' values related to learning mathematics with struggle in juxtaposition to their response to student struggle, provides important insight regarding a potential relationship between their values and responses to student struggle in learning mathematics. Presenting participants with an opportunity for to respond to authentic, visible examples of student struggle in learning will provide the opportunity for this relationship to be illuminated. Examples of student struggles that included: struggle to engage with a mathematical task; struggle to complete a mathematical task; students' inability to explain or justify a solution; and inaccurate completion of a task provided context for the struggle in teaching and learning mathematics that teachers must consider. Each type of student struggle presented to participants has been defined.

Struggle to Engage with a Mathematical Task

The process of learning mathematics often involves engagement with tasks that reach beyond a student's immediate ability to recognize, analyze, and understand the goals of a task, requiring exploration and application of previously learned concepts (Warshauer, 2014). For engagement with a mathematical task to commence, a student must first determine a potential solution pathway, isolating important aspects of the prompt to determine a meaningful starting point. Determining a starting point is critical in learning mathematics, however, can be observed as a barrier to learning and growth when a student is unable to get started.

When a student is observed struggling with initial engagement in a mathematical task, they might express uncertainty or confusion, ask for guidance or help, or simply not do anything. These signs of struggle to engage with a mathematical task provide opportunity for a teacher to respond, supporting students in productive struggle towards engagement with the task and learning. The struggle encountered as students determine how to begin a task is an essential consideration in the first standard for mathematical practice, make sense of problems and persevere in solving them (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010). Pasquale (2015) reflected on the productive struggle we could anticipate with initial task engagement, noting that "when students labor and struggle but continue to try to make sense of a problem, they are engaging in *productive struggle*" (p. 2). Without students having the opportunity to engage in this way, learning opportunity diminishes.

Struggle to Complete a Mathematical Task

At times students will be unable to apply previous mathematical knowledge and understanding to a task, finding themselves at an impasse when application of knowledge reaches beyond a basic understanding. Students who struggle to complete a mathematical task will demonstrate similar observable behaviors to those struggling to begin, including asking questions or voicing confusion or doubt, as they "ultimately reac[h] an impasse in carrying out [a] plan" (Warshauer, 2015, p. 835). While a student's struggle to progress in a task may elicit a teachers' inclination to diminish the task or funnel a student towards an accurate response, such a response can be detrimental. Instead, a teacher should support the opportunity for a student to persevere in determining a meaningful solution pathway, with the opportunity to struggle productively. Ultimately, supporting students in productive struggle provides an opportunity to learn with understanding.

Students Inability to Explain or Justify Solution Strategies

At times, students will solve a math problem with an inability to clearly explain the solution strategies they have chosen, a model they have created, or how they derived an answer regardless of the accuracy of the response. With a goal of students learning math with understanding, teachers must hold the students accountable for the proof and justification of their work (Hanna, 2000; Knuth, 2002). Providing the opportunity and expectation for students to engage in the explanation and justification of their solutions with their peers is one way to ensure that these deeper connections result (Borasi, 1996; Hoffman et al., 2009; Sherin et al., 2000).

Inaccurate Completion of a Mathematical Task

At times, students will solve a mathematical task, discovering at the end that their solution is inaccurate. Discovery of an inaccurate solution can occur because of class discussion, teacher questioning, or revisiting the initial prompt to address the original question. Students who solve a mathematical task inaccurately often believe that they have found an accurate solution, however, are unable to "construct viable arguments" to defend their thinking and reasoning, an expectation of third standard for mathematical practice (NGA & CCSSO, 2010, p. 8). Struggles that follow often emerge as students struggle to determine how to revise current efforts towards an accurate solution pathway.

Significance of Struggle in Learning Mathematics

The goal of this study was to better understand the synergistic impacts of productive struggle in teaching and learning, specifically how each participant's anticipation of struggle in learning, planning opportunity for and potential responses to these struggles, and teacher responses to struggle in learning fractions impact the potential for learning with productive struggle. To better understand the phenomena experienced by each participant, a consideration of their beliefs related to the role of struggle in learning mathematics was considered.

Investigating the implications of productive struggle in the elementary grades provides important insights, as a student's long-term academic success is often predicated on the acquisition of foundational skills acquired in the elementary grades (Duncan et al., 2007; Hiebert & Wearne, 1996; Hunsader & Thompson, 2014; Kilpatrick et al., 2001). Understanding the importance of providing access to learning and understanding of mathematics with productive struggle is imperative. Polya (as cited in Stanic and Kilpatrick, 1988) believed "the same mathematics should be taught to all students because no one can know in advance which students will eventually use mathematics professionally" (p. 16). This belief illuminates an important truth which is often disregarded. We -- society, educators, and parents -- cannot definitively predict what the future holds for any individual. We presume to hold the knowledge of what the future holds, when indeed our bias can foster the reality. While the research here does not discuss what happens to students who are denied the opportunity to learn mathematics with productive struggle, we know that many students do not finish high school, go on to college or a technical school, and lack skills for professional and technical workforce opportunities.

Summary

Realizing the vision and charge set forth by NCTM almost four decades ago to teach mathematics in a manner that affords students the opportunity to problem-solve, grappling with concepts as they make connections between prior knowledge and novel tasks while building understanding, requires continuous reflection, analysis, and adjustment to our instructional practices. We have learned that in order to achieve mathematics proficiency for all, teachers must be willing to refocus their efforts, often times changing their practices. With a goal of developing mathematically proficient students, additional considerations including the use of meaningful mathematical tasks that elicit productive struggle are an essential consideration. Implementing tasks that afford the opportunity for struggle, with the goal of productive struggle in learning mathematics, requires intentional teaching practices such as those found in the eight teaching practices recommended by NCTM (2014). Understanding the learning opportunity afforded to students learning mathematics with struggle, how to support them productively in this struggle, and how one's beliefs impact their practices, can support efforts to continually improve practice in an effort to make this vision a reality.

CHAPTER 3

METHODOLOGY

This chapter begins by reviewing the purpose of this research study, a restatement of the research questions, and a descriptive explanation of the theoretical framework guiding the methodology. The subsequent sections provide an explanation and justification for the research methodology and methods, including the researcher's role, methods of data collection, and piloting efforts informing development of study instruments. An explanation and justification of the procedures in the analysis of data, interpretation of findings, and limitations demonstrates how the data collection process allowed for understanding of the research questions and phenomenon explored. Personal ideology and reflexivity in this qualitative study follow, with a final consideration reflecting on the credibility and trustworthiness of the findings.

Purpose, Problem Statement, and Research Questions Restated

The purpose of this study is to understand the potential impacts of anticipating, planning, teaching, and responding to struggle in learning mathematics. Understanding the synergistic impacts of anticipating struggle, planning strategies to support this struggle in a manner that fosters productive struggle, and implementing these strategies as struggles arise during learning, holds the potential to further inform our practices related to teaching and learning mathematics.

NCTM (2014) highlighted the need to support productive struggle as one of the eight highly effective teaching practices, suggesting that "effective teaching of mathematics consistently provides students, individually and collectively, with opportunities to engage in productive struggle as they grapple with mathematical ideas and relationships" (p. 21). The fostering of productive struggle in learning must be explored. Situating a teacher's stated practices and instructional responses to student struggle in learning in relation to their beliefs regarding the role of productive struggle in learning mathematics supported the analysis and interpretation of these responses.

A teacher's response to student struggle can either be an affordance that facilitates student ownership of the learning with struggle, while making progress toward understanding, or a constraint that removes student ownership of the learning process, diminishing the opportunity for students to grapple with mathematics, and undermining the efforts of students as they struggle to develop understanding (Ball, 1993; Doyle, 1988; Stein et al., 2000). The dynamic nature of teaching and learning compels one to investigate this process synergistically, with the opportunity to understand the relationship between a teacher's anticipation of student struggle, intentional instructional design that responds to/leverages this struggle, lesson implementation that fosters opportunity for productive struggle, and teacher reaction to examples of student struggle. Situating a teacher's instructional design, lesson implementation, and responses to examples of student struggle within their stated beliefs framed the analysis and interpretation of the data in this study, mitigating potential bias of data interpretation devoid of this perspective.

Previous research on the impacts of instructional design, with the consideration of teaching practices that include questioning, discourse, and productive struggle related to student learning, have been investigated. The research reported herein utilizes previous understanding gained, to illuminate the impacts and opportunity for learning mathematics with productive struggle. The study was designed to address the following research questions:

Research question:

What role does productive struggle play in the design and implementation of mathematics lessons?

Attendant questions:

How do teachers perceive their role and the role of students as it relates to learning with productive struggle?

How do teachers prepare for anticipated struggle when planning for mathematics instruction?

How do teachers respond to evidence of struggle in student learning? Do the response(s) have the potential to invoke a productive struggle for students?

Each of these questions guided the semi-structured interviews, data collection, and data analysis.

Theoretical Framework

The theoretical framework guiding this study is grounded in interpretivist theory. With a goal of understanding the experiences of the participants, in the context in which the phenomenon of productive struggle operates, an interpretivist framework is advantageous. This theory supports the ontological perspective that "reality is socially constructed, complex, and ever-changing," (Glesne, 2011, p. 8) and is dependent upon how each individual experiences the world around them. This study was designed with an interpretivist framework, allowing for multiple perspectives, experiences, and perceptions shaping a teacher's interpretation and incorporation of productive struggle in their mathematical teaching to be explored.

Interpretive research is best suited for situations where a researcher "seeks to understand the world in which [a participant] lives and work" (Creswell, 2007, p. 20) with the recognition and appreciation of individual experiences to build a deeper understanding of a phenomenon. Constructivist in nature, interpretivist approaches to researching a phenomenon must then ensure that the research efforts "provid[e] an understanding of direct lived experience instead of abstract generalizations" predicated on work that "elicit[s] and describe[s] these meanings and contradictions" (Glesne, 2011, p. 35). The value in this approach lies in the opportunity to explore the phenomenon in a manner that privileges participant experiences and the meaning they bring to this phenomenon.

The interpretivist framework has provided guidance in the design of this qualitative semi-structured interview study, ensuring that the reality and lived experience of each participant was interpreted and explained. Achieving a thorough understanding of the synergistic relationship of a teacher's anticipation, planning, teaching, and responding to examples of student struggle within these "socially constructed, complex, and ever-changing" (Glesne, 2011, p. 8) realities must then include opportunity for "description that goes beyond the mere reporting of an act (thin description), but describes and probes the intentions, motives, meanings, contexts, situations, and circumstances of an action" (Denzin, 1989, p. 39). This interpretivist approach provided the opportunity to understand how teachers define productive struggle, interpret examples of student struggle or impasse, and respond to this struggle in a productive manner.

Qualitative Research Methodology

Qualitative research is the most appropriate methodology to answer the questions in this study and provide understanding of the phenomenon. Qualitative studies utilize a systematic approach to exploring the qualities and nature of a phenomenon, within a context, to build understanding (e.g., Brantlinger et al., 2005; Creswell & Creswell, 2018; Merriam, 2009). Additionally, qualitative studies are an effective approach to understanding individuals' experiences and perspectives related to a phenomenon, an essential consideration when "qualitative researchers are interested in understanding how people interpret their experiences, how they construct their worlds, and what meaning they attribute to their experiences;" (Merriam, 2009, p. 5) an explanation that captures the essence of this study. In educational research, qualitative studies allow for the exploration of participants' attitudes, beliefs, opinions, and personal reactions to a contextualized phenomenon (Brantlinger et al., 2005). While determining the most appropriate qualitative research approach for the study herein, three common approaches were considered as potential approaches to understand the phenomenon: case study, phenomenology, and a generic approach.

Case study is most appropriate when the opportunity exists to study a phenomenon in its real-life context. Case studies utilize in-depth description and analysis, with multiple data collection methods, in a bounded system, over a period (e.g., Merriam, 2009; Miles & Huberman, 1994; Moss & Haertel, 2016). Initial efforts with this study included observation of the teaching and learning process, with preliminary conversations with three of the four participants in this setting. The Covid-19 pandemic interrupted efforts of the study, preventing continued observation of the phenomenon in its real-life context. As a result, as well as the time frame to conduct this investigation, case study was removed as a research option.

A phenomenological approach can also be used in qualitative research that investigates individuals lived experience. Phenomenology is a meaningful approach when the opportunity to understand a phenomenon as a shared experience is described by participants, utilizing in-depth interview techniques to build understanding (e.g., Creswell & Creswell, 2018; Merriam, 2009; Patton, 2015). While worthwhile as a methodological approach, the lack of a wholly shared experience limited the efficacy of this approach in its truest form.

A generic approach to qualitative inquiry is most appropriate when the research questions and phenomenon being investigated lacks exclusive alignment with a single approach (e.g., Blandford, 2013; Merriam, 1997; Patton, 2015). Qualitative inquiry of this nature is used to build understanding of a phenomenon in a manner like a case study or phenomenology, with the flexibility of gathering data from a variety of sources such as "in-depth interviewing, fieldwork observations, and document analysis…without framing the inquiry" (Patton, 2015, p. 155). With generic inquiry, the opportunity exists for the researcher to determine the most appropriate methods to answer the questions being investigated, utilizing a theoretical approach that informs the methodology rather than defining the structure (e.g., Blandford, 2013; Creswell, 2013; Kostere & Kostere, 2021; Merriam, 2009; Patton, 2015). Generic qualitative inquiry was determined to be most appropriate for this study.

Semi-Structured Qualitative Interview Study

Generic qualitative inquiry provided a meaningful approach to define the methods to study the phenomenon and investigate the research questions. To gain insights otherwise unknown to the researcher, the use of semi-structured interviewing supported the goal to "explore meaning and perceptions to gain a better understanding" (DiCicco-Bloom & Crabtree, 2006, p. 314) of the role of productive struggle in their teaching practices. The use of semi-structured interviews afforded the opportunity to engage with participants in discussion of the complex choices, actions, reactions, beliefs, and feelings related to the phenomenon that are not easily captured in other qualitative methodologies (e.g., DiCicco-Bloom & Crabtree, 2006; Budd & Kandemir, 2018; Lieberman, 1987). Utilizing semi-structured interviews afforded the opportunity to provide consistency between participants, while simultaneously providing the flexibility to respond to individual responses, pursuing interesting and unexpected ideas from participants (e.g., Blandford, 2013; Glesne, 2011; Merriam, 2009; Patton, 2015).

Each semi-structured interview provided a clear progression through the topic of conversation. The beginning of each interview provided reminders of the purpose, intent, and rights of the participant as we started our time together. This was followed by the opportunity for participants to ask any questions as we began, as well as for my own clarification of any unclear ideas shared in the previous interview. The progression of opening questions, to more in-depth core questions, provided a gradual immersion into the theme of each interview. Finally, each interview provided closure that included the opportunity for participants to ask any questions, and I provided insight regarding the focus of the next interview. This approach allowed for knowledge of each participant's experience with the phenomenon to deepen through conversation, founded on the trust and established relationship, as I gained their insight and wisdom on each topic discussed (Legard et al., 2003).

The creation of the semi-structured interview questions was also predicated on a semi-structured interview design, allowing for consistency in the open-ended questions used, with the potential for probing questions to illuminate unique aspects of each participant. Ultimately, the multiple phases of the interview guide development contributed to the credibility and confirmability of the study (Lincoln & Guba, 1985). Interviewing teachers in this manner ensured the opportunity to gain insight related to the questions guiding these efforts, providing a means to (a) explore how teachers perceive their role and the role of students in learning with productive struggle; (b) identify how teachers plan for anticipated struggle; (c) identify how teachers respond to examples of student struggle, with the potential to afford or constrain student learning with productive struggle; (d) and identify a potential relationship between teacher beliefs regarding the role of productive struggle in learning and their actions. The use of three semi-structured interviews with a stimulus prompt, as well as questions that illuminated this phenomenon from a variety of participant perspectives (knowledge, experiences, behavior, opinions, and values), supported the interpretation of the synergistic impacts of productive struggle on teaching and learning.

Development of Study Instruments

Study instruments were an essential consideration in gathering data that illuminated each participants experiences and engagement with the phenomenon. The initial design of this study provided the opportunity for in-person observation of teaching, interaction between teachers and their students, and the opportunity to debrief this observed engagement. In March of 2020, when the pandemic was beginning to spread rapidly and schools were closing, my opportunity to study this phenomenon in-person was eliminated.

The pandemic created an unanticipated barrier to the completion of my work, making it necessary for me to pivot to a different methodology. While interviews provide greater insight into the beliefs, thoughts, and actions of my participants in a more in-depth manner than observation alone can provide, I needed to determine additional structures to understand teacher response to student struggle in a more authentic manner. To better understand how teachers responded to student struggle, I created a structure that provided a similar opportunity to produce data for analysis. The stimulus texts I created mimicked "the phenomenon under study as interaction" (Toronnen, 2002, p. 347), orienting the teachers to the struggle in a manner that required them to interpret and respond to examples of student struggle in plausible scenarios. In this section I provide a detailed description of the development of the stimulus prompt and semi-structured interview protocols used in the study herein.

Development of the Stimulus Prompts

The stimulus prompt is an approach used in research that "selectively stimulate[s] elements of the research topic under study" (Hughes & Huby, 2002, p. 383). With a goal of understanding how teachers respond to student struggles in learning fractions, the stimulus prompts provided the opportunity for teachers to respond to authentic student struggles by sharing their thinking and reasoning related to their hypothetical responses. Interviewing teachers whose experience encompassed the teaching of fifth grade fraction concepts required the presentation of trustworthy examples to illuminate “crucial situations and/or events of the subject matter in their constituent contexts” (Törrönen, 2002, pp. 344-345).

Development of the stimulus prompt for this study required: the selection of a worthwhile fraction task; recruitment of students to engage with the task; selection of student responses to pilot with teachers; writing of interview questions to field-test; and field-testing of the interview questions in a manner that would provide an opportunity to “channel the interviewees to mimetic action, to identify and interpret whether the stimulus text represents the phenomenon under investigation truthfully and credibly…mak[ing] visible and concrete invisible feelings and layers of reality” (p. 354). These initial efforts informed the study reported herein, allowing for a more authentic context to engage participants, improved interview questions, and practice with interviewing strategies to illuminate ideas.

Fraction Task Selection

Selecting a fraction task to elicit student responses was the first step in contextualizing the opportunity for participant engagement. With a focus on intermediate level fraction knowledge and understanding, fifth grade standards provided the opportunity to select a task that required both conceptual understanding and procedural fluency of fraction concepts. When selecting a fraction task, consideration of the fifth-grade standards related to fraction multiplication, a cognitive demand analysis, and a task evaluation were utilized to determine the potential efficacy.

Tasks that require problem solving often elicit struggle or points of impasse for students, leading to the opportunity for teacher response to this struggle to be documented. An authentic mathematical task called *the Cornbread task* (see Figure 1) designed, and tested by Rumsey et al. (2016), was selected for this purpose. This task was evaluated using Smith and Stein's (1998) guide for examining the cognitive demand of tasks, as well as a task evaluation and selection guide. The use of these strategies supported selection of a task that required problem-solving and student engagement in *doing mathematics*, the highest level of cognitive demand noted by Smith and Stein.

The task was completed by students in grades four, five, and six to provide authentic representations of student struggles or points of impasse. The student responses used in the final interview protocol were selected from their work samples. Student work examples and verbal responses were selected based upon a clear representation of struggle, impasse, or misconception.

Identification of Students

Students eligible to participate in this portion of the study were fourth, fifth, or sixth grade students whose current math experience included working with fractions. The students selected for this portion of the study were determined through a convenience sampling process (Creswell & Creswell, 2018). While hypothetical student work samples and responses could have been created through previous experience of the interviewer, this authentic approach was deemed most meaningful.

The target population for this portion of the data generation were minors, requiring that the initial communication was sent to parents or guardians. The email sent to parents of eligible students explained this study and requested consideration of their child's participation. Included in this communication was a permission form (see Appendix A) required to be completed by parents interested in having their children participate. Additionally, a child assent form (see Appendix B) for students interested in participating in the study was provided.

Figure 1

The Cornbread Task

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

- How much does she pay for the cornbread she bought? Explain your reasoning.

The Cornbread Task adapted from Bussey, et. al., *Assessing Student's Understanding of Fraction Multiplication*, TCM, 2016.

Note: This figure shows the portion of *The Cornbread Task* ultimately selected for inclusion in the study.

Parents with interested students shared an address where consent forms were mailed, including a copy of the task that was completed at the time of the interview. To expedite the return of signed permission forms, a self-addressed and stamped envelope was provided. Once the forms were received, participants' parents or guardians were contacted via email to schedule a time for the student interview.

While the fraction task targeted fifth grade learning standards related to the multiplication of fractions, a broader range of potential participants supported the collection of a variety of work samples and responses. A total of ten families responded returning parent consent and student assent forms. Ultimately, eight students participated in this portion of the study with two students rescinding. The group of eight student participants was comprised of one female fourth grade student, four female and two male fifth grade students, and one female sixth grade student. Student data was collected individually during thirty-minute Zoom-recorded interviews (see Appendix C). Zoom recorded interviews were utilized in lieu of in-person interviews due to the limitations of the pandemic and provided the opportunity to collect real-time data while engaging with students as they completed the task. Parents were provided a stamped envelope to return the originals but had the option to share photos of their child's work via email if this was more convenient.

The selection of student responses used in the study was predicated on Warshauer's (2011) investigation that identified categories of visible student struggle including: getting started, carrying out a process, providing a mathematical explanation, and expressing a misconception or error. The incorporation of student struggles representative of the categories identified by Warshauer was intentional, allowing the opportunity for teacher participants to engage with a variety of student struggles, as well as providing an opportunity to elicit a variety of teacher responses to the struggles shared. Only a portion of the student work samples, and verbal responses were selected for use in the final stimulus prompts.

Development of the Semi-Structured Interview Protocol

The goal of understanding how a teacher anticipates potential student struggles, plans for those struggles, reacts to student struggle during teaching and learning, and how their beliefs about the role of struggle in learning mathematics framed the writing of the interview questions. With a qualitative approach to uncovering participants synergistic engagement with struggles in the noted areas, a focus on participants' experience, behaviors, knowledge, opinions, and values (Merriam, 2009; Patton, 2015) guided the development of questions to build understanding of the research questions.

The interview questions were informed by a small field-test, conducted with retired and in-service fifth grade teachers, in the winter of 2020. The decision to target fifth grade teachers was due in part to the goals of *The Cornbread Task* (Rumsey et al., 2016) which focused on fifth grade mathematics standards in fraction multiplication. All the teachers participating in the field-testing process had a minimum of five years teaching experience, with the group comprised of one male and two female teachers. The field-test simulated the real interview, informing final interview protocols and the amount of time required for each session (Krauss et al., 2009; Chenail, 2011). The field-testing led to more open-ended questions, with improved follow-up questions. The specific interview protocols are discussed in subsequent sections.

Participant Selection: Teachers

This study aimed to identify three to five teacher participants. The decision to investigate this phenomenon with multiple participants was predicated on the desire to increase confidence in the findings, by "looking at a range of similar and contrasting cases [in order] to understand single-case findings" (Miles & Huberman, 1994, p. 33) as well as holding the potential to understand *how*, *what*, *when*, and *why* the phenomenon emerges in these contexts. Participant selection was guided by purposeful sampling, best fitting the design and intent of this research study. Patton (2002) argued that "the logic and power of purposeful sampling lies in selecting *information-rich* cases for study in-depth…those from which one can learn a great deal about issues of central importance to the purpose of the inquiry, thus the term *purposeful* sampling" (p. 230, emphasis in the original). The essential criteria for the purpose of this study were that a minimum of three teachers from grade four, five, or six, with experience teaching fraction concepts, be included in this work. Ultimately, four teachers participated in this study.

Selection of potential candidates involved a three-step process. To determine opportunity and interest of potential participants, the selection process included (a) survey; (b) initial conversation; and (c) email with participant invitation or dismissal.

Survey

Potential participants were invited to participate in the study in an email communication. Given the nature of this work, approximately 25 teachers from five districts received the initial email communication with an attached interest survey link, created in Qualtrix. The number of potential participants was predicated on the need to elicit adequate potential participants (see Appendix D). The use of a survey in this manner allowed for self-selection into the process, decreasing the potential for loss of participants during the study.

Initial conversation

Once potential participants were determined from their responses to the survey, an email request for an initial conversation was sent (see Appendix E). The purpose of the conversation was to connect personally with potential participants and inquire further about their interest, discuss the use of recorded Zoom meetings for data collection, gain initial insights into their experience and teaching of mathematics, and to determine if schedules will match for the purpose of this work. Interested participants who met the criteria for participation and were accepting of the requirements for participation were invited into the study and interview schedules were established.

Participant invitation

The final step in participant engagement was garnering formal permission for participation. Potential participants were sent an email that provided a brief overview of the research commitments discussed in the initial conversation. Additional attachments to this email (see Appendix G) included an active permission form that the participants were required to sign to engage in this work (see Appendix F). While participants who committed to participating in the study received the form electronically, they were also presented a paper version of the form with a self-addressed, stamped envelope for ease of return.

Data Collection

The effort to build understanding of the synergistic impacts of participant's anticipation of student struggle in learning mathematics, planning with and for this opportunity, responding to examples of student struggle in learning mathematics, and participant's beliefs regarding the role of struggle in learning mathematics was best supported through semi-structured interviewing techniques (e.g., Blandford, 2013; DiCicco-Bloom & Crabtree, 2006; Kallio et al., 2016; Patton, 2015). Utilizing semi-structured, in-depth interviews "[allows] us to enter into the other person's perspective," predicated on the belief that this perspective is "meaningful, knowable and can be made explicit" (Patton, 2015, p. 426), providing the opportunity to access the thoughts, actions, and beliefs of the participants in their own words. Utilizing interviewing techniques in an interpretivist theoretical perspective, allowed for the “advance[ment of] tentative explanations or interpretations of what the respondent ha[d] been saying” with the opportunity to probe further and build a deeper understanding of the phenomenon. The addition of a stimulus text during the second interview supported the opportunity to reveal subtleties not apparent otherwise (Lieberman, 1987), allowing for explication of the pedagogical decisions the teachers shared. Representing a microcosm of student struggle, the prompts provided the opportunity for participants to “construct a sufficiently solid and enclosing description” (Törrönen, 2002, p. 354) of their responses to examples of student struggles when getting started, carrying out a process, providing a mathematical explanation, and expressing a misconception or error.

Gathering evidence from a variety of participants facilitated a deeper understanding of the phenomenon being studied, providing a rich exploration of the dynamic and synergistic impacts of productive struggle in learning mathematics. Data collection was strengthened by the opportunity to explore the phenomenon at a variety of stages in the teaching and learning process, from the anticipation of struggle in learning, to planning with and for struggle, to teaching and responding to struggle experienced by students. Additional insight gained from teacher’s shared beliefs regarding the role of struggle in learning mathematics illuminated the synergistic impacts being investigated. Merriam (2009) noted the power of an interview as a means of building understanding of a phenomenon, providing opportunity to “find out from [participants] what we cannot directly observe…[the] feelings, thoughts, and intentions” (p. 88) guiding their actions. This study isolated specific considerations impacting the opportunity for productive struggle in learning mathematics at various times in the instructional design and lesson implementation process, providing the opportunity for triangulation of corroborating evidence for each participant (Merriam, 2009; Stake, 1995; Yin, 2018). Throughout this research, data was collected and analyzed in a simultaneous process, where interview notes included reflective and analytic memos, with emphasis on pedagogical practices impacted by student struggle.

Interviews: Semi-Structured Protocols

Throughout the research process, semi-structured, in-depth interviews were utilized. The three interviews provided the opportunity to build trust and rapport with participants in a manner that simultaneously provided a progressive and deeper understanding of the role of productive struggle in their teaching and learning environments. The use of semi-structured, in-depth interviews supported the inclusion of "specific information…desired from all respondents" in order that data collection between participants maintained consistency, while having the flexibility to be responsive (Merriam, 2009, p. 90). These efforts required a more flexible approach to data collection that included probing questions, as well as the use of personal responses to connect and relate ideas shared to build understanding, as each participant's unique story emerged. The structured, yet open-ended interview questions provided the necessary means to gain related insight with broader and more personal understanding of the phenomenon.

Each interview started with a reminder of participant rights in this process, as well as agreements regarding data collection. This component of the protocol was essential, as participants could rescind their permission for the use of their data at any time, leaving the study. The use of recorded Zoom interviews was predicated on the limitations of social distancing at the time the data was collected. The recordings allowed for accurate transcription of the data. Discussion of the use of Zoom for data collection, contingency of permission to record, and the safeguards to ensure confidentiality were discussed prior to participants agreeing to be in the study. Additionally, each interview started with a brief recap of ideas discussed in the previous interview and focus for the current interview, providing a connection to the overall efforts of the study.

Interviews: A Qualitative and Interpretivist Approach

The consideration of interviewing participants with an interpretivist approach was intentional. In order to gather data that "capture[d] how those being interviewed view their world, to learn *their* terminology and judgments, and to capture *their* individual perceptions and experiences" (Patton, 2015, p. 442, emphasis in original) the types and structures of questions were an essential consideration in the three interviews that occurred. The questions were designed to gather evidence from a variety of perspectives ranging from experiences to actions, to beliefs, ensuring a more complete depiction of each participant's engagement with the phenomenon (Patton, 2015). In addition to the consideration of the types of questions that would illuminate participants' lived experience, the sequencing was an important consideration as well. The first interview provided insight related to a teacher's anticipation, planning, and teaching mathematics with the potential for struggle to emerge. The second interview expanded upon this understanding, exploring these ideas in the context of *The Cornbread Task* (Rumsey et al., 2016), providing the opportunity for teachers to respond hypothetically to examples of student struggle. The final interview provides perspective on participant's beliefs related to the role of struggle in learning mathematics.

Interview I: Anticipating Struggle in Learning

The first interview provided the opportunity to gain insight into the participant's teaching and learning of mathematics, with an opportunity to gain insight into the question: *How do teachers prepare for anticipated struggle when planning for mathematics instruction?* as an initial consideration (see Appendix H). This perspective is an essential foundation, providing a foundation from which to build understanding of this phenomenon.

Guided by the goals of learning, a teacher must consider their “planning for instruction, including the selection of tasks, the plan for implementing the task, and the preparation of key questions to check on students’ prior understanding related to the goal and to help move students toward the goal” (Huinker & Bill, 2017, p. 24). A consideration within this is that of anticipation (NCTM, 2014). While not asked directly, the anticipation of struggle is an essential consideration in instructional design with the element of struggle being a necessary component for students to learn with understanding (Hiebert & Grouws, 2007). As the interview progresses, participants are asked to expand upon their initial reflections related to instructional design of mathematics lessons. This progression of questions for Interview I (see Table 1) provides the opportunity for a connected story, as well as the opportunity to illuminate initial understanding related to the second portion of the research question *What role does productive struggle play in the instructional design and implementation of mathematics lessons?* This connection and exploration are an essential consideration, as Ball and Forzani (2011) posit that a student’s learning of mathematics “depends fundamentally on what happens inside the classroom as teachers and learners interact over the curriculum” (p. 17), which is illuminated in a teacher’s description of this iterative interaction of teaching and learning.

The interview concludes with a narrowed focus on the teaching of fractions. Fractions were chosen as a topic to explore directly “because of their inherent role in more advanced mathematics, the strong predictive relation between earlier knowledge of them and later mathematics achievement, and the difficulty many children and adults have in learning about them” (Siegler et al., 2013, p. 13). The questions in this final portion of the interview provide a foundation for the subsequent interview, while also adding to the understanding of the question, *How do teachers prepare for anticipated struggle when planning for mathematics instruction?* with the presentation of a fraction multiplication task.

Table 1

Questions and Supporting Rationale for Interview I

Question	Purpose	Research Base
Could you please describe how you plan for a typical mathematics lesson? Are there any strategies that you consider essential in your planning efforts?	How do teachers prepare for anticipated struggle when planning for mathematics instruction? These questions provide insight into participants' experience and behaviors.	Hiebert & Wearne, 1993 Kilpatrick et al., 2001 Kilpatrick, 2003 NCTM, 2014 Patton, 2015 Smith & Stein, 2011 Stein et al., 2000 Superfine, 2008
Could you please describe a typical mathematics lesson for me? Potential probing questions, as needed. What would I expect to see you doing? What would I expect to see students doing? As you reflect on student engagement in the classroom, what specific expectations related to this engagement do you consider to be essential for learning?	How do teachers prepare for anticipated struggle when planning for mathematics instruction? These questions provide insight into participants' experience, behavior, and values.	Dewey, 1938 Franke et al., 2007 Hiebert & Grouws, 2007 Kilpatrick & Stanic, 1995 NCTM, 2014 NRC, 2001 Stein & Lane, 1996

Could you please share the goals of your instruction related to fractions in [fourth, fifth, sixth]-grade? Potential probing questions, as needed. As you compare your preparation for fraction instruction to other topics in [fourth, fifth, sixth]-grade mathematics, is there anything that makes this topic unique? Earlier we discussed what a typical mathematics lesson would look, sound, and feel like. As you reflect specifically on the instruction of fractions, are there any differences that I would note if I were observing a lesson on fractions?	How do teachers prepare for anticipated struggle when planning for mathematics instruction? These questions provide insight into participants' knowledge, experience, and behavior.	Dewey, 1938 Franke et al., 2007 Hiebert & Grouws, 2007 Kilpatrick & Stanic, 1995 NCTM, 2014 NRC, 2001 Stein & Lane, 1996
While I understand that you have only had a few minutes to review this task, what are your initial thoughts on implementing this task in your classroom? Are there any specific strategies that you would use to facilitate student engagement with this task? What aspects of the task stood out to you? Potential probing questions, as needed. Why did these aspects of the task stand out to you? Are there any aspects of the task that you anticipate might cause impasse for students? How would you anticipate supporting students with this task if they experience impasse? Do you have any other initial thoughts related to this task that you would like to share?	What role does productive struggle play in the design and implementation of mathematics lessons? These questions provide insight into participants' experience, behavior, and opinion.	Ball, 1993 Cohen et al., 2002 Henningsen & Stein, 1997 Hiebert & Grouws, 2007 Kapur, 2012 Kilpatrick et al., 2001 Lampert, 2001 Schoenfeld, 1998 Stein et al.. 1996 Stigler & Hiebert, 1999

Interview II: Lesson Implementation and Response to Student Struggle in Learning

The purpose of the second interview (see Appendix I) was to provide insight and understanding related to a participant's teaching practices when a student demonstrates struggle or an impasse in their learning, answering the questions: *How do teachers respond to evidence of struggle in student learning?* and *Do the response(s) have the potential to invoke a productive struggle for students?* During this interview, the use of stimulus prompts provides the opportunity for participants to explain "their complex interactions, their inner feelings and thoughts" (Lieberman, 1987, p. 3) that are otherwise unknown. The use of a multiplication fraction task was intentional, considering the knowledge that "difficulty with learning fractions is pervasive and is an obstacle to further progress in mathematics and other domains dependent on mathematics, including algebra" (National Mathematics Panel 2008, p. 28).

The second interview provided an additional opportunity for participants to engage with the task, following a brief reflection at the end of the first interview. During the time between the first and second interviews, participants were asked to review the task, recording any thoughts, questions, or reflections on how they would plan to implement this task in their own classrooms. The interview started with this discussion, providing a starting point to move forward and review worked student samples. With an established perspective, teachers were then asked to provide some insight into their reflection on the potential for using *The Cornbread Task* (Rumsey et al., 2016) in their own classroom (see Table 2).

Presentation of the five student stimulus prompts followed the opening questions for Interview II. The intent of the stimulus prompts was to provide an opportunity for participants to engage with authentic student responses that indicated a variety of struggles. The stimulus prompts contextualized the responses to a hypothetical implementation of the task in each teacher's classroom, drawing from and incorporating ideas each had shared.

The examples of student struggles represented in the study comprise five of the eight student participants. The work samples selected represent a variety of the struggles demonstrated by the students, including the expression of misconceptions and errors, difficulty getting started, difficulty carrying out a process, and difficulty giving a mathematical explanation. While a portion of the stimulus prompts (see Table 3) provided examples of students written responses, additional verbal responses collected during the interviews provided a more authentic opportunity for teachers to engage hypothetically with the prompts.

Table 2

Questions and Supporting Rationale for Interview II

<table>
<tr><th>Question / Number Before Engagement with Stimulus Prompts</th><th>Purpose</th><th>Research Base</th></tr>
<tr><td>1. As we begin today, could you please share additional reflections from your engagement with this task, in preparation for our conversation.

Potential probing question, as needed.

That is interesting. Could you tell me more about [revoice the response indicated by the teacher].</td><td rowspan="2">How do teachers prepare for anticipated struggle when planning for mathematics instruction?

These questions provide insight into participants' experience and behavior</td><td rowspan="2">Dewey, 1938
Franke et al., 2007
Henningsen & Stein, 1997
Hiebert & Grouws, 2007
Kilpatrick & Stanic, 1995
NCTM, 2014
NRC, 2001
Stigler & Hiebert, 1999
Stein & Lane, 1996</td></tr>
<tr><td>2. Based on your experience, could you please describe how you would use this task with your students.

Potential probing questions, as needed.

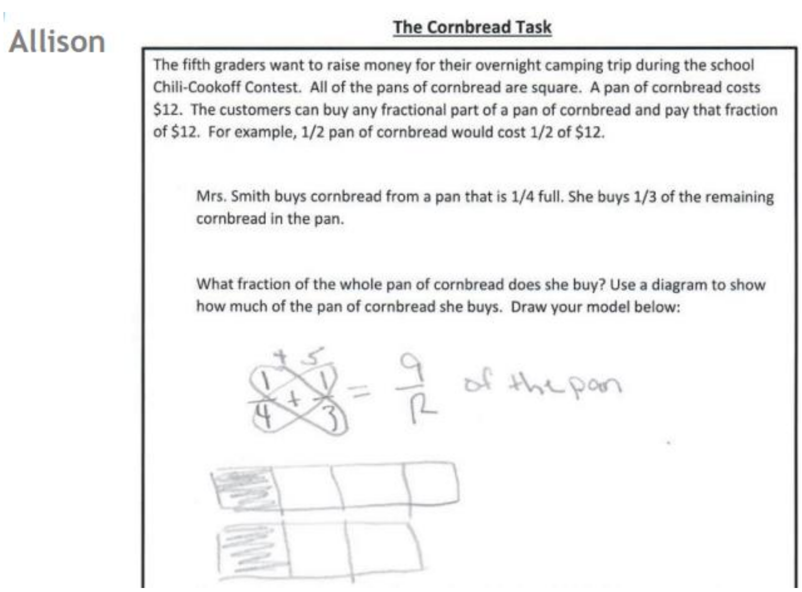
As you were thinking about your students, were there important considerations related to engagement with this task?

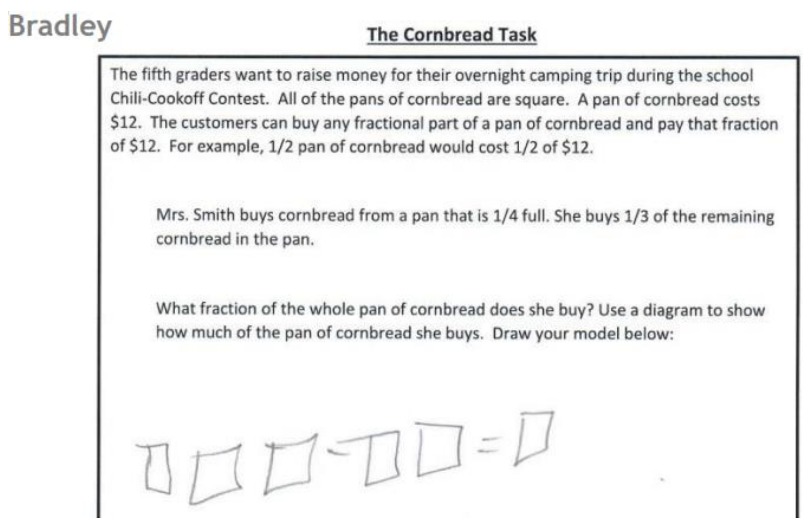
Could you tell me more about that?</td></tr>
</table>

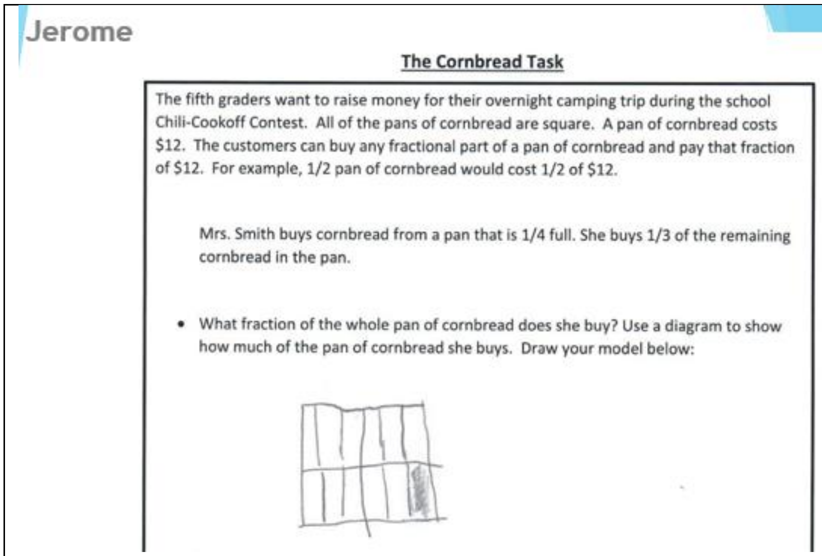
So, you noted that as a teacher that [revoice the response indicated by the teacher] was an important consideration and thinking of your students' [revoice the response indicated by the teacher] was important. With these in mind, could you describe what using this task would be like in your classroom?		

Table 3

Interview II: Student Struggle Stimulus Prompts

Stimulus Prompt	Interview Questions
Stimulus Prompt 1: ***Allison*** In this scenario, it is assumed that you are working with your students and you have [Answer from lesson structure previously indicated]. To ensure that all students have equal opportunity to engage with the prompt, you read the story context aloud for them and then ask if there are any questions about the story context. Seeing none, you ask students to go ahead and start in on the task. Following a few minutes for students to work on the task, you begin circulating around your classroom. The first student whose work catches your attention is Allison and she is done. Here is what you see:  You ask Allison to explain her response to the first prompt. Allison responds, "I drew two, like, the squares – I don't know what to call	Based on Allison's interpretation of the task, how would you respond? Potential probing questions, as needed. Why would you choose [Answer / approach]? With [revoice the response indicated by the teacher], how would you anticipate [student] would proceed? What would you consider as a potential cause of the student's difficulties? If Allison responded, "Well my teacher last year told me that we should always find a common denominator and so I cross-multiplied so I could add them together" how would you move forward from this? In the event that the student indicated an inability to

them, but I drew one-fourth and one-third and then I added one-fourth and one-third together to get nine-twelfths and then I labeled it ‘of the pan.’”	make sense of the task following your [revoice the response indicated by the teacher], what would be your next response?
Stimulus Prompt 2 – Part I: ***Bradley*** After leaving Allison, you come upon Bradley, you see that is just staring at his paper. As you pause by his desk, you ask him why he has not started. He responds “I can’t think. I’m trying to think of what I am going to draw.” Stimulus Prompt 2 – Part II: ***Bradley*** After Bradley (draws the square / reads the story problem aloud again / begins to solve the problem), you return to his desk where he is beginning to draw a model to move towards a solution. This is what you see:  Bradley The Cornbread Task The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12. Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan. What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below: You ask him to share his thinking with you. He says “Oh, yeah – squares” and then as he is drawing this [show image below] he is talking aloud “So, 1, 2, 3, 4…so one-fourth. Oh, wow, I’m surprised I didn’t think of that. So, [revisiting the problem] what fraction of the cornbread does she buy? Uh…let’s see. Uh, so I think she will buy one piece of the cornbread from the one-fourth pan full.”	Stimulus Prompt 2 – Part I: ***Bradley*** Based upon Bradley’s response, how would you support him? Potential probing questions, as needed: What makes [answer] come to mind? With [revoice the response indicated by the teacher], how would you anticipate the student would proceed? Stimulus Prompt 2 – Part II: ***Bradley*** With this new insight into Bradley’s thinking and reasoning, how would you proceed? Potential probing questions, as needed. Why would you choose [Answer / approach]? With [revoice the response indicated by the teacher], how would you anticipate [student] would proceed?
Stimulus Prompt 3 – Part I: ***Jerome*** You continue walking around the room and come upon Jerome. He is done with his work and this is what you see on his paper for the first task.	Stimulus Prompt 3 – Part I: ***Jerome*** Based upon Jerome’s response, what would you say to him?


Jerome

The Cornbread Task

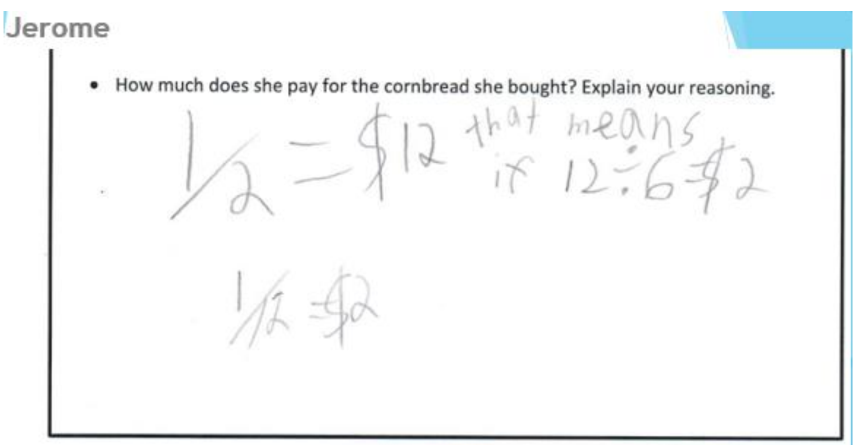
The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Stimulus Prompt 3 – Part II: ***Jerome***

While Jerome shares an accurate model, answer, and explanation of his thinking, you also noticed that he has answered the second part of prompt A with this answer.


Jerome

- How much does she pay for the cornbread she bought? Explain your reasoning.

1/2 = $12 that means if 12÷6 = $2

1/12 = $2

Potential probing questions/prompts, as needed.

You could acknowledge his response, would there be anything else you might ask him or you would want to say to him?

That is what I asked and Jerome responded, “Because all the pans are square I made a square and then I divided it into four because it says she buys cornbread from the pan that is one-fourth full, but she wanted one third. So, I divided them all into thirds and shaded in one; so, she bought one-twelfth of the pan.”

Stimulus Prompt 3 – Part II: ***Jerome***

How would you respond?

Potential probing questions, as needed.

Why would you choose [revoice the response indicated by the teacher] as an approach?

With [revoice the response indicated by the teacher], how would you anticipate [student] would proceed?

Stimulus Prompt 4 – Part I: ***Maria***

Maria is sitting near Jerome and her work catches you eye next. She has started creating a drawing to solve the problem but has not progressed beyond her first image. This is what Maria has drawn on her paper.

Stimulus Prompt 4 – Part I: ***Maria***

Given your experience, how would you engage with Maria?

Potential probing questions, as needed.

Maria

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Stimulus Prompt 4 – Part II: ***Maria***

(In response to teacher response, one of the following was used)

You provided Maria some more time to work with this task and when you came back around, this is what you saw:

After [revoice the response indicated by the teacher] Maria, she draws a new model from which to build and solve the problem. Here is her updated work:

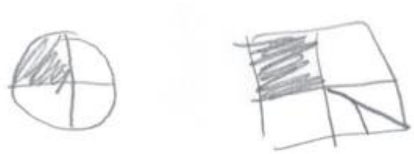
Maria

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:



Could you tell me more about your decision to use [revoice the response indicated by the teacher] approach?

With [revoice the response indicated by the teacher], how would you anticipate a student would respond?

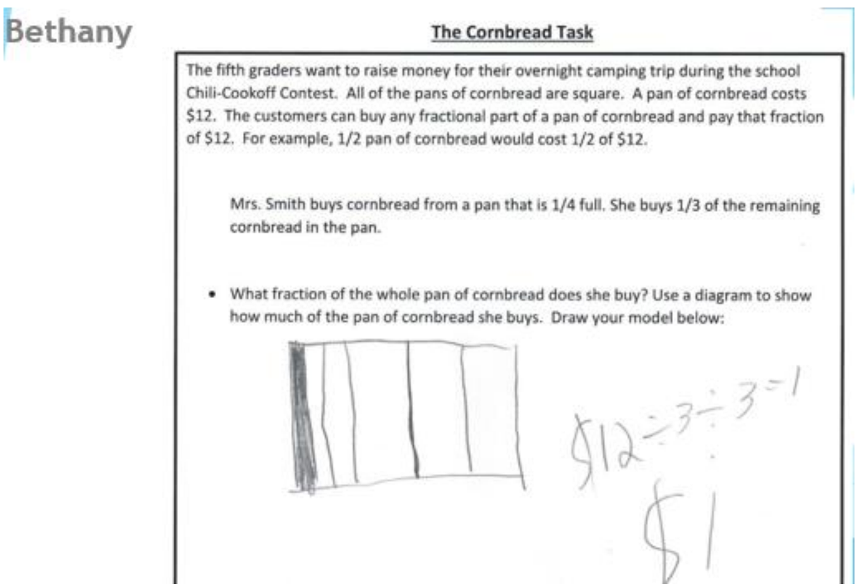
Stimulus Prompt 4 – Part II: ***Maria***

How would you support Maria, based on her current efforts?

Potential probing questions, as needed.

Why would you choose [revoice the response indicated by the teacher] as an approach?

With [revoice the response indicated by the teacher], how would you anticipate [student] would proceed?

Stimulus Prompt 5: ***Bethany*** Upon leaving Maria's desk, you take one final loop around the room. While Bethany has not finished everything, you notice that her solution pathway is different than other students, which piques your interest. Here is her work:  Bethany **The Cornbread Task** The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12. Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan. • What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below: $12 ÷ 3 ÷ 3 = 1 $1	Based upon Bethany's response, what would you say to her? If the teacher states that they would want to hear her solution strategy, share Bethany's response below: If Bethany responded "So, ummm…the whole entire thing. So, I just made a square for the pan and I cut it into fourths – well, into halves and then half again. [and] Then I cut one of the fourths into thirds and then I said so there's four pieces divided by twelve, which is the whole thing which would be $3, because twelve divided by three is four and then – No. Twelve divided by four is three – yeah, and then there would be three pieces and she wants one of the thirds, so it costs $1." Now how would you respond to Bethany? Potential probing questions, as needed. Why would you choose [revoice the response indicated by the teacher] as an approach? What would you ask her? / Would you ask her anything? You could acknowledge her response, would there be anything else you might ask her or you would want to say to her?

Interview III: Defining the Role of Productive Struggle in Learning Mathematics

The final interview (see Appendix J) played an important role, providing insight and perspective into a participant's definition of productive struggle in learning mathematics, the role of productive struggle in learning mathematics, and how these beliefs inform their instructional practices, in an effort to answer the question: *How do teacher's stated beliefs regarding the role of productive struggle in learning impact their instructional decisions?* The heart of understanding this phenomenon lies in the synergistic definition, interpretation, value, application, and purpose participants shared related to productive struggle in the teaching and learning of mathematics. The interviews leading up to this final interview require the insight of how participants define struggle and productive struggle in the teaching and learning of mathematics. While these questions could have been used to begin the efforts of this work, the intentional placement of this inquiry at the end was to prevent bias of participant responses throughout.

The interpretivist questions in this final interview will play an essential role in the overall interpretation of the data (see Table 4). These final responses will act as a vehicle to interpret all data, "provid[ing] a check on what [I] think [I am] understanding, as well as offer an opportunity for yet more information, opinions, and feelings to be revealed" (Merriam, 2009, p. 98), from which a more thorough and accurate analysis of data can transpire. This final interview provides additional opportunity to connect and extend upon previous responses shared by the participants, providing a clearer understanding of their lived experience.

Table 4

Questions and Supporting Rationale for Interview III

Question	Purpose	Research Base
How do you define struggle in learning mathematics? How do you define productive struggle in learning mathematics? How do you see the role of productive struggle in learning mathematics?	These questions will provide essential understanding of how a participant defines and values productive struggle in learning. These questions provide insight into participants' experience, opinion, and knowledge.	AMTE, 2017 Chazen & Ball, 1999 Cleary, 2009 Hiebert & Grouws, 2007 NCTM, 1991, 2014, 2020 NRC, 2001 Smith & Stein, 2011 Stein et al., 2007 Warshauer, 2011, 2014
When reflecting on your planning and implementation of fraction instruction, what role does productive struggle play in this process? If I were to observe you teaching with [stated role of productive struggle], what would I see? What would I hear you saying to students? What would I see students doing in this productive struggle? You mentioned that productive struggle ________________________ in your planning of fraction instruction. Related to this, in our last interview you indicated that for the fraction task shared, you would anticipate _________________ struggle(s) as a cause of impasse. With this student struggle in learning mathematics as a focus, how do you typically respond when a student demonstrates struggle in learning mathematics?	How do teacher's stated beliefs regarding the role of productive struggle in learning impact their instructional decisions? These questions provide insight into participants' experience, behavior, and values.	AMTE, 2017 Chazen & Ball, 1999 Cleary, 2009 Henningsen & Stein, 1997 Hiebert & Grouws, 2007 NCTM, 1991, 2014, 2020 NRC, 2001 Smith & Stein, 2011 Stein et al., 2007 Warshauer, 2011, 2014

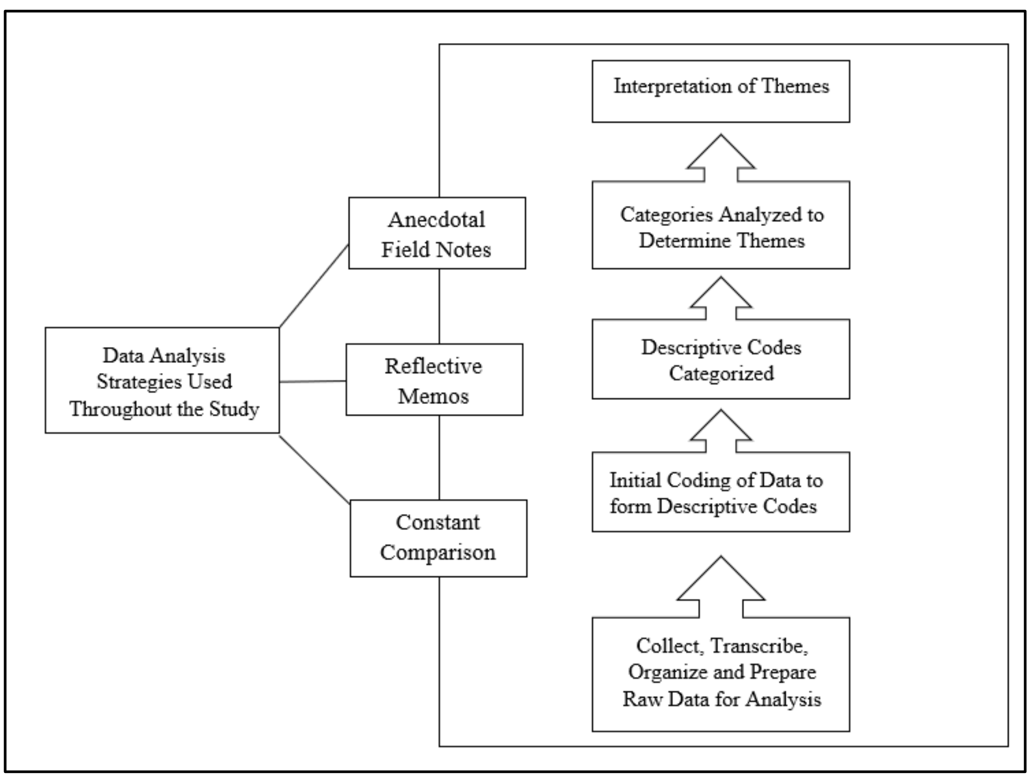
How would you explain your role in the teaching of mathematics? How would you explain the role of the students in learning mathematics?	How do teacher's stated beliefs regarding their role in the teaching of mathematics impact their instructional decisions? These questions provide insight into participants' values and Opinions	Hiebert & Carpenter, 1992 Hiebert & Grouws, 2007 NCTM, 2020 Schoen & LaVenia, 2019

Data Organization and Analysis

Organization and analysis of the data in this study comprises seven distinct steps, some of which were simultaneous in nature. Figure 2 displays the process for data analysis implemented during the study. This figure demonstrates the progression of steps, capturing when each process was employed. The remainder of this section provides a description of each step in the data analysis strategy and how they were implemented during the study.

Figure 2

Data Analysis Procedure



Note: This figure demonstrates the progression of data analysis from collection, transcription, and organization, to coding and categorizing, to theme development.

The data collected came from a variety of sources, including an electronic survey, transcribed interviews, teacher planning notes, and a personal research journal. Although the data sets were distinct, the data from the surveys, interviews, planning notes, and my personal reflections were analyzed with a coding process to interpret the data. Each participant was analyzed individually for patterns, themes, and relationships within and across completed interviews.

Collection of Data

Data was derived from a survey, participant planning notes related to *the Cornbread task* (Rumsey et al., 2016), and three semi-structured interviews. The survey data provided demographic information on each participant, providing valuable context for the study. Participant planning notes were a result of a request that participants reflect on how they would implement *the Cornbread task* in their own classrooms, prior to the second interview. While each participant approached this reflection and preparation differently, the documents they shared provided meaningful insight into their planning and anticipation. The three semi-structured interviews provided the greatest amount of data and insight into each participants experiences related to the anticipation of student struggle, planning and teaching that responded to and/or leveraged this struggle in a productive way, and teacher responses to examples of student struggle in learning mathematics. Throughout the data collection process, anecdotal field notes, reflective memos, and constant comparison of the data occurred to ensure trustworthiness.

Transcribing Data for Initial Analysis

Each of the three semi-structured interviews was audio and video recorded through a teleconferencing software called Zoom. The Zoom recordings provided the opportunity for accurate and verbatim transcription of each interview, as well as the opportunity to review and compare anecdotal field notes to participant engagement. Field notes provided hand-written details regarding teacher responses, reactions to questions, and noted apparently supporting and conflicting responses within each interview for further reflection. The transcribing process was completed by me, with initial reflective memos and notes of constant comparison considered in-between each interview. This was an integral part of the data collection process, allowing for any necessary elaboration or clarification to be requested from participants in a timely manner. Each participants' interviews were fully transcribed within three weeks of the completion of each interview.

Organization, Condensation, and Preparation of Data for Analysis

Interview transcription started approximately two weeks following the first interview with each participant. In this initial effort, the focus was on accurately recording all questions and responses during each interview verbatim. Following the full transcription of each interview, I read the transcript while listening to the audio, to ensure that all of the transcription was completed fully and accurately. A third opportunity to listen to and view recorded interviews once again provided another opportunity to revisit my field notes and reflective memos. This effort was strengthened through the writing and reflection upon anecdotal field notes, reflective memos, and a constant comparison of data within and across each participants' responses.

Throughout the data organization process, the need for data condensation was also an important consideration. In this qualitative study, the volume of information collected was large, while not every piece of information recorded was consequential. To ensure a meaningful management and analysis of the data collected, a "process of selecting, focusing, simplifying, abstracting, and/or transforming the data that appears in the full corpus (body) of written up field notes, interview transcripts, documents, and other empirical materials" (Miles et al., 2014, p. 12) occurred concurrently with the data preparation process. Doing so provided the opportunity to focus on the flow of the conversation, questions and response, and participant reactions that might lead to additional reflective memos to be added.

Initial and Descriptive Coding: A First Cycle Coding Process

The data coding occurred in two levels, as a first and second cycle coding process ensued. The first cycle coding method was elemental in nature and a form of descriptive coding as the data's content was summarized (Saldaña, 2013). This approach provided the framework for second cycle coding. The goal in implementing a two-cycle coding methodology was to provide a broader perspective and opportunity for reflection as "data analysis begins by identifying segments in your data set that are responsive to your research question…a unit of data which is a potential answer of part of an answer to the question(s) you have asked" (Merriam, 2009, p. 176). The initial data analysis process entailed multiple iterations of elemental coding and interpretation, leading to the descriptive codes (listed in Table 5). In this initial coding process, I was able to break down the data into discrete parts, determining similarities and differences (Strauss & Corbin, 1998) and determining descriptive codes that captured the essence of the data collected. The initial step of descriptive coding provides the opportunity for the reader of this study "to see what [I] saw and to hear what [I] heard in general" (Wolcott, 1994, p. 55). This initial work reflects a pattern matching process, while subsequent iterations of coding support an explanation that builds throughout, with analysis focused on synthesis of findings for each participant. The data derived from surveys, interviews, and artifacts was coded in this manner.

Table 5

Descriptive Codes Identified During First Cycle Coding

Initial Codes	Descriptive Code
• Understand equal parts • Understand breaking down parts • Understand what numerator and denominator represent in a fraction • Using models to demonstrate understanding of fractions • Using models to demonstrate fraction operations • Students drawing picture to deepen understanding	Relational understanding of fractions Conceptual understanding of fractions
• Knowing basic facts • Knowing where the numerator and denominator are located • Knowing rules to apply for accurate solutions (ex. PEMDAS – order of operations; keep-change-flip) • Students share "Fist of Five" to indicate understanding of procedure	Procedural knowledge of fractions Knowing and memorizing rules or mental tricks Procedural understanding
• Students communicate accurately about mathematics • Students work collaboratively • Students explain their thinking, reasoning, and problem-solving strategies with peers • Students defend their thinking, reasoning, and problem-solving strategies with class • Students draw visual models to represent solution process • Students work in rotating groups to increase learning opportunity • Students discuss math with others to further their thinking • Students problem-solve independently; discuss solution strategies with peer • Students work with assigned peer (weak/strong match)	Expectation of student collaboration and communication for learning Expectation of student communication of learning

• Students use manipulatives to demonstrate understanding • Students demonstrate multiple solution pathways and/or models • Students explain what a fraction means (ex. 7/8) • Students share different models for problem-solving and share benefits or drawbacks for each • Students listen, watch, and record as teacher models the procedure to solve • Students practice procedures independently and check solutions against teacher's answers; correct errors • Struggling students redirected to color-coded notes • Struggling student directed to use models to support thinking • Struggling students directed to show visuals to correct errors • Students use jingles, acronyms, "tricks" to remember the steps to solving problems	Expectation of student modeling thinking, reasoning, and problem-solving processes Expectation of student justification of solution strategies Expectation of student listening, recording, practicing procedure modeled, check solution for accuracy Expectation of students to use models when struggling to solve Expectation of students to use memory tricks to accurately solve
• Teacher provides tools and visual resources for student independence • Teacher elicits student ideas for strategic guidance • Teacher models strategies shared by students • Teacher provides learning goals to focus students • Teacher relates learning to real-life • Teacher shares one strategy to avoid "student struggle" with multiple options • Teacher models procedural process, modeling this in a think aloud, color-coding each step • Teacher directs problem-solving of student task • Teacher practices skills with students until confident they are ready to try on their own	Teacher practice includes providing tools and resources for student Teacher practice involves privileging student strategies for problem-solving Teacher practice involves focusing student learning

• Teach students jingles, acronyms, “tricks” to remember the steps to solving problems • Teacher supports struggling students with manipulatives, visuals, and/or decomposing • Teacher determines the need for visual support based on low, self-rated student confidence • Teacher places students in groups to work when struggle is anticipated	Teacher practice involves mitigation of error and preventing struggle Teacher practice indicates response to struggle with visual and concrete models Teacher practice indicates group work to prevent struggle
• Teacher proactively prevents struggle through telling when experience warrants this • Teacher experience yields confidence and ability to respond flexibly, without planning • Teacher notes of procedural steps complete and revisited each year with new students • Kids want to do the instantaneous multiply across the top, multiply across the bottom…developing that understanding and that visual model is not as easy as knowing facts • Key concept of parts within parts and equal parts • Using visual models • Slowing down the pace • Student providing multiple representations • Making students think beyond a role memorization strategy • Getting students thinking about the whole • Getting students to think about multiple models and approaches	Teacher responds to anticipated struggle with planning that involves telling Teacher experience and previous preparation presumes planning Teacher experience indicates planning to build understanding beyond procedural fluency

• Revisit previous learning – vocabulary review – Teacher Modeling – Student Practice • LESRA Model – Launch, Explore, Summarize, Reflect, Apply • 5 E Model / Indirect E – Engage, Explore, Explain, Extend, Evaluate • Students watch, listen, take color-coded notes, follow along as teacher models (new procedure day 1) • Students watch, listen, share “next steps,” practice procedure, check answers as teacher models solution, fix mistakes (day 2)	Teaching follows predictable routine
Cornbread Task Initial Thoughts • Students apply prior knowledge in new learning (ex. unit fraction knowledge in multiplication task) • Teacher anticipates need for modeling “on this task” • Determining how much support and what type of support might be needed • Teacher anticipates potential struggle and how to redirect • Teacher anticipates need to extend task for “advanced students” • Teacher anticipates the potential need for “easier” fractions to support student access • Teacher anticipates ease of task and student success • Teacher anticipates potential student solutions / models • Teacher indicates resistance to modeling and explaining solution process (throughout)	Cornbread Task Initial Thoughts Teacher anticipates requisite knowledge, skills, and understanding when planning Teacher anticipates potential struggle and plans strategies for support Teacher anticipates the potential need to extend task with more complex fractions Teacher anticipates potential student solutions Teacher anticipates student resistance to expectations of task
Response to Student Struggle	Response to Student Struggle – Cornbread Task

• Why questions • What questions • Eliciting student explanation and understanding of task • Focusing students on terminology that might be helpful in interpreting the task • Can you show me more? Can you add a bit more for me? • Can you explain your model? • How can you label your model? • Asking “What does this really mean?”	Supports student engagement in productive struggle (Probing)
• Teacher draws model for student • Teacher completes task for student, explaining their work in each step • Teacher uses concrete manipulatives to represent the task “literally,” using a whole number approach	Teacher removes opportunity for student engagement with productive struggle (Doing)
• Teacher dictates student engagement when prompted • Teacher provides step-by-step guidance for student to complete task • Teacher reads task, pausing to prompt exact process (ex. draw a square) • Teacher provides model of the solution and walks student through the problem	Teacher funnels student engagement to mitigate struggle and lead students to a correct solution (Telling)
• Teacher asks questions that focus student on key information in the task • Teacher asks questions that result in student completing the task step-by-step • Teacher asks questions that funnel student to a correct response	Teacher funnels student engagement through questions that lead student to a correct solution (Guiding)
• Student willingness to verbalize thinking, engage with task, but might need a little nudging or redirection to persevere • Students demonstrating willingness to continue working when [math task] is difficult and they need to struggle a little is productive • Students need to be supported through struggle – determine what they know to move them forward; not frustrate	Positive beliefs depicted through teacher’s stated purpose and potential of struggle in learning mathematics
• Students who struggle will not ask questions or advocate for themselves • Students who struggle need the teacher to give them tools and tricks at their disposal (ex. the rap keep – change – flip for diving fractions) • Students who struggle in math feel like they’re not good at anything else	Negative beliefs depicted through teachers stated barriers and detrimental impacts of struggle in learning mathematics

• When students struggle in math it puts a bad attitude in their mind about doing math and their ability in math • When students struggle, tell them “you know, some people are just better at math than others, you know there are other subjects you are better at”	
• Make struggle productive, without taking over – give a nudge or a hint • Build from what students know – elicit their thinking and reasoning to move them forward • Every child has the ability to learn with productive struggle; teacher needs to support this • If they are not struggling, it is too easy for them • I do believe for some kids it’s a nice challenge for them • Productive struggle is important for students to go through in math with support and guidance	Struggle as a productive aspect of learning mathematics
• You don’t want them to struggle in math • Students need to advocate for themselves, not just sit there • You have to have your pocket full of those tricks for those students that struggle • If there are struggles, it’s definitely in the math area • When students struggle, that is when they are building the conceptual understanding	Struggle as a negative aspect of learning mathematics
• With struggle in learning, if kids are able to verbalize their thinking, they’re willing to engage with the task but might need some redirection • In those instances [of struggle], we need to support students in connecting what they know to build confidence to move forward • Having students do things that are hard for them like proving why, giving and explanation, providing a model is needed for understanding • Students need to reflect on their work and figure out what’s wrong, ask questions • Students who are willing to struggle are willing to learn • When students struggle, with support, you can push their thinking • We have to teach students to use what they know, to figure it out, with productive struggle – but stop it before frustration sets in • Struggle in math relates to growth mindset – students need to know that when your brain struggles, your	Productive struggle in learning mathematics conceptualized as being positive – purpose and potential focus

brain is making new connections. If you are not struggling, you are not exercising your brain • Good struggle [is supported] by a lot of communication, talking between students, sharing ideas • A perfect good struggle would be a student answering something and another student jumping in and saying I agree, but what if we try it this way; that back-and-forth conversation	
• Kids who are struggling will not ask questions, so they will not learn • Kids who are struggling will avoid math (ex. go to the bathroom) • When kids feel like they're not good in math, they feel like they're bad at everything • Struggle in learning math puts a negative attitude in [students'] minds and it carries with them for a long time • When they struggle, I feel like they learn quite a bit from that struggle sometimes, but some kids don't react, you know – well – to that struggle	Struggle in learning mathematics conceptualized as being detrimental – creating a barrier to learning
• When kids give up and put their heads down it is not good struggle • When kids don't even start it's a sign that they need something more • When kids don't advocate for themselves and they're just staring off • If there's struggle, it's definitely in math • Struggle comes from [students not knowing] the basic facts • Struggle happens because students won't ask questions • Students who struggle avoid engagement	Negative struggle depicted in a lack of student engagement when struggle emerges
• When kids can verbalize their thinking and are willing to engage with the task it's good struggle • When it's good struggle you can lean in to give hints or guide students to what they know • When a student is struggling but still working through it, asking questions, actively trying to figure out what's wrong that's a productive struggle • If kids are putting an effort forth • When it's not a complete meltdown, it's a good struggle	Positive struggle depicted in classroom engagement (teacher and student) that supports struggle in learning

Note: This table depicts the first cycle coding from initial to descriptive codes

Descriptive Coding to Categories: A Second Cycle Coding Process

The goal of the second cycle coding process was to provide the opportunity to reorganize and reanalyze the data, in a process Boije (as cited in Saldaña, 2013) explained as being one "to determine which [codes] in the research are dominant ones and which are the less important ones…[and to] reorganize the data set: synonyms are crossed out, redundant codes are removed and the best representative codes are selected" (p. 218). This process was most appropriate as an extension of the first cycle coding as it provided the opportunity to define more specific categories, considering the properties and dimensions of the data, while exploring the interrelatedness of subcategories (Saldaña, 2009). During the second cycle coding process, a focused coding approach supported the reorganization of data into related clusters, defining tentative categories (see Table 6), as the stories began to emerge.

Table 6

Categories Identified During Second Cycle Coding

Descriptive Codes	Categories
Relational understanding of fractions Conceptual understanding of fractions	Teacher description of student understanding is conceptual
Procedural knowledge of fractions Knowing and memorizing rules or mental tricks Procedural understanding	Teacher description of student understanding is procedural
Expectation of student collaboration and communication for learning Expectation of student communication of learning Expectation of student modeling thinking, reasoning, and problem-solving processes Expectation of student justification of solution strategies	Expectation for student engagement in learning is active and requires student ownership of learning

Expectation of student listening, recording, practicing procedure modeled, check solution for accuracy Expectation of students to use models when struggling to solve Expectation of students to use memory tricks to accurately solve	Expectation for student engagement is passive and demonstrates teacher ownership of learning
Teacher practice includes providing tools and resources for student Teacher practice involves privileging student strategies for problem-solving Teacher practice involves focusing student learning Teacher practice involves mitigation of error and preventing struggle Teacher practice indicates group work to prevent struggle Teacher practice indicates response to struggle with visual and concrete models (indicated as "below" level engagement)	Classroom culture fosters autonomy, situating students as knowers and doers of mathematics Classroom culture prevents struggle Classroom culture reinforces that struggle is indicated by the inability to demonstrate procedural fluency, requiring a more "primary" approach
Teacher responds to anticipated struggle with planning that involves telling Teacher experience and previous preparation presumes planning Teacher experience indicates planning to build understanding beyond procedural fluency Teaching follows predictable routine	Learning is supported through teaching that prevents the opportunity for struggle Learning is supported through activities that foster the opportunity for student connections to emerge

	Routine supports student engagement in learning
Cornbread Task Initial Thoughts Teacher anticipates requisite knowledge, skills, and understanding when planning Teacher anticipates potential struggle and plans strategies for support Teacher anticipates the potential need to extend task with more complex fractions Teacher anticipates potential student solutions Teacher anticipates student resistance to expectations of task	Anticipation involves reflection on prior learning and potential needs to support student engagement Anticipation involves reflection on student resistance to engagement
Response to Student Struggle – Cornbread Task Supports student engagement in productive struggle	Students own learning with productive struggle
Teacher removes opportunity for student engagement with productive struggle	Teacher owns thinking and reasoning
Teacher funnels student engagement to mitigate struggle and lead students to a correct solution	Teacher directs student through a process, with a procedural approach
Teacher funnels student engagement through questions that lead student to a correct solution	Teacher directs student thinking and reasoning through questions
Positive beliefs depicted through teacher’s stated purpose and potential of struggle in learning mathematics Struggle as a productive aspect of learning mathematics Productive struggle in learning mathematics conceptualized as being positive – purpose and potential focus	Struggle viewed as an affordance to learning, when productive, that must be fostered in the classroom

Negative beliefs depicted through teachers stated barriers and detrimental impacts of struggle in learning mathematics Struggle as a negative aspect of learning mathematics Struggle in learning mathematics conceptualized as being detrimental – creating a barrier to learning	Struggle viewed as a constraint to learning, that must be mitigated in the classroom
Negative struggle depicted in a lack of student engagement when struggle emerges	Negative struggle results from a lack of student self-advocacy
Positive struggle depicted in classroom engagement (teacher and student) that supports struggle in learning	Positive struggle dependent on classroom culture

Note: This table depicts the second cycle coding from descriptive codes to categories

Dependability and Credibility

Credibility and dependability were key considerations in this qualitative semi-structured interview study, guided by the goals to understand what others feel and think in an effort to understand their personal experiences related to the phenomenon being investigated. In addition to the considerations of credibility and trustworthiness, quality indicators were considered.

Lincoln and Guba (1985) indicated that one way to ensure credibility of an interview study was to develop the interview guide in phases. These phases required that I research the phenomenon and interviewing techniques thoroughly, resulting in a preliminary interviewing guide to be field tested and modified as needed. I completed both of these steps, providing valuable insight to inform my final interview guides used in this study. Additionally, according to Shenton (2004) credibility in a qualitative study refers to the accurate recording of the phenomenon being studied. In the case of this study, all interviews were recorded with Zoom, to ensure an accurate and verbatim transcription of ideas shared by participants.

While the initial formulation of interview questions and thorough and accurate recording of the data was consequential, the interpretation of the data collected was of the utmost importance in maintaining credibility. Merriam (2009) reminds us of this caveat, as the data must be interpreted and represent the data collected. This essential reflection ensures that a reliable, cohesive, and compelling story provides confidence in the interpretations and conclusions (Creswell & Miller, 2000). Maxwell (2013) refers to this as providing findings that make sense to the reader and researcher alike. Creswell and Creswell (2018) suggested that researchers use eight specific strategies to ensure a credible and accurate findings (see Figure 3). In this study, I employed the following strategies: (1) triangulation of data; (2) use rich, thick description to communicate findings; (3) disclose bias; and (4) present negative and discrepant information.

Figure 3

Creswell & Creswell's (2018) Eight Qualitative Research Validity Strategies

- *Triangulate* different data sources, checking for consistency in what people say about the same thing
- *Use member checking* to determine accuracy of findings
- *Use a rich, thick description* to convey findings
- *Clarify personal bias* the researcher brings to the study
- *Present negative or discrepant information* that contradicts themes
- *Spend prolonged time in the field* gathering evidence
- *Use peer debriefing* to increase the accuracy of the account
- *Use an external auditor* to review the project in its entirety

Note: While this presents eight strategies, the recommendation is to use multiple strategies. The strategies are listed from those used most frequently to least frequently.

Triangulation requires an examination of evidence from multiple sources to build a coherent understanding. This study considered the perspective of how participants would interpret the role of struggle in a variety of teaching practices, including the anticipation of struggle in learning mathematics, planning, teaching, and responding to examples of student struggle, and teacher's stated beliefs regarding the role of struggle in learning mathematics. The twelve interview transcripts, and four participant planning documents produced during the data collection phase of this study provided the multiple sources of data to meet this criteria. The interview data collected provided different opportunities for participants to share how they engage with the phenomenon, in a variety of contexts, as well as through their personal reflections of the role of struggle in earning mathematics.

The use of rich, thick descriptions provides the reader with the opportunity to become more closely connected to the study. Through the careful presentation of participant quotes, field notes, and documents readers can form an impression of the overall findings. Navigating the findings from each participants' experience, I connected and related the participants' pedagogical practices and stated beliefs in the findings. The use of anecdotal field notes and reflective memos ensured the accuracy of the information presented.

Disclosure of my personal bias was also an important consideration in this study. My ideological position exists on my ascribed incremental learning theory, that all students can learn mathematics when taught by competent and confident mathematics educators who utilize meaningful pedagogical practices. This belief is supported by relationships found between teacher competence in mathematics for teaching and student learning outcomes (Hill et al., 2005).

Aligned with my incremental learning theory are my teaching philosophies and practices that align with constructivism and support productive struggle in learning. As an elementary educator for 14 years prior to my current efforts, I have always been interested in the role struggle played in learning mathematics. While the phraseology productive struggle did not exist at this time, my master's research focused on an *Error Analysis Procedure* that I created and implemented into my daily teaching routine with students. The process required that students engaged in the analysis of their errors, determining where and why these errors occurred, in an effort to learn and grow. While this initially elicited a struggle that required my support for a productive engagement, students learning increased and their expectation that this "productive struggle" was a part of learning became standard.

To minimize bias in my interpretation of participants stated beliefs, responses to student struggle, and intentional preparation with productive struggle, I set aside my personal experiences, preconceived ideas, and judgments during the data analysis phase. An example of how I mitigated the impacts of my personal bias can be found in my analytic memos taken during interviews. These memos not only served as an opportunity to stop and reflect on any unanticipated personal responses to participants, but also ensured that I asked probing questions that would serve to clarify participant responses. This elaboration allowed for a more complete description and explanation from participants, rather than relying on initial interpretations of the responses. One example comes from my first interview with Jessica, when I asked:

> So thinking about those visuals you mentioned - in particular you talked about the need for students to understand that fractions are a part of something. With all of this in mind, when you think about planning a fraction lesson, is there an essential strategy that factors into that planning?

Jessica’s response indicated:

> Using the Common Core, so like all of our lessons have to – you know – line up with the Common Core…Math can be done lots of different ways, so the hardest part is, I can’t, I don’t want to show them like four ways [of] how to do something.

At this point in Jessica’s response I noted “While I understand that this could be hard, this seems contradictory to the Common Core guidance alluded to in this response. Follow-up.” As Jessica’s response continued, she stated:

> I’ve done that before. That was hard. Kids really struggled with that…So what I did was to put it down to the main one that has worked over the years for the kids and then for the kids that didn’t - you know – quite get it, then I would say *Okay, here’s another way*.

In my memos I wrote “main way reminds me of the standard, traditional algorithm as being the “one approach that works.” Clarify.

After Jessica finished her thoughts, I asked the following probing question “You mentioned one thing that I would like to clarify. You mentioned “one main method,” to kind of streamline this for students. So, in fractions - you happen to be in fraction multiplication. What would be the main method?” Jessica responded:

> As long as they are both fractions - you know if you had a whole number times a fraction you have to make that whole number [into] a fraction - you’re multiplying the numerators, multiplying the denominators. You know then you have to simplify.

Following this response, I wrote another analytic memo that said “Clarification supports previous thoughts. But now I am unsure about visuals. I will need to probe further.” This ongoing written and reflective analytic memo process was utilized throughout all interviews with all participants, in an effort to ensure that I was interpreting their responses with an accurate interpretation and probing when clarification was needed. This process was necessary, as "the uniqueness of our experience and personality means that each of us will develop a somewhat different arrangement or pattern of learning theory to serve as our basis for our behavior as educators" (Lindgren, 1959, p. 335). My goal was to understand each participants experience and engagement with struggle in learning mathematics, unfettered by any preconceived notions.

Maintaining a conscious awareness of my reflexivity was also necessary in this endeavor. Reflexivity in research relies heavily on our past experiences and biases, as we define and reflect on how these understandings potentially occlude our ability to discern the "truths" in our findings. Maintaining a conscious awareness of "[my] perspectives…[and] how those perspectives might lead[me] to ask certain questions (and not others) and to make certain interpretations (and not others) of interactions within the research setting" (Glesne, 2011, p.54) was an important consideration throughout the research process. Personal experiences as a student and educator have implicit implications in my reflexivity as a researcher. This understanding required an analysis of the questions I answered, consideration of the emotions that impact my reality and acknowledgement of my values (Glesne, 2011). This reflexive approach and examination mediated my relationship as a researcher with my participants, allowing for a more accurate analysis of the questions.

Trustworthiness was another important consideration in this qualitative study. Lincoln and Guba (1986) provide the guidance that the researcher must be balanced, fair, and conscientious in taking account of the multiple perspectives of their participants. Trustworthiness and credibility are present in the presentation of discrepant findings of the data, an important consideration in this study. While much of the data collected from participants provided consistency throughout, important discrepancies existed. Reporting of this evidence is an essential consideration in assuring the reader of the credibility of the findings in this study.

Limitations of the Study

While this qualitative semi-structured interview study provided greater opportunity to understand this phenomenon, limitations exist. Identified limitations include the investigation of the phenomenon experienced by participants in an indirect manner, using interviews and teacher planning notes in lieu of direct observation of teaching practice due to the pandemic, and the potential for partner rescindment (Glesne, 2011). While the noted limitations could impact the trustworthiness of data obtained, insights gained about this phenomenon provide the opportunity to explore emerging relationships on a larger scale in the future, in a classroom setting.

The participants being investigated in the exploration of the research questions provided an initial understanding of the synergistic relationship being considered. Initial findings illuminate the potentially synergistic effects of anticipation, planning, teaching, and response to student struggle in learning mathematics. Findings within the evidence among and between participants provides opportunity for building deeper understanding of this phenomenon, with the potential of further exploration of the resulting themes.

The potential for inconsistency of teacher availability, while a potential barrier, was addressed proactively. All interviews were scheduled according to participant availability, with flexibility and autonomy in their selection of location for our Zoom interviews. When participants encountered schedule conflicts, interviews were rescheduled in a timely manner.

My belief is that this study will inform future researchers and educators who consider the role of productive struggle in learning mathematics, and the potential synergistic impacts of a teacher's anticipation of student struggle, intentional instructional design that responds to and leverages this student struggle, lesson implementation that fosters opportunity for productive struggle, and teacher reaction to examples of student struggle. In this analysis, consideration of a teacher's personal beliefs was an essential consideration.

CHAPTER 4

FINDINGS

The goal of this qualitative semi-structured interview study was to understand the potentially synergistic impacts of anticipating, planning, teaching, and responding to struggle in learning mathematics. Understanding the synergistic impacts of anticipating struggle, planning strategies to support this struggle in a manner that fosters productive struggle, and implementing these strategies as struggles arise during learning, holds the potential to further inform our practices related to teaching and learning mathematics. The study was informed by the following questions:

> Research question: What role does productive struggle play in the design and implementation of mathematics lessons?
>
> Attendant question 1: How do teachers perceive their role and the role of students as it relates to learning with productive struggle?
>
> Attendant question 2: How do teachers prepare for anticipated struggle when planning for mathematics instruction?
>
> Attendant question 3: How do teachers respond to evidence of struggle in student learning? Do the response(s) have the potential to invoke a productive struggle for students?

The remaining sections of this chapter provide the context of the investigation, beginning with a brief description of the participants and the research setting. An explanation of the data analysis process follows, providing the description, analysis, and interpretation required in determining six overarching themes and fifteen sub-themes.

Participants

The participants included four female teachers who expressed an interest in participating in this study. The group comprises three teachers in grade five and one teacher in grade six, teaching in districts representing a variety of demographics in a Midwestern state. While all participants indicated experience teaching multiple grade-levels during their teaching career, Katherine, Jessica, and Melissa were teaching fifth grade at the time of the study and Ashleigh's responses focused on her sixth-grade experience (see Table 7).

Table 7

Study Participants Teaching Experience and Training

Participant Pseudonym	Race / Ethnicity	Grade Levels Taught	Years of Teaching Experience	Training in Mathematics Pedagogy	District Classification	Teaching Responsibilities
Katherine	White	3 & 5	17	Yes	Suburban District	All Content
Jessica	White	2 & 5	21	No	Rural District	All Content
Ashleigh	White	6 - 8	7	Yes	Rural District	Content Specialist
Melissa	White	1, 2, & 5	5	Yes	Urban District	All Content

Three of the participants involved in the study were recruited pre-pandemic. This allowed for classroom visits and informal conversations with Katherine, Jessica, and Ashleigh. The fourth participant, Melissa, was recruited after the pandemic started, preventing a classroom visit.

Katherine is an experienced teacher working in a suburban school district in the Midwest. Her students represent a diverse demographic ethnically and socioeconomically, with a small percentage of the students being English Language Learners (ELL). Katherine is very active in her district, serving in leadership roles on district committees, especially in curriculum adoption and instructional practices. When Katherine was asked what piqued her interest about this study, she indicated:

> Of all subject areas, I am most passionate about math. We use the Bridges curriculum which has some huge benefits for our students in the areas of application to life and conceptual understanding. I spend a great deal of time in my classroom talking about how mistakes are valued and that we can learn so much through the mistakes that are made. The idea of productive struggle discussed in this study is one I feel passionately about. It is also an area where I would like to learn more to better improve my teaching.

Katherine currently holds a teaching license for grades one through eight and a master's degree in Educational Leadership and Administration. She has participated in additional training in teaching mathematics, however, indicated that these courses were approximately fifteen years ago.

Jessica is the most experienced teacher in the group of participants. She teaches in a rural school district in the Midwest. She noted that her students are not very diverse demographically, with the greatest diversity being their socioeconomic status, though she indicated an increased number of ELL students in recent years. Despite this, her ELL student population is typically one to two students in a class of twenty-five students. Jessica participates in staff development opportunities provided by the district and summer training when time permits. When Jessica was asked what piqued her interest about this study, she stated "Math has really changed over the years and many students struggle, so if there is anything that I can do to help them, I am willing to do so." Jessica holds a teaching license in grades one through eight.

Ashleigh has less experience than the first two participants described, however, she is considered an experienced teacher with seven years in the classroom. Ashleigh teaches in a small, rural school district in the Midwest where all grades from 4K through twelve share a single building. She teaches all of the middle school mathematics classes, though the study focused on her experiences with her sixth-grade students. The student population Ashleigh serves is not diverse ethnically nor in their ELL status, however, diversity exists in the socioeconomic status of her students. When Ashleigh was asked what piqued her interest and desire to participate in the study, she indicated:

> I have sat in on what you've talked about in regard to mindsets, have had conversations with you before, so I thought that working with you, again, would be a great experience. Also, I teach one of the grades that you are studying and would like to share what we do in my classroom.

Ashleigh has participated in summer training supported by Elementary and Secondary Education Act (ESEA) grants, that focused on teaching practices in Science, Technology, Engineering, and Mathematics (STEM) education. She also participates in district training opportunities related to mathematics education.

Melissa has the least amount of experience with five years of teaching and would be referred to as an experienced teacher, though at the beginning end of this spectrum. Melissa works in an urban district in the Midwest, teaching ethnically and socioeconomically diverse students, as well as a small percentage of English Language Learners. Melissa was in her second year of teaching fifth grade during the time of the study. She indicated the following when asked what piqued her interest about participating in the study:

> I saw that you were doing this study and I am always interested in learning more about teaching. I am currently in my master’s program and taking courses on mathematics

teaching methods, so it seemed like a good fit. I am worried because this is only my second year of teaching fifth grade, but I have been teaching for five years now.

Melissa's current work in mathematics teaching, as she works towards her master's degree, is in addition to her participation in district-level training opportunities.

Overview of Themes

Six overarching themes and fifteen sub-themes emerged from the data analysis. The first three themes and their related sub-themes are closely tied to a teacher's beliefs about the role of struggle in teaching mathematics, including:

(1) teacher beliefs about the role of struggle in learning mathematics

(1a) positive beliefs about struggle in learning mathematics

(1b) negative beliefs about struggle in leaning mathematics

(2) teacher anticipation of struggle in learning mathematics

(2a) anticipated struggle as an affordance in learning mathematics

(2b) anticipated struggle as a constraint in learning mathematics

(3) teacher conceptualization of student understanding in learning mathematics

(3a) students constructing connections between prior learning and new learning experiences

(3b) students knowing mathematical facts, procedures, and ideas taught by the teacher

The next two themes provide further insight into how one's conceptualization of struggle in learning mathematics, as well as their conceptualization of student understanding, emerge in:

(4) teacher and student roles in learning mathematics

(4a) student-centered, constructivist teaching philosophy

(4b) teacher-centered, transmissionist teaching philosophy

(5) teacher responses to struggle in learning mathematics

(5a) doing the mathematics and mathematical thinking for the student

(5b) directing the student through the steps

(5c) leading the student to a correct response with funneling questions

(5d) probing the student to elicit their understanding

The final theme considers the relationship between teachers' stated beliefs and actions regarding the role of struggle in learning mathematics, drawing from the juxtaposition of identified themes:

(6) Alignment of beliefs and actions related to struggle in learning mathematics

(6a) Unconscious dissonance between stated beliefs and actions

(6b) Conscious dissonance between stated beliefs and actions

(6c) Concurrence between stated beliefs and actions

While differences emerged among the four participants, common perspectives and experiences were also apparent in each theme.

Theme 1: Teacher Beliefs About the Role of Struggle in Learning Mathematics

A major theme that emerged focused on teacher's beliefs about the role of struggle in learning mathematics. Teachers beliefs about the role of struggle in learning mathematics emerged in their descriptions of good and bad struggle in learning mathematics and in their depiction of student engagement in learning. The teacher beliefs that emerged proved to be essential in determining a relationship to their anticipation, planning, teaching, and response to student struggle. In order to elicit these beliefs more authentically, teacher participants were prompted to respond to other teachers' hypothetically positive and negative beliefs regarding the role of struggle in learning mathematics.

In this study, teacher beliefs regarding the role of struggle in learning mathematics were varied. While the use of the term productive struggle was never shared in my interviewing, some teachers described the expectation for this type of struggle in learning mathematics. Analysis of data from participants indicated the sub-themes of positive beliefs about struggle in mathematics and negative beliefs about struggle in learning mathematics.

Sub-theme 1a: Positive beliefs about struggle in learning mathematics

Positive beliefs regarding the role of struggle in learning mathematics emerged in depictions of learning mathematics with productive struggle. Two participants, Katherine and Melissa, exhibited positive beliefs that indicated an opportunity to learn with productive struggle. This opportunity was fostered by participants' eliciting student knowledge and understanding to scaffold task completion, as students grappled with concepts and built understanding, without taking over student thinking. Katherine described how she has experienced productive struggle in her classroom. She noted:

> Productive struggle does get kids a long way…I think when kids are able to verbalize their thinking, that tells you that they're at least willing to engage with the task and that what really just needs to be done is maybe some redirection or some filling in the gaps…maybe [the student] has just done it incorrectly and giving a gentle nudge or a reminder - could be referring to something you've already done, could be to highlighting some points within the problem, whether it's the numbers, or the questions, or the wording.

Katherine's description of productive struggle alluded to key characteristics of student engagement that included the opportunity for students to learn with struggle when an impasse is identified, and scaffolding is provided. While the direct connection to productive struggle was made by Katherine, a more thorough description helped to illuminate how the opportunity for students to engage with productive struggle in learning mathematics. She said:

> And so that's - that's like the big conversation is like, [Katherine] “How do you do this?” And [the student will] be like “Well I know this” and I’m like “See, you totally know this!” and I’ll be like “Okay, now what would you do?” [the student might respond] “Well, I don't know.” And so, it's this conversation [Katherine] “Well, let's think about it. What could we look at? Remember in your notebook yesterday when we did this? Let's pull it back out. Let's take a peek. Let's notice…” It could be “ I see you did this model. How could you use this model and show me a different way?” and giving those little nudges and then backing away.

Katherine’s description of navigating struggle with students demonstrates a purpose and potential in this struggle and the value she places on the role of struggle in her students’ learning. Katherine indicated that she offers support in the emerging struggle, for example she noted that she would ask “How do you do this?” and when the student indicated what they know, reinforced this by saying “See, you totally know this!” Her approach demonstrates a strategy that has the potential to reengage students who lack confidence in their ability to complete a mathematics task successfully. She continued the conversation sharing questions and prompts that provide the opportunity for a student to grapple on their own following some redirection, for example “Okay, now what would you do?...Well, let’s think about it. What could we look at?” The description she provides exemplifies what Hiebert and Wearne (2007) describe when students struggle within reason while being expected to make connections to prior knowledge and learn more deeply, without the teacher taking over the thinking. Katherine’s response also depicts three of the strategies recommended by Warshauer (2015) – asking questions, providing encouragement, and providing time - in support of productive struggle.

Katherine's positive belief about the role of struggle is depicted in her description of supporting a struggling student. In this example she does not take over the thinking or the struggle, but rather the student is asked to think, make connections, and reflect on previous learning as they navigate this struggle. Coupled with her positive belief regarding the role of struggle in learning, Katherine demonstrated the importance of helping build students' self-confidence to navigate this struggle, in a manner that affords them the opportunity to own the struggle. In another example, Katherine described how she would support a student experiencing impasse, sharing how the conversation might evolve. She said she would tell them:

> "We're going to do math - the two of us together - for the next 15 minutes or however long it takes to figure out What do *you* know?" And there's a lot of praising…like "See - like *you* know how to do this. *You've totally* got this." Because I think sometimes it's like - we make a blanket statement and kids make a blanket statement like "I'm so bad at math; like I just can't do it, I just don't understand it." And it's really - it's - you don't understand *it*, you don't …maybe understand a certain small part of it, but you do understand like 90% of it. But it's that 10% that all of a sudden is what's stopping you from really being able to feel success, because you haven't arrived at the final answer.

The depiction of struggle by Katherine in both examples requires her engagement and support to help the student navigate through the struggle, while eliciting and validating the student's knowledge and understanding, in a productive manner. In this example Katherine emphasizes how she supports the student in defining their role in the learning during struggle, noting "See – like *you* know how to do this. *You've totally* got this…you understand like 90% of it. But it's that

10% that all of a sudden is what’s stopping you” reframing the struggle to emphasize how small this is in relation to their understanding. She recognizes that when students struggle, they often believe that they do not know anything, however, uses the struggle as an opportunity for student learning. By eliciting what students know, Katherine is able to increase their confidence and willingness to struggle productively while reinforcing to students that struggle is an important aspect of learning.

Melissa was another participant who depicted positive beliefs about struggle in learning mathematics. She stated this in a more direct manner, noting struggle as an essential aspect of learning. When responding hypothetically to a colleague about struggle in learning mathematics, she said:

> If I had a classroom where I put up problems and the kids just knew what to do right away, like I don't understand how that would work for teaching, I mean what would I be teaching them then, right? So, I feel like the struggle - you know I've done…like growth mindset work with students at the beginning of the year, so I think about it that way. Like when your brain struggles, your brain is making new connections and growing. If it's not struggling, you're not really growing your brain, you're not exercising your brain, so I think about it that way - I feel like struggle is good.

Melissa clearly depicts her belief that struggle is a necessary part of learning when she said “[W]hen your brain struggles, your brain is making new connections and growing. If it’s not struggling, you’re really not growing your brain.” An important consideration in this message is her intentional effort to support students in seeing the connection between struggling and learning. In order for the struggle inherent in learning to be productive, students must be provided the opportunity to engage with problematic situations, as they grapple through and make connections in their learning. Melissa expanded upon her initial ideas noting:

> I think productive struggle is very important for students to go through with math and I think it can be…introduced to students in a way that doesn't have to be scary. So - for example, thinking about - you know not showing them - you know multiplying fractions and [just saying] figure this out and walk away, but rather showing them almost like math games or…like math puzzles where you have to struggle for a while and then you know you work together, and you get the answer. I feel like promoting those type of activities in the classroom can really help students be okay with that initial struggle and problems.

Melissa explains how struggle can be positive and productive when students are motivated, such as when being challenged to solve a puzzle. She also indicates that this does not mean leaving students to grapple with a concept while feeling unsupported, which can lead to an unproductive struggle. The clear delineation between struggle that is unproductive and productive was explained when Melissa acknowledged that not all struggle is beneficial, but that struggle has the potential to be productive. She stated that:

> If it's so much struggle and there's no end to the struggle for the student or no end in sight, that's just going to discourage them…and they're not going to want to keep trying because they don't see any hope in continuing to try…I think the other thing too with math that makes it special is when I think about…the struggle in math. I feel like that's where the students really build the like the conceptual understanding or like they build those different strategies. So, I just feel like if there wasn't any struggle, they would learn one way to do math and that would be the only way they would learn it, and I feel like

> they've missed out on a lot of other opportunities to really apply that math in the real world and apply it in - you know - multiple situations.

Melissa's thoughtful reflection on the role of productive struggle that "I feel like that's where the students really build the conceptual understanding or like they build those different strategies…to really apply that math in the real world" demonstrates both a purpose in creating more proficient mathematics students, but also the potential to empower students in real world application of this knowledge and understanding. Similar productive beliefs about teaching and learning mathematics are recommended by NCTM (2014) when they note, "All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures" (p. 11). The belief indicated by Melisa holds the potential to influence a learning environment that fosters mathematics learning with productive struggle.

Katherine and Melissa both convey the belief that productive struggle in learning mathematics is beneficial for student learning. While the examples shared depict struggle in learning mathematics as being positive, nuanced differences between the two are also apparent. One discrepancy appears in the way Katherine and Melissa approach the topic of struggle in learning mathematics with their students. Katherine depicts an indirect approach of attending to students struggle, deemphasizing the struggle as a small consideration in students ability to complete a task, while emphasizing what they know and understand. In this approach, she tells students "You don't understand *it*, you don't …maybe understand a certain small part of it, but you do understand like 90% of it. But it's that 10% that all of a sudden is what's stopping you from really being able to feel success." This indirect approach to supporting students in the moment of struggle focuses their attention on what they do know that can help them to be successful. Melissa also focuses on struggle in learning mathematics in her approach with students, however, in a more direct manner. She shared the direct approach of teaching growth mindset, where students learn that "Like when your brain struggles, your brain is making new connections and growing if it's not struggling you're not really growing your brain, you're not exercising your brain." In Melissa's direct approach, students are taught to see struggle as an expected aspect of learning mathematics, without which your brain is not growing. Both approaches indicate that the struggle is not the endpoint in learning, but rather a consideration in learning mathematics with struggle.

Sub-theme 1b: Negative beliefs about struggle in learning mathematics

Contrary to the first two participant's positive beliefs, a sub-theme of negative beliefs about struggle in learning mathematics emerged in the data analysis for Jessica and Ashleigh. Jessica and Ashleigh depicted struggle in mathematics as being problematic to learning and impacting students' self-efficacy in learning mathematics. In the descriptions of their learning environments, struggle impacted students' perceptions of learning mathematics and the teachers worked to alleviate this negative association.

Jessica's negative experience with students' struggles in learning mathematics was apparent in her response to a hypothetical conversation with a colleague regarding struggle in learning mathematics. She depicted this struggle as a barrier to student learning and engagement, with potentially long-term implications. In particular, Jessica stated:

> I do feel definitely if there are any struggles, it is definitely in the math area…I feel like that math [struggle] is a huge one…[struggle in learning math] is hard, because I think it puts a negative attitude in their minds for the kids when they do struggle, [then] they are very negative when it comes to doing [math]. So, when they do struggle, now they say

> they're not good at math and that carries with them, I think, for a long time…I'll tell them, *you know just some people are better at math than others, you know there's other subjects you're better at*. Or you know…I tell them you know you're not all good at the same sport…you know I really relate it back to sports; I really do because a lot of those kids can make that - you know - connection.

Jessica reflected that "if there are any struggles, it's definitely in the math area" implying that struggle appears to be the norm in learning mathematics, however, not in a positive way. The struggle in learning mathematics perceived by Jessica as a barrier to student learning is consequential in that "[W]hen they do struggle, now [students] say they're not good at math and that carries with them…for a long time. This belief leads Jessica to respond to students demonstrating struggle with comments such as, "you know just some people are better at math than others, you know there's other subjects you're better at;" a response that is intended to buffer their negative self-concept. This approach to navigating student struggle has potentially detrimental consequences for students, impacting "not only their mathematical thinking but also their own sense of their ability to do and persist with mathematics, the way they are viewed as competent in mathematics, and their ability to perform successfully in school" (Franke et al., 2007, p. 226). Thus, when Jessica tries to mediate the negative self-concept of her students, in reality she is more likely to reinforce their negative perceptions about their ability to be successful in learning mathematics.

These negative experiences related to students struggles in learning mathematics were informed by the impacts Jessica has observed in students who "kind of check out a little bit sooner. So, I definitely see over the last 20 years – there definitely have been struggles in math for sure." In spite of this perspective, Jessica did indicate:

> I like the challenges it sometimes brings for some - not very many. Normally it's - if there are struggles, there's more of the breakdown and then it’s hard to bring them back. But I do believe, for some kids, it's a nice challenge for them; the ones I know do well enough and that can handle it, you know.

While Jessica stated that challenge can be beneficial in learning mathematics, she reserves the opportunity to learn mathematics with struggle for “some, not many.” The potential exists for students to recognize this discrepancy in learning opportunity, reinforcing their identity as a student capable of struggling productively in learning mathematics or as a student incapable of learning with productive struggle.

Ashleigh was another participant whose beliefs related to the role of struggle in learning mathematics were depicted negatively. Initially, Ashleigh’s reflection on struggle appeared contradictory to Jessica, as she reflected on the difference between productive struggle and struggle in learning mathematics in general. In her experience, the difference between productive struggle and unproductive struggle was dependent upon student self-advocacy. She noted:

> I think there's a difference between like a productive struggle and a struggle itself, where a productive struggle is when a student - yes they're struggling, but they're still working through it, they're asking questions, and they are actively trying to figure out what's wrong – *Why am I* having a hard time with this? and they’re being more like an advocate for themselves…[Struggle] should not be labeled as a bad thing…I mean it’s part of being a student and an expected part of that whole learning spectrum in, in my opinion…I would say if students are not going to put forth an effort period, if they’re just going to sit there and not ask questions and not advocate for themselves, [struggle is] not a good thing.

Ashleigh's description of productive struggle relies fully on student self-advocacy and perseverance through the struggle, in her depiction that "they're still working through it, they're asking questions, they are actively trying to figure out what's wrong – *Why am I* having a hard time with this?" a depiction contrary to the recommendation for active engagement by teachers in supporting students in their struggle. Hiebert and Grouws (2007) noted "In fact, it seems plausible that students' struggle should be sufficiently bounded and directed so that it centers on the important mathematical ideas. This requires some level of teacher guidance" (p. 390) a consideration that is missing from Ashleigh's description. Her reflection that "I would say if students are not going to put forth an effort period, if they're just going to sit there and not ask questions and not advocate for themselves, [struggle is] not a good thing" demonstrates her belief that navigating this struggle productively is the responsibility of the student. While students have an important role in struggling productively in their learning, teachers must support this opportunity through asking questions, providing encouragement, giving adequate time, and acknowledging student contributions (Warshauer, 2015).

The conclusion could be drawn that Ashleigh believes that struggle in learning mathematics, in and of itself, indicates the potential for productive struggle if the student chooses to own this opportunity. However, additional insight was gained as Ashleigh continued to describe her learning environment. In this, she noted that as a mathematics teacher you do not want your students to struggle, recognizing that mathematics is not a subject all students will like. This is an important reflection to consider, as "teachers greatly influence how students perceive and approach struggle in the mathematics classroom" (NCTM, 2014, p. 50). She stated:

> Yes, students do struggle and I mean as a math teacher you don't want them to struggle, but I think it's okay for them to be a struggler. I recognize as a math teacher that my

> subject is not going to be liked by everybody, but I ask my students still to try. So, I wouldn't say it's bad for them to struggle, though I feel like if they're not struggling it's probably just too easy for them or they made it too easy for themselves because they haven't done it correctly - like they skipped some steps.

With Ashleigh's previous statements that "[Struggle] should not be labeled as a bad thing…I mean it's part of being a student and an expected part of that whole learning spectrum…Yes, students do struggle and I mean as a math teacher you don't want them to struggle" contradictory messages emerged regarding her beliefs about struggle in learning mathematics. Further probing provided insight into Ashleigh's perspective on struggle in learning when she shared how she would respond to a student who had made a mistake while renaming a mixed number as an improper fraction. She shared:

> So, depending on what her mistake was - maybe she turned a mixed number improper incorrectly. I'd be like "Let's check our math on this one real quick, because I don't get this number when I do the work to turn this into an improper fraction." And so, I would go through her work because she's got the proof that she's tried; there's just like a little bit of a roadblock. I would try and find where is that mistake and then guide her and be like "I think if you can get this part fixed then you're going to see that you're going to be getting the right answer."

Ashleigh's previous responses indicated that the struggle is productive if students are trying and advocating for themselves. However, she ultimately responds to student struggle in a manner that removes the opportunity for the student to engage in productive struggle. Ashleigh's example reinforces her previous statement that she does not want to see her students struggle.

In both Jessica and Ashleigh's depictions of student struggle, a negative attitude about mathematics manifests in students as a result. Jessica depicted this when she noted "I think it puts a negative attitude in their minds for the kids when they do struggle, [then] they are very negative when it comes to doing [math] as students," while Ashleigh stated "Yes, students do struggle...I recognize as a math teacher that my subject is not going to be liked by everybody, but I ask my students still to try." Inherent in both is the acknowledgement that struggle is expected in learning mathematics, leading to students' dislike of mathematics.

While both participants recognize struggle as an expected aspect of learning mathematics, the struggle is not presented with a positive potential in learning. Instead, both teachers demonstrate an approach to alleviate this struggle, with nuanced differences. Jessica's approach demonstrates an attempt to alleviate the negative attitude that students might develop, telling them "You know just some people are better at math than others, you know there's other subjects you're better at." While well intended, this response might instead serve to demotivate students and lead to future low performance (Rattan et al., 2011). Ashleigh approaches the alleviation of struggle differently, taking ownership of finding the mistake and leading students in how to overcome the barrier. She explained "I would try and find where is that mistake and then guide her and be like "I think if you can get this part fixed then you're going to see that you're going to be getting the right answer." While Ashleigh's approach to alleviate the negative attitude in students is different, the removal of opportunity for the student to struggle with important mathematics will diminish their opportunity to improve their understanding (Hiebert & Grouws, 2007). Both participants negative depictions and responses to struggle in learning mathematics indicate that while struggle in learning mathematics is anticipated, it is not a beneficial aspect of learning.

Theme 2: Teacher Anticipation of Struggle in Learning Mathematics

A second major theme that emerged as an important consideration in the data related to teachers' anticipation of struggle in learning mathematics. Through the analysis of teachers' descriptions of anticipated struggle in student learning, two sub-themes emerged indicating the anticipated struggle as an affordance to learning mathematics or as a constraint to learning mathematics.

Sub-theme 2a. Anticipated struggle as an affordance in learning mathematics

One sub-theme that emerged in the data depicted anticipated struggle in learning mathematics as an affordance to learning. Struggle as an affordance in learning mathematics was depicted in data from two participants, Katherine and Melissa, in descriptions of their mathematics learning environments. In these descriptions, the positive opportunity that struggle afforded learning mathematics was depicted indirectly, through expectations of student engagement where students demonstrated multiple solutions to a problem (e.g., models and algorithms); explained their thinking, reasoning, and problem solving strategies to others; engaged in making connections and building understanding between prior knowledge and new learning; and held the expectation for students to use their resources as they struggled productively.

The first example of struggle as an affordance to learning was shared by Katherine. In her description of fraction multiplication, a topic that students were learning at the time of our discussion, she shared her anticipation of student struggle in developing and understanding visual models. In spite of this anticipated struggle, Katherine maintained the expectation of students to demonstrate their understanding in this way. Katherine explained:

> The work we're doing right now is actually [multiplying] a fraction by a fraction and it's interesting because kids want to just do like the instantaneous like *Oh - I multiply across the top, but then multiply across the bottom and then I'm done* and like developing that understanding and that visual model has been not as easy as just knowing your facts… and trying to like get them to think beyond a rote memorization strategy… *How can you show me?* And trying to really get them to see there are multiple models, there's multiple ways that you can show the same thing. Even beyond the "there's two ways, show me three ways, show me four ways" so that *I truly understand that you understand* what it is that you need to do…And as we get to more complex models, making sure that they're transferring it - that it's not a skill that they have learned in isolation and memorized because they've seen one example.

While Katherine's expectation for students to demonstrate understanding by modeling was indicated as challenging for students, the purpose and potential to build understanding was a beneficial outcome. The expectations in the learning environment described by Katherine require students to make connections between the algorithmic, step-by-step process, and modeling solutions in multiple ways. Furthermore, she reflects on how student understanding of models is important as students transfer and apply prior knowledge in more complex problems. Learning environments like the one described by Katherine foster productive struggle and promote student thinking, reasoning, problem-solving, and sense-making (e.g., Hiebert et al., 1996; Kapur, 2009; NCTM, 2014; Stein & Lane, 1996). While the opportunity to learn with struggle is inferred in the previous response, Katherine provided a specific example of an interaction with a student demonstrating struggle. She shared how she navigated this situation, saying:

> I'm envisioning right now one of my students who – we were taking the checkpoint and she broke down in tears and she's like "I don't remember how to do this" and I'm like "It's okay. Where can *you* look? What can *you* do to help you get there?" I also knew she totally knew how to do it, but it was like that freeze moment [because] the question was phrased in a different way than she was used to having it phrased and like trying to get kids so they're not at that point of like that fight or flight and like it's that panic moment that has just set in - it's like, how can we reassure kids that you are able to do this and you do have the tools and you have what you need?

While the examples provide a broad depiction of Katherine's anticipation and expectation of struggle in learning mathematics, this was also apparent in her expectations for student engagement with the Cornbread task (Rumsey et al., 2016). In her anticipation of student needs to engage with this task, Katherine shared the need to provide more complex numbers due to the ease of unit fractions. She said:

> One of my initial thoughts was this is exactly where we are at in the year right now for math. Like a question like this, I could very easily see that my kids would solve this week; like the timing is exactly where we're at and what's going on…My other initial thought was they're pretty easy numbers, because they're unit fractions and the cost is $12…you know you could just change the cost - the cost could be $24, the cost could then be $18 and looking at different ways that we can get it, or you can also look at making the fractions different fractional pieces and how would that come into play, and then…like you can have the kids make it - like have the students actually be the ones to come up with the numbers…It depends on your population - your group of kiddos and

> kind of where they're at skill-wise.

The anticipation of struggle with the task shared by Katherine did not depict how students might find difficulty, but rather considered how she could afford opportunity for struggle in learning. Katherine considered how modification of the task could meet student learning needs, depicting the learning potential afforded when you "just change the cost…you can also look at making the fractions different fractional pieces…you can have the kids make it." The anticipated task modifications that Katherine believes are necessary for her students to engage in deep reflection and problem-solving, in turn lead to an opportunity for learning with productive struggle.

In addition to her anticipation of student engagement with the task, Katherine also considered how students might model the solution to this task, with the possibility for multiple representations to emerge. In this process of anticipation, she indicated flexibility in the expectations of student modeling to demonstrate understanding. Katherine reflected:

> It would be fascinating for me to think of how they would set up their model. What would they divide it into? Would they draw a square divided into four and then would they divide just the one chunk into three? I would like to think that they would do that. I don't know if they would…I thought, could you have kids show their model in more than one way?…Who says that your pan can't be a one by twelve and it's a really long wonky pan? Right? Like how do we think about different ways that we can make shapes to represent the fraction pieces? So, it's all about that model and do they truly understand how to create the model to express what it's asking.

Anticipating multiple approaches to solving this task and modeling the solutions in a variety of ways depicts Katherine's expectation that students will draw upon previous knowledge to determine how to solve the tasks on their own (Carpenter et al., 2014). Throughout Katherine's descriptions of her learning environment, hypothetical use of the Cornbread task (Rumsey et al., 2016), and the anticipation and assurance of struggle in learning, Katherine conveys struggle as an affordance and provides opportunity for students to learn with productive struggle.

Similar to Katherine, Melissa's description of her classroom expectations depicts the anticipation of struggle in learning mathematics. She describes the anticipated need for teaching approaches that support students in building understanding and making connections in a subject that might feel less intuitive to them, while maintaining the expectation that students own this opportunity to struggle. She shared:

> Students need to be able to understand mathematics which can feel foreign to them. Manipulatives can help to build understanding and help students make connections…I also have students make their own fraction strips to understand fractions and unit fractions better…So in general, when I think about planning I want students to begin by exploring a problem on their own and then share their solution strategies with partners. When they have had enough time, I begin eliciting strategies [from the students] for solving [the task]…So – Like that thinking about *What kind of questions could I ask?* is important so I am sure students own their engagement.

Melissa shares her anticipation that that learning mathematics can be difficult and "feel foreign" to students yet demonstrates a commitment to having students build their understanding and make connections. It is important to note that this acknowledgement of difficulty does not become an excuse to remove the opportunity for students to build their own understanding, but rather informs her anticipation and planning of teaching strategies, such as using "manipulatives to build understanding and help students make connections…hav[ing] students make their own fraction strips to understand fractions and unit fractions better" and planning questions that will ensure the sense-making and ownership of learning remains with the students.

Similar to her anticipation of student struggle while learning mathematics, Melissa's reflection on using the Cornbread task (Rumsey et al., 2016) indicated the anticipation of connections and potential challenges. During her initial reflection at the end of the first interview, Melissa shared how she might navigate anticipated student struggle to engage with the task. She noted:

> I feel – I believe that students would be ready to begin the task. They understand one-fourth and one-third and they could draw a cornbread pan. I think they might…they might not be ready to model one-third of one-fourth. I might just show them the beginning of the task first…No - I might start with another example and show them the model. Then I would have them interpret and explain the model in the example to me before doing the Cornbread task.

Melissa's reflection and anticipation of student engagement with the task demonstrates her anticipation of potential student struggle and reflection on how to scaffold and provide access to the task. While the anticipation of struggle in her initial reflection led Melissa to the potential approach of modeling a different task and engaging students in making connections, her response to this anticipated struggle yielded a different approach when more time to reflect was afforded. The anticipation of struggle in her updated approach leads to the potential for students to make sense of the task. Melissa shared:

> So, I kind went through the different steps of the LESRA (Launch, Explore, Summarize, Review, Apply) model…so I thought this would be a good launching activity…because it kind of leads into the exploring and then we can practice how we would solve more

> problems like this. So, in the launch, I’d just kind of ask them to think about [the task] because it doesn't really give them a question - it's just kind of giving them some information, so I’d ask them like *What kind of questions do you think we could ask about this cornbread?*

Melissa notes having the students reflect on the task, considering “What kind of questions do you think we could ask about the cornbread?” before engaging in a solution process, the opportunity for student connections and questions emerge to build upon. In the next phase of her lesson, Melissa anticipated that students would benefit from exploring the fraction relationship depicted in the task in a concrete manner, folding paper to represent fractional shares, before moving to a solution or representational model. She said:

> So, it says that the cornbread pan is square, so my thought was to pass out little colored construction [paper] squares and have them just explore - like how they could even cut this apart, to make even size pieces? And I would give students a bunch of them, so they have a whole bunch to work with. After they kind of think on their own they could work with partners and then we would start to discuss it as a class…then [students] would begin working through the questions and they could be using those cutouts as they think about it. So when [students] think about how a cornbread pan was one quarter full they could actually fold the cutout and go through that process. Then when it says she buys one third, then they could fold it again.

Melissa’s initial anticipation of students’ struggle indicated that “[students] might not be ready to model one-third of one-fourth” with her reflection that “I might just show them the beginning of the task first” removing student ownership of learning. However, following the opportunity to reflect more thoughtfully, the anticipation of student struggle engaging with the task led Melissa to consider an alternative approach. Students would explore fractional relationships by folding paper, while making connections and applying understanding from prior learning to this novel task. The latter approach has an increased potential for productive struggle to emerge.

In both Katherine and Melissa's anticipated struggle in learning mathematics, this struggle was depicted as an affordance to learning with understanding. Their expectations for student engagement have the potential to afford student learning with productive struggle. In spite of this parallel depiction, there were nuanced differences. Katherine did not anticipate student struggle in engaging with the Cornbread task (Rumsey et al., 2016), however, anticipated the need to extend the task providing greater opportunity to learn with productive struggle. Melissa did not anticipate the need to extend the task, however, anticipated the potential for student struggle to emerge in students' initial engagement with the task. Both participants responses to the anticipated need for struggle as well as struggle with engagement resulted in strategies that afford the opportunity for students to learn with productive struggle.

While the term struggle was not used directly by either Katherine or Melissa when describing the challenges in learning they anticipated in their mathematics classrooms, the learning environments described embrace struggle as a learning opportunity. The teaching practice of anticipation in both Katherine and Melissa's descriptions of student engagement demonstrate their conceptualization of struggle as an affordance in learning mathematics, with a clear purpose and potential for students to make connections and build understanding (e.g., Ball, 1993; Kapur, 2012; Lampert, 2001; & Schoenfeld, 1998).

Sub-theme 2b. Anticipated struggle as a constraint in learning mathematics

An opposite sub-theme that emerged in two participants' anticipation of struggle in learning mathematics depicted struggle as a constraint to learning. The depictions of struggle as a constraint in learning mathematics was apparent in responses from Jessica and Ashleigh. These responses demonstrated how anticipated struggle resulted in teaching practices devoid of the opportunity for problem-solving or student engagement in making connections that build understanding. In these classrooms, the anticipated struggle was prevented or removed, with learning focused on students learning specific algorithms and strategies to remember them.

The first depiction of struggle in learning mathematics as a constraint was shared in the description of mathematics teaching from Jessica. In her reflection, the anticipated struggle in learning mathematics was described as a barrier that made learning difficult for students. Jessica indicated that students' struggle in learning mathematics was a constraint to their engagement, resulting in her intentional efforts to remove the potential for student struggle. She noted:

> Math can be done lots of different ways, so the hardest part is, I can't – I don't want to show them like four ways [of] how to do something. I've done that before – like okay – show [students] these four ways to do this. That was hard; kids really struggled with that…But you know – that's what the book told me to do, is to do all those methods. So, what I did was put it down to the main one that has worked over the years for kids. And then for the kids that didn't – you know quite get it – then I would say okay – here's another way…But basically any way they can show me how to do it, I'm okay with it – whether it's a model, whether it's just doing the math – I'm okay with however they do it as long as they understand and can do it correctly.

In this depiction of teaching mathematics, Jessica's anticipation of student struggle in learning a variety of methods to solve a problem has resulted in her conscious decision to "put it down to the main one that has worked over the years for kids" in an effort to prevent the potential for struggle. Jessica's response to focus on one "main" strategy aligns with previous investigations indicating a similar response from teachers who feared that students would struggle with navigating multiple approaches to solving a task (Leikin et al., 2006). While the intent to prevent struggle in learning from occurring is perceived as helpful, this approach is unproductive for student learning. Demonstrating an example of what NCTM (2014) noted as an unproductive belief that “students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems” (p. 11), a student’s opportunity for productive struggle is limited. The removal of anticipated struggle in learning was reinforced later in the conversation, when Jessica provided an example from a fraction multiplication lesson she taught the day of our interview. In this description, the anticipation of struggle resulted in strategies to prevent this. Jessica stated:

> Like when we did fractions today - when I said [multiply] the two and one fourth times three and two thirds, [students] were multiplying the numerators, multiplying the denominators, and multiplying the whole numbers. *And I knew they were going to do that because kids have done that in the past.* So, I told [the] kids we're not just multiplying, you have to change [the mixed numbers] into [improper fractions] first. So, knowing that ahead of time really helps me.

While Jessica did not note that the student struggle depicted here was a constraint to learning, by removing the opportunity for students to struggle productively, the perception of learning constraint is apparent and mirrors her previous remarks that this is “hard” for students. Instead, the teaching approach shared by Jessica relies on the memorization of a process without the opportunity for students to understand why or how the mathematics works. While Jessica depicts the anticipation of student struggle as helpful to herself, the resulting instructional approach lacks opportunity for students to make connections, build understanding, and learn with productive struggle. Her anticipation of student struggle in learning as a constraint results in a direct instruction approach and diminished opportunity for students to make their own connections and build understanding.

Similar to Jessica's response to anticipated student struggle in familiar learning, her response to anticipated struggle on the Cornbread task (Rumsey et al., 2016) was to remove the potential for struggle to emerge. In her reflection on areas of this task where she felt students might get stuck, she shared the following reflection and how she would navigate this struggle. She said:

> I think the part that maybe you'll get some hung up on it is where - when it says "Mrs. Smith buys cornbread from a pan that is a quarter of the way full. Now she buys a third of the remaining." I think they'll get the fourth of it, you know. They'll get that quarter of it full, but they should be able to - now that we've been talking about one third "of that remaining." Again, they [need to] just kind of look for those clues. If they get that clue there, they'll get it. I mean they've got to learn when they do any kind of word problems they have to break it apart. So, I might say "you're going to start with…the quarter full first, so draw your square pan of cornbread…make it into a fourth, and then focus on that one fourth and she only wants one third of that remaining piece and so on."

While Jessica's response indicated her anticipation that students would struggle to interpret the portion of the story problem "Now she buys a third of the remaining," she stated that students should be able to find success if they "look for those clues." In spite of her own recognition that students should be able to "get it," she proceeds to describe how she would tell students what to do, in a step-by-step approach. This response to anticipated struggle precedes additional student struggle with modeling the fraction relationship that Jessica anticipates. Again, her depiction of the struggle and anticipated response that would remove the opportunity for students to struggle productively, is shared. She noted:

> I think if they draw that picture, they will get it. But again, that picture there - when they see "draw a diagram" they're like - they kind of moan and groan. But then when I show them the diagram and show them how easy it is, I think - I think they'll wonder why they're moaning and groaning.

In this reflection, Jessica acknowledges that drawing a picture to represent the fraction multiplication will help students to understand, however, also indicates that students dislike of modeling could be a barrier. The anticipation of students' dislike of modeling is reframed as anticipated struggle, when she indicates that by showing students "how easy it is" their concerns leading to their "moaning and groaning" are removed. Throughout Jessica's descriptions of anticipated student struggles in learning mathematics, this struggle is depicted as a constraint that she would prevent or remove, diminishing the opportunity for students to learn with struggle.

Ashleigh shared similar descriptions of her teaching with anticipated struggle in learning fractions and the benefit of a hands-on approach yet teaching that is highly structured and procedural in nature. Ashleigh shares her anticipation that fraction concepts will become increasingly more difficult for students, requiring students to successfully complete steps in the future. In her reflection, Ashleigh indicates that the anticipated struggle in learning fraction concepts can become a constraint to future fraction learning, resulting in teaching practices intended to mitigate future student struggles. She shared:

> What I feel is important is different than what I actually do, and I think part of it is…especially when it comes to the fraction concepts, that is a skill that I see a struggle in sixth grade, so I feel like it would be good for them to have more like a hands-on

> approach, but I don't necessarily do a hands-on approach. I give them like the steps to solve…because you're going to see it in the next grade and it's going to get harder, and in the next grade it's going to get harder, and so trying to find that piece to help them find like the deeper understanding is something that I'm still working on.

While there is an inferred value in mathematics activities that provide opportunity for students to make connections and grapple with mathematical concepts, this is not how Ashleigh teaches or engages students in learning. Ashleigh's recognition that hands-on learning is a valuable approach and would help learners conceptualize fraction concepts is important, but only if students are afforded this opportunity. The anticipation of future student struggle in learning fractions being a constraint to learning is met with removal of the opportunity for students to engage in hands-on learning that could lead to the deeper understanding for students that she seeks. Ashleigh's decision to use a step-by-step approach appears to be guided by an unproductive belief that in order to teach mathematics effectively, learning should focus on practicing and memorizing procedures (NCTM, 2014), as the application in subsequent grades will only become more difficult.

Further into the interview, the use of a hands-on approach was revisited directly to ensure that the initial response shared by Ashleigh fully depicted the reality of her instruction. In her initial response, Ashleigh anticipated that a hands-on approach to learning would be beneficial in supporting her sixth graders during fraction instruction but deferred to a procedural approach when teaching fraction concepts. When asked about how hands-on learning might inform her fraction instruction, Ashleigh reflected:

> I think it would probably more if I knew they were more of a visual [learner] and that's how they got it, I would probably try and incorporate more visuals. If they were more of

> a hands-on [learner] and they showed that, I would maybe do more of a hands-on approach to things. I should be doing those things more often it's just I don't…So then that one on one support would come after the fact, where I would present the lesson to them and then it would be "Okay, so you're struggling and let's work through some problems together" or "I'll help you solve this one - that's a whole group problem." It's more so once I see that there's a struggle - that's when I really start working with them a little bit more.

While the opportunity to learn with a hands-on approach could exist in this learning environment, it is only presented as a hypothetical in response to a learning style, rather than as a valued instructional practice. Additionally, while struggle was previously noted as an expectation in learning, here it is depicted as a constraint in learning that must be addressed reactively. Neither of these examples depict the potential for productive struggle in learning mathematics, paralleling Ashleigh's previous statement that she does not want her students to struggle.

While Ashleigh's anticipation of student struggle in learning fractions resulted in teaching steps to solving problems, the Cornbread task (Rumsey et al., 2016) was different than instructional opportunities described, affording an additional opportunity to understand Ashleigh's experience. When Ashleigh was asked to describe how she would anticipate using this task in her classroom, her reflections focused on learner engagement. She said:

> …I think depending on the level of the students, I think some would really thoroughly enjoy something like this. I could see if I put them in…partners, but then you will have the one person who is really good at math and the one who struggles and they'll be like "oh yeah that's right, yeah that's right" and I would almost want…Conversation is so important, I would almost want to just have them try it independently and then see how

> they do, and then have them work with maybe a partner and share their ideas with each other and see if they can maybe figure out any hiccups.

Ashleigh's initial anticipation of student engagement with the task indicated her belief that only some students would enjoy an open-ended task like this, while others may struggle and rely on their peers for the answers. While this response does not indicate a clear consideration of the struggle that would likely emerge, her perception that only some students would enjoy this required more insight. When Ashleigh was asked which portions of the task she anticipated that students would do well on, she instead described areas that might be challenging. She said:

> Okay. I think the part that might be the most difficult is probably then the breakdown of like the cost. Where we look at this and we're like well half of 12 is six, but then in A they take a pan that is one-fourth full and then she buys one third of that amount, so it would be that whole breaking down that price barrier. They might be able to easily draw and model - What is that supposed to look like? But then figuring out the cost, from that point would be a little bit difficult for them.

While the intent of the prompt was to determine areas Ashleigh anticipated student success, her anticipation of potential constraints indicates a deficit approach to the potential for learning with this task. With the anticipated struggle, Ashleigh shared the following approach to supporting students, saying:

> We would probably talk through it initially and [I would say] *Okay well what is half of 12? Now I need a quarter of 12. Well, if half is six, what's a quarter? It's a half of a half. What would then be a quarter? Oh - that's going to be three* and so I think we would try and figure out what those fractional pieces would be of the whole $12, so that they can

see that as their pieces get smaller…the price that they're going to pay is going to decrease.

This reflection provides insight into Ashleigh's apparent hesitation with this type of mathematics task. Due to the open-ended nature, there is an implied anticipation of struggle that would need to be addressed proactively, and a perception that students would need structure to engage successfully. This approach mirrors Ashleigh's description of her classroom teaching, reinforcing that while she expects struggle, the anticipated student difficulty is perceived as a constraint to learning and diminishes the opportunity for productive struggle to emerge.

Jessica and Ashleigh's anticipated struggle in learning mathematics was depicted with similar responses to struggle as a constraint to learning with understanding. Their expectations for student engagement in times of anticipated struggle resulted in the removal of the potential of the struggle through a funneled and step-by-step approach to teaching. Similarly, both participants recognized the value of models in teaching fraction concepts. In spite of their agreement on the benefits of teaching fractions with models, there were nuanced differences. Jessica anticipated student struggle in modeling the cornbread in the task, removing the expectation for students to grapple with this independently and instead showing them "how easy" it is to model. Ashleigh also recognized the potential in hands-on learning and modeling of fraction concepts noting "I should be doing those things more often it's just I don't;" choosing to teach using a procedural approach instead.

While each participant's anticipation of struggle in learning mathematics depicts their experience and engagement with the phenomenon, the experiences depict important similarities and differences. An interesting finding within participant's responses was that Katherine and Melissa's description of anticipated struggle in learning was depicted as an affordance to learning. Contrary to this, Jessica and Ashleigh's description of anticipated struggle was depicted as a constraint and responses to the struggle that prevented or removed the opportunity to learn with struggle.

Theme 3: Teacher Conceptualization of Student Understanding of Mathematics

Another major theme that emerged in the data was teachers' conceptualization of student understanding. This conceptualization was apparent in teachers expectations related to students demonstration of learning and understanding in their classrooms. In the descriptions of student understanding provided, important insight into how each participant's definition of understanding provided the opportunity for student learning with productive struggle emerged.

The two sub-themes of student understanding that emerged described students constructing connections between new and prior learning experiences, or students knowing mathematical facts, procedures, rules, and memory tricks taught by the teacher. The experiences shared by participants were essential in determining how each conceptualized and determined student understanding of mathematics in their classrooms. The experiences and conceptions of mathematics teaching and learning shared by participants demonstrated the potential influence of struggle in learning mathematics on their teaching and expected learner engagement (e.g., Kilpatrick et al., 2001; Leinhardt & Greeno, 1986).

Sub-theme 3a: Students constructing connections between new and prior learning

Students construction of connections between new and prior learning experiences provided one depiction of student understanding that emerged from participants. This conceptualization of students understanding emerged in the data from Katherine and Melissa. These teachers expected students to construct understanding, making connections between prior learning and new learning, while modeling, explaining their thinking and reasoning, engaging in problem-solving strategies, and representing the mathematics in multiple ways.

One of the two participants whose conceptualization of student understanding as constructing was Katherine. Katherine viewed learning mathematics as an experience where students construct their understanding through learning experiences. She shared her expectations of how students must conceptualize fractions, demonstrate relational understanding of fractions, and use visuals to make connections beyond an algorithm. She reflected:

> With fractions - like do [students] understand that a fraction has to be an equal part, and do [students] understand that when you create those equal parts those parts can become parts? Like that whole idea of taking the whole and breaking it down. So that key concept of parts within parts and equal parts…as well as just understanding what does the numerator *represent*, what does the denominator *represent*, and how do we use that information to help us?

The expectation of Katherine's students demonstrates conceptual connections of a fraction that require understanding of fractions that extends beyond knowing a procedure or fact. Katherine reflected on her current unit of multiplying fractions, indicating that "it's interesting because kids want to just do like the instantaneous *Oh, I multiply across the top and then multiply across the bottom, and then I'm done*. But developing that understanding and that visual model has not been as easy as just knowing your facts." In this reflection she recognizes that developing fraction understanding in this manner is not as easy as knowing facts, however, maintains the expectation that students develop fraction understanding conceptually as well.

An additional consideration that Katherine noted when determining students' understanding was students' ability to construct models to represent the mathematics in multiple ways. Students are expected to construct multiple visuals, algorithmic approaches, or written explanations to represent the mathematics to prove they have constructed new knowledge, connecting and relating ideas in novel contexts, rather than simply mimicking one approach. Katherine provided an explanation of her expectations for students when demonstrating understanding. She said:

> You know [with visual models] it's interesting…the kids are like yeah, yeah we get it, we know it, and I'm like yeah, but do you *really*? Like yeah, you can give me an answer, but can you think about multiple ways in which to do this? So, [a] perfect example we just did a checkpoint this last week of [multiplying] a whole number by a fraction - like seven times one third. And so my question is "Well how did you get it?" And what I value is all the different ways that they could show me. And the thing is once they get a model, they generally stick with one model – *Well this is the model I'm going to use, because this is the one that works.* [So] trying to really get them to see there are multiple models – there's multiple ways that you can show the same thing. So, "show me two ways, show me three ways, show me four ways" so that *I truly understand that you understand what it is you need to do*. So, I think having that visual representation [with] all of the different examples and ways to do it is a really good approach I try to use each day.

Katherine describes how that students often project an attitude of "yeah, yeah we get it, we know it" but remains dissatisfied until students can prove their understanding. The teaching response described by Katherine engages students in reflection with questions like "Well how did you get it?" as well as learning opportunity that has the potential to deepen student understanding as they connect and relate "multiple models" to see that there are "multiple ways you can do the same thing." This approach to teaching requires students to build understanding as they connect mathematical facts, procedures, models and ideas through active learning engagement (e.g., Cobb, 1988; Hiebert & Grouws, 2007; Richards, 1991; Simon, M.A., 1995). By engaging her students in connecting and relating the models and algorithms, Katherine's students construct understanding of the concepts they learn.

Melissa held a similar expectation for students to construct understanding of fractions by conceptualizing and drawing visual models, constructing fraction strips, and using manipulatives to represent fractions while connecting fraction concepts. Melissa described how engaging students in constructing understanding in this way helps students to conceptualize fraction magnitude and develop relational understanding. She shared:

> What I see as essential in [learning] fractions is making it visual. Students need to understand the mathematics which can feel foreign to them. Manipulatives can help to build understanding and help students make connections…Students also *have* to draw a picture. Especially with fractions, it helps them to show a fraction. For example, what does 7/8 *mean?* What does this look like? How big or small is this? When students have to draw the fraction and shade the parts to show that seven out of eight parts are shaded, they can think "Oh, that's almost one whole." The visuals are a must in fractions for sure...I also have students make their own fraction strips to understand fractions and unit fractions better.

Melissa clearly describes a learning environment where the expectation for students' engagement in the construction of visual models, use of manipulatives, and construction of fraction strips builds understanding of fractions in a relational way. Hiebert and Carpenter (1992) describe this construction of understanding, noting that "children construct relationships which yield a connected network containing representations of the materials and their interaction with them"

(p. 70). The expectation for students to use prior knowledge and experiences, connecting and relating fraction representations, while attending to these relationships were described in both Katherine and Melissa's depiction of how students build understanding in their classrooms.

Another way that Katherine and Melissa expected students to demonstrate understanding was in their ability to explain their thinking, reasoning, and problem-solving related to mathematics tasks. Katherine emphasized the expectation of her students to construct and communicate accurate explanations of their solution process, while making connections to their partners ideas. She described:

> Asking student to share with one another – "Okay, if I show you this [fraction], what could represent the one? What If I change [the fraction] to this? What does [the fraction] mean this time?" Like the whole goal of – well, not the whole goal, but one of our main goals is like being able to accurately communicate about the math and the work that we're doing...It's interesting because sometimes you put them in small groups or with partners and it's like *well I'm not going to talk to my partner, I'm just going to sit by my partner*. And – that's not the point. The point of this is that you have a conversation about the work that is being done. And you know I think of our mathematical practices and having that ability to talk about math is key. And if we don't get kids to do that, it's never gonna get better.

Katherine's expectation for students to construct, communicate, and justify their conclusions with others while building understanding is an important consideration in her learning environment. Similar to Katherine, Melissa shared her classroom expectation that students construct and communicate mathematical ideas with peers. Student communication and collaboration were an expectation of Melissa as her students learn mathematics. She described how this dynamic supported students in problem-solving and justification of their solution strategies in a collaborative learning dynamic. She said:

> Students are expected to turn and talk about solutions to problems I have on the board, or if they are not done, where they are at. They need to share their thinking, so others understand how they solved a problem…Partner and group work is an essential strategy in [my teaching] – this collaborative and cooperative learning, where students have to think about problems and solve them together - this makes them think about why their strategy is meaningful and then they have to tell others. Then there's discussion in the classroom where the students share their responses and I have other students explain the strategies in their own words to learn from each other.

The expectation in Melissa classroom for students to construct knowledge collaboratively, with the additional expectation that students revoice problem-solving strategies shared by peers, creates a learning dynamic to deepen student understanding. Hiebert and Carpenter (1992) note the efficacy of student engagement in this way, as "similarities and differences between alternate representations of the same information are relationships that can stimulate the construction of useful connections at all levels of expertise" emerge (p. 68). Both Katherine and Melissa's facilitation of mathematics discourse in their classrooms that engages students in purposeful sharing of ideas, with the expectation of students to explain their thinking, demonstrate important teacher and student actions recognized to build a shared understanding with the opportunity for productive struggle.

While demonstration of understanding in Katherine and Melissa's classrooms both depicted the expectation that students construct connections between prior learning and new learning, demonstrate conceptual connections in interpreting fractions, use visual models to connect and relate learning, and communicate about mathematics, nuances in their classroom structures that supported this dynamic were apparent. In Katherine's classroom, the expectation to explain conceptual connections between the visual model and fraction number required that students understand "that key concept of parts within parts and equal parts…as well as just understanding what does the numerator *represent*, what does the denominator *represent*, and how do we use that information to help us?" with an expectation that students to explain how this information helps them. Similar, yet demonstrating a nuanced difference, Melissa described the expectation that her students understand a fraction in these terms "What does 7/8 *mean?* What does this look like? How big or small is this? When students have to draw the fraction and shade the parts to show that seven out of eight parts are shaded, they can think *Oh, that's almost one whole*" with the ability to make connections apparent, but without the requirement for students to explain the potential in these understandings.

Katherine and Melissa also described the importance of students communicating about mathematics. In her description of this engagement, Katherine indicated that "one of our main goals is like being able to accurately communicate about the math and the work that we're doing," including the interpretation and explanation of fractional relationships. Melissa shared a similar expectation, adding the expectation that students "students share their responses and I have other students explain the strategies in their own words to learn from each other," in the act of revoicing. While a few nuanced differences existed between these participants, they depicted the same overarching expectations for student demonstration of understanding.

Sub-theme 3b: Students knowing mathematical facts, procedures, rules, and ideas

Contrary to student understanding being constructed through the connections they make, an additional conceptualization of student understanding as knowing mathematical facts, procedures, rules, and ideas taught by the teacher emerged in the data. Understanding as knowing was depicted in responses from two participants, Jessica and Ashleigh.

Jessica was one of the participants whose conceptualization of understanding mathematics included students knowing facts. She emphasized the importance of this knowledge a few times during our interviews as an essential consideration for being successful in learning fractions. The importance of students knowing facts as a component of understanding initially emerged during our first interview. She said:

> I guess my biggest thing is, especially with fractions, is knowing the basic facts - just making sure they know those basic facts. And basically, do they understand those fractions – just that visual part of it, like what does it mean to have a part of something?... Just like I said, more of a visual knowing where the numerator is at, where the denominator is at.

While the majority of Jessica's student expectations in learning fractions focused on "just knowing those basic facts," she also alluded to "that visual part of it." As Jessica referenced visuals in a context that could include drawing, in addition to visualizing where the numerator and denominator are located, additional probing was necessary to ensure a correct interpretation. When probed about the use of visual models, she noted:

> Yes, I have some visual models. A lot of kids don't - you know, there are some kids that are like *I don't need those types of things*. So, it's not necessarily for all of them. I started out - when I was first teaching fractions I did all kinds and the students were like *Yeah, I know – just the multiply the numerators, multiply the denominators* and then they just didn't like [creating visual models]. So, a lot of times now it's just for the kids that I feel need that visual - like some kids are like *No I can get it*, they don't want that [visual]

> piece of it. So…I break them up into groups for those kids that really need those visuals…but like I said it's not - it's not for all of them, some of them get bored with that pretty easy because they're like *I already know what to do*.

While visual models are explained here as a representation of the mathematics, Jessica noted "A lot of kids don't - you know, there are some kids that are like *I don't need those types of things*. So, it's not necessarily for all of them" and reserved "just for the kids that I feel need that visual." Throughout our interviews Jessica noted students not liking or enjoying the expectation of creating visuals or drawings with the use of visuals described as a response to students struggles, rather than as a means to build understanding of the fraction concepts. This approach has the potential to frame student thinking related to the purpose and potential of models as negative, being reserved as a strategy "for those kids that really need those visuals" as they struggle. As the interview progressed, Jessica expanded upon her definition of student understanding, connecting this to a procedural approach. Jessica valued teaching that ensured student success by providing assurance to students that if they followed "this" process, it would "work every time." She said:

> So, I'd say "If you do it this way, you know it will work every time" is what I kind of tell them…As long as they are both fractions - you know if you had a whole number times a fraction you have to make that whole number a fraction, and you're multiplying the numerators, multiplying the denominators, you know, then you have to simplify…They can do it - like I was teaching the kids "Okay, you could multiply - add - same denominator" we kind of have that…mindset like "Oh, we multiply the denominator times the whole number, and then we add the numerator - you know over the same denominator."

While Jessica explained the efficiency of students learning a procedure that “works every time,” she also described students for whom this approach is not effective in building understanding. In these instances, she noted a different teaching approach that relied on conceptual connections, but where understanding was demonstrated through memorized steps. She said:

> That works for some kids, but otherwise sometimes I have to pull it apart and be like okay two and one fourth is nine fourths. Or you can do four fourths, plus four fourths, plus one fourth, which is still the nine fourths so just showing them it's you're still going to get the same answer…So what I do is when I call on those kids, they have to walk me through problems. And I really like that, because it gives them - they have to explain the steps in words. I know that they understand it if they can explain [the steps], you know from beginning to end if they can put those steps into words and explain it to somebody else, then they know they’ve got it. That is my biggest thing really if they can explain it - those steps - then I know that they got it.

The expectation for students to demonstrate that “they know it” by using a procedure deemed to be efficient and effective by Jessica aligns with her previously stated teaching beliefs that it is “hard” for students to learn more than one way to approach solving a problem. In this example, the exception to this was when students failed to demonstrate understanding through the ability to share the steps in solving the problem at which point she shared the conceptual connection to unit fractions. Ultimately, student understanding in Jessica’s classroom was measured by their ability to complete and share the procedural steps with others.

While Jessica’s procedural description of student understanding indicated a strong conviction, Ashleigh’s descriptions depicted inconsistency between her beliefs and actions. Ashleigh described her teaching practice as being different than what she believes to be important for student learning, resigning herself to a procedural approach to define student understanding. She describes her teaching and students' engagement in this way:

> What I feel is important [for student learning] is different than what I actually do…So, I feel like it would be good for them to have more like a hands-on approach, but I don't necessarily do a hands-on approach, I give them like the steps to solve. We do a lot of practice together as a whole, they get a lot of practice by themselves, they get practice with a partner and they just they see it over and over and over again. So, I mean if I teach the chapter on the fraction decimal unit, then they're going to see it on our quizzes - after [practice] we do a Friday quiz. So, they're going to see it over and over and over again, to keep them knowing that you need to know this.

Ashleigh's description of her teaching and student engagement provides insight into her expectation of students to know the procedural steps, practicing "over and over again" in preparation for quizzes. Her depiction of students knowing "the steps to solve" in preparation for quizzes mirrors unproductive beliefs of teaching and learning mathematics (NCTM, 2014). Both Jessica and Ashleigh's focus on students' knowing procedures diminishes the opportunity for student engagement in reasoning, problem-solving, and communication; recommended strategies to support learning with productive struggle.

Additional insights were provided when rules and memory tricks to support student use and application of procedures emerged. Jessica shared some specific strategies and approaches she has found to be effective in supporting students in their learning of mathematics. She has found that mnemonic devices, acronyms, visual structures like "the butterfly method," and other memory tricks can be helpful for her students. She explained:

> I do a lot with mnemonic devices, too…like the PEMDAS – Please Excuse My Dear Aunt Sally when it comes to doing any kind of work. I like anything to do with like *Okay you guys - remember* ***denominator*** *is* ***down***, you know, *so obviously the numerator is on top*. Anything we do like the butterfly technique to compare numbers. So, if you have two thirds and one fourth you do the two and the four - multiply those, and then [multiply] the one and the three. That's another way for them to compare numbers without getting common denominators. So, I try to try to show them lots of techniques that are kind of fun, and they can see the butterfly figure type of thing.

The focus of the strategies depicted by Jessica is to provide students reminders of specific rules or strategies, devoid of the expectation that students develop a deep understanding of the mathematics. Additionally, the emphasis placed on making mathematics fun implies that the process of engaging in mathematics is not enough to motivate students. Jessica indicates the addition of videos as well, to help teach students strategies in a manner that will serve to spark their memory. She shared:

> Sometimes I even find a video if there's a video that's kind of a more of a visual as well, I really like using those, especially when we get to dividing fractions there's a "keep – change – flip" [video]. And it's just more of like it keeps it in their mind so like when they go on to middle school they're like "We still remember keep – change – flip" and just those key terms that they can remember, or just a visual that they can remember.

While Jessica noted that she liked to use mnemonic devices, acronyms, visual structures like "the butterfly method" and other memory tricks and found them to be effective, the effectiveness was not consistent in all of her learners. She noted that in particular populations of learners, including students with IEPs, their ability to recall the memory trick was not demonstrated in their ability to apply the strategy. She reflected:

> Like even today I would say “keep” and they’d say “change – flip” which is fine, but the - some of those students with the IEPs and things like that they want that more of the visual. They like the cute saying and things like that, and they remember it, and they can say it and say it and say it, but they can't apply it. So, we do still have to go back to the more of a visual with *them* and the manipulatives - anything we can use for fractions versus just doing the math.

This additional insight into how Jessica defines student understanding as knowing is illuminated as being procedural in nature, supported by memory tricks, mnemonic devices, and memorized steps that allow students to demonstrate their mathematical knowledge by following steps. She also reinforces a negative association of visuals for those students who struggle, singling out a specific population of students who need this strategy to be successful in her mathematics classroom.

A similar approach of using a memory trick was shared by Ashleigh in a strategy she uses to help students remember the steps for dividing fractions. In her teaching she models a step-by-step procedure, using invented terms like “re-flip-rocal” that she creates, to help students remember the steps. She said:

> You would hear the repetition of *So when you're dividing fractions* I don't teach the keep-multiply-flip, I teach *You put the first fraction on top, the second on the bottom. They have to be improper to begin with, you have to multiply the bottom by the reciprocal, the “re-flip-rocal.”* So, you would hear me trying to come up with ways to help them remember the terms. It would be *What you do to the bottom, you do to the top* and so you

would hear a lot of like repetition, a lot of repetition and trying to point out like the more like important things trying to make connections to other things.

Ashleigh's description of a specific strategy she uses to help students remember procedures for dividing fractions relates back to her focus on students knowing the procedures. Using memory tricks or unique terms, she is supporting students in knowing the steps to complete a procedure.

Both Jessica and Ashleigh's expectation for students to demonstrate understanding mathematics by knowing mathematical facts, procedures, rules, and ideas taught by them removes the opportunity for student to build understanding and connections on their own (e.g., Hiebert et al., 2003; Silver & Smith, 1996; Stigler & Hiebert, 1997). While both Jessica and Ashleigh referenced visuals in teaching and learning fractions, their reflections depicted differences. Ashleigh noted that the use of visuals would be beneficial in supporting student learning of fraction concepts but that she did not use them. Contrary to this, Jessica noted the use of these visuals but that this was reserved for the students she felt needed them, in times when they struggled. Ultimately, the teaching and learning strategies shared by Jessica and Ashleigh to support students' learning of mathematics can lead to unproductive beliefs about learning mathematics.

Theme 4: Teacher and Student Roles in Learning Mathematics

The fourth theme that emerged in the data related to teachers' depictions of their roles and the roles of students in learning mathematics. Participants' descriptions of their teaching and the engagement of their students in learning mathematics provided valuable insight into the role of teachers and students in each of their classrooms. Among the four participants, two sub-themes depicting teacher and student roles emerged that illuminated teachers philosophies. The sub-themes demonstrated a student-centered and constructivist teaching and learning structure or demonstrated a teacher-centered and transmissionist teaching and learning structure.

Sub-theme 4a: Student-centered, constructivist teaching philosophy

The first sub-theme that emerged in participants' descriptions of the teaching and learning of mathematics was student-centered and constructivist in nature. In a student-centered learning environment, the conceptualization of teacher and student roles in learning mathematics emerged in descriptions of student-centered, constructivist classrooms. In these learning environments, teachers described student learning opportunities that both provided for and expected active engagement, provided resources to support independence, and valued student ideas and mathematical thinking.

The first example of a student-centered, constructivist classroom appeared in Katherine's description of a typical lesson. Her lessons begin with an opportunity for students to revisit prior learning as a foundation to engage in new learning. Following this, students are presented a mathematical task to engage in with their peers. In her description of a typical lesson, Katherine depicted how this engagement would look, noting:

> You would see the kids - the kids would be working. I try not to do the stand up and teach for too long type of mentality, right? Like it's - Okay I'm going to give you a little snippet and then you're going to go off, because the more I can get them up and engaging with their classmates and the less I talk, I feel like the better that is…like that whole idea of [students] having a conversation and working with partners, it really does further their thinking, whether there's somebody who is struggling or whether there's somebody who's doing really well. Either way, it gives them that opportunity…and then [they're] going to come back and I'm going to see what [they] did and give [them] a little bit more…So, the

> more that I can have kids take on that lead when I put out a problem and then they figure it out, the better off they are.

The learning engagement in Katherine's classroom holds the expectation for students to make sense of a mathematical task as they reason, problem solve, and communicate with peers to "further their thinking." In this learning opportunity, time is provided for student engagement, check-ins allow students to determine their progress, and students continue to connect their ideas as they move forward towards a solution. Katherine's belief regarding learner engagement can summed up as "the more that I can have students take on that lead when I put out a problem and then they figure it out, the better off they are." This teaching approach supports a student-centered, constructivist learning environment.

Additional considerations relating to the structure of Katherine's classroom and how she supports student engagement provided important insight into her conceptualization of students' roles in their learning. Katherine shared specific practices that were essential strategies that she incorporates into her mathematics classroom, including the opportunity for student collaboration and providing visual resources as a reference for students. saying:

> Varying who [students] work within those collaboration conversations is key, because while there is a comfort that comes with working with certain people there's a lot of learning that's lost if you always work with the same person. And definitely having things that kids can use and look at, because it's like one of those things - with like a toolbox, right - you're not going to use it unless it's available and [students] have to have things that are close by that [they] can refer to, and that [they] can use.

Katherine's description of her mathematics classroom demonstrates expectations for student collaboration, active engagement in problem-solving, and student-directed use of tools and resources. All of these expectations are characteristic of a student-centered, constructivist learning environment where students are expected to build their own knowledge and develop new ways of thinking (Lesh et al., 2003).

Similar to Katherine's conceptualization of teacher and student roles in the classroom, Melissa shared learning expectations that would foster a student-centered learning environment. Students in Melissa's classroom are provided focus for their learning engagement, opportunity to grapple with new learning in relation to prior learning, and the opportunity and expectation to share their ideas with peers. She said:

> So, if you were to see my lessons…I will talk to the students about their goals for learning. This helps them to get thinking about the problem. Then…always give them a problem to work on to get their brains thinking about the math we're learning – this is like the launch. I talk with them about the problem and make sure that they are ready to work on it. Next is the explore - I make sure that they have 2 – 3 minutes to work on the problem, so they do not feel rushed. Then I have students turn and talk with others about their solution or if they are not done, where they are at…they need to share their thinking, so others understand how they solved the problem. When they have had some time with that, I start the summarize part where students are asked to share their solutions and we talk about them.

The instruction described by Melissa follows a known model LESRA (Launch, Explore, Summarize, Reflect, Apply). This approach affords students the opportunity to engage with a task in a manner that requires them to make connections as they think about previous learning, problem-solve, and communicate their ideas on their own prior to formal and explicit instruction on a topic. This instructional approach fosters students' critical thinking as they work to build their own knowledge and understanding through active engagement (Polly et al., 2014).

The instructional approach described by Melissa is constructivist in nature, with the students' role in learning being the construction of knowledge through problem-solving, while her role is to structure the opportunity and facilitate this navigation (Cobb, 1988). With this insight, the role of students was further defined in her expectations and intentional planning with collaboration in mind. She noted:

> So, like I said before, partner and group work where students have to solve problems together is a must. Students know that they are expected to turn and talk about problems I share on the board. So, "What is the first step? Second step?..." and then they have to explain how to use the strategy we are learning to solve the problem. This makes them think about why the strategy is meaningful and so the discussion in the classroom – helps them to make those connections. This is something you would see in every lesson as an essential strategy.

The student discussion described by Melissa carries the expectation that students are able to describe both the process for solving a problem, as well as how the strategy is used. Melissa's intentional student engagement and facilitation of learning, as well as student presentation and explanation of ideas, provides the opportunity for meaningful mathematical discourse to ensue (NCTM, 2014).

In both Katherine and Melissa's descriptions of their classroom practices, a constructivist teaching philosophy emerges as the teachers provide both the opportunity and expectation for students' active engagement in building new mathematical understanding informed by prior learning and experiences. Positioning the students as engaged builders of knowledge whose ideas and strategies are valued, establishes the expected role of their learners as sense-makers and doers of mathematics. The learning environments described in Katherine and Melissa's classrooms afford the opportunity for students to learn mathematics with productive struggle.

Sub-theme 4b: Teacher-centered, transmissionist teaching philosophy

An additional sub-theme that emerged in the data was a teacher-centered, transmissionist teaching philosophy. In this conceptualization of teacher and student roles, the description of learning environments was more transmissionist in nature, where the teachers' conveyed information about mathematics and students passively received the information to learn procedures, rules, and facts (e.g., Cobb, 1988; Franke et al, 2007; Schoen & LaVenia, 2018).

Jessica provided insight into this teaching and learning dynamic in our first interview, when describing what I would see if I were to observe her teaching a lesson. In her description of a typical lesson, a clear delineation of roles emerged. She said:

> So, a lesson…I always walk them through the first…I do some problems, do the think aloud - we might do six or so problems where I'm pretty much doing it and showing them how to do it…then I'll ask for input like "Okay, so what do we do next here? And then I slowly give them kind of the reigns. So, I…I do it all, and then I have them kind of [tell me] *Okay, what do we do at this spot?* and then I have a kid tell me *How do you do a problem from beginning to end?* and then the kids are you know, on their own.

In Jessica's depiction of her typical mathematics class, she transmits knowledge to her students through her words and actions (Cobb, 1988), in a teacher-centered approach. In her own description, the curricular resource provides a daily opportunity for student problem-solving, leading into a lesson. This opportunity is removed, as students watch and listen to her solution strategy, recording the solution in their workbooks. Following multiple opportunities for watching, listening, and recording, students are prompted to share the next steps in the solution process, in order to determine if they are prepared to work independently.

An opportunity to work in groups is afforded when the teacher determines the mathematics is hard and prolonged struggling of students supported through teacher modeling of a mathematical solution using manipulatives. The purpose of partner work at these times is for students to “walk each other through,” implying an emphasis on a procedural process. In the learning environment depicted by Jessica, student success is predicated on the ability for students to follow along and remember the steps the teacher has taught them. In an effort to clarify what the small group learning opportunities would look like in the classroom, further probing was necessary. This additional information provided insight into how students would engage with one another. Jessica explained when this happens and how groups and partnerships are determined. She said:

> Sometimes when - when I feel like it's a harder chapter [I have] them go into small groups, so they can walk each other through it…sometimes they don't pick - you know - the right partner, because sometimes like - I'll say to them *Now remember when you pick a partner, pick someone that you know is going to help you*. They're starting to catch on to that - some of the kids know which kids to go to…They have learned - and I go around and listen and ask questions - like *How's it going?* or *How did you get that answer?* making sure that both kids contribute…If they don't, then I may pull out, at that point and I've done that even this week - if a group was stuck then I will pull out the manipulatives for me to use with them.

An opportunity to work in groups is afforded when the teacher determines the mathematics is hard, with struggling students supported with teacher modeling using manipulatives. Even at these times, the purpose is for students to “walk each other through,” implying an emphasis on a procedural process. In the learning environment depicted by Jessica, student success is predicated on their ability for students to follow along and remember the steps the teacher has taught them. Jessica reiterated the purpose of specific teaching strategies, such as the use of manipulatives, noting that “even this week - if a group was stuck then I will pull out the manipulatives for me to use with them” reinforcing to students the purpose and potential for this tool to support learning during times of struggle, directed by the teacher.

A similar teaching approach was described by Ashleigh. She depicted a typical lesson that involved her actively sharing a step-by-step process with students, as they listened and copied this into their notes. The mathematical thinking, reasoning, and making connections is done by the teacher with the students acting as passive recipients of the knowledge. It is only after many examples of the teacher completing the mathematics and mathematical thinking that the students are afforded the opportunity to practice what has been demonstrated on their own. Ashleigh’s description of her classroom indicates a top-down teaching methodology in which a teacher-centered approach dominates. In her description of a typical lesson, she explained:

> So, I teach the step-by-step process and solving and as said I do a color-coded version of steps…[because] I feel it's important so that when I'm doing my notes…the students can see - O*kay, this step was written in words in black, this is what it looks like*. *And then this step was written in words in blue, here's what it actually looks like*. So, I start whole group and I give the directions and then we do as many example problems as we can in the time frame that we have [that first day]. And then the next day…we would do a lot of examples, mostly led by me talking through each of the steps, modeling it up on the board for them so that they can see what it looks like, and they're hearing it over and over

> and over again. And then, once we've done that for a bit I will release them to do some practice on their own. Then I will go through, and I will solve the problems after the fact. So, I'd be like "Okay, you're going to try this one by yourself" and then after so much time I will go through the steps and talk through them so then [students] can see *How did I do? What steps did I get confused on?* Then if we have time, they might do some partner work at the end.

The transmissionist teaching described by Ashleigh depicts students learning procedures and practicing skills, with a teacher-directed approach. While there is an opportunity for student self-reflection following practice, the expectation is for students to compare their work to the teacher's work, check for accuracy, and determine any steps they were "confused on." This approach reinforces the transmissionist perspective that the teacher's role is to transmit knowledge to students (Schoen & LaVenia, 2019). In an effort to ensure the depiction of teacher and student roles in the classroom were interpreted correctly, I probed Ashleigh further to clarify student roles. When I asked her to explain what I would see and hear students doing, she described the following:

> They would be writing during the presentation of like the notes. They follow what I do - they're given the option of you can do this step-by-step colors like I do, or you can do it all one color. I encourage them to do this - step by step with the different colors - because then they can see "Oh, this is written in blue, what does it look like in blue?"…But they would be listening, and they would be writing, and then we would be going through the problems. And depending on the skill, if it's something that they might have some prior knowledge [about], I would be like "Okay - well, what do we do for this?" and then they

> would be able to maybe make some of those connections on their own, with a little bit of [guidance] by me.

In both descriptions, the role of Ashleigh's students is one of a passive participant. While she is actively engaged and providing clearly defined step-by-step approaches to solving problems, students are expected to listen, take notes, and learn the steps of the process for accurate application later. While this helped me to better envision the role of students in her classroom, I probed further about the reference to students working in partners. She clarified how this engagement would evolve during a lesson. She said:

> Depending on the skill and how many steps it is they work from like five to ten problems where they will solve the problem by themselves, and once their partner is done, they will compare their answers. If their answers match, they'll move to the next one. If they don't match, they will then have a conversation and try and figure out where the mistake is. So, if they can't find their mistake…I would bop between each person and determine *I like these steps*. And then, if their answer - if their work looks good, I'll be like *Okay, this is the one that we're trying to get* and then I would go to the other one who had a different answer and I would [ask] *Okay, let's see -what was your first step? What did you do? Look at this math again*…So, we would go through it like step-by-step together, the partner set and myself, and we would talk through the steps to solve and then see if they could figure out where the error is.

The engagement of student partners in sharing their answers and determining potential mistakes is meaningful. However, in a learning environment that promotes procedural knowledge as understanding, the discussion described is guided by the step-by-step process the teacher has taught in an effort to produce the outcome that the teacher expects (Cobb, 1988). Ashleigh's explanation of how she would support students with different answers supports her "persistent directions to solve tasks in prescribed ways" (Cobb, 1988, p. 96) which would deter students from considering alternative approaches.

Theme 5: Teacher Responses to Struggle in Learning Mathematics

The fifth theme that emerged related to teachers responses to student struggles in learning mathematics. The teacher responses to struggle that were identified in the study provide meaningful insight into how teachers might respond to student struggle and were elicited using authentic representations of student struggle in the Cornbread task (Rumsey et al., 2016). Through data analysis, the four sub-themes of (a) doing the mathematics and mathematical thinking for the student, (b) directing the student through the steps or providing the strategies, (c) leading the student with funneling questions, and (d) probing the student to elicit their understanding emerged. These sub-themes represent a progression of responses by teachers that move from the removal of opportunity for the student to struggle towards strategies that engage students more actively, ultimately promoting the opportunity for productive struggle.

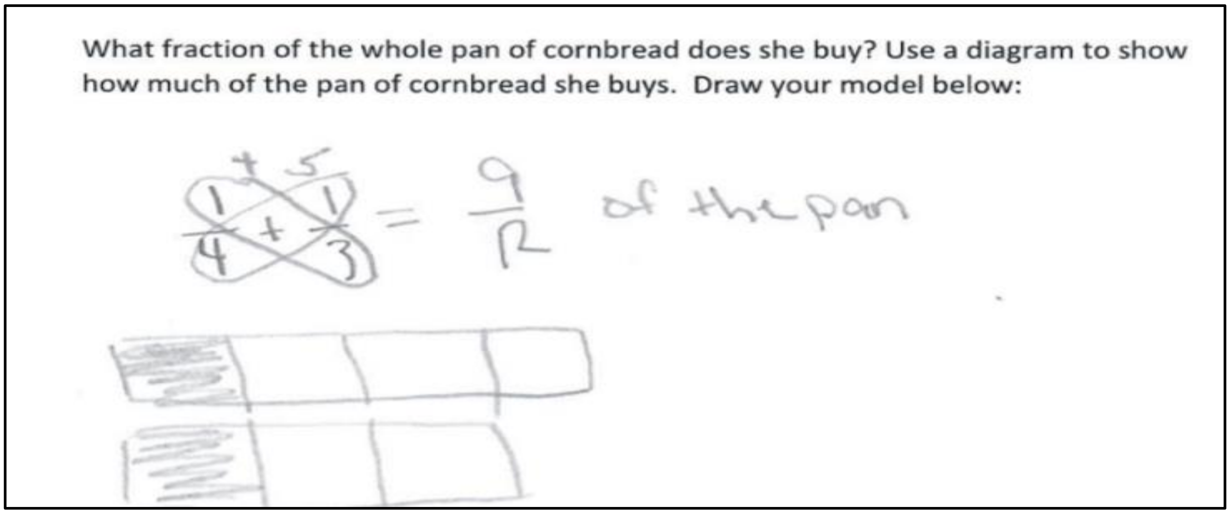
Sub-theme 5a: Doing the mathematics and mathematical thinking for the student

The first sub-theme of *doing the mathematics and mathematical thinking for the student* refers to teacher indicated responses to student struggles where the teacher would draw, write, and/or complete the mathematical task for the student, while the student sat and observed their process. While this response was not as prevalent, there was no opportunity for a student to own their struggle in these examples.

The first example of a teacher response to student struggle that resulted in *doing the mathematics and mathematical thinking* was Katherine's response to Alison in the first stimulus prompt (see Figure 4). In this stimulus prompt, the teacher sees Allison's student work and is presented with the student explanation that “I drew two – like the squares – I don’t know what to call them. But I drew one-fourth and one-third, and then I added one-fourth and one-third together to get nine-twelfths and then I labeled if of the pan.”

Figure 4

Allison's Work



Katherine shared that she would focus Allison’s attention on the model, drawing the visual to ensure that the mathematical relationship in the task could be clearly represented. She also indicated how she would respond to Allison while focusing on the model as a starting point. She said:

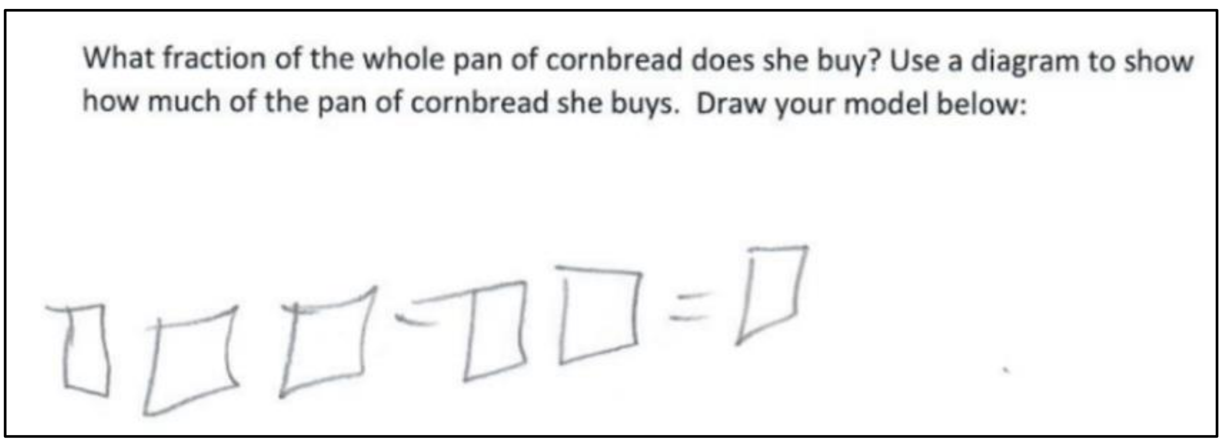
> So, I would go back to that idea of the model - I don't even think I would try to broach the equation piece of it yet - because for [some] kids…visuals and like that understanding through a model is going to make a much bigger impact than it is writing out an equation. So, I would honestly like draw out a rectangle and I would say, *Okay, this is your rectangle.* And I would be the one to draw a rectangle because I wouldn't want her to draw such a long skinny rectangle - I would want to make a wider one, so that we could

truly divide it into thirds after we have split it into fourths.

While the act of drawing a model for a student can appear trivial, this approach removes the opportunity for productive student engagement in the action of making choices about the model that would help her to make sense of the problem situation. Katherine's response of *doing the mathematics and mathematical thinking for the student* in her hypothetical engagement with Allison was similar to her response to the student in the second stimulus prompt. In this stimulus prompt, the teacher has hypothetically observed a student, Bradley, who initially struggled to get started but has begun to work drawing the image in Figure 5 to represent the mathematics. In this portion of the stimulus prompt, the teacher hypothetically returns to Bradley's desk after supporting him in starting the task.

Figure 5

Bradley's Work



The teacher is presented the work sample to review in addition to Bradley's explanation of his thinking and reasoning related to the model. His thinking reflected:

> Oh, squares because the problem said squares. So, 1, 2, 3, 4 - so one-fourth. Oh, wow. I'm surprised I didn't think of that." So, and then he looks back up at the problem he's pointing and he's looking. And he says, "What fraction of the cornbread does she buy?

> Let's see – oh, so I think she'll buy one piece of the cornbread from the one-fourth pan full.

Katherine's response to Bradley's model and verbal explanation indicated that she would engage in *doing the mathematics and mathematical thinking* as a response to his struggle, removing the opportunity for Bradley to make connections through a model he would create. Katherine explained how she would model the solution to the task in a concrete form, while completing a math think aloud, in response to Bradley's struggle. She said:

> I honestly might get out some tiles...So, let's get out some square tiles and let's talk about what this could look like…Do I tell him how many squares to start with? At that point I might. I might actually say let's pretend that this is, this is the pan and literally put out the 12 squares for him and say *Okay like let's create this*. And if we have it set up this way and we're going to put them in a line and now like *literally* like put the three rows with the four [rows] - you know - like to like really, really set it up just to try to visualize and then actually remove and take things away. *So, if you have one-fourth of this [square pan], this is what it is. We're going to take away three-fourths* - *we're going to move nine tiles all the way away. How many tiles do you have left? Okay, if you have three tiles left she wants one third, that would be one out of the three tiles. We're going to get rid of the other two tiles we're going to move them away, how many tiles does she have left?* So, I think he could easily say he has one tile left. The question is, then you got to pull all those tiles back together to get that denominator out and talk about how it's 12.

In Katherine's attempt to support Bradley through *doing the mathematics and mathematical thinking*, she has removed Bradley's opportunity to think, reason, problem-solve, and make connections while struggling productively. Additionally, in her approach to explaining the mathematical relationship using the model, a whole number relationship focusing on the cost of each piece emerges instead of the intended fraction multiplication relationship of one-third of one-fourth of the pan of cornbread.

Sub-theme 5b: Directing the student through the steps

The second sub-theme that emerged in the data was apparent as a response to all stimulus prompts. The response of *directing the student through the steps* emerged as teachers supported struggling students by providing step-by-step directions or a model to funnel student thinking to a correct response. In the teacher responses that indicated *directing the student through the steps*, the most common approach was for teachers to provide a step-by-step process for students to follow, leading to a correct response.

There were two teachers, Jessica and Katherine, who responded similarly to the second portion of the stimulus prompt for Maria. In this hypothetical interaction, Maria had an initial model that failed to provide an exact representation of the square cornbread model implied in the task. Both of the teacher participants initially redirected Maria back to the context in an effort to encourage her to modify her model. This response mirrored the actual student engagement depicted in the stimulus prompt, with additional time provided for Maria to continue her efforts before teachers returned to Maria to check on her progress. In this portion of the stimulus prompt, teachers were shown Maria's updated model shown beside her original model (see Figure 6).

Figure 6

Maria's Work

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

In the first response from Jessica, the response indicates an initial reflection on the two models created by Maria, with an indication that she prefers the square model. As she focuses on the square, she begins to notice specific aspects of the model that are inaccurate. While Jessica initially indicates that she might elicit the Maria's thinking around the task, she quickly moves in a different direction. As her response progresses, she indicates that she would tell Maria to remove the inaccurately partitioned square, progressing to a step-by-step approach to redirect her thinking. She said:

> I would, I would definitely say I like the square better than the circle when it comes to these ones…I guess, I can see the one for maybe without the diagonal line on that last one, but I don't see - I still don't see the one third per se…I mean that diagonal on that line on that bottom right hand corner I'm not sure what was going on right there - I'm not sure what her thinking was there…So, I'm going to have her explain that to me…maybe I can say *Okay let's erase those little lines that [you] have in the bottom right-hand corner - let's kind of get rid of those for right now.* So, we can still kind of salvage the one fourth. And then, once we have the one for now, can we do the third of that. I'd say *Now let's put those each into thirds and go from there.* I don't want to completely say *oh let's just do a whole other shape* because she's already got two shapes going on there. I would

think I would try my best at keeping shape number two. On the rest [she] could start making - do the one third on there, and I'd see - you know - how she can do with that.

Jessica's response demonstrates her desire to maintain the accurate aspects of the second model, eliminating the need for a third model, and potentially decreasing the student's frustration. Jessica's decision to tell Maria a step-by-step process provides a pathway towards an accurate model, however, does not support Maria in understanding why she is completing the steps. Jessica's response indicates a belief that "an effective teacher makes the mathematics easy for students by leading them step-by-step through problem-solving to ensure that they are not frustrated or confused" (NCTM, 2014, p. 11) while simultaneously sending the message to the student that the teacher is responsible for making connections and representing solutions.

Another form of *directing the student through the steps* was apparent in participant responses where the teacher indicated that they would provide a tool or model to ensure student success. Katherine shared a response that aligned with this second approach. While her initial response to support Maria indicated the opportunity to reengage Maria with prior learning experiences, such as working with Geoboards and other rectangular diagrams, she ultimately described providing Maria with a model that ensured an accurate response. In her reflection, she indicated that some students just need to be provided a model to find success. She said:

> And I would probably pull out some models of things that we've used in the past to get her really started and thinking about how to use those…We do - we've done some work with GEO boards, we've done some work with some other like rectangular type diagrams, and I might see if I could give her one that wouldn't be filled in but to see if she could show me how to use it in a different way…And so [Maria] could very easily show me fourths and then do one out of three on there, so it would depend on her comfort with

> that. Ideally, I'd love to pull her back to that three by four grid and really try to push in more with that model. I mean voice and choice is great but at some kids just need to have a model that's going to be the best, the most efficient, the easiest to understand and so I think probably giving her something to draw with, rather than expecting her to come up with it on her own, it's going to be important.

While the response from Katherine holds clear indications of directing students to a specific model to use to ensure success, the potential existed for the student to struggle productively if the student was to draw a model. While probing Katherine further regarding what would be provided to Maria to "draw with" Katherine said:

> So, I'm thinking - so we have a lot of like grid paper - that are like a template type paper - that would already have models on it that she could choose from, and that we would have a conversation about *Why would this be a good model? Why would this not be a good model?* rather than her trying to come up with them on her own.

The use of a template that provides an accurate model for the student to choose from could engage the student in the opportunity to engage more fully. However, in Katherine's clarification she chooses the type of tool for Maria to use and asks questions to narrow the options of available models represented, removing the opportunity for Maria to determine the best option to represent the mathematics.

Sub-theme 5c: Leading the student to a correct response with funneling questions

The third sub-theme of teacher response to student struggles that emerged was when a teacher indicated that they would *lead the student to a correct response with funneling questions.* This response was found in two participant's responses to student struggle, as they utilized questioning in a manner that funneled students toward an accurate answer. Ashleigh and Melissa both used this strategy, however, Melissa used this response to student struggles consistently. The leading questions shared by participants do increase the engagement of students with the task but the questions they ask are stated in a progression that has the potential to funnel students to an accurate response.

The first participant whose data demonstrated the strategy of *leading the student to a correct response with funneling questions* as a response to student struggle was Ashleigh. While responding to Maria's struggle to model the fraction multiplication (see Figure 3), she stated that she would begin by validated what Maria had done accurately, partitioning the square into fourths. Her approach that followed was to revisit the task, focus Maria on one fourth of the model, and ask questions that would prompt Maria to model the task accurately. Ashleigh said:

> I want to validate the fact that she did do it correctly - she started it well after the prompt; she was able to translate splitting a circle into fourths, into a square into fourths and she did that all correctly. But I would want to understand why [she chose] to split a completely different piece into three chunks. I don't know if she would recognize that they're not equal pieces. So, then I would probably proceed with kind of rereading it to her and seeing if we could find a way to pull out using like I don't know, like, I want to use colors but emphasize "Okay, she buys one - she buys from a pan that's one fourth full, so let's focus on this piece that you shaded if that is the one fourth full pan. She buys one third of that. Can you show me what one third of that one shaded chunk looks like?

While it is possible that Maria will be able to partition the one-fourth share of the cornbread pan into thirds and identify one share of this, there is no relational understanding of the fractional portion to the whole. The response shared by Ashleigh can be perceived as productive yet depicts the unproductive belief that an effective teacher tells students exactly how to demonstrate the mathematics in a problem (NCTM, 2014).

Melissa also demonstrated the teacher response of *leading the student to a correct response with funneling questions* as a response to student struggle. This was first apparent in her response to Allison's work (see Figure 4). Contrary to responses by other participants, when Melissa was presented with Allison's work and explanation that "I drew two – like the squares – I don't know what to call them. But I drew one-fourth and one-third, and then I added one-fourth and one-third together to get nine-twelfths and then I labeled if of the pan," Melissa used a response of *leading the student to a correct response with funneling questions.* Melissa indicated that she would begin by directing Allison to reread the task prompt. In her description of how she would support Allison, Melissa embedded questions and inflection to guide Allison as they reread the prompt together. She said:

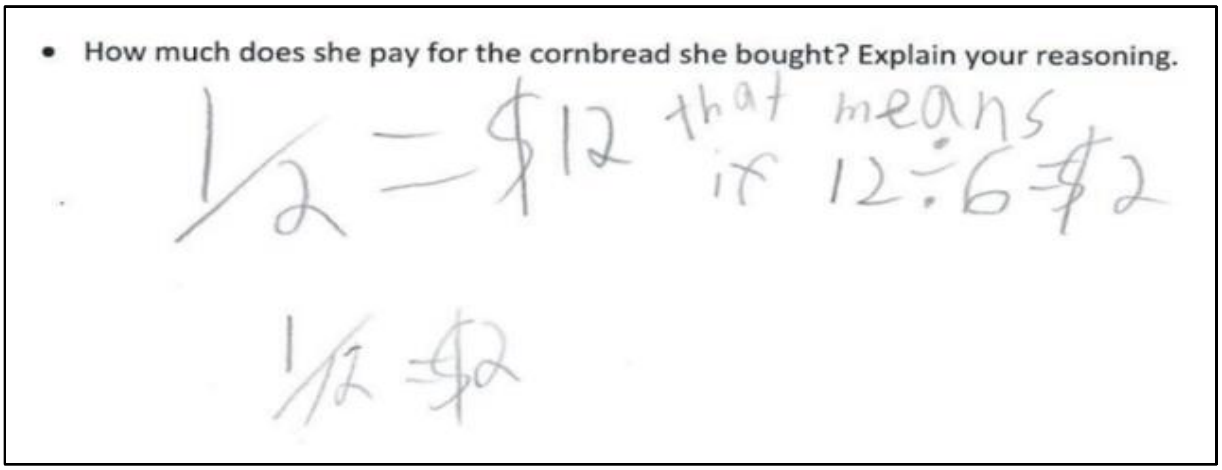
> So, then I would say, well let's take a look at the problem again and I want you to imagine that this cornbread is in front of us. And then I would kind of read the problem with her…so then we read through it again and when I'm reading it with her, I would emphasize the *one fourth full* and *she buys one third of that* and like - stretch it out…And I would say "Can you show me what that would look like?" And I think what I would try to have her do is [to] think about how she could show me that part. And if she was stuck immediately, I would ask her to draw the square pan like it says at the top and then show me the one fourth full…then I would stop there and I would ask her to read that she buys one third of the remaining cornbread in the pan…I guess what I'm trying to get at is I would try to get her to realize it's only one third of this little square I filled in not the

whole pan. So that's kind of how I would do it, but I'd use more specific questions to go back and kind of fill in, if those questions were needed, too.

Melissa depicts the use of questions that emphasized the steps required for Allison to create an accurate model to represent the task. This type of questioning that is *leading the student to a correct response with funneling questions* in this manner removes the opportunity for students to struggle productively while making connections and building understanding.

Figure 7

Jerome's Work



Melissa also used the strategy of *leading the student to a correct response with funneling questions* in her response to Jerome. In this second portion of this stimulus prompt, the teacher sees Jerome's written response in Figure 7 and is asked how they would respond. Melissa's questioning holds the specific purpose of engaging Jerome in reflecting on the meaning of the task and the question being asked related to cost. Her initial approach is to have Jerome share his thinking and reasoning, before noting specific questions that would guide their discussion. She said:

Okay I'm not following his - I understand that he misread the prompt, but I'm not following what - when he says, "That means if twelve divided by six equals two," so I

> would first ask him to just kind of explain his thinking here, to see what how he responds. Hopefully, I can get more insight into like what he means by twelve divided by six…So I don't understand where he's going with that okay. *So, Jerome would you please tell me what your thinking is here? So where in the prompt do you see this, Jerome?* and then he would probably show me that sentence. And then I would reread it with him and say let's think about what this means. So, it says a pan of cornbread costs $12. *So how much of the pan is $12?* And then he would say the whole pan. So, then I'd say *So, okay the whole pan costs $12. What if I just bought half the pan?* I think I would try to break it down like that, kind of how it did in the problem. So, I would go back and explain that part, then I'd say *Okay, now that you know this, What if they only bought 1/12 of the pan? So, if the whole pan is $12. And we have 12 sections, what's each section worth?* I'd say think about that and I'll come back to check on you and, hopefully, he works through that.

In Melissa's approach, Jerome is engaged in responding to her questions as they revisit the task together. In her approach with Jerome, *leading the student to a correct response with funneling questions,* the questions asked are intended to focus Jerome's thinking, yet result in funneling him to the correct response at the same time. In this approach, Jerome is simply providing answers to questions that lead him progressively towards a solution. While questioning is a meaningful strategy that has the potential to afford learning with productive struggle, the questioning must build upon student thinking rather than taking this over for them (NCTM, 2014).

Sub-theme 5d: Probing the student to elicit their understanding

A final sub-theme of teacher response to student struggle that emerged in the data was *probing the student to elicit their understanding.* This theme was apparent in participant responses to student struggle that indicated questions that would elicit the student's knowledge or understanding, and/or when a teacher responded in a manner that focused student thinking. Responses shared by teachers that demonstrated *probing the student to elicit their understanding* held the expectation that students focus on what they knew and/or understood related to a task in order to progress productively though the struggle. An interesting finding related to teacher's use of *probing the student to elicit their understanding* was that this response emerged as an initial response to student struggle but was abandoned when subsequent student responses indicated continued struggle with the task.

The first example of *probing the student to elicit their understanding* was shared by Jessica in response to the first part of Bradley's stimulus prompt. In this part of the stimulus prompt, teachers are hypothetically leaving one student and walking around their room when they come upon Bradley and notice he's just staring at his paper. Seeing this, the teacher pauses to ask him why he hasn't started, and he responds "I can't think. I'm trying to think of what I'm going to draw." In response to how she would engage with Bradley, Jessica said:

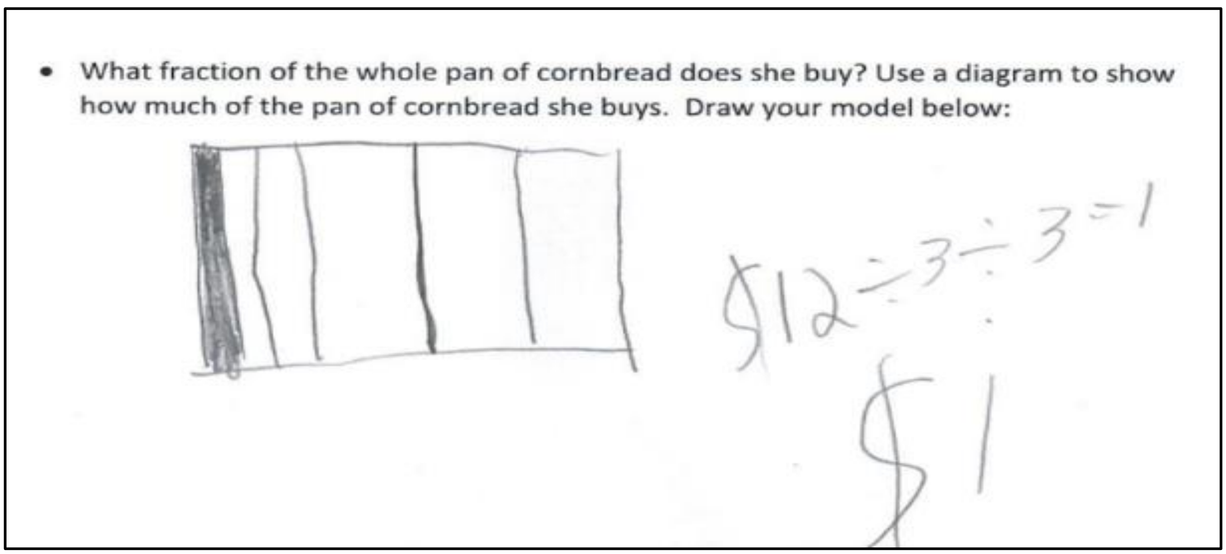
> So - either I read it, or he reads it again and he [has to] tell me what information…Like *What do you know?* - Whether it's terminology or the number -*What does one fourth mean? What does one third mean?* basically just *What do you know so far about this question?* And then *What are we trying to find out?* So, especially with story problems that's when I always use the saying *Okay, what do we know? What is the question we need to answer?* So, this is what you know, so let's write that down…Hopefully, that would help - you know - get [him] in the right direction.

The strategy of *probing the student to elicit their understanding* used by Jessica in this example to determine what the student knows, provides important insight for her to determine meaningful strategies to support the student. Simultaneously, she is “mak[ing] the mathematics more visible and accessible for student examination and discussion” (NCTM, 2014, p. 41).

In most responses that indicated a *probing the student to elicit their understanding* approach, participants only shared a few initial questions or thoughts regarding the information they would want to determine, in an effort to illuminate student thinking. An example of this was demonstrated in all participants initial responses to Bethany’s work (see Figure 8), where the *probing the student to elicit their understanding* approach was intended to illuminate student thinking and determine what the student knew and understood.

Figure 8

Bethany's Work



Ashleigh’s initial response related to Bethany’s work was intended to determine what Bethany understood related to the task and how the model represented the mathematics. She said “I would ask her to explain it to me. *Can you explain your thought process to me? Why did you do 12 divided by three and then divide that by three again?*” in an effort to make sense of Bethany’s solution strategy. In a similar consideration, Melissa said “I would ask her to explain her thinking so that - you know - I don't make assumptions based on just the drawing” as an initial response to determine Bethany's thinking and reasoning related to her model and solution process. Katherine's approach was similar to Ashleigh and Melissa, with a response that was intended to gain insight and understanding of Bethany's work. She said:

> I think I would ask her to explain to me the model by adding labels and telling me what she's actually showing… I want to be able to understand how [she] arrived at [her] answer. So, I would ask *How can you go back? How can you change what you've done so that if I'm checking it at home tonight, or this weekend or whenever I can still get that same understanding of what you just showed [and] what you just shared?*

Jessica's initial response to Bethany did not indicate a specific question, however, demonstrated a similar desire to understand her thinking. Jessica said:

> Well, she's [going to] have to explain herself. I don't see much here – I don't see a one fourth – I don't see a one third…That one's really interesting, because I mean she's got a game plan [but] it doesn't even work.

In all of the participant responses that depicted a form of *probing the student to elicit their understanding* the teachers demonstrate a desire to elicit student thinking and reasoning before responding in a manner that takes over the students' thinking. While the use of questioning in this manner promotes productive struggle in learning, this approach was not sustained by any participant when they were presented additional student explanations and/or additional student modeling.

Theme 6: Alignment of Beliefs and Actions Related to Struggle in Learning Mathematics

The final theme that emerged in the data demonstrated dissonance or concurrence between teachers stated beliefs, practices, and responses to struggle in learning mathematics.

This theme demonstrates unanticipated findings, as the majority of participant data initially indicated a relationship between positive or negative beliefs about the role of struggle in learning mathematics; conceptualization of struggle as an affordance or constraint in learning mathematics; conceptualization of roles in learning mathematics; and conceptualization of student understanding in learning mathematics.

The uneven findings between participants stated beliefs, practices, and response to struggle in learning mathematics emerged in two participants, Katherine, and Ashleigh. This data resulted in the two sub-themes (a) unconscious dissonance, apparent in responses from Katherine and (b) conscious dissonance, apparent in responses from Ashleigh. The data from Jessica and Melissa provided a more seamless and predictable pattern of responses between stated beliefs, practices, and response to struggle in learning mathematics. This analysis of their data resulted in the identification of the final sub-theme, (c) concordance.

Sub-theme 6a: Unconscious dissonance between stated beliefs and responses to student struggle

The sub-theme of *unconscious dissonance* emerged in responses to student struggle that were contrary to a Katherine's stated beliefs and anticipated actions. The designation of this dissonance as being unconscious draws from her matter-of-fact approach and presentation of these responses with the same fluidity and conviction as previous responses that indicated an opposite message. This dissonance appeared to be unknown in Katherine's responses to student struggle, which demonstrated contradiction to her stated beliefs and descriptions of teaching practices.

In the analysis of data across interviews, Katherine demonstrated opposite messages when she described how she would respond to a student struggling to engage with a task. During the first interview, Katherine described an example of how she navigated a student's inability to engage with a task. In her description, the student was distressed to the point of tears and yet she described an interaction that had the potential to elicit productive struggle. She said:

> I'm envisioning right now one of my students who – we were taking the checkpoint and like broke down in tears and she's like I don't remember how to do this and I'm like *It's okay* like Where can *you* look? What can *you* do to help *you* get there?…and I also knew she totally knew how to do it, but it was like that freeze moment of the question was phrased in a different way than she was used to having it phrased and like trying to get kids so they're not at that point of like that fight or flight and like it's that panic moment that has just set in - it's like, how can we reassure kids that you are able to do this and you do have the tools and you have what you need.

In this example, Katherine engages with a student who is demonstrating struggle and frustration while attempting to get started with a task. In spite of the student's apparent frustration, she described using a probing response which afforded her student the opportunity to own their learning. By redirecting the student to the tools and resources they have to be successful, she is able to reassure the student that she could find success.

In contrast, when Katherine is presented the stimulus prompt demonstrating Bradley's struggle to begin and draw a model of the mathematics, she hypothetically responds in a manner that diminishes the opportunity for the student to own their learning. Katherine responded by *leading the student to a correct response with funneling questions*. She said:

> I would start by saying *Well, it says that it's going to be a pan - like what does a pan look like? What's the shape of a pan?* And so, I mean he may say it's a circle, he may say it's a rectangle. So, I'm like okay - like let's get something down…And then, after he draws the

> pan, then we'll say *Okay, so we know that it says it's going to split into fractional parts*. I might highlight on his paper if that would be something that would help to pull things out and just say - *Okay, it says that it is one-fourth full. Like what do we know about these numbers, how do we draw something that's one-fourth?*...Just trying to like get him to verbalize before, because sometimes - it's the idea of putting it on paper that's the scary part, because they don't want to be wrong…So, if you can talk it out first and have that conversation before you have to draw it out, I feel like sometimes kids think that's a lot more successful than having to draw and erase and draw and erase. So, breaking it apart piece by piece and then giving that opportunity for a discussion before he does anything to kind of talk him through what he's gonna do.

The response to Bradley shared by Katherine indicates a *leading* approach that provides the student with questions that will funnel the student to a correct response. This response also implies that the process of a student attempting and revising a solution pathway as they "draw and erase, draw and erase" should be avoided by finding the correct pathway before "committing." With previous reflections that productive struggle is important and can be supported by the teacher, this was dissonant from previous examples and statements.

An additional example of unconscious dissonance emerged in Katherine's data when comparing her anticipated implementation of the Cornbread task (Rumsey et al., 2014) with the expectation and anticipation of students modeling this in a variety of ways, to a response that limits and structures the model in response to student struggle. At the beginning of the second interview, Katherine indicated that she would want students to model the task in more than one way, without limiting expectations of the model. She shared:

> I also saw a little bit about how you said - like draw your model below - and I thought *Could you have kids show their model in more than one way?* Because I feel like we - like the model I drew I can pretty much guarantee is the model that my kids would draw because that's the model we use in class all the time. And it's like a three by four grid because it just is nice and neat and perfect, but what even if your pan - *Like who says a pan has to be a rectangle that's a three by four? Who says that your pan can't be a one by 12 and it's a really long wonky pan, right? Like how do we think about different ways that we can make shapes to represent the fraction pieces?*

With Katherine's description of anticipated task implementation that included an open-ended approach to modeling, as well as the expectation of students to create multiple models, the uneven response to Allison that demonstrated *doing the mathematics and mathematical thinking for the student* was surprising. In response to the stimulus prompt that depicted Allison's inaccurate model, Katherine said:

> I would honestly like draw out a rectangle and I would say *Okay, this is your rectangle*. And I would be the one to draw a rectangle because I wouldn't want her to draw such a long skinny rectangle - I would want to make a wider one, so that we could truly divide it into thirds after we have split it into fourths. Umm, I'd say *Okay, if this is our rectangle show me one-fourth again.*

Katherine's unanticipated hypothetical response of drawing the model and directing Allison through the mathematical thinking of the task demonstrated an opposite message from previous teaching practices. Katherine's demonstrated dissonance in her anticipation of engaging students with the task in an open-ended manner and her response when a student demonstrated struggle. It was additionally ironic that she provided the specific example of students drawing a pan that might be “a one by twelve and it’s a really long, wonky pan” to model the mathematics, yet this was the same type of model she wanted to avoid in working with Allison. Instead, I would have anticipated a teaching approach that depicted scaffolding in the event a student chose this model.

Sub-theme 6b: Conscious dissonance between stated beliefs and responses to student struggle

The sub-theme of *conscious dissonance* emerged in responses from participants that fully acknowledged a discrepancy between their stated beliefs and responses to student struggle. The designation of this dissonance being conscious draws from participant’s matter of fact approach and presentation of these responses, with a clear acknowledgement of the underlying discrepancy.

As was discussed previously with Ashleigh, conscious dissonance emerged in her own acknowledgment of this discrepancy. The first indication of conscious dissonance in Ashleigh’s responses occurred during a reflection of her stated beliefs and response to student struggle. This dissonance emerged in her first interview when asked what she felt was important in fraction instruction. She said:

> *Especially when it comes to the fraction concepts that is a skill that I see a struggle in [sixth graders]…*So, I feel like it would be good for them to have more like a hands-on approach, but I don't necessarily do a hands-on approach, I give them like the steps to solve.

Ashleigh response clearly acknowledged that sixth graders struggle with fractions and that she believes that a hands-on approach would be more beneficial in supporting this struggle. In spite of this she shared her more direct, step-by-step approach to her instruction; a clear indication of conscious dissonance between her stated beliefs and practices related to struggle in learning fractions.

While this initial response only provided an initial indication of a conscious dissonance between Ashleigh's beliefs and practices, an example that reinforced this dissonance emerged later in our discussion. As the interview progressed, Ashleigh mentioned providing one-on-one support with struggling students. When probed about how this impacted her planning, she said:

> If I knew [a student] were more of a visual [learner] and that's how they got it, I would probably try and incorporate more visuals. If they were more of a hands-on [learner] and they showed that, I would need to do more of a hands-on approach to things. I should be doing those things more often it's just I don't…It's what I *should* be doing, it's just - I don't do it. So then that one-on-one support would come after the fact, where I would present the lesson to them and then it would be "Okay, so you're struggling - let's work through some problems together" or "I'll help you solve this one; that's a whole group problem" or whatever…It's more so once they see that there's a struggle - that's when I really start working with them a little bit more.

Ashleigh acknowledges that there are opportunities to support students in learning fractions with teaching methodologies including the use of visuals and providing the opportunity for students to learn in a "hands-on approach." In spite of her recognition that this is what she "*should* be doing" in her instruction, she shares that the one-on-one support for struggling students is reactive and implies that she will work through the struggle in a step-by-step manner. The recognition that there are teaching practices that Ashleigh believes she should use was reinforced a third time during the first interview. She said:

> Student needs when it comes to fractions – they need visuals. They need repetition, I feel. Again, they need to get their hands on something – they need to be able to connect it to like a real-world thing…I should be doing those things more often it's just I don't.

The responses shared by Ashleigh directly indicate student struggles in learning fractions, identifies strategies she believes that they need in order to learn fraction concepts, nonetheless acknowledges that she does not provide this learning opportunity.

Sub-theme 6c: Concurrence between stated beliefs and responses to student struggle

The sub-theme of *concurrence between stated beliefs and responses to student struggle* emerged in responses from participants whose stated beliefs, practices, and responses to student struggle were consistent. The designation of this concordance draws from participant's reflections on their beliefs about struggle in learning and responses to student struggle. Concordance emerged in the data from two participants, Jessica and Melissa.

While this final theme depicted both predictable and unexpected alignment of participants stated beliefs, practices, and responses to student struggle, the data suggests that participants' inability to describe examples of productive struggle in learning could influence these outcomes. Exploring this emerging relationship more directly might yield a more complete understanding of these relationships.

Summary

This chapter identified the six major themes and fifteen sub-themes that emerged from the twelve interviews conducted with participants. Three of the four participants were fifth-grade teachers, while one teacher was a sixth-grade teacher. All teachers had experience teaching mathematics, with their teaching experience ranging from five to twenty-one years.

The qualitative, semi-structured interview study proved to be a meaningful approach to investigating this phenomenon and provided rich data to understand each participants experiences with student struggle in learning mathematics. Table 9 demonstrates the emerging relationships between the themes, as they interact in participants positive or negative beliefs about the role of struggle in learning mathematics; positive or negative conceptualization of struggle in learning mathematics; conceptualization of student understanding in learning mathematics; conceptualization of roles in learning mathematics; and teacher response to struggle in learning mathematics.

Table 8

Themes and Emerging Sub-themes in Participant Data

Participant	Theme 1	Theme 2	Theme 3	Theme 4	Theme 5	Theme 6
	Teacher Beliefs about Struggle in Learning Mathematics	Teacher Conceptualization of Struggle in Learning Mathematics	Teacher Conceptualization of Student Understanding of Mathematics	Teacher Conceptualization of Roles in Learning Mathematics	Teacher Response to Struggle in Learning Mathematics	Dissonance in Beliefs and Actions Related to Struggle in Learning Mathematics
Katherine	Struggle in learning mathematics is good for students.	Struggle is an affordance to learning mathematics, benefitting students, and helping them to build understanding.	Understanding mathematics involves students constructing connections between new and prior learning experiences.	Student-centered and constructivist classrooms depicted students as actively involved in building mathematical knowledge and understanding in a community of learners. Teachers in these classrooms were depicted as facilitators this learning dynamic.	Doing the mathematics and mathematical thinking removed all opportunity for productive struggle in learning mathematics. Directing a student through the steps or providing strategies resulted in a diminished opportunity for productive struggle in	Unconscious dissonance in stated beliefs and actions was depicted in responses to student struggle and impacted by an inability to describe examples of productive struggle in learning.

					learning mathematics. Probing a student to elicit their understanding provided the greatest opportunity for learning mathematics with productive struggle.	
Jessica	Struggle in learning mathematics is bad for students.	Struggle is a constraint to learning, resulting in long term implications that become a barrier to future success.	Understanding mathematics involves students knowing mathematical facts, processes, and ideas taught by the teacher.	Teacher-centered and transmissionist classrooms depicted students as passive recipients of knowledge, expected to listen, copy, and practice the skills being taught. Teachers in these classrooms were depicted as transmitters of knowledge and ideas.	Directing Probing	Concordance in stated beliefs and actions were depicted in responses to student struggle. The potential exists that this relationship was impacted by an inability to describe examples of productive struggle in learning.
Ashleigh	Struggle in learning mathematics is bad for students.	Struggle is a constraint to learning, resulting in long term implications that become a barrier to future success.	Understanding mathematics involves students knowing mathematical facts, processes, and ideas taught by the teacher.	Teacher-centered and transmissionist classrooms depicted students as passive recipients of knowledge, expected to listen, copy, and practice the skills being taught. Teachers in these classrooms were depicted as transmitters of knowledge and ideas.	Directing Probing	Conscious dissonance in stated beliefs and actions were depicted in responses to student struggle and impacted by an inability to describe examples of productive struggle in learning.
Melissa	Struggle in learning mathematics is good for students.	Struggle is an affordance to learning mathematics, benefitting students, and helping them to build understanding.	Understanding mathematics involves students constructing connections between new and prior learning experiences.	Student-centered and constructivist classrooms depicted students as actively involved in building mathematical knowledge and	Leading students to a correct response with funneling questions results in a diminished opportunity	Concordance in stated beliefs and actions were depicted in responses to student struggle. The potential

				understanding in a community of learners. Teachers in these classrooms were depicted as facilitators this learning dynamic.	for productive struggle in learning mathematics Probing	exists that this relationship was impacted by an inability to describe examples of productive struggle in learning.

In the first theme *Teacher Beliefs about the Role of Struggle in Learning Mathematics,* two sub-themes emerged, providing insights regarding participants' positive or negative beliefs related to struggle in learning mathematics. With participants who viewed struggle in a positive manner, struggle was depicted as having a purpose and providing potential for learning. Depictions of how struggle impacted learning in these classrooms clearly indicated both an opportunity and expectation of student engagement that would elicit struggle. With participants who viewed struggle in a negative manner, struggle was depicted as being problematic, creating barriers to learning, and being detrimental to students' long-term success in mathematics. Depictions of the impacts of struggle in these learning environments clearly indicated practices to prevent or remove the opportunity for struggle.

In the second theme *Teacher Conceptualization of Struggle in Learning Mathematics*, the two sub-themes that emerged provided insight into how anticipated struggle was conceptualized as an affordance or constraint in learning mathematics. With participants who conceptualized struggle as an affordance with purpose and a positive potential for learning, a variety of common practices emerges. In the descriptions from participants who indicated struggle as an affordance in learning mathematics, students were expected to demonstrate multiple solutions to a problem (e.g., models and algorithms) and explain their thinking, reasoning, and problem-solving strategies to others. In these learning environments productive struggle in learning was depicted as an expectation and opportunity for students to learn and build connections as students used their resources and drew upon previous knowledge and understanding. In the descriptions of learning environments from participants who conceptualized struggle as a constraint in learning mathematics, participants' depiction of their mathematics learning environments was devoid of the opportunity for students to engage in problem-solving or in making connections that build understanding. In these learning environments, the participants described students as passively engaged in listening to the teacher, taking notes, and practicing the steps of the algorithms that were taught. There was little opportunity for student collaboration other than comparing answers to determine potential mistakes in a process.

The third theme *Teacher Conceptualization of Student Understanding of Mathematics,* depicted two sub-themes, providing insight into teachers' definition of student understanding as being an act of constructing or knowing. In participants' classrooms where student understanding was defined as an act of constructing connections between prior learning and new learning experiences, students were expected to construct conceptual connections through modeling, explain their thinking and reasoning, engage in problem-solving strategies, and represent the mathematics in multiple ways. Learner engagement in constructing understanding of mathematics in the ways described by participants holds the greatest opportunity for students to learn with productive struggle. Contrary to this, in participants' classrooms where student understanding was defined as an act of knowing mathematical facts, procedures, and ideas taught by the teacher, students were expected to listen, copy, and learn from the teacher's modeling of mathematical processes and mathematical thinking. Students' knowing was supported by memory tricks in each classroom setting.

The fourth theme *Teacher and Student Roles in Learning Mathematics* depicted two sub-themes of student-centered, constructivist learning environments or teacher-centered, transmissionist learning environments. In participants' classrooms that depicted a student-centered approach, a constructivist philosophy emerged as teachers described student learning opportunities that both provided and expected active engagement, provided resources to support independence, and valued student ideas and mathematical thinking. In an opposite fashion, teacher-centered classrooms depicted by participants' described learning environments that were more transmissionist in nature, where teachers conveyed information about mathematics and students passively received the information to learn procedures, rules, and facts. In the student-centered, constructivist classrooms, the greatest potential for students to learn with productive struggle emerged.

The fifth theme *Teacher Response to Struggle in Learning Mathematics,* yielded four sub-themes. The sub-themes that emerged included (a) doing the mathematics and mathematical thinking for the student, (b) directing the student with specific steps or strategies, (c) leading the student to a correct response with funneling questions, and (d) probing the student to elicit their understanding. The sub-theme of *doing the mathematics and mathematical thinking for the student* was depicted in participants' descriptions of themselves taking over the mathematical engagement as they completed the drawing, writing, or a task for a student who was struggling. This approach to supporting struggle removes the potential for productive struggle to emerge. The second sub-theme *directing the student with specific steps* was depicted in participants descriptions of themselves providing students with step-by-step directions, tools, or a model to funnel students to a correct response. Similar to *doing,* this approach removes the opportunity for learning with productive struggle. The third sub-theme *leading the student to a correct response* *with funneling questions* was depicted in participants descriptions of themselves utilizing questioning in a manner that funnels a student to a correct response. While this approach demonstrates an increase in engagement from the struggling student, the opportunity for a student to struggle productively is diminished by the funneling approach. The final sub-theme that emerged was *probing the student to elicit their understanding*. Participant depictions of this strategy included descriptions of themselves asking questions or responding to students in a manner that focused student thinking on what they know or understand related to a task. This approach to supporting a struggling student holds the greatest opportunity for supporting productive struggle in learning by the student.

Theme six emerged as an indication of *Dissonance in Teachers Stated Beliefs and Actions Related to Struggle in Learning Mathematics.* This theme resulted in three sub-themes that included (6a) Unconscious dissonance between stated beliefs and actions, (6b) Conscious dissonance between stated beliefs and actions, and (6c) Concurrence between stated beliefs and actions. The sub-theme of *unconscious dissonance between stated beliefs and actions* emerged in data from one participant whose stated beliefs and practices depicted a positive conceptualization of struggle in learning mathematics, with specific examples of responses from their teaching that matched this. Contrary to this initial depiction, however, were their responses to student struggle with the Cornbread task (Rumsey et al., 2016) and their responses that removed and diminished the opportunity for productive struggle in learning mathematics. The presentation of the uneven responses appeared to be an unconscious discrepancy. This unconscious dissonance was only apparent in one participant. The sub-theme *conscious dissonance between stated beliefs and actions* was apparent in stated practices and responses to student struggle that were in direct opposition to a participant's beliefs. This conscious dissonance was only apparent in one participant. The final sub-theme of *concurrence between stated beliefs and actions* was apparent in the responses from two participants. In their responses, a predictable pattern between their positive or negative beliefs regarding the role of struggle in learning mathematics was apparent.

The five themes provide important insight into the potentially synergistic impacts of teacher beliefs on their anticipation, planning, teaching, and response to student struggle. Chapter Five provides a discussion of the findings as they relate to the research questions: What role does productive struggle play in the design and implementation of mathematics lessons? How do teachers perceive their role and the role of students as it relates to learning with productive struggle? How do teachers prepare for anticipated struggle when planning for mathematics instruction? How do teachers respond to evidence of struggle in student learning? Does the response have the potential to invoke a productive struggle for students?

CHAPTER 5

DISCUSSION AND CONCLUSIONS

The purpose of this study was to understand the potential impacts of anticipating, planning, teaching, and responding to struggle in learning. Understanding the synergistic impacts of anticipating struggle, planning strategies to support this struggle in a manner that fosters productive struggle, and implementing these strategies as struggles arise during learning, holds the potential to further inform our practices related to teaching and learning mathematics.

The major findings in this qualitative semi-structured interview study were derived from three interviews conducted with the four participants. The first interview focused on participant's anticipation of student struggle in learning fraction concepts, prompting participants to share how they plan instruction and teach fraction concepts. At the end of this interview, time was provided for participants to review and share their initial reflections and anticipation of student engagement with the Cornbread task (Rumsey et al., 2016). This task was shared with participants with the request that they engage more thoughtfully with the task, as if planning for its use in their classrooms. Teacher engagement with this task provided focus and context for the second interview. The second interview featured five stimulus prompts that provided examples of student struggles with the fraction multiplication task, while also providing the opportunity for participants to share how they would implement the task in their own classrooms. The stimulus prompts situated participants in their own classroom setting, eliciting their hypothetical responses to students written and verbalized responses to demonstrated struggles. The third interview provided the opportunity to elicit participants beliefs about the role of struggle in learning mathematics, through their responses to other teachers hypothetical beliefs. The data gathered from the interviews informed the six themes and fifteen sub-themes, in turn providing insight into the seven findings related to the phenomenon of productive struggle in learning mathematics.

Discussion of the Findings

This study examined how the opportunity to learn with productive struggle emerges in a teacher's beliefs, anticipation, planning, teaching, and response to struggle in in learning mathematics. A critical consideration in students development of mathematical proficiency is the learning opportunities they are afforded. Expanding upon important research on productive struggle, the findings of this study suggest that understanding the relationship between our beliefs, practices, and ability to identify productive struggle has a direct impact on students opportunity to learn with productive struggle. The findings from this study contribute to the emerging research efforts on productive struggle in learning mathematics.

Through analysis of the six identified themes and fifteen sub-themes, seven major findings emerged that impact a student's opportunity to learn with productive struggle (see Table 10). The first two findings were determined through the analysis of the themes *teacher beliefs about struggle in learning mathematics* and *teacher anticipation of struggle in learning mathematics*. These findings relate to participants descriptions of their internalized beliefs related to struggle in learning mathematics and the impact of these beliefs on a student's opportunity to learn with productive struggle. The following four findings were determined through the analysis of the themes of *teacher conceptualization of roles in learning mathematics*, *teacher conceptualization of student understanding in learning mathematics,* and *teacher response to struggle in learning mathematics.* These findings relate to participants depictions of their externalized practice related to struggle in learning mathematics, including how their anticipation of student struggle impacts their planning, teaching, and response to student struggle. The last finding was informed by analysis of the final theme, *alignment of teacher's stated beliefs and actions*, and provides insight into how teachers beliefs and actions can be influenced by their ability to define productive struggle in learning mathematics. This finding demonstrated both dissonance and concurrence of participant data.

Table 9

Major Findings Informed by Themes and Sub-themes

	Major Finding	Theme	Related Sub-themes
Teachers Internalized Beliefs	Teachers beliefs provide a strong indication of an opportunity or lack of opportunity for their students to learn with productive struggle.	Teacher beliefs about the role of struggle in learning mathematics	Struggle in learning mathematics is good for students
			Struggle in learning mathematics is bad for students
	Teachers who believe that struggle is a benefit to student learning create this opportunity, while teachers who believe that struggle is a barrier remove or diminish this opportunity.	Teacher anticipation of struggle in learning mathematics	Struggle as an affordance to learning mathematics
			Struggle as a constraint to learning mathematics
Teachers Externalized Practice	Teachers describing a student-centered and constructivist learning environment were more likely to indicate an opportunity for learning that fostered productive struggle.	Teacher and student roles in learning mathematics	Student-Centered and Constructivist
	Teachers describing a teacher-centered and transmissionist learning environment were more likely to indicate a diminished opportunity for		Teacher-Centered and Transmissionist

	learning with productive struggle.		
	Teacher descriptions of students' mathematical understanding indicated a relationship to their teaching philosophy.	Teacher conceptualization of student understanding in learning mathematics	Constructing connections between new and prior learning experiences
			Knowing mathematical facts, procedures, and ideas taught by the teacher
	Teacher inability to recognize productive struggle among students in their classrooms impacted their responses to evidence of student struggle.	Alignment of beliefs and actions related to struggle in learning mathematics	Unconscious dissonance in stated beliefs and actions
			Conscious dissonance in stated beliefs and actions
			Concordance in stated beliefs and actions
	Teachers removed the opportunity for learning with productive struggle when students demonstrated a prolonged struggle.	Teacher responses to struggle in learning mathematics	Doing the mathematics and mathematical thinking for the student
			Directing the student through the steps or providing strategies
			Leading the student to a correct response with funneling questions
			Probing the student to elicit their understanding

Note. This table depicts the seven major findings, delineated by their categorical alignment to internalized beliefs or externalized practices. The findings depict their relationship to the themes and sub-themes that informed them.

Research Question: What role does productive struggle play in the design and implementation of mathematics lessons?

The findings of this study provide compelling evidence that the phenomenon of productive struggle in learning mathematics is directly impacted by teachers beliefs about the role of struggle in learning mathematics. When teachers described anticipated student struggle, their beliefs framed this struggle as an opportunity or barrier. Teachers who anticipated student struggle as an opportunity for learning described planning that would foster and support this struggle. Teachers who anticipated struggle in learning as a barrier described planning that removed or diminished the opportunity for students to learn with productive struggle. Teacher descriptions of their learning environments provided a strong indication of how their beliefs and practices aligned in a manner that would afford or constrain the opportunity for students to learn with productive struggle.

While teachers responses demonstrated a close relationship between their beliefs and practices related to student struggle in learning mathematics, inconsistencies emerged. The unanticipated finding related to agreement between a teacher's beliefs and practices, with an indication that a teacher's inability to describe productive struggle impacted their response to student struggle. This relationship was depicted in a diminished opportunity for students to learn with productive struggle when prolonged struggle emerged following a teacher response to an initial student struggle.

Table 8 shares each participant's initial response to a student demonstrating struggle with the Cornbread task (Rumsey et al., 2016), as well as their subsequent response to students written and verbalized responses demonstrating prolonged struggle. In each example, participants whose initial response to student struggle depicted probing, with the potential for a student to learn with productive struggle, indicated subsequent responses to prolonged struggle that diminished or removed the opportunity for students to learn with productive struggle. Similarly, teachers depicting an initial response that limited the opportunity for a student to learn with productive struggle maintained this level of support, indicating the removal or a diminished opportunity for students to learn with productive struggle. In one instance, Katherine started her hypothetical response to Allison with complete removal of the opportunity for productive struggle and intervened by doing the mathematics and mathematical thinking for the student in the first portion of the task, followed by directing her through the steps of the second portion of the task. An additional pattern that emerged through the analysis of teacher responses was the frequency of a specific response to student struggle. The most frequently used strategy by teachers was directing, which positions the teacher as the problem-solver and sense-maker, while the student is simply recording the teacher's thinking and reasoning.

Table 10

Student Struggles and Successive Teacher Responses

Student Work	Type of Struggle	Doing the Mathematics and Mathematical Thinking	Directing the Student Through the Steps	Leading the Student to a Correct Response With Funneling Questions	Probing the Student to Elicit their Understanding
Allison – Part I	Expressing a Misconception or Error	K	J		Initial – A & M
Allison – Part II	Expressing a Misconception or Error		K, J, & A	M	
Bradley – Part I	Struggle to get Started		A	M	Initial – K & J
Bradley – Part II	Struggle Providing Mathematical Explanation	K	J & A	M	
Jerome – Part I	Struggle Providing Mathematical Explanation				Initial – K, J, A, & M
Jerome – Part II	Expressing a Misconception or Error		K, J, & A	M	

Maria – Part I	Struggle to Carry out a process		A	M	Initial – K & J
Maria – Part II	Expressing a Misconception or Error		K	J, A, & M	
Bethany – Part I	Struggle Providing a Mathematical Explanation		J		Initial – K, A, & M
Bethany – Part II	Struggle to Carry out a process		K, J, & A	M	

Note. This table depicts each participant's initial response to the student struggles presented as well as their subsequent response to the prolonged struggle demonstrated in the student work or response shared. The data is disaggregated by student, type of struggle demonstrated, and teacher response to the initial struggle (Part I) or prolonged struggle (Part II). Each participant is represented with the first initial of their pseudonym: K – Katherine; J – Jessica; A – Ashleigh; and M – Melissa.

Recognizing how teachers respond to student struggle is an essential step in supporting the opportunity for productive struggle in learning mathematics. The opportunity for students to engage in productive struggle in learning mathematics is considered an essential consideration in learning with understanding (NCTM, 2014). Thus, consideration of teaching practices that sustain this opportunity are consequential and must be considered intentionally.

Attendant question 1: How do teachers perceive their role and the role of students as it relates to learning with productive struggle?

Finding 1: Teachers describing a student-centered and constructivist learning environment were more likely to indicate an opportunity for learning that fostered productive struggle.

This finding suggested that the opportunity for a student to learn with productive struggle is greatly influenced by the expectation of student engagement in making sense of the mathematics, with teaching practices that support a mathematical community of learners. In these learning environments students are expected to be resourceful and collaborative contributors of mathematical knowledge, thinking, and reasoning as they build understanding together. In these environments, learning and understanding are an active and collaborative interaction between students and the teacher.

This finding supports previous investigations that assert learning opportunity as being predicated on the expectations that students are active and engaged learners, presented with meaningful mathematical tasks, and provided the opportunity and expectation to share their collective understanding (e.g., Handa, 2003; Hiebert & Wearne, 2007; Silver & Stein, 1996; Stein & Lane, 1996). This finding suggests that a student-centered, constructivist classroom provides greater opportunity for students to learn with productive struggle.

Finding 2: Teachers describing a teacher-centered and transmissionist learning environment were more likely to indicate a diminished opportunity for learning with productive struggle.

This finding suggested that the opportunity for a student to learn with productive struggle was diminished or removed by virtue of the expectation of passive student engagement. In these learning environments, student learning was predicated on their listening, copying, rote memorization, and recall of memory tricks (like mnemonic devices) to learn. The learning focus in these environments depicts an emphasis on skill acquisition and efficiency.

The teacher-centered and transmissionist learning environments described by teachers depicted the teachers as conveyors of knowledge and their students as passive recipients. The focus in these learning environments is skill efficiency, diminishing the opportunity for students to learn with productive struggle. This finding is supported by examples of teaching practices that focus on the skill development versus conceptual understanding (e.g., Ball, 1993; Cobb, 1988; Cobb et al., 1991; Hiebert & Wearne, 1993; Stein et al., 2007). This basic finding is consistent with investigations that demonstrate a decreased opportunity for productive struggle in learning when students lack engagement in problem-solving, sense-making, reasoning, and communicating about mathematics.

Finding 3: Teacher descriptions of students' mathematical understanding indicated a relationship to their teaching philosophy.

This finding demonstrated a relationship between how teachers described student understanding and their teaching philosophy depicted in their described practices. Teachers who conceptualized understanding mathematics to mean students constructing connections between new and prior knowledge demonstrated a student-centered and constructivist philosophy. These teachers provided time for students to grapple with new ideas, reminders of the resources available for students to complete novel tasks, and the expectation that students construct and explain these connections in relation to previous learning.

Teachers who conceptualized understanding to mean students knowing mathematical facts, procedures, and ideas taught by the teacher demonstrated a teacher-centered and transmissionist philosophy. These teachers provided time for students to practice the step-by-step procedures they had taught, but only when the teachers felt confident that students were demonstrating success with a process. Success in this regard was described by teachers as students' ability to share the next step in a process being demonstrated by the teacher, or to accurately verbalize all of the steps that lead to a correct solution.

Attendant question 2: How do teachers prepare for anticipated struggle when planning for mathematics instruction?

Finding 4: Teachers beliefs provide a strong indication of an opportunity or lack of opportunity for their students to learn with productive struggle.

Teachers describing student struggle in learning mathematics as being good, provided examples and descriptions of learning opportunities in their classrooms that afforded productive struggle in learning. The teachers in these classrooms described struggle as an expected consideration in learning that supported students in constructing connections and building conceptual understanding. Teachers describing student struggle in learning mathematics as being bad provided examples and descriptions of learning opportunities in their classrooms that constrained productive struggle in learning. The teachers in these classrooms shared examples of anticipating struggle and proactively planning strategies to remove the struggle through a direct instruction approach or to support the struggle reactively. This finding suggests that our beliefs about the role of struggle in learning mathematics impact whether a teacher will create the opportunity for students to learn with productive struggle.

Finding 5: Teachers who believe that struggle is a benefit to student learning create this opportunity, while teachers who believe that struggle is a barrier remove or diminish this opportunity.

Teachers describing struggle in learning mathematics as being good also described strategies to ensure the opportunity for learning with struggle was present, while considering strategies to support the struggle without taking over students thinking and reasoning during learning. Teachers with this belief structured learning opportunities that would elicit struggle, requiring students to engage in problem solving, grapple with new ideas while constructing connections to previous learning, and communicate their ideas.

Teachers describing struggle in learning mathematics as being bad, described strategies to prevent or remove the opportunity for student struggle in learning mathematics. These teachers described the anticipation of student struggles in learning mathematics and specific strategies to prevent this or respond to this with a direct teaching approach. These teachers described student struggle as something to react or respond to as it emerged. Both teachers who depicted struggle in learning negatively indicated a conditional support of struggle in learning for select students, at select times, with select tasks. This struggle was limited in nature and anticipated to be problematic, even for students they believed could navigate the struggle. Consequentially, struggle was not considered an opportunity for learning mathematics that was planned, but rather an expectation in learning that was perceived as problematic for most students.

Together, these findings align tangentially with studies that demonstrated a relationship between a teacher's beliefs and practices (e.g., Anderson et al., 2005; Ernest, 1991; Fennema et al., 1989; Romberg, 1984) as well as building upon recent research that investigated and defined productive struggle in learning (Hiebert & Wearne, 2007; Warshauer, 2014). This finding implies that a teacher's perception of student struggle in learning mathematics impacts the affordance or constraint of the opportunity for students to learn with productive struggle, as well their response to the anticipated struggle.

Attendant question 3: How do teachers respond to evidence of struggle in student learning? Do the response(s) have the potential to invoke a productive struggle for students?

Finding 6: Teacher inability to recognize productive struggle among students in their classrooms impacted their responses to evidence of student struggle.

One teacher's hypothetical responses to student struggles demonstrated examples of unconscious dissonance. The responses to student struggle shared indicated a removal or diminished opportunity for a student to struggle productively in the Cornbread task (Rumsey et al., 2016), whereas described beliefs and practices indicated the affordance of this opportunity. The fluidity and "matter of fact" presentation of these responses, directly opposing previously shared beliefs, demonstrated an unconscious dissonance.

Another teacher demonstrated a conscious dissonance in their beliefs and practices. She recognized and anticipated student struggle in learning, shared teaching strategies she believed would support students in more productive engagement yet declared "I know I should do these things – I just don't." Underlying this reaction is the potential relationship of her inability to define struggle that is productive, as she noted that there is "no bad struggle" and that this is expected in learning mathematics. While the responses she indicated when asked to respond to student struggle were not unexpected, it is my belief that this conscious dissonance is greatly influenced by her inability to recognize and support productive struggle in learning mathematics.

This finding recognizes the results of previous investigations demonstrating alignment between a teacher's beliefs and practices (e.g., Putnam et al., 1992; Remillard, 1999; Thompson, 1992), considering the ability for teachers to move beyond a textbook definition of productive struggle to identify, describe, and define this phenomenon. It is my belief that a potential factor in a teacher's ability to support productive struggle in learning is their inability to recognize it.

Teachers struggled to provide specific examples of student struggle that was productive, instead describing extreme examples of what it is not. It is my belief that in order to respond to student struggle in learning mathematics in a productive manner, a teacher must be able to recognize productive student struggle and employ practices that afford and sustain this opportunity. Teachers must also acknowledge their own struggle with student struggle in learning mathematics.

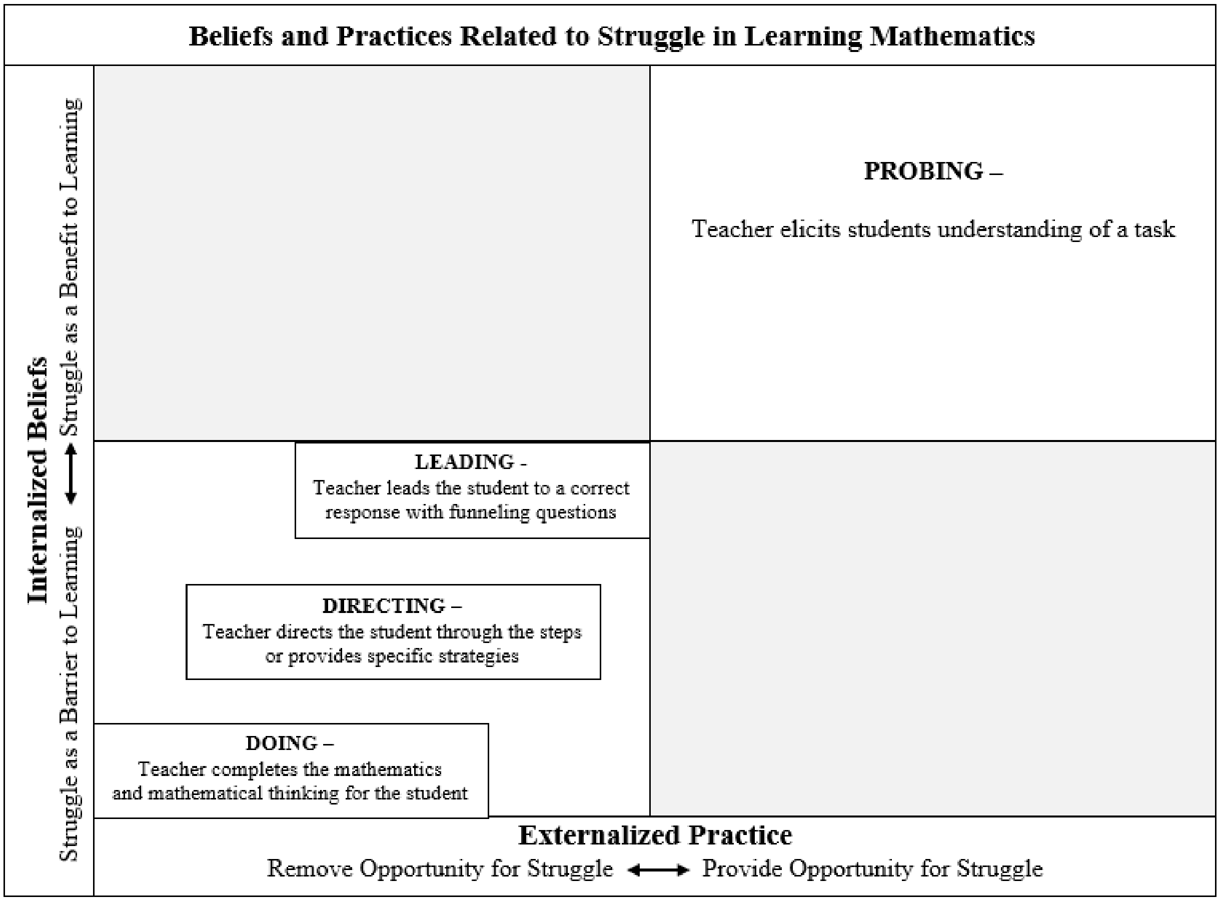
Finding 7: Teachers removed the opportunity for learning with productive struggle when students demonstrated a prolonged struggle following probing.

While some of the teachers shared internalized beliefs and externalized practices that predicted a productive response to support a student demonstrating struggle, the opportunity for productive struggle was limited. Figure 9 represents the relationship of teachers beliefs and practices, with the intersection defined by teachers hypothetical responses to the student struggles they were presented. The axes in Figure 9 demonstrate a continuum of a teacher's beliefs and practices related to struggle in learning mathematics. On the vertical axis, the continuum depicts teachers internal beliefs regarding struggle in learning mathematics from being a barrier to a benefit. The horizontal axis depicts a continuum of teachers external practices related to struggle in learning mathematics, moving from the removal of this opportunity to providing it.

The figure depicts that probing as a response to student struggle would align with positive beliefs and practices, affording the opportunity for students to engage in productive struggle in learning mathematics. The response of *probing the student to elicit their understanding* appeared as an initial response when first presented with examples of student struggle in completing the Cornbread task (Rumsey et al., 2016).

Figure 9

The Impact of Teacher Struggle to Define Productive Struggle on Their Beliefs and Practices



The remaining teacher responses to student struggle identified in this study indicated the removal of student struggle or a diminished opportunity for a student to struggle productively. These responses typically occurred when teachers were presented with examples of prolonged student struggle in completing the Cornbread task (Rumsey et al., 2016). It is important to recognize that the responses are in a quadrant that depicts the intersection of negative beliefs and the removal of the opportunity for students to struggle productively in learning mathematics. While it is recognized that all of these responses limit the opportunity for students to struggle productively, they also depict a progression from a complete removal of this opportunity to a diminished opportunity.

In the lowest portion of the left corner, the response of *doing – teacher completes the mathematics and mathematical thinking for the student* depicts a complete disengagement of the student from the learning. With this teacher response, student engagement involves passive listening and watching the teacher doing the mathematics. This response was only demonstrated by one teacher, as both an initial and subsequent response to student struggle. Recognizing the potential for this response to completely remove the opportunity for productive struggle in learning is essential.

Moving slightly up and to the right, in the center of this box was the most prevalent teacher response to evidence of prolonged student struggle. The response of *directing the student through the steps* was the most prevalent response when teachers encountered this prolonged struggle. This strategy maintains teacher ownership of the mathematical thinking and reasoning, while the student engagement involves completion of steps or strategies as directed.

Moving to the top right corner of the lower left quadrant, the teacher response of *leading the student to a correct response with funneling questions* emerged. This response requires a slight increase in student engagement as they answer questions, however, the questions funnel the student to an accurate response. In spite of this, the student is not required to think, reason, or problem-solve to complete the task successfully.

Both *directing* and *leading* responses to student struggle diminish the opportunity for productive struggle in learning mathematics. These strategies appeared in teacher responses to all types of student struggles. Ultimately, as teachers were provided evidence of prolonged student struggle, their response to the student struggle further diminished or removed the opportunity for students to learn with productive struggle.

This finding is consequential in a few ways. The first consideration relates to the specific teacher response to prolonged student struggle (see Figure 9). As teachers encountered prolonged student struggle, their preferred strategy to navigate this struggle was to *direct the student through the steps.* This finding provides a strong indication that teachers perceive this strategy as helpful, as the student records what the teacher directs them to record, uses a strategy or model that the teacher directs them to use, and it depicts students making positive progress towards an accurate solution. In juxtaposition, it is the teacher completing the mathematical thinking, reasoning, and problem solving with the student acting as a conduit, simply recording the teachers efforts. Additionally, recognizing that this strategy is a preferred option when students demonstrate prolonged struggle provides insight regarding the need to support teachers in the identification of productive struggle and training in strategies that will help to sustain this learning dynamic.

The hypothetical responses to student struggle shared by participants and captured in this figure demonstrate that teacher responses to student struggle in learning mathematics falls on a continuum. Ultimately, a teacher’s responses to the student struggle they encounter represent a complex relationship of beliefs, experiences, and decision making that is unique to each individual. Unraveling the complexities of each experience was not the intent of this study, however, noting that the continuum that exists and that there is an apparent relationship between a teacher’s beliefs and actions related to struggle in learning mathematics when prolonged student struggle emerges is important. The apparent relationship of prolonged student struggle to teacher responses that diminish the opportunity for learning with productive struggle is noteworthy, with learning implications that should be explored more intentionally.

This study has the potential to add to the field of knowledge on the phenomenon of productive struggle in learning mathematics, demonstrating that a teacher's ability to recognize and support productive struggle as important factors in determining a productive response. In this study a teacher's response to student struggle, in juxtaposition with their stated beliefs, practices, and inability to define productive struggle, resulted in responses that consistently removed or diminished the opportunity for productive struggle.

This finding goes beyond previous studies on productive struggle in learning mathematics (e.g., Polly, 2017; Russo et al., 2020; Warshauer, 2014) indicating an important consideration of the total interaction between a teacher and student during an episode of struggle. While teachers often indicated an initial response to student struggle that demonstrated the opportunity for productive struggle to emerge through probing, teachers failed to maintain student engagement in productive struggle when students demonstrated prolonged struggle. This is an important finding in the understanding of students opportunity to learn with productive struggle.

Limitations

This study demonstrates recognized limitations. The study focused specifically on an intermediate developmental level of mathematics, affording the opportunity to consider struggle in learning fraction concepts. While the purpose and intent were not to draw conclusions, exploring this phenomenon with participants representing a variety of grade-levels, in multiple domains of mathematics, would have yielded additional data and allowed for different understandings to emerge.

An additional limitation related to my biases of mathematics instruction. With this knowledge as I designed and implemented my study, I incorporated the intentional practices of writing analytic and reflective memos with each interview to attend to these biases proactively, acknowledging that my own theoretical perspectives and experiences influence my data analysis process. In order to communicate potential biases and assumptions to readers, I disclosed my own theoretical perspective in Chapter 3. In order to minimize this bias, I engaged epochē (Moustakas, 1994), bracketing or setting aside my own prejudgments, when analyzing the data and communicating results of this study.

Student struggles were an essential element in this qualitative semi-structured interview study. While the opportunity for participants to respond to authentic examples of student struggle were contextualized in as realistic a manner as possible, observing student struggle that emerges in a classroom setting would provide additional insight into this phenomenon. In a related consideration, observing teaching provides a different insight than hearing teachers describe their practices.

The limitations I have shared were considered during the design and implementation of the study herein. While the acknowledged limitations could impact findings if the study were conducted through observation, additional opportunity to gain insight into teachers thinking, reasoning, and motivations emerged in the semi-structured interview process that might not have been illuminated otherwise. The data depicts individual experiences with the phenomenon that while at times were similar, were additionally unique and nuanced. This expected finding frames recommendations for practice and further research, providing an opportunity to continue gaining insight into this phenomenon and how productive student struggle can impact the teaching and learning of mathematics.

Future Research

Previous research on struggle in learning mathematics suggests the importance of struggle in mathematics learning opportunities (e.g., Hiebert et al., 1996; Hiebert & Wearne, 2004; Kapur, 2009; NCTM, 2014), through student engagement in problem-solving (e.g., Ball, 1993; Lampert, 2001; NCTM, 1980, 2000; 2014; Schoenfeld, 1998; Stein & Lane, 1996), with teaching strategies that support students engagement with productive struggle (e.g., NCTM, 2014; Smith, 2000; Warshauer, 2015). Identification of the type of student struggle (Warshauer, 2011) provides a starting point to support students while maintaining the opportunity for productive struggle (e.g., Hiebert & Grouws, 2007; Hiebert & Wearne, 2004; Permatasari, 2016; Warshauer, 2015).

In this study, I explored the role productive struggle played in the design and implementation of mathematics lessons. Further research should explore how a teacher's ability to define student struggle that is productive impacts their ability to choose strategies that support and maintain productive struggle. Identifying the type and level of student struggle provides an essential starting point in supporting students as they struggle in learning mathematics, while choosing strategies that afford this opportunity must be considered intentionally. Additional insights could be gained by focusing efforts on the struggles that emerge and if there are patterns around teacher responses to different types of student struggle.

Expanding upon the efforts of this study and focusing future research on productive struggle that identifies what in-service teachers notice when viewing a student's written work that demonstrates struggle, as well as identifying what in-service teachers notice in a student's verbal response that indicates a misconception, error, or apparent struggle, could provide important insight into this phenomenon. This insight would be beneficial in understanding a potential relationship between what a teacher notices in the student's struggle, their targeted response, and the ability of the response to afford the opportunity for a student to learn with productive struggle. Warshauer (2021) conducted a study with pre-service teachers in which the intent was to "focus PTs [pre-service teachers] toward noticing forms of *student struggle, teaching responses,* and *resolutions to struggle"* (p. 91, emphasis in the original). This study provides important insight into strategies that could meaningfully support pre-service teachers, though the study I describe would begin by identifying what in-service teachers notice in student's struggle to create a framework for this population.

An additional opportunity for future research that emerged in this study is to explore how elementary teachers anticipate student struggle, planning for and with this struggle in mind. Some of the data suggests that when unproductive struggle is not anticipated that this impacts how a teacher responds to student struggle in learning mathematics. While research provides recommendations that teachers anticipate how students might respond or answer a specific task, potential misconceptions, and gaps in learning, it was my experience that anticipation of unproductive struggle needed to be prompted. Understanding how anticipation of student struggle impacts a teacher's ability to respond productively to student struggle could provide guidance on how to approach this anticipation in an intentional manner.

Finally, it might be beneficial to replicate this study in other elementary grade-levels and in additional domains of mathematics. While semi-structured interviews demonstrate limitations, the opportunity to gain insight into teachers beliefs, motivations, and experiences can inform how to move forward, ensuring the opportunity to learn with productive struggle is ubiquitous. The findings of this study indicated that the relationship between a teachers internalized beliefs related to struggle in learning mathematics and their externalized practices are further influenced by their ability to recognize productive struggle in learning mathematics. Thus, understanding how this relationship emerges in different contexts can inform how to support teachers in providing the opportunity for students to learn with productive struggle.

Recommendations for Practice

This study contributes to three distinct areas of mathematics education. First, from an empirical standpoint, the major contribution of this study is the relationship that emerged between a student's prolonged struggle in learning mathematics and a teacher's response to this struggle that diminished the opportunity for learning with productive struggle. This response emerged in all participant's hypothetical responses to prolonged student struggle, regardless of the type of struggle demonstrated in the student work or verbal response shared. The data indicated that while teachers' initial responses to student struggle often had the potential to promote productive struggle in learning, the prolonged student struggle resulted in teaching responses that diminished or removed the opportunity for student learning with productive struggle. This suggests that teachers personal struggle with prolonged student struggle impacts their pedagogical decisions to support students encountering struggle. Further, understanding how teachers conceptualize the role of struggle in learning mathematics, how student struggle is supported in an effort to promote learning and understanding, and the efficacy of these responses enables the field to respond to this intentionally. Future research can build on this knowledge by identifying potential patterns of teacher response to identified student struggle, such as a student *expressing a misconception or error,* in order to support teachers with training that equips them with targeted strategies to support productive struggle in learning.

The second contribution of this study is to the field of teacher education. Knowing that a teacher's beliefs about the role of struggle in learning mathematics provides an indication of the opportunity for learning with productive struggle can inform teacher education programs and professional development. This study informs both mathematics content and methodology courses, indicating a need to provide the opportunity for our pre-service educators to experience learning the mathematics content with productive struggle, contributing to their conceptualization and development of pedagogical knowledge in their methodology courses that support this effort. As this study highlights the interconnectedness of our beliefs and practices related to struggle in learning mathematics, supporting pre-service educators in identifying their own beliefs while demonstrating the efficacy of practices that support learning with productive struggle is an essential consideration.

Finally, this study has methodological implications related to teaching with productive struggle. In particular, the study suggests that anticipation of and providing opportunity for productive struggle in learning, as well as planning intentionally to support this, is necessary. Training pre-service teachers and in-service teachers to anticipate this struggle will require their understanding of learning progressions and potential areas of unfinished learning that could impact emerging student struggles. With this knowledge, teachers can plan opportunities for students to engage in problem-solving tasks that elicit struggle and feel confident in implementing teaching strategies and responses that support and sustain a learning dynamic with productive struggle.

Pre-service teachers are learning methodologies and practices that support learning opportunities to build student understanding, using problem-solving tasks; however, the strategies alone do not ensure this opportunity. This was apparent in the findings of this study, as some in-service teachers described the use of problem-solving tasks, the expectation that students use and connect a variety of mathematical models, the use of questioning, an expectation for discourse, and the eliciting and sharing of student thinking; yet their own descriptions of response to student struggle did not demonstrate an opportunity for productive struggle to emerge. Consequently, working with pre-service and in-service teachers to frame, focus, and structure these efforts around the opportunity for students to learn with productive struggle is imperative.

Concluding Comments

Expanding upon our current understandings related to the role productive struggle plays in learning mathematics demonstrates a meaningful potential to ensure mathematics proficiency for all. Important research related to productive struggle has provided important recommendations to guide this effort yet more can be learned about how our responses to this struggle afford or constrain this opportunity.

Some of the findings from this study reinforce existing understanding of our teaching practices that have the potential to afford or constrain the opportunity for productive struggle, as well as how our beliefs about struggle in learning mathematics impacts our teaching practices. While these recommendations exist, supporting teachers in identifying their own potential biases regarding struggle in learning mathematics is an essential starting point. With the acknowledgment that teachers struggle with student struggle, we can support teachers in identifying and incorporating intentional practices of anticipating struggle, planning the opportunity for and support during this struggle, and meaningful responses to this struggle that can ensure the opportunity for productive struggle in learning mathematics.

Teaching practices that support productive struggle in learning mathematics requires a learning culture that fosters and expects students willingness to grapple with mathematics, as they struggle productively to build understanding. Carter (2008) noted that "although [students] have not yet mastered a particular mathematics concept, students can feel success if they can understand that struggle is an expected, essential part of learning" (p. 135). Realizing the potential of productive struggle in learning mathematics will require intentional efforts by teachers as they recognize and respond to their own personal biases, anticipate student struggle, and plan opportunity for and responses to these struggles. With this intentional approach, teaching, learning, and responding to student struggle in a productive manner can be realized. Only then will this vision become a reality for all learners.

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Appendix A:
Parent Request and Permission for Student Participation

Good Afternoon,

My name is Erin Edgington and I am a doctoral candidate in the UW-Milwaukee Urban Education Doctoral Program, emphasizing in mathematics education. I am reaching out to you because you have a son or daughter in 5th grade (or another grade close in proximity) and have expressed that you and your child are interested in supporting my efforts.

I am asking for the opportunity to Zoom with your child for approximately 30 minutes, as they engage in a fraction task. During this time, I will interact with your child but will ask that parents not offer any assistance or guidance in this process. I am also asking for permission to record these Zoom sessions, in order to better understand how students engage with this task, the types of questions that they ask, and support(s) they might require in order to be successful. These sessions will be deleted following my opportunity to reflect on how students engaged with the task.

Prior to the date we schedule for me to meet with your child via Zoom, I will ask that you send me your mailing address in order for me to send you the required parent and student permission forms required for participation, as well as a copy of the mathematical task. At the end of our time together, I will ask that you send me a photo of your child's work to this email, as their work will help me to understand how your child approached their completion of the task, in addition to the discussion that we have as they complete it.

Your child's work samples and responses will be considered collectively with other students' responses as I approach my formal research. In this work, I am investigating how teachers support students during the administration of mathematical tasks with a high cognitive demand. While this might sound intimidating, the task I have chosen is grade appropriate for your child and I will be there to support them as they work.

If you are still interested in your child participating, please let me know a good day and time to connect with your child. With my teaching schedule this semester, the best days for me are Tuesday or Thursday or Friday after 11:00 am, or any evening after 5:00 pm. Ideally, I would like to finish this process by ________________.

If you have any questions that would help you to make a decision, I would be happy to have a phone conversation with you. I appreciate your child and your willingness to support my efforts and I look forward to hearing from you.

Kindly,

Erin Edgington
Doctoral Student Researcher
edgingt3@uwm.edu 608.553.0635

Parent/Guardian Consent Form for Student Participation

Introduction

Your child is being invited to participate in a doctoral research study conducted by Erin Edgington, a student in the Urban Education doctoral program at the University of Wisconsin-Milwaukee and an Assistant Professor of Education at the University of Wisconsin-Platteville. The purpose of this research is to gain insights into teaching and learning fractions when students struggle. This form provides detailed information on the research to help you decide if you would like to participate in the study. Please read it carefully.

Procedures

Your child's participation in this study will involve approximately one thirty-minute interview by the researcher. During the interview, your child will complete a fraction task while the researcher will act as an observer, asking questions as your child works, as well as taking field notes of the interaction. Copies of your child's work (with names redacted) will be requested.

Risks & Benefits

This is a minimal risk research study. Safeguards will be in place to maintain confidentiality of student responses collected, including the use of pseudonyms in the presentation of findings.

While there is no direct benefit to participating in this research there is the potential to enhance our understanding of teaching and learning mathematics when struggles occur.

Confidentiality

The researcher will make every effort to ensure that the information collected as a part of this study remains confidential. Names of participants, schools, and communities will be protected with the use of pseudonyms. All data collected will be securely stored in an encrypted USB; stored in a restricted-access folder on box.com, an encrypted, cloud-based storage system; or stored in a locked drawer in a restricted-access office.

Voluntary Participation

Your child's participation in this research is completely voluntary. If at any time during the study you choose to rescind this permission, you have the right to do this without recourse.

IRB Review

The Institutional Review Board (IRB) for the protection of human research participants at the University of Wisconsin – Milwaukee has reviewed and approved this study. If you have questions about the study, please contact the investigator at (608) 553-0635 or edgingt3@uwm.edu. If you have questions about your rights or would simply like to speak to someone other than the researcher about questions or concerns, you can contact the University of Wisconsin – Milwaukee IRB at (414) 229-3182 or email: irbinfo@uwm.edu

____

Erin Edgington
Doctoral Student Researcher
Edgingt3@uwm.edu
608.553.0635

Parental / Guardian Permission for Child Participation in Study

I **give my permission** for my child to participate in this research.

___________________________________	________________________
Signature of Parent or Guardian	Relationship to child
___________________________________	________________________

Name of Child (Please print) Date

____

Parent Email contact for meeting link

Appendix B:
Student Assent for Participation

We're inviting you to be in a research study. A research study is a way to learn new things. We are trying to learn more about teaching and learning mathematics when students struggle.

If you agree to be in this study, we will ask you to be videotaped during a Zoom session as you complete a math task. The researcher will observe you as you complete the task, ask you some questions and take notes, to better understand how students might work on this task. The Zoom video will help the researcher to take the most accurate notes possible but will not be shared with anyone else in these formats.

When important interactions are observed, or important conversations during learning happen, the researcher might want to share what was said. In order to protect your privacy and confidentiality, pseudonyms (fake names) will be used to protect your identity.

A risk is something bad that could happen to you. Being in this study might have some risks. The biggest risk is that you might feel nervous at first with a new person in the classroom when you learn math. Another risk is that you might have friends in other classes who see a stranger in your classroom and might ask you questions, so they might know you are in the study.

A benefit is something good that happens. Being in this study will not have any benefits for you, but it could help other kids learning math someday. The researcher hopes to learn more about teaching and learning math, by observing your teacher, you, and your classmates learning math and that this study will help other kids someday, too.

You don't have to be in this study. It is up to you. No one will be mad, no matter what you decide. If you say yes now, but change your mind later, that's ok too. Just let me know.

When we are finished with this study, we will write a report about what we learned. This report won't have your name in it, or that you were in the study.

Signatures

If you decide you want to be in this study, write your name on the line below.

____ __________________

Name of Participant Date

____

Name of Researcher obtaining assent (print)

____ __________________

Signature of Researcher obtaining assent Date

Appendix C:
Stimulus Prompt - Student Protocol

Below I have created a script to follow during the student interview portion of the research process, to ensure standardization of protocols from the inception of the meeting until the end of the interview. While this interview protocol shares consistent questions that I seek to understand, the potential probing questions will vary between participants as determined to be appropriate. Following the script, I have included the adapted task upon which the interview questions are based.

I begin the interview by sharing my screen with the student participant, which shows the same task that they have received in the mail. Throughout the interview, I have the student read each prompt aloud before completing the work expected in each problem, without preemptive interventions. In each portion of the task, the first step is for students to create a diagram to represent the problem and then solve. If a student asks a question, I will provide positive support and reinforcement without funneling them in this process. If a student is showing frustration, I will end the interview and thank them for their time.

Each prompt also elicits student thinking and reasoning, with the expectation that this is explained in writing. If I sense that a student is hesitant to write their ideas once the question has been read, I will ask if they prefer to share their thoughts with me. As my goal is to capture student thinking and reasoning regarding the task, this will potentially remove a barrier if writing is something that a student participant avoids.

Interview:

As we begin, I want to revisit the letter you received regarding the purpose of our work together. I would like to read this to you and make sure to answer any questions you might have. [Read letter and respond to any questions]

Thank you for agreeing to do some math with me today. I also want to remind you that we discussed the use of Zoom to record our conversation today, to ensure that my notetaking is accurate and complete. As we work together today, I will observe you as you complete the math task, ask you some questions, and take to notes to help me understand how students might work on this task. The Zoom video will help me to take the most accurate notes possible but will not be shared with anyone else in this format. When I am watching you work and I see you using strategies for solving the problems, or you share important ideas, I might want to share how you solved a problem or something that you said. To protect your privacy and confidentiality, pseudonyms (fake names) will be used to protect your identity.

As a reminder, your participation is voluntary, you may stop at any time, and there is no consequence for choosing to stop or decide you no longer want to be involved. Before beginning, do I still have your permission to record this conversation? I have recorded your response as ____. Do you have any questions for me, or are you ready to begin? Great.

- Reestablishing purpose, providing an overview of what the student can expect, and reminding them of their rights as a participant is important to ensuring that they still want to work with me.

Before we get started, I would like to know a little about you. I know you are a fifth grader, but I do not know much else. Could you please tell me a little but about yourself? Thank you! I also [connect personally to something I heard them tell me]. I want you to know a little about me, too. I taught fourth graders for most of my life, so I really enjoy working with students your age. I also taught middle school and high school students, but now I teach college students who want to be teachers. Do you have any questions before we work on some math?

- With only one formal interaction with students, it is essential I try to connect with them in a personal manner before delving into our work together. I want students to feel safe and comfortable sharing their mathematical strategies and ideas with a stranger.

Okay, I am ready to begin, too. Will you please take out the packet I mailed to you? I will share this on my screen, too, but you can do your work right on the page. Let me know when you are ready to begin. Super! As we move through the math problems on each page, I will ask that you read the problem aloud and solve this before moving on to the next item. I will also ask you to hold up your work to the camera, so I can see your diagrams as we talk. Does this make sense? Good. Let's begin.

The following script will be used for all problems in the task, with identified modifications as needed.

Please read the problem. [Student reads prompt] Go ahead and solve the problem but let me know if you need anything. When you are done, please let me know so we can talk about your work.

- While reading the prompt does not indicate understanding of the prompt, it will ensure that any student in need of support with reading the prompt is identified and supported in this manner. In the event a student requests that I read the prompts aloud, demonstrates hesitation that inhibits progress, or demonstrates an inability to read the problems, I will read these aloud for them.

You are done? Thank you for working so hard. Could you please show me your work by holding your paper up the camera? Thank you.

Can you please describe your diagram for me?

How did you find your answer?

- While some written work can clearly depict a solution strategy, I am interested in student thinking and reasoning behind the creation of their diagrams and solution strategies. The ideas shared from students who demonstrate impasse provide data for the stimulus text to be used in interviews with teachers. The potential exists to use some, all, or none of these responses.

Thank you for sharing your ideas. Let's move on to the next question. Would you please read the problem. [Student reads prompt]. Thank you. Please write down your ideas but let me know if you need anything. When you are done, please let me know so we can talk about what you wrote.

- While some written work can clearly articulate student ideas, I need to be sure that the written thinking and intended explanation match. The ideas shared from students who demonstrate impasse provide data for the stimulus text to be used in interviews with teachers. The potential exists to use some, all, or none of these responses.

In the event a student demonstrates hesitation or an inability to capture their reasoning in writing, the following script will be used:

So, I notice that you have been thinking about [refer to prompt] for a little while. Would it be easier to share your ideas by talking about them? Great.

Let me read the question to you, and then you can share your thoughts. [Read question that has caused impasse]

Thank you for those ideas. Let's move to the next question. Would you read this for me, please?

Potential probing questions, as needed.

Can you explain how you knew to do that?

You said _______________, could you please explain that to me?

Can you explain how that your model helped you to solve this? [If student references the use of their model]

You said that you do not see multiplication in the problem. What type of math [operation] do you see?

Thank you for spending some of your time doing math with me today. I really appreciate you sharing your super strategies and ideas, so I can learn more about how students will complete this math task. Do you have any questions for me? Great. Just know that if you have any questions in the future, you can have your [mom, dad, guardian] email me and I will answer them for you.

<u>The Cornbread Task</u>

Task form modified to fit this document. Original spans three pages to allow for workspace and increased readability.

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

A. Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:
- How much does she pay for the cornbread she bought? Explain your reasoning.
- Where do you see multiplication in this problem? Explain your reasoning.

B. The next customer is the school principal. He buys cornbread from a different pan that is 1/2 full. He buys 5/6 of the remaining cornbread.

- What fraction of the whole pan of cornbread does he buy? Use a diagram to show how much of the pan of cornbread he buys. Draw your model below:
- How much does he pay for the cornbread he bought? Explain your reasoning.
- Where do you see multiplication in this problem? Explain your reasoning.

C. The principal wants to buy cornbread from a different pan that is 1/2 full. He buys 5/6 of the remaining cornbread from each of the pans below.

What would be the cost of the cornbread the principal bought if the price of the entire pan changed to one of the following? Use a diagram and explanation to show how much he would pay in each situation. Explain your reasoning:

- $24
- $60
- $18

Adapted and modified from a task in Rumsey, R., Guarino, J., Beltramini, J., Cole, S., Farmer, A., Gray, K., & Saxby, M. (2016). Assessing students' understanding of fraction multiplication. *Teaching Children Mathematics 23*(3): 166-173.

Appendix D:
Teacher Participant Recruitment Letter and Survey Questions

Good Morning,

You are receiving this email because you are an elementary or middle level mathematics educator in grades four, five or six, and I am conducting research related to the teaching and learning of mathematics with this focus. In this study, I seek to understand the teaching and learning of mathematics from your perspective and experience as an intermediate elementary or middle school teacher. Without the opportunity to observe this dynamic in-person, I would like to utilize interviews to gain your perspective and provide me with insight related to your experiences. The study will provide information to the educational community on how teachers support students during instruction of a mathematics lesson.

If you are interested in being considered as a participant in this research study, I ask that you complete an electronic interest survey, using the link provided. It is anticipated that the survey will take no more than ten minutes to complete. It asks for basic information regarding your interest and your contact information.

Once you complete the survey, I will email you to establish a time to reach you by phone to discuss more details and answer any questions you may have about the study and your potential involvement.

If you are selected as a participant, you can anticipate that we will schedule three separate Zoom interviews, preferably completed over a fifteen-day span of time. These interviews will be scheduled on days and times that are most convenient to you.

I look forward to hearing from you.

Erin Edgington
Doctoral Student Researcher
edgingt3@uwm.edu
608.553.0635

Online Survey Tool Questions: Establishing Demographics and Background

What is your name?

- This is important for the purpose of reaching out to potential candidates formally.

Where do you teach?

- This information provides the specific school where a teacher works, allowing for comparison to other respondents, as well as for the purpose of visiting the location.

What grade do you teach?

- This question is asked to ensure that the respondent is a viable candidate for this work, teaching in grade 4, 5, or 6.

How many years have you been teaching in this grade?

- This question provides insight on a teacher's experience with the content and could provide potential insight into the instructional decisions of a teacher participant.

What piqued your interest about this this study? Please provide a brief explanation about your interest in participating.

- This question will provide important insight into the potential participant's interest level and motivation for engagement.

What is your preferred method of contact? Please provide an email, phone number or both indicating which is the most effective for you.

- This information will be used to establish an initial connection with participants. During the first interview, participant preference for communication will be determined.

Appendix E:
Teacher Participant Recruitment Follow-Up Email

Good Morning,

Thank you for completing the survey and expressing an interest in being considered as a participant in this research study.

I am reaching out to you to schedule a time that we can connect by Zoom to discuss more details and answer any questions you may have about the study and your potential involvement. Additionally, I would like to ask more questions to learn about your mathematics teaching.

Upon the conclusion of this initial conversation, your continued interest and viability as a research participant will be determined.

I look forward to hearing from you.

Erin Edgington
Doctoral Student Researcher
edgingt3@uwm.edu
608.553.0635

Appendix F:
Invitation Email / Alternate Request Email / Rejection Email

Invitation Email

Good Morning,

I would like to begin by thanking you for the time you have invested in this process thus far, responding to my survey and initial questions. I would like to invite you to join my research work as a case study participant, beginning __________, 2021. As we discussed, the study will require three separate interviews in close succession, over a 15-day period of time. The first interview is anticipated to last approximately 60 minutes, the second interview is anticipated to last approximately 90 minutes, and the final interview is anticipated to last approximately 45 minutes.

As we concluded our meeting, we agreed that this process would begin at _____ am/pm on ____________ , 2021. As a reminder, you are not required to participate and can rescind your permission for research at any time in the process. Thank you for your consideration and I look forward to working with you.

Sincerely,

Erin Edgington
Doctoral Student Researcher
Edgingt3@uwm.edu
608.553.0635

Alternate Request Email

Good Morning,

I would like to begin by thanking you for the time you have invested in this process thus far, responding to my survey and initial questions. At this time, I have had more teachers express an interest in engaging in this research than I anticipated, with many viable candidates emerging. I would appreciate it if you would consider being placed on an alternate list, if a participant is unable to continue in this study.

Please contact me by _____________, 2021 with your decision on moving forward in this work. Thank you again for sharing your time, talent, and expertise thus far. I look forward to hearing from you.

Sincerely,

Erin Edgington
Doctoral Student Researcher
Edgingt3@uwm.edu
608.553.0635

Rejection Email

Good Morning,

I would like to begin by thanking you for the time you have invested in this process thus far, responding to my survey and initial questions. At this time, I have had more teachers express an interest in engaging in this research than I anticipated, with many viable candidates emerging. I have had to make the difficult decision of selecting a small number of candidates for this work and your engagement will not be needed at this time.

Thank you again for sharing your time, talent, and expertise. Best wishes for an excellent school year.

Sincerely,

Erin Edgington
Doctoral Student Researcher
Edgingt3@uwm.edu
608.553.0635

Appendix G:
Permission for Teacher Participation

Introduction

You are being invited to participate in a doctoral research study conducted by Erin Edgington, a student in the Urban Education doctoral program at the University of Wisconsin-Milwaukee and an Assistant Professor of Education at the University of Wisconsin-Platteville. The purpose of this research is to gain insights into teaching and learning fractions when students struggle. This form provides detailed information on the research to help you decide if you would like to participate in the study.

Procedures

Your participation in this study will involve approximately three in-depth interviews by the researcher. The length of time of each interview varies, based upon the purpose and content of the interview. The first interview is anticipated to last approximately 60 minutes, the second interview is anticipated to last approximately 90 minutes, and the final interview is anticipated to last approximately 45 minutes. During these interviews, the researcher will ask a variety of questions related to the teaching and learning of mathematics, video and audiotaping these interviews using Zoom, as well as taking anecdotal notes of the discussion.

Risks & Benefits

This is a minimal risk research study. The foreseeable risks are that a participant's identity and participation in the research will not be completely confidential, due to the nature of working in an educational system. Outside of this system, however, safeguards will be in place to maintain confidentiality including the use of pseudonyms in the presentation of findings. Additionally, while there are time commitments involved in a participant's engagement in this work, every effort will be made to ensure a meaningful balance.

While there is no direct benefit to participating in this research there is the potential to enhance our understanding of teaching and learning mathematics when struggles occur.

Confidentiality

The researcher will make every effort to ensure that the information collected as a part of this study remains confidential. Names of participants, schools, and communities will be protected with the use of pseudonyms. All data collected will be securely stored in an encrypted USB or stored in a restricted-access folder on box.com, an encrypted, cloud-based storage system, or stored in a locked drawer in a restricted-access office.

Voluntary Participation

Your participation in this research is completely voluntary. If at any time during the study you choose to rescind this permission, you have the right to do this without recourse. As this is an opt-in process, you will need to provide active consent by signing the form below.

IRB Review

The Institutional Review Board (IRB) for the protection of human research participants at the University of Wisconsin – Milwaukee has reviewed and approved this study. If you have questions about the study, please contact the investigator at (608) 553-0635 or edgingt3@uwm.edu. If you have questions about your rights or would simply like to speak to someone other than the researcher about questions or concerns, you can contact the University of Wisconsin – Milwaukee IRB at (414) 229-3182 or email: irbinfo@uwm.edu

Erin Edgington
Doctoral Student Researcher
Edgingt3@uwm.edu
608.553.0635

Agreement to Participate in Research

I am interested in participating in the research described above, giving permission for Erin Edgington to observe and document the mathematics instruction in my classroom. I understand my rights and commitments in this endeavor.

________________________________ ____________________________
Signature of Participant Date

Appendix H:
First Interview Protocol and Questions

Below I have created a script to follow during the research process, to ensure standardization of protocols from the inception of the meeting until the end of the interview. While this interview protocol shares consistent questions that I seek to understand, the potential probing questions will vary between participants as determined to be appropriate.

Interview:

As we begin, I want to remind you that we discussed the use of Zoom to record our conversation today, to ensure that my notetaking is accurate and complete. Following this interview, our conversation will be transcribed, and all video files will be destroyed. Additionally, all information will remain confidential, your participation is voluntary, you may stop at any time, and there is no intention to inflict any harm. The only individuals privy to any of this information are researchers on this project. Before proceeding, do I still have your permission to record this conversation? I have recorded your response as ____.

Thank you for agreeing to participate and answer some questions about the process of teaching and learning mathematics. You have been invited to participate in this research because you are an elementary or middle level educator of mathematics in grade four, five, or six and have demonstrated an interest in being involved in this work. In this work, I seek to understand the teaching and learning of mathematics from your perspective and experience. Without an opportunity to observe this, I would like to ask you some questions that will help provide me with this insight. As we proceed, I would ask that you elaborate as much or as little as you feel comfortable in doing. Are you ready to begin? Great.

I would like to begin by thanking you for your consideration in being a participant in this research. Today, I would like to learn more about your math class, your planning process, and your teaching. One topic of particular interest to me is the instruction of fractions; this topic will focus our discussion.

Q1: As you plan your fraction instruction, what do you feel is most important in your [nth]-grade classroom?

[*** NOTE: This question will provide insight …]

Q2: With (… Answer to Q1…) in mind, what do you think about as you plan a lesson?

Q3: Could you tell me how that (… Answer to Q2…) factors into your planning?

[* NOTE: Do they mention: task, struggle, engagement, or student thinking explicitly or implicitly?]**

Potential probing questions:

- Are there specific routines you follow?
- Please tell me more about (… Answer …).
- How does (… Answer …) impact your planning process?

Q4: Are there any strategies that you consider essential in your planning efforts?

Potential probing questions:

- You mentioned (… Answer to Q4…) as an essential strategy. Could you tell me more about this?
  - What would this look like in your classroom?
- Are there any other strategies you consider essential?
  - What makes (… Answer …) essential?

[* NOTE: The questions are asked to understand …]**

Q5: You have shared many ideas about your planning. I am wondering if you could talk to me [more] about how your students factor into your planning process?

Q5a: You mentioned __________, __________, and __________ as ways that students impact your planning process. Could you tell me more about (… Answer …)?

Q5b: Does **_____________** impact how you plan for student engagement?

[***** NOTE: Do they mention: task, struggle, or student thinking explicitly or implicitly?]**

Potential Probing Questions

- Could you provide examples of the types of student needs that impact your planning?

[* NOTE: Related to the previous question, …]**

Thank you for this thorough description of your planning process for your fraction instruction; it helps me to build an understanding of your work.

If time allows, incorporate the next question. There needs to be a minimum of 10 minutes to explore the Cornbread Task.

As we continue, I would like to hear about a typical lesson on fractions.

Q6. Could you please describe a typical fraction lesson for me?

Potential Probing Questions

- What type of engagement would I see during the lesson? What would I hear during the lesson?
- What type of work might students be doing?

This insight is really helpful in allowing me to get a glimpse into your planning and teaching – thank you. During the last part of our time together today, I want to introduce you to a fifth-grade task that aligns nicely to the standards called *The Cornbread Task*. This is a task that we will be looking at more in-depth during our next session together, but I would like you to take a few minutes to look over this task, with a focus on your initial thoughts of how you might use this task in your classroom (see Appendix I).

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

A. Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

- How much does she pay for the cornbread she bought? Explain your reasoning.

The Cornbread Task adapted from Bussey, et. al., *Assessing Student's Understanding of Fraction Multiplication*, TCM, 2016.

While I understand that you have only had a few minutes to review this task, I would appreciate you sharing some of your initial thoughts.

Q7. If you were to use this task with students, how do you think they might respond or engage with the task?

Potential Probing Questions:

- Are there parts of the task you think students would do well on?
  - Could you tell me more about that?
- Are there parts of the task where you think students might get "stuck"?
  - Would you elaborate on that?
  - I would appreciate a bit more detail. How would you anticipate supporting these students?

Q8. Are there any specific strategies that you would use to support student engagement with this task?

Potential probing question, as needed.

- That's interesting. Could you tell me more about ______

Q9. Do you have any other initial thoughts related to this task that you would like to share?

Thank you for your time today, sharing insight into your classroom and teaching [fourth, fifth, sixth]-grade math, as well as your initial thoughts on how you would use this task in your classroom. As we plan for the second interview, I will ask that you to take some time to work through the task, reviewing the task more in-depth, and recording any reflections or questions related to this task. I ask that you approach this reflection as if you were preparing to implement this task in your classroom. Please do not implement this task in your classroom, as the purpose of the task is to focus our discussion on fraction teaching and learning. I have noted that our next scheduled time together is [agreed upon date and time]. Do you have any questions for me at this time? Great! I look forward to our next discussion on _______________ at _________.

Appendix I:
Second Interview Protocol and Questions

Thank you for meeting again to discuss the teaching and learning of mathematics. As a reminder, I am requesting that this interview be recorded in Zoom, for me to accurately transcribe our conversation. Your identity will remain confidential, as I will use a pseudonym to protect your identity and you have the right to stop the interview at any time. Do I have your permission to record our interview today? I have recorded your response as ___.

When we last met, we discussed what a lesson would look, sound, and feel like in your classroom. Additionally, at the end of our time together you shared your initial thoughts and reactions to a task that is aligned with fifth-grade fraction standards. At the time that I shared this task, I had asked for you to work through the task, treating this process as if you were planning to use this with your own students.

Grounding from Interview I and Task Engagement:

P1: As we begin today, could you please share additional reflections from your engagement with this task, in preparation for our conversation.

Potential probing questions:

- That is interesting. Could you tell me more about _(…Answer…)__.

P2: Based on your experience, could you please describe how you would use this task with your students.

Potential probing questions:

- As you were thinking about your students, were there important considerations related to engagement with this task?
    - Could you tell me more about this?

- So, you noted that as a teacher that __(…Answer…)_ was an important consideration

  and thinking of your students' __(…Answer…)_ was important. With these in mind, could you describe what using this task would be like in your classroom?

Thank you for sharing these thoughtful reflections on the implementation of this task in your classroom. As we move ahead, I am going to share student responses to this prompt and ask how you would respond to each student if they were in your classroom working on this task.

Before we begin, would you like to take a short break or are you ready to proceed? [Break if requested] Great, we will proceed.

Stimulus Prompt Context:

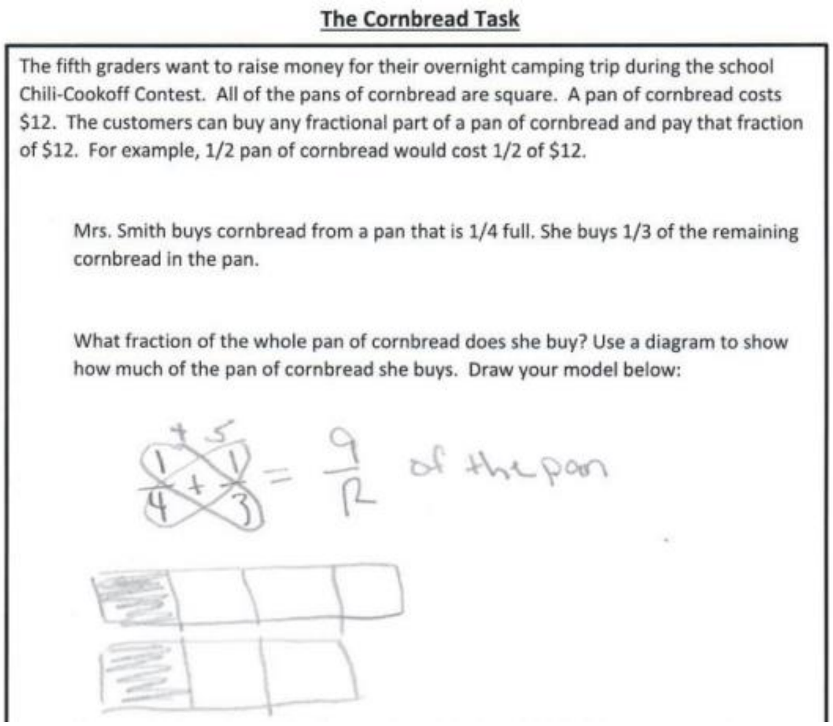
P1: In this scenario, it is assumed that you are working with your students and you have

__(…Answer…)_ (lesson structure previously indicated). To ensure that all students have equal opportunity to engage with the prompt, you read the story context aloud for them and then ask if there are any questions about the story context. Seeing none, you ask students to go ahead and start in on the task. Following a few minutes for students to work on the task, you begin circulating around your classroom.

Stimulus Task 1 - Allison

The first student whose work catches your attention is Allison and she is done. Here is what you see: (show slide 1)

Allison



The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

You ask Allison to explain her response to the first prompt. Allison responds, "I drew two, like, the squares – I don't know what to call them, but I drew one-fourth and one-third and then I added one-fourth and one-third together to get nine-twelfths and then I labeled it 'of the pan.'"

Q1: Based on Allison's interpretation of the task, how would you respond?

Potential probing questions:

- Why would you choose __(…Answer…)_ approach?
- With [revoice the response indicated by the teacher], how would you anticipate a student would proceed?
- What would you consider as a potential cause of the student's difficulties?
- If Allison responded, "Well my teacher last year told me that we should always find a common denominator and so I cross-multiplied so I could add them together" how would you move forward from this?

Q2: In the event that the student indicated an inability to make sense of the task following your [revoice the response indicated by the teacher], what would be your next response?

Stimulus Task 2 - Bradley

P2: After leaving Allison, you come upon Bradley, you see that is just staring at his paper. As you pause by his desk, you ask him why he has not started. He responds "I can't think. I'm trying to think of what I am going to draw."

Q1: Based upon Bradley's response, how would you support him?

Potential probing questions:

- What makes __(Answer)____come to mind first?
- With [revoice the response indicated by the teacher], how would you anticipate Bradley would proceed?

(Show Bradley's work)

Anticipated follow-up prompt: After Bradley (draws the square / reads the story problem aloud again / begins to solve the problem), you return to his desk where he is beginning to draw a model to move towards a solution. You ask him to share his thinking with you. He says "Oh, yeah – squares" and then as he is drawing this [show image below] he is talking aloud "So, 1, 2, 3, 4…so one-fourth. Oh, wow, I'm surprised I didn't think of that. So, [revisiting the problem] what fraction of the cornbread does she buy? Uh…let's see. Uh, so I think she will buy one piece of the cornbread from the one-fourth pan full."

Bradley

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

P2b. After hearing Bradley's "think aloud" process and seeing his current work, you ask him to explain his drawing. Bradley tells you, "It's [three] squares minus a square equals two; but it doesn't make sense because I know he paid $3 for the cornbread – I know this because she bought one-fourth of a pan of cornbread and that costs $3."

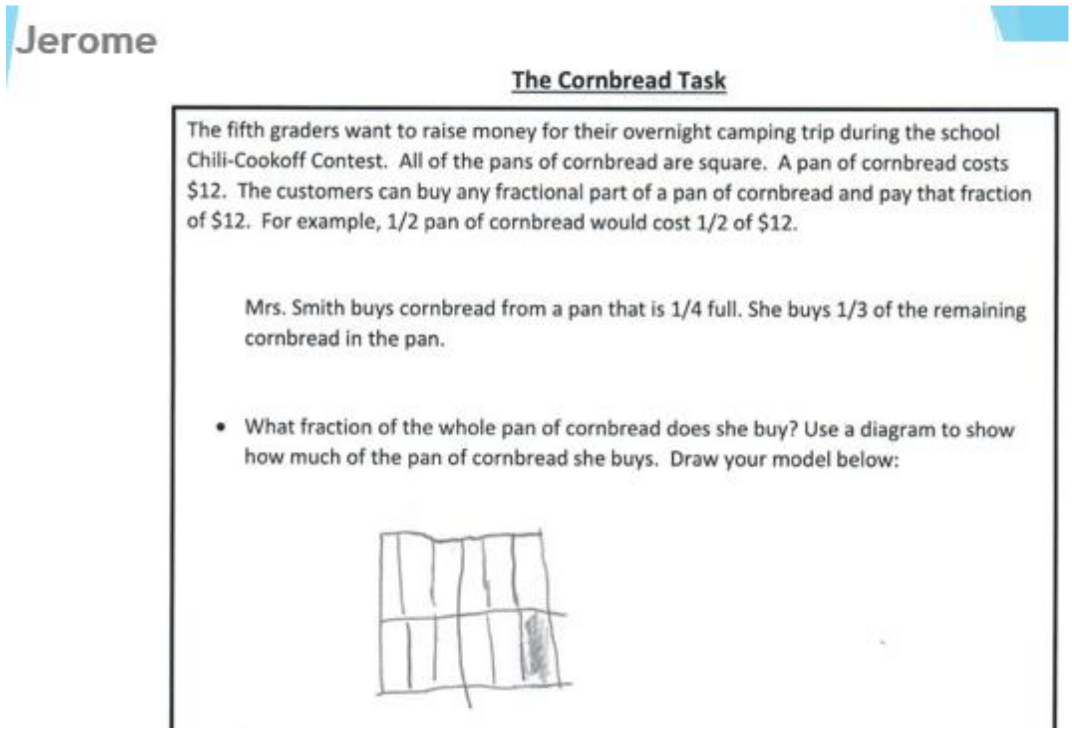
Q2. With this new insight into Bradley's thinking and reasoning, how would you proceed?

Potential probing questions:

- Why would you choose __(…Answer…)_ / approach?
- With [revoice the response indicated by the teacher], how would you Bradley would proceed?

Stimulus Task 3 - Jerome

P3. You continue walking around the room and come upon Jerome. (Show Jerome's work) He is done with his work and this is what you see on his paper for the first task.


Jerome

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Q1: Based upon Jerome's response, what would you say to him?

Potential probing questions:

You could acknowledge his response, would there be anything else you might ask him or you would want to say to him?

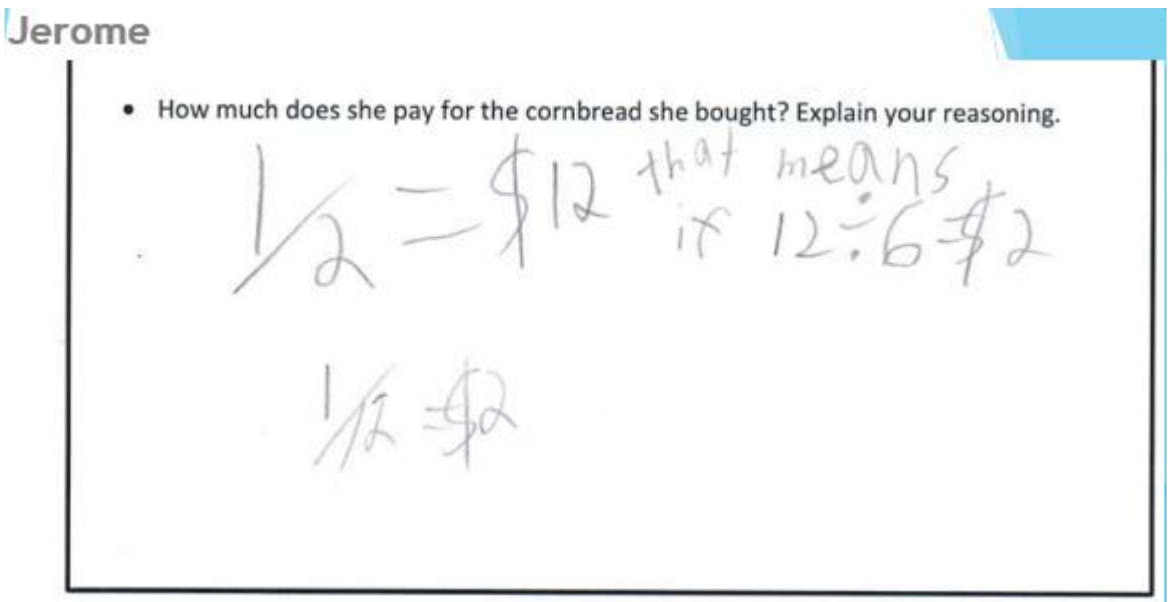
That is what I asked Jerome / I asked Jerome to share his solution strategy and he responded, "Because all the pans are square I made a square and then I divided it into four because it says she buys cornbread from the pan that is one-fourth full, but she wanted one third. So, I divided them all into thirds and shaded in one; so, she bought one-twelfth of the pan."

- Now that you have heard his response, how would you respond to Jerome now?

P3a. While Jerome shares an accurate model, answer, and explanation of his thinking, you also noticed that he has answered the second part of the prompt with this answer.

(Show Jerome Slide 2

Q2. When looking at Jerome's second answer, how would you respond?



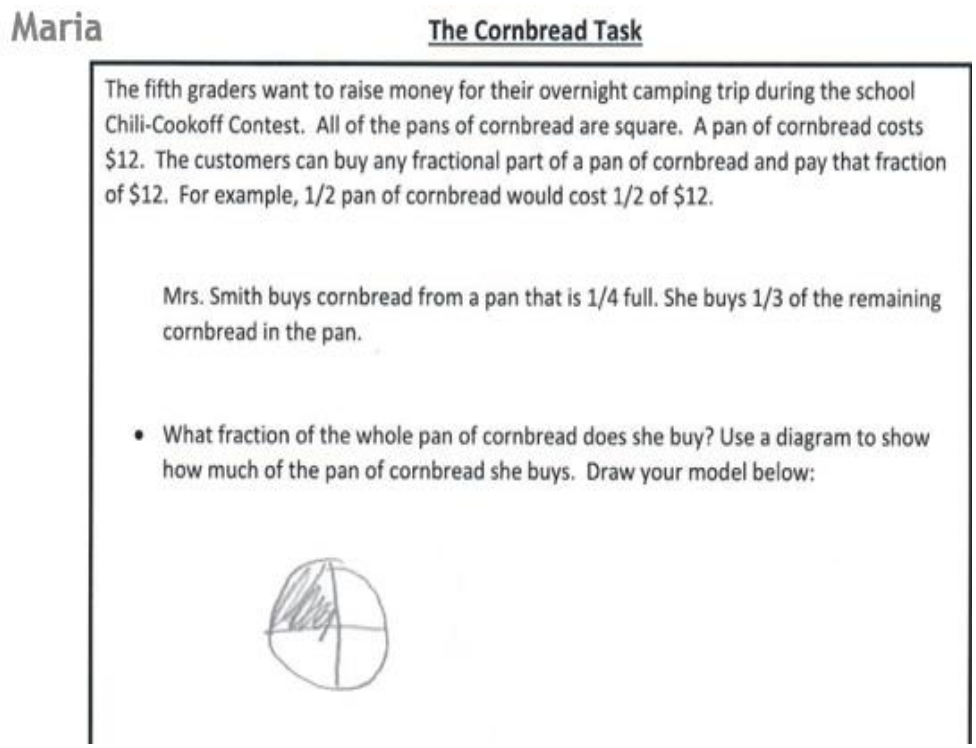
Potential probing questions:

- Given your experience, with [revoice the response indicated by the teacher], how would you anticipate Jerome would proceed?

Stimulus Task 4 – Maria

(Show Maria's Work)

P4. Maria is sitting near Jerome and her work catches you eye next. She has started creating a drawing to solve the problem but has not progressed beyond her first image. This is what Maria has drawn on her paper.



Maria

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Given your experience, how would you engage with Maria?

Potential probing questions:

- Could you tell me more about your decision to use __(…Answer…)_ approach?
- With [revoice the response indicated by the teacher], how would you anticipate a student would proceed?

Anticipated follow-up prompt:

In the classroom, the teacher did provide Maria some more time to work with this and when she came back around, she saw that Maria had drawn the second image.

Potential Follow-Up Prompt: After __(…Answer…)_ Maria, she draws a new model from which to build and solve the problem. Here is her updated work:

Maria

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

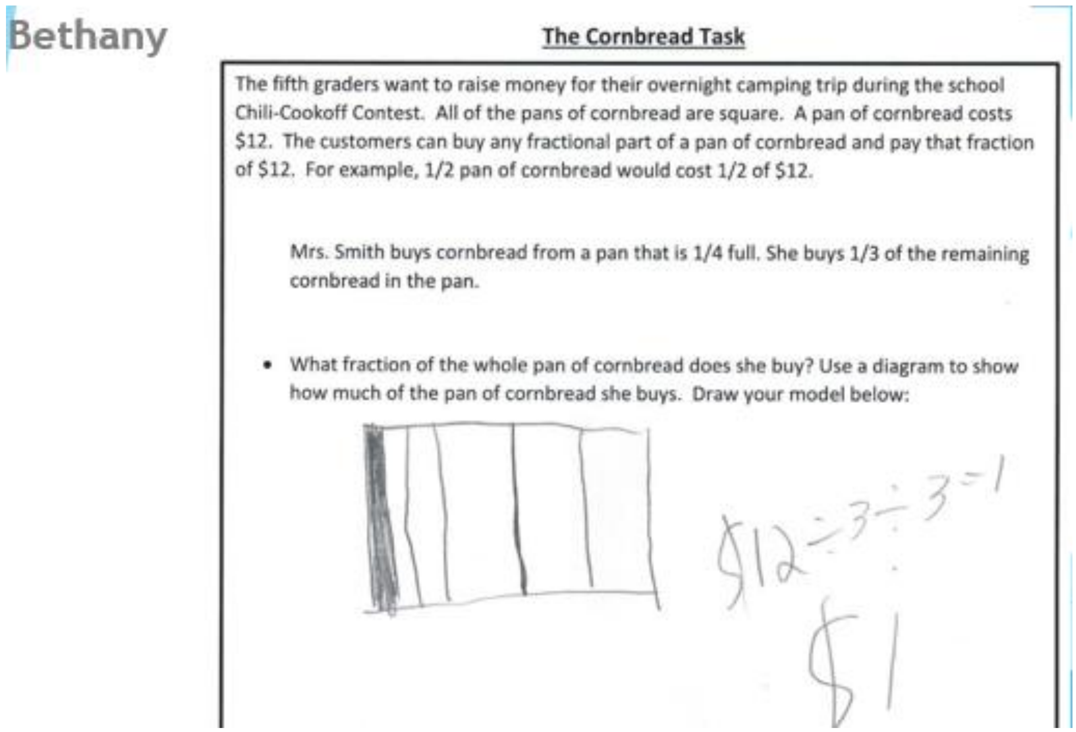
- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Q3. How would you support Maria, based upon her current efforts?

Potential probing questions:

- Why would you choose __(…Answer…)_ approach?
- With [revoice the response indicated by the teacher], how would you anticipate a student would proceed?

P5. Upon leaving Maria's desk, you take one final loop around the room. While Bethany has not finished everything, you notice that her solution pathway is different than other students, which piques your interest. Here is her work:


Bethany

The Cornbread Task

The fifth graders want to raise money for their overnight camping trip during the school Chili-Cookoff Contest. All of the pans of cornbread are square. A pan of cornbread costs $12. The customers can buy any fractional part of a pan of cornbread and pay that fraction of $12. For example, 1/2 pan of cornbread would cost 1/2 of $12.

Mrs. Smith buys cornbread from a pan that is 1/4 full. She buys 1/3 of the remaining cornbread in the pan.

- What fraction of the whole pan of cornbread does she buy? Use a diagram to show how much of the pan of cornbread she buys. Draw your model below:

Q1. Based upon Bethany's response, what would you say to her?

Potential probing questions:

- You could acknowledge her response, would there be anything else you might ask her or you would want to say to her?
- If teacher states that they would want to hear her solution strategy/ prompt that this was asked for: If Bethany responded "So, ummm…the whole entire thing. So, I just made a square for the pan and I cut it into fourths – well, into halves and then half again. [and] Then I cut one of the fourths into thirds and then I said so there's four pieces divided by twelve, which is the whole thing which would be $3, because twelve divided by three is four and then – No. Twelve divided by four is three – yeah, and then there would be three pieces and she wants one of the thirds, so it costs $1."

- *Utterance overheard as she worked: "Umm…so it's twelve dollars divided by four – ok, so three dollars. So, it's telling me how much money it would cost for one-third of one-fourth of the cornbread."*

- How would you respond to Bethany?
- Why would you choose __(…Answer…)_ approach?

I appreciate all of your reflections on the student responses to The Cornbread Task.
Is there anything else you would want to tell me about the Cornbread Task that we have not discussed today?

Thank you for these thoughtful and thorough explanations of how you would interact with students while using this task, in the event that your students demonstrated some of the responses shared.

I have learned a great deal while discussing this fraction task with you today – thank you. Do you have any questions for me at this time? If not, we have our final interview scheduled for ______________. In this final interview, we are going to talk about struggle in the teaching and learning of math. Does this date and time still work for you? Great! I look forward to our discussion.

Appendix J:
Third Interview Protocol and Questions

Thank you for meeting to discuss the teaching and learning of mathematics over these past few weeks. As a reminder, I am requesting that this interview be recorded in Zoom, for me to accurately transcribe our conversation.

During our time together, we have discussed how that you plan for mathematics instruction, sharing what a lesson would look, sound, and feel like for an observer, as well as how you would engage students with the fraction task I shared.

Q1. After reflecting more on our conversation from __(note day/last week)__, do you have any questions for me or reflections you would like to add?

Potential follow-up question(s):

- In our last conversation, you noted that you would __(Answer)__ if you saw __(Name student)'s_ work. Could you elaborate for me on_____________?
- In our last conversation, you mentioned _______________ could you please clarify what this means?

Today I want to talk more about struggle in learning mathematics.

Q1. Many teachers talk about how struggle in learning math is good for kids. What do you think about this?

Potential probing questions:

- How do you identify good struggle?

Q2. Could you share an example of how struggle in learning math is good for kids?

Potential probing questions:

- How would you respond to __(Answer 2 struggle)__?

Q3. Many teachers talk about how struggle in learning math is not good for kids. What do you think about this?

Potential probing questions:

- How do you identify struggle that is not good?

Q4. Could you share an example of how struggle in learning math can be bad for kids?

Potential probing questions:

- How would you respond to __(Answer 4 struggle)__?

Q5. Thinking of struggle in learning math, are there specific types of struggle that are more common than others?

Potential probing question(s):

- What are the types of struggles you see students having in your math classroom?

Q6. Do you have any final thoughts about struggle in learning math?

Thank you again for this insight into your planning and teaching of mathematics! I do not have any further questions at the moment, but I want to thank you for all of your time and engagement in this work. Do you have any questions for me? If a question or concern arises in the future, please reach out to me in order me to address this. As this is our last interview together, I will ask if it is possible to reach out for any further questions or clarifications that might arise, as I begin to transcribe the data. May I reach out to you for this purpose? Thank you. I truly appreciate you sharing your time, experience, and teaching expertise.

CURRICULUM VITAE
Erin G. Edgington

EDUCATION

Ph.D in Urban Education
Anticipated: Fall 2021
Mathematics Education emphasis
University of Wisconsin – Milwaukee

M.S in Education
July 2010
Mathematics Education emphasis

University of Wisconsin – Platteville, Platteville, WI

B.S. in Education

Illinois State University, Normal, IL
December 1996

- BS Education

PROFESSIONAL LICENSING / AFFILIATIONS

Wisconsin Educator License

- 2005 – PDP Team Member
  2006 - 2016

License Type: T001 Teacher – Lifetime License
Present

- 1088 – Elementary / Middle Level Education Grades 1 – 8

License Type: T001 Teacher – Lifetime License

- 1365 – Spanish Grades 1 - 9

License Type: T001 Teacher – Lifetime License

- 1300 – English Grades 1 - 9

Memberships

- National Council of Teachers of Mathematics (NCTM)
- National Council of Supervisors of Mathematics (NCSM)

- Association for Supervision and Curriculum Development (ASCD)
- Wisconsin Mathematics Council (WMC)
- Wisconsin Mathematics Leadership Council (WiMLC)
- Association of Mathematics Teacher Educators (AMTE)

HIGHER EDUCATION EXPERIENCE

Assistant Professor, School of Education Fall 2015 - Present

University of Wisconsin, Platteville – Platteville, WI

- Design instruction and assessment that engages students in authentic learning experiences, related to developmentally appropriate instructional design and pedagogy
- Coordinate practicum experiences for elementary education majors
- Supervise and mentor pre-service educator practicum experiences
- Supervise and mentor student-teachers
- Advise students in the School of Education
- Courses taught:
    - TCHG 1230 – Introduction to Education
    - TCHG 2130 – Human Growth and Development
    - TCHG 3240 – Pre-K Methods of Cognitive Development
    - TCHG 3730 – Assessment and Management
    - TCHG 3730 – Instructional Design and Assessment (Changed Fall 2021)
    - TCHG 4160 – Elementary Science Methods (Grades 3-5)
    - TCHG 4170 – Teaching Elementary Mathematics
    - TCHG 4250 – Senior Seminar
    - TCHG 4080 – Intermediate Elementary Practicum
    - TCHG 4240 – Early Childhood Practicum

edTPA Coordinator Spring, 2016 – Summer, 2019

- Provide in-service training for student teachers on edTPA expectations during drive-in seminar
- Organize and disseminate calendar and reminders related to this process
- Participate in bi-monthly state and national edTPA Coordinator calls, disseminating pertinent information to faculty
- Provide writing workshops on three separate weekends, aligned to the edTPA tasks
- Maintain records related to student completion of edTPA tasks, including filming permission
- Oversee edTPA submission and response when students fail to meet the state mandated score
    - Mentor students and/or align faculty mentors and disseminate communications related to resubmission process when warranted
    - Disseminate communications related to an exception process
    - Review exception application requests

STEM Coordinator, School of Education Fall 2014 – Spring 2016
University of Wisconsin, Platteville – Platteville, WI

- Collaborate with colleagues in the School of Education regarding meaningful changes and additions to the new 1 – 8 major
- Collaborate with faculty from Science, Technology, Engineering, Agriculture and Mathematics (STEAM) to gain insight and expertise in the design of the new MC-EA STEM 1 – 8 degree
- Establish the STEAM Advisory Board to inform each step of the process
- Communicate and record changes, new courses and complete the various steps related to this

<u>K-12 TEACHING / LEADERSHIP EXPERIENCE</u>

K-12 District Math Coach / Middle School Multi-Tiered Systems of Support (MTSS) Coach
2012 - 2014
Dodgeville School District, Dodgeville, WI

- Research and implement a meaningful classroom coaching model, driven by data and teacher determined student needs, support the implementation of a formative assessment/reflective teaching practice cycle, model and train teachers on best practice instructional methodologies
- Provide weekly communication with administration about the duties performed as they relate to both coaching and the MTSS process
- Provide weekly communication with 6-8 staff regarding the MTSS process, student plans and collaborations
- Provide weekly and monthly communications with teams regarding instruction, collaborations, and data
- Organize and disseminate information related to weekly meetings, student plans, and data collection
- Analysis of data and compiled reports related to this
- Research on meaningful interventions to support growth and achievement, as well as behavior interventions to support student success
- Provide staff training on mathematics instructional practices
- Create resources and present to staff regarding the RtI process, framework, and implementation as determined by our district's guidelines

K-12 RtI (Response to Intervention) Coordinator 2011 – 2012

- Using the framework and model determined meaningful for the Dodgeville School District, continue to support further growth and/or implementation of a meaningful RtI process, at the various levels, according to teachers' level of knowledge and prior experience with the RtI process

- Provide weekly communication with administration about the duties performed as they relate to the RtI process
- Provide weekly communication with K – 12 staff regarding the RtI process, student plans, and collaborations
- Organize and disseminate information related to weekly meetings, student plans and data collection
- Provide analysis of data and research on meaningful interventions to support growth and achievement
- Create resources and present to staff regarding the RtI process, framework, and implementation as determined by our district's guidelines
- Parent communication

4th grade classroom teacher 1997 – 2003; 2004 – 2011

- Served a diverse population of learners as a classroom teacher at Dodgeville Elementary School
- Successfully integrated state standards into the curriculum
- Modified curriculum to meet the needs of all learners
- Served as the grade-level representative for our Staff Effectiveness Team
- Served as the grade-level rep for our RtI committee

9th grade English teacher 2003 - 2004

- Served a diverse population of learners as a classroom teacher at Dodgeville High School
- Successfully integrated state standards into the curriculum
- Modified curriculum to meet the needs of all learners

Brookhill Institute for Mathematics 2012 – Spring 2016

- Spring 2016
    - Consult on module modifications
- Summer 2014
    - Facilitated professional development modules for teachers in *Operations and Algebraic Thinking Grades 3-5* module (10 – 3-hour sessions)
- Fall 2012 – Summer 2013
    - Co-authored and facilitated professional development modules for *Operations and Algebraic Thinking Grades 3-5* (10 – 3-hour sessions)
    - Trained future facilitators
- Summer 2012
    - Facilitated professional development modules for teachers in *Number and Operations Fractions Grades 3-5* (10 – 3-hour sessions)

PRESENTATIONS

Edgington, E. (July 2018). Productive Struggle: A Mindset for STEM Learning. *I^2 STEM Institute,* University of Wisconsin Platteville, Platteville, WI.

Edgington, E. (May 2018). Habits and Practices: Supporting a Mindset for STEM Learning. *Wisconsin Mathematics Council State Annual Conference*, Green Lake, WI.

Collins, J. & **Edgington, E.** (November 2017). Breaking Down Silos: Teacher Education + Social Sciences = Success. *National Council for the Social Studies* in San Francisco, CA.

Edgington, E. (October 2017). Mindsets and Learning – Unlocking Student Potential. *JAM Consortium Fall In Service,* Juda, WI.

Edgington, E. (September 2017). STEM Forward: Building Bridges to Increase Educator Capacity. *Midwest STEM Forum,* BTC Institute, Fitchburg, WI.

Edgington, E. (July 2017). Mindsets and Learning: Session III and IV. *I^2 STEM Institute,* University of Wisconsin-Platteville, Platteville, WI.

Edgington, E. (March 2017). Involve, Engage, Empower – Building Professional Identity Through Local, State and National Engagement. *National Association of Professional Development School's Annual Conference* in Washington, D.C.

Edgington, E. (August 2016). Mindsets and Learning: Session I and II. *I^2 STEM Institute,* University of Wisconsin-Platteville, Platteville, WI.

Edgington, E. (July 2016). Writing Effective Student Learning Outcomes: A Roadmap for Success. Tech Mash-Up, University of Wisconsin-Platteville, Platteville, WI.

Hollingsworth, L., Brogley, J., & **Edgington, E.** (2016, February). edTPA the UW-Platteville Way. *UW-System edTPA Conference* in Wisconsin Dells, WI.

Hollingsworth, L., Brogley, J., & **Edgington, E.** (2016, March). Preparing students for edTPA: Design of a mini-edTPA Portfolio. *National Association of Professional Development School's Annual Conference* in Washington, D.C.

Hollingsworth, L., **Edgington, E**., Monhardt, L., Stinson, K. (2015, September). New 1-8 Major in School of Education. *University of Wisconsin-Platteville Faculty & Staff Research Day,* Platteville, WI. (Poster)

Edgington, E. (March 2013). Involve, Engage, Empower: How Daily Formative Assessment, Through Error Analysis, Can Transform Student Learning and Understanding. *DPI Focus Schools Conference,* Oconomowoc, WI.

Edgington, E. (April 2013). Involve, Engage, Empower: How Daily Formative Assessment, Through Error Analysis Can Transform Student Learning and Understanding. *DPI Focus Schools Conference*, Appleton, WI.

Edgington, E. (December 2012, January 2013). Number and Operations Fractions – Using Fraction Strips. *Mathematics Proficiency for Every Student.*

Edgington, E. (May 2012). Involve, Engage, Empower: How Placing the Onus of Learning on Students Translates into Increased Understanding. *Wisconsin State Mathematics Conference,* Green Lake, WI.

Edgington, E., (2012, July). Value Added Instruction: How Constructed Responses and Error Analysis Procedure Lead to Increased Math Understanding. *CESA #11: Creating Math Learning Communities*

Edgington, E., & Polglaze, C. (2011, November). Value Added Instruction: How Constructed Responses and Error Analysis Procedure Lead to Increased Math Understanding. *Wisconsin Mathematics Council – Mathematics Proficiency for Every Student*

Edgington, E. (2011, July). Value Added Instruction: How Constructed Responses and Error Analysis Lead to Increased Math Understanding. *ESEA Summer grant course – UW-Platteville*

Edgington, E., Piper, J., Rhode, L., Tranel, A., & Whitford, T. (2011, March). The Dodgeville RtI Diamond –

Interventions for All. *Wisconsin RtI Summit: Strategies for Increasing Academic and Behavioral Success for All Students*

PROFESSIONAL, SCHOLARLY, and CREATIVE ACTIVITY

Professional Affiliations

Association of Mathematics Teacher Educators

WI Association of Mathematics Teacher Educators (Elected Board Member – Fall 2021)

National Council of Teachers of Mathematics

National Council of Supervisors of Mathematics

Wisconsin Mathematics Council

Creative Activity

On the Farm STEM Experience (May 2018)

- Selected by the American Farm Bureau as one of 30 participants to attend training in Portland, Oregon focused on the incorporation of Agricultural concepts in STEM instruction.

PUBLICATIONS

Edgington, E., Grunow, J. (Fall, 2010). Using Interventions to Build Mathematical Proficiency: Lessons from the Classroom. *Wisconsin Teacher of Mathematics.*

AWARDS and HONORS

2020 College of Liberal Arts & Education Distinguished Educator Award

Liberal Arts and Education Teamwork Award (2015)

PROFESSIONAL SERVICE

University Service

Institutional Review Board for Human Subjects Research (IRB) (LAE: Fall 2021 – 2024)

University Undergraduate Curriculum Commission (UUCC) (LAE: Fall 2020 – 2023)

LAE / SOE Scholarship Committee (2020 – 2021)

UW-System Math Initiative – UW-Platteville Institutional Change Team (Spring 2019 – Present)

Teaching Observation Team (Spring 2018 – Present)

Improvement of Learning Committee (Fall 2018 – Spring 2020)

Co-Pi on Noyce Grant – Engineers to Teachers (Summer 2017 – Summer 2018)

Member, Library Committee (2015 - Spring 2018)

Member, OIEA Advisory Board (2015 - 2017)

Member, College of EMS STEP Grant Internal Advisory Board (2015)

<u>School of Education Service</u>

School of Education Faculty Hiring Committee (Fall 2021)

K – 9 Admission and Retention Policy Committee (Summer 2020)

Curriculum Committee (Fall 2018 – Present)

Member, Elementary Education Redesign Committee (2014 - 2020)

Supported Allied Faculty in the edTPA

- Modeled student support in the retake process (Spring 2019)
- Provided an edTPA overview for students in the Music Education program (Fall 2019)

Recruitment Committee (Fall 2018 – Spring 2019)

edTPA Coordinator (Spring 2016 – Summer 2019)

Evaluator, Pre-Professional Days (2014-present)

Educators Rising Initiatives (Fall 2018)

<u>Professional / Community Service</u>

Closing the Fraction Gap Teacher Professional Development Series for WI-AMTE (September, October, November, and December 2021)

Mathematics Resource Review for New Curriculum Adoption – Dodgeville School District – Spring / Summer 2021

Book Study for WI-AMTE and WMC *Catalyzing Change in Early Childhood and Elementary Mathematics* (October 2020)

Wisconsin Presidential Award for Excellence in Mathematics and Science Teaching (PAEMST) Selection Committee (Fall 2020)

Family Math Night Coordinator (Fall and Spring Events) – Dodgeville School District (Fall 2015 – Spring 2018)

Family Math Night Coordinator (Fall Event) – Mineral Point School District (Fall 2017, 2018, 2019)

Volunteer Math Intervention and Extension Collaboration/Consultation – Mineral Point Elementary School (Fall, 2017)

Research and Teaching Interests

Equitable mathematics teaching practices

Elementary mathematics teacher preparation

Elementary mathematics education

Impacts of mindset on the teaching and learning of mathematics

Mathematical content knowledge, pedagogical knowledge, and pedagogical content knowledge needed for teaching

Productive struggle in learning mathematics