2022

Geometric Properties of Weighted Projective Space Simplices

Derek Hanely
University of Kentucky, dhanely17@gmail.com
Author ORCID Identifier:
https://orcid.org/0000-0001-9548-3866
Digital Object Identifier: https://doi.org/10.13023/etd.2022.129

STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained needed written permission statement(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine) which will be submitted to UKnowledge as Additional File.

I hereby grant to The University of Kentucky and its agents the irrevocable, non-exclusive, and royalty-free license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless an embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's thesis including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Derek Hanely, Student

Dr. Benjamin Braun, Major Professor

Dr. Benjamin Braun, Director of Graduate Studies

Geometric Properties of Weighted Projective Space Simplices

DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
Derek W. Hanely
Lexington, Kentucky

Director: Dr. Benjamin Braun, Professor of Mathematics
Lexington, Kentucky
2022

https://orcid.org/0000-0001-9548-3866

ABSTRACT OF DISSERTATION

Geometric Properties of Weighted Projective Space Simplices

A long-standing conjecture in geometric combinatorics entails the interplay between three properties of lattice polytopes: reflexivity, the integer decomposition property (IDP), and the unimodality of Ehrhart h^*-vectors. Motivated by this conjecture, this dissertation explores geometric properties of *weighted projective space simplices*, a class of lattice simplices related to weighted projective spaces.

In the first part of this dissertation, we are interested in which IDP reflexive lattice polytopes admit regular unimodular triangulations. Let $\Delta_{(1,\mathbf{q})}$ denote the simplex corresponding to the associated weighted projective space whose weights are given by the vector $(1, \mathbf{q})$. Focusing on the case where $\Delta_{(1,\mathbf{q})}$ is IDP reflexive such that $\mathbf{q}$ has two distinct parts, we establish a characterization of the lattice points contained in $\Delta_{(1,\mathbf{q})}$ and verify the existence of a regular unimodular triangulation of its lattice points by constructing a Gröbner basis with particular properties.

In the second part of this dissertation, we explore how a necessary condition for IDP that relaxes the IDP characterization of [11] yields a natural parameterization of an infinite class of reflexive simplices associated to weighted projective spaces. This parametrization allows us to check the IDP condition for reflexive simplices having high dimension and large volume, and to investigate h^*-unimodality for the resulting IDP reflexives in the case that $\Delta_{(1,\mathbf{q})}$ is 3-supported.

KEYWORDS: lattice polytope, Ehrhart theory, simplices, projective space, reflexivity, integer decomposition property

Derek W. Hanely

April 27, 2022

Geometric Properties of Weighted Projective Space Simplices

By
Derek W. Hanely

Dr. Benjamin Braun
Director of Dissertation

Dr. Benjamin Braun
Director of Graduate Studies

April 27, 2022
Date

Dedicated to my mother, Paula.

ACKNOWLEDGMENTS

The development of this thesis would not have been possible without the assistance of many people. Firstly, I would like to thank my advisor, Benjamin Braun, for his support, guidance, patience, and dedication to the projects contained within. He picked me up when I was feeling down and helped me become a better mathematician. Without his assistance and constant encouragement, this thesis would not have been possible. I could not have asked for a better advisor.

I also wish to thank my other thesis committee members for their helpful insight and suggestions.

Next, I thank my collaborators and coauthors, Ben, Rob, Liam, Morgan, Martha, Khrystyna, Julie, Andrés, Matias, Katie, and Zach, for our work together. Thank you for constantly reminding me about the joys of math.

Lastly, I would like to thank my family and friends for their support throughout this entire process. Specifically, my mother, Cole, Will, Vasily, Justin, and Matias provided the reassurance necessary to make my graduate goals a reality. I cannot imagine having done this project without their emotional support. The work contained herein is truly a testament to the support and encouragement I received from everyone mentioned above and so many others. Thank you all so much.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1 Introduction & Background

1.1 Introduction

This dissertation consists of four sections, the latter half of which combine the results of two related projects. These two projects, both of which extend the literature on geometric properties of so-called *weighted projective space simplices*, are

- identifying a regular unimodular triangulation of 2-supported weighted projective space simplices, and
- providing a classification of 3-supported reflexive weighted projective space simplices possessing the integer decomposition property.

Chapter 1 delineates all the pertinent background material and preliminaries required to understand the aforementioned projects. Specifically, it provides an overview of the Ehrhart theory of lattice polytopes, the primary framework motivating my work, and triangulations of lattice polytopes. In Chapter 2, we introduce the protagonist of the rest of the document, namely, weighted projective space simplices. The results in Chapter 3 establish the existence of a regular unimodular triangulation of certain 2-supported weighted projective space simplices. Finally, Chapter 4 presents a complete characterization of 3-supported weighted projective space simplices that are simultaneously reflexive and posses the integer decomposition property.

1.2 Lattice Polytopes

Polytopes, in the most basic and informal sense, are geometric objects with flat sides. In essence, polytopes can be thought of as generalizations of polygons (e.g., triangles, quadrilaterals, pentagons) in two dimensions and polyhedra (e.g., tetrahedra, cubes, icosahedra) in three dimensions. The study of polytopes is very much active as they possess combinatorially rich properties and have a number of applications in a wide variety of fields including optimization, physics, algebraic geometry, and topology.

The integer points $\mathbb{Z}^d$ form a lattice in $\mathbb{R}^d$, and we refer to integer points as *lattice points*. For our purposes, we will only be interested in polytopes whose *vertices* are lattice points. We now give a formal definition of such polytopes.

Definition 1.2.1. A subset $P \subset \mathbb{R}^n$ is a *d-dimensional (convex) lattice polytope* if it is the convex hull of finitely many points $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^n$, called *vertices* of P, that collectively span a d-dimensional affine subspace of $\mathbb{R}^n$. In this case,

$$P = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{v}_i \,:\, \lambda_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\} \subseteq \mathbb{R}^n$$

is the *vertex description*, or *$\mathcal{V}$-description*, of P, and we write $P = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$.

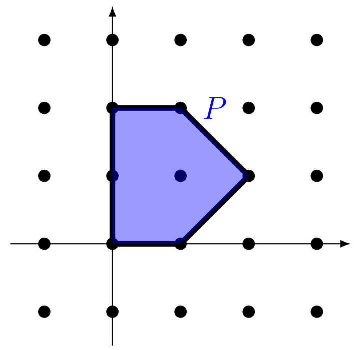

Figure 1.1: A 2-dimensional lattice polytope with 5 vertices.

Equivalently, a lattice polytope P can be described as the intersection of finitely many closed halfspaces. This description is referred to as the *hyperplane description*, *halfspace description*, or *$\mathcal{H}$-description* of P. A fundamental result of convex geometry is that every polytope has both a $\mathcal{V}$-description and an $\mathcal{H}$-description. However, it is a cumbersome task to prove the equivalence of these definitions, and it is, in general, algorithmically nontrivial to obtain one description from the other (see, e.g., [36]). We will be primarily concerned with $\mathcal{V}$-descriptions of lattice polytopes moving forward.

Example 1.2.2. Consider the lattice polytope P depicted in Figure 1.1. A $\mathcal{V}$-description of P is given by the convex hull of its five vertices, i.e.,

$$P = \operatorname{conv}\{(0,0),(1,0),(2,1),(1,2),(0,2)\}.$$

An irredundant (i.e., no redundant constraints) $\mathcal{H}$-description of P is given by the following five inequalities that cut out the polytope:

$$P = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, y \leq 2, y \leq 3-x, y \geq x-1\}.$$

For a fixed dimension d, we can define an important class of polytopes that have the minimum number of vertices possible. Such polytopes are called *simplices*.

Definition 1.2.3. A polytope formed by the convex hull of exactly $d+1$ affinely independent points is called a *d-simplex*. For example, line segments are 1-simplices, triangles form the set of all 2-simplices, and the 3-dimensional simplices are tetrahedra.

For a polytope P, recall that the smallest affine subspace of $\mathbb{R}^n$ containing P is called the *affine hull* of P. We denote this subspace by $\operatorname{aff}(P)$, and it is given by $\operatorname{aff}(P) = \left\{\sum_{i=1}^{\ell} \lambda_i \mathbf{x}_i : \mathbf{x}_i \in P, \sum_{i=1}^{\ell} \lambda_i = 1\right\}$. If a polytope can be obtained from another via certain transformations, such as reflections or translations, the two polytopes effectively carry the same combinatorial and geometric data. Thus, for our purposes, we will consider lattice polytopes up to the following equivalence. Given two lattice polytopes $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^{n'}$, we say P and P' are *unimodularly equivalent* if there exists an affine map from the affine span of P to the affine span of P' that maps P to P' and maps $\mathbb{Z}^n \cap \operatorname{aff}(P)$ bijectively onto $\mathbb{Z}^{n'} \cap \operatorname{aff}(P')$. Due to this equivalence, it will be convenient for us to simply consider lattice polytopes that are *full-dimensional*, that is, lattice polytopes living in their affine span so that their dimension is equivalent to the dimension of their affine span.

Further recall from geometry that a *supporting hyperplane* of a set Q in $\mathbb{R}^n$ is a hyperplane that satisfies both of the following conditions:

(i) Q is entirely contained in one of the two closed halfspaces bounded by the hyperplane, and

(ii) Q has at least one boundary point on the hyperplane.

The following definition provides further polytopal terminology that will be useful later on.

Definition 1.2.4. Let P be a lattice polytope.

- The intersection of P with a supporting hyperplane H is called a k-dimensional face or k-face if $\dim(\operatorname{aff}(P) \cap H) = k$.
- If $\dim(P) = d$, the $(d-1)$-faces are called *facets* and the 0-faces are the vertices of P.
- If $\dim(P) = d$, its *normalized volume*, denoted $\operatorname{Vol}(P)$, is defined to be $d!$ times the Euclidean volume of P.

For reasons that will become more apparent in the following section, combinatorialists are usually interested in normalized volume since it is always an integer for any lattice polytope. Therefore, the term volume hereinafter will be taken to be the normalized volume rather than Euclidean volume, unless otherwise stated explicitly.

1.3 Ehrhart Theory

Ehrhart theory is a fundamental area of discrete geometry concerned with counting lattice points in polytopes and their dilates. Before considering the general case for any dimension, we first state the following classical result due to Georg Pick (1859-1942) which describes an interesting relationship between the lattice points of lattice polygons and their areas. This provides some context for why the enumeration of lattice points is of interest.

Theorem 1.3.1 (Pick's Theorem, [30])**.** *For a lattice polygon $P \subseteq \mathbb{R}^2$, let A denote the area of P. Then,*

$$A = i + \frac{b}{2} - 1,$$

where i denotes the number of lattice points in the interior of P and b denotes the number of lattice points on the boundary of P.

Example 1.3.2. Consider the lattice polygon given in Figure 1.1. Observe that there is 1 interior lattice point (namely, $(1,1)$) and 6 boundary lattice points. According to the theorem, this means that

$$A = 1 + \frac{1}{2}(6) - 1 = 3.$$

Indeed, this value coincides with the area of the pentagon obtained via basic geometry.

Unfortunately, higher-dimensional analogues for the volume of lattice polytopes using only interior and boundary points do not exist. The quintessential counterexample in three dimensions is the *Reeve tetrahedron*: a 3-dimensional polytope defined as the convex hull of its vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,r)$, where r is a positive integer. Indeed, for any r, the Reeve tetrahedron has 0 interior lattice points and 4 boundary lattice points (namely, its vertices), but the arbitrariness of r allows one to generate infinitely many possible volumes. This motivates the following theory for lattice point enumeration in the general case (beyond two dimensions).

For a nonnegative integer t and polytope P, the t-th *dilate of* P, denoted tP, is given by $tP := \{t\mathbf{p} : \mathbf{p} \in P\}$. Many interesting geometric and algebraic properties of P are revealed by considering the *cone over* P, which provides an interesting means of interpreting dilates of polytopes.

Definition 1.3.3. If $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{Z}^n$ are the vertices of a lattice polytope $P = \operatorname{conv}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$, we lift the vertices into $\mathbb{R}^{n+1}$ by appending a 1 as the last coordinate of each vertex of P and consider the nonnegative span of the resulting vectors. The *cone over* P is

$$\operatorname{cone}(P) := \operatorname{span}_{\mathbb{R}_{\geq 0}}\{(\mathbf{v}_i, 1) : i = 1, \ldots, m\}.$$

Note that for each $t \in \mathbb{Z}_{>0}$, we can recover tP by simply slicing the cone at height t, i.e., considering $\operatorname{cone}(P) \cap \{z_{n+1} = t\}$. Now, let P be a d-dimensional lattice polytope. To P, we associate its *Ehrhart function* (or *lattice point enumerator*), denoted $\mathcal{L}_P$, which enumerates the lattice points contained in nonnegative integral dilates of P, i.e., $\mathcal{L}_P(t) := \left|tP \cap \mathbb{Z}^d\right|$. This function (amazingly) is a polynomial in t of degree d [18], and thus is more commonly referred to as the *Ehrhart polynomial of* P. Moreover, due to Stanley [32], it is known that the generating function encoding this polynomial, the *Ehrhart series of* P, is a rational function

$$\operatorname{Ehr}_P(t)(z) := \sum_{t \geq 0} \mathcal{L}_P(t) z^t = \frac{\sum_{j=0}^{d} h_j^* z^j}{(1-z)^{d+1}},$$

where $h_0^* = 1$ and $h_j^* \geq 0$ for all j. We call the numerator of the Ehrhart series the *h^*-polynomial of* P, and the vector of its coefficients, $h^*(P) = (h_0^*, h_1^*, \ldots, h_d^*)$, the *$h^*$-vector*. Note that the Ehrhart series of two lattice polytopes P and Q that are unimodularly equivalent are equal.

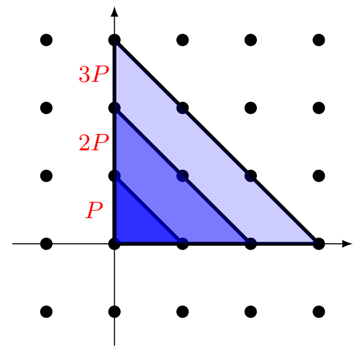

Figure 1.2: The unit triangle $P = \operatorname{conv}\{(1,0),(0,1),(0,0)\}$ and some of its dilates.

Example 1.3.4. Consider the unit triangle $P = \operatorname{conv}\{(1,0),(0,1),(0,0)\}$ depicted in Figure 1.2. Observing that the t-th dilate of P adds $t+1$ additional lattice points not already contained in the $(t-1)$-st dilate, we can conclude that the lattice point enumerator is the sum of the first $t+1$ positive integers. This is counted by the binomial coefficient $\binom{t+2}{2}$, so using techniques from enumerative combinatorics, we have that

$$\mathrm{Ehr}_P(z) = \sum_{t\geq 0} \binom{t+2}{2} z^t = \frac{1}{(1-z)^3}.$$

Therefore, in this case, $h^*(P;z) = 1$.

Much work has been done to identify combinatorial interpretations of the coefficients of h^*-polynomials in the context of a given polytope. For example, it is known that the sum of the coefficients of the h^*-polynomial of a polytope P yields the normalized volume of P. Moreover, h_1^* is always given by $\left|P \cap \mathbb{Z}^d\right| - d - 1$, and h_d^* is given by the number of strictly interior lattice points contained in P. Recently, there has been a significant focus on determining when $h^*(P)$ is unimodal [8]. We say $h^*(P)$ is *unimodal* when there exists an index k such that $h_i^* \leq h_{i+1}^*$ for all $0 \leq i < k$ and $h_i^* \geq h_{i+1}^*$ for all $k \leq i \leq d$. Unimodality results for h^*-vectors are of interest because their proofs frequently suggest some underlying structure or other interesting properties of the polytope that are not readily discernible. However, verifying unimodality of h^*-vectors is a very challenging task in general, and identifying sufficient conditions for h^*-unimodality remains a mystery. This is even the case for highly structured classes of polytopes.

Two important properties associated with lattice polytopes (and unimodality of their h^*-vectors) are the integer decomposition property and reflexivity.

Definition 1.3.5. We say a lattice polytope P has the *integer decomposition property*, or *is IDP*, if for every $m \in \mathbb{Z}_{\geq 1}$ and $\mathbf{p} \in mP \cap \mathbb{Z}^d$, there exist $\mathbf{p}_1, \ldots, \mathbf{p}_m \in P \cap \mathbb{Z}^d$ such that $\mathbf{p} = \mathbf{p}_1 + \cdots + \mathbf{p}_m$.

If we consider $\operatorname{cone}(P)$, we can reformulate the previous definition as follows: P is IDP if each integral point at height t in $\operatorname{cone}(P)$ can be expressed as a sum of t height 1 integral points.

Table 1.1: The number of reflexive polytopes (up to unimodular equivalence) in low dimensions.

Reflexive Polytopes	
Dimension	# of equivalence classes
1	1
2	16
3	4,319
4	473,800,776
5	?

Definition 1.3.6. Letting K° denote the topological interior of a space K, P is *reflexive* if, possibly after translation by an integer vector, the origin is contained in P° and the (polar) dual of P is also a lattice polytope. The *polar dual* of P is given by

$$P^* = \{\mathbf{x} \in \mathbb{R}^d \,|\, \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in P\}.$$

In particular, we have that $(P^*)^* = P$.

There are many interesting questions about polytopes that are IDP and/or reflexive, and they have been the subject of a copious amount of research [7, 9, 10, 11, 12, 19, 29]. Lagarias and Ziegler [24] demonstrated that there are only finitely many reflexive polytopes (up to unimodular equivalence) in each dimension. Indeed, Table 1.1 provides the results of Kreuzer and Skarke who used the assistance of computational software to classify all refelxive polytopes up to dimension four [23]. The number of equivalence classes of reflexive polytopes in dimension five and higher remains a very difficult open problem. Furthermore, reflexive polytopes have a number of applications in toric geometry, combinatorial mirror symmetry, and the theory of error-correcting codes and random walks. In fact, reflexive polytopes were coined by Victor Batyrev in [3] when he discovered applications to mirror symmetry in physical string theory. Since reflexive polytopes, via Definition 1.3.6, come in dual pairs, reflexive polytopes describe mirror families of Calabi-Yau manifolds and can consequently be used to compute invariants of the associated Calabi-Yau varieties. The following result demonstrates one reason why reflexive polytopes are of interest in the context of Ehrhart theory.

Theorem 1.3.7 (Hibi, [21])**.** *A d-dimensional lattice polytope $P \subset \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(P)$ satisfies $h^*_i = h^*_{d-i}$.*

This result gives us a new technique for identifying reflexive polytopes. Hibi [20] originally conjectured that every reflexive polytope has a unimodal h^*-vector, but Mustaţă and Payne [26, 29] discovered counterexamples in dimensions 6 and higher. While these counterexamples arise as reflexive simplices, none of these simplices are IDP, lending itself to the following long-standing conjecture frequently attributed to Hibi and Ohsugi.

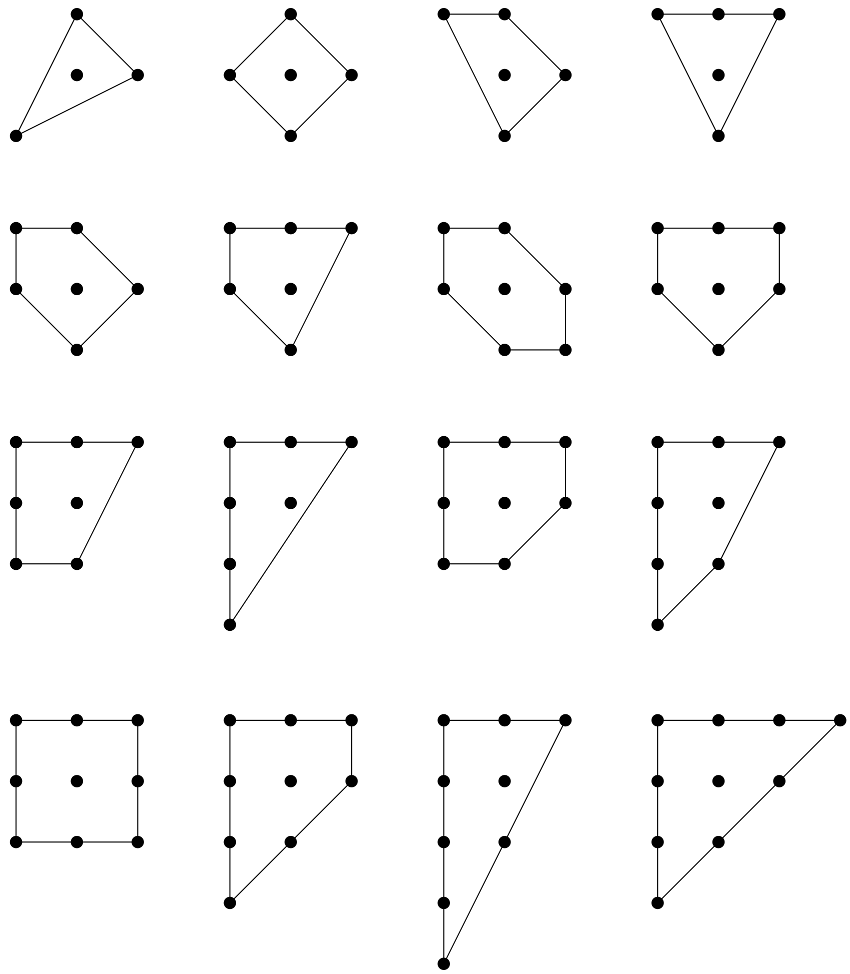

Figure 1.3: All 16 reflexive polytopes (up to unimodular equivalence) in 2 dimensions.

Conjecture 1.3.8 (Hibi and Ohsugi, [28])**.** *If P is a lattice polytope that is reflexive and IDP, then P has a unimodal Ehrhart h^*-vector.*

The interplay between reflexivity, IDP, and h^*-unimodality is not well-understood [8], due in part to an inability to test the IDP condition for general polytopes living in high dimensions. Moreover, it is a very difficult open problem to determine the number of reflexive polytopes of a fixed dimension (up to unimodular equivalence). Given these challenges, much of the work done to explore these properties in the context of Conjecture 1.3.8 has focused on certain classes of lattice polytopes. In particular, restricting to families of lattice simplices has been shown to form a rich source of examples in recent investigations [9, 10, 11] as one can leverage their geometric and algebraic properties. Indeed, there is an algorithmic classification of reflexive simplices [15], making this family a natural starting point for testing Conjecture 1.3.8. One such class of reflexive simplices I have studied with Braun, Davis, Lane, and Solus [10, 12] that also satisfy IDP (under certain conditions) is that of *weighted projective space simplices*, defined and discussed in detail in the next chapter.

Another related result to this line of investigation regarding h^*-unimodality deals with *regular unimodular triangulations*. Given that some of my work with weighted projective space simplices verifies and exploits this property, we elaborate on such triangulations in the following section.

1.4 Polytopal Triangulations

Stanley conjectured and Athanasiadis [1] verified that the h^*-vector of the Birkhoff polytope, i.e., the polytope of doubly-stochastic matrices, is unimodal. The Birkhoff polytope is both *Gorenstein* (i.e., has a symmetric h^*-vector) and IDP, making it a source of interesting problems regarding unimodality [17]. A key property of the Birkhoff polytope that Athanasiadis's argument relies on is that it admits a *regular unimodular triangulation*.

Definition 1.4.1. Let $P \subset \mathbb{R}^n$ be a lattice polytope, and let $\mathbf{A}_P$ be the point configuration consisting of the lattice points of P. A *triangulation* of $\mathbf{A}_P$ is a collection $\mathcal{T}_P$ of d-simplices all of whose vertices are points in $\mathbf{A}_P$ that satisfies the following two properties:

(i) The union of all these simplices equals the convex hull of $\mathbf{A}_P$.

(ii) Any pair of these simplices intersects in a (possibly empty) common face.

Moreover, a triangulation is *unimodular* if every simplex has normalized volume one. A triangulation is *regular* if it can be obtained by projecting the lower envelope of a lifting of $\mathbf{A}_P$ from $\mathbb{R}^{d+1}$.

Bruns and Römer, [14] verified the following well-known result regarding the relationship between h^*-unimodality and regular unimodular triangulations of reflexive polytopes.

Theorem 1.4.2 (Bruns and Römer [14])**.** *If P is reflexive and admits a regular unimodular triangulation, then P has a unimodal Ehrhart h^*-vector.*

Additionally, it is well known that if a lattice polytope admits a regular unimodular triangulation, then the polytope is IDP. Therefore, it is of interest to determine whether or not lattice polytopes that are both IDP and reflexive have a regular unimodular triangulation, motivating the investigation described in Chapter 3.

Chapter 2 Weighted Projective Space Simplices

2.1 Defining $\Delta_{(1,\mathbf{q})}$.

As mentioned in Chapter 1, lattice simplices provide a nice testing ground for vetting Conjecture 1.3.8. In this section, we formally define *weighted projective space simplices*, the chief objects of interest in my work in [10, 12]. First, for context, we recall the definition of a weighted projective space from algebraic geometry.

Definition 2.1.1. Given positive integers $\lambda_1, \ldots, \lambda_n$ such that $\gcd\{\lambda_1, \ldots, \lambda_n\} = 1$, we define the polynomial algebra $S(\lambda_1, \ldots, \lambda_n) := \mathbb{C}[x_1, \ldots, x_n]$ graded by $\deg x_i := \lambda_i$. A *weighted projective space* with weights $\lambda_1, \ldots, \lambda_n$ is the projective variety $\mathbb{P}(\lambda_1, \ldots, \lambda_n) := \mathrm{Proj}(S(\lambda_1, \ldots, \lambda_n))$.

Now, consider an integer partition $\mathbf{q} \in \mathbb{Z}^d_{\geq 1}$ where $q_1 \leq \cdots \leq q_d$.

Definition 2.1.2. The lattice simplex associated with $\mathbf{q}$ is

$$\Delta_{(1,\mathbf{q})} := \mathrm{conv}\left\{\mathbf{e}_1, \ldots, \mathbf{e}_d, -\sum_{i=1}^{d} q_i \mathbf{e}_i\right\} \subset \mathbb{R}^d,$$

where $\mathbf{e}_i$ denotes the i-th standard basis vector in $\mathbb{R}^d$. Such lattice simplices are referred to as *weighted projective space simplices*.

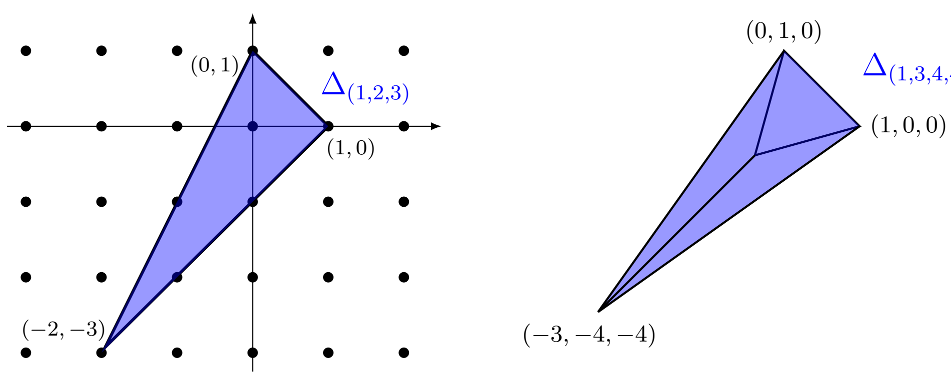

Figure 2.1: Two weighted projective space simplices, $\Delta_{(1,2,3)}$ (left) and $\Delta_{(1,3,4,4)}$ (right).

Set $N(\mathbf{q}) := 1 + \sum_i q_i$. One can show, as for instance in [27, Proposition 4.4], that $N(\mathbf{q})$ is the normalized volume of $\Delta_{(1,\mathbf{q})}$. Let $\mathcal{Q}$ denote the set of all lattice simplices of the form $\Delta_{(1,\mathbf{q})}$. The family of simplices $\mathcal{Q}$ are the focus of active study [2, 9, 11, 13, 25, 31], particularly in regard to Conjecture 1.3.8. The simplices in $\mathcal{Q}$ correspond to a subset of the simplices defining weighted projective spaces [15]. Specifically, the vector $(1, \mathbf{q})$ gives the weights of the projective coordinates of the associated weighted projective space. Given a vector of distinct positive integers $\mathbf{r} = (r_1, \ldots, r_t)$, we write

$$(r_1^{x_1}, r_2^{x_2}, \ldots, r_t^{x_t}) := (\underbrace{r_1, r_1, \ldots, r_1}_{x_1 \text{ times}}, \underbrace{r_2, r_2, \ldots, r_2}_{x_2 \text{ times}}, \ldots, \underbrace{r_t, r_t, \ldots, r_t}_{x_t \text{ times}}) .$$

There is a natural stratification of $\mathcal{Q}$ based on the distinct entries in the vector $\mathbf{q}$, leading to the following definition.

Definition 2.1.3. If $\mathbf{q} = (q_1, \ldots, q_d) = (r_1^{x_1}, r_2^{x_2}, \ldots, r_t^{x_t})$, we say that both $\mathbf{q}$ and $\Delta_{(1,\mathbf{q})}$ are *supported* by the vector $\mathbf{r} = (r_1, \ldots, r_t)$ with *multiplicity* $\mathbf{x} = (x_1, \ldots, x_t)$. We write $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ in this case, and say that $\mathbf{q}$ is t*-supported.*

Figure 2.1 depicts two examples of weighted projective space simplices. Observe that both $\Delta_{(1,2,3)}$ and $\Delta_{(1,3,4,4)}$ are 2-supported.

2.2 Important Results

A well-known result providing a number-theoretic basis for studying reflexive simplices in $\mathcal{Q}$ is given in the following theorem.

Theorem 2.2.1 (Conrads [15]). *The simplex $\Delta_{(1,\mathbf{q})} \in \mathcal{Q}$ is reflexive if and only if*

$$q_i \textit{ divides } 1 + \sum_{j=1}^{n} q_j \quad \textit{for all } 1 \leq i \leq n \ .$$

Equivalently, if $\mathbf{q} = (\mathbf{r}, \mathbf{x})$, then $\Delta_{(1,\mathbf{q})}$ is reflexive if and only if $\operatorname{lcm}(r_1, \ldots, r_d)$ *divides* $1 + \sum_{i=1}^{d} x_i r_i$.

Example 2.2.2. Let $\mathbf{q}_1 = (2, 3)$ and $\mathbf{q}_2 = (3, 4, 4)$. Note that $\Delta_{(1,\mathbf{q}_1)}$ and $\Delta_{(1,\mathbf{q}_2)}$ are precisely the weighted projective space simplices depicted in Figure 2.1. Since each weight divides the total sum of weights for both simplices, Theorem 2.2.1 affords that $\Delta_{(1,\mathbf{q}_1)}$ and $\Delta_{(1,\mathbf{q}_2)}$ are both reflexive polytopes.

In [11], Braun, Davis, and Solus gave a characterization of the reflexive IDP $\Delta_{(1,\mathbf{q})}$, and a formula for their h^*-polynomials (given in Theorem 2.2.3 below), both based in elementary number theory. By relaxing this characterization to a necessary condition for IDP, Braun, Davis, Lane, Solus, and I [10] were able to derive a natural parameterization of an infinite class of reflexive simplices associated to weighted projective space (see Chapter 4). This machinery yields an efficient method for checking the IDP and unimodality conditions for a relatively broad family of reflexive simplices having large dimension and volume, thereby addressing some of the aforementioned challenges associated with detecting IDP in the previous chapter.

Theorem 2.2.3 (Braun, Davis, Solus [11]). *The h^*-polynomial of $\Delta_{(1,\mathbf{q})}$ is*

$$h^*(\Delta_{(1,\mathbf{q})}; z) = \sum_{b=0}^{q_1+\cdots+q_n} z^{w(\mathbf{q},b)},$$

where

$$w(\mathbf{q}, b) := b - \sum_{i=1}^{n} \left\lfloor \frac{q_i b}{1 + q_1 + \cdots + q_n} \right\rfloor .$$

Example 2.2.4. Let $\mathbf{q} = (3, 4, 4)$. Then, $w(\mathbf{q}, b) = b - \left\lfloor \frac{b}{4} \right\rfloor - 2 \left\lfloor \frac{b}{3} \right\rfloor$, and thus by Theorem 2.2.3,

$$h^*(\Delta_{(1,\mathbf{q})}; z) = 1 + 5z + 5z^2 + z^3.$$

Observe that the sum of the coefficients of $h^*(\Delta_{(1,\mathbf{q})}; z)$ (i.e., the normalized volume of $\Delta_{(1,3,4,4)}$) is 1+5+5+1 = 12 which precisely corresponds with $N(3, 4, 4) = 1+3+4+4$, i.e., the sum of the projective weights.

Chapter 3 Triangulations of 2-Supported $\Delta_{(1,\mathbf{q})}$

In this chapter, we study triangulations of IDP reflexive $\Delta_{(1,\mathbf{q})}$ that are 2-supported. This is based on joint work with Benjamin Braun. The results have been accepted for publication and will appear in *Annals of Combinatorics*. A preprint of this work can be found here [12].

3.1 Overview of Main Results

The following theorem due to Braun, Davis, and Solus [11] provides a complete classification of 2-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$.

Theorem 3.1.1 (Braun, Davis, Solus, [11]). *Let $r_1 < r_2$ be positive integers, and let $\mathbf{q} = (r_1^{x_1}, r_2^{x_2})$ for $x_1, x_2 \in \mathbb{Z}_{\geq 1}$. Then, $\Delta_{(1,\mathbf{q})}$ is IDP and reflexive if and only if either*

$$\begin{array}{ll} (1) & r_1 > 1 \textit{ with } r_2 = 1 + r_1 x_1 \textit{ and } x_2 = r_1 - 1, \textit{ or} \\ (2) & r_1 = 1 \textit{ with } r_2 = 1 + x_1 \textit{ and } x_2 \textit{ arbitrary.} \end{array}$$

As motivated by the discussion at the end of Section 1.4 , we seek to establish the existence of a regular unimodular triangulation of 2-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$.

To begin, it has been shown [11] that each 2-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$ with $\mathbf{q} = (1^{x_1}, (1+x_1)^{x_2})$ arises as an affine free sum (defined in Chapter 4) of $\Delta_{(1,1^{x_1})}$ and $\Delta_{(1,1^{x_2})}$. Thus, every $\Delta_{(1,\mathbf{q})}$ of this form admits a regular unimodular triangulation, for example the triangulation arising as the join of the boundary of $\Delta_{(1,1^{x_1})} \times (0^{x_2})$ with the unique unimodular triangulation of $(0^{x_1}) \times \left(\Delta_{(1,1^{x_2})} - \mathbf{e}_{x_1+1}\right)$ in $\mathbb{R}^{x_1+x_2}$. (Note that this latter simplex has two triangulations, one being the entire simplex and the other being the cone of the interior point with the boundary complex, and only one of these is unimodular.)

Given this fact, we need only consider the other 2-supported case. Thus, for the remainder of this chapter, we assume that $\mathbf{q} = (r_1^{x_1}, (1 + r_1 x_1)^{r_1 - 1})$ with $r_1 > 1$. Observe that $\dim \Delta_{(1,\mathbf{q})} = d = x_1 + x_2 = r_1 + x_1 - 1$. Define $\mathcal{A}'(\mathbf{q}) := \{\mathbf{a}'_1, \ldots, \mathbf{a}'_{r_1+3}, \mathbf{b}'_1, \ldots, \mathbf{b}'_d\} \subset \mathbb{Z}^d$, where:

$$\begin{aligned} &\mathbf{a}'_{r_1+1} = ((-1)^{x_1}, (-x_1)^{r_1-1}) \\ &\mathbf{a}'_{r_1+2} = (0^{x_1}, (-1)^{r_1-1}) \\ &\mathbf{a}'_{r_1+3} = (0^{x_1}, 0^{r_1-1}) \\ &\mathbf{a}'_i = (r_1 - i + 1)\mathbf{a}'_{r_1+1} + \mathbf{a}'_{r_1+2} \text{ for } 1 \leq i \leq r_1 \\ &\mathbf{b}'_j = \mathbf{e}_{d-j+1} \text{ for } 1 \leq j \leq d \end{aligned}$$

Observe that $\mathbf{a}'_1 = -\mathbf{q}$, so all vertices of $\Delta_{(1,\mathbf{q})}$ are contained in $\mathcal{A}'(\mathbf{q})$. Note that later we will use the notation $\mathcal{A}(\mathbf{q})$ to denote the set of these vectors where each vector has a 1 appended. Thus, we use $\mathcal{A}'(\mathbf{q})$ for the vectors defined above.

$$
\begin{array}{cccccccccccccccccc}
\mathbf{a}'_1 & \mathbf{a}'_2 & \mathbf{a}'_3 & \mathbf{a}'_4 & \mathbf{a}'_5 & \mathbf{a}'_6 & \mathbf{a}'_7 & \mathbf{a}'_8 & \mathbf{a}'_9 & \mathbf{b}'_1 & \mathbf{b}'_2 & \mathbf{b}'_3 & \mathbf{b}'_4 & \mathbf{b}'_5 & \mathbf{b}'_6 & \mathbf{b}'_7 & \mathbf{b}'_8 & \mathbf{b}'_9 \\
\end{array}
$$
$$
\begin{pmatrix}
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-6 & -5 & -4 & -3 & -2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-25 & -21 & -17 & -13 & -9 & -5 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Figure 3.1: $\mathcal{A}'(\mathbf{q})$ for $\mathbf{q} = (6^4, 25^5)$.

Example 3.1.2. Let $r_1 = 6$ and $x_1 = 4$, so $\mathbf{q} = (6^4, 25^5) \in \mathbb{Z}^9$. The elements of $\mathcal{A}'(\mathbf{q})$ are given by the columns of the matrix in Figure 3.1.

In this chapter, we prove the following theorems.

Theorem 3.1.3. *For* $\mathbf{q} = (r_1^{x_1}, (1 + r_1x_1)^{r_1-1})$ *with* $r_1 > 1$*, the lattice points of the IDP simplex* $\Delta_{(1,\mathbf{q})}$ *are given by* $\mathcal{A}'(\mathbf{q})$*.*

Theorem 3.1.4. *For* $\mathbf{q} = (r_1^{x_1}, (1+r_1x_1)^{r_1-1})$ *with* $r_1 > 1$*, there exists a lexicographic squarefree initial ideal of the toric ideal associated with* $\Delta_{(1,\mathbf{q})}$*.*

Corollary 3.1.5. *Every 2-supported IDP reflexive simplex* $\Delta_{(1,\mathbf{q})}$ *admits a regular unimodular triangulation. When* $\mathbf{q} = (r_1^{x_1}, (1 + r_1x_1)^{r_1-1})$ *with* $r_1 > 1$*, this triangulation is induced by a lexicographic term order* $<_{lex}$*.*

The remainder of this chapter is structured as follows. In Section 3.2 we prove Theorem 3.1.3. In Section 3.3 we introduce needed algebraic machinery and prove Theorem 3.1.4. In Section 3.4 we describe the facets of the resulting triangulation and discuss connections to the Ehrhart h^*-vector of $\Delta_{(1,\mathbf{q})}$.

3.2 Proof of Theorem 3.1.3

Our strategy is to determine the number of lattice points in $\Delta_{(1,\mathbf{q})}$, show that this value equals the number of columns of $\mathcal{A}'(\mathbf{q})$, and then show that all of the columns of $\mathcal{A}'(\mathbf{q})$ are contained in $\Delta_{(1,\mathbf{q})}$.

Proposition 3.2.1. *For* $\mathbf{q}$ *as given in Theorem 3.1.3, we have* $\left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right| = r_1 + d + 3$*.*

Proof. Using Theorem 2.2.3, we know the *Ehrhart h^*-polynomial* of $\Delta_{(1,\mathbf{q})}$, denoted $h^*(\Delta_{(1,\mathbf{q})}; z) := h_0^* + h_1^* z + \cdots + h_d^* z^d$, is given by

$$h^*(\Delta_{(1,\mathbf{q})}; z) = \sum_{b=0}^{r_1(x_1 r_1+1)-1} z^{w(\mathbf{q},b)},$$

where

$$w(\mathbf{q}, b) := b - x_1 \left\lfloor \frac{b}{1 + x_1 r_1} \right\rfloor - (r_1 - 1) \left\lfloor \frac{b}{r_1} \right\rfloor .$$

It is well known (see, e.g., [5]) that the coefficient h_1^* is given by the formula

$$h_1^* = \left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right| - (\dim \Delta_{(1,\mathbf{q})} + 1) . \tag{3.1}$$

To compute h_1^*, we must determine all b for which $w(\mathbf{q}, b) = 1$. Since $0 \le b \le r_1(x_1 r_1 + 1) - 1$, the division algorithm allows us to write $b = \alpha(1 + x_1 r_1) + \beta$, where $0 \le \alpha < r_1$ and $0 \le \beta < 1 + x_1 r_1$. Hence,

$$\begin{aligned} w(\mathbf{q}, b) &= w(\mathbf{q}, \alpha(1 + x_1 r_1) + \beta) \\ &= \alpha(1 + x_1 r_1) + \beta - x_1 \left\lfloor \frac{\alpha(1 + x_1 r_1) + \beta}{1 + x_1 r_1} \right\rfloor - (r_1 - 1) \left\lfloor \frac{\alpha(1 + x_1 r_1) + \beta}{r_1} \right\rfloor \\ &= \alpha(1 + x_1 r_1) + \beta - \alpha x_1 - (r_1 - 1) \left(\alpha x_1 + \left\lfloor \frac{\alpha + \beta}{r_1} \right\rfloor \right) \\ &= \alpha + \beta - (r_1 - 1) \left\lfloor \frac{\alpha + \beta}{r_1} \right\rfloor . \end{aligned}$$

Therefore, the equation $w(\mathbf{q}, b) = 1$ becomes

$$\alpha + \beta - (r_1 - 1) \left\lfloor \frac{\alpha + \beta}{r_1} \right\rfloor = 1 \quad \Longleftrightarrow \quad \alpha + \beta = 1 + (r_1 - 1) \left\lfloor \frac{\alpha + \beta}{r_1} \right\rfloor .$$

Now, let $\ell = \left\lfloor \frac{\alpha+\beta}{r_1} \right\rfloor$. By the previous equation, $\alpha + \beta = 1 + (r_1 - 1)\ell$. Substituting this equivalent expression for $\alpha + \beta$ into both sides of the previous equation, it follows that solving $w(\mathbf{q}, b) = 1$ is equivalent to finding all pairs (α, β) such that

$$\begin{aligned} 1 + (r_1 - 1)\ell &= 1 + (r_1 - 1) \left\lfloor \frac{1 + (r_1 - 1)\ell}{r_1} \right\rfloor \\ &= 1 + (r_1 - 1) \left(\ell + \left\lfloor \frac{1 - \ell}{r_1} \right\rfloor \right) . \end{aligned}$$

Rearranging this equation yields

$$(r_1 - 1) \left\lfloor \frac{1 - \ell}{r_1} \right\rfloor = 0.$$

Therefore, since $r_1 > 1$, this implies

$$\left\lfloor \frac{1-\ell}{r_1} \right\rfloor = 0 \quad \Longrightarrow \quad \ell = \begin{cases} 0 \\ 1 \end{cases} \quad \Longrightarrow \quad \alpha + \beta = \begin{cases} 1 \\ r_1 \end{cases} .$$

If $\alpha + \beta = 1$, then $(\alpha, \beta) = (1, 0)$ or $(\alpha, \beta) = (0, 1)$. Otherwise, in the case that $\alpha + \beta = r_1$, there are r_1 possible pairs (α, β) where $\alpha \in \{0, \ldots, r_1 - 1\}$ and $\beta = r_1 - \alpha$. Thus,

$$h_1^* = |\{b : w(\mathbf{q}, b) = 1\}| = r_1 + 2.$$

Consequently, (3.1) implies

$$\left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right| = r_1 + d + 3,$$

as desired. □

Proposition 3.2.2. *For* $\mathbf{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$*, define*

$$\lambda_k(\mathbf{t}) := \begin{cases} \displaystyle\sum_{\substack{j=1 \\ j \neq k}}^{d} t_j - x_1 r_1 t_k, & \text{if } 1 \leq k \leq x_1 \\ \displaystyle\sum_{\substack{j=1 \\ j \neq k}}^{d} t_j - (r_1 - 1) t_k, & \text{if } x_1 + 1 \leq k \leq d \\ \displaystyle\sum_{j=1}^{d} t_j, & \text{if } k = d + 1 \, . \end{cases}$$

An irredundant $\mathcal{H}$*-description of* $\Delta_{(1,\mathbf{q})}$ *is given by* $\lambda_k(\mathbf{t}) \leq 1$ *for all* $1 \leq k \leq d + 1$*.*

Proof. Observe that for all $1 \leq j \leq d$, $\mathbf{e}_j$ satisfies all of the given inequalities tightly except when $k = j$ (i.e., $\lambda_k(\mathbf{e}_j) = 1$ for all $k \neq j$ and $\lambda_j(\mathbf{e}_j) < 1$). Moreover, $-\mathbf{q}$ satisfies the first d inequalities tightly (i.e., $\lambda_k(-\mathbf{q}) = 1$ for all $1 \leq k \leq d$), but not $\sum_j t_j \leq 1$. Thus, as each vertex of the simplex $\Delta_{(1,\mathbf{q})}$ satisfies exactly d of the given inequalities with equality, the inequalities necessarily constitute an $\mathcal{H}$-description of $\Delta_{(1,\mathbf{q})}$. □

Proof of Theorem 3.1.3. To begin, observe that $|\mathcal{A}'(\mathbf{q})| = r_1 + d + 3 = \left|\Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d\right|$. Therefore, as each element in $\mathcal{A}'(\mathbf{q})$ is an integer vector, it suffices to show that each point satisfies the inequalities in Proposition 3.2.2. To this end, let λ_k be defined as in Proposition 3.2.2; we evaluate each vector in $\mathcal{A}'(\mathbf{q})$ on λ_k. For each $1 \leq i \leq r_1$, note that

$$\mathbf{a}'_i = (r_1 - i + 1)\mathbf{a}'_{r_1+1} + \mathbf{a}'_{r_1+2} = \left((-(r_1 - i + 1))^{x_1}, ((-(1 + (r_1 - i + 1)x_1))^{r_1 - 1}\right).$$

Therefore, we have that

$$\lambda_k(\mathbf{a}'_1) = 1 \text{ if } 1 \leq k \leq d \quad \text{and} \quad \lambda_{d+1}(\mathbf{a}'_1) < 1,$$

and for each $i \in \{2, \dots, r_1\} \cup \{r_1 + 2\}$,

$$\lambda_k(\mathbf{a}'_i) = 1 \text{ if } x_1 + 1 \le k \le d \quad \text{and} \quad \lambda_k(\mathbf{a}'_i) < 1 \text{ otherwise.}$$

Also,

$$\lambda_k(\mathbf{a}'_{r_1+1}) = 1 \text{ if } 1 \le k \le x_1 \quad \text{and} \quad \lambda_k(\mathbf{a}'_{r_1+1}) < 1 \text{ otherwise,}$$

and

$$\lambda_k(\mathbf{a}'_{r_1+3}) < 1 \text{ for all } 1 \le k \le d + 1.$$

Lastly, for all $1 \le j \le d$,

$$\lambda_k(\mathbf{b}'_j) = 1 \text{ if } k \ne d - j + 1 \quad \text{and} \quad \lambda_k(\mathbf{b}'_j) < 1 \text{ if } k = d - j + 1.$$

Thus, $\mathcal{A}'(\mathbf{q}) \subseteq \Delta_{(1,\mathbf{q})} \cap \mathbb{Z}^d$, and the result follows. $\square$

3.3 Proof of Theorem 3.1.4

We next seek to prove the existence of a regular unimodular triangulation of the convex hull of these points. Given a field K, there are natural parallels between properties of lattice polytopes and algebraic objects such as semigroup algebras, toric varieties, and monomial ideals. The following one-to-one correspondence between lattice points and Laurent monomials plays a central role:

$$\mathbf{a}' = (a_1, \dots, a_d) \in \mathbb{Z}^d \quad \longleftrightarrow \quad \mathbf{t}^{\mathbf{a}'} := t_1^{a_1} \cdots t_d^{a_d} \in K[t_1^{\pm 1}, \dots, t_d^{\pm 1}].$$

For details regarding the significance of this correspondence, see [34, Chapter 8]. Furthermore, for all notation related to combinatorial commutative algebra, we refer the reader to [16].

Let K be a field, and define $\mathcal{A}(\mathbf{q}) = (\mathbf{a}_1, \dots, \mathbf{a}_{r_1+3}, \mathbf{b}_1, \dots, \mathbf{b}_d) \subset \mathbb{Z}^{(d+1)\times(r_1+d+3)}$ to be the homogenization of $\mathcal{A}'(\mathbf{q})$ where $\mathbf{a}_i = (\mathbf{a}'_i, 1)$ and $\mathbf{b}_j = (\mathbf{b}'_j, 1)$; that is, $\mathcal{A}(\mathbf{q})$ is the matrix associated with the point configuration consisting of all vectors in $\mathcal{A}'(\mathbf{q})$ lifted to height 1. (Note that we can view the columns of $\mathcal{A}(\mathbf{q})$ as the intersection of $\mathbb{Z}^{d+1}$ with the degree 1 slice of the polyhedral cone over $\Delta_{(1,\mathbf{q})}$.) Let $K[\mathcal{A}(\mathbf{q})] := K[z_1, \dots, z_{r_1+3}, y_1, \dots, y_d]$ be the polynomial ring associated with the columns of $\mathcal{A}(\mathbf{q})$ in $r_1 + d + 3$ variables over K. Moreover, let $\mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ denote the set of monomials contained in $K[\mathcal{A}(\mathbf{q})]$, and let $\mathcal{R}_K[\mathcal{A}(\mathbf{q})]$ be the K-subalgebra of the Laurent polynomial ring $K[\mathbf{t}^{\pm 1}] := K[t_1^{\pm 1}, \dots, t_{d+1}^{\pm 1}]$ generated by all monomials $\mathbf{t}^{\mathbf{a}}$ with $\mathbf{a} \in \mathcal{A}(\mathbf{q})$, where $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_{d+1}^{a_{d+1}}$ if $\mathbf{a} = (a_1, \dots, a_{d+1})$. The toric ideal $I_{\mathcal{A}(\mathbf{q})}$ is the kernel of the surjective ring homomorphism $\pi : K[\mathcal{A}(\mathbf{q})] \to \mathcal{R}_K[\mathcal{A}(\mathbf{q})]$ defined by

$$\pi(z_i) = \mathbf{t}^{\mathbf{a}_i}, \text{ for } 1 \le i \le r_1 + 3$$
$$\pi(y_j) = \mathbf{t}^{\mathbf{b}_j}, \text{ for } 1 \le j \le d.$$

A generating set for $I_{\mathcal{A}(\mathbf{q})}$ is given by the set of all homogeneous binomials $f-g$ with $\pi(f)=\pi(g)$ and $f,g \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$, see [34, Lemma 4.1]. We fix the lexicographic term order $<_{lex}$ on $K[\mathcal{A}(\mathbf{q})]$ induced by the ordering of the variables

$$z_1 > z_2 > \cdots > z_{r_1+3} > y_1 > y_2 > \cdots > y_d.$$

Moreover, for $f = z_1^{\gamma_1}\cdots z_{r_1+3}^{\gamma_{r_1+3}} y_1^{\delta_1}\cdots y_d^{\delta_d} \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$, we introduce the notation

$$\operatorname{supp}_{\mathbf{z}}(f) := \{i \in \{1,\dots,r_1+3\} : \gamma_i > 0\}.$$

Given this setup, we restate Theorem 3.1.4. Note that Corollary 3.1.5 follows immediately from Theorem 3.3.1 as it indicates the existence of a squarefree initial ideal of the toric ideal $I_{\mathcal{A}(\mathbf{q})}$ [34, Corollary 8.9].

Theorem 3.3.1 (Restatement of Theorem 3.1.4)**.** *Let B be the set of all $(i,j) \in \mathbb{N}^2$ satisfying the following conditions:*

(i) $j - i \geq 2$

(ii) $1 \leq i \leq r_1$

(iii) $j \leq r_1 + 3$

(iv) $j \neq r_1 + 1$

(v) $(i,j) \neq (r_1, r_1+2)$

Given $(i,j) \in B$, define (k,ℓ) as follows:

$$\begin{aligned}
&k = \left\lfloor \frac{i+j}{2} \right\rfloor, \ell = \left\lceil \frac{i+j}{2} \right\rceil && \text{if } j < r_1+1\\
&k = \left\lfloor \frac{i+j-1}{2} \right\rfloor, \ell = \left\lceil \frac{i+j-1}{2} \right\rceil && \text{if } j = r_1+2\\
&k = i+1,\ \ell = r_1+1 && \text{if } j = r_1+3, i \neq r_1\\
&k = r_1+1,\ \ell = r_1+2 && \text{if } j = r_1+3, i = r_1.
\end{aligned}$$

If $x_1 \geq r_1 - 2$, then the set of binomials $\mathcal{G}$ given by

$$z_i z_j - z_k z_\ell, \qquad (i,j) \in B \tag{3.2}$$

$$z_{k+1}\prod_{\ell=1}^{r_1-1} y_\ell - z_{r_1+1}^{r_1-k} z_{r_1+3}^{k}, \quad 0 \leq k \leq r_1 - 1 \tag{3.3}$$

$$z_{r_1-k}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1}^{k} z_{r_1+2}^{x_1+1-k}, \quad 0 \leq k \leq r_1 - 1 \tag{3.4}$$

$$z_{r_1+2}\prod_{\ell=1}^{r_1-1} y_\ell - z_{r_1+3}^{r_1}, \tag{3.5}$$

$$z_{r_1+1}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1+2}^{x_1} z_{r_1+3} \tag{3.6}$$

is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to the lexicographic term order $<_{lex}$ as specified above. In the case that $x_1 < r_1 - 2$, replace (3.4) *above with*

$$\begin{cases} z_{r_1-k}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1}^{k} z_{r_1+2}^{x_1+1-k}, & 0 \le k \le x_1+1 \\ z_{r_1-k}\prod_{\ell=r_1}^{d} y_\ell - z_{r_1-1}^{k-x_1-1} z_{r_1}^{2x_1+2-k}, & x_1+2 \le k \le r_1-1. \end{cases} \tag{3.4*}$$

Note that regardless of case (either $x_1 \geq r_1 - 2$ or $x_1 < r_1 - 2$), the initial terms of the k-th binomial in (3.4) and (3.4*) are identical. Therefore, whenever we are considering only leading terms of these polynomials, we can ignore any relationship between x_1 and $r_1 - 2$.

Remark 3.3.2. The intuition for most of these binomial relations is that they are encoding the additive structure on the columns of $\mathcal{A}(\mathbf{q})$. Specifically, in the definition of $\mathcal{A}'(\mathbf{q})$, we see that $\mathbf{a}'_i = (r_1 - i + 1)\mathbf{a}'_{r_1+1} + \mathbf{a}'_{r_1+2}$ for $1 \leq i \leq r_1$, and there are natural syzygies that result from this structure. We require the replacement of (3.4) with (3.4*) in the case that $x_1 < r_1 - 2$ since otherwise, the exponent of z_{r_1+2}, namely $x_1 + 1 - k$, would fail to be positive when $x_1 + 2 \leq k \leq r_1 - 1$.

To prove Theorem 3.3.1, we employ the following well-known lemma, e.g. [22, (0.1)], for proving a finite subset of the toric ideal $I_{\mathcal{A}(\mathbf{q})}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$. For a finite set of polynomials $\mathcal{G}$ in a polynomial ring with a term order $<$, let $in_<(\mathcal{G})$ denote the ideal generated by the set of initial terms of elements of $\mathcal{G}$.

Lemma 3.3.3 ([22])**.** *A finite set $\mathcal{G}$ of $I_{\mathcal{A}(\mathbf{q})}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to the term order $<$ if and only if $\{\pi(f) : f \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})]), f \notin in_<(\mathcal{G})\}$ is linearly independent over K; i.e., if and only if $\pi(f) \neq \pi(g)$ for all $f \notin in_<(\mathcal{G})$ and $g \notin in_<(\mathcal{G})$ with $f \neq g$.*

We will also require the following fact which provides an upper bound on the supported z-variables for any monomial outside the initial ideal generated by the binomials in Theorem 3.3.1 with respect to $<_{lex}$.

Lemma 3.3.4. *Let $\mathcal{G}$ be the set of binomials given in Theorem 3.3.1. Suppose*

$$f = z_1^{\gamma_1} \cdots z_{r_1+3}^{\gamma_{r_1+3}} y_1^{\delta_1} \cdots y_d^{\delta_d} \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$$

with $f \notin in_{<_{lex}}(\mathcal{G})$ and $|\mathrm{supp}_{\mathbf{z}}(f)| \geq 1$. Let m denote the minimal index such that z_m divides f. Then, $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we are restricted to the following possibilities:

(1) if $1 \leq m \leq r_1 - 1$, then $\gamma_{m+1}, \gamma_{r_1+1} \geq 0$ and $\gamma_i = 0$ for all $i \in \{1, \ldots, r_1+3\} \setminus \{m, m+1, r_1+1\}$.

(2) if $m = r_1$, then $\gamma_{r_1+1}, \gamma_{r_1+2} \geq 0$ and $\gamma_i = 0$ for all $i \in \{1, \ldots, r_1 - 1\} \cup \{r_1 + 3\}$.

(3) if $m \in \{r_1+1, r_1+2, r_1+3\}$, then $\gamma_i = 0$ for all $i < m$ and $\gamma_i \geq 0$ for all $i > m$.

Proof. Suppose $1 \leq m \leq r_1 - 1$. Since $z_m z_{m+1} \notin in_{<_{lex}}(\mathcal{G})$ and $z_m z_{r_1+1} \notin in_{<_{lex}}(\mathcal{G})$, z_{m+1} and z_{r_1+1} possibly divide f. However, given the structure of B as defined in Theorem 3.3.1, it follows that $z_m z_{r_1+2}, z_m z_{r_1+3}, z_m z_n \in in_{<_{lex}}(\mathcal{G})$ for all n with $n > m+1, n \neq r_1+1$. Therefore, since m is minimal, $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we precisely satisfy the conditions of Lemma 3.3.4(1).

Now, suppose $m = r_1$. By the minimality of m, we need only consider indices greater than r_1. Observe that $z_{r_1} z_{r_1+1} \notin in_{<_{lex}}(\mathcal{G})$, $z_{r_1} z_{r_1+2} \notin in_{<_{lex}}(\mathcal{G})$, and $z_{r_1} z_{r_1+3} \in in_{<_{lex}}(\mathcal{G})$. Thus, we have that $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$ and we end up in Lemma 3.3.4(2).

Finally, for $m \in \{r_1+1, r_1+2, r_1+3\}$, minimality of m immediately implies $|\mathrm{supp}_{\mathbf{z}}(f)| \leq 3$. To see that this case yields Lemma 3.3.4(3), observe that $z_m z_n \notin in_{<_{lex}}(\mathcal{G})$ for $m, n \in \{r_1+1, r_1+2, r_1+3\}$ with $m \neq n$. $\square$

Proof of Theorem 3.3.1. One easily checks that each binomial $h = m_1 - m_2 \in \mathcal{G}$ is contained in $I_{\mathcal{A}(\mathbf{q})}$ by showing $\pi(m_1) = \pi(m_2)$. To show $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$, we employ Lemma 3.3.3. Suppose $f, g \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ with $f \neq g$, $f \notin in_{<_{lex}}(\mathcal{G})$, and $g \notin in_{<_{lex}}(\mathcal{G})$. Write

$$f = z_1^{\alpha_1} \cdots z_{r_1+3}^{\alpha_{r_1+3}} y_1^{\beta_1} \cdots y_d^{\beta_d} \quad \text{and} \quad g = z_1^{\alpha_1'} \cdots z_{r_1+3}^{\alpha_{r_1+3}'} y_1^{\beta_1'} \cdots y_d^{\beta_d'},$$

where $\alpha_i, \alpha_i', \beta_j, \beta_j' \geq 0$. We may assume f and g are relatively prime (since otherwise, we could simply factor out the common variables and consider the images of the reduced monomials). Further assume to the contrary that $\pi(f) = \pi(g)$, and without loss of generality, assume $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$. For convenience, let $\mathbf{f}^\pi, \mathbf{g}^\pi \in \mathbb{Z}^{d+1}$ denote the exponent vectors associated with $\pi(f)$ and $\pi(g)$, respectively, and let $\mathbf{f}^\pi[k]$ (resp. $\mathbf{g}^\pi[k]$) denote the k-th entry of $\mathbf{f}^\pi$ (resp. $\mathbf{g}^\pi$). With this notation, observe that $\pi(f) = \pi(g)$ if and only if $\mathbf{f}^\pi[k] = \mathbf{g}^\pi[k]$ for all $1 \leq k \leq d+1$.

The general structure for the rest of the proof is to consider cases based on the size of the $\mathbf{z}$-support for monomials g and f. Throughout, we identify the minimal indices of the $\mathbf{z}$-variables dividing both g and f, and we repeatedly apply Lemma 3.3.4 to deduce a contradiction in each of the resulting cases.

<u>Case 1</u>: $|\mathrm{supp}_{\mathbf{z}}(g)| = 0$. By definition, it follows that $\alpha_i' = 0$ for all $1 \leq i \leq r_1+3$. Therefore, we know that

$$\mathbf{g}^\pi = \left(\beta_d', \ldots, \beta_1', \textstyle\sum_j \beta_j'\right).$$

<u>Subcase 1.1</u>: $|\mathrm{supp}_{\mathbf{z}}(f)| = 0$. Thus,

$$\mathbf{f}^\pi = \left(\beta_d, \ldots, \beta_1, \textstyle\sum_j \beta_j\right).$$

Since $\pi(f) = \pi(g)$, this implies $\beta_j = \beta_j'$ for all $1 \leq j \leq d$, and consequently, $f = g$, a contradiction.

<u>Subcase 1.2</u>: $|\mathrm{supp}_{\mathbf{z}}(f)| \geq 1$. Let m denote the minimal index such that z_m divides f (i.e., $\alpha_m > 0$ and $\alpha_i = 0$ for all $i < m$).

(a) Suppose $1 \leq m \leq r_1 + 1$. Since $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4) and (3.6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell \in \{r_1, \ldots, d\}$ such that $\beta_\ell = 0$. Hence,

$$\mathbf{f}^\pi[d-\ell+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-\ell+1,r_1+3+j}}_{=0} < 0.$$

However, $\mathbf{g}^\pi[d-\ell+1] = \beta'_\ell \geq 0$, a contradiction.

(b) Suppose $m = r_1 + 2$. Since $z_{r_1+2} y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.5)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $k \in \{1, \ldots, r_1 - 1\}$ such that $\beta_k = 0$. Since $k < r_1$, it follows that $d-k+1 > x_1$. Therefore, $\mathcal{A}(\mathbf{q})_{d-k+1,r_1+2} = -1$. Hence,

$$\mathbf{f}^\pi[d-k+1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k+1,i}}_{=-\alpha_{r_1+2}<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-k+1,r_1+3+j}}_{=0} < 0.$$

However, $\mathbf{g}^\pi[d-k+1] = \beta'_k \geq 0$, a contradiction.

(c) Suppose $m = r_1+3$. Since m is minimal, we know $\alpha_i = 0$ for all $1 \leq i \leq r_1+2$. Since $\mathcal{A}(\mathbf{q})$ is homogenized, we also know $\sum_i \alpha_i + \sum_j \beta_j = \sum_i \alpha'_i + \sum_j \beta'_j$ (this can be seen directly from $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$). Hence, in this case, the equation simplifies to $\alpha_{r_1+3} + \sum_j \beta_j = \sum_j \beta'_j$, and moreover,

$$\mathbf{f}^\pi = (\beta_d, \ldots, \beta_1, \alpha_{r_1+3} + \textstyle\sum_j \beta_j).$$

Since $\pi(f) = \pi(g)$, $\beta_j = \beta'_j$ for all $1 \leq j \leq d$. Therefore, substituting into the above equation,

$$\alpha_{r_1+3} + \textstyle\sum_j \beta_j = \sum_j \beta'_j = \sum_j \beta_j,$$

but $\alpha_{r_1+3} > 0$, a contradiction.

__Case 2:__ $|\text{supp}_{\mathbf{z}}(g)| \geq 1$. Let n denote the minimal index such that z_n divides g (i.e., $\alpha'_n > 0$ and $\alpha'_i = 0$ for all $i < n$). Since $|\text{supp}_{\mathbf{z}}(f)| \geq |\text{supp}_{\mathbf{z}}(g)|$ and $|\text{supp}_{\mathbf{z}}(g)| \geq 1$, we know $\text{supp}_{\mathbf{z}}(f) \neq \emptyset$. Hence, let m denote the minimal index such that z_m divides f. Via Lemma 3.3.4, this case naturally lends itself to the following subcases of consideration.

__Subcase 2.1:__ $n \in \{1, \ldots, r_1 - 1\}$. By Lemma 3.3.4, we know $\alpha'_n > 0$, $\alpha'_{n+1} \geq 0$, $\alpha'_{r_1+1} \geq 0$, and $\alpha'_i = 0$ for all $i \in \{1, \ldots, r_1 + 3\} \setminus \{n, n+1, r_1+1\}$. Since $z_n y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.3)), $z_n y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $g \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_1 \in \{1, \ldots, r_1 - 1\}$ and $\ell_1 \in \{r_1, \ldots, d\}$ such that

$\beta'_{k_1} = \beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^\pi[d-k_1+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i} + \beta_{k_1} \tag{3.7}$$

$$\mathbf{g}^\pi[d-k_1+1] = -(1+(r_1-n+1)x_1)\alpha'_n - (1+(r_1-n)x_1)\alpha'_{n+1} - x_1\alpha'_{r_1+1} \tag{3.8}$$

$$\mathbf{f}^\pi[d-\ell_1+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{3.9}$$

$$\mathbf{g}^\pi[d-\ell_1+1] = -(r_1-n+1)\alpha'_n - (r_1-n)\alpha'_{n+1} - \alpha'_{r_1+1}. \tag{3.10}$$

Note that $\pi(f) = \pi(g)$ implies (3.7) = (3.8) and (3.9) = (3.10). Now, we claim $m \in \{1, \ldots, r_1+1\}$. Indeed, assume otherwise, that is, $\text{supp}_{\mathbf{z}}(f) \subseteq \{r_1+2, r_1+3\}$. Then, $\mathbf{f}^\pi[d-\ell+1] = \beta_\ell \geq 0$ for all $\ell \in \{r_1, \ldots, d\}$, but from (3.10), $\mathbf{g}^\pi[d-\ell_1+1] < 0$ since $\alpha'_n > 0$ and $\alpha'_{n+1}, \alpha'_{r_1+1} \geq 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.3.4, we consider the following subsubcases.

(a) $m \in \{1, \ldots, r_1-1\}$. Since $z_m y_1 \cdots y_{r_1-1}$ (by (3.3)), $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \ldots, r_1-1\}$ and $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d-k_2+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{3.11}$$

$$\begin{aligned}\mathbf{g}^\pi[d-k_2+1] = &-(1+(r_1-n+1)x_1)\alpha'_n - (1+(r_1-n)x_1)\alpha'_{n+1} \\ &- x_1\alpha'_{r_1+1} + \beta'_{k_2}\end{aligned} \tag{3.12}$$

$$\mathbf{f}^\pi[d-\ell_2+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.13}$$

$$\mathbf{g}^\pi[d-\ell_2+1] = -(r_1-n+1)\alpha'_n - (r_1-n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.14}$$

where (3.11) = (3.12) and (3.13) = (3.14) as $\pi(f) = \pi(g)$. Since $1 \leq k_i \leq r_1 - 1$ for $i \in \{1, 2\}$, subtracting the equation (3.11) = (3.12) from (3.7) = (3.8) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, since $r_1 \leq \ell_i \leq d$ for $i \in \{1, 2\}$, subtracting equation (3.13) = (3.14) from (3.9) = (3.10) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{k_1} = \beta'_{k_2} = \beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4, we know $\alpha_m > 0$, $\alpha_{m+1}, \alpha_{r_1+1} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1+3\} \setminus \{m, m+1, r_1+1\}$. Consequently, equations (3.7) and (3.9) simplify to

$$\mathbf{f}^\pi[d-k_1+1] = -(1+(r_1-m+1)x_1)\alpha_m - (1+(r_1-m)x_1)\alpha_{m+1} - x_1\alpha_{r_1+1} \tag{3.15}$$

$$\mathbf{f}^\pi[d-\ell_1+1] = -(r_1-m+1)\alpha_m - (r_1-m)\alpha_{m+1} - \alpha_{r_1+1}. \tag{3.16}$$

Since $\pi(f) = \pi(g)$, (3.15) = (3.8) and (3.16) = (3.10), thereby implying $x_1(3.10) - (3.8) = x_1(3.16) - (3.15)$. Observe that $x_1(3.10) - (3.8) = x_1(3.16) -$

(3.15) is the following

$$\alpha_m + \alpha_{m+1} = \alpha'_n + \alpha'_{n+1}. \tag{3.17}$$

Now, consider the equation (3.16) = (3.10):

$$\begin{aligned}-(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} - \alpha_{r_1+1} = -(r_1 - n + 1)\alpha'_n \\ - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1}.\end{aligned}$$

Adding (3.17) to this equation r_1 times yields

$$(m-1)\alpha_m + m\alpha_{m+1} - \alpha_{r_1+1} = (n-1)\alpha'_n + n\alpha'_{n+1} - \alpha'_{r_1+1}.$$

Either $m < n$ or $m > n$ (note that $m \neq n$ since f and g are relatively prime). First, suppose $m < n$. Subtracting (3.17) from our previous equation $m - 1$ times gives

$$\alpha_{m+1} - \alpha_{r_1+1} = (n-m)\alpha'_n + (n-m+1)\alpha'_{n+1} - \alpha'_{r_1+1} \tag{3.18}$$

As $m < n$, we have that

$$\begin{aligned}\alpha_{m+1} - \alpha_{r_1+1} &= \underbrace{(n-m)}_{>0}\underbrace{\alpha'_n}_{>0} + \underbrace{(n-m+1)}_{>0}\underbrace{\alpha'_{n+1}}_{\geq 0} - \alpha'_{r_1+1} \\ &> \alpha'_n + \alpha'_{n+1} - \alpha'_{r_1+1} \\ &\overset{(3.17)}{=} \alpha_m + \alpha_{m+1} - \alpha'_{r_1+1},\end{aligned}$$

which implies

$$\alpha'_{r_1+1} > \underbrace{\alpha_m}_{>0} + \alpha_{r_1+1} \quad \Longrightarrow \quad \alpha'_{r_1+1} > 0.$$

Since f and g are relatively prime, this forces $\alpha_{r_1+1} = 0$. Thus, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{m, m+1\}$. Moreover, $\alpha'_{n+1} = 0$ since $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$ and we have found $\alpha'_n, \alpha'_{r_1+1} > 0$. Consequently, (3.17) reduces to $\alpha'_n = \alpha_m + \alpha_{m+1}$ and (3.18) reduces to

$$\alpha'_{r_1+1} = (n-m)\alpha_m + (n-m-1)\alpha_{m+1}. \tag{3.19}$$

Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_n + \alpha'_{r_1+1} + \sum_j \beta'_j.$$

Since $\alpha'_n = \alpha_m + \alpha_{m+1}$ and $\alpha'_{r_1+1} > 0$, this implies $\sum_j \beta_j > \sum_j \beta'_j$. For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = (r_1 - n + 1)\alpha'_n + \alpha'_{r_1+1} - \beta'_j.$$

Solving for α'_{r_1+1} and substituting $\alpha'_n = \alpha_m + \alpha_{m+1}$ yields

$$\alpha'_{r_1+1} = (n-m)\alpha_m + (n-m-1)\alpha_{m+1} + \beta'_j - \beta_j.$$

Adding these equations for each $r_1 \le j \le d$ gives

$$\begin{aligned}(d-r_1+1)\alpha'_{r_1+1} = (d-r_1+1)\left[(n-m)\alpha_m + (n-m-1)\alpha_{m+1}\right] \\ + \sum_{r_1\le j\le d} (\beta'_j - \beta_j).\end{aligned} \tag{3.20}$$

Similarly, for each $1 \le j \le r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(1+(r_1-m+1)x_1)\alpha_m + (1+(r_1-m)x_1)\alpha_{m+1} - \beta_j = (1+(r_1-n+1)x_1)\alpha'_n + x_1\alpha'_{r_1+1} - \beta'_j.$$

Solving for $x_1\alpha'_{r_1+1}$ and making the appropriate substitutions yields

$$x_1\alpha'_{r_1+1} = (n-m)x_1\alpha_m + (n-m-1)x_1\alpha_{m+1} + \beta'_j - \beta_j.$$

Adding these equations for each $1 \le j \le r_1 - 1$ gives

$$\begin{aligned}(r_1-1)x_1\alpha'_{r_1+1} = (r_1-1)\left[(n-m)x_1\alpha_m + (n-m-1)x_1\alpha_{m+1}\right] \\ + \sum_{1\le j\le r_1-1} (\beta'_j - \beta_j).\end{aligned} \tag{3.21}$$

Combining (3.20) and (3.21) gives

$$r_1x_1\alpha'_{r_1+1} = r_1x_1 \underbrace{\left[(n-m)\alpha_m + (n-m-1)\alpha_{m+1}\right]}_{=\alpha'_{r_1+1} \text{ by (3.19)}} + \underbrace{\sum_{j=1}^{d}(\beta'_j - \beta_j)}_{<0},$$

a contradiction. Now, suppose $m > n$. In this case, rather than subtracting $m-1$ copies of (3.17), we instead subtract $n-1$ copies of (3.17) yielding

$$(m-n)\alpha_m + (m-n+1)\alpha_{m+1} - \alpha_{r_1+1} = \alpha'_{n+1} - \alpha'_{r_1+1}.$$

Then, since $m - n > 0$, the same argument from the $m < n$ case will follow through by appropriately replacing each occurrence of α'_n with α_m, α_m with α'_n, α'_{n+1} with α_{m+1}, α_{m+1} with α'_{n+1}, α'_{r_1+1} with α_{r_1+1}, and α_{r_1+1} with α'_{r_1+1}.

(b) $m = r_1$. Since $z_{r_1}y_1\cdots y_{r_1-1}$ (by (3.3)), $z_{r_1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1,\ldots,r_1-1\}$ and $\ell_2 \in \{r_1,\ldots,d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d-k_2+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{3.22}$$

$$\begin{aligned}\mathbf{g}^\pi[d-k_2+1] = -(1+(r_1-n+1)x_1)\alpha'_n - (1+(r_1-n)x_1)\alpha'_{n+1} \\ - x_1\alpha'_{r_1+1} + \beta'_{k_2}\end{aligned} \tag{3.23}$$

$$\mathbf{f}^\pi[d-\ell_2+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.24}$$

$$\mathbf{g}^\pi[d-\ell_2+1] = -(r_1-n+1)\alpha'_n - (r_1-n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.25}$$

where (3.22) = (3.23) and (3.24) = (3.25) as $\pi(f) = \pi(g)$. Subtracting the equation (3.22) = (3.23) from (3.7) = (3.8) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, subtracting equation (3.24) = (3.25) from (3.9) = (3.10) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{k_1} = \beta'_{k_2} = \beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4, we know $\alpha_{r_1} > 0$, $\alpha_{r_1+1}, \alpha_{r_1+2} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1 - 1\} \cup \{r_1 + 3\}$. Consequently, equations (3.7) and (3.9) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -(1 + x_1)\alpha_{r_1} - x_1\alpha_{r_1+1} - \alpha_{r_1+2} \tag{3.26}$$
$$\mathbf{f}^\pi[d - \ell_1 + 1] = -\alpha_{r_1} - \alpha_{r_1+1}. \tag{3.27}$$

Since $\pi(f) = \pi(g)$, (3.26) = (3.8) and (3.27) = (3.10), thereby implying $x_1(3.10)-(3.8) = x_1(3.27)-(3.26)$. Observe that $x_1(3.10)-(3.8) = x_1(3.27)-(3.26)$ is the following

$$\alpha_{r_1} + \alpha_{r_1+2} = \alpha'_n + \alpha'_{n+1}. \tag{3.28}$$

Now, consider the equation $-(3.27) = -(3.10)$:

$$\alpha_{r_1} + \alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} + \alpha'_{r_1+1}.$$

Substituting (3.28) into this equation yields

$$\alpha'_n + \alpha'_{n+1} - \alpha_{r_1+2} + \alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} + \alpha'_{r_1+1}.$$

Rearranging by subtracting $\alpha'_n + \alpha'_{n+1}$ on both sides yields

$$\alpha_{r_1+1} - \alpha_{r_1+2} = \underbrace{(r_1 - n)\alpha'_n}_{>0} + \underbrace{(r_1 - n - 1)\alpha'_{n+1}}_{\geq 0} + \alpha'_{r_1+1}. \tag{3.29}$$

Observe that (3.29) implies $\alpha_{r_1+1} > 0$, so since f and g are relatively prime, this forces $\alpha'_{r_1+1} = 0$. Therefore, subtracting $r_1 - n$ copies of (3.28) from (3.29) gives

$$\alpha_{r_1+1} - (r_1 - n)\alpha_{r_1} - (r_1 - n + 1)\alpha_{r_1+2} = -\alpha'_{n+1},$$

which implies

$$\alpha'_{n+1} = (r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}. \tag{3.30}$$

Now, $\mathbf{f}^\pi[d + 1] = \mathbf{g}^\pi[d + 1]$ gives that

$$\alpha_{r_1} + \alpha_{r_1+1} + \alpha_{r_1+2} + \sum_j \beta_j = \alpha'_n + \alpha'_{n+1} + \sum_j \beta'_j.$$

Since $\alpha'_n = \alpha_{r_1} + \alpha_{r_1+2} - \alpha'_{n+1}$ by (3.28) and $\alpha_{r_1+1} > 0$, this implies $\sum_j \beta_j < \sum_j \beta'_j$. For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$\alpha_{r_1} + \alpha_{r_1+1} - \beta_j = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} - \beta'_j.$$

Solving for α'_{n+1} and substituting $\alpha'_n = \alpha_{r_1} + \alpha_{r_1+2} - \alpha'_{n+1}$ yields

$$\alpha'_{n+1} = (r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1} + \beta_j - \beta'_j.$$

Adding these equations for each $r_1 \le j \le d$ gives

$$\begin{aligned}(d - r_1 + 1)\alpha'_{n+1} = (d - r_1 + 1)\left[(r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}\right] \\ + \sum_{r_1 \le j \le d} (\beta_j - \beta'_j).\end{aligned} \tag{3.31}$$

Similarly, for each $1 \le j \le r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$(1 + x_1)\alpha_{r_1} + x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \beta'_j.$$

Solving for $x_1\alpha'_{n+1}$ and making the appropriate substitutions yields

$$x_1\alpha'_{n+1} = (r_1 - n)x_1\alpha_{r_1} + (r_1 - n + 1)x_1\alpha_{r_1+2} - x_1\alpha_{r_1+1} + \beta_j - \beta'_j.$$

Adding these equations for each $1 \le j \le r_1 - 1$ gives

$$\begin{aligned}(r_1 - 1)x_1\alpha'_{n+1} = (r_1 - 1)\left[(r_1 - n)x_1\alpha_{r_1} + (r_1 - n + 1)x_1\alpha_{r_1+2} - x_1\alpha_{r_1+1}\right] \\ + \sum_{1 \le j \le r_1 - 1} (\beta_j - \beta'_j).\end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) gives

$$r_1x_1\alpha'_{n+1} = r_1x_1 \underbrace{\left[(r_1 - n)\alpha_{r_1} + (r_1 - n + 1)\alpha_{r_1+2} - \alpha_{r_1+1}\right]}_{=\alpha'_{n+1} \text{ by } (3.30)} + \underbrace{\sum_{j=1}^{d} (\beta_j - \beta'_j)}_{<0},$$

a contradiction.

(c) $m = r_1 + 1$. Since $z_{r_1+1}y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.33}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.34}$$

where (3.33) $=$ (3.34) as $\pi(f) = \pi(g)$. Subtracting the equation (3.33) $=$ (3.34) from (3.9) $=$ (3.10) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \ge 0$ for all $1 \le j \le d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4, we know $\alpha_{r_1+1} > 0$, $\alpha_{r_1+2}, \alpha_{r_1+3} \ge 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Consequently, since $\beta_{\ell_1} = 0$, equations (3.7) and (3.9) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -x_1\alpha_{r_1+1} - \alpha_{r_1+2} + \beta_{k_1} \tag{3.35}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -\alpha_{r_1+1}. \tag{3.36}$$

Furthermore, since f and g are relatively prime, $\alpha_{r_1+1} > 0$ implies $\alpha'_{r_1+1} = 0$, so equations (3.8) and (3.10) simplify to

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + (r_1 - n + 1)x_1)\alpha'_n - (1 + (r_1 - n)x_1)\alpha'_{n+1} \quad (3.37)$$
$$\mathbf{g}^\pi[d - \ell_1 + 1] = -(r_1 - n + 1)\alpha'_n - (r_1 - n)\alpha'_{n+1}. \quad (3.38)$$

Since $\pi(f) = \pi(g)$, (3.35) = (3.37) and (3.36) = (3.38). Therefore, we have that $-$(3.35) = $-$(3.37) and $-$(3.36) = $-$(3.38), that is,

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_{k_1} = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} \quad (3.39)$$

and

$$\alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1}. \quad (3.40)$$

Now, $\mathbf{f}^\pi[d + 1] = \mathbf{g}^\pi[d + 1]$ gives that

$$\alpha_{r_1+1} + \alpha_{r_1+2} + \alpha_{r_1+3} + \sum_j \beta_j = \alpha'_n + \alpha'_{n+1} + \sum_j \beta'_j.$$

Substituting (3.40) and since $(r_1 - n)\alpha'_n > 0$, we obtain

$$\sum_j \beta_j < \sum_j \beta'_j\,. \quad (3.41)$$

For each $r_1 \le j \le d$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$\alpha_{r_1+1} - \beta_j = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} - \beta'_j,$$

which readily implies

$$\alpha_{r_1+1} = (r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1} + \beta_j - \beta'_j.$$

Adding these equations for each $r_1 \le j \le d$ gives

$$(d - r_1 + 1)\alpha_{r_1+1} = (d - r_1 + 1)\left[(r_1 - n + 1)\alpha'_n + (r_1 - n)\alpha'_{n+1}\right] + \sum_{r_1 \le j \le d} (\beta_j - \beta'_j).$$

Using (3.40), this simplifies to

$$0 = \sum_{r_1 \le j \le d} \left(\beta_j - \beta'_j\right). \quad (3.42)$$

Similarly, for each $1 \le j \le r_1 - 1$, $-\mathbf{f}^\pi[d - j + 1] = -\mathbf{g}^\pi[d - j + 1]$ is given by

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1 + (r_1 - n + 1)x_1)\alpha'_n + (1 + (r_1 - n)x_1)\alpha'_{n+1} - \beta'_j,$$

which implies

$$x_1\alpha_{r_1+1} = (1+(r_1-n+1)x_1)\alpha'_n + (1+(r_1-n)x_1)\alpha'_{n+1} - \alpha_{r_1+2} + \beta_j - \beta'_j.$$

Adding these equations for each $1 \le j \le r_1 - 1$ gives

$$(r_1-1)x_1\alpha_{r_1+1} = (r_1-1)\big[(1+(r_1-n+1)x_1)\alpha'_n + (1+(r_1-n)x_1)\alpha'_{n+1} - \alpha_{r_1+2}\big] + \sum_{1\le j\le r_1-1} (\beta_j - \beta'_j)\,.$$

Using (3.39), this simplifies to

$$0 = -(r_1-1)\beta_{k_1} + \sum_{1\le j\le r_1-1} (\beta_j - \beta'_j). \tag{3.43}$$

Combining (3.42) and (3.43), and observing (3.41), gives

$$0 = -(r_1-1)\beta_{k_1} + \underbrace{\sum_{j=1}^{d}(\beta_j - \beta'_j)}_{<0},$$

which implies $(r_1-1)\beta_{k_1} < 0$, a contradiction.

<u>Subcase 2.2:</u> $n = r_1$. By Lemma 3.3.4, we know $\alpha'_{r_1} > 0$, $\alpha'_{r_1+1}, \alpha'_{r_1+2} \ge 0$, and $\alpha'_i = 0$ for all $i \in \{1,\dots,r_1-1\} \cup \{r_1+3\}$. Since $z_{r_1}y_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.3)), $z_{r_1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $g \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_1 \in \{1,\dots,r_1-1\}$ and $\ell_1 \in \{r_1,\dots,d\}$ such that $\beta'_{k_1} = \beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^{\pi}[d-k_1+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i} + \beta_{k_1} \tag{3.44}$$

$$\mathbf{g}^{\pi}[d-k_1+1] = -(1+x_1)\alpha'_{r_1} - x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} \tag{3.45}$$

$$\mathbf{f}^{\pi}[d-\ell_1+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{3.46}$$

$$\mathbf{g}^{\pi}[d-\ell_1+1] = -\alpha'_{r_1} - \alpha'_{r_1+1}. \tag{3.47}$$

Note that $\pi(f) = \pi(g)$ implies (3.44) = (3.45) and (3.46) = (3.47). Now, we claim $m \in \{1,\dots,r_1-1\} \cup \{r_1+1\}$ (we need not consider $m = r_1$ since f and g are relatively prime and $n = r_1$ in this case). Indeed, assume otherwise, that is, $\operatorname{supp}_{\mathbf{z}}(f) \subseteq \{r_1+2, r_1+3\}$. Then, $\mathbf{f}^{\pi}[d-\ell+1] = \beta_\ell \ge 0$ for all $\ell \in \{r_1,\dots,d\}$, but from (3.47), $\mathbf{g}^{\pi}[d-\ell_1+1] < 0$ since $\alpha'_{r_1} > 0$ and $\alpha'_{r_1+1} \ge 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.3.4 and since m cannot be r_1, we consider the following subsubcases.

(a) $m \in \{1,\dots,r_1-1\}$. Since $z_m y_1\cdots y_{r_1-1}$ (by (3.3)), $z_m y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1,\dots,r_1-1\}$ and

$\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{3.48}$$
$$\mathbf{g}^\pi[d - k_2 + 1] = -(1 + x_1)\alpha'_{r_1} - x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{3.49}$$
$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.50}$$
$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.51}$$

where (3.48) = (3.49) and (3.50) = (3.51) as $\pi(f) = \pi(g)$. Subtracting the equation (3.48) = (3.49) from (3.44) = (3.45) implies $\beta_{k_1} = -\beta'_{k_2}$. Similarly, subtracting equation (3.50) = (3.51) from (3.46) = (3.47) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{k_1} = \beta'_{k_2} = \beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4, we know $\alpha_m > 0$, $\alpha_{m+1}, \alpha_{r_1+1} \geq 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1 + 3\} \setminus \{m, m+1, r_1+1\}$. Consequently, equations (3.44) and (3.46) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -(1 + (r_1 - m + 1)x_1)\alpha_m - (1 + (r_1 - m)x_1)\alpha_{m+1} - x_1\alpha_{r_1+1} \tag{3.52}$$
$$\mathbf{f}^\pi[d - \ell_1 + 1] = -(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} - \alpha_{r_1+1}. \tag{3.53}$$

Since $\pi(f) = \pi(g)$, (3.52) = (3.45) and (3.53) = (3.47), thereby implying $x_1(3.47) - (3.45) = x_1(3.53) - (3.52)$. Observe that $x_1(3.47) - (3.45) = x_1(3.53) - (3.52)$ is the following

$$\alpha_m + \alpha_{m+1} = \alpha'_{r_1} + \alpha'_{r_1+2}. \tag{3.54}$$

Now, consider the equation $-(3.53) = -(3.47)$:

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} + \alpha_{r_1+1} = \alpha'_{r_1} + \alpha'_{r_1+1}.$$

Substituting (3.54) into this equation and solving for α'_{r_1+1} yields

$$\alpha'_{r_1+1} = (r_1 - m)\alpha_m + (r_1 - m - 1)\alpha_{m+1} + \alpha_{r_1+1} + \alpha'_{r_1+2}. \tag{3.55}$$

Observe that (3.55) implies $\alpha'_{r_1+1} > 0$, so since f and g are relatively prime, this forces $\alpha_{r_1+1} = 0$. Thus, $\operatorname{supp}_{\mathbf{z}}(f) \subseteq \{m, m+1\}$. Moreover, since $|\operatorname{supp}_{\mathbf{z}}(f)| \geq |\operatorname{supp}_{\mathbf{z}}(g)|$, $\alpha_{r_1+1} = 0$, and we have $\alpha'_{r_1}, \alpha'_{r_1+1} > 0$, it follows that $\alpha_{m+1} > 0$ and $\alpha'_{r_1+2} = 0$. Consequently, (3.54) reduces to $\alpha'_{r_1} = \alpha_m + \alpha_{m+1}$ and (3.55) reduces to

$$\alpha'_{r_1+1} = (r_1 - m)\alpha_m + (r_1 - m - 1)\alpha_{m+1}.$$

Summing these reduced equations yields

$$\alpha'_{r_1} + \alpha'_{r_1+1} = (r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1}. \tag{3.56}$$

Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_{r_1} + \alpha'_{r_1+1} + \sum_j \beta'_j.$$

Since $\alpha'_{r_1} = \alpha_m + \alpha_{m+1}$ and $\alpha'_{r_1+1} > 0$, this implies

$$\sum_j \beta_j > \sum_j \beta'_j\,. \tag{3.57}$$

For each $r_1 \le j \le d$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = \alpha'_{r_1} + \alpha'_{r_1+1} - \beta'_j,$$

which, via (3.56), implies $\beta_j = \beta'_j$. Similarly, for each $1 \le j \le r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$\begin{aligned}(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} - \beta_j = (1 + x_1)\alpha'_{r_1}&\\ + x_1\alpha'_{r_1+1} - \beta'_j&,\end{aligned}$$

which, via (3.56), implies $\beta_j = \beta'_j$. Thus, we have that $\beta_j = \beta'_j$ for all $1 \le j \le d$, but we had in (3.57) that $\sum_j \beta_j > \sum_j \beta'_j$, a contradiction.

(b) $m = r_1 + 1$. Since $z_{r_1+1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.58}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1} - \alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.59}$$

where (3.58) = (3.59) as $\pi(f) = \pi(g)$. Subtracting the equation (3.58) = (3.59) from (3.46) = (3.47) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \ge 0$ for all $1 \le j \le d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4, we know $\alpha_{r_1+1} > 0$, $\alpha_{r_1+2}, \alpha_{r_1+3} \ge 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Consequently, since $\beta_{\ell_1} = 0$, equations (3.44) and (3.46) simplify to

$$\mathbf{f}^\pi[d - k_1 + 1] = -x_1\alpha_{r_1+1} - \alpha_{r_1+2} + \beta_{k_1} \tag{3.60}$$

$$\mathbf{f}^\pi[d - \ell_1 + 1] = -\alpha_{r_1+1}. \tag{3.61}$$

Furthermore, since f and g are relatively prime, $\alpha_{r_1+1} > 0$ implies $\alpha'_{r_1+1} = 0$, so equations (3.45) and (3.47) simplify to

$$\mathbf{g}^\pi[d - k_1 + 1] = -(1 + x_1)\alpha'_{r_1} - \alpha'_{r_1+2} \tag{3.62}$$

$$\mathbf{g}^\pi[d - \ell_1 + 1] = -\alpha'_{r_1}. \tag{3.63}$$

Since $\pi(f) = \pi(g)$, (3.60) = (3.62) and (3.61) = (3.63). Therefore, we have that $-(3.60) = -(3.62)$ and $-(3.61) = -(3.63)$, that is,

$$(1 + x_1)\alpha'_{r_1} + \alpha'_{r_1+2} = x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_{k_1} \tag{3.64}$$

and

$$\alpha'_{r_1} = \alpha_{r_1+1}. \tag{3.65}$$

Plugging (3.65) into (3.64) and solving for β_{k_1} gives

$$\beta_{k_1} = \alpha_{r_1+2} - \alpha_{r_1+1} - \alpha'_{r_1+2}. \tag{3.66}$$

Note that if $\alpha'_{r_1+2} > 0$, the relatively prime condition would force $\alpha_{r_1+2} = 0$, thereby implying $\beta_{k_1} < 0$, a contradiction. Hence, we may assume $\alpha'_{r_1+2} = 0$, and since $\beta_{k_1} \geq 0$, it must be that $\alpha_{r_1+2} > 0$. Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_{r_1+1} + \alpha_{r_1+2} + \alpha_{r_1+3} + \sum_j \beta_j = \alpha'_{r_1} + \sum_j \beta'_j.$$

Substituting (3.65) and since $\alpha_{r_1+2} > 0$, this implies

$$\sum_j \beta_j < \sum_j \beta'_j\,. \tag{3.67}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$\alpha_{r_1+1} - \beta_j = \alpha'_{r_1} - \beta'_j,$$

which, via (3.65), implies $\beta_j = \beta'_j$. Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$x_1\alpha_{r_1+1} + \alpha_{r_1+2} - \beta_j = (1+x_1)\alpha'_{r_1} - \beta'_j,$$

which, via (3.64), implies $\beta_{k_1} = \beta_j - \beta'_j$. Therefore,

$$0 < \sum_{j=1}^{d}(\beta'_j - \beta_j) = \sum_{j=1}^{r_1-1}(\beta'_j - \beta_j) + \sum_{j=r_1}^{d}(\beta'_j - \beta_j) = -(r_1-1)\beta_{k_1} \leq 0\,,$$

a contradiction.

<u>**Subcase 2.3:**</u> $n = r_1 + 1$. By Lemma 3.3.4, we know $\alpha'_{r_1+1} > 0$, $\alpha'_{r_1+2}, \alpha'_{r_1+3} \geq 0$, and $\alpha'_i = 0$ for all $i \in \{1, \ldots, r_1\}$. Since $z_{r_1+1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.6)) and $g \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_1 \in \{r_1, \ldots, d\}$ such that $\beta'_{\ell_1} = 0$. Then,

$$\mathbf{f}^\pi[d-\ell_1+1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i} + \beta_{\ell_1} \tag{3.68}$$

$$\mathbf{g}^\pi[d-\ell_1+1] = -\alpha'_{r_1+1}, \tag{3.69}$$

where (3.68) = (3.69) as $\pi(f) = \pi(g)$. Now, we claim $m \in \{1, \ldots, r_1\}$ (we need not consider $m = r_1 + 1$ since f and g are relatively prime and $n = r_1 + 1$ in this case). Indeed, assume otherwise, that is, $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1+2, r_1+3\}$. Then, $\mathbf{f}^\pi[d-\ell+1] = \beta_\ell \geq 0$ for all $\ell \in \{r_1, \ldots, d\}$, but from (3.69), $\mathbf{g}^\pi[d-\ell_1+1] < 0$ since $\alpha'_{r_1+1} > 0$. This contradicts $\pi(f) = \pi(g)$. Hence, given the structure of Lemma 3.3.4 and since m cannot be r_1+1, we consider the following subsubcases.

(a) $m \in \{1, \ldots, r_1 - 1\}$. Since $z_m y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.3)), $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \ldots, r_1 - 1\}$ and $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{3.70}$$

$$\mathbf{g}^\pi[d - k_2 + 1] = -x_1 \alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{3.71}$$

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.72}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.73}$$

where (3.70) = (3.71) and (3.72) = (3.73) since $\pi(f) = \pi(g)$. Subtracting the equation (3.72) = (3.73) from (3.68) = (3.69) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. Also, by Lemma 3.3.4 and since $n = r_1 + 1$, we know $\alpha_m > 0$, $\alpha_{m+1} \geq 0$, $\alpha_{r_1+1} = 0$, and $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1 + 3\} \setminus \{m, m+1\}$. Consequently, since $\beta_{\ell_1} = 0$, the equation (3.68) = (3.69) simplifies to

$$-(r_1 - m + 1)\alpha_m - (r_1 - m)\alpha_{m+1} = -\alpha'_{r_1+1},$$

which implies

$$\alpha'_{r_1+1} = (r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1}. \tag{3.74}$$

Furthermore, the equation $-$(3.70) = $-$(3.71) simplifies to

$$(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2}.$$

Via (3.74), this equation is equivalent to

$$\alpha_m + \alpha_{m+1} + x_1\alpha'_{r_1+1} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2},$$

which implies

$$\beta'_{k_2} = \alpha'_{r_1+2} - \alpha_m - \alpha_{m+1}. \tag{3.75}$$

Note that if $\alpha'_{r_1+2} = 0$, $\beta'_{k_2} < 0$ by (3.75), a contradiction. Hence, we may assume $\alpha'_{r_1+2} > 0$. Also, since $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$ and $\alpha_{r_1+1} = 0$, it follows that $\alpha_{m+1} > 0$ and $\alpha'_{r_1+3} = 0$. Now, $\mathbf{f}^\pi[d+1] = \mathbf{g}^\pi[d+1]$ gives that

$$\alpha_m + \alpha_{m+1} + \sum_j \beta_j = \alpha'_{r_1+1} + \alpha'_{r_1+2} + \sum_j \beta'_j.$$

Substituting (3.74), this implies

$$\sum_j \beta_j > \sum_j \beta'_j\,. \tag{3.76}$$

For each $r_1 \leq j \leq d$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(r_1 - m + 1)\alpha_m + (r_1 - m)\alpha_{m+1} - \beta_j = \alpha'_{r_1+1} - \beta'_j,$$

which, via (3.74), implies $\beta_j = \beta'_j$. Similarly, for each $1 \leq j \leq r_1 - 1$, $-\mathbf{f}^\pi[d-j+1] = -\mathbf{g}^\pi[d-j+1]$ is given by

$$(1 + (r_1 - m + 1)x_1)\alpha_m + (1 + (r_1 - m)x_1)\alpha_{m+1} - \beta_j = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_j,$$

which, via (3.74) and (3.75), implies

$$\beta'_{k_2} = \beta'_j - \beta_j. \tag{3.77}$$

Therefore, by (3.76) and (3.77),

$$\begin{aligned} 0 < \sum_{j=1}^{d}(\beta_j - \beta'_j) &= \sum_{j=1}^{r_1-1}(\beta_j - \beta'_j) + \sum_{j=r_1}^{d}(\beta_j - \beta'_j) \\ &= \sum_{j=1}^{r_1-1}(\beta_j - \beta'_j) \\ &= -(r_1 - 1)\beta'_{k_2} \\ &\leq 0, \end{aligned}$$

a contradiction.

(b) $m = r_1$. Since $z_{r_1}y_1 \cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.3)), $z_{r_1}y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4)), and $f \notin in_{<_{lex}}(\mathcal{G})$, there exist indices $k_2 \in \{1, \ldots, r_1 - 1\}$ and $\ell_2 \in \{r_1, \ldots, d\}$ such that $\beta_{k_2} = \beta_{\ell_2} = 0$. Then, we have that

$$\mathbf{f}^\pi[d - k_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_2+1,i} \tag{3.78}$$

$$\mathbf{g}^\pi[d - k_2 + 1] = -x_1\alpha'_{r_1+1} - \alpha'_{r_1+2} + \beta'_{k_2} \tag{3.79}$$

$$\mathbf{f}^\pi[d - \ell_2 + 1] = \sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_2+1,i} \tag{3.80}$$

$$\mathbf{g}^\pi[d - \ell_2 + 1] = -\alpha'_{r_1+1} + \beta'_{\ell_2}, \tag{3.81}$$

where (3.78) = (3.79) and (3.80) = (3.81) since $\pi(f) = \pi(g)$. Subtracting the equation (3.80) = (3.81) from (3.68) = (3.69) implies $\beta_{\ell_1} = -\beta'_{\ell_2}$. Since $\beta_j, \beta'_j \geq 0$ for all $1 \leq j \leq d$, this implies $\beta_{\ell_1} = \beta'_{\ell_2} = 0$. We know $\alpha_{r_1+1} = 0$ since $\alpha'_{r_1+1} > 0$. Also, by Lemma 3.3.4, we know $\alpha_{r_1} > 0$ and $\alpha_{r_1+2} \geq 0$, so it follows that $\alpha_i = 0$ for all $i \in \{1, \ldots, r_1 + 3\} \setminus \{r_1, r_1 + 2\}$. Consequently, since $\beta_{\ell_1} = 0$, the equation (3.68) = (3.69) simplifies to

$$\alpha_{r_1} = \alpha'_{r_1+1}. \tag{3.82}$$

Furthermore, the equation $-(3.78) = -(3.79)$ simplifies to

$$(1 + x_1)\alpha_{r_1} + \alpha_{r_1+2} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2}.$$

Via (3.82), this equation is equivalent to

$$(1 + x_1)\alpha'_{r_1+1} + \alpha_{r_1+2} = x_1\alpha'_{r_1+1} + \alpha'_{r_1+2} - \beta'_{k_2},$$

which implies

$$\beta'_{k_2} = \alpha'_{r_1+2} - \alpha'_{r_1+1} - \alpha_{r_1+2}. \tag{3.83}$$

Note that if $\alpha'_{r_1+2} = 0$, $\beta'_{k_2} < 0$ by (3.83), a contradiction. Hence, it must be that $\alpha'_{r_1+2} > 0$. However, by the relatively prime condition, this implies $\alpha_{r_1+2} = 0$. As a consequence, since $\alpha_{r_1+1} = \alpha_{r_1+2} = 0$ and $\alpha'_{r_1+1}, \alpha'_{r_1+2} > 0$, we have that

$$|\mathrm{supp}_{\mathbf{z}}(f)| = 1 < 2 \leq |\mathrm{supp}_{\mathbf{z}}(g)|,$$

contradicting our assumption that $|\mathrm{supp}_{\mathbf{z}}(f)| \geq |\mathrm{supp}_{\mathbf{z}}(g)|$.

Subcase 2.4: $n \in \{r_1 + 2, r_1 + 3\}$. In this case, $\mathrm{supp}_{\mathbf{z}}(g) \subseteq \{r_1 + 2, r_1 + 3\}$. Consequently, for $1 \leq j \leq d$, we have that

$$\mathbf{g}^{\pi}[d - j + 1] = \begin{cases} -\alpha'_{r_1+2} + \beta'_j, & \text{for } 1 \leq j \leq r_1 - 1 \\ \beta'_j, & \text{for } r_1 \leq j \leq d. \end{cases} \tag{3.84}$$

Now, we consider the possibilities for m.

(a) $m \in \{1, \ldots, r_1 + 1\}$. Since $z_m y_{r_1} \cdots y_d \in in_{<_{lex}}(\mathcal{G})$ (by (3.4) or (3.6)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $\ell_1 \in \{r_1, \ldots, d\}$ such that $\beta_{\ell_1} = 0$. Therefore, since $\alpha_m > 0$, we have that

$$\mathbf{f}^{\pi}[d - \ell_1 + 1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-\ell_1+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-\ell_1+1,r_1+3+j}}_{=0} < 0,$$

but this contradicts $\pi(f) = \pi(g)$ since $\mathbf{g}^{\pi}[d - \ell_1 + 1] = \beta'_{\ell_1} \geq 0$ from (3.84).

(b) $m \in \{r_1 + 2, r_1 + 3\}$. Note that since the relatively prime condition implies $m \neq n$, it follows that $|\mathrm{supp}_{\mathbf{z}}(f)| = |\mathrm{supp}_{\mathbf{z}}(g)| = 1$ in this case. Therefore, we may assume without loss of generality that $m = r_1 + 2$ and $n = r_1 + 3$. Since $z_{r_1+2} y_1 \ldots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ (by (3.5)) and $f \notin in_{<_{lex}}(\mathcal{G})$, there exists an index $k_1 \in \{1, \ldots, r_1 - 1\}$ such that $\beta_{k_1} = 0$. Therefore, since $\alpha_{r_1+2} > 0$, we have that $\alpha'_{r_1+2} = 0$ and

$$\mathbf{f}^{\pi}[d - k_1 + 1] = \underbrace{\sum_{i=1}^{r_1+3} \alpha_i \mathcal{A}(\mathbf{q})_{d-k_1+1,i}}_{<0} + \underbrace{\sum_{j=1}^{d} \beta_j \mathcal{A}(\mathbf{q})_{d-k_1+1,r_1+3+j}}_{=0} < 0.$$

However, this contradicts $\pi(f) = \pi(g)$ since $\mathbf{g}^{\pi}[d-k_1+1] = -\underbrace{\alpha'_{r_1+2}}_{=0} + \beta'_{k_1} \geq 0$ from (3.84).

Since each of the above cases (which together cover all possible pairs (m, n)) yields a contradiction, Lemma 3.3.3 implies that $\mathcal{G}$ forms a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to $<_{lex}$, as required. $\square$

In sum, since we have demonstrated that $\mathcal{G}$ is a Gröbner basis of $I_{\mathcal{A}(\mathbf{q})}$ with respect to $<_{lex}$, we know $in_{<_{lex}}(\mathcal{G}) = in_{<_{lex}}(I_{\mathcal{A}(\mathbf{q})})$. Therefore, since we can clearly see $in_{<_{lex}}(\mathcal{G})$ is squarefree, Theorem 3.1.4 holds and [34, Corollary 8.9] proves Corollary 3.1.5. As such, there exists a regular unimodular triangulation of the points in $\mathcal{A}'(\mathbf{q})$, as desired.

3.4 Facets of the Triangulation

For $\mathbf{q} = (r_1^{x_1}, (1+r_1x_1)^{r_1-1})$ with $r_1 > 1$, let $\mathcal{T}(\mathbf{q})$ denote the regular unimodular triangulation induced by the lexicographic term order $<_{lex}$ used in the previous section. This triangulation is identical to the placing triangulation obtained by placing the columns of $\mathcal{A}(\mathbf{q})$ from left to right in the order as given in Figure 3.1. Throughout this section, we will abuse notation in that the variable in $K[\mathcal{A}(\mathbf{q})]$ associated with each vertex of the triangulation $\mathcal{T}(\mathbf{q})$ will represent that vertex. The Gröbner basis $\mathcal{G}$ for $I_{\mathcal{A}(\mathbf{q})}$ in Theorem 3.3.1 indicates which elements of $\mathcal{M}(K([\mathcal{A}(\mathbf{q})]))$ generate the minimal non-faces (i.e., minimal subsets of vertices that are not faces) of $\mathcal{T}(\mathbf{q})$. From this, we can deduce the facets of $\mathcal{T}(\mathbf{q})$ as outlined in the following corollary. More specifically, the facets correspond to the squarefree monomials of degree $d+1$ in $K[\mathcal{A}(\mathbf{q})]$ that are not contained in $in_{<_{lex}}(\mathcal{G})$.

Corollary 3.4.1. *Let $f \in \mathcal{M}(K[\mathcal{A}(\mathbf{q})])$ be squarefree with $f \notin in_{<_{lex}}(\mathcal{G})$. Let m denote the minimal index such that z_m divides f. Then, f defines a facet of $\mathcal{T}(\mathbf{q})$ when it is one of the following possible forms (the notation $\widehat{y_k}$ indicates the variable y_k is omitted):*

(i) if $1 \leq m \leq r_1 - 1$, then $f = z_m z_{m+1} z_{r_1+1} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1 - 1$ and $r_1 \leq j \leq d$;

(ii) if $m = r_1$, then $f = z_{r_1} z_{r_1+1} z_{r_1+2} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1 - 1$ and $r_1 \leq j \leq d$;

(iii) if $m = r_1 + 1$, then $f = z_{r_1+1} z_{r_1+2} z_{r_1+3} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $1 \leq i \leq r_1 - 1$ and $r_1 \leq j \leq d$ or $f = z_{r_1+1} z_{r_1+3} y_1 \cdots y_{r_1-1} y_{r_1} \cdots \widehat{y_j} \cdots y_d$ for any $r_1 \leq j \leq d$;

(iv) if $m = r_1 + 2$, then $f = z_{r_1+2} z_{r_1+3} y_1 \cdots \widehat{y_i} \cdots y_{r_1-1} y_{r_1} \cdots y_d$ for any $1 \leq i \leq r_1 - 1$;

(v) if $m = r_1 + 3$, then $f = z_{r_1+3} y_1 \cdots y_d$.

Proof. The normalized volume of $\Delta_{(1,\mathbf{q})}$, denoted $N(\mathbf{q})$, is given by $N(\mathbf{q}) = 1 + \sum_{i=1}^{d} q_i = 1 + x_1r_1 + (r_1-1)(1+r_1x_1) = r_1(1+r_1x_1)$. Since $\mathcal{T}(\mathbf{q})$ is unimodular, we know the number of facets of $\mathcal{T}(\mathbf{q})$ should equal $N(\mathbf{q})$. Indeed, since $d = r_1+x_1-1$, it is straightforward to verify that there are precisely $r_1(1+r_1x_1)$ squarefree monomials given by the forms (i)-(v) above. Moreover, note that any facet of $\mathcal{T}(\mathbf{q})$ will require the inclusion of at least one z-variable since facets must consist of $d+1$ points and there are a total of d y-variables.

Now, suppose $1 \le m \le r_1 - 1$. By Lemma 3.3.4, we know $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{m, m+1, r_1+1\}$. Since $z_my_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (3.3) and $z_my_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ by (3.4), there exist indices $1 \le i \le r_1-1$ and $r_1 \le j \le d$ such that $y_i \nmid f$ and $y_j \nmid f$. As facets of $\mathcal{T}(\mathbf{q})$ must contain exactly $d+1$ points, this forces the inclusion of all other y-variables, z_{m+1}, and z_{r_1+1}. With no further restriction on i and j, we obtain form (i).

Now suppose $m = r_1$. By Lemma 3.3.4, we know $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1, r_1+1, r_1+2\}$. Again, (3.3) and (3.4) indicate that $z_{r_1}y_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ and $z_{r_1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$, so there exist indices $1 \le i \le r_1-1$ and $r_1 \le j \le d$ such that $y_i \nmid f$ and $y_j \nmid f$. Thus, to have a collection of $d+1$ points, it must be that f is of the form $f = z_{r_1}z_{r_1+1}z_{r_1+2}y_1\cdots\widehat{y_i}\cdots y_{r_1-1}y_{r_1}\cdots\widehat{y_j}\cdots y_d$, giving form (ii).

Next, suppose $m = r_1+1$. Lemma 3.3.4 gives that $\mathrm{supp}_{\mathbf{z}}(f) \subseteq \{r_1+1, r_1+2, r_1+3\}$, and we have that $z_{r_1+1}y_{r_1}\cdots y_d \in in_{<_{lex}}(\mathcal{G})$ by (3.6). Therefore, there exists some index $r_1 \le j \le d$ such that $y_j \nmid f$. Now, suppose $z_{r_1+2} \mid f$. Since $z_{r_1+2}y_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (3.5), there exists some index $1 \le i \le r_1-1$ such that $y_i \nmid f$. The exclusion of y_i and y_j necessarily requires the inclusion of all other y-variables and z_{r_1+3} to have a total of $d+1$ points. As such, f is of the form $f = z_{r_1+1}z_{r_1+2}z_{r_1+3}y_1\cdots\widehat{y_i}\cdots y_{r_1-1}y_{r_1}\cdots\widehat{y_j}\cdots y_d$. Otherwise, if $z_{r_1+2} \nmid f$, then the exclusion of y_j forces the inclusion of all other y-variables and z_{r_1+3} to have a total of $d+1$ points. Thus, f is of the form $f = z_{r_1+1}z_{r_1+3}y_1\cdots y_{r_1-1}y_{r_1}\cdots\widehat{y_j}\cdots y_d$. Combining these two possibilities gives form (iii).

Next, suppose $m = r_1+2$. Since $z_{r_1+2}y_1\cdots y_{r_1-1} \in in_{<_{lex}}(\mathcal{G})$ by (3.5), there exists some index $1 \le i \le r_1-1$ such that $y_i \nmid f$. To have a total of $d+1$ points, this forces the inclusion of all other y-variables and z_{r_1+3}. Therefore, f is of the form $f = z_{r_1+2}z_{r_1+3}y_1\cdots\widehat{y_i}\cdots y_{r_1-1}y_{r_1}\cdots y_d$, giving (iv).

Finally, suppose $m = z_{r_1+3}$. Then, for f to be supported on $d+1$ points, we must necessarily include all y-variables, yielding the form $f = z_{r_1+3}y_1\cdots y_d$. Note that $f \notin in_{<_{lex}}(\mathcal{G})$, so we obtain form (v). $\square$

Given that we know an explicit description of the facets of the unimodular triangulation $\mathcal{T}(\mathbf{q})$, a natural problem is to find a shelling of the facets from which we can recover the Ehrhart h^*-polynomial using standard techniques [6, 33]. This would provide another proof of Ehrhart h^*-unimodality, and give an explicit combinatorial interpretation to the coefficients of the h^*-polynomial. It is not clear how to construct a shelling in which both the shelling and the restriction sets admit a reasonable description. For example, one natural way to list the facets is to list them in lexicographic order; however, while this works for some small values of r_1 and x_1, computations with SageMath [35] show that this is not a shelling order when x_1 is sufficiently large compared to r_1.

It would be of interest to describe the regular unimodular triangulations of the 2-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$, and to connect these shellings explicitly to the Ehrhart theory of these simplices. However, the most important aspect of the existence of the regular unimodular triangulation given in this work is to establish that the h^*-unimodality of these simplices falls within the framework of Theorem 1.4.2.

Chapter 4 Classification of 3-Supported IDP Reflexive $\Delta_{(1,\mathbf{q})}$

This chapter extends known classification results of IDP reflexive $\Delta_{(1,\mathbf{q})}$ in the 2-supported case to the 3-supported case. This chapter is based on joint work with Benjamin Braun, Robert Davis, Morgan Lane, and Liam Solus. The results contained within have been submitted for publication and can be found here [10].

4.1 Integer Decomposition Property & Reflexivity

Throughout this chapter, we seek to identify a classification result (similar to Theorem 3.1.1) of $\Delta_{(1,\mathbf{q})}$ that are both IDP and reflexive in the 3-supported case. Recall Theorem 2.2.1 which provided a number-theoretic basis for studying reflexive simplices in $\mathcal{Q}$. The following setup provides us with another means of studying reflexive $\Delta_{(1,\mathbf{q})}$.

Setup 4.1.1. Let $\mathbf{q}$ be reflexive and supported by the vector $\mathbf{r} = (r_1, \ldots, r_d) \in \mathbb{Z}^d_{\geq 1}$ with multiplicity $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{Z}^d_{\geq 1}$. Let $\ell = \ell(\mathbf{q})$ be the integer defined by

$$1 + \sum_{i=1}^{d} x_i r_i = \ell \cdot \operatorname{lcm}(r_1, r_2, \ldots, r_d) \, . \tag{4.1}$$

Finally, we define $\mathbf{s} := (s_1, \ldots, s_d)$ where

$$s_i := \frac{\operatorname{lcm}(r_1, \ldots, r_d)}{r_i} \tag{4.2}$$

for each $1 \leq i \leq d$.

This setup provides useful restrictions on $\mathbf{q}$, such as the following lemma.

Lemma 4.1.2 (Braun, Liu, [13]). *In Setup 4.1.1, we have that* $\gcd(r_1, \ldots, r_d) = 1$ *and thus*

$$\operatorname{lcm}(s_1, \ldots, s_d) = \operatorname{lcm}(r_1, \ldots, r_d) \, . \tag{4.3}$$

4.1.1 Restrictions on IDP reflexive $\Delta_{(1,\mathbf{q})}$

For each $\mathbf{r}$-vector, it is known [11] that there are infinitely many reflexive $\Delta_{(1,\mathbf{q})}$'s supported on $\mathbf{r}$. Given a pair $\Delta_{(1,\mathbf{q})}$ and $\Delta_{(1,\mathbf{p})}$, both reflexive and IDP, Braun and Davis [9] proved that a new reflexive IDP $\Delta_{(1,\mathbf{y})}$ can be constructed as shown in the following theorem. Suppose that $\Delta_{(1,\mathbf{q})} \subset \mathbb{R}^n$ and $\Delta_{(1,\mathbf{p})} \subset \mathbb{R}^m$ are reflexive and the vertices of $\Delta_{(1,\mathbf{p})}$ are labeled as $v_0, v_1, \ldots, v_m$. For every $i = 0, 1, \ldots, m$, define the *affine free sum*

$$\Delta_{(1,\mathbf{q})} *_i \Delta_{(1,\mathbf{p})} := \operatorname{conv}\left\{(\Delta_{(1,\mathbf{q})} \times 0^m) \cup (0^n \times \Delta_{(1,\mathbf{p})} - v_i)\right\} \subset \mathbb{R}^{n+m}.$$

The notion of an affine free sum can be generalized significantly [4], but in this article it will not be necessary.

Theorem 4.1.3 (Braun, Davis [9]). *The simplex $\Delta_{(1,\mathbf{q})}$ is reflexive and arises as a free sum $\Delta_{(1,\mathbf{p})} *_0 \Delta_{(1,\mathbf{w})}$ if and only if $\Delta_{(1,\mathbf{p})}$ and $\Delta_{(1,\mathbf{w})}$ are reflexive and $\mathbf{q} = (\mathbf{p}, (1+\sum_i p_i)\mathbf{w})$. If $\Delta_{(1,\mathbf{p})}$ is IDP reflexive and $\Delta_{(1,\mathbf{q})}$ is IDP, then $\Delta_{(1,\mathbf{q})} *_0 \Delta_{(1,\mathbf{p})}$ is IDP. Further, if $\Delta_{(1,\mathbf{p})}$ and $\Delta_{(1,\mathbf{q})}$ are reflexive, IDP, and h^*-unimodal, then so is $\Delta_{(1,\mathbf{q})} *_0 \Delta_{(1,\mathbf{p})}$.*

Thus, there are infinitely many reflexive IDP $\Delta_{(1,\mathbf{q})}$'s that arise as a result of the affine free sum operation, and these all satisfy Conjecture 1.3.8. However, the support vector for $\Delta_{(1,\mathbf{q})} *_0 \Delta_{(1,\mathbf{p})}$ is distinct from that of $\mathbf{p}$ and $\mathbf{q}$, so this operation does not respect the stratification of $\mathcal{Q}$ given by support vectors. In fact, for many $\mathbf{r}$-vectors, it is impossible to generate infinitely many reflexive IDP $\Delta_{(1,\mathbf{q})}$'s supported on $\mathbf{r}$, as the following theorem shows.

Theorem 4.1.4 (Braun, Davis, Solus [11]). *Given a support vector $\mathbf{r} \in \mathbb{Z}^d_{\geq 1}$, if there exists some $j < d$ such that $r_j \nmid r_d$, then only finitely many reflexive IDP $\Delta_{(1,\mathbf{q})}$'s are supported on $\mathbf{r}$.*

Computational experiments suggest that $\Delta_{(1,\mathbf{q})}$ satisfying the criteria in Theorem 4.1.4 are rare. Specifically, consider all $\mathbf{r}$-vectors that are partitions of $M \leq 75$ with distinct entries, such that there exist some r_j such that $r_j \nmid r_d$. Table 4.1 shows that only 509 IDP reflexives are supported on $\mathbf{r}$-vectors of this type. While this suggests that IDP reflexive $\Delta_{(1,\mathbf{q})}$'s are rare, it is important to keep in mind that this represents a relatively small sample set of simplices. For example,

$$\mathbf{q} = (210, 211, 211, 211, 211, \underbrace{1055, 1055, \ldots, 1055}_{41 \text{ times}})$$

is not among this sample, but it is both IDP and reflexive with $210 \nmid 1055$.

Table 4.1: Experimental results.

# of $\mathbf{r}$-vectors with some $r_j \nmid r_d$	# of IDP reflexives supported by these
501350	509

Fortunately, the following theorem provides a number-theoretic characterization of the IDP property for reflexive $\Delta_{(1,\mathbf{q})}$.

Theorem 4.1.5 (Braun, Davis, Solus [11]). *The reflexive simplex $\Delta_{(1,\mathbf{q})}$ is IDP if and only if for every $j = 1, \ldots, n$, for all $b = 1, \ldots, q_j - 1$ satisfying*

$$b\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i \neq j} \left\lfloor \frac{bq_i}{q_j} \right\rfloor \geq 2 \tag{4.4}$$

there exists a positive integer $c < b$ satisfying the following equations, where the first is considered for all $1 \leq i \leq n$ with $i \neq j$:

$$\left\lfloor \frac{bq_i}{q_j} \right\rfloor - \left\lfloor \frac{cq_i}{q_j} \right\rfloor = \left\lfloor \frac{(b-c)q_i}{q_j} \right\rfloor, \text{ and} \tag{4.5}$$

$$c\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j}\left\lfloor \frac{cq_i}{q_j}\right\rfloor = 1. \tag{4.6}$$

The next corollary of Theorem 4.1.5 provides a necessary condition for a reflexive $\Delta_{(1,\mathbf{q})}$ to be IDP. This condition is an essential tool in our study of IDP reflexive elements of $\mathcal{Q}$.

Corollary 4.1.6 (Braun, Davis, Solus [11])**.** *If $\Delta_{(1,\mathbf{q})}$ is reflexive and IDP, then for all $j = 1, 2, \ldots, n$,*

$$1 + \sum_{i=1}^{n} (q_i \bmod q_j) = q_j$$

or equivalently

$$1 + \sum_{i=1}^{n} x_i(r_i \bmod r_j) = r_j\,.$$

Definition 4.1.7. Any $\mathbf{q}$ satisfying one (hence both) of these equations for all $j = 1, \ldots, n$ is said to satisfy *the necessary condition for IDP*.

Note that if $\mathbf{q}$ satisfies the necessary condition for IDP, then $\Delta_{(1,\mathbf{q})}$ is reflexive.

4.1.2 Stratifying by multiplicity instead of support

The necessary condition for IDP allows us to produce the following new refinement of Theorem 4.1.4. It has the added benefit of giving a geometric interpretation of IDP reflexive simplices as lattice points contained in a d-dimensional box with boundary determined by the coordinates of the support vector $\mathbf{r} \in \mathbb{Z}^d_{\geq 1}$.

Theorem 4.1.8. *Let $(\mathbf{r}, \mathbf{x}) = \mathbf{q} \in \mathbb{Z}^n_{\geq 1}$, where $\mathbf{q}$ has at least two distinct entries and $r_1 < r_2 < \cdots < r_d$. If $\Delta_{(1,\mathbf{q})}$ is reflexive and IDP, then*

$$x_i \leq r_{i+1}/r_i$$

for all $i \leq d-1$. Further, if there exists some $j < d$ such that $r_j \nmid r_d$, then

$$x_d \leq r_j/(r_d \bmod r_j)\,.$$

Thus, if there exists some $j < d$ such that $r_j \nmid r_d$, then there are at most finitely many IDP reflexives supported on $\mathbf{r}$.

Proof. Let $j < d$, and assume that $\Delta_{(1,\mathbf{q})}$ is reflexive and IDP. Then by Corollary 4.1.6, we have

$$x_j r_j \leq 1 + \sum_{i=1}^{d} x_i(r_i \bmod r_{j+1}) = r_{j+1}\,,$$

from which the first inequality follows. Similarly, if $r_j \nmid r_d$, then

$$x_d(r_d \bmod r_j) \leq 1 + \sum_{i=1}^{d} x_i(r_i \bmod r_j) = r_j\,,$$

from which the second inequality follows. $\square$

Theorem 4.1.8 indicates that there are important relationships between the multiplicity vector $\mathbf{x}$ of $\mathbf{q}$ and the support vector $\mathbf{r}$. By shifting our primary focus to the multiplicity vector, we are able to give a complete classification of all reflexive IDP $\Delta_{(1,\mathbf{q})}$'s that are supported on up to 3 distinct entries. If $\mathbf{x}$ has one entry, it is straightforward to prove the following.

Proposition 4.1.9. *For $\mathbf{q} = (r_1^{x_1})$, if $\Delta_{(1,\mathbf{q})}$ is IDP reflexive, then $\mathbf{q} = (1, 1, \ldots, 1)$.*

If $\mathbf{x}$ has two entries, meaning that $\mathbf{r}$ has two distinct entries, the following theorem applies.

Theorem 4.1.10 (Braun, Davis, Solus [11])**.** *For the vector $\mathbf{q} = (r_1^{x_1}, r_2^{x_2})$, $\Delta_{(1,\mathbf{q})}$ is IDP reflexive if and only if it satisfies the necessary condition. The following is a classification of all such vectors, for $x_1, x_2 \geq 1$:*

1. $q = (1^{x_1}, (1 + x_1)^{x_2})$
2. $q = ((1 + x_2)^{x_1}, (1 + (1 + x_2)x_1)^{x_2})$

Note that in the first case $r_1 \mid r_2$ while in the second case $r_1 \nmid r_2$. We next extend these known results using Theorem 4.1.8 and Corollary 4.1.6 to the 3-supported case.

Theorem 4.1.11. *Consider a 3-supported vector $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ such that $\Delta_{(1,\mathbf{q})}$ satisfies the necessity condition for IDP given in Corollary 4.1.6. If $\mathbf{x} = (x_1, x_2, x_3)$ is the multiplicity vector, then $\mathbf{r}$ is of one of the following forms.*

(i) $\mathbf{r} = (1, 1 + x_1, (1 + x_1)(1 + x_2))$.

(ii) $\mathbf{r} = (1 + x_2, 1 + x_1(1 + x_2), (1 + x_1(1 + x_2))(1 + x_2))$.

(iii) $\mathbf{r} = ((1 + x_2)(1 + x_3), 1 + x_1(1 + x_2)(1 + x_3), (1 + x_1(1 + x_2)(1 + x_3))(1 + x_2))$.

(iv) $\mathbf{r} = (1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$.

(v) $\mathbf{r} = (1+(1+x_3)x_2, (1+x_3)(1+x_1(1+(1+x_3)x_2)), (1+(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2))$.

(vi) $\mathbf{r} = ((1+x_3)(1+(1+x_3)x_2), (1+x_3)(1+x_1(1+x_3)(1+(1+x_3)x_2)), (1+(1+x_3)(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2))$.

(vii) $\mathbf{r} = (1 + x_3, (1 + x_3)(1 + x_1(1 + x_3)), (1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2))$.

(viii) *There exists some $k, s \geq 1$, where*

$$\mathbf{r} = (1+kx_2, (skx_2+s+k)(1+x_1(1+kx_2)), (1+x_1(1+kx_2))(1+x_2(skx_2+s+k)))$$

and

$$\mathbf{x} = (x_1, x_2, skx_2 + s - k + 1)\,.$$

Further, the first seven $\mathbf{r}$*-vectors produce IDP* $\Delta_{(1,\mathbf{q})}$*'s, while (viii) does not.*

As the necessary condition for IDP (see Definition 4.1.7) implies reflexivity, one immediate consequence of Proposition 4.1.9 together with Theorems 4.1.10 and 4.1.11 is a complete classification of all IDP reflexive simplicies corresponding to weighted projective spaces with one projective coordinate having weight unity and the others limited to three options. Note that the first seven $\mathbf{r}$-vectors in Theorem 4.1.11 each correspond to a unique divisibility criteria for $\mathbf{r} = (r_1, r_2, r_3)$, as follows:

(i) $r_1 \mid r_2, r_1 \mid r_3, r_2 \mid r_3$

(ii) $r_1 \nmid r_2, r_1 \mid r_3, r_2 \mid r_3$

(iii) $r_1 \nmid r_2, r_1 \nmid r_3, r_2 \mid r_3$

(iv) $r_1 \mid r_2, r_1 \mid r_3, r_2 \nmid r_3$

(v) $r_1 \nmid r_2, r_1 \mid r_3, r_2 \nmid r_3$

(vi) $r_1 \nmid r_2, r_1 \nmid r_3, r_2 \nmid r_3$

(vii) $r_1 \mid r_2, r_1 \nmid r_3, r_2 \nmid r_3$

(viii) $r_1 \nmid r_2, r_1 \mid r_3, r_2 \nmid r_3$

We see that *(v)* and *(viii)* share the same divisibility pattern, yet of these two families only *(v)* contains IDP simplices. Note that for each positive integer vector $\mathbf{x}$ of length at most three, and for each divisibility condition on the support vector $\mathbf{r}$, there is at most one support vector $\mathbf{r}$ such that $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ is reflexive IDP. It would be of interest to determine if this is true for $\mathbf{x}$ of arbitrary length.

4.2 Ehrhart Unimodality in the 3-Supported Case

Prior to discussing the case of reflexive IDP $\Delta_{(1,\mathbf{q})}$, it is worth asking whether or not h^*-unimodality is expected for an arbitrary $\Delta_{(1,\mathbf{q})}$. Based on experiments conducted via SageMath [35], there appears to be a trend in the overall frequency of h^*-unimodality of $\Delta_{(1,\mathbf{q})}$ as the normalized volume $N(\mathbf{q})$ grows. Specifically, let $V(M) := \{\Delta_{(1,\mathbf{q})} : N(\mathbf{q}) = M\}$ and define $\mathrm{un}(M)$ to be the fraction of the simplices in $V(M)$ having a unimodal h^*-vector. The values of $\mathrm{un}(M)$ for $M \leq 75$ have been computed exactly, involving 61,537,394 simplices. A plot of approximate values of $\mathrm{un}(M)$ using random samples is given in Figure 4.1.

The limiting behavior of $\mathrm{un}(M)$ is not known; various random sampling of $\mathbf{q}$ with fixed values of $N(\mathbf{q})$ suggests that $\mathrm{un}(M)$ continues to decrease as $M \to \infty$, but the rate of decrease also appears to be approaching zero. This leads us to the following question.

Question 4.2.1. *Is* $\lim_{M\to\infty} \mathrm{un}(M) = 0$*? If not, does the limit exist? If not, what are the values of* $\limsup_{M\to\infty} \mathrm{un}(M)$ *and* $\liminf_{M\to\infty} \mathrm{un}(M)$*?*

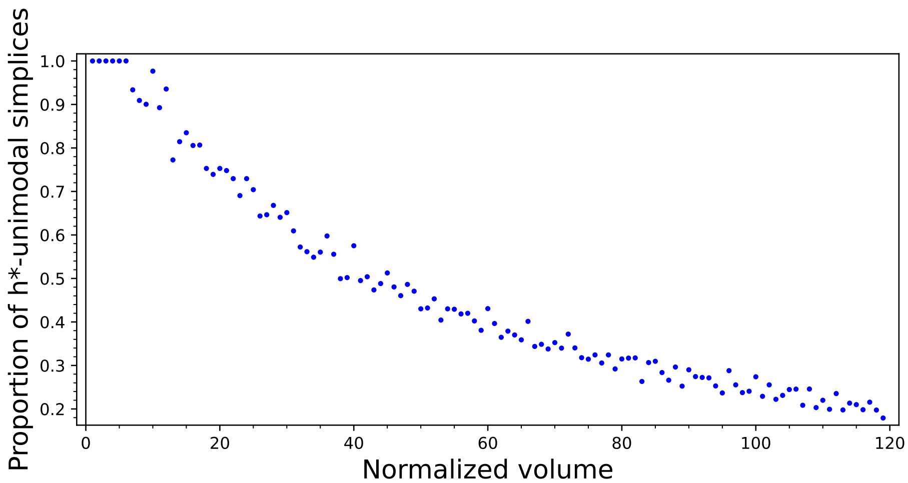

Figure 4.1: Approximate values of $\mathrm{un}(M)$ for $M \leq 120$, based on randomly sampled partitions.

Based on this empirical data, it seems reasonable to suspect that those $\Delta_{(1,\mathbf{q})}$ having unimodal h^*-polynomials are uncommon, and thus Conjecture 1.3.8 is suggesting that reflexive IDP $\Delta_{(1,\mathbf{q})}$'s are unusual in this regard. Returning our focus to the case where $\Delta_{(1,\mathbf{q})}$ is IDP reflexive, we will later need to know that the h^*-polynomial of $\Delta_{(1,\mathbf{q})}$ often admits a geometric series as a factor, as the following definition and theorem demonstrate.

Definition 4.2.2. Suppose $\mathbf{r}, \mathbf{x}$, ℓ and $\mathbf{s}$ are as given in Setup 4.1.1. We define

$$g_{\mathbf{r}}^{\mathbf{x}}(z) := \sum_{0 \leq \alpha < \mathrm{lcm}(r_1, \ldots, r_d)} z^{u(\alpha)}$$

where

$$u(\alpha) = u_{\mathbf{r}}^{\mathbf{x}}(\alpha) := \alpha\ell - \sum_{i=1}^{d} x_i \left\lfloor \frac{\alpha}{s_i} \right\rfloor .$$

Theorem 4.2.3 (Braun, Liu [13])**.** *Assuming Setup 4.1.1, we have that*

$$h^*(\Delta_{(1,\mathbf{q})}; z) = \left(\sum_{t=0}^{\ell-1} z^t \right) \cdot g_{\mathbf{r}}^{\mathbf{x}}(z) .$$

Example 4.2.4. For $\mathbf{q} = (1^7, 3^4, 5^5)$, we have

$$(z^2 + z + 1)(x^{14} + x^{11} + 2x^{10} + 2x^8 + 3x^7 + 2x^6 + 2x^4 + x^3 + 1) .$$

Note that in this case, $\ell = 3$ and a factor of $z^2 + z + 1$ appears in the h^*-polynomial.

4.2.1 h^*-unimodality and $\Delta_{(1,\mathbf{q})}$

It has been previously shown [11] that all 1-supported and 2-supported IDP reflexive $\Delta_{(1,\mathbf{q})}$ are h^*-unimodal. We consider here several of the 3-supported classes given in Theorem 4.1.11.

Theorem 4.2.5. *For $\mathbf{x} = (x_1, x_2, x_3)$ a positive integer vector, if $\mathbf{q} = (\mathbf{r}, \mathbf{x})$ where $\mathbf{r}$ is one of the following forms, then $\Delta_{(1,\mathbf{q})}$ is h^*-unimodal.*

(i) $\mathbf{r} = (1, 1 + x_1, (1 + x_1)(1 + x_2))$.

(ii) $\mathbf{r} = (1 + x_2, 1 + x_1(1 + x_2), (1 + x_1(1 + x_2))(1 + x_2))$.

(iv) $\mathbf{r} = (1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$.

Proof. As shown in the proof of Theorem 4.1.11, these cases all arise as affine free sums. Thus, by Theorem 4.1.3, they are h^*-unimodal. □

Remark 4.2.6. Based on experimental evidence using SageMath [35], we conjecture that the remaining four cases, listed here, are also h^*-unimodal.

(iii) $\mathbf{r} = ((1 + x_2)(1 + x_3), 1 + x_1(1 + x_2)(1 + x_3), (1 + x_1(1 + x_2)(1 + x_3))(1 + x_2))$.

(v) $\mathbf{r} = (1+(1+x_3)x_2, (1+x_3)(1+x_1(1+(1+x_3)x_2)), (1+(1+(1+x_3)x_2)x_1)(1+(1 + x_3)x_2))$.

(vi) $\mathbf{r} = ((1+x_3)(1+(1+x_3)x_2), (1+x_3)(1+x_1(1+x_3)(1+(1+x_3)x_2)), (1+(1+x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2))$.

(vii) $\mathbf{r} = (1 + x_3, (1 + x_3)(1 + x_1(1 + x_3)), (1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2))$.

However, these do not arise as affine free sums. While a direct proof of h^*-unimodality for these cases might be possible, it is not clear how to carry this out other than by ad hoc arguments given case-by-case, similar to the proof of Theorem 4.1.11.

It is interesting that among the $\Delta_{(1,\mathbf{q})}$ simplices, some can be found that are "on the boundary" of both h^*-unimodality and the IDP condition. One example is the following. Recall that for a lattice polytope P, the *Hilbert basis* of $\mathrm{cone}\,(P)$ is the minimal generating set of $\mathrm{cone}\,(P) \cap \mathbb{Z}^{n+1}$. Thus, P is IDP if and only if the Hilbert basis of $\mathrm{cone}\,(P)$ consists of the elements at height 1 in $\mathrm{cone}\,(P)$, i.e. $(1, P) \cap \mathbb{Z}^{n+1}$.

Theorem 4.2.7. *For $n \geq 1$, define $\mathbf{r}(n) = (1, 3n, 10n, 15n)$ and $\mathbf{x}(n) = (2n - 1, 1, 1, 1)$. Thus, $\mathbf{q}(n) := (\mathbf{r}(n), \mathbf{x}(n)) = \mathrm{rs}((3n, 10n, 15n))$. For $n \geq 2$ with $\mathbf{q} = (\mathbf{r}(n), \mathbf{x}(n))$, we have*

$$h^*(\Delta_{(1,\mathbf{q})}; z) = (1 + z^2 + z^4 + z^6 + \cdots + z^{2n-2}) \cdot (1 + 7z + 14z^2 + 7z^3 + z^4)\,,$$

which can be verified to be non-unimodal. For $n \geq 1$ with $\mathbf{q} = (\mathbf{r}(n), \mathbf{x}(n))$, let $V(n) = \{(1, \mathbf{v}) : \mathbf{v}$ a vertex of $\Delta_{(1,\mathbf{q})}\}$. The Hilbert basis for $\mathrm{cone}\left(\Delta_{(1,\mathbf{q})}\right)$ consists of

$V(n)$ and the columns of the following matrix (where the height coordinate is the first entry):

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & -1 & -1 & -2 & -3 & -4 & -7 & -10 \\ 0 & -1 & -1 & -2 & -3 & -5 & -6 & -10 & -15 \end{bmatrix}$$

Thus, there are exactly two elements in the Hilbert basis of height greater than 1*, both of which are at height* 2*.*

As another example, for $n \geq 2$, let

$$\mathbf{q} = ((n, (2n-1)(n+1), 2n(n+1)), (1, 1, 2(n-1)))\,.$$

For $n \leq 20$, it has been verified that

$$\begin{aligned} h^*(\Delta_{(1,\mathbf{q})}; z) = (1, (n+1)^2,&(2n+1)(n+1)+1, (2n+1)(n+1), (2n+1)(n+1)+1, \ldots, \\ &(2n+1)(n+1)+1, (2n+1)(n+1), (2n+1)(n+1)+1, (n+1)^2, 1) \end{aligned}$$

and that the Hilbert basis of cone$\left(\Delta_{(1,\mathbf{q})}\right)$ consists of the points $(1, \Delta_{(1,\mathbf{q})}) \cap \mathbb{Z}^{2n+1}$ together with following the lattice point at height two (as given by the first coordinate):

$$(2, -1, -2n-1, -2(n-1), -2(n-1), -2(n-1), \ldots, -2(n-1))^T\,.$$

Thus, this family of simplices is another example of polytopes on the boundary of both IDP and h^*-unimodality; this family is more arithmetically complicated than the one given in Theorem 4.2.7.

4.2.2 Proof of derivation of (i)-$(viii)$ in Theorem 4.1.11

We first suppose that $(\mathbf{r}, \mathbf{x})$ satisfies the necessity condition, and show that the resulting $\mathbf{r}$-vectors must be of one of the eight types listed. Since $r_1 < r_2 < r_3$, reducing modulo r_3 gives $r_3 = 1 + r_1x_1 + r_2x_2$. If we next consider the modulo r_2 necessary condition, then substituting the above for r_3 and simplifying gives $r_2 = 1 + x_1r_1 + x_3((1 + r_1x_1) \bmod r_2)$. The challenge here is that we would like to specify r_2 using this formula, but it involves a remainder which could fluctuate. The key observation is to recall that if the necessary condition for IDP holds, then Theorem 4.1.8 implies $x_1 \leq r_2/r_1$. Thus, we have

$$1 + x_1r_1 \leq 1 + (r_2/r_1)r_1 = 1 + r_2\,. \tag{4.7}$$

There are now three cases to consider.

Case 1: Suppose we have equality in (4.7). It is immediate that in this case

$$(1 + r_1x_1) \bmod r_2 = (1 + r_2) \bmod r_2 = 1\,,$$

and thus $r_2 = 1 + x_1 r_1 + x_3 = r_2 + 1 + x_3$. As $x_3 \geq 1$, this yields a contradiction, and thus this case does not occur.

Case 2: Consider if $1 + x_1 r_1 = r_2$ in (4.7). Then

$$(1 + r_1 x_1) \bmod r_2 = r_2 \bmod r_2 = 0\,.$$

Thus, we have $(\mathbf{r}, \mathbf{x}) = ((r_1, 1 + x_1 r_1, r_3), (x_1, x_2, x_3))$. However, we know that

$$r_3 = 1 + r_1 x_1 + r_2 x_2 = (1 + x_1 r_1)(1 + x_2)\,,$$

and thus $(\mathbf{r}, \mathbf{x}) = ((r_1, 1 + x_1 r_1, (1 + x_1 r_1)(1 + x_2)), (x_1, x_2, x_3))$. If $r_1 = 1$, then the result is $(\mathbf{r}, \mathbf{x}) = ((1, 1 + x_1, (1 + x_1)(1 + x_2)), (x_1, x_2, x_3))$. which corresponds to (i) in our theorem statement. If $r_1 \geq 2$, we consider our necessary condition modulo r_1 and obtain $r_1 = 1 + x_2 + x_3((1 + x_2) \bmod r_1)$. Since (by this equality) we have $1 + x_2 \leq r_1$, it follows that there are two subcases.

Subcase 2.1: Suppose $1 + x_2 = r_1$. Then our vector is

$$(\mathbf{r}, \mathbf{x}) = ((1 + x_2, 1 + x_1(1 + x_2), (1 + x_1(1 + x_2))(1 + x_2)), (x_1, x_2, x_3))\,,$$

yielding (ii) in our theorem statement.

Subcase 2.2: Suppose $1 + x_2 < r_1$. Then

$$r_1 = 1 + x_2 + x_3((1 + x_2) \bmod r_1) = 1 + x_2 + x_3(1 + x_2) = (1 + x_2)(1 + x_3)\,.$$

Then our vector is

$$\begin{aligned}(\mathbf{r}, \mathbf{x}) = &(((1 + x_2)(1 + x_3),\\ &1 + x_1(1 + x_2)(1 + x_3),\\ &(1 + x_1(1 + x_2)(1 + x_3))(1 + x_2)), (x_1, x_2, x_3))\,,\end{aligned}$$

yielding (iii) in our theorem statement.

Case 3: If $1 + x_1 r_1 < r_2$ in (4.7), it is immediate that $(1 + r_1 x_1) \bmod r_2 = 1 + r_1 x_1$, and thus $r_2 = 1 + x_1 r_1 + (1 + r_1 x_1) x_3 = (1 + x_3)(1 + x_1 r_1)$. Combining this with

$$r_3 = 1 + r_1 x_1 + r_2 x_2 = 1 + r_1 x_1 + (1 + x_3)(1 + x_1 r_1) x_2 = (1 + r_1 x_1)(1 + (1 + x_3) x_2)\,,$$

we obtain that

$$(\mathbf{r}, \mathbf{x}) = ((r_1, (1 + x_3)(1 + x_1 r_1), (1 + r_1 x_1)(1 + (1 + x_3) x_2)), (x_1, x_2, x_3))\,,$$

which is a function of the multiplicities and the value r_1. If $r_1 = 1$, then we obtain

$$(1, (1 + x_1)(1 + x_3), (1 + x_1)(1 + x_2(1 + x_3)))$$

which corresponds to item (iv) in our theorem statement. If $r_1 \geq 2$, we again consider the necessary condition modulo r_1, for which we obtain

$$\begin{aligned}r_1 &= 1 + x_2((1 + x_3)(1 + x_1 r_1) \bmod r_1) + x_3((1 + r_1 x_1)(1 + (1 + x_3) x_2) \bmod r_1)\\ &= 1 + x_2((1 + x_3) \bmod r_1) + x_3((1 + (1 + x_3) x_2) \bmod r_1).\end{aligned}$$

We now have three subcases to consider.

Subcase 3.1: If $1+x_3 < r_1$ then

$$r_1 = 1 + x_2(1+x_3) + x_3((1+(1+x_3)x_2) \bmod r_1)\,.$$

This requires two subsubcases.

Subsubcase 3.1.1: Suppose $(1+(1+x_3)x_2) = r_1$. Then x_3 is arbitrary, and we have

$$\begin{aligned}(\mathbf{r},\mathbf{x}) =&((1+(1+x_3)x_2, (1+x_3)(1+x_1(1+(1+x_3)x_2)),\\ &(1+(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2)), (x_1,x_2,x_3))\,,\end{aligned}$$

establishing item (v) in the theorem statement.

Subsubcase 3.1.2: Suppose $(1+(1+x_3)x_2) < r_1$. Then

$$r_1 = 1 + x_2(1+x_3) + x_3(1+(1+x_3)x_2) = (1+x_3)(1+(1+x_3)x_2)\,,$$

and thus it follows that for $\mathbf{x} = (x_1,x_2,x_3)$, we have

$$\begin{aligned}\mathbf{r} =&((1+x_3)(1+(1+x_3)x_2),\\ &(1+x_3)(1+x_1(1+x_3)(1+(1+x_3)x_2)),\\ &(1+(1+x_3)(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2))\,,\end{aligned}$$

establishing item (vi) in the theorem statement.

Subcase 3.2: Suppose $1+x_3 = r_1$. Then for $\mathbf{x} = (x_1,x_2,x_3)$ we have

$$\mathbf{r} = (1+x_3, (1+x_3)(1+x_1(1+x_3)), (1+(1+x_3)x_1)(1+(1+x_3)x_2))\,,$$

establishing item (vii) in the theorem statement.

Subcase 3.3: Suppose $1+x_3 > r_1$. If $r_1 \mid (1+x_3)$, then we have

$$r_1 = 1 + x_2((1+x_3) \bmod r_1) + x_3((1+(1+x_3)x_2) \bmod r_1) = 1 + x_3\,.$$

This is a contradiction, and thus it follows that $r_1 \nmid 1+x_3$. If $r_1 \mid x_2$, then

$$r_1 = 1 + x_2((1+x_3) \bmod r_1) + x_3\,,$$

which implies that $r_1 \mid (1+x_3)$, again a contradiction. Thus, we must have that $r_1 \nmid x_2$, and since $r_1 \nmid (1+x_3)$ we also know $r_1 > x_2$ since

$$r_1 = 1 + x_2((1+x_3) \bmod r_1) + x_3((1+(1+x_3)x_2) \bmod r_1) > x_2\,.$$

We now consider two subsubcases.

Subsubcase 3.3.1: Suppose $r_1 \mid (1+x_2(1+x_3))$. Then

$$r_1 = 1 + x_2((1+x_3) \bmod r_1)\,.$$

Thus, there exists some $k \geq 1$ where $r_1 = 1 + x_2 k$, and we are forced to have

$$r_1 = 1 + x_2 k = 1 + x_2((1+x_3) \bmod r_1)$$

implying that $k = (1 + x_3) \bmod 1 + x_2k$. Thus, for some $s \geq 1$ we set

$$x_3 = skx_2 + s - k + 1$$

and we have the case

$$\begin{aligned}\mathbf{r} =& (1 + kx_2, (2 + skx_2 + s - k)(1 + x_1(1 + kx_2)),\\ & (1 + x_1(1 + kx_2))(1 + x_2(2 + skx_2 + s - k)))\end{aligned}$$

with

$$\mathbf{x} = (x_1, x_2, skx_2 + s - k + 1)\,,$$

corresponding to item $(viii)$ in our theorem statement.

Subsubcase 3.3.2: Suppose $r_1 \nmid (1 + x_2(1 + x_3))$, so we have

$$r_1 = 1 + x_2((1 + x_3) \bmod r_1) + x_3((1 + (1 + x_3)x_2) \bmod r_1)\,.$$

We also have that $1 + x_3 > r_1 > x_2$. However, $r_1 \nmid (1 + x_2(1 + x_3))$ implies that

$$r_1 = 1 + x_2((1 + x_3) \bmod r_1) + x_3((1 + (1 + x_3)x_2) \bmod r_1) \geq 1 + x_3\,,$$

yielding a contradiction.

This completes our analysis of possible cases based on the necessary condition. In particular, all of the types listed yield reflexive simplices.

4.2.3 Proof of IDP for types (i), (ii), and (iv) and of non-IDP for type $(viii)$ in Theorem 4.1.11

We now show that type $(viii)$ $\mathbf{r}$-vectors yield non-IDP simplices, where we apply Theorem 4.1.5 and the notation therein. Let $q_j = r_2 = (skx_2+s+k)(1+x_1(1+kx_2))$, and set $b = skx_2 + s + k$ which is strictly less than r_2 as needed for Theorem 4.1.5. It is tedious but straightforward to reduce the left-hand side of (4.4) to the form $x_3 + (x_3 - 1)x_2(skx_2 + s + k)$. Note that by the assumption of type $(viii)$, we have

$$x_3 = s + 1 + k(sx_2 - 1) \geq 2$$

and thus

$$x_3 + (x_3 - 1)x_2(skx_2 + s + k) \geq 2 + x_2(skx_2 + s + k) \geq 2\,,$$

satisfying (4.4). Setting $q_i = r_3$, we now ask if there is a solution $0 < c < b$ satisfying (4.5) and (4.6). Using the fact that we must have $0 < c < b = skx_2 + s + k$, we obtain that the left-hand side of 4.5 is: (

$$\begin{aligned}&\left\lfloor \frac{(skx_2 + s + k)(1 + x_1(1 + kx_2))(1 + x_2(skx_2 + s + k))}{(skx_2 + s + k)(1 + x_1(1 + kx_2))} \right\rfloor - \left\lfloor \frac{c(1 + x_2(skx_2 + s + k))}{skx_2 + s + k} \right\rfloor\\ &= 1 + x_2(skx_2 + s + k) - cx_2 - \left\lfloor \frac{c}{skx_2 + s + k} \right\rfloor\\ &= 1 + x_2(skx_2 + s + k - c)\end{aligned}$$

On the other hand, using $0 < c < b = skx_2 + s + k$, we find that the right-hand side of 4.5 is:

$$\begin{aligned}
&\left\lfloor \frac{(skx_2+s+k-c)(1+x_2(skx_2+s+k))}{(skx_2+s+k)} \right\rfloor \\
&= x_2(skx_2+s+k-c) + \left\lfloor 1 - \frac{c}{skx_2+s+k} \right\rfloor \\
&= x_2(skx_2+s+k-c)
\end{aligned}$$

Thus, there is no c value in this range that satisfies (4.5), and hence we find that simplices of type $(viii)$ are not IDP.

We next show that simplices of types (i)-(vii) are IDP. Types (i), (ii), and (iv) all follow from affine free sum decompositions as follows. For (i), observe that

$$\mathbf{r} = (1^{x_1}) *_0 (1^{x_2}) *_0 (1^{x_3})\,,$$

and thus Theorem 4.1.3 applies. For (ii), observe that

$$\mathbf{r} = ((1+x_1)^{x_2}, (1+x_1(1+x_2))^{x_2}) *_0 (1^{x_3})\,,$$

and thus Theorem 4.1.10 and Theorem 4.1.3 apply to finish this case. Finally, for (iv) observe that

$$\mathbf{r} = (1^{x_3}) *_0 ((1+x_2)^{x_3}, (1+x_2(1+x_3))^{x_3})\,,$$

and thus again Theorem 4.1.10 and Theorem 4.1.3 apply to finish this case.

Types (iii), (v), (vi), and (vii) do not follow from affine free sum decompositions, and thus we must use Theorem 4.1.5 directly. Throughout the remainder of this proof, we use the notation

$$h(b) := b\left(\frac{1+\sum_{i\neq j} q_i}{q_j}\right) - \sum_{i\neq j} \left\lfloor \frac{bq_i}{q_j} \right\rfloor$$

to denote the left-hand side of (4.4).

4.2.4 Proof of IDP for type (iii) in Theorem 4.1.11

We next verify IDP for $\mathbf{r}$-vectors of type (iii) using Theorem 4.1.5. We must consider three cases corresponding to three possible values of q_j.

Case: $q_j = (1+x_2)(1+x_3)$. It is straightforward to verify that

$$h(b) = b - x_3 \left\lfloor \frac{b}{(1+x_3)} \right\rfloor\,,$$

and using this formula one can check that

$$h(k(1+x_3)) = k\,.$$

Combining these two observations, it follows that $h(b) = 1$ only when $b = 1$ and $b = (1 + x_3)$, thus identifying the b-values we are required to check in (4.4). To verify that (4.5) always has the desired solution, we consider three cases. If $q_i = (1 + x_2)(1 + x_3)$, the result is trivial. If $q_i = x_1(1 + x_2)(1 + x_3) + 1$, then we may select $c = 1$, from which it follows that both sides of (4.5) are equal to $x_1(b - 1)$. If $q_i = (1 + x_2)(x_1(1 + x_2)(1 + x_3) + 1)$, then we set $c = (1 + x_3)$, from which it is straightforward to compute that both sides of (4.5) are equal to $b(1+x_2)x_1 - x_1(1+x_2)(1 + x_3) - 1 + \lfloor b/(1 + x_3)\rfloor$. This completes our first case.

Case: $q_j = x_1(1 + x_2)(1 + x_3) + 1$. It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1 + x_2)(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor ,$$

where the values of b range from 1 to $x_1(1 + x_2)(1 + x_3)$. To verify that (4.5) always has the desired solution, we consider three cases. If $q_i = x_1(1 + x_2)(1 + x_3) + 1$, the result is trivial. If $q_i = (1+x_2)(1+x_3)$, then we write $b = \alpha x_1 + \beta$, where $0 \leq \beta < x_1$ and $0 \leq \alpha \leq (1 + x_2)(1 + x_3)$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, we have

$$\begin{aligned}
\left\lfloor \frac{b(1 + x_2)(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor &= \left\lfloor \frac{(\alpha x_1 + \beta)(1 + x_2)(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor \\
&= \alpha + \left\lfloor \frac{\beta(1 + x_2)(1 + x_3) - \alpha}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor \\
&= \alpha + \left\{ \begin{array}{rl} 0, & \beta > 0 \\ -1, & \beta = 0 \end{array} \right\} .
\end{aligned}$$

In the case that $\beta > 0$, we have that $h(b) = b - x_1\alpha = \beta$. Hence, the viable candidates for c-values from 1 to $x_1(1 + x_2)(1 + x_3)$ that satisfy equation (4.6) are precisely those c such that $c \equiv 1 \bmod x_1$. Therefore, the b-values we are required to check in (4.4) are all $b = \alpha x_1 + \beta$, where $2 \leq \beta \leq x_1 - 1$. In this case, we may choose $c = \alpha x_1 + 1$, from which it follows that both sides of (4.5) are equal to 0, giving the desired result. On the other hand, if $\beta = 0$, then we have that $h(b) = h(\alpha x_1) = \alpha x_1 - x_1(\alpha - 1) = x_1$. If $x_1 = 1$, then $h(b) = 1$. Thus, we need only consider when $x_1 > 1$. In order to satisfy (4.4), it must be that $\alpha > 0$. Given that $x_1 > 1$ and $\alpha > 0$, it is straightforward to verify that both sides of (4.5) when $c = 1$ are equal to $\alpha - 1$. Finally, if $q_i = (1 + x_2)(x_1(1 + x_2)(1 + x_3) + 1)$, then we can set $c = 1$ and the result is immediate. This completes our second case.

Case: $q_j = (1 + x_2)(x_1(1 + x_2)(1 + x_3) + 1)$. We first identify those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor - ((1 + x_2) - 1) \left\lfloor \frac{b}{(1 + x_2)} \right\rfloor .$$

Writing $b = m(1 + x_2) + t$ where $0 \leq m \leq x_1(1 + x_2)(1 + x_3)$ and $0 \leq t \leq (1 + x_2)$, it follows that

$$h(b) = h(m(1 + x_2) + t) = m + t - x_1 \left\lfloor \frac{m(1 + x_2)(1 + x_3) + t(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor .$$

We can now further divide into cases: either we have $m = x_1(1+x_2)(1+x_3)$ or we have $m = kx_1 + w$ where $0 \le k < (1+x_2)(1+x_3)$ and $0 \le w < x_1$, which yields

$$h(b) = h((kx_1 + w)(1+x_2) + t) = w + t - x_1 \left\lfloor \frac{w(1+x_2)(1+x_3) + t(1+x_3) - k}{x_1(1+x_2)(1+x_3)+1} \right\rfloor .$$

For $m \neq x_1(1+x_2)(1+x_3)$, observe that since $0 \le w(1+x_2)(1+x_3) \le x_1 - 1$ and $0 \le t(1+x_3) < (1+x_2)(1+x_3)$, with $0 \le k < (1+x_2)(1+x_3)$, we have that $0 \le w(1+x_2)(1+x_3) + t(1+x_3) < x_1(1+x_2)(1+x_3)$. Thus, $\left\lfloor \frac{w(1+x_2)(1+x_3)+t(1+x_3)-k}{x_1(1+x_2)(1+x_3)+1} \right\rfloor$ is equal to either 0 or -1.

Subcase 1 of 3: Suppose $m = x_1(1+x_2)(1+x_3)$. Since $0 \le t(1+x_3) < (1+x_2)(1+x_3)$, we have

$$h((1+x_2)(1+x_3)x_1(1+x_2) + t) = t - x_1 \left\lfloor \frac{t(1+x_3) - (1+x_2)(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor = t + x_1 .$$

If this is equal to 1, then it must be that $t = 0$ and $x_1 = 1$. Thus, if $x_1 = 1$, we have that $h((1+x_2)(1+x_3)(1+x_2)) = 1$.

Subcase 2 of 3: Suppose now that $m \neq x_1(1+x_2)(1+x_3)$ and that

$$\left\lfloor \frac{w(1+x_2)(1+x_3) + t(1+x_3) - k}{x_1(1+x_2)(1+x_3)+1} \right\rfloor = -1 .$$

Then since $w, t \ge 0$ and $x_1 \ge 1$, we have $h(b) = w + t + x_1 = 1$ which forces $w = t = 0$ and $x_1 = 1$. In this case, $h(k(1+x_2)) = 1$ any time that $k > 0$. Thus, if $x_1 = 1$, we have that $h(k(1+x_2)) = 1$ when $0 < k < (1+x_2)(1+x_3)$.

Subcase 3 of 3: Suppose again that $m \neq x_1(1+x_2)(1+x_3)$ and that

$$\left\lfloor \frac{w(1+x_2)(1+x_3) + t(1+x_3) - k}{x_1(1+x_2)(1+x_3)+1} \right\rfloor = 0 .$$

Then $0 \le k \le w(1+x_2)(1+x_3) + t(1+x_3)$, which implies that either (A) $0 < w < x_1$ with $0 \le t < (1+x_2)$ or (B) $w = 0$ with $k \le t(1+x_3)$. If (A) holds, then $h(b) = w + t = 1$ forces $w = 1$ and $t = 0$ since $w > 0$, which means that $h(b) = 1$ when $b = (kx_1 + 1)(1+x_2)$ for $0 \le k < (1+x_2)(1+x_3)$. If (B) holds, then our same equation forces $w = 0$ and $t = 1$ when $k \le (1+x_3)$, which means that $h(b) = 1$ when $b = kx_1(1+x_2) + 1$ for $0 \le k < (1+x_2)(1+x_3)$ and $k \le (1+x_3)$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$ and $0 < k \le (1+x_2)(1+x_3)$ we have $b = k(1+x_2)$.
- If $x_1 \ge 1$ and $0 \le k < (1+x_2)(1+x_3)$, we have $b = (kx_1 + 1)(1+x_2)$.
- If $x_1 \ge 1$ and $0 \le k < (1+x_2)(1+x_3)$ and $k \le (1+x_3)$, we have $b = kx_1(1+x_2) + 1$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1+x_2)(x_1(1+x_2)(1+x_3)+1)$. If $q_i = (1+x_2)(x_1(1+x_2)(1+x_3)+1)$, then (4.5) is trivially satisfied. If $q_i = x_1(1+x_2)(1+x_3)+1$, we write $b = m(1+x_2)+t$ where $0 \le m < x_1(1+x_2)(1+x_3)+1$ and $0 \le t < (1+x_2)$. Substituting this form of b into (4.5) yields the equation

$$-\left\lfloor \frac{c}{(1+x_2)} \right\rfloor = \left\lfloor \frac{t-c}{(1+x_2)} \right\rfloor .$$

If $b > (1+x_2)$, we set $c = (1+x_2)$ and the equation is satisfied. If $2 < b < (1+x_2)$, then we set $c = 1$ and the equation is satisfied.

If $q_i = (1+x_2)(1+x_3)$, the analysis becomes more complicated. We write $b = m(1+x_2)+t$ where $0 \le m < d$ and $0 \le t < (1+x_2)$. Our argument will proceed by considering $x_1 = 1$ and $x_1 > 1$ separately.

Suppose $x_1 = 1$. Then the left-hand-side of (4.5) is reduced to

$$\left\lfloor \frac{t(1+x_3)-m}{(1+x_2)(1+x_3)+1} \right\rfloor - \left\lfloor \frac{c(1+x_3)}{(1+x_2)(1+x_3)+1} \right\rfloor$$

and the right-hand-side to

$$\left\lfloor \frac{t(1+x_3)-m-c(1+x_3)}{(1+x_2)(1+x_3)+1} \right\rfloor .$$

Since $m < (1+x_3)$, if $t(1+x_3)-m < 0$ this forces $t = 0$ and $0 < m$, thus b is a multiple of $(1+x_2)$, and we found earlier that $h(m(1+x_2)) = 1$. Thus, we need proceed no further. If $t(1+x_3)-m \ge 0$, then since $m < (1+x_3)$ we must have $t \ge 1$, and we also have $t(1+x_3)-m < (1+x_2)(1+x_3)$. Thus, $\left\lfloor \frac{t(1+x_3)-m}{(1+x_2)(1+x_3)+1} \right\rfloor = 0$, from which it follows that (4.5) reduces to

$$-\left\lfloor \frac{c(1+x_3)}{(1+x_2)(1+x_3)+1} \right\rfloor = \left\lfloor \frac{t(1+x_3)-m-c(1+x_3)}{(1+x_2)(1+x_3)+1} \right\rfloor .$$

If $m = 0$, set $c = 1$ and this equation is solved. If $m \ge 1$, set $c = m(1+x_2)+1$ which is less than b in this case, and this equation is again satisfied. This completes our proof for $x_1 = 1$.

We next consider when $x_1 \ge 2$, maintaining our previous notation of $b = m(1+x_2)+t$. Write $m = fx_1 + g$ where $0 \le f \le \lfloor (1+x_3)/x_1 \rfloor$ and $0 \le g < x_1$ except in the case where $f = \lfloor (1+x_3)/x_1 \rfloor$ in which case g is bounded above by $(1+x_3)-(1+x_3)\lfloor (1+x_3)/x_1 \rfloor$. This leads to the left-hand-side of (4.5) having the form

$$f + \left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f}{x_1(1+x_2)(1+x_3)+1} \right\rfloor - \left\lfloor \frac{c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor$$

while the right-hand-side has the form

$$f + \left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f-c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor .$$

We thus need to solve the equation

$$\left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f}{x_1(1+x_2)(1+x_3)+1} \right\rfloor - \left\lfloor \frac{c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor$$
$$= \left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f-c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor$$

subject to the constraints $0 \le g < x_1$ (with the exception mentioned above), $0 \le t \le (1+x_2)$, and $0 \le f \le \lfloor (1+x_3)/x_1 \rfloor$. Note that the first two inequalities imply that $0 \le g(1+x_2)(1+x_3)+t(1+x_3) < x_1(1+x_2)(1+x_3)$, and also $f \le \lfloor (1+x_3)/x_1 \rfloor \le (1+x_2)(1+x_3)$, hence

$$\left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f}{x_1(1+x_2)(1+x_3)+1} \right\rfloor = \begin{cases} 0 & \text{if } g(1+x_2)(1+x_3)+t(1+x_3) \ge f \\ -1 & \text{if } g(1+x_2)(1+x_3)+t(1+x_3) < f \end{cases}$$

Subcase 1 of 2: Suppose $g(1+x_2)(1+x_3)+t(1+x_3)-f \ge 0$. Then (4.5) reduces to

$$-\left\lfloor \frac{c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor = \left\lfloor \frac{g(1+x_2)(1+x_3)+t(1+x_3)-f-c(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor .$$

Note that $f \le \lfloor (1+x_3)/x_1 \rfloor < (1+x_3)$, and thus we can set $c = fx_1(1+x_2)+1$ which is less than b. The left-hand-side of our above equation is given by

$$\begin{aligned} & -\left\lfloor \frac{(fx_1(1+x_2)+1)(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor \\ &= -\left\lfloor \frac{fx_1(1+x_2)(1+x_3)+(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor \\ &= -\left\lfloor \frac{fx_1(1+x_2)(1+x_3)+f+(1+x_3)-f}{x_1(1+x_2)(1+x_3)+1} \right\rfloor \\ &= -f\,. \end{aligned}$$

Similarly, the right-hand-side of our equation is given by

$$\begin{aligned} & -f + \left\lfloor \frac{f-(1+x_3)+g(1+x_2)(1+x_3)+t(1+x_3)-f}{x_1(1+x_2)(1+x_3)+1} \right\rfloor \\ &= -f + \left\lfloor \frac{-(1+x_3)+g(1+x_2)(1+x_3)+t(1+x_3)}{x_1(1+x_2)(1+x_3)+1} \right\rfloor . \end{aligned}$$

Since $h(b)$ is assumed to be at least 2, we have that one or both of g and t are non-zero. Combining this observation with $g(1+x_2)(1+x_3)+t(1+x_3)-f \ge 0$ it follows that $g(1+x_2)(1+x_3)+t(1+x_3) > 0$. Note that $(1+x_3) \mid (g(1+x_2)(1+x_3)+t(1+x_3))$, and thus $x_1(1+x_2)(1+x_3)+1 > g(1+x_2)(1+x_3)+t(1+x_3)-(1+x_3) \ge 0$, which forces the right-hand-side of our equation to equal $-f$, satisfying (4.5).

Subcase 2 of 2: Suppose $g(1+x_2)(1+x_3)+t(1+x_3)-f < 0$. Note that since $g, (1+x_2)(1+x_3), t, (1+x_3) \ge 0$, it follows that $f \ge 1$ and thus $b = (fx_1+g)(1+$

$x_2) + t \geq (1 + x_2)$. Set $c = (1 + x_2)$, which is less than b. With these conditions, the left-hand-side of (4.5) is easily seen to equal -1. The right-hand-side of (4.5) is given by

$$\left\lfloor \frac{g(1 + x_2)(1 + x_3) + t(1 + x_3) - f - (1 + x_2)(1 + x_3)}{x_1(1 + x_2)(1 + x_3) + 1} \right\rfloor .$$

Since $g(1+x_2)(1+x_3)+t(1+x_3)-f < 0$ and $-(1+x_2)(1+x_3) < 0$, the numerator above is strictly negative. Also, since $g, (1+x_2)(1+x_3), t, (1+x_3) \geq 0$, the numerator is minimized by $-f-(1+x_2)(1+x_3) > -(1+x_3)-(1+x_2)(1+x_3) \geq -2(1+x_2)(1+x_3)$. But, since we assumed that $x_1 \geq 2$, it follows that $x_1(1+x_2)(1+x_3)+1 > 2(1+x_2)(1+x_3)$ and thus the floor function above is equal to -1, satisfying equality for (4.5). This completes the proof establishing IDP for **r**-vectors of type (iii).

4.2.5 Proof of IDP for type (v) in Theorem 4.1.11

We next verify IDP for **r**-vectors of type (v) using Theorem 4.1.5. Again, we must consider three cases corresponding to three possible values of q_j.

Case: $q_j = 1 + (1 + x_3)x_2$. It is straightforward to verify that

$$h(b) = b - x_2 \left\lfloor \frac{b(1 + x_3)}{1 + (1 + x_3)x_2} \right\rfloor ,$$

where $1 \leq b \leq (1 + x_3)x_2$. To verify that (4.5) always has the desired solution, we consider three cases. If $q_i = 1 + (1 + x_3)x_2$, the result is trivial. If $q_i = (1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))$, then we write $b = \alpha x_2 + \beta$, where $0 \leq \beta < x_2$ and $0 \leq \alpha \leq 1 + x_3$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, observe that

$$\left\lfloor \frac{b(1 + x_3)}{1 + (1 + x_3)x_2} \right\rfloor = \left\lfloor \frac{(\alpha x_2 + \beta)(1 + x_3}{1 + (1 + x_3)x_2} \right\rfloor = \alpha + \left\lfloor \frac{\beta(1 + x_3) - \alpha}{1 + (1 + x_3)x_2} \right\rfloor = \alpha + \left\{ \begin{array}{rl} 0, & \beta > 0 \\ -1, & \beta = 0 \end{array} \right\} .$$

In the case that $\beta > 0$, this formula implies that $h(b) = b - x_2\alpha = \beta$. Hence, the viable candidates for c-values that satisfy (4.6) are precisely those c such that $c \equiv 1 \bmod x_2$. Therefore, the b-values we are required to check in (4.4) are all $b = \alpha x_2 + \beta$, where $2 \leq \beta \leq x_2 - 1$. In this case, we may choose $c = \alpha x_2 + 1$, from which it follows that both sides of (4.5) are equal to $(\beta - 1)(1 + x_3)x_1$. On the other hand, if $\beta = 0$, then we have that $h(b) = h(\alpha x_2) = \alpha x_2 - x_2(\alpha - 1) = x_2$. If $x_2 = 1$, then $h(b) = 1$. Thus, we need only consider when $x_2 > 1$. Given that $x_2 > 1$ and $0 \leq \alpha \leq 1 + x_3$, it is straightforward to verify that both sides of (4.5) when $c = 1$ are equal to $(\alpha x_2 - 1)(1 + x_3)x_1 + \alpha - 1$. If $q_i = (1 + (1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$, then we may again set $c = 1$, from which it is straightforward to compute that both sides of (4.5) are equal to $(b - 1)(1 + (1 + (1 + x_3)x_2)x_1)$. This completes our first case.

Case: $q_j = (1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))$. We first identify those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1 + (1 + x_3)x_2)}{(1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))} \right\rfloor - x_3 \left\lfloor \frac{b}{1 + x_3} \right\rfloor .$$

Writing $b = \alpha(1+x_3)+\beta$, where $0 \le \beta < 1+x_3$ and $0 \le \alpha \le x_1(1+(1+x_3)x_2)$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1+x_3)+\beta) \\ &= \alpha+\beta - x_1 \left\lfloor \frac{\alpha(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor . \end{aligned}$$

We can now further divide into cases: either we have $\alpha = x_1(1+(1+x_3)x_2)$ or we have $\alpha = \gamma x_1 + \delta$ where $0 \le \gamma < 1+(1+x_3)x_2$ and $0 \le \delta < x_1$, which yields

$$\begin{aligned} h(b) &= h((\gamma x_1+\delta)(1+x_3)+\beta) \\ &= \delta+\beta - x_1 \left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2)-\gamma(1+x_3)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor . \end{aligned}$$

For $\alpha \ne x_1(1+(1+x_3)x_2)$, observe that since $0 \le \delta(1+x_3)(1+(1+x_3)x_2) \le (x_1-1)(1+x_3)(1+(1+x_3)x_2)$ and $0 \le \beta(1+(1+x_3)x_2) < (1+x_3)(1+(1+x_3)x_2)$ with $0 \le \gamma(1+x_3) < (1+x_3)(1+(1+x_3)x_2)$, we have that $0 \le \delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2) < x_1(1+x_3)(1+(1+x_3)x_2)$. Thus, $\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma(1+x_3)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor$ is equal to either 0 or -1.

Subcase 1 of 3: Suppose $\alpha = x_1(1+(1+x_3)x_2)$. Since $0 \le \beta(1+(1+x_3)x_2) < (1+x_3)(1+(1+x_3)x_2)$, we have

$$\begin{aligned} h((1+x_3)x_1(1+(1+x_3)x_2)+\beta) &= \beta - x_1 \left\lfloor \frac{\beta(1+(1+x_3)x_2)-(1+x_3)(1+(1+x_3)x_2)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor \\ &= \beta + x_1. \end{aligned}$$

If this is equal to 1, then it must be that $\beta = 0$ and $x_1 = 1$. Thus, if $x_1 = 1$, we have that $h((1+x_3)(1+(1+x_3)x_2)) = 1$.

Subcase 2 of 3: Suppose now that $\alpha \ne x_1(1+(1+x_3)x_2)$ and that

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma(1+x_3)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor = -1.$$

Then, writing $\alpha = \gamma x_1 + \delta$ where $0 \le \gamma < 1+(1+x_3)x_2$ and $0 \le \delta < x_1$, we have $h(b) = \delta+\beta+x_1$. Since $\delta, \beta \ge 0$ and $x_1 \ge 1$, for this to equal 1, we must have $\delta = \beta = 0$ and $x_1 = 1$. In this case, $h(\gamma(1+x_3)) = 1$ whenever $\gamma > 0$. Thus, if $x_1 = 1$, we have that $h(\gamma(1+x_3)) = 1$ when $0 < \gamma < 1+(1+x_3)x_2$.

Subcase 3 of 3: Suppose now that $\alpha \ne x_1(1+(1+x_3)x_2)$ and that

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma(1+x_3)}{(1+x_3)(1+x_1(1+(1+x_3)x_2))} \right\rfloor = 0.$$

Then, it follows that $0 \le \gamma(1+x_3) \le \delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2)$, which given the bounds on γ, δ, and β, implies that either (A) $0 < \delta < x_1$ or (B) $\delta = 0$ with $\gamma(1+x_3) \le \beta(1+(1+x_3)x_2)$. If (A) holds, then $h(b) = \delta+\beta = 1$ forces $\delta = 1$ and $\beta = 0$ since $\delta > 0$. Therefore, $h(b) = 1$ when $b = (\gamma x_1+1)(1+x_3)$ for $0 \le \gamma < 1+(1+x_3)x_2$. If (B) holds, then our same equation forces $\delta = 0$ and $\beta = 1$ when $\gamma(1+x_3) \le 1+1(1+x_3)x_2$, which further implies $0 \le \gamma \le x_2$. This means $h(b) = 1$ when $b = \gamma x_1(1+x_3)+1$ for $0 \le \gamma \le x_2$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$ and $0 < \gamma \leq 1 + (1 + x_3)x_2$, we have $b = \gamma(1 + x_3)$.
- If $x_1 \geq 1$ and $0 \leq \gamma < 1 + (1 + x_3)x_2$, we have $b = (\gamma x_1 + 1)(1 + x_3)$.
- If $x_1 \geq 1$ and $0 \leq \gamma < 1 + (1 + x_3)x_2$ and $\gamma < x_2$, we have $b = \gamma x_1(1 + x_3) + 1$.

Next, we establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1+x_3)(1+x_1(1+(1+x_3)x_2))$. If $q_i = (1+x_3)(1+x_1(1+(1+x_3)x_2))$, then (4.5) is trivially satisfied. Now suppose $q_i = (1 + (1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$. Note that $b \neq 1 + x_3$ since $h(1 + x_3) = 1$. If $b > 1 + x_3$, we set $c = 1 + x_3$ from which it is straightforward to compute that both sides of (4.5) are equal to $bx_2 - (1 + (1 + x_3)x_2) + \left\lfloor \frac{b}{1+x_3} \right\rfloor$. Otherwise, if $b < 1 + x_3$, we may choose $c = 1$ from which it follows that both sides of (4.5) are equal to $(b - 1)x_2$ since $1 \leq b < 1 + x_3$ implies $0 \leq b - 1 < 1 + x_3$. Finally, if $q_i = 1 + (1 + x_3)x_2$, we write $b = \alpha(1 + x_3) + \beta$, where $0 \leq \beta < 1 + x_3$ and $0 \leq \alpha < 1 + x_1(1 + (1 + x_3)x_2)$. Moreover, we write $\alpha = \gamma x_1 + \delta$, where $0 \leq \delta < x_1$ and $0 \leq \gamma \leq 1 + (1 + x_3)x_2$. We consider the following possible cases:

Subcase 1 of 4: Suppose $\beta > 0$ and $\delta > 0$. Then, choosing $c = (\gamma x_1+1)(1+x_3) < (\gamma x_1 + \delta)(1 + x_3) + \beta = b$, it follows that both sides of (4.5) are equal to 0.

Subcase 2 of 4: Suppose $\beta = 0$ and $\delta > 0$. Note that $\delta \neq 1$ since $h((\gamma x_1 + 1)(1 + x_3)) = 1$. Therefore, $2 \leq \delta < x_1$, so we may consider $c = (\gamma x_1 + 1)(1 + x_3) < (\gamma x_1 + \delta)(1 + x_3) = b$. Since $2 \leq \delta < x_1$ and $0 \leq \gamma < 1 + (1 + x_3)x_2$, both sides of (4.5) are equal to 0.

Subcase 3 of 4: Suppose $\beta > 0$ and $\delta = 0$. If $0 \leq \gamma \leq x_2$, then $\beta \neq 1$ since $h(\gamma x_1(1 + x_3) + 1) = 1$. Thus, it must be that $\beta > 1$, thereby allowing us to consider $c = \gamma x_1(1 + x_3) + 1 < \gamma x_1(1 + x_3) + \beta = b$. With this choice of c, it is straightforward to verify both sides of (4.5) will again be equal to 0. Otherwise, if $x_2 < \gamma \leq 1 + (1 + x_3)x_2$, consider $c = x_2x_1(1 + x_3) + 1 < \gamma x_1(1 + x_3) + \beta = b$. Then, the left-hand side of (4.5) simplifies to

$$\gamma - x_2 + \left\lfloor \frac{\beta(1 + (1 + x_3)x_2) - \gamma(1 + x_3)}{(1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))} \right\rfloor,$$

and the right-hand side of (4.5) simplifies to

$$\gamma - x_2 + \left\lfloor \frac{\beta(1 + (1 + x_3)x_2) - \gamma(1 + x_3) - 1}{(1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))} \right\rfloor.$$

Indeed, these two quantities are equivalent because $\beta(1 + (1 + x_3)x_2) \neq \gamma(1 + x_3)$. To see this, assume otherwise. Then, it would follow that $\beta = (\gamma - \beta x_2)(1 + x_3)$. However, this is a contradiction as we assumed $0 < \beta < 1 + x_3$. Therefore, we have that (4.5) is satisfied.

Subcase 4 of 4: Suppose $\beta = \delta = 0$. In this case, $b = \gamma x_1(1 + x_3)$. Moreover, note that $\gamma > 0$ since otherwise, $b = 0$ contradicting our bounds on b. If $x_1 = 1$, then $h(b) = h(\gamma(1 + x_3)) = 1$. Hence, we need only consider when $x_1 > 1$. Since $\gamma > 0$ and $x_1 > 1$, we may consider $c = 1 + x_3 < \gamma x_1(1 + x_3) = b$, from which it is straightforward to find that both sides of (4.5) are equal to $\gamma - 1$.

As these cases cover all possible values for β and δ, this completes our second case.

Case: $q_j = (1+(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2)$. Thus, we consider $1 \leq b \leq (1+(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2)-1$. Again, we will start by identifying those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b}{1+(1+(1+x_3)x_2)x_1} \right\rfloor - x_2 \left\lfloor \frac{b(1+x_3)}{1+(1+x_3)x_2} \right\rfloor.$$

Writing $b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \leq \beta < 1+(1+x_3)x_2$ and $0 \leq \alpha < 1+(1+(1+x_3)x_2)x_1$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1+(1+x_3)x_2)+\beta) \\ &= \alpha+\beta - x_1 \left\lfloor \frac{\alpha(1+(1+x_3)x_2)+\beta}{1+(1+(1+x_3)x_2)x_1} \right\rfloor - x_2 \left\lfloor \frac{\beta(1+x_3)}{1+(1+x_3)x_2} \right\rfloor. \end{aligned}$$

Now, writing $\beta = \gamma x_2 + \delta$, where $0 \leq \delta < x_2$ and $0 \leq \gamma \leq 1+x_3$, and $\alpha = \varepsilon x_1 + \eta$, where $0 \leq \eta < x_1$ and $0 \leq \varepsilon \leq 1+(1+x_3)x_2$, it follows that

$$h(b) = \eta+\delta - x_1 \left\lfloor \frac{\eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon}{1+(1+(1+x_3)x_2)x_1} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor.$$

Since $0 \leq \delta < x_2$ and $0 \leq \gamma \leq 1+x_3$, observe that

$$\left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor = \begin{cases} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Given the bounds on ε, η, γ, and δ, note that $-(1+(1+x_3)x_2) \leq \eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon < 1+(1+(1+x_3)x_2)x_1$. Consequently, it follows that

$$|\eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon| < 1+(1+(1+x_3)x_2)x_1,$$

and this implies that $\left\lfloor \frac{\eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon}{1+(1+(1+x_3)x_2)x_1} \right\rfloor$ is equal to either 0 or -1. To resolve this floor function, we consider the following subcases which analyze the sign of the numerator of its argument. Let n denote that numerator, i.e., $n = \eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon$.

Subcase 1 of 3: Suppose $\eta = 0$ and $\gamma x_2 + \delta \geq \varepsilon$. Then, we have that $n \geq 0$, implying

$$\left\lfloor \frac{\eta(1+(1+x_3)x_2)+\gamma x_2+\delta-\varepsilon}{1+(1+(1+x_3)x_2)x_1} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to

$$h(b) = \delta - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\}.$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2) = x_2$. Thus, if $x_2 = 1$, we have that $h(b) = h(\varepsilon x_1(2 + x_3) + \gamma) = 1$ whenever $\gamma \geq \varepsilon$ with $\gamma > 0$. Otherwise, $h(b) = \delta$, which forces $\delta = 1$, i.e., $h(b) = h(\varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+\delta) = 1$ whenever $\gamma x_2 + \delta \geq \varepsilon$.

Subcase 2 of 3: Suppose $\eta = 0$ and $\gamma x_2 + \delta < \varepsilon$. Then, we have that $n < 0$, implying

$$\left\lfloor \frac{\eta(1 + (1 + x_3)x_2) + \gamma x_2 + \delta - \varepsilon}{1 + (1 + (1 + x_3)x_2)x_1} \right\rfloor = -1.$$

Therefore, our equation for $h(b)$ simplifies to

$$h(b) = \delta + x_1 - x_2 \cdot \begin{Bmatrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{Bmatrix}.$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2) = x_1 + x_2 > 1$. Otherwise, $h(b) = \delta + x_1$, which forces $\delta = 0$ and $x_1 = 1$ since $x_1 \geq 1$, i.e., $h(b) = h(\varepsilon(1 + (1 + x_3)x_2) + \gamma x_2) = 1$ whenever $\gamma x_2 < \varepsilon$.

Subcase 3 of 3: Suppose $\eta \geq 1$. Then, we have that $n \geq 0$, implying

$$\left\lfloor \frac{\eta(1 + (1 + x_3)x_2) + \gamma x_2 + \delta - \varepsilon}{1 + (1 + (1 + x_3)x_2)x_1} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to

$$h(b) = \eta + \delta - x_2 \cdot \begin{Bmatrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{Bmatrix}.$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = h((\varepsilon x_1+\eta)(1+(1+x_3)x_2)+\gamma x_2) = \eta+x_2 \geq 1+x_2 > 1$. Otherwise, $h(b) = \eta + \delta$, which forces $\delta = 0$ and $\eta = 1$ since we assume $\eta \geq 1$, i.e., $h(b) = h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2) = 1$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$, $\eta = \delta = 0$, and $0 \leq \gamma x_2 < \varepsilon \leq 1 + (1 + x_3)x_2$, we have $b = \varepsilon(1 + (1 + x_3)x_2) + \gamma x_2$.
- If $x_2 = 1$, $\eta = \delta = 0$, and $0 \leq \varepsilon \leq \gamma \leq 1 + x_3$ with $\gamma > 0$, we have $b = \varepsilon x_1(2 + x_3) + \gamma$.
- If $x_1, x_2 \geq 1$, $\eta = 0$, $\delta = 1$, and $0 \leq \varepsilon \leq \gamma x_2 + 1 \leq 1 + (1 + x_3)x_2$, we have $b = \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1$.
- If $x_1, x_2 \geq 1$, $\eta = 1$, $\delta = 0$, $0 \leq \gamma \leq 1 + x_3$, and $0 \leq \varepsilon \leq 1 + (1 + x_3)x_2$, we have $b = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1 + (1 + (1 + x + 3)x_2)x_1)(1 + (1 + x_3)x_2)$. If $q_i = (1 + (1 + (1 + x + 3)x_2)x_1)(1 + (1 + x_3)x_2)$, the result is trivial. If $q_i = 1 + (1 + x_3)x_2$, we write

$b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \le \beta < 1+(1+x_3)x_2$ and $0 \le \alpha < 1+(1+(1+x_3)x_2)x_1$. Furthermore, we write $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma \le 1 + x_3$, and we write $\alpha = \varepsilon x_1 + \eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon \le 1 + (1 + x_3)x_2$. We consider the following possible cases:

Subcase 1 of 4: Suppose $\eta > 0$ and $\delta > 0$. We consider $c = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2 < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 + \delta = b$. Since $0 < \eta < x_1$ and $0 \le \delta < x_2$, it follows that $0 < (\eta - 1)(1 + (1 + x_3)x_2) + \delta < x_1(1 + (1 + x_3)x_2)$. Therefore, it is straightforward to verify that both sides of (4.5) are equal to 0.

Subcase 2 of 4: Suppose $\eta = 0$ and $\delta > 0$. If $\gamma x_2 + 1 \ge \varepsilon$, note that $\delta \ne 1$ since otherwise, $h(b) = h(\varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+1) = 1$. Thus, we have that $\delta > 1$, and we consider $c = \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+1 < \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+\delta = b$. Given that $1 < \delta < x_2$ implies $\gamma x_2 + \delta > \gamma x_2 + 1 \ge \varepsilon$, it is straightforward to verify that our choice of c gives that both sides of (4.5) are equal to 0. Otherwise, if $\gamma x_2 + 1 < \varepsilon$ (and hence, $\varepsilon > 1$), we consider $c = 1 + x_1(1 + (1 + x_3)x_2) \le \delta + x_1(1 + (1 + x_3)x_2) < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + \delta = b$. With this choice of c, it is straightforward to verify both sides of (4.5) are equal to

$$\varepsilon - 1 + \left\lfloor \frac{\gamma x_2 + \delta - \varepsilon}{1 + (1 + (1 + x_3)x_2)x_1} \right\rfloor .$$

Subcase 3 of 4: Suppose $\eta > 0$ and $\delta = 0$. Note that $\eta \ne 1$ since otherwise, $h(b) = h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2) = 1$. Thus, we have that $\eta > 1$, and we consider $c = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2 < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 = b$. Since $1 < \eta < x_1$, it is straightforward to verify that both sides of (4.5) are equal to 0.

Subcase 4 of 4: Suppose $\eta = \delta = 0$. Further suppose $\gamma x_2 < \varepsilon$ (and hence, $\varepsilon > 0$). If $x_1 = 1$, then $h(b) = h(\varepsilon(1 + (1 + x_3)x_2) + \gamma x_2) = 1$, so we may assume $x_1 > 1$. Consider $c = 1 + (1 + x_3)x_2 < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 = b$. Since $x_1 > 1$ and $\gamma x_2 < \varepsilon$, it follows that $-x_1(1 + (1 + x_3)x_2) \le \gamma x_2 - \varepsilon - (1 + (1 + x_3)x_2) < 0$. Given this, it is straightforward to verify that both sides of (4.5) are equal to $\varepsilon - 1$. Now, suppose $\gamma x_2 > \varepsilon$. Note that $\gamma \ne 0$ since otherwise, $\varepsilon < 0$ contradicting our initial bounds on ε. Thus, we have that $\gamma > 0$. Moreover, if $x_2 = 1$, it follows that $h(b) = h(\varepsilon x_1(2 + x_3) + \gamma) = 1$, so we may assume $x_2 > 1$. Taking $c = 1$, the inequality $\gamma x_2 > \varepsilon$ readily implies that both sides of (4.5) are equal to ε. Finally, suppose $\gamma x_2 = \varepsilon$. Note that neither γ nor ε can be equal to 0 since otherwise, we would have $\eta = \delta = \gamma = \varepsilon = 0$, implying $b = 0$. This, of course, contradicts the bounds on b. Moreover, we may again assume $x_2 > 1$ (and thus, $\varepsilon > 1$) since otherwise, $h(b) = h(\varepsilon x_1(2+x_3)+\gamma) = 1$. Since $\gamma x_2 = \varepsilon$, observe that $b = \varepsilon(1+x_1(1+(1+x_3)x_2))$. As such, we may consider $c = 1 + x_1(1 + (1 + x_3)x_2)$ which is strictly less than b since $\varepsilon > 1$. This choice of c readily gives that both sides of (4.5) are equal to $\varepsilon - 1$.

Finally, if $q_i = (1 + x_3)(1 + x_1(1 + (1 + x_3)x_2))$, we begin again by writing $b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \le \beta < 1+(1+x_3)x_2$ and $0 \le \alpha < 1+(1+(1+x_3)x_2)x_1$. Furthermore, we write $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma \le 1 + x_3$, and we write $\alpha = \varepsilon x_1 + \eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon \le 1 + (1 + x_3)x_2$. We consider the following possible cases:

Subcase 1 of 3: Suppose $\delta > 1$. Combined with our bounds on γ, this implies that $0 \leq (\delta - 1)(1 + x_3) - \gamma < 1 + (1 + x_3)x_2$. Taking $c = 1$, it follows that both sides of (4.5) are equal to $\alpha(1 + x_3) + \gamma$.

Subcase 2 of 3: Suppose $\delta = 1$. If $\gamma x_2 + 1 \geq \varepsilon$, note that $\eta \neq 0$ since otherwise, $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1) = 1$. Thus, $\eta \geq 1$, and we may consider $c = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 + 1 = b$. Since $\gamma \leq 1 + x_3$, it is straightforward to show both sides of (4.5) are equal to $(\eta - 1)(1 + x_3) + \gamma$. On the other hand, if $\gamma x_2 + 1 < \varepsilon$ (and hence, $\varepsilon > 1$), we may consider $c = 1 + (1 + x_3)x_2 < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 + 1 = b$, from which it follows that both sides of (4.5) are equal to $(\varepsilon x_1 + \eta - 1)(1 + x_3) + \gamma$.

Subcase 3 of 3: Suppose $\delta = 0$. Note that $\eta \neq 1$ since otherwise, $h(b) = 1$. This lends itself to two possibilities: (A) $\eta = 0$ or (B) $\eta > 1$. If (A) holds, we first suppose $\gamma x_2 < \varepsilon$ (and hence, $\varepsilon > 0$). If $x_1 = 1$, then $h(b) = 1$, so we may assume $x_1 > 1$. Since $\varepsilon > 0$ and $x_1 > 1$, we may consider $c = 1 + (1 + x_3)x_2 < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 = b$. For this choice of c, it readily follows that both sides of (4.5) are equal to

$$(\varepsilon x_1 - 1)(1 + x_3) + \left\lfloor \frac{\gamma x_2}{1 + (1 + x_3)x_2} \right\rfloor .$$

Now, suppose $\gamma x_2 > \varepsilon$. Since $\varepsilon \geq 0$ by construction, this inequality implies $\gamma \neq 0$. Moreover, note that if $x_2 = 1$, then $h(b) = 1$. Thus, we may assume $x_2 > 1$, and we simply consider $c = 1$. Given that $0 < \gamma \leq 1 + x_3$ and $x_2 > 1$, it follows that $0 < \gamma + (1 + x_3) < 1 + (1 + x_3)x_2$. Therefore, with $c = 1$, we find that both sides of (4.5) are equal to $\varepsilon x_1(1 + x_3) + \gamma - 1$. Finally, suppose $\gamma x_2 = \varepsilon$. Since $b \neq 0$, this equality implies that both γ and ε cannot be 0. Moreover, we may again assume $x_2 > 1$ since $x_2 = 1$ would imply $h(b) = 1$. Since $\gamma x_2 = \varepsilon$, it follows that $b = \varepsilon(1 + x_1(1 + (1 + x_3)x_2))$, and we also get that $\varepsilon > 1$ since we assume $x_2 > 1$. Combining, we may consider $c = 1 + x_1(1 + (1 + x_3)x_2) < \varepsilon(1 + x_1(1 + (1 + x_3)x_2)) = b$. As before, the inequality $0 < \gamma + (1 + x_3) < 1 + (1 + x_3)x_2$ still holds in this case, from which it is straightforward to verify both sides of (4.5) are equal to $(\varepsilon - 1)x_1(1 + x_3) + \gamma - 1$.

On the other hand, if (B) holds, we have that $\eta > 1$. Therefore, we consider $c = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2 < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 = b$, from which it follows that both sides of (4.5) simplify to $(\eta - 1)(1 + x_3)$. In any case, we find that both sides of (4.5) are equivalent for each possible q_i, thereby completing our third and final case. Thus, we have established IDP for **r**-vectors of type (v).

4.2.6 Proof of IDP for type (vi) in Theorem 4.1.11

We next verify IDP for **r**-vectors of type (vi) using Theorem 4.1.5. Again, we must consider three cases corresponding to three possible values of q_j.

Case: $q_j = (1 + x_3)(1 + (1 + x_3)x_2)$. We first identify those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_2 \left\lfloor \frac{b}{1 + (1 + x_3)x_2} \right\rfloor - x_3 \left\lfloor \frac{b}{1 + x_3} \right\rfloor ,$$

where $1 \leq b \leq (1+x_3)(1+(1+x_3)x_2)-1$. Writing $b = \alpha(1+x_3)+\beta$, where $0 \leq \beta < 1+x_3$ and $0 \leq \alpha < 1+(1+x_3)x_2$ for $\alpha, \beta \in \mathbb{Z}$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1+x_3)+\beta) \\ &= \alpha + \beta - x_2 \left\lfloor \frac{\alpha(1+x_3)+\beta}{1+(1+x_3)x_2} \right\rfloor. \end{aligned}$$

We can now further divide into cases: either we have $\alpha = (1+x_3)x_2$ or we have $\alpha = \gamma x_2 + \delta$ where $0 \leq \gamma < 1+x_3$ and $0 \leq \delta < x_2$, which yields

$$\begin{aligned} h(b) &= h((\gamma x_2 + \delta)(1+x_3)+\beta) \\ &= \delta + \beta - x_2 \left\lfloor \frac{\delta(1+x_3)+\beta-\gamma}{1+(1+x_3)x_2} \right\rfloor. \end{aligned}$$

For $\alpha \neq (1+x_3)x_2$, let $n = \delta(1+x_3)+\beta-\gamma$. Observe that since $0 \leq \delta(1+x_3)+\beta < 1+(1+x_3)x_2$ and $0 \leq \gamma < 1+x_3$, it follows that $|n| < 1+(1+x_3)x_2$. Thus, $\left\lfloor \frac{\delta(1+x_3)+\beta-\gamma}{1+(1+x_3)x_2} \right\rfloor$ is equal to either 0 or -1.

Subcase 1 of 4: Suppose $\alpha = (1+x_3)x_2$. Since $0 \leq \beta < 1+x_3$, we have

$$\begin{aligned} h((1+x_3)^2 x_2 + \beta) &= (1+x_3)x_2 + \beta - (1+x_3)x_2 - x_2 \left\lfloor \frac{\beta-(1+x_3)}{1+(1+x_3)x_2} \right\rfloor \\ &= \beta + x_2. \end{aligned}$$

If this is equal to 1, then it must be that $\beta = 0$ and $x_2 = 1$. Thus, if $x_2 = 1$, we have that $h((1+x_3)^2) = 1$.

For the next three subcases, we assume $\alpha \neq (1+x_3)x_2$, so we may write $\alpha = \gamma x_2 + \delta$ where $0 \leq \delta < x_2$ and $0 \leq \gamma < 1+x_3$.

Subcase 2 of 4: Suppose $\delta = 0$ and $\beta \geq \gamma$. Then, we have that $n \geq 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)+\beta-\gamma}{1+(1+x_3)x_2} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \beta$, which forces $\beta = 1$, i.e., $h(b) = h(\gamma x_2(1+x_3)+1) = 1$ whenever $0 \leq \gamma \leq 1$.

Subcase 3 of 4: Suppose $\delta = 0$ and $\beta < \gamma$. Then, we have that $n < 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)+\beta-\gamma}{1+(1+x_3)x_2} \right\rfloor = -1.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \beta + x_2$, which forces $\beta = 0$ and $x_2 = 1$ since $x_2 \geq 1$, i.e., $h(b) = h(\gamma(1+x_3)) = 1$ for $0 < \gamma < 1+x_3$.

Subcase 4 of 4: Suppose $\delta \geq 1$. Then, since $0 \leq \gamma < 1+x_3$, we have that $n \geq 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)+\beta-\gamma}{1+(1+x_3)x_2} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \delta + \beta$, which forces $\beta = 0$ and $\delta = 1$ since we assume $\delta \geq 1$, i.e., $h(b) = h((\gamma x_2 + 1)(1+x_3)) = 1$ for $0 \leq \gamma < 1+x_3$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_2 = 1$ and $0 < \gamma \leq 1 + x_3$, we have $b = \gamma(1 + x_3)$.
- If $x_2 \geq 1$, $\delta = 0$, $\beta = 1$, and $0 \leq \gamma \leq 1$, we have $b = \gamma x_2(1 + x_3) + 1$.
- If $x_2 \geq 1$, $\delta = 1$, $\beta = 0$, and $0 \leq \gamma < 1 + x_3$, we have $b = (\gamma x_2 + 1)(1 + x_3)$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1 + x_3)(1 + (1 + x_3)x_2)$. If $q_i = (1 + x_3)(1 + (1 + x_3)x_2)$, the result is trivial. If $q_i = (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$, we write $b = \alpha(1 + x_3) + \beta$, where $0 \leq \beta < 1 + x_3$ and $0 \leq \alpha < 1 + (1 + x_3)x_2$. Note that $b \neq 1 + x_3$ since $h(1 + x_3) = 1$. If $b > 1 + x_3$, we set $c = 1 + x_3$ from which it is straightforward to compute that both sides of (4.5) are equal to $((\alpha - 1)(1 + x_3) + \beta)(1 + (1 + x_3)x_2)x_1 + \alpha - 1$. Otherwise, if $b < 1 + x_3$, note that $\alpha = 0$ and hence $b = \beta$. To satisfy (4.4), we need only consider $1 < \beta < 1 + x_3$. Thus, we may choose $c = 1$ from which it follows that both sides of (4.5) are equal to $(\beta - 1)(1 + (1 + x_3)x_2)x_1$ since $1 < b < 1 + x_3$ implies $1 \leq \beta - 1 < 1 + x_3$. Finally, if $q_i = (1 + x_3)(1 + x_1(1 + x_3)(1 + (1 + x_3)x_2))$, we again write $b = \alpha(1 + x_3) + \beta$, where $0 \leq \beta < 1 + x_3$ and $0 \leq \alpha < 1 + (1 + x_3)x_2$. Moreover, in the case that $\alpha \neq (1 + x_3)x_2$, we write $\alpha = \gamma x_2 + \delta$, where $0 \leq \delta < x_2$ and $0 \leq \gamma < (1 + x_3)$. We consider the following possible cases:

Subcase 1 of 5: Suppose $\alpha = (1+x_3)x_2$. If $x_2 = 1$, then $\beta \neq 0$ since $h((1+x_3)^2) = 1$. Thus, we may consider $c = (1 + x_3)^2 < (1 + x_3)^2 + \beta = b$. With this choice of c, it is straightforward to verify that both sides of (4.5) are equal to $\beta x_1(1 + x_3)$. Otherwise, if $x_2 > 1$, we consider $c = 1 + x_3 < (1 + x_3)^2 x_2 + \beta = b$. Since $x_2 > 1$ implies $-x_2(1 + x_3) < \beta - 2(1 + x_3) < 0$, it is straightforward to verify that both sides of (4.5) are equal to $(b - (1 + x_3))x_1(1 + x_3) + x_3$.

Subcase 2 of 5: Suppose $\alpha \neq (1 + x_3)x_2$ with $\delta > 0$ and $\beta > 0$. Then, choosing $c = (\gamma x_2 + 1)(1 + x_3) < (\gamma x_2 + \delta)(1 + x_3) + \beta = b$, it follows that both sides of (4.5) are equal to $((\delta - 1)(1 + x_3) + \beta)x_1(1 + x_3)$.

Subcase 3 of 5: Suppose $\alpha \neq (1 + x_3)x_2$ with $\delta > 0$ and $\beta = 0$. Note that $\delta \neq 1$ since $h((\gamma x_2 + 1)(1 + x_3)) = 1$. Therefore, $2 \leq \delta < x_2$, so we may consider $c = (\gamma x_2 + 1)(1 + x_3) < (\gamma x_2 + \delta)(1 + x_3) = b$. Given this choice, we find that both sides of (4.5) are equal to $(\delta - 1)(1 + x_3)x_1(1 + x_3)$.

Subcase 4 of 5: Suppose $\alpha \neq (1 + x_3)x_2$ with $\delta = 0$ and $\beta > 0$. If $0 \leq \gamma \leq 1$, then $\beta \neq 1$ since $h(\gamma x_2(1 + x_3) + 1) = 1$. Thus, it must be that $\beta > 1$, thereby allowing us to consider $c = \gamma x_2(1 + x_3) + 1 < \gamma x_2(1 + x_3) + \beta = b$. With this choice of c, it is straightforward to verify both sides of (4.5) are equal to $(\beta - 1)x_1(1 + x_3)$. Otherwise, if $1 < \gamma < 1 + x_3$, consider $c = 1 + (1 + x_3)x_2 < \gamma x_2(1 + x_3) + \beta = b$. Then, both sides of (4.5) are equal to

$$((\gamma - 1)x_2(1 + x_3) + \beta - 1)x_1(1 + x_3) + \gamma - 1 + \left\lfloor \frac{\beta - \gamma}{1 + (1 + x_3)x_2} \right\rfloor.$$

Subcase 5 of 5: Suppose $\alpha \neq (1 + x_3)x_2$ with $\delta = \beta = 0$. In this case, $b = \gamma x_2(1 + x_3)$. Moreover, note that $\gamma > 0$ since otherwise, $b = 0$ contradicting our bounds on b. If $x_2 = 1$, then $h(b) = h(\gamma(1 + x_3)) = 1$. Hence, we need only consider when $x_2 > 1$. Since $\gamma > 0$ and $x_2 > 1$, we may take $c = 1 + x_3 < \gamma x_2(1 + x_3) = b$, from which it is straightforward to find that both sides of (4.5) are equal to $(\gamma x_2 - 1)(1 + x_3)^2 x_1 + \gamma - 1$.

This completes our first case.

Case: $q_j = (1 + x_3)(1 + x_1(1 + x_3)(1 + (1 + x_3)x_2))$. We first identify those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1 + (1 + x_3)x_2)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor - x_3 \left\lfloor \frac{b}{1 + x_3} \right\rfloor .$$

Writing $b = \alpha(1 + x_3) + \beta$, where $0 \leq \beta < 1 + x_3$ and $0 \leq \alpha \leq 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1 + x_3) + \beta) \\ &= \alpha + \beta - x_1 \left\lfloor \frac{(\alpha(1 + x_3) + \beta)(1 + (1 + x_3)x_2)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor . \end{aligned}$$

We can now further divide into cases: either we have $\alpha = x_1(1 + x_3)(1 + (1 + x_3)x_2)$ or we have $\alpha = \gamma x_1 + \delta$ where $0 \leq \gamma < (1 + x_3)(1 + (1 + x_3)x_2)$ and $0 \leq \delta < x_1$. If $\alpha = x_1(1 + x_3)(1 + (1 + x_3)x_2)$, then since $0 \leq \beta < 1 + x_3$, we have

$$\begin{aligned} h(b) &= \beta - x_1 \left\lfloor \frac{\beta(1 + (1 + x_3)x_2) - (1 + x_3)(1 + (1 + x_3)x_2)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor \\ &= \beta + x_1. \end{aligned}$$

If this is equal to 1, then it must be that $\beta = 0$ and $x_1 = 1$. Thus, if $x_1 = 1$, we have that $h((1 + x_3)^2(1 + (1 + x_3)x_2)) = 1$. Otherwise, if $\alpha \neq x_1(1 + x_3)(1 + (1 + x_3)x_2)$, we write $\alpha = \gamma x_1 + \delta$ where $0 \leq \gamma < (1 + x_3)(1 + (1 + x_3)x_2)$ and $0 \leq \delta < x_1$. Thus, $h(b)$ simplifies as follows

$$\begin{aligned} h(b) &= h((\gamma x_1 + \delta)(1 + x_3) + \beta) \\ &= \delta + \beta - x_1 \left\lfloor \frac{(\delta(1 + x_3) + \beta)(1 + (1 + x_3)x_2) - \gamma}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2} \right\rfloor . \end{aligned}$$

For $\alpha \neq x_1(1 + (1 + x_3)x_2)$, observe that since $0 \leq \delta(1 + x_3)(1 + (1 + x_3)x_2) \leq (x_1 - 1)(1 + x_3)(1 + (1 + x_3)x_2)$ and $0 \leq \beta(1 + (1 + x_3)x_2) < (1 + x_3)(1 + (1 + x_3)x_2)$ with $0 \leq \gamma < (1+x_3)(1+(1+x_3)x_2)$, we have that $0 \leq (\delta(1+x_3)+\beta)(1+(1+x_3)x_2) < x_1(1+x_3)(1+(1+x_3)x_2)$. Let $n = (\delta(1+x_3)+\beta)(1+(1+x_3)x_2)-\gamma$. The inequalities above readily imply $|n| < 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)$. Thus,

$$\left\lfloor \frac{\delta(1 + x_3)(1 + (1 + x_3)x_2) + \beta(1 + (1 + x_3)x_2 - \gamma}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor$$

is equal to either 0 or -1. We further write $\gamma = \varepsilon(1 + (1 + x_3)x_2) + \eta$, where $0 \leq \eta < 1 + (1 + x_3)x_2$ and $0 \leq \varepsilon < 1 + x_3$. Then, $h(b)$ becomes

$$h(b) = \delta + \beta - x_1 \left\lfloor \frac{(\delta(1 + x_3) + \beta)(1 + (1 + x_3)x_2) - (\varepsilon(1 + (1 + x_3)x_2) + \eta)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor .$$

Subcase 1 of 4: Suppose $\delta = 0$ and $\beta > \varepsilon$. Then, we have that $n > 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \beta$, which forces $\beta = 1$. Note that $\beta = 1$ implies $\varepsilon = 0$ as $\beta > \varepsilon$. Thus, we have $h(b) = h(\eta x_1(1 + x_3) + 1) = 1$ whenever $0 \leq \eta \leq (1 + x_3)x_2$.

Subcase 2 of 4: Suppose $\delta = 0$ and $\beta < \varepsilon$. Note that $\varepsilon > 0$ since $\beta < \varepsilon$. Then, we have that $n < 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = -1.$$

Consequently, our equation for $h(b)$ simplifies to $h(b) = \beta + x_1$, which forces $\beta = 0$ and $x_1 = 1$ since $x_1 \geq 1$, i.e., $h(b) = h((\varepsilon(1 + (1 + x_3)x_2) + \eta)(1 + x_3)) = 1$ whenever $0 < \varepsilon < 1 + x_3$ and $0 \leq \eta \leq (1 + x_3)x_2$.

Subcase 3 of 4: Suppose $\delta = 0$ and $\beta = \varepsilon$. If $\eta > 0$, then we have that $n < 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = -1.$$

In this case, $h(b)$ simplifies to $h(b) = \beta + x_1$, which again forces $\beta = 0$ and $x_1 = 1$. Since $\beta = \varepsilon$, it follows that $\varepsilon = 0$, and so we have that $h(b) = h(\eta(1 + x_3) = 1$ whenever $0 < \eta \leq (1+x_3)x_2$. Otherwise, if $\eta = 0$, then we have that $n = 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \beta$, which forces $\beta = 1$ (and thus, $\varepsilon = 1$), i.e., $h(b) = h((1 + (1 + x_3)x_2)x_1(1 + x_3) + 1) = 1$.

Subcase 4 of 4: Suppose $\delta \geq 1$. Then, we have that $n \geq 0$, implying

$$\left\lfloor \frac{\delta(1+x_3)(1+(1+x_3)x_2)+\beta(1+(1+x_3)x_2-\gamma}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0.$$

Therefore, our equation for $h(b)$ simplifies to $h(b) = \delta + \beta$, which forces $\delta = 1$ and $\beta = 0$ since $\delta \geq 1$. That is, we have $h(b) = h(((\varepsilon(1+(1+x_3)x_2)+\eta)x_1+1)(1+x_3)) = 1$ whenever $0 \leq \varepsilon < 1 + x_3$ and $0 \leq \eta \leq (1 + x_3)x_2$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$, $\beta = 0$, and $\alpha = (1 + x_3)(1 + (1 + x_3)x_2)$, we have $b = (1 + x_3)^2(1 + (1 + x_3)x_2$.
- If $x_1 = 1$, $\beta = \delta = 0$, $0 < \varepsilon < 1 + x_3$, and $0 \leq \eta \leq (1 + x_3)x_2$, we have $b = (\eta(1 + (1 + x_3)x_2) + \eta)(1 + x_3)$.
- If $x_1 = 1$, $\beta = \delta = \varepsilon = 0$, and $0 < \eta \leq (1 + x_3)x_2$, we have $b = \eta(1 + x_3)$.

- If $x_1 \geq 1$, $\delta = \varepsilon = 0$, $\beta = 1$, and $0 \leq \eta \leq (1+x_3)x_2$, we have $b = \eta x_1(1+x_3)+1$.
- If $x_1 \geq 1$, $\delta = \eta = 0$, and $\beta = \varepsilon = 1$, we have $b = (1+(1+x_3)x_2)x_1(1+x_3)+1$.
- If $x_1 \geq 1$, $\delta = 1$, $\beta = 0$, $0 \leq \varepsilon < 1+x_3$, and $0 \leq \eta \leq (1+x_3)x_2$, we have $b = ((\varepsilon(1+(1+x_3)x_2)+\eta)x_1+1)(1+x_3)$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1+x_3)(1+x_1(1+x_3)(1+(1+x_3)x_2))$. If $q_i = (1+x_3)(1+x_1(1+x_3)(1+(1+x_3)x_2))$, the result is trivial. If $q_i = (1+(1+x_3)(1+(1+x_3)x_2)x_1)(1+(1+x_3)x_2)$, note that $b \neq 1+x_3$ since $h(1+x_3) = 1$. If $b > 1+x_3$, we set $c = 1+x_3$ from which it is straightforward to compute that both sides of (4.5) are equal to $bx_2 - (1+(1+x_3)x_2) + \left\lfloor \frac{b}{1+x_3} \right\rfloor$. Otherwise, if $b < 1+x_3$, we may choose $c = 1$ from which it follows that both sides of (4.5) are equal to $(b-1)x_2$ since $1 < b < 1+x_3$ implies $1 \leq b-1 < 1+x_3$. Finally, if $q_i = (1+x_3)(1+(1+x_3)x_2)$, we again write $b = \alpha(1+x_3)+\beta$, where $0 \leq \beta < 1+x_3$ and $0 \leq \alpha < 1+x_1(1+x_3)(1+(1+x_3)x_2)$. Moreover, in the case that $\alpha \neq x_1(1+x_3)(1+(1+x_3)x_2)$, we write $\alpha = \gamma x_1 + \delta$ with $\gamma = (\varepsilon(1+(1+x_3)x_2)+\eta)$, where $0 \leq \delta < x_1$, $0 \leq \varepsilon < (1+x_3)$, and $0 \leq \eta \leq (1+x_3)x_2$. We consider the following possible cases:

Subcase 1 of 5: Suppose $\alpha = x_1(1+x_3)(1+(1+x_3)x_2)$. If $x_1 = 1$, note that $\beta \neq 0$ since $h((1+x_3)^2(1+(1+x_3)x_2)) = 1$. Thus, we may consider $c = (1+x_3)^2(1+(1+x_3)x_2) < (1+x_3)^2(1+(1+x_3)x_2)+\beta = b$. With this choice of c, it is straightforward to verify that both sides of (4.5) are equal to 0. Otherwise, if $x_1 > 1$, we consider $c = 1$. Since $x_1 > 1$ implies $-x_1(1+x_3)(1+(1+x_3)x_2) < (\beta-x_3-2)(1+(1+x_3)x_2) < 0$, it is straightforward to verify that both sides of (4.5) are equal to $(1+x_3)(1+(1+x_3)x_2)-1$.

Subcase 2 of 5: Suppose $\alpha \neq x_1(1+x_3)(1+(1+x_3)x_2)$ with $\delta > 0$ and $\beta > 0$. Then, choosing $c = (\gamma x_1+1)(1+x_3) < (\gamma x_1+\delta)(1+x_3)+\beta = b$, it follows that both sides of (4.5) are equal to 0 since $0 \leq \delta - 1 < x_1 - 1$ and $0 < \beta < 1+x_3$ together imply $0 < (\delta-1)(1+x_3)+\beta < x_1(1+x_3)$.

Subcase 3 of 5: Suppose $\alpha \neq x_1(1+x_3)(1+(1+x_3)x_2)$ with $\delta > 0$ and $\beta = 0$. Note that $\delta \neq 1$ since $h((\gamma x_1+1)(1+x_3)) = 1$. Therefore, $2 \leq \delta < x_2$, so we may consider $c = (\gamma x_1+1)(1+x_3) < (\gamma x_1+\delta)(1+x_3) = b$. Given this choice, we find that both sides of (4.5) are equal to 0 since $0 < \delta < x_1$.

Subcase 4 of 5: Suppose $\alpha \neq x_1(1+x_3)(1+(1+x_3)x_2)$ with $\delta = 0$ and $\beta > 0$. If $\beta > \varepsilon$, then $\beta \neq 1$ since otherwise, $\beta = 1$ would force $\varepsilon = 0$ and $h(\eta x_1(1+x_3)+1) = 1$. Thus, it must be that $\beta > 1$, thereby allowing us to consider $c = \eta x_1(1+x_3) < (\varepsilon(1+(1+x_3)x_2)+\eta)x_1(1+x_3)+\beta = b$. With this choice of c and since $\beta - 1 \geq \varepsilon$ with $\beta > 1$, it is straightforward to verify both sides of (4.5) are equal to $\varepsilon(1+(1+x_3)x_2)$. Now, if $\beta < \varepsilon$, we have that $\varepsilon > 1$ since we assumed $\beta > 0$. Therefore, we consider $c = (1+(1+x_3)x_2)x_1(1+x_3)+1 < (\varepsilon(1+(1+x_3)x_2)+\eta)x_1(1+x_3)+\beta = b$. Then, both sides of (4.5) are equal to $(\varepsilon-1)(1+(1+x_3)x_2)+\eta-1$. Lastly, if $\beta = \varepsilon$, we consider two possibilities. If $\eta > 0$, we may again choose $c = (1+(1+x_3)x_2)x_1(1+x_3)+1 < (\varepsilon(1+(1+x_3)x_2)+\eta)x_1(1+x_3)+\beta = b$ and find that both sides of (4.5) are equal to $(\varepsilon-1)(1+(1+x_3)x_2)+\eta-1$. Otherwise, if $\eta = 0$, note that $\varepsilon = \beta \neq 1$ since otherwise, $h(b) = h((1+(1+x_3)x_2)x_1(1+x_3)+1) = 1$.

Therefore, the same value of c, namely $c = (1 + (1 + x_3)x_2)x_1(1 + x_3) + 1$, will again be strictly less than b, from which it follows that both sides of (4.5) are equal to $(\varepsilon - 1)(1 + (1 + x_3)x_2) + \eta - 1$.

Subcase 5 of 5: Suppose $\alpha \neq x_1(1 + x_3)(1 + (1 + x_3)x_2)$ with $\delta = 0$ and $\beta = 0$. In this case, $b = \gamma x_1(1 + x_3)$. Moreover, note that $\gamma > 0$ since otherwise, $b = 0$ contradicting our bounds on b. If $x_1 = 1$, then $h(b) = h(\gamma(1 + x_3)) = 1$. Hence, we need only consider when $x_1 > 1$. Since $\gamma > 0$ and $x_2 > 1$, we may take $c = 1 + x_3 < \gamma x_1(1 + x_3) = b$, from which it is straightforward to find that both sides of (4.5) are equal to $\gamma - 1$.

This completes our second case.

Case: $q_j = (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$. Thus, we consider $1 \leq b \leq (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2) - 1$. Again, we will start by identifying those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1 + x_3)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{b(1 + x_3)}{1 + (1 + x_3)x_2} \right\rfloor .$$

Writing $b = \alpha(1 + (1 + x_3)x_2) + \beta$, where $0 \leq \beta < 1 + x_3$ and $0 \leq \alpha < 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)$ for $\alpha, \beta \in \mathbb{Z}$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1 + (1 + x_3)x_2) + \beta) \\ &= \alpha + \beta - x_1 \left\lfloor \frac{(\alpha(1 + (1 + x_3)x_2) + \beta)(1 + x_3)}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\beta(1 + x_3)}{1 + (1 + x_3)x_2} \right\rfloor . \end{aligned}$$

There are two different possibilities for both α and β: either $\alpha = x_1(1 + x_3)(1 + (1 + x_3)x_2)$ or $\alpha = \varepsilon x_1 + \eta$ where $0 \leq \eta < x_1$ and $0 \leq \varepsilon < (1 + x_3)(1 + (1 + x_3)x_2)$, and either $\beta = (1 + x_3)x_2$ or $\beta = \gamma x_2 + \delta$ where $0 \leq \delta < x_2$ and $0 \leq \gamma < 1 + x_3$. We consider the following subcases.

Subcase 1 of 4: Suppose $\alpha = x_1(1 + x_3)(1 + (1 + x_3)x_2)$ and $\beta = (1 + x_3)x_2$. Then,

$$\begin{aligned} h(b) &= x_1(1 + x_3)(1 + (1 + x_3)x_2) + (1 + x_3)x_2 \\ &\quad - x_1 \left\lfloor \frac{x_1(1 + x_3)^2(1 + (1 + x_3)x_2)^2 + (1 + x_3)^2 x_2}{1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{(1 + x_3)^2 x_2}{1 + (1 + x_3)x_2} \right\rfloor \\ &= x_1 + x_2 \\ &> 1. \end{aligned}$$

Subcase 2 of 4: Suppose $\alpha = x_1(1 + x_3)(1 + (1 + x_3)x_2)$ and $\beta \neq (1 + x_3)x_2$.

Writing $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma < 1 + x_3$, we have that

$$\begin{aligned}
h(b) &= x_1(1+x_3)(1+(1+x_3)x_2) + \gamma x_2 + \delta \\
&\quad - x_1 \left\lfloor \frac{(x_1(1+x_3)(1+(1+x_3)x_2)^2 + \beta)(1+x_3)}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{(\gamma x_2 + \delta)(1+x_3)}{1+(1+x_3)x_2} \right\rfloor \\
&= \delta - x_1 \left\lfloor \frac{\beta(1+x_3) - (1+x_3)(1+(1+x_3)x_2)}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3) - \gamma}{1+(1+x_3)x_2} \right\rfloor \\
&= \delta + x_1 - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\} .
\end{aligned}$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = x_1 + x_2 > 1$. Otherwise, we have that $h(b) = \delta + x_1$. For this to be equal to 1, it must be the case that $\delta = 0$ and $x_1 = 1$ since $x_1 \ge 1$. Moreover, note that $\beta = 0$ here since $\delta = 0$ implies $\gamma = 0$ (otherwise, we are in the previous case that $\delta = 0$ and $\gamma > 0$). Therefore, when $x_1 = 1$, we have that $h((1+x_3)(1+(1+x_3)x_2)^2) = 1$.

Subcase 3 of 4: Suppose $\alpha \ne x_1(1+x_3)(1+(1+x_3)x_2)$ and $\beta = (1+x_3)x_2$. Writing $\alpha = \varepsilon x_1 + \eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon < (1+x_3)(1+(1+x_3)x_2)$, we have that

$$\begin{aligned}
h(b) &= \varepsilon x_1 + \eta + (1+x_3)x_2 - x_1 \left\lfloor \frac{((\varepsilon x_1 + \eta)(1+(1+x_3)x_2) + (1+x_3)x_2)(1+x_3)}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor \\
&\quad - x_2 \left\lfloor \frac{(1+x_3)^2 x_2}{1+(1+x_3)x_2} \right\rfloor \\
&= \eta - x_1 \left\lfloor \frac{(\eta(1+(1+x_3)x_2) + (1+x_3)x_2)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor + x_2 \,.
\end{aligned}$$

Given that $0 \le \eta < x_1$ and $0 \le \varepsilon < (1+x_3)(1+(1+x_3)x_2)$, observe that

$$|(\eta(1+(1+x_3)x_2) + (1+x_3)x_2)(1+x_3) - \varepsilon| < 1 + x_1(1+x_3)(1+(1+x_3)x_2).$$

Therefore, $\left\lfloor \frac{(\eta(1+(1+x_3)x_2)+(1+x_3)x_2)(1+x_3)-\varepsilon}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor$ is equal to either 0 or -1. If

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + (1+x_3)x_2)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = -1 \,,$$

then $h(b) = \eta + x_1 + x_2 > 1$. Otherwise, if

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + (1+x_3)x_2)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0 \,,$$

then either (A) $0 < \eta < x_1$ or (B) $\eta = 0$ with $0 \le \varepsilon \le (1+x_3)^2 x_2$. If (A) holds, then $h(b) = \eta + x_2 > 1$ since $\eta > 0$. If (B) holds, then our same equation forces $\eta = 0$ and $x_2 = 1$ when $\varepsilon \le (1+x_3)^2$, which means that $h(\varepsilon x_1(2+x_3) + (1+x_3)) = 1$ whenever $0 \le \varepsilon \le (1+x_3)^2$.

Subcase 4 of 4: Suppose $\alpha \neq x_1(1+x_3)(1+(1+x_3)x_2)$ and $\beta \neq (1+x_3)x_2$. Writing $\alpha = \varepsilon x_1 + \eta$ where $0 \leq \eta < x_1$ and $0 \leq \varepsilon < (1+x_3)(1+(1+x_3)x_2)$ and $\beta = \gamma x_2 + \delta$ where $0 \leq \delta < x_2$ and $0 \leq \gamma < 1 + x_3$, we have that

$$\begin{aligned} h(b) &= \varepsilon x_1 + \eta + \gamma x_2 + \delta - x_1 \left\lfloor \frac{((\varepsilon x_1 + \eta)(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3)}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor \\ &\quad - x_2 \left\lfloor \frac{(\gamma x_2 + \delta)(1+x_3)}{1+(1+x_3)x_2} \right\rfloor \\ &= \eta + \delta - x_1 \left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3) - \gamma}{1+(1+x_3)x_2} \right\rfloor \\ &= \eta + \delta - x_1 \left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\} . \end{aligned}$$

Given the bounds on η, γ, δ, and ε, observe that

$$|(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon| < 1 + x_1(1+x_3)(1+(1+x_3)x_2).$$

Therefore,

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor$$

is equal to either 0 or -1.

(i) Suppose $\delta = 0$ and $\gamma > 0$. If

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = -1\,,$$

then $h(b) = \eta + x_1 + x_2 > 1$. Otherwise, if

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0\,,$$

then either (A) $0 < \eta < x_1$ or (B) $\eta = 0$ with $0 \leq \varepsilon \leq \gamma x_2(1+x_3)$. If (A) holds, then $h(b) = \eta + x_2 > 1$ since $\eta > 0$ and $x_2 \geq 1$. If (B) holds, then our same equation forces $\eta = 0$ and $x_2 = 1$ when $\varepsilon \leq \gamma(1+x_3)$, which means that $h(\varepsilon x_1(2+x_3)+\gamma) = 1$ whenever $0 \leq \varepsilon \leq \gamma(1+x_3)$.

(ii) Suppose otherwise, i.e., $\delta = 0$ and $\gamma > 0$ does not hold. If

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = -1\,,$$

note that $\eta = 0$. Then, $h(b) = \delta + x_1$. For this to equal 1, it must be the case that $\delta = 0$ (and hence $\gamma = 0$ since otherwise, we would be in the previous case) and $x_1 = 1$. Since $\gamma = \delta = 0$, we have that $\beta = 0$. Therefore, given that $\eta = 0$ as well, it must be that $\varepsilon > 0$ since otherwise, $b = 0$ contradicting our bounds on b. In this case, we have that $h(\varepsilon(1+(1+x_3)x_2)) = 1$ whenever $0 < \varepsilon < (1+x_3)(1+(1+x_3)x_2)$. Otherwise, if

$$\left\lfloor \frac{(\eta(1+(1+x_3)x_2) + \gamma x_2 + \delta)(1+x_3) - \varepsilon}{1 + x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor = 0\,,$$

then either (A) $0 < \eta < x_1$ or (B) $\eta = 0$ with $0 \le \varepsilon \le (\gamma x_2 + \delta)(1 + x_3)$. If (A) holds, then $h(b) = \eta + \delta$, which setting equal to 1 forces $\eta = 1$ and $\delta = 0$ since $\eta > 0$. Note that $\delta = 0$ forces $\gamma = 0$ since otherwise, we would be in the previous case. Therefore, we have that $h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2)) = 1$ whenever $0 \le \varepsilon < (1 + x_3)(1 + (1 + x_3)x_2)$. On the other hand, if (B) holds, then our same equation forces $\eta = 0$ and $\delta = 1$ when $0 \le \varepsilon \le (\gamma x_2 + 1)(1 + x_3)$, which means that $h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1) = 1$ whenever $0 \le \gamma < 1 + x_3$ and $0 \le \varepsilon \le (\gamma x_2 + 1)(1 + x_3)$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$, $\beta = 0$, and $0 < \varepsilon \le (1 + x_3)(1 + (1 + x_3)x_2)$, we have $b = \varepsilon(1 + (1 + x_3)x_2)$.
- If $x_2 = 1$, $\delta = \eta = 0$, $0 < \gamma \le 1 + x_3$, and $0 \le \varepsilon \le \gamma(1 + x_3)$, we have $b = \varepsilon x_1(2 + x_3) + \gamma$.
- If $\beta = 0$, $\eta = 1$, and $0 \le \varepsilon < (1 + x_3)(1 + (1 + x_3)x_2)$, we have $b = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2)$.
- If $\eta = 0$, $\delta = 1$, $0 \le \gamma < 1 + x_3$, and $0 \le \varepsilon \le (\gamma x_2 + 1)(1 + x_3)$, we have $b = \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$. If $q_i = (1 + (1 + x_3)(1 + (1 + x_3)x_2)x_1)(1 + (1 + x_3)x_2)$, the result is trivial. If $q_i = (1 + x_3)(1 + x_1(1 + x_3)(1 + (1 + x_3)x_2))$, we write $b = \alpha(1 + (1 + x_3)x_2) + \beta$, where $0 \le \beta < 1 + (1 + x_3)x_2$ and $0 \le \alpha < 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)$. Note that $b \ne 1 + (1 + x_3)x_2$ since $h(1 + (1 + x_3)x_2) = 1$. If $b > 1 + (1 + x_3)x_2$, we set $c = 1 + (1 + x_3)x_2$ from which it is straightforward to compute that both sides of (4.5) are equal to

$$(\alpha - 1)(1 + x_3) + \left\lfloor \frac{\beta(1 + x_3)}{1 + (1 + x_3)x_2} \right\rfloor .$$

Otherwise, if $b < 1 + (1 + x_3)x_2$, note that $\alpha = 0$ and $b = \beta$. Therefore, to ensure we satisfy (4.4), we consider $2 \le \beta < 1 + (1 + x_3)x_2$. Moreover, note that $\beta \not\equiv 1 \bmod x_2$ since otherwise $h(b) = h(\beta) = 1$. Now, suppose $\beta = (1 + x_3)x_2$. If $x_2 = 1$, then $h(\beta) = h(1 + x_3) = 1$, so we may assume $x_2 > 1$. Setting $c = 1$ and since $x_2 > 1$, it is straightforward to compute that both sides of (4.5) are equal to x_3. Otherwise, if $\beta \ne (1 + x_3)x_2$, we write $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma < 1 + x_3$. Note that $\delta \ne 1$ since $h(\gamma x_2 + 1) = 1$ for $0 \le \gamma < 1 + x_3$. Suppose $\delta > 1$. Then, choosing $c = \gamma x_2 + 1 < \gamma x_2 + \delta = b$, it is straightforward to show that both sides of (4.5) are equal to 0. On the other hand suppose $\delta = 0$, so $\beta = \gamma x_2$, where $\gamma > 0$. If $x_2 = 1$, then $h(\beta) = 1$, so we may assume $x_2 > 1$. Taking $c = 1$ and observing that $x_2 > 1$ implies $-x_2(1 + x_3) < -(1 + x_3) - \gamma$, it is straightforward to compute that both sides of (4.5) are equal to $\gamma - 1$.

Finally, if $q_i = (1+x_3)(1+(1+x_3)x_2)$, the analysis becomes a bit more complicated. We start by again writing $b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \le \beta < 1+(1+x_3)x_2$ and $0 \le \alpha < 1+x_1(1+x_3)(1+(1+x_3)x_2)$. Suppose $\alpha = x_1(1+x_3)(1+(1+x_3)x_2)$. If $x_1 = 1$, then $\beta \ne 0$ since otherwise $h(b) = 1$. Thus, we may consider $c = (1+x_3)(1+(1+x_3)x_2)^2 < (1+x_3)(1+(1+x_3)x_2)^2+\beta = b$ from which it is straightforward to compute that both sides of (4.5) are equal to 0. If $x_1 > 1$, observe that

$$-x_1(1+x_3)(1+(1+x_3)x_2) < -(1+x_3)(2+(1+x_3)x_2) \le (\beta-1)(1+x_3)-(1+x_3)(1+(1+x_3)x_2) < 0\,.$$

Choosing $c = 1$, the previous inequality readily gives that both sides of (4.5) are equal to $(1+x_3)(1+(1+x_3)x_2)-1$.

Now, suppose $\alpha \ne x_1(1+x_3)(1+(1+x_3)x_2)$. Then, we may write $\alpha = \varepsilon x_1+\eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon < (1+x_3)(1+(1+x_3)x_2)$, and so $b = (\varepsilon x_1+\eta)(1+(1+x_3)x_2)+\beta$. Suppose $\beta = (1+x_3)x_2$. If $\eta \ge 1$, then we may consider $c = (\varepsilon x_1+1)(1+(1+x_3)x_2) < (\varepsilon x_1+\eta)(1+(1+x_3)x_2)+(1+x_3)x_2 = b$ from which it is straightforward to show that both sides of (4.5) are equal to 0. Otherwise, if $\eta = 0$, note that we may assume $x_2 > 1$ since $x_2 = 1$ gives that $h(b) = h(\varepsilon x_1(2+x_3)+(1+x_3)) = 1$.

We consider two possible cases, namely when $0 \le \varepsilon < 1+x_3$ and when $1+x_3 \le \varepsilon < (1+x_3)(1+(1+x_3)x_2)$. If $0 \le \varepsilon < 1+x_3$, we consider $c = \varepsilon x_1(1+(1+x_3)x_2)+x_2x_3+1$ which is strictly less than $b = \varepsilon x_1(1+(1+x_3)x_2)+(1+x_3)x_2$ as $x_2 > 1$. With this choice of c and since $0 \le \varepsilon < 1+x_3$, it is straightforward to show that both sides of (4.5) are equal to 0. On the other hand, if $1+x_3 \le \varepsilon < (1+x_3)(1+(1+x_3)x_2)$, we may consider $c = 1+x_1(1+x_3)(1+(1+x_3)x_2) < \varepsilon x_1(1+(1+x_3)x_2)+(1+x_3)x_2 = b$ from which it is straightforward to compute that both sides of (4.5) are equal to

$$\varepsilon-(1+x_3)+\left\lfloor \frac{(1+x_3)^2x_2-\varepsilon}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor\,.$$

Now, suppose $\beta \ne (1+x_3)x_2$. Then, we may write $\beta = \gamma x_2+\delta$, where $0 \le \delta < x_2$ and $0 \le \gamma < 1+x_3$, and so b can be written as $b = (\varepsilon x_1+\eta)(1+(1+x_3)x_2)+\gamma x_2+\delta$. We consider the following possible subcases.

Subcase 1 of 4: Suppose $\eta > 0$ and $\delta > 0$. We consider $c = (\varepsilon x_1+1)(1+(1+x_3)x_2) < (\varepsilon x_1+1)(1+(1+x_3)x_2)+\gamma x_2+\delta = b$. Since $0 < \eta < x_1$ and $0 < \gamma x_2+\delta < (1+x_3)x_2$, it follows that $0 < (\eta-1)(1+(1+x_3)x_2)+\gamma x_2+\delta < x_1(1+(1+x_3)x_2)$. Therefore, it is straightforward to verify that both sides of (4.5) are equal to 0.

Subcase 2 of 4: Suppose $\eta = 0$ and $\delta > 0$. If $\varepsilon \le (\gamma x_2+1)(1+x_3)$, note that $\delta \ne 1$ since otherwise, $h(b) = h(\varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+1) = 1$. Thus, we have that $\delta > 1$, and we consider $c = \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+1 < \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+\delta = b$. Given that $\varepsilon \le (\gamma x_2+1)(1+x_3)$ and $1 < \delta < x_2$, it is straightforward to verify that our choice of c gives that both sides of (4.5) are equal to 0. Otherwise, if $(\gamma x_2+1)(1+x_3) < \varepsilon$ (and hence, $\varepsilon > 1+x_3$), we consider $c = 1+x_1(1+x_3)(1+(1+x_3)x_2) < \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+\delta = b$. With this choice of c, it is straightforward to verify that both sides of (4.5) are equal to

$$\varepsilon-(1+x_3)+\left\lfloor \frac{(\gamma x_2+\delta)(1+x_3)-\varepsilon}{1+x_1(1+x_3)(1+(1+x_3)x_2)} \right\rfloor\,.$$

Subcase 3 of 4: Suppose $\eta > 0$ and $\delta = 0$. If $\gamma > 0$, we may consider $c = (\varepsilon x_1+1)(1+(1+x_3)x_2) < (\varepsilon x_1+\eta)(1+(1+x_3)x_2)+\gamma x_2 = b$ from which it is straightforward to compute that both sides of (4.5) are equal to 0. On the other hand, if $\gamma = 0$, note that $\eta \neq 1$ since otherwise, $h(b) = h((\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2)) = 1$. Therefore, we may again choose $c = (\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) < (\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) = b$. Since $1 < \eta < x_1$, it is straightforward to verify that both sides of (4.5) are equal to 0.

Subcase 4 of 4: Suppose $\eta = \delta = 0$. Further suppose $\gamma x_2(1 + x_3) < \varepsilon$. If $\gamma > 0$, then it follows that $\gamma x_2 \geq 1$. Therefore, our assumed inequality implies $\varepsilon > 1 + x_3$, so we may consider $c = 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2) < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 = b$. Since $\gamma x_2(1+x_3) < \varepsilon$, our choice of c readily gives that both sides of (4.5) are equal to $\varepsilon - x_3 - 2$. Otherwise, if $\gamma = 0$ (and hence, $\varepsilon > 0$ since we assumed $\gamma x_2(1+x_3) < \varepsilon$), we may assume $x_1 > 1$ since otherwise, we would have that $h(b) = h(\varepsilon(1+(1+x_3)x_2)) = 1$. Thus, since $x_1 > 1$, we may take $c = 1 + (1 + x_3)x_2 < \varepsilon x_1(1 + (1 + x_3)x_2) = b$. Observe that the bounds on ε and $x_1 > 1$ imply $-x_1(1 + x_3)(1 + (1 + x_3)x_2) \leq -2(1 + x_3)(1 + (1 + x_3)x_2) < -(1 + x_3)(1 + (1 + x_3)x_2) - \varepsilon < 0$. Consequently, it is straightforward to verify that both sides of (4.5) are equal to $\varepsilon - 1$. Now, suppose $\gamma x_2(1 + x_3) > \varepsilon$. Note that $\gamma \neq 0$ since otherwise, $\varepsilon < 0$ contradicting our initial bounds on ε. Thus, we have that $\gamma > 0$. Moreover, if $x_2 = 1$, it follows that $h(b) = h(\varepsilon x_1(2 + x_3) + \gamma) = 1$, so we may assume $x_2 > 1$. Given the addition restriction that $\varepsilon \leq (\gamma x_2 - 1)(1 + x_3)$, we may choose $c = 1$ from which the inequality $\varepsilon \leq (\gamma x_2 - 1)(1 + x_3)$ readily implies both sides of (4.5) are equal to ε. However, for $(\gamma x_2 - 1)(1 + x_3) < \varepsilon < \gamma x_2(1 + x_3)$, we consider $c = 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2)$. Note that $x_2 > 1$ and $\gamma > 0$ together with our restriction on ε imply that $\varepsilon > 1 + x_3$. Therefore, we satisfy $c < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 = b$, and a straightforward computation gives that both sides of (4.5) are equal to $\varepsilon - (1 + x_3)$. Finally, suppose $\varepsilon = \gamma x_2(1 + x_3)$. Given this equality, note that neither γ nor ε can be equal to 0 since otherwise, we would have $\eta = \delta = \gamma = \varepsilon = 0$, implying $b = 0$. This, of course, contradicts the bounds on b. Moreover, we may again assume $x_2 > 1$ (and thus, $\varepsilon > 1 + x_3$) since otherwise, $h(b) = h(\varepsilon x_1(2 + x_3) + \gamma) = 1$. Since $\varepsilon > 1 + x_3$, we may consider $c = 1 + x_1(1 + x_3)(1 + (1 + x_3)x_2) < \varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 = b$. This choice of c readily gives that both sides of (4.5) are equal to $\varepsilon - (1 + x_3)$.

In any case, we find that both sides of (4.5) are equivalent for each possible q_i, thereby completing our third and final case. Thus, we have established IDP for $\mathbf{r}$-vectors of type (vi).

4.2.7 Proof of IDP for type (vii) in Theorem 4.1.11

Here, we verify IDP for $\mathbf{r}$-vectors of type (vii) using Theorem 4.1.5. Again, we must consider three cases corresponding to three possible values of q_j.

Case: $q_j = 1 + x_3$. Since $1 \leq b \leq x_3$ in this case, it is straightforward to verify that $h(b) = b$. Hence, the b-values we are required to check in (4.4) are $2 \leq b \leq x_3$. To verify that (4.5) always has the desired solution, we consider three cases. If $q_i = 1 + x_3$, the result is trivial. If $q_i = (1 + x_3)(1 + x_1(1 + x_3))$, then we may select $c = 1$, from which it follows that both sides of (4.5) are equal to $(b - 1)(1 + x_1(1 + x_3))$. If $q_i = (1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)$, then we may again set $c = 1$, from which it is straightforward to compute that both sides of (4.5) are equal to $(b - 1)(x_1 + x_2 + x_1x_2(1 + x_3))$. This completes our first case.

Case: $q_j = (1+x_3)(1+x_1(1+x_3))$. It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b}{1+x_1(1+x_3)} \right\rfloor - x_3 \left\lfloor \frac{b}{1+x_3} \right\rfloor ,$$

where the values of b range from 1 to $(1+x_3)(1+x_1(1+x_3))-1$. To verify that (4.5) always has the desired solution, we consider three cases. If $q_i = (1+x_3)(1+x_1(1+x_3))$, the result is trivial. If $q_i = 1+x_3$, then we write $b = \alpha(1+x_1(1+x_3))+\beta$, where $0 \le \beta < 1+x_1(1+x_3)$ and $0 \le \alpha < 1+x_3$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, we have that

$$\begin{aligned} h(b) &= h(\alpha(1+x_1(1+x_3))+\beta) \\ &= \alpha(1+x_1(1+x_3)) + \beta - \alpha x_1 - \alpha x_1 x_3 - x_3 \left\lfloor \frac{\alpha+\beta}{1+x_3} \right\rfloor \\ &= \alpha + \beta - x_3 \left\lfloor \frac{\alpha+\beta}{1+x_3} \right\rfloor . \end{aligned}$$

If $\beta > 0$, we may select $c = 1$, from which it follows that both sides of (4.5) are equal to α. If $\beta = 0$, then our formula for $h(b)$ reduces to

$$h(b) = h(\alpha(1+x_1(1+x_3))) = \alpha$$

since $0 \le \alpha < 1+x_3$. Thus, to satisfy (4.4), it must be that $\alpha > 1$, implying $b = \alpha(1+x_1(1+x_3)) > 1+x_1(1+x_3)$. In this case, taking $c = 1+x_1(1+x_3)$, it is straightforward to verify that both sides of (4.5) are equal to $\alpha-1$, and (4.6) is satisfied as $h(1+x_1(1+x_3)) = 1$. Finally, if $q_i = (1+(1+x_3)x_1)(1+(1+x_3)x_2)$, then we write $b = \alpha(1+x_3)+\beta$, where $0 \le \beta < 1+x_3$ and $0 \le \alpha < 1+x_1(1+x_3)$ for $\alpha, \beta \in \mathbb{Z}$. Consequently, since $0 \le \beta < 1+x_3$, we have that

$$\begin{aligned} h(b) &= h(\alpha(1+x_3)+\beta) \\ &= \alpha(1+x_3) + \beta - x_1 \left\lfloor \frac{\alpha(1+x_3)+\beta}{1+x_1(1+x_3)} \right\rfloor - \alpha x_3 - x_3 \left\lfloor \frac{\beta}{1+x_3} \right\rfloor \\ &= \alpha + \beta - x_1 \left\lfloor \frac{\alpha(1+x_3)+\beta}{1+x_1(1+x_3)} \right\rfloor . \end{aligned}$$

If $\beta > 0$, we may select $c = 1$, from which it is straightforward to verify that both sides of (4.5) are equal to $\alpha(1+(1+x_3)x_2) + (\beta-1)x_2$ (since $0 \le \beta - 1 < x_3$). On the other hand, if $\beta = 0$, then our formula for $h(b)$ reduces to

$$h(b) = h(\alpha(1+x_3)) = \alpha - \left\lfloor \frac{\alpha(1+x_3)}{1+x_1(1+x_3)} \right\rfloor .$$

In order to satisfy (4.4), it must be that $\alpha > 1$, which implies $b = \alpha(1+x+3) > 1+x_3$. Thus, in this case, we consider $c = 1+x_3$. Clearly, $h(1+x_3) = 1$, giving (4.6), and moreover, it is straightforward to verify that both sides of (4.5) when $c = 1+x_3$ are equal to $(\alpha-1)(1+(1+x_3)x_2)$. This completes our second case.

Case: $q_j = (1+(1+x_3)x_1)(1+(1+x_3)x_2)$. We first identify those values of b that satisfy (4.4) and (4.6). It is straightforward to verify that

$$h(b) = b - x_1 \left\lfloor \frac{b(1+x_3)}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{b(1+x_3)}{(1+(1+x_3)x_2)} \right\rfloor .$$

Writing $b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \le \beta \le (1+x_3)x_2$ and $0 \le \alpha \le (1+x_3)x_1$, it follows that

$$\begin{aligned} h(b) &= h(\alpha(1+(1+x_3)x_2)+\beta) \\ &= \alpha+\beta - x_1 \left\lfloor \frac{\alpha(1+x_3)(1+(1+x_3)x_2)+\beta(1+x_3)}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\beta(1+x_3)}{1+(1+x_3)x_2} \right\rfloor . \end{aligned}$$

Now, writing $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma \le 1+x_3$, it follows that

$$h(b) = \alpha+\delta - x_1 \left\lfloor \frac{\alpha(1+x_3)(1+(1+x_3)x_2)+(\gamma x_2+\delta)(1+x_3)}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor .$$

Since $0 \le \delta < x_2$ and $0 \le \gamma \le 1+x_3$, observe that

$$\left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor = \begin{cases} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{cases} .$$

We further write $\alpha = \varepsilon x_1 + \eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon \le 1+x_3$. Then,

$$\begin{aligned} h(b) &= \eta+\delta - x_1 \left\lfloor \frac{(\eta(1+x_3)-\varepsilon)(1+(1+x_3)x_2)+(\gamma x_2+\delta)(1+x_3)}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor \\ &= \eta+\delta - x_1 \left\lfloor \frac{(\eta(1+x_3)-\varepsilon+\gamma)(1+(1+x_3)x_2)+\delta(1+x_3)-\gamma}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor - x_2 \left\lfloor \frac{\delta(1+x_3)-\gamma}{1+(1+x_3)x_2} \right\rfloor . \end{aligned}$$

Given the bounds on ε, η, γ, and δ, note that $-(1+x_3) \le \delta(1+x_3)-\gamma < 1+(1+x_3)x_2$ and $-(1+x_3) \le \eta(1+x_3)-\varepsilon+\gamma \le (1+x_3)x_1$. Consequently, it follows that

$$|(\eta(1+x_3)-\varepsilon+\gamma)(1+(1+x_3)x_2)+\delta(1+x_3)-\gamma| < (1+(1+x_3)x_1)(1+(1+x_3)x_2),$$

and this implies that $\left\lfloor \frac{(\eta(1+x_3)-\varepsilon+\gamma)(1+(1+x_3)x_2)+\delta(1+x_3)-\gamma}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor$ is equal to either 0 or -1. To resolve this floor function, we consider the following subcases which analyze the sign of the numerator of its argument.

Subcase 1 of 6: Suppose $\eta = 0$ and $\varepsilon > \gamma$. Then, since $\delta(1+x_3)-\gamma < 1+(1+x_3)x_2$, the numerator above will be negative, implying

$$\left\lfloor \frac{(\eta(1+x_3)-\varepsilon+\gamma)(1+(1+x_3)x_2)+\delta(1+x_3)-\gamma}{(1+(1+x_3)x_1)(1+(1+x_3)x_2)} \right\rfloor = -1.$$

Therefore, our equation for $h(b)$ simplifies to

$$h(b) = \delta + x_1 - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\} .$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2) = x_1 + x_2 > 1$. If $\delta = \gamma = 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2)) = x_1$. Thus, if $x_1 = 1$, we have that $h(\varepsilon x_1(1 + (1 + x_3)x_2)) = 1$ whenever $\varepsilon > 0$. If $\delta > 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + \delta) = \delta + x_1 > 1$.

Subcase 2 of 6: Suppose $\eta = 0$ and $\varepsilon < \gamma$. Then, $\eta(1 + x_3) - \varepsilon + \gamma > 0$, and consequently, the numerator of our floor function argument will be positive. Hence,

$$\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)} \right\rfloor = 0,$$

simplifying our formula for $h(b)$ to

$$h(b) = \delta - x_2 \cdot \left\{ \begin{array}{ll} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{array} \right\}.$$

If $\delta = 0$ and $\gamma > 0$, then $h(b) = h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2) = x_2$. Thus, if $x_2 = 1$, we have that $h(b) = h(\varepsilon x_1(2 + x_3) + \gamma) = 1$ whenever $\varepsilon < \gamma$. Otherwise, $h(b) = \delta$, which forces $\delta = 1$, i.e., $h(\varepsilon x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1) = 1$ whenever $\varepsilon < \gamma$.

Subcase 3 of 6: Suppose $\eta = 0$ and $\varepsilon = \gamma$. Then, $\eta(1 + x_3) - \varepsilon + \gamma = 0$, so the numerator of our floor function argument reduces to $\delta(1 + x_3) - \gamma$. If $\delta = 0$ and $\gamma > 0$, then $\delta(1 + x_3) - \gamma < 0$ which implies

$$\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)} \right\rfloor = -1.$$

Hence, for $\varepsilon = \gamma > 0$, $h(b) = h(\gamma x_1(1+(1+x_3)x_2)+\gamma x_2) = x_1+x_2 > 1$. If $\delta = \gamma = 0$, then $\delta(1 + x_3) - \gamma = 0$ which implies

$$\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)} \right\rfloor = 0.$$

Hence, $h(b) = h(0) = 0$. If $\delta > 0$, then $\delta(1 + x_3) - \gamma > 0$ since $0 \leq \gamma \leq (1 + x_3)$. Therefore,

$$\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)} \right\rfloor = 0.$$

As such, we have that $h(b) = h(\gamma x_1(1+(1+x_3)x_2)+\gamma x_2+\delta) = \delta$, which forces $\delta = 1$, i.e., $h(b) = h(\gamma x_1(1 + (1 + x_3)x_2) + \gamma x_2 + 1) = 1$ for $0 \leq \gamma \leq 1 + x_3$.

Subcase 4 of 6: Suppose $\eta = 1$ and $0 \leq \varepsilon < 1 + x_3$. Then, it follows that $\eta(1 + x_3) - \varepsilon + \gamma > 0$. Consequently, since $\delta(1 + x_3) - \gamma < 1 + (1 + x_3)x_2$, we have that the numerator of our floor function argument is positive, implying

$$\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)} \right\rfloor = 0$$

for any $0 \leq \gamma \leq 1 + x_3$ and $0 \leq \delta < x_2$. As a result, if $\delta = 0$ and $\gamma > 0$, then $h(b) = h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2) = 1 + x_2 > 1$. If $\delta = \gamma = 0$, then

$h(b) = h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2)) = \eta = 1$ whenever $0 \le \varepsilon < 1 + x_3$. If $\delta > 0$, then $h(b) = h((\varepsilon x_1 + 1)(1 + (1 + x_3)x_2) + \gamma x_2 + \delta) = 1 + \delta > 1$.

Subcase 5 of 6: Suppose $\eta = 1$ and $\varepsilon = 1 + x_3$. Then, the numerator of our floor function argument reduces to $\gamma(1+x_3)x_2 + \delta(1+x_3)$, which is certainly nonnegative. Therefore, it follows that

$$\left\lfloor \frac{(\eta(1+x_3) - \varepsilon + \gamma)(1 + (1+x_3)x_2) + \delta(1+x_3) - \gamma}{(1 + (1+x_3)x_1)(1 + (1+x_3)x_2)} \right\rfloor = 0,$$

which simplifies our formula for $h(b)$ to

$$h(b) = h((1 + (1+x_3)x_1)(1 + (1+x_3)x_2) + \gamma x_2 + \delta) = 1 + \delta - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\}.$$

Therefore, if $\delta = 0$ and $\gamma > 0$, it follows that $h(b) = h((1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2) + \gamma x_2) = 1 + x_2 > 1$. Otherwise, $h(b) = h((1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2) + \gamma x_2 + \delta) = 1 + \delta$. So, for this to be equal to 1, it must be that $\delta = \gamma = 0$, i.e., $h((1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)) = 1$.

Subcase 6 of 6: Suppose $\eta > 1$. Then, it follows that $\eta(1 + x_3) - \varepsilon + \gamma > 0$, and consequently, since $\delta(1+x_3) - \gamma < 1 + (1+x_3)x_2$, we have that the numerator of our floor function argument is positive. Therefore, we have that

$$\left\lfloor \frac{(\eta(1+x_3) - \varepsilon + \gamma)(1 + (1+x_3)x_2) + \delta(1+x_3) - \gamma}{(1 + (1+x_3)x_1)(1 + (1+x_3)x_2)} \right\rfloor = 0,$$

which simplifies our formula for $h(b)$ to

$$h(b) = h((\varepsilon x_1 + \eta)(1 + (1+x_3)x_2) + \gamma x_2 + \delta) = \eta + \delta - x_2 \cdot \left\{ \begin{matrix} -1, & \delta = 0, \gamma > 0 \\ 0, & \text{otherwise} \end{matrix} \right\}.$$

Therefore, if $\delta = 0$ and $\gamma > 0$, it follows that $h(b) = h((\varepsilon x_1 + \eta)(1 + (1+x_3)x_2) + \gamma x_2) = \eta + x_2 > 1$. Otherwise, $h(b) = h((\varepsilon x_1 + \eta)(1 + (1 + x_3)x_2) + \gamma x_2 + \delta) = \eta + \delta > 1$.

We summarize the values of b for which $h(b) = 1$ that were just derived:

- If $x_1 = 1$, $\eta = \delta = \gamma = 0$, and $0 < \varepsilon \le 1 + x_3$, we have $b = \varepsilon(1 + (1 + x_3)x_2)$.
- If $x_2 = 1$, $\eta = \delta = 0$, and $0 \le \varepsilon < \gamma \le 1 + x_3$, we have $b = \varepsilon x_1(2 + x_3) + \gamma x_2$.
- If $\eta = 0$, $\delta = 1$, and $0 \le \varepsilon \le \gamma \le 1+x_3$, we have $b = \varepsilon x_1(1+(1+x_3)x_2)+\gamma x_2+1$.
- If $\eta = 1$, $\delta = \gamma = 0$, and $0 \le \varepsilon \le 1 + x_3$, we have $b = (1 + \varepsilon x_1)(1 + (1 + x_3)x_2)$.

Our next goal is to establish that (4.5) is always satisfied; recall that we are in the case where $q_j = (1+(1+x_3)x_1)(1+(1+x_3)x_2)$. If $q_i = (1+(1+x_3)x_1)(1+(1+x_3)x_2)$, the result is trivial. If $q_i = (1+x_3)(1+(1+x_3)x_1)$, we write $b = \alpha(1+(1+x_3)x_2)+\beta$, where $0 \le \beta < 1 + (1 + x_3)x_2$ and $0 \le \alpha < 1 + (1 + x_3)x_1$. If $b > 1 + (1 + x_3)x_2$ (and thus, $\alpha \ge 1$), we can take $c = 1 + (1 + x_3)x_2$ as this makes both sides of (4.5) equal to $(\alpha - 1)(1 + x_3) + \left\lfloor \frac{\beta(1+x_3)}{1+(1+x_3)x_2} \right\rfloor$. If $2 \le b < 1 + (1 + x_3)x_2$, note that α must be 0 and hence $b = \beta$. We write $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma \le 1 + x_3$. In the case that $\delta > 1$, we set $c = \gamma x_2 + 1$ as this makes both sides of (4.5) equal to 0. Moreover, note that we need not consider the case where $\delta = 1$ since $h(\gamma x_2 + 1) = 1$. Therefore, it only remains to find a c-value when $\delta = 0$. If $\delta = 0$, then $b = \beta = \gamma x_2$. Observe that $\gamma > 0$ since $\gamma = 0$ would imply $h(b) = 0 \not\geq 2$. In this case, we set $c = (\gamma - 1)x_2 + 1$. Since $1 \le \gamma \le 1 + x_3$ implies that $1 \le 2 + x_3 - \gamma \le 1 + x_3$, it is straightforward to check that both sides of (4.5) will again be equal to 0.

Finally, if $q_i = 1 + x_3$, the analysis becomes slightly more complicated. As in the previous case, we begin by writing $b = \alpha(1 + (1 + x_3)x_2) + \beta$, where $0 \le \beta < 1 + (1 + x_3)x_2$ and $0 \le \alpha < 1 + (1 + x_3)x_1$. Furthermore, we write $\beta = \gamma x_2 + \delta$, where $0 \le \delta < x_2$ and $0 \le \gamma \le 1 + x_3$, and we write $\alpha = \varepsilon x_1 + \eta$, where $0 \le \eta < x_1$ and $0 \le \varepsilon \le 1 + x_3$. If $b > 1 + (1 + x_3)x_2$, we consider $c = 1$. Substituting $c = 1$ and the alternate form for b into the left-hand side of (4.5), yields

$$\varepsilon + \underbrace{\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + 1 + x_3)x_2)} \right\rfloor}_{=: F_1}.$$

On the other hand, substituting into the right-hand side of (4.5) yields

$$\varepsilon + \underbrace{\left\lfloor \frac{(\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + (\delta - 1)(1 + x_3) - \gamma}{(1 + (1 + x_3)x_1)(1 + 1 + x_3)x_2)} \right\rfloor}_{=: F_2}.$$

We must show that $F_1 = F_2$. To this end, let $n_1 = (\eta(1+x_3) - \varepsilon + \gamma)(1 + (1+x_3)x_2) + \delta(1 + x_3) - \gamma$ and $n_2 = (\eta(1 + x_3) - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + (\delta - 1)(1 + x_3) - \gamma$, that is, n_1 and n_2 are the numerators of the arguments in F_1 and F_2, respectively. Given the bounds on ε, η, γ, and δ, note that $-(1 + x_3) \le \delta(1 + x_3) - \gamma < 1 + (1 + x_3)x_2$, $-2(1+x_3) \le (\delta - 1)(1+x_3) - \gamma < 1 + (1+x_3)x_2$, and $-(1+x_3) \le \eta(1+x_3) - \varepsilon + \gamma \le (1 + x_3)x_1$. Consequently, it follows that $|n_k| < (1 + (1 + x_3)x_1)(1 + (1 + x_3)x_2)$ for $k \in \{1, 2\}$, and this implies that F_k is equal to either 0 or -1. Therefore, to achieve our goal, we must verify that either $n_1, n_2 < 0$ or $n_1, n_2 \ge 0$. Now, observe that $b > 1 + (1 + x_3)x_2$ implies that either (A) $\alpha = 1$ and $\beta > 0$, or (B) $\alpha > 1$. For each of these scenarios, we consider subcases. First assume (A) holds, i.e., $\alpha = 1$ and $\beta > 0$.

Subcase 1 of 2: Suppose $x_1 = 1$. Then, since $0 \le \eta < x_1 = 1$, it follows that $\eta = 0$. Consequently, as $1 = \alpha = \varepsilon x_1 + \eta$, we have that $\varepsilon = 1$. Thus, n_1 and n_2 reduce to

$$n_1 = (\gamma - 1)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma, \text{ and}$$
$$n_2 = (\gamma - 1)(1 + (1 + x_3)x_2) + (\delta - 1)(1 + x_3) - \gamma$$

If $\gamma = 0$, then the numerators $n_1, n_2 < 0$ and hence $F_1 = F_2 = -1$. If $\gamma = 1$, note that $\delta \neq 1$ (since $\eta = 0, \varepsilon = \gamma$, and $\delta = 1$ imply $h(b) = 1$). So, if $\delta = 0$, then $n_1, n_2 < 0$ and hence $F_1 = F_2 = -1$. Otherwise, if $\delta > 1$, then $n_1, n_2 > 0$ and thus $F_1 = F_2 = 0$. If $\gamma > 1$, then $n_1, n_2 > 0$, implying $F_1 = F_2 = 0$.

Subcase 2 of 2: Suppose $x_1 > 1$. Then, given that $\alpha = 1$, it must be the case that $\varepsilon = 0$ and $\eta = 1$. Therefore, it immediately follows that $F_1 = F_2 = 0$ since $n_1, n_2 > 0$.

Thus, we can conclude that $F_1 = F_2$ in situation (A). Now, we must consider situation (B), i.e., when $\alpha > 1$. We again consider subcases.

Subcase 1 of 3: Suppose $\eta = 0$. Then, it follows that $\varepsilon > 0$, and our numerators reduce to

$$\begin{aligned} n_1 &= (\gamma - \varepsilon)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma, \text{ and} \\ n_2 &= (\gamma - \varepsilon)(1 + (1 + x_3)x_2) + (\delta - 1)(1 + x_3) - \gamma. \end{aligned}$$

If $\gamma > \varepsilon$, then the numerators of both arguments will be positive, implying $F_1 = F_2 = 0$. If $\gamma = \varepsilon$, note that $\delta \neq 1$ (since $\eta = 0$, $\varepsilon = \gamma$, and $\delta = 1$ imply $h(b) = 1$). So, if $\delta = 0$, we have that $n_1, n_2 < 0$, and hence $F_1 = F_2 = -1$. Otherwise, if $\delta > 1$, it follows that $n_1, n_2 \geq 0$ which implies $F_1 = F_2 = 0$. Finally, if $\gamma < \varepsilon$, it follows that $F_1 = F_2 = -1$ since $n_1, n_2 < 0$.

Subcase 2 of 3: Suppose $\eta = 1$. Again, since $\alpha > 1$, this implies $\varepsilon > 0$. As a result, we have the following reduction of n_1 and n_2:

$$\begin{aligned} n_1 &= (1 + x_3 - \varepsilon + \varepsilon)(1 + (1 + x_3)x_2) + \delta(1 + x_3) - \gamma, \text{ and} \\ n_2 &= (1 + x_3 - \varepsilon + \gamma)(1 + (1 + x_3)x_2) + (\delta - 1)(1 + x_3) - \gamma. \end{aligned}$$

If $\varepsilon < 1 + x_3$ then we have $n_1, n_2 > 0$, and thus $F_1 = F_2 = 0$. Otherwise, $\varepsilon = 1 + x_3$. Note that since $\eta = 1$, δ and γ cannot both be 0 as this would imply $h(b) = 1$. Therefore, if $\gamma = 0$, it must be that $\delta > 0$ which implies $n_1, n_2 > 0$ and $F_1 = F_2 = 0$. Otherwise, if $\gamma > 0$, $n_1, n_2 > 0$, and thus $F_1 = F_2 = 0$.

Subcase 3 of 3: Suppose $\eta > 1$. Then, it immediately follows that $n_1, n_2 > 0$, and we have that $F_1 = F_2 = 0$.

Thus, we find that $F_1 = F_2$. Therefore, we have that (4.5) is satisfied with $c = 1$ for $b > 1 + (1 + x_3)x_2$. It remains to consider $2 \leq b < 1 + (1 + x_3)x_2$. If $2 \leq b < 1 + (1 + x_3)x_2$, note that α must be 0 and hence $b = \beta$. Therefore, to ensure we satisfy (4.4), we consider $2 \leq \beta < 1 + (1 + x_3)x_2$. Since $\beta > 1$ in this case, we may take $c = 1$ from which it is straightforward to verify that both sides of (4.5) are equal to 0. This completes our third and final case, thereby establishing IDP for **r**-vectors of type (vii).

Bibliography

[1] C. A. Athanasiadis. Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley. *J. Reine Angew. Math.*, 583:163–174, 2005.

[2] G. Balletti, T. Hibi, M. Meyer, and A. Tsuchiya. Laplacian simplices associated to digraphs. *Arkiv fr matematik*, 56, 12 2018.

[3] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi–Yau hypersurfaces in toric varieties. *J. Algebraic Geom.*, 3(3):493–535, 1994.

[4] M. Beck, P. Jayawant, and T. B. McAllister. Lattice-point generating functions for free sums of convex sets. *J. Combin. Theory Ser. A*, 120(6):1246–1262, 2013.

[5] M. Beck and S. Robins. *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra*. Undergraduate Texts in Mathematics. Springer New York, 2007.

[6] M. Beck and S. Robins. *Computing the continuous discretely.* Undergraduate Texts in Mathematics. Springer, New York, second edition, 2015. Integer-point enumeration in polyhedra, With illustrations by David Austin.

[7] B. Braun. An Ehrhart series formula for reflexive polytopes. *Electron. J. Combin.*, 13(1):Note 15, 5 pp. (electronic), 2006.

[8] B. Braun. Unimodality problems in Ehrhart theory. In *Recent trends in combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 687–711. Springer, [Cham], 2016.

[9] B. Braun and R. Davis. Ehrhart series, unimodality, and integrally closed reflexive polytopes. *Ann. Comb.*, 20(4):705–717, 2016.

[10] B. Braun, R. Davis, D. Hanely, M. Lane, and L. Solus. Projective normality, the integer decomposition property, and Ehrhart unimodality for weighted projective space simplices. 2021. Preprint available at arXiv:2103.17156v1.

[11] B. Braun, R. Davis, and L. Solus. Detecting the integer decomposition property and Ehrhart unimodality in reflexive simplices. *Advances in Applied Mathematics*, 100, 08 2016.

[12] B. Braun and D. Hanely. A regular unimodular triangulation of reflexive 2-supported weighted projective space simplices. 2020. Preprint available at arXiv:2010.13720v1.

[13] B. Braun and F. Liu. h^*-polynomials with roots on the unit circle. *Experimental Mathematics*, 0(0):1–17, 2019.

[14] W. Bruns and T. Römer. h-vectors of Gorenstein polytopes. *J. Combin. Theory Ser. A*, 114(1):65–76, 2007.

[15] H. Conrads. Weighted projective spaces and reflexive simplices. *Manuscripta Math.*, 107(2):215–227, 2002.

[16] D. A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra, 3/e (Undergraduate Texts in Mathematics)*. Springer-Verlag, Berlin, Heidelberg, 2007.

[17] R. Davis. Ehrhart series of polytopes related to symmetric doubly-stochastic matrices. *Electron. J. Combin.*, 22(2):P2.17, (electronic), 2015.

[18] E. Ehrhart. Sur les polyèdres rationnels homothétiques à n dimensions. *C. R. Acad. Sci. Paris*, 254:616–618, 1962.

[19] C. Haase and I. V. Melnikov. The reflexive dimension of a lattice polytope. *Ann. Comb.*, 10(2):211–217, 2006.

[20] T. Hibi. *Algebraic Combinatorics on Convex Polytopes*. Carslaw Publications, Australia, 1992.

[21] T. Hibi. Dual polytopes of rational convex polytopes. *Combinatorica*, 12(2):237–240, 1992.

[22] T. Hibi and H. Ohsugi. Quadratic initial ideals of root systems. *Proc. Amer. Math. Soc.*, 130(7):1913–1922, 2001.

[23] M. Kreuzer and H. Skarke. Complete classification of reflexive polyhedra in four dimensions. *Adv. Theor. Math. Phys.*, 4(6):1209–1230, 2000.

[24] J. C. Lagarias and G. M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canad. J. Math.*, 43(5):1022–1035, 1991.

[25] F. Liu and L. Solus. On the relationship between Ehrhart unimodality and Ehrhart positivity. *Ann. Comb.*, 23(2):347–365, 2019.

[26] M. Mustaţă and S. Payne. Ehrhart polynomials and stringy Betti numbers. *Math. Ann.*, 333(4):787–795, 2005.

[27] B. Nill. Volume and lattice points of reflexive simplices. *Discrete Comput. Geom.*, 37(2):301–320, 2007.

[28] H. Ohsugi and T. Hibi. Special simplices and Gorenstein toric rings. *J. Combin. Theory Ser. A*, 113(4):718–725, 2006.

[29] S. Payne. Ehrhart series and lattice triangulations. *Discrete Comput. Geom.*, 40(3):365–376, 2008.

[30] G. Pick. Geometrisches zur zahlenlehre. *Sitzenber. Lotos (Prague)*, 19:311–319, 1899.

[31] L. Solus. Simplices for numeral systems. *Transactions of the American Mathematical Society*, 371, 06 2017.

[32] R. P. Stanley. Decompositions of rational convex polytopes. *Ann. Discrete Math.*, 6:333–342, 1980. Combinatorial mathematics, optimal designs and their applications (Proc. Sympos. Combin. Math. and Optimal Design, Colorado State Univ., Fort Collins, Colo., 1978).

[33] R. P. Stanley. *Combinatorics and commutative algebra*, volume 41 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[34] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.

[35] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.2)*, 2020. `https://www.sagemath.org`.

[36] G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

Vita

Derek W. Hanely

Education

University of Kentucky (UK), Lexington, KY
Ph.D. in Mathematics, May 2022 (expected)

- Advisor: Benjamin Braun, Ph.D.

M.A. in Mathematics, December 2019

Indiana University of Pennsylvania (IUP), Indiana, PA
M.S. in Applied Mathematics, May 2017
B.S. in Mathematics (*Summa cum laude*), May 2016
B.A. in Economics (Honors Program, *Summa cum laude*), May 2016

Publications & Preprints

- B. Braun & **D. Hanely**. *A Regular Unimodular Triangulation of Reflexive 2-supported Weighted Projective Space Simplices.* 2020. Accepted for publication in Annals of Combinatorics, arXiv:2010.13720 [math.CO].
- M. von Bell, B. Braun, **D. Hanely**, K. Serhiyenko, J. Vega, A. Vindas-Meléndez, & M. Yip. *Triangulations, Order Polytopes, and Generalized Snake Posets*. 2021. Submitted, arXiv:2102.11306 [math.CO].
- **D. Hanely**, J. Martin, D. McGinnis, D. Miyata, G. Nasr, A.R. Vindas-Meléndez, & M. Yin. *Ehrhart Positivity for Paving Matroids.* 2022. Submitted, arXiv:2201.12442 [math.CO].
- M. von Bell, B. Braun, K. Bruegge, **D. Hanely**, Z. Peterson, K. Serhiyenko, & M. Yip. *Triangulations of Flow Poltyopes, Ample Framings, and Gentle Algebras*. 2022. Submitted, arXiv:2203.01896 [math.CO].

Professional Positions

- **Tenure-Track Assistant Professor**, Penn State Behrend, Fall 2022 –
- **Graduate Teaching Assistant**, University of Kentucky, Fall 2017 – Spring 2022
- **Graduate Research Assistant**, Indiana University of Pennsylvania, Fall 2016 – Spring 2017

Awards & Honors

- Summer Research Fellowship, University of Kentucky
- Distinguished Alumnus Award, West Forest High School
- Certificate for Outstanding Teaching, College of Arts & Sciences, University of Kentucky
- Max Steckler Fellowship, Mathematics Department, University of Kentucky
- Scholarships–Creating Opportunities for Applying Mathematics (S-COAM), Mathematics Department, Indiana University of Pennsylvania