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EXPLORATION OF TRIANGLES IN EUCLIDEAN, SPHERICAL, AND HYPERBOLIC SPACES FOR THE HIGH SCHOOL GEOMETRY CLASSROOM

John Rowell

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An Essay Submitted to the
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for the Degree of
Master of Arts

By
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Introduction

I have been teaching geometry for seven years. In those seven years I have seen a change in the direction of the way that geometry is being taught. I have found that there has been a push to apply algebraic topics into geometric applications in every place that one can. In doing so, I have found that geometry has lost its emphasis on being an axiomatic system that is created from the bottom up and instead is a mathematical system full of given properties used to practice algebra.

After taking both Euclidean and non-Euclidean geometry classes, I have found a need to create a unit plan that introduces students to the axiomatic system of geometry while allowing them to build it from the bottom up. I also have found there are many extensions to Euclidean geometry that can be taught at a highschool level that would allow students to see the creation of new geometries through a new and exciting lens.

My first objective in this unit plan is to demonstrate and review the origin of geometry. By first researching Euclid and his Elements, students will begin to understand geometry's origin. My next series of lessons introduce the definitions, postulates, and propositions of his elements. By introducing these topics as Euclid did in his Elements, students will become mathematicians by understanding why each element is built upon previous propositions instead of just how to apply these elements.

My second objective in this unit plan is to recreate the excitement and wonder that comes from seeing a new system of mathematics. By taking a look at planar, spherical and hyperbolic geometry, I hope to bring adventure and discovery into the classroom. I will use the topics of area of triangles and sum of angles in a triangle in each of these geometries to lead to the goals of each lesson. By reviewing the axioms of absolute geometry in each space, students will review the first objective in seeing what it takes to build a geometry. They will be able to then explore and develop geometric systems they have never seen before.

While these objectives may seem ambitious, very few tools are needed for the unit plan. In the first four lessons, the only materials needed are a straightedge and compass. If not, students can also go online using desmos.com/geometry as a free online application to work on constructions. All planar constructions shown in this unit plan have been created using this application [1]. In the final four lessons, it is advised that students use a lacrosse ball or other blank sphere canvas to help visualize spherical and hyperbolic geometry [3] [4]. I have used premade geogebra applications to help students visualize the spheres needed for the guided notes and homework.

The unit plan is structured as eight separate lesson plans. Each lesson plan is designed to be taught in a class that has 70 minutes each time they meet. Each lesson contains guided notes and homework for students along with filled out guided notes and answer keys to the homework for the teachers. It is designed as a unit for the end of the year as a way to review what students have learned and explore past the given curriculum

Lesson 1: Euclid's Elements

Teacher Overview**:**

This lesson is an introduction to Euclidean Geometry. The class starts off with exploration research to find information on both Euclid and Euclid's Elements. The class will then watch a video that goes further into depth on both Euclid and Euclid's Elements. The middle portion of the class will be a time for students to define basic terms used throughout Euclid's Elements along with some of The Elements' definitions. The end of the class will be used to have students start working on becoming more comfortable with compass and straightedge constructions.

Geometric Overview:

Euclid's Elements is a series of books that march through postulates and theorems of planar geometry. Euclid started with five postulates and developed all theorems after that using only the five postulates, a straightedge and compass. These theorems build off of each other and grow into the whole system of geometry. This lesson stresses the history of Euclid and the difference between postulates and propositions. This lesson acts as a set up for taking a deeper look at the postulates followed by looking the development of propositions that are logically deduced from the compass, straightedge and postulates.

Learning Targets:

- 85% of students will be able to recall basic historic facts on Euclid and Euclid's Elements.
- 85% of students will be able to compare Postulates and Propositions.
- 85% of students will be able to classify types of triangles.
- 85% of students will be able to define a circle.
- 85% of students will be able to construct/draw an image using only a compass and straightedge.

Prerequisite Knowledge:

- None

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout
    - Computer
    - Straightedge

    - Compass
    - Student homework assignment
- For teachers
    - Annotated student note sheet
    - Answer key for homework assignment

<u>Lesson Overview:</u>

- Do Now: (10 minutes)
    - Research basic facts on Euclid.
- Notes: (40 minutes)
    - Students will watch video over Euclid’s Elements
    - Students will fill out note sheet on definitions from Euclid’s Elements
- Exit Ticket: (20 Minutes)
    - Students will create emoji art using only a straightedge and compass.

STUDENT HANDOUT

Do Now

Take some time to do some research on basic facts of the mathematician Euclid. Use the following worksheet as a guide.

Name: ____________________________

Birth Year: ________________________

Nationality: _________________________

What was his most important work:

How many books were in the works:

What were the books about:

Find a fun fact not already mentioned:

Notes

- Euclid is known to be the ___________________ of _____________________.

- Geometry has greek roots: Geo means _________________ and metry means __________________.

- Euclid's Elements was a ________ volume textbook on three major topics ____________,___________, and _________.

- In Euclid's Elements he started with basic assumptions also called _________ or ______________. He then proved and deduced other statements known as ____________ and _____________.

- Euclid's Elements was the _________ most printed book in the western world.

- Abraham Lincoln stated, "You can never make a lawyer if you do not understand what ______________ means. and I left my situation in Springfield, went home to my fathers house and stayed there til I could give any ______________ in the ________ __________ of ___________ at sight. I then found out what demonstrate means and went back to my law studies.

Definitions

Postulate: ____

Proposition: ____

____

Straightedge:____

Compass:____

____

Point:____

Line:____

Circle:____

____

____

Angle: ____

____

Triangles: ____

Equilateral Triangle: ____

Isosceles Triangle:____

Scalene Triangle: ____

Right-angled Triangle: ____

Obtuse-angled Triangle: ____

Acute-angled Triangle: ____

Square:____

Parallel: ____

____

____

Equilateral Triangle	Isosceles Triangle	Scalene Triangle	Circle
Right Triangle	Obtuse Triangle	Acute Triangle	Square

Exit Ticket

Use a straightedge and compass to draw the following emojis:

ANNOTATED STUDENT HANDOUT

Do Now

Take some time to do some research on basic facts of the mathematician Euclid. Use the following notes as a guide.

Name: Euclid of Alexandria

Birth Year: 300 BC

Nationality: Greece

What was his most important work:
Elements: books written by Euclid that range from planar geometry, number theory, solid figures and platonic solids.

How many books were in the works: 13

What were the books about:

Book 1:Fundamentals of geometry, theories of triangles parallels and area
Book 2: Geometric algebra
Book 3: Theory of circles
Book 4: Constructions for inscribed and circumscribed figures
Book 5: Theory of abstract proportions
Book 6: Similar figures and proportions in geometry
Book 7-9: Number theory
Book 10: Classification of incommensurables
Book 11: Solid geometry
Book 12: Measurements of figures
Book 13: Regular solids

Find a fun fact not already mentioned:

- Founded the Alexandrian School of Mathematics
- Euclid was taught from Plato's School
- Euclid's optics was another work that was on light and vision.

Lesson Notes

Show the class this eight minute video while having them fill out the guided notes.

https://www.khanacademy.org/math/geometry/hs-geo-transformations/hs-geo-intro-euclid/v/euclid-as-the-father-of-geometry

- Euclid is known to be the Father of Geometry.
- Geometry has Greek roots: Geo means earth and metry means measuremememnt.
- Euclid's Elements was a 13 volume textbook on three major topics Geometry, number theory, and solid geometry.
- In Euclid Elements he started with Basic assumptions also called axioms or postulates. He then proved and deduced other statements known as theorems and propositions.
- Euclid's Elements was the second most printed book in the western world.
- Abraham Lincoln stated, "You can never make a lawyer if you do not understand what demonstrate means. and I left my situation in Springfield, went home to my fathers house and stayed there til I could give any proposition in the six books of Euclid at sight. I then found out what demonstrate means and went back to my law studies.

Definitions:

Postulate: A statement which is accepted without proof. Also known as an axiom.

Proposition: A statement that is accepted with proof. This proof can contain definitions, postulates, common notions, or earlier propositions. Also known as a theorem.

Straightedge: A mathematical tool used to draw straight lines without measurement.

Compass: Mathematical tool used to draw circles or arcs. Original compass was collapsible (loses radius length when picked up). Euclid constructed a proof that shows the ability to copy length.

Point: A point is that which has no part.

Line: A breadthless length. A straight line which lies evenly with the points on itself.

Circle: A plane figure contained by one line such that all the straight lines falling upon it from one point (center) among those lying within the figure equal one another.

Angle: The inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.

Triangles: A polygon with three edges and three vertices.

Equilateral triangle: A triangle which has three sides equal.

Isosceles triangle: A triangle which has two sides alone equal.

Scalene triangle: A triangle which has its three sides unequal.

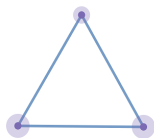
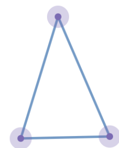
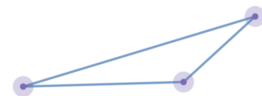
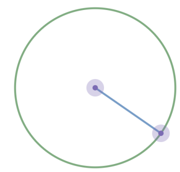
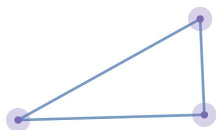
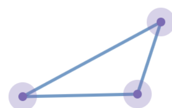
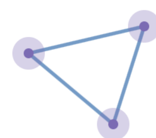
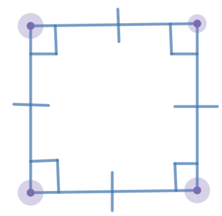
Right angled triangle: A triangle which has a right angle.

Obtuse-angled triangle: A triangle which has an obtuse angle.

Acute-angled triangle: A triangle which has three acute angles.

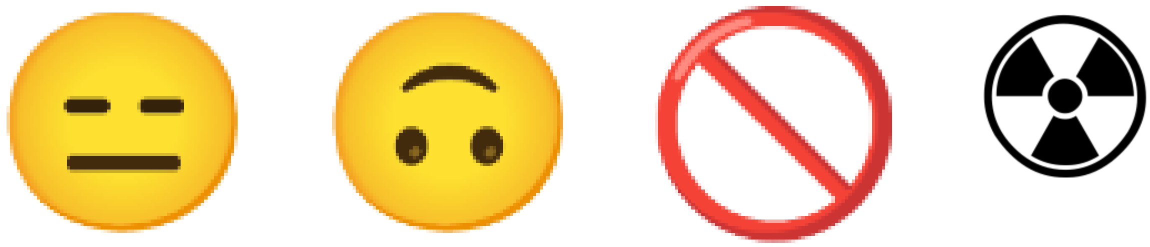
Square:A polygon with four congruent sides and 4 congruent right angles.

Parallel: Straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another either direction.

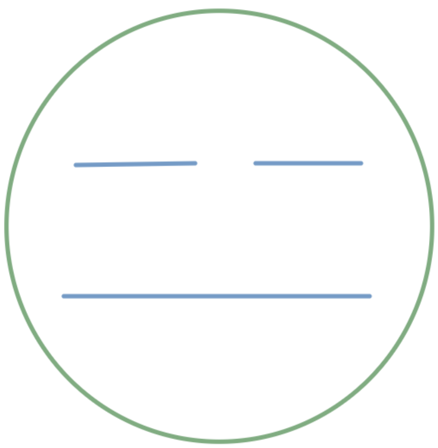
Equilateral Triangle 	**Isosceles Triangle** 	**Scalene Triangle** 	**Circle** 
RIght Triangle 	**Obtuse Triangle** 	**Acute Triangle** 	**Square** 

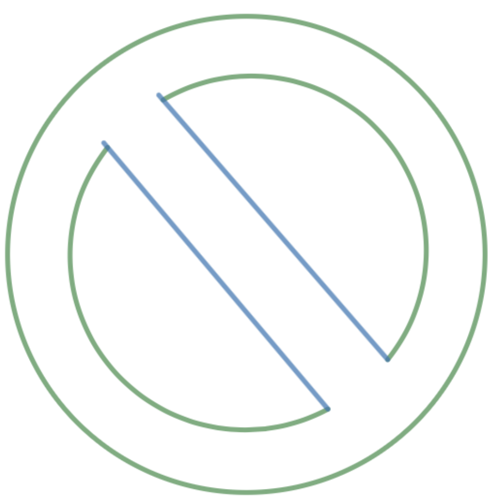
** These diagrams and all planar diagrams after were constructed using desmos.com/geometry*

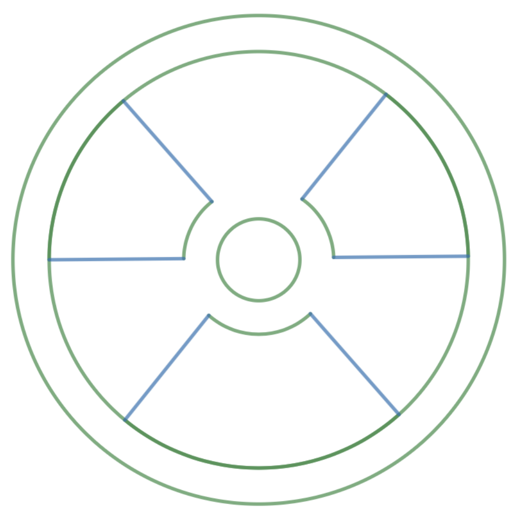
Exit Ticket
Use a straightedge and compass to draw the following emojis:



Examples:







Euclid’s Elements Homework

A compass and straightedge will be used throughout our time in this unit. They are important tools Euclid used to be precise. When Euclid builds the Elements from the postulates and propositions, the precision of each figure is important to keep his work logically sound. Let's start by getting a better understanding of how they work. Remember, a straightedge can be used to connect to points or extend a segment but not copy a measure. A compass(collapsible) is used to create circles or arcs with a specific radius and center. By having a collapsible compass, the radius length is lost after being picked up. Create a piece of artwork only using a straightedge and a compass. All lines must be a straight line made with the stratedge or an arc/circle made with the compass.

Euclid's Elements Homework Answer Key

A compass and straightedge will be used throughout our time in this unit. They are important tools Euclid used to logically build his propositions. As we use these tools to build our logic, let's start by getting a better understanding of how they work. Remember, a straightedge can be used to connect to points or extend a segment but not copy a measure. A compass(collapsible) is used to create circles or arcs with a specific radius and center. By having a collapsible compass, the radius length is lost after being picked up. Create a piece of artwork only using a straightedge and a compass. All lines must be a straight line made with the stratedge or an arc/circle made with the compass.

Sample Art:

Lesson 2: Euclid's Postulates

Teacher Overview**:**

This lesson introduces Euclid's five postulates. These postulates are the building blocks of all propositions that follow in his books. Because of this, it is important to not only go through each one, but also have an understanding of why he introduced each one. By going through and using them on beginning propositions and constructions, students will gain a better understanding of what they are and why we have them.

Geometric Overview:

The main topic of this lesson is Euclid's Postulates. By going through each of the five postulates, we make the connection between the postulates and the axioms of constructions using a straightedge and compass. By looking at each postulate individually and making its connection to constructions we are able to get a better understanding of the building blocks of Euclidean geometry.

Learning Targets:

- 85% of students will be able to recall and recite all 5 of Euclid's postulates.
- 85% of students will be able to construct an equilateral triangle.
- 85% of students will be able to construct a perpendicular bisector.
- 85% of students will be able to construct a square.

Prerequisite Knowledge:

- Lesson 1
- Definition of a square and properties of parallelograms.

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout:
    - Straightedge
    - Compass
    - Student homework assignment
- For teachers
    - Annotated student note sheet
    - Answer key for homework assignment

Lesson Overview:

- Do Now: (5 minutes)

    - Similarities and differences of postulate vs proposition
- Lecture: (25 minutes)
    - Euclid’s postulates
- Exploration:(30 Minutes)
    - Constructions
        - Perpendicular bisector
        - Equilateral triangle
- Exit Ticket: (10 Minutes)
    - Restate Euclid’s postulate with a diagram for each one.

STUDENT HANDOUT

Do Now

Fill out the following chart:

Postulate	**Proposition**
Synonym:	Synonym:
Definition:	Definition:
Similarities:	
Differences:	

Notes

In Euclid's Elements, Book one starts with a list of definitions and a list of five postulates. We looked at some definitions yesterday. Today we will look at the five postulates.

Euclids's Postulates:

1) To draw a straight line from any point to any point.

2) To produce a finite straight line continuously in a straight line.

3) To describe a circle with any center and radius.

4) That all right angles equal one another.

*5) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

We can use these postulates along with a compass and straightedge to start constructing and proving propositions.

Proposition 1: To construct an equilateral triangle on a given finite straight line.

Extension: Using the diagram drawn for proposition 1, we can create a perpendicular bisector. Try redoing the construction and finding the perpendicular bisector for AB:

Exit Ticket

Without looking at notes, can you name all five postulates? Create a construction that illustrates each postulate.

ANNOTATED STUDENT HANDOUT

Do Now

Fill out the following chart:

Postulate	**Proposition**
Synonym: Axiom	Synonym:Theorem
Definition: A statement understood to be true without proof.	Definition: A statement understood to be true given a proof.

Similarities: Both are statements that can be taken as true.
Differences: Postulate is taken to be true without proof. A Proposition needs a proof to be taken as true.

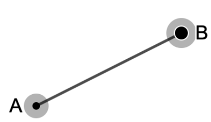
Notes

In Euclid's Elements, Book one starts with a list of definitions and a list of five postulates. We looked at some definitions yesterday, today we will look at the five postulates.

Euclids's Postulates:

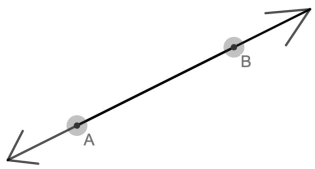
1) To draw a straight line from any point to any point.

Start by drawing two points. Connect the points using a straightedge. One must use a straightedge to connect these dots to make a true line. It is important to note in Euclidean geometry a line is the straight line which is the shortest path from point to point. Another name for this is a ***geodesic***.



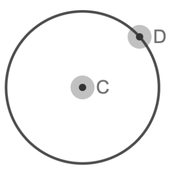
2) To produce a finite straight line continuously in a straight line.

A line can be extended into a line that is infinite by using a straight edge. Although we can not draw a line infinitely, we can add arrows to the end of line segments to signal the line continues indefinitely.



3) To describe a circle with any center and radius.

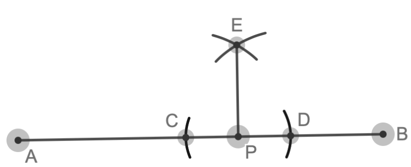
To draw and label a circle use the compass. The end without a pencil will be the center of the circle (point C) and the length between points on the circle and the center will be the radius (line CD).



4) That all right angles equal one another.

All right angles are 90˚. Although this seems like a given, having this postulate allows Euclid to talk about angles and the way they are measured. Let's take the time to construct a right angle:

Start with a line AB and a point P between A and B. Using a compass, mark the same distance on either side of P. Label those points C and D. Then use that compass with the point on C to create an arc directly above P whose radius has length CD. Repeat this process with the compass point on D. The intersection of these arcs should be labeled point E. Use straightedge to create line EP. We have now created right angles ∠ CPE and ∠ DPE.



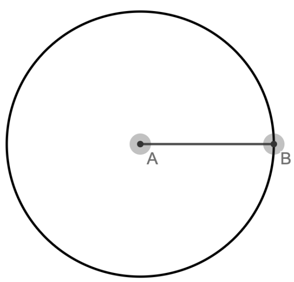
*5) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.**

The first four postulates make up absolute geometry. They are true even in non-Euclidean geometries. We will talk more about non-Euclidean geometries later. For now, know that this postulate is what defines Euclidean geometry. We will take the next lesson to prove other statements are equivalent to this and what it means for our geometry landscape.

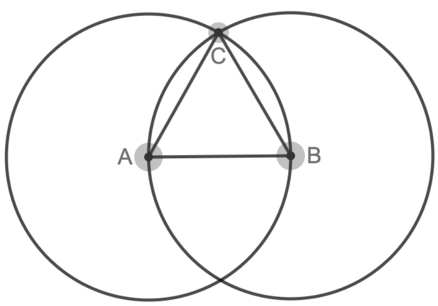
We can use these postulates along with a compass and straightedge to start constructing and proving propositions.

Proposition 1: To construct an equilateral triangle on a given finite straight line.

Start with line AB. Using a compass, create a circle with radius AB and center A.



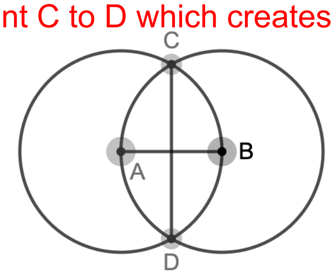
Then make a second circle with radius AB and center B. The two circles will intersect at a point that we will call point C.



Since AC and AB are both the radius for circle A, then AC=AB. Similarly, AB and BC are both the radius for circle B. Thus, AB=BC. Therefore, AB=BC=AC.The points ABC have created an equilateral triangle with side length AB.

Extension: Using the diagram drawn for proposition 1, we can create a perpendicular bisector. Try redoing the construction and finding the perpendicular bisector for AB:

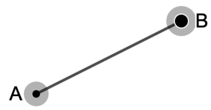
Using the diagram above, Name point D, the second point where the two circles intersect. Using a straightedge, connect point C to D which creates the line CD which is the perpendicular bisector of AB.



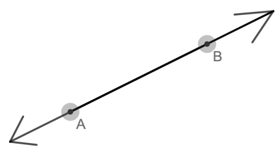
<u>Exit Ticket:</u>

Without looking at notes, can you name all five postulates. Create a construction that illustrates each postulate.

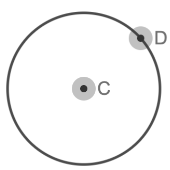
1) To draw a straight line from any point to any point.



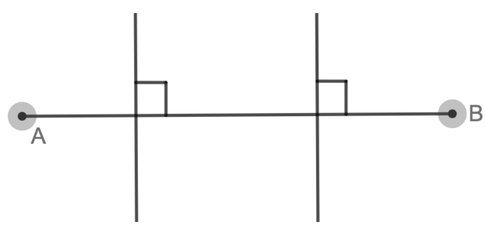
2) To produce a finite straight line continuously in a straight line.



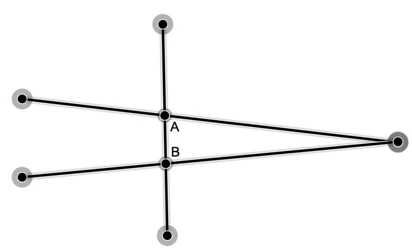
3) To describe a circle with any center and radius.



4) That all right angles equal one another.



*5) That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles. (∠ A+∠ B<180˚)



Euclid's Postulates Homework

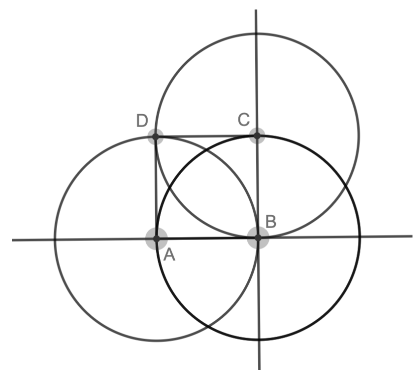
1) Using a similar process to proposition1 and constructing a right angle, construct a square.

2) Using Euclid's postulates, explain how we know if any of the sides of the square are parallel.

Euclid’s Postulates Homework Answer Key

1) Using a similar process to proposition1 and constructing a right angle, construct a square.

Start with line segment AB. Using a straight edge, extend line AB past B. Then use steps from today's notes to create a line perpendicular to AB at point B. Extend the perpendicular line from B. Using a compass, create a circle centered at B with radius AB. The intersection of the circle and perpendicular line is point C. Using the compass, create a circle centered at C with length BC and a circle centered at A with length AB. The intersection of these two circles are points B and D. Therefore, we have square ABCD.



2) Using Euclid's postulates, explain how we know if any of the sides of the square are parallel.

Because we know this is a square, all angles are right angles. Since we have interior angles on the same side that add up to 180, then the two pairs of sides are parallel.

Lesson 3: Euclid’s Propositions

Teacher Overview**:**

This lesson serves as an introduction to Euclid’s Propositions. It has students connect theorems they have learned throughout their geometry class to the propositions in Euclid’s Elements. The lesson helps students look at different features of proofs for propositions. It shows students that some propositions use constructions, while other propositions build off previously proven propositions. By proving three specific propositions (13, 16, 27), students will get a better understanding of the framework of Euclid’s Elements and their connection to the theorems learned in the geometry class. The propositions chosen will set up a proof for equivalent statements to Euclid's fifth postulate.

Geometric Overview:

This lesson takes time to put a spotlight on some of Euclid's most important propositions in his first book. While the postulates might be the building blocks of the Euclidean Elements, propositions 1, 16, and 27 are all key propositions that unlock important information in Euclidean geometry. These postulates not only help build the next propositions listed but they also are key in proving congruent statements to the fifth postulate. This fifth postulate is the postulate that makes Euclidean geometry different from spherical or hyperbolic geometry. By putting an emphasis on these three propositions, we are able to better understand not only why but how Euclidean built this axiomatic system.

Learning Targets:

- 85% of students will be able to understand the framework of Euclid’s first book.
- 85% of students will be able to prove the exterior angle theorem.
- 85% of students will be able to prove the alternate interior angles theorem.
- 85% of students will be able to compare propositions to theorems learned in class.

Prerequisite Knowledge:

- Lessons 1-2
- A background of geometry class with ability to recall theorems from class.

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout:

    - Straightedge
    - Compass
    - Student homework assignment
- For Teachers
    - Annotated student note sheet
    - Answer key for homework assignment

<u>Lesson Overview:</u>
- Do Now: (5 minutes)
    - Construction of equilateral triangle
    - State the exterior angle inequality theorem

- Lecture: (55 minutes)
    - Introduction to propositions
    - Prove the following propositions
        - Proposition 13
        - Proposition 16
        - Proposition 27

- Exit Ticket: (10 Minutes)
    - What proposition proof was most challenging
    - Answer clarification questions from proofs

<u>STUDENT HANDOUT</u>

<u>Do Now</u>

Last class we constructed an equilateral triangle, this construction helps us show we can copy lengths, as well as use our compass and straightedge to build propositions. Try constructing an equilateral triangle on your own now:

Define a linear pair:

What is the exterior angle inequality theorem?

<u>Notes</u>

Euclid's Propositions:

Some propositions are proved through constructions. Euclid's Proposition 1 is a perfect example of this. In this proposition we used a compass and straightedge to construct an equilateral triangle.

Some other examples of this are:

- **Proposition 2: ____**
- **Proposition 9: ____**
- **Proposition 10:____**
- **Proposition 12:____**

Many of the propositions are equivalent to theorems we have learned in the class.

Proposition 13: If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles. We can restate this: If a line intersects another line, the two adjacent angles created are supplementary.

There are many propositions that build off from each other. Euclid is very precise in all of his definitions and ordering of propositions to make sure the logic is sound. This proposition 16 is important on its own but will also be used in proposition 27 to prove congruent alternate interior angles implies that the lines are parallel.

Proposition 16: In a triangle, if one of the sides is extended, then the exterior angle is greater than either of the interior and opposite angles.

The next proposition is an important one.Not only does it use the previous proposition in its own proof but it also is a key component in better understanding the fifth postulate. This postulate will be proven equivalent to many other statements including the sum of a triangle’s angles is 180˚. Many of these equivalent statements need the 27th proposition in their equivalence proofs. We will now take time to not only understand what this proposition is saying but also how to prove it is true.

Proposition 27: In a straight line falling on two straight lines makes the alternate angles equal to one another, then straight lines are parallel to one another.

<u>Exit Ticket</u>

Which of the three propositions was the hardest to prove? What was challenging about it?

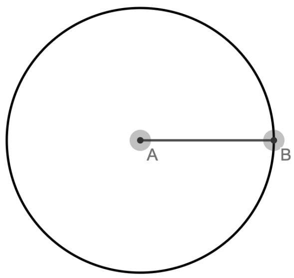
What questions do you still have about the proof?

ANNOTATED STUDENT HANDOUT

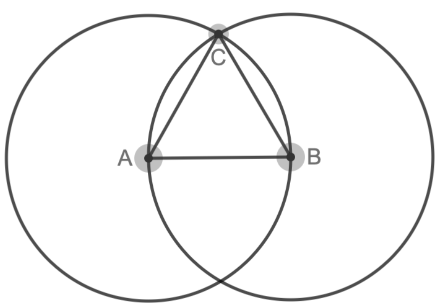
Do Now

Last class we constructed an equilateral triangle, this construction helps us show we can copy lengths, as well as use our compass and straightedge to build propositions. Try constructing an equilateral triangle on your own now:

Start with line AB. Using a compass, create a circle with radius AB and center A.



Then make a second circle with radius AB and center B. The two circles will intersect at a point that we will call point C.



Since AC and AB are both the radius for circle A, then AC=AB. Similarly, AB and BC are both the radius for circle B. Thus, AB=BC. Therefore, AB=BC=AC.The points ABC have created an equilateral triangle with side length AB.

Define a linear pair:

A pair of adjacent supplementary angles formed when two lines intersect.

What is the exterior angle inequality theorem?

The measure of an exterior angle is greater than either of the opposite interior angles.

Notes

Euclid Propositions:

Some propositions are proved through constructions. Euclid's Proposition 1 is a perfect example of this. In this proposition we use a compass and straightedge to construct an equilateral triangle.

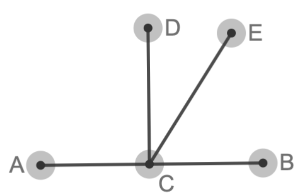
Some other examples of this are:

- **Proposition 2:** To place a straight line equal to a given straight line with one end at a given point.
- **Proposition 9:** To bisect a given rectilinear angle.
- **Proposition 10:** To bisect a finite straight line.
- **Proposition 12:** To draw a line perpendicular to a given infinite straight line from a given point not on it.

Many of the propositions are equivalent to theorems we have learned in the class.

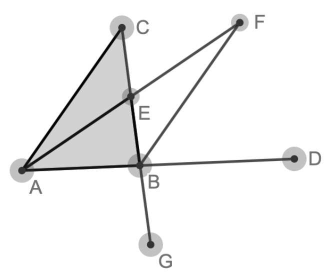
Proposition 13: If a straight line stands on a straight line, then it makes either two right angles or angles whose sum equals two right angles.

This is our linear pair theorem. To prove it, we use two cases. Case 1: ∠ ACD and ∠DCB are right angles. Therefore ACB is 180˚ and its sum is two right angles. Case 2: Assume the straight line EC is not perpendicular to AB. Then construct a perpendicular line DC. Then we know ∠ ACD and ∠DCB are right angles. We also know that ∠ DCE + ∠ECB =90. Then we know that those three angles, ∠ DCE, ∠ECB, and ∠ACD add up to 180.



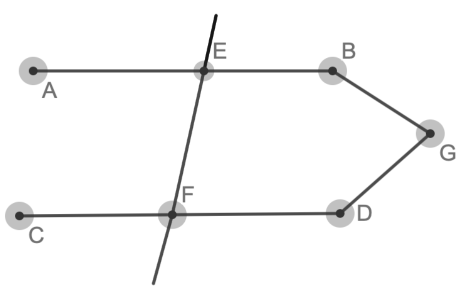
There are many propositions that build off from each other. Euclid is very precise in all of his definitions and ordering of propositions to make sure the logic is sound. This proposition 16 is important on its own but will also be used in proposition 27 to prove congruent alternate interior angles implies that the lines are parallel.

Proposition 16: In a triangle, if one of the sides is extended, then the exterior angle is greater than either of the interior and opposite angles.
Start with triangle ABC and extend side AB to D. Bisect side BC and label it point E, therefore CE =EB. Draw line AE and extend the line to F such that AE= EF. We also know that ∠AEC = ∠BEF because they are vertical angles. Therefore we know△AEC and △BEC are congruent by SAS(Proposition 4). Thus, ∠ACF = ∠BAC. Since ∠CBD= ∠CBF + ∠FBD, and∠CBF = ∠BCA, we can conclude ∠CBD > ∠BCA. We can repeat this process with a bisector on AC, therefore proving ∠ABG > ∠BAC.



The next proposition is an important one.Not only does it use the previous proposition in its own proof but it also is a key component in better understanding the fifth postulate. This postulate will be proven equivalent to many other statements including the sum of a triangle's angles is 180˚. Many of these equivalent statements need the 27th proposition in their equivalence proofs. We will now take time to not only understand what this proposition is saying but also how to prove it is true.

Proposition 27: In a straight line falling on two straight lines makes the alternate angles equal to one another, then straight lines are parallel to one another.
Start with lines AB and CD, and EF such that line EF intersects lines AB and CD making congruent alternate interior angles, so ∠AEF = ∠EFD. We will say that AB is parallel to CD. If it was not the case, AB and CD would then intersect on either the side containing B and D or the side containing A and C. Let assume lines AB and CD intersect at point G on the side containing B and D. Therefore there is a triangle EFG. By proposition 16, we know that ∠AEF > EFG but this can't be true because ∠AEF = ∠EFD. We then know G must be on the side containing AC. But by the same reasoning we know ∠AEF = ∠EFD and ∠AEF < ∠EFD can not be true. Thus AB is parallel to CD.



<u>Exit Ticket</u>
Which of the three propositions was the hardest to prove?
Take a poll to see which proposition was hardest for students.

What questions do you still have about the proof?
Take time to review questions students have from the proofs above. Use the notes from each proposition to help answer any questions.

Euclid's Propositions Homework

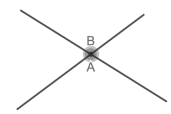
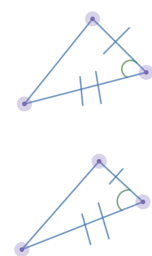
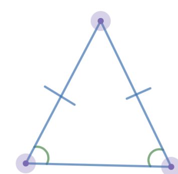
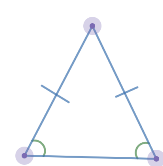
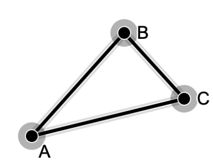
Find five propositions that have been used in our class this year, then fill out the chart:

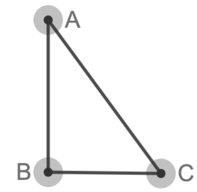
Proposition Number	Proposition	Name of Theorem	Illustrated Example

<u>Euclid's Postulates Homework Answer Key</u>

Find five propositions that have been used in our class this year, then fill out the chart:

Here are some examples:

Proposition Number	Proposition	Name of Theorem	Illustrated Example
15	If two starting lines cut one another, then they make the vertical angles equal to one another.	Vertical Angles Theorem	 ∠A=∠B
4	If two triangles have two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, then they also have the base equal to the base, the triangle equals the triangle, and the remaining angles equal the remaining angles respectively, namely those opposite the equal sides.	SAS Triangle Congruence Theorem	
5	In isosceles triangles the angles at the base equal one another, and, if the equal straight lines are produced further, then the angles under the base equal one another.	Isosceles Triangle Theorem	
6	If in a triangle two angles equal one another, then the sides opposite the equal angles also equal one another.	Base Angle Theorem	
20	In any triangle the sum of any two sides is greater than the remaining one.	Triangle Inequality Theorem	 AB+BC>AC AB+AC>BC BC+AC>AB

48	If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right.	Converse to Pythagorean Theorem	 If $AB^2 + BC^2 = AC^2$, then ABC is a right triangle.

Lesson 4: The Parallel Postulate

Teacher Overview**:**

This lesson helps students develop the parallel postulate and its equivalent statements. We will focus on Euclid's Postulate 5 and the Playfair's Postulate. By proving one implies the other and vice versa, we will show that the two statements are equivalent. We will then show how this leads to and is equivalent to the triangle sum postulate.

Geometric Overview:

This lesson takes an in depth look at Euclid's Fifth Postulate. This postulate is the defining postulate of Euclidean Geometry. By taking a look at the fifth postulate and its congruent statements (Playfair's Postulate and Triangle Angle Sum) we get a better understanding of its importance to planar geometry. In proving their congruence, we use previous propositions that help solidify Euclid's Fifth Postulate and its role in planar geometry. Proving the fifth postulate will also set up the ability for students to ask the question what does the geometry system look like if we do not have this postulate in place.

Learning Targets:

- 85% students will be able to state Euclid's Postulate 5.
- 85% of students will be able to state Playfair's Postulate.
- 85% of students will understand equivalence statements.
- 85% of students will be able to find the sum of the angles in any Euclidean triangle.
- 85% of students will get an introduction into college level proof writing.

Prerequisite Knowledge:

- Lessons 1-3

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
  - Student note handout:
  - Straightedge
  - Compass
  - Student homework assignment
- For Teachers
  - Annotated student note sheet
  - Answer key for homework assignment

Lesson Overview:

- Do Now: (5 minutes)
    - Review questions on postulate vs theorem
    - What is their definition of lines being parallel
- Lecture: (60 minutes)
    - Go through the following proofs as students follow along with their note page. Allow students to give input on proofs as you are writing them.
        - Euclid’s Postulate 5 => Playfair’s Postulate
        - Playfair’s Postulate => Euclid’s Postulate 5
        - Playfair’s Postulate => Triangle Sum
- Exit Ticket: (5 minutes)
    - How has your definition of a parallel postulate changed?

STUDENT HANDOUT

Do Now

What are the differences between a definition, postulate and theorem?

What does it mean for two lines to be parallel? Which theorems have we used to prove parallel lines in class?

Notes

Euclid’s Postulate 5:

Playfair’s Postulate:

Logic Map:

We will now take some time to prove some of these implications. Although all statements are congruent, In the following proofs we will prove that Euclid’s Postulate is congruent to Playfair’s Postulate and that Playfair’s Postulate implies the Triangle Angle Sum.

Proofs

Euclid’s Postulate 5 (with help from postulates 1-4) => Playfair’s Postulate:

Playfair’s Postulate=> Euclid’s Postulate 5

Playfair’s Postulate=> Triangle Angle Sum

Exit Ticket

How has your definition of a parallel postulate changed?

ANNOTATED STUDENT HANDOUT

Do Now

What are the differences between a definition, postulate and theorem?

A definition is a statement that describes what the term means.
A postulate (axiom) is a statement understood to be true without proof.
A proposition or theorem is a statement understood to be true given a proof.

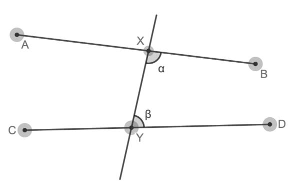
What does it mean for two lines to be parallel? Which theorems have we used to prove parallel lines in class?

Two infinite lines on a plane that do not intersect each other. Ways to prove lines are parallel will vary. Some examples of theorems would be converse to corresponding congruent angles theorem or converse to alternate interior angles theorem.

Notes

Euclid's Postulate 5:

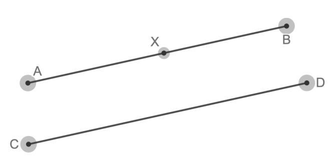
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



If $\alpha+\beta<180$, then AB intersects CD on the side of α and β.

Playfair's Postulate:

Given a line CD and a point not on the line X, there is a unique line AB that contains X and is parallel to CD.



Logic Map:

Euclid's Postulate 5

Playfair's Postulate ⟷ Triangle Angle Sum

We will now take some time to prove some of these implications. Although all statements are congruent, In the following proofs we will prove that Euclid's Postulate is congruent to Playfair's Postulate and that Playfair's Postulate implies the Triangle Angle Sum.

Proofs

Euclid’s Postulate 5 (with help from postulates 1-4) =>Playfair’s Postulate

Assuming Euclid’s Postulate 5 is true. Let the line CD be given and let X be the point not on CD. It will be shown that there is a unique line through X that is parallel to CD. Let Y be any point on the line CD, and let AB be a line containing X such that the alternating angles $\angle DYX$ and $\angle AXY$ are equal. By Euclid’s proposition 27, the line AB is necessarily parallel to the line CD. If A’B’ is any other line through X then either

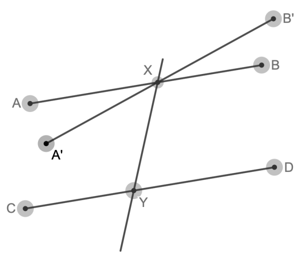
$\angle A'XY < \angle AXY = \angle DYX = 180^\circ - \angle CYX$

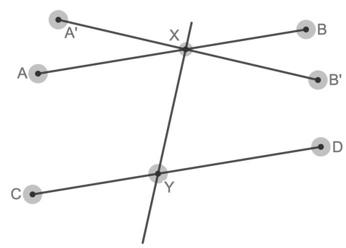
or

$\angle B'XY < \angle BXY = \angle CYX = 180^\circ - \angle DYX$

In other words, either $\angle A'XY + \angle CYX < 180$ Or $\angle A'XY + \angle CYX < 180$.

In either case it follows from Euclid’s Postulate 5 that the line A’B’ intersects the line CD.





Playfair’s Postulate=>Euclid’s Postulate 5

Assume Playfair’s Postulate is true. Let the line XY intersect the line A’B’ and the line CD such that $\angle DYX + \angle B'XY < 180$. Let Y be any point on the line CD, and AB be a line containing X such that the alternate interior angles $\angle DYX$ and $\angle AXY$ are congruent. By Euclid’s proposition 27, the line AB is necessarily parallel to the line CD. Using Playfairs’s Postulate we know AB is a unique line through X parallel to the line CD. Therefore, the line A’B’ must intersect the line CD.

We now must show that the line A’B’ intersects the line CD on the side containing $\angle DYX$ and $\angle B'XY$. To prove this, we use proof by contradiction. Let's assume that the line A’B doesn’t intersect on side D but instead at a point E on the line CD.

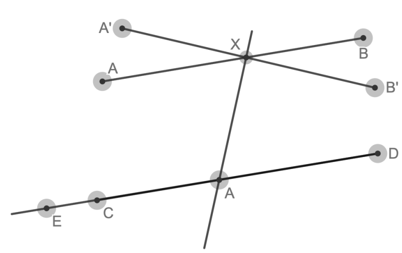
It was assumed:

$\angle DYX + \angle B'XY < 180$.

By definition of linear pair:

$\angle DYX + \angle CYX = 180$ and $\angle A'XY + \angle B'XY = 180$

Thus $\angle CYX + \angle A'XY < 180$.

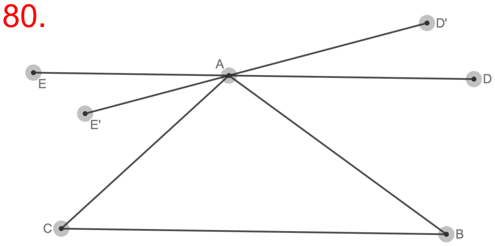


This contradicts Euclid’s Proposition 17, that two angles in the same triangle can’t add up to more than 180°. Thus the line A’B’ must intersect on the same side as $\angle DYX$ and $\angle B'XY$.

Playfair's Postulate => Triangle Angle Sum

Assume Playfair's Postulate, and let △ABC be given. Let DE be the unique line through point A and parallel to BC. Let D'A be a straight line such that ∠BAD'= ∠ABC. By Euclids's Proposition 27, D'A is parallel to BC and hence it must be identical to DE. Thus ∠BAD= ∠ABC and similarly ∠EAC= ∠ BCA. Consequently,

∠ABC + ∠BCA+ ∠CAB= ∠BAD + ∠EAC+ ∠CAB= 180.



Exit Ticket

How has your definition of a parallel postulate changed?

Use this question as a think, pair, share. Start with having the students write their answers. Then have students share with a partner, then have students share with class.

The Parallel Postulate Homework

1) What does it mean for statements to be equivalent?

2) Rewrite Euclid's Postulate 5. Make sure to create a diagram:

3) Rewrite Playfair's Postulate. Make sure to create a diagram:

4) Rewrite the Triangle Angle Sum. Make sure to create a diagram:

5) Recreate the logic map of Euclid's Postulate 5 that shows the congruent statements that we have proven in our lesson:

6) Which of the equivalent statements makes the most sense to you. Explain why.

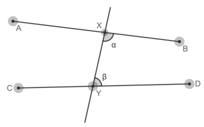
Parallel Postulate Homework Answer Key

1) What does it mean for statements to be equivalent?

For two statements a and b to be equivalent, one must prove a implies b and b implies a.

2) Rewrite Euclid's Postulate 5. Make sure to create a diagram:

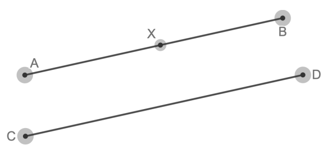
That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



If $\alpha+\beta<180$, then AB intersects CD on the side of α and β.

3) Rewrite Playfair's Postulate. Make sure to create a diagram:

Given a line CD and a point not on the line X, there is a unique line AB that contains X and is parallel to CD.



4) Rewrite the Triangle Angle Sum. Make sure to create a diagram:

The sum of the angles in every triangle is 180˚.

5) Recreate the logic map of Euclid's Postulate 5 that shows the congruent statements that we have proven in our lesson:

Playfair's Postulate, Triangle Angle Sum, Converse to the Alternate Interior Angles Theorem.

6) Which of the equivalent statements makes the most sense to you. Explain why.

Various Answers can be accepted.

Lesson 5: Circles in Euclidean Geometry

Teacher Overview:

This lesson is an introduction to circles in Euclidean geometry. In the lesson, students are introduced to circumference and the number π through archimedes method of exhaustion. By finding the perimeter of inscribed and circumscribed regular polygons, students get an accurate estimation of pi. The lesson then introduces the idea of radians as a form of angle measurement. It is introduced through looking at radians around a circle to help connect the circumference that they just learned to angle measurement radians.

Geometric Overview:

This lesson serves as an exploration into Euclidean circles. By introducing π the same way that Archimedes did,we will see a better connection between π and the Euclidean circle. As we move to spherical and hyperbolic geometry it will be important to have a good foundation of circles and their geometric definition. This lesson also introduces radians as a measurement for angles which will be vital when looking at triangles in both spherical and hyperbolic geometries.

Learning Targets:

- 85% of students will be able to find the perimeter of an inscribed regular polygon.
- 85% of students will be able to find the perimeter of a circumscribed regular polygon.
- 85% of students will be able to develop the definition of π.
- 85% of students will be able to find the circumference of a circle.
- 85% of students will be able to convert radians into degrees.
- 85% of students will be able to convert degrees into radians.

Prerequisite Knowledge:

- Lessons 1-4

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout
    - Student homework assignment
    - String
- For Teachers
    - Annotated student note sheet

    - Answer Key for homework assignment:

<u>Lesson Overview:</u>

- Do Now: (10 minutes)
    - Definition of circle in geometry
    - Definition of circle in algebra

  Exploration: (30 minutes)
    - Circumference and π definitions
- Notes(5 minutes)
    - π and circumference definitions
- Exploration(10 Minutes)
    - Radians introduction
- Notes: (5 minutes)
    - Angle measurements-Radians and Degrees
- Exit Ticket: (10 minutes)
    - Find the circumference of a circle
    - Convert angle from radians into degrees

STUDENT HANDOUT

Do Now

What is the geometric definition of a circle?

What is the geometric definition of regular polygon?

What is the geometric definition of inscribed and circumscribed polygons?

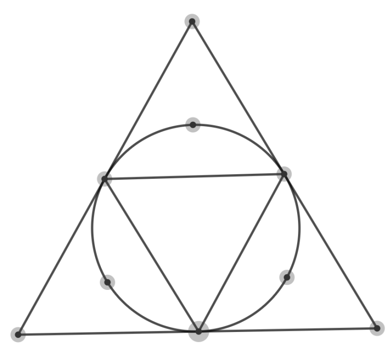
What is the algebraic equation for a circle with center (h,k) and radius length r?

Exploration

Here is a circumscribed and inscribed triangle with circle radius 1 cm. Find the perimeter of both and then find the average of the two.

Inscribed triangle perimeter =

Circumscribed triangle perimeter=



Fill in the chart:

	3	**4**	**6**	**8**	**N**
Inscribed Diagram					
Circumscribed Diagram					
Inscribed Perimeter					
Circumscribed Perimeter					
Average Perimeter					

Notes

Using the formula we found for the average of the circumscribed and inscribed perimeters we can find the estimate for polygons with many more sides:

Formula =

10 sides:

20 sides:

50 sides:

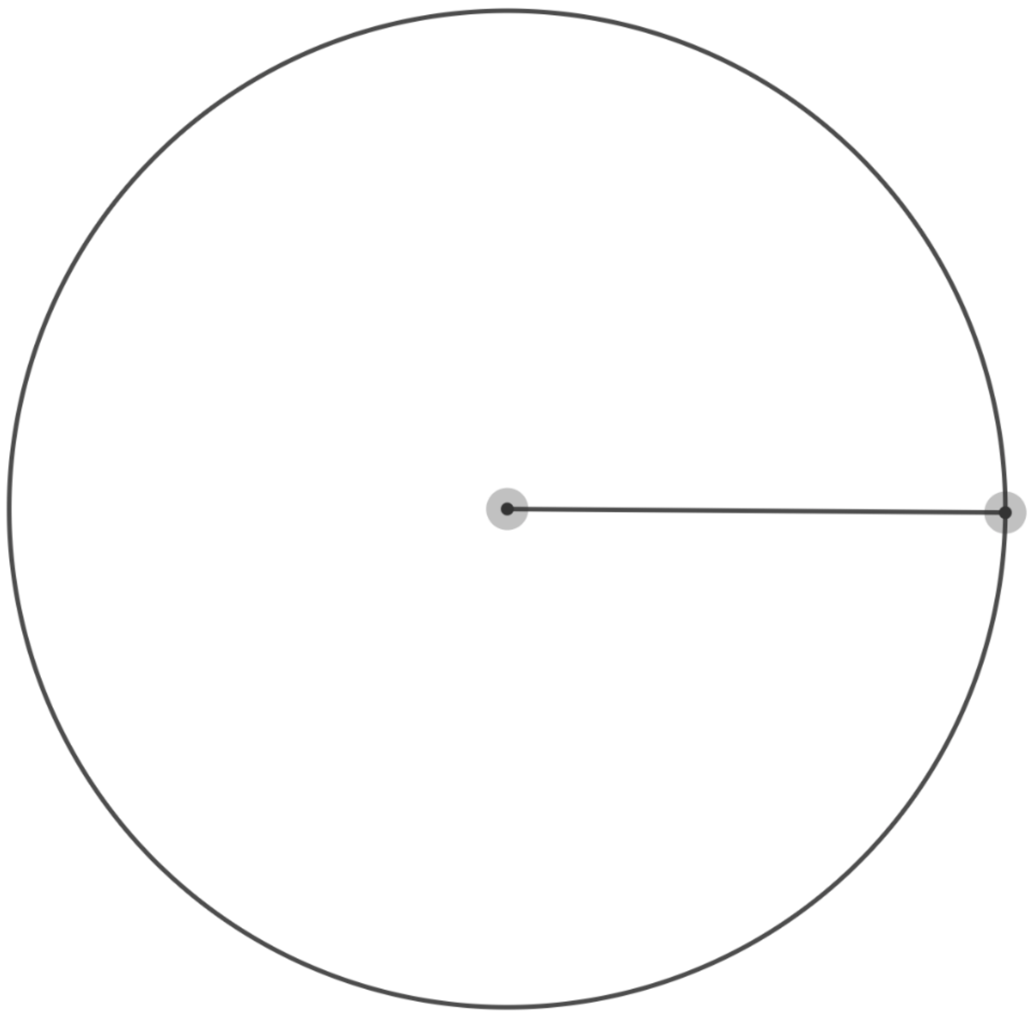
100 sides:

Archimedes found that as the numbers of sides got larger and larger, the estimation of the circumference of the circle became more and more accurate. When taking the circumference and dividing it by the diameter of the circle, in our case 2, he found the number π. We can define the ratio as

_____ = _____________≈_________

Exploration

Take a piece of string that is the length of the length of the radius and wrap it around the edge of the circle and mark how many radii it takes to go completely around the circle



How many radii did it take to go around the circle?

How does this relate to the circumference of the radius?

Notes

Mathematicians use this as a measurement for angles where 1 radius around the circumference of the circle is equal to 1 radian. This means there are 2π radians in the circumference of a circle (360˚) or 1 radian every 180˚. Thus we can use the following conversion:

__________ = ___________

We can convert degrees into radians by multying by ______.

We can convert radians into degrees by multying by ______.

Exit Ticket

Find the circumference of a circle whose radius is:

a) 10 b) 9

Convert the following angles from degrees into radians:

a) 360˚ b) 180˚ c) 60˚ d) 70˚

Convert the following angles from degrees into radians:

a) 2π b) π c) $\frac{\pi}{4}$ d) 1 radian

<u>**ANNOTATED STUDENT HANDOUT**</u>

Do Now

What is the geometric definition of a circle?

A plane figure contained by one line such that all the straight lines falling upon it from one point (center) among those lying within the figure equal one another.

What is the geometric definition of regular polygon?

A polygon in which all angles are equal in measurement and all sides are equal in measurement.

What is the geometric definition of inscribed and circumscribed polygons?

A inscribed polygon is a polygon whose vertices are all points of the circle.
A circumscribed polygon is a polygon whose sides all intersect the circle at one point.

What is the algebraic equation for a circle with center (h,k) and radius length r?

$(x - h)^2 + (y - k)^2 = r^2$

Exploration

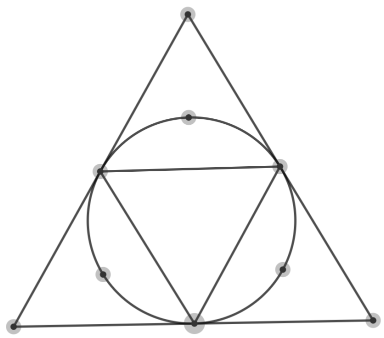
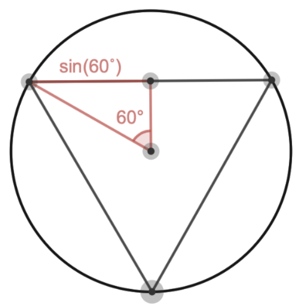
Here is a circumscribed and inscribed triangle with circle radius 1 cm. Find the perimeter of both and then find the average of the two.

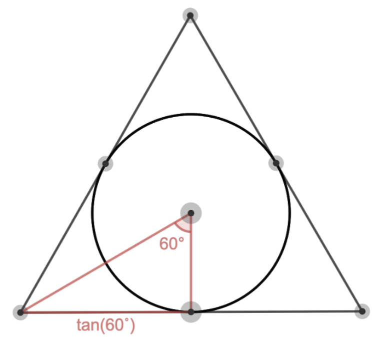
Inscribed triangle perimeter $=(2 * sin\,(\frac{360}{2*3}))\ *\ 3$

=5.20 cm.

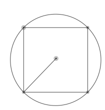
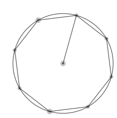
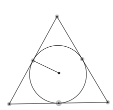
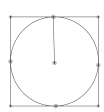
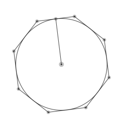
Circumscribed triangle perimeter$=(2 * tan\,(\frac{360}{2*3}))\ *\ 3$

=10.39 cm.



Given the same circle with radius 1 cm is given, fill in the chart

	3	**4**	**8**	**N**
Inscribed Diagram				
Circumscribed Diagram				
Inscribed Perimeter	$(2 * sin\,(\frac{360}{2*3}))\ *\ 3$ =5.20 cm.	$(2 * sin\,(\frac{360}{2*4}))\ *\ 4$ =5.66 cm.	$(2 * sin\,(\frac{360}{2*8}))\ *\ 8$ =6.12 cm.	$(2 * sin\,(\frac{360}{2*n}))\ *\ n$
Circumscribed Perimeter	$(2 * tan\,(\frac{360}{2*3}))\ *\ 3$ =10.39 cm.	$(2 * tan\,(\frac{360}{2*4}))\ *\ 4$ =8 cm.	$(2 * tan\,(\frac{360}{2*3}))\ *\ 8$ =6.63 cm.	$(2 * tan\,(\frac{360}{2*n}))\ *\ n$
Average Perimeter	$\frac{5.2+10.39}{2}$ =7.79 cm.	$\frac{5.66+8}{2}$ =6.83 cm.	$\frac{6.12+6.63}{2}$ =6.38 cm.	$\frac{(2*sin\,(\frac{360}{2*n}))*n\ +\ (2*tan\,(\frac{360}{2*n}))*n}{2}$

Notes

Using the formula we found for the average of the circumscribed and inscribed perimeters we can find the estimate for polygons with many more sides:

Formula = $\frac{(2*sin\,(\frac{360}{2*n}))*n+\ (2*tan\,(\frac{360}{2*n}))*n}{2}$

10 sides:=$\frac{(2*sin\,(\frac{360}{2*10}))*10+\ (2*tan\,(\frac{360}{2*10}))*10}{2}$=6.498 cm.

20 sides:$\frac{(2*sin\,(\frac{360}{2*20}))*20+\ (2*tan\,(\frac{360}{2*20}))*20}{2}$=6.335 cm.

50 sides:$\frac{(2*sin\,(\frac{360}{2*50}))*50+\ (2*tan\,(\frac{360}{2*50}))*50}{2}$=6.291cm.

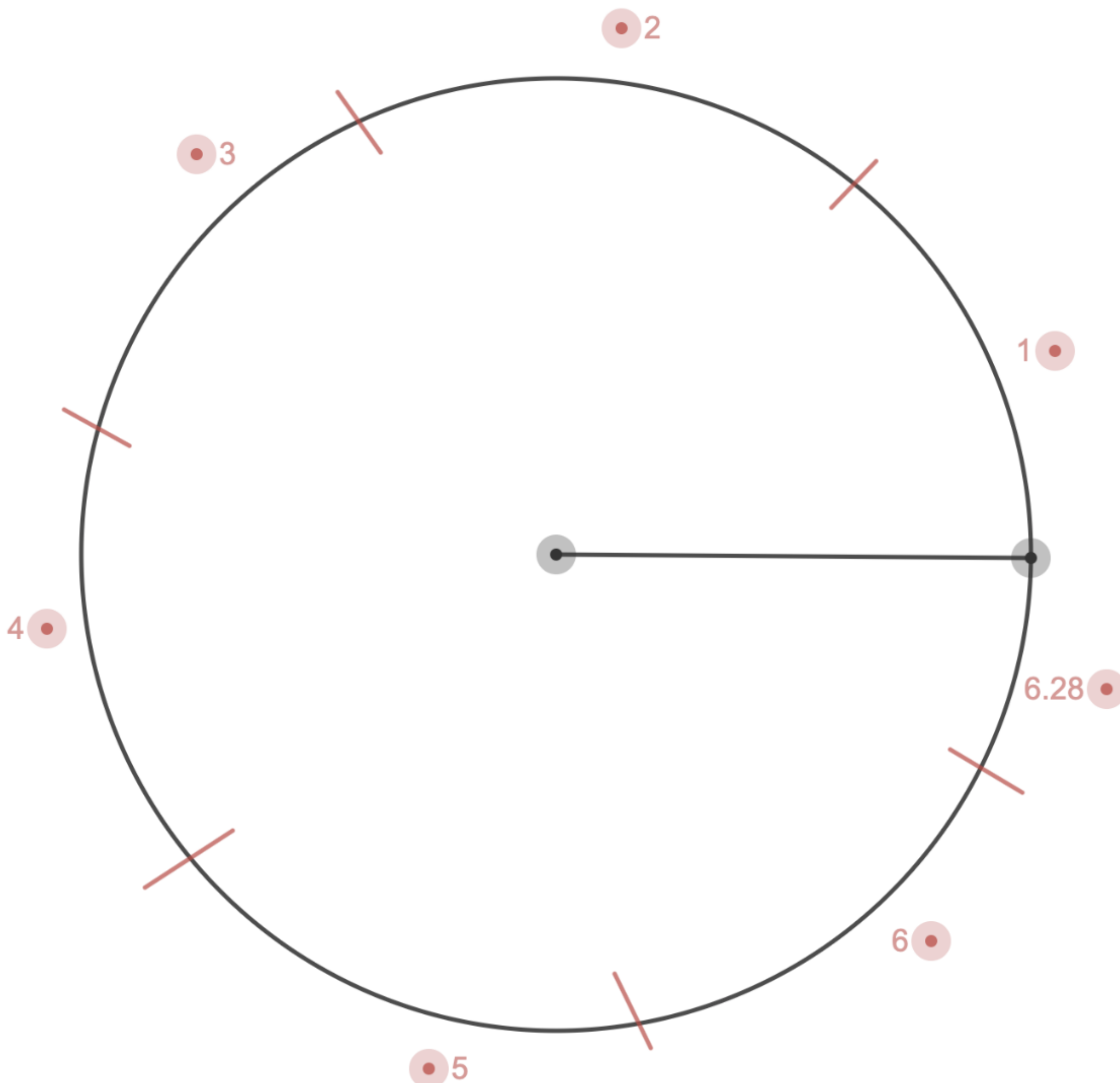
100 sides:$\frac{(2*sin\,(\frac{360}{2*100}))*100+\ (2*tan\,(\frac{360}{2*100}))*100}{2}$=6.285 cm.

Archimedes found that as the numbers of sides got larger and larger, the estimation of the circumference of the circle became more and more accurate. When taking the circumference and dividing it by the diameter of the circle, in our case 2, he found the number π. We can define the ratio as

π = $\frac{circumference}{diameter}$≈3.14159

Exploration

Take a piece of string that is the length of the length of the radius and wrap it around the edge of the circle and mark how many radii it takes to go completely around the circle



How many radii did it take to go around the circle?

6.28

How does this relate to the circumference of the radius?

Circumference= $2\pi * r$

This shows there are 2π radii that go around the circle.

Notes

Mathematicians use this as a measurement for angles where 1 radius around the circumference of the circle is equal to 1 radian. This means there are 2π radians in the circumference of a circle (360˚) or 1 radian every 180˚. Thus we can use the following conversion:

π radians = 180˚

We can convert degrees into radians by multying by $\frac{\pi}{180}$.

We can convert radians into degrees by multying by $\frac{180}{\pi}$.

Exit Ticket

Find the circumference of a circle whose radius is:

a) 10 in.

circumference=2π*r
=2π*10
=20π
≈62.83 in.

b) 9 in.

circumference= 2π*r
=2π*9
=18π
≈ 56.55 in.

Convert the following angles from degrees into radians:

a) 360˚

=2π radians

b) 180˚

=π radians

c) 60˚

$= \frac{60}{180} * \pi$ radians

$= \frac{\pi}{3}$ radians

d) 70˚

$= \frac{70}{180} * \pi$ radians

$= \frac{7\pi}{18}$ radians

Convert the following angles from degrees into radians:

a)2π

=360˚

b) π

=180˚

c) $\frac{\pi}{4}$

$= \frac{\pi}{4} * \frac{180}{\pi}$

$= \frac{180}{4}°$

$= 45°$

d) 1 radian

$= 1 * \frac{180}{\pi}$

$= \frac{180}{\pi}°$

= 56.30˚

Euclidean Spherical Geometry Homework

1) Find the circumference of a circle whose diameter is:

a) 5 ft.   b) 12 ft.

2) Find the circumference of a circle whose radius is:

a) 3 cm.   b) 4.5 cm.

3) Find the radius of a circle whose circumference is:

a) 20π in.   b) 40 in.

4) Convert the following angles from degrees into radians:

a) 240° b) 310° c) 58°

5) Convert the following angles from degrees into radians:

a) 3π b) $\frac{5\pi}{6}$ c) 3 radians

Euclidean Circle Homework Answer Key

1) Find the circumference of a circle whose diameter is:

a) 5 ft.

circumference=d*π

=5π

≈15.71 ft.

b) 12 ft.

circumference= d*π

=12π

≈37.70 ft.

2) Find the circumference of a circle whose radius is:

a) 3 cm.

circumference=2π*r

=2π*3

=6π

≈18.85 cm.

b) 4.5 cm.

circumference= 2π*r

=2π*4.5

=9π

≈ 28.27 cm.

3) Find the radius of a circle whose circumference is:

a. 20π

circumference=2π*r

20π=2π*r

10 in. =r

b. 40

circumference = 2π*r

40 =2π*r

$\frac{40}{2\pi}$ =r

r ≈ 6.37 in.

4) Convert the following angles from degrees into radians:

a) 240˚

$=\frac{240}{180} * \pi$ radians

$= \frac{4\pi}{3}$ radians

b) 310˚

$=\frac{310}{180} * \pi$ radians

$=\frac{31\pi}{18}$ radians

c) 58˚

$= \frac{58}{180} * \pi$ radians

≈1.01 radians

5) Convert the following angles from degrees into radians:

a) 3π

$=3\pi * \frac{180}{\pi}$

$= 540^\circ$

b) $\frac{5\pi}{6}$

$=\frac{5\pi}{6} * \frac{180}{\pi}$

$= 150^\circ$

c) 3 radians

$= 2 * \frac{180}{\pi}$

$= \frac{180}{\pi}^\circ$

$\approx 171.89^\circ$

Lesson 6: Spherical Geometry

Teacher Overview:

This lesson introduces spherical geometry to students. It first has them define what a sphere is and looks at the definition of a geodesic in both Euclidean and spherical geometry. Students will then go through an exploration in spherical geometry and its postulates. Students will take the second half of class to define lunes and learn how to find their area.

Geometric Overview:

By introducing spherical geometry students begin to see a geometry outside of the planar world. To introduce spherical geometry, we look at the axioms of absolute geometry to make sure they still hold in the spherical space. After showing the axioms hold, we start looking at how area works in spherical geometry. This sets us up for proving the area of a triangle in spherical geometry along with proving the angle sum of a triangle in spherical geometry is greater than π.

Learning Targets:

- 85% of students will be able to define a sphere.
- 85% of students will be able to compare geodesic
- 85% of students will be able to develop the postulates in spherical geometry
- 85% of students will be able to apply the formula of area of a lune.

Prerequisite Knowledge:

- Lessons 1-5

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout
    - Student homework assignment
    - Lacrosse ball
    - Rubberbands
- For Teachers
    - Annotated student note sheet
    - Answer Key for homework assignment:

Lesson Overview:

- Do Now: (10 minutes)

- What is the shortest distance between two lines?
- What is a sphere?
- greatcirclemap.com

- Definitions: (10 minutes)
  - Spheres, geodesics,
- Exploration: (15 minutes)
  - Drawing great circles on a sphere
- Notes: (25 minutes)
  - Definitions of antipodal points and lunes
  - Formula for area of lune
- Exit Ticket: (10 minutes)
  - Use a lacrosse ball and rubber bands to create a lune that has a measurement of 90˚. Find the area of the lune.

STUDENT HANDOUT

Do Now

Given two points in a plane, what is the shortest distance between the two points? What about the shortest distance between a point and a line? How do you know it's the shortest distance?

What is the geometric definition of a sphere?

Go to www.greatcirclemap.com:

Check the flight path to any two cities of your choosing. What do you notice? What do you wonder?

Check the flight from Cleveland to Tokyo. Make sure to compare the flight on the satellite map vs the 3D Globe. What do you notice? What do you wonder?

Definitions

A______________is the set of points in three dimensional space that are ____________ from one fixed point (the center).

A ______________is the shortest distance between any two points.

In the geometry flat space, known as______________ ______________, the ______________ is the straight line between those points.

In ______________ ______________ the geodesic is a____________ __________.

Exploration

A rubber band is an elastic band that when taught will mold to the shortest distance or geodesic.

- Wrap the rubber band around a book. Notice when wrapped around a book (flat space) the rubber band takes the path of a ______________ ___________.
- Wrap the rubberband around the lacrosse ball. Notice that on a lacrosse ball whose surface is a sphere, the rubberband will only stay on the ball when the rubber band is on a ______________ that cuts the lacrosse ball in half. This is a ______________ ______________.
- Using a pen to trace the path of the great circle and then restart. Answer the following questions
    - How many great circles can you make?
    - Can you extend a line segment (part of a great circle, connecting two points) into an infinite line?
    - Can you make a circle that's not a great circle?
    - Can you create a 90 degree angle between two circles?
    - Can you create a closed figure using only geodesic lines?

Notes

______________ ____________ are points that correspond to opposite ends of a diameter of the sphere.

A _________ is one of the regions between two great circles and the antipodal points they cross.

A ______________ is given by the angle θ at the center of the sphere between the two great circles.

In order to talk more about lunes and other figures on a sphere we will assume the following properties of a sphere:

1) The sphere of radius R has area ______________.

2) The area of a union of _____-______________ regions is the _______ of their areas.

3) The areas of ______________ regions are __________.

4) The ratio of the __________ enclosed by two great circles to the ____________ of the whole sphere is the same as the ratio of the __________ ______________ __________ to 2π˚.

By assuming the fourth property we can use the following formula:

Area of lune=

Exit Ticket

Use two and rubber bands to create a lune that is 90˚.

Assuming the radius of the lacrosse ball is 1 inch, what is the area of the lune you have created?

ANNOTATED STUDENT HANDOUT

Do Now

Given two points in a plane, what is the shortest distance between the two points? What about the shortest distance between a point and a line? How do you know it's the shortest distance?

The shortest distance between two points is known as a geodesic. In Euclidean geometry the geodesic is made with a straight line between the two points. The shortest distance between a point and a line is the straight line that goes through the point and is perpendicular to the given line.

What is a sphere?

A sphere is the set of points in three dimensional space that are equidistant from one fixed point (the center).

Go to www.greatcirclemap.com :

Check the flight path to any city of your choosing. What do you notice? What do you wonder?

Students should notice that flight paths are not straight lines on the satellite map but look straight on the globe. Students should wonder why the planes don't take a straight line path.

Check the flight from Cleveland to Tokyo. Make sure to compare the flight on the satellite map vs the 3D Globe. What do you notice? What do you wonder?

This flight path shows an exaggerated example of the path not being a straight line allowing students to clearly see that the geodesic is not a straight line on a satellite map.

Definitions

A sphere is the set of points in three dimensional space that are equidistant from one fixed point (the center).

A geodesic is the shortest distance between any two points.

In the geometry flat space, known as Euclidean geometry, the geodesic is the straight line between those points.

In spherical geometry the geodesic is a great circle.

Exploration of Great Circle

Give each student a rubber band and lacrosse ball to help fill out the following exploration.

A rubber band is an elastic band that when taught will mold to the shortest distance or geodesic.

- Wrap the rubber band around a book. Notice when wrapped around a book (flat space) the rubber band takes the path of a straight line.
- Wrap the rubberband around the lacrosse ball. Notice that on a lacrosse ball(sphere), the rubberband will only stay on the ball when the rubber band is on a circle that cuts the lacrosse ball in half. This is a great circle.
- Using a pen trace the path of the great circle and then restart. Answer the following questions
    - How many great circles can you make?
      Infinite great circles can be drawn.
    - Can you extend a line segment (part of a great circle, connecting two points) into an infinite line?
      If you draw a segment of a great circle, it can be extended to go around the entire sphere.
    - Can you make a circle that's not a great circle?
      Yes, these circles are not geodesics.
    - Can you create a 90 degree angle between two circles?
      Yes
    - Can you create a closed figure using only geodesic lines?
      Yes, this shape is called a lune.

Notes

Antipodal points are points that correspond to opposite ends of a diameter of the sphere.

A lune is one of the regions between two great circles and the antipodal points they cross.

A lune is given by the angle θ at the center of the sphere between the two great circles. In order to talk more about lunes and other figures on a sphere we will assume the following properties of a sphere:

1) The sphere of radius R has area $4\pi R^2$.
2) The area of a union of non-overlapping regions is the sum of their areas.
3) The areas of congruent regions are equal.
4) The ratio of the area enclosed by two great circles to the area of the whole sphere is the same as the ratio of the angle between them to $2\pi^{\circ}$.

By assuming the fourth property we can use the following formula:

Area of lune= $2\theta R^2$, where θ is in radians

Exit Ticket

Use two and rubber bands to create a lune that is 90˚.

If students are having a hard time with this, rephrase the question to have them cut the circle into quarters using the rubber bands.

Assuming the radius of the lacrosse ball is 1 inch, what is the area of the lune you have created?

Area of lune= $2\theta R^2$

$= 2\frac{90}{180}\pi(1)^2$

$= \pi(1)^2$

$= \pi$

≈ 3.14 inches^2

Spherical Geometry Homework

1) Find the area of a sphere whose radius is 20 in.

2) Find the radius of a sphere whose area is 196π in^2.

3) Given a sphere with radius 10 ft, find the area of the lunes with following angle θ:

a) $\theta = 180$ b) $\theta=30$ c) $\theta=120$ d) $\theta=135$

4) Using the answers from above find the ratio of area of lune to area of the sphere.

a) $\theta = 180$ b) $\theta=30$ c) $\theta=120$ d) $\theta=135$

5) Find the radius of a sphere whose lune with $\theta = 120$ has an area of 96π m^2.

6) Find the angle of a lune with area 12π in^2 whose sphere has $R=6$ in.

Spherical Geometry Homework Answer Key

1) Find the area of a sphere whose radius is 20 in.

   Area = $4\pi 20^2$ = 1600π in^2

2) Find the radius of a sphere whose area is 196π in^2.

   Area = $4\pi R^2$

   196π = $4\pi R^2$

   49=R^2

   R=7 in.

3) Given a sphere with radius 10 ft, find the area of the lunes with following angle θ:

   a) θ= 180 b) θ=30 c) θ=120 d) θ=135

   Area = $2\theta R^2$ Area = $2\theta R^2$ Area = $2\theta R^2$ Area = $2\theta R^2$

   $=2\pi 10^2$ $=2 * \frac{\pi}{6} * 10^2$ $=2 * \frac{2\pi}{3} * 10^2$ $=$

   $2 * \frac{135}{180}\pi * 10^2$

   = 200π ft^2 $= \frac{100\pi}{3}$ ft^2 $= \frac{400\pi}{3}$ ft^2 = 150π ft^2

4) Using the answers from above find the ratio of area of lune to area of the sphere.

   a) θ= 180 b) θ=30 c) θ=120 d) θ=135

   ratio=1/2 ratio=1/18 ratio=1/3 ratio=3/8

5) Find the radius of a sphere whose lune with θ=$\frac{2\pi}{3}$ has an area of 96π m^2.

   Area = $2\theta R^2$

   48π = $2\frac{2\pi}{3}R^2$

   36 = R^2

   R = 6 m.

6) Find the angle of a lune with area 12π in^2 whose sphere has R=6 in .

   Area = $2\theta R^2$

   12π = $2\theta * 6^2$

   $\frac{\pi}{6} = \theta$

   $\theta = 30$

Lesson 7: Spherical Triangles

Teacher Overview**:**

This lesson takes a deeper look at triangles in spherical geometry. It starts by having students compare triangle and parallel postulates from Euclidean and spherical geometries. Students then are guided through a proof of the area of spherical triangles. This also proves the triangle angle sum of a spherical triangle.The lesson ends with example problems of finding area and angle excess of spherical triangles.

Geometric Overview:

This is the second lesson that looks at spherical geometry. After proving the absolute geometry axioms hold and finding out the area of a sphere, we take a deeper look at the area of a triangle in spherical geometry. To do this, we must accept that the fifth postulate does not hold and there are no parallel lines. Instead when looking at great circles and the triangles they create, we find they have an area of (angle access)*R^2. Since each spherical triangle has area this proves that all spherical triangles have an angle sum that is greater than π.

Learning Targets:

- 85% of students will be able to find the angle sum of a Euclidean triangle.
- 85% of students will be able to find the angle sum of a spherical triangle.
- 85% of students will be able to find the area of a spherical triangle.
- 85% of students will be able to find angle excess of a spherical triangle given the area.

Prerequisite Knowledge:

- Lessons 1-6

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
  - Student Note Handout
  - Lacrosse Ball
  - Colored Pens, markers,crayons or pencils
  - Student Homework Assignment
  - Protractor
- For teachers
  - Annotated Student Note Sheet
  - Answer Key for Homework Assignment

Lesson Overview:

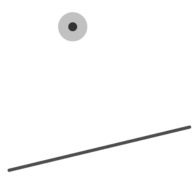
- Do Now(15 minutes)
    - In Euclidean space: draw a triangle and a parallel line through a point and a line not on that point.
    - On a lacrosse ball, try to draw a triangle with three right angles, and a parallel line through a point and a line not on that point.
- Notes:(40 minutes)
    - Parallel lines don’t exist in spherical geometry
    - A triangle angle sum is greater than 180.
    - Proof for area of a triangle
- Exit Ticket: (15 minutes)
    - Find the area of spherical triangles

STUDENT HANDOUT

Do Now

In the space below, draw a triangle in the Euclidean plane. Then using a protractor, find the sum of its angles.

Given a line and a point not on that line, draw a line parallel to the given line and on the given point.



Get out your lacrosse ball:

Draw a triangle that has three right angles. What is the angle sum of this triangle?

Now draw a great circle and a point not on the great circle. Is it possible to draw a line parallel to the great circle that goes through the point? *Remember parallel lines are great circles which, being in the same sphere are produced indefinitely in either direction and do not meet one another in either direction*

Lecture

In Euclidean Geometry:

- The sum of the angles in a triangle is always equal to ___________.

- Given a line and a point not on the line, there is ___________ ___________ line that contains the point and is ___________ to the given line.

In Spherical Geometry:

- The sum of the angles in a triangle is always greater than 180˚ or π.

- Given a line and a point not on the line, there is ______ line that contains the point and is ___________ to the given line.

Proofs

The area of spherical triangles = (angle excess)* R^2

Assume the triangle lies on one _____________. If not you can break the triangle up into smaller triangles that do lie in one hemisphere. A ___________ is the intersection of three __________. The lune given by $\overline{AB} \cap \overline{AC}$ (_________ _____), the lune given by $\overline{AB} \cap \overline{BC}$ (_________ ________), and the lune given by $\overline{BC} \cap \overline{AC}$(_________ ______). Each of these three lunes have an antipodal lune and the intersection of these three lunes is an antipodal triangle $\Delta A'B'C'$. Consider the following depiction of the sphere, from Notes on Lines and Triangles in Spherical Geometry by Doug Norris [2].

We will denote the lunes by the angle that includes them, BAC(_____), ABC(___________) and ACB (________). Starting with ΔABC add the lune ABC, add to them the lune ACB. This covers ΔABC three times. Now add in the antipodal lune to CAB and the antipodal triangle $\Delta A'B'C'$. This covers the hemisphere determined by $\overline{BC}$, counting the triangle three times.

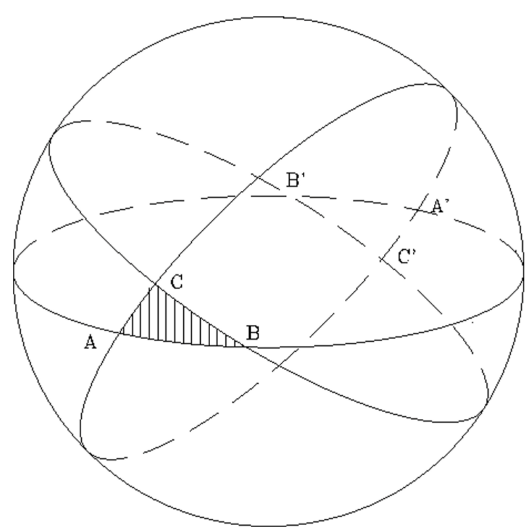
We know the following areas are:

Hemisphere = ______________________

Lune BAC= ________________________

LuneABC= _________________________

Lune ACB= ________________________



Thus:

$2\pi R^2 = 2\,\theta_{BAC} * R^2 + 2\,\theta_{ABC} * R^2 + 2\,\theta_{ACB} * R^2 - 3 * area(\Delta ABC) + area \Delta ABC$

$2\pi R^2 = 2(\theta_{BAC} + \theta_{ABC} + \theta_{ACB}) * R^2 - 2 * area\ (\Delta ABC)$

$2 * area\ (\Delta ABC = 2(\theta_{BAC} + \theta_{ABC} + \theta_{ACB}) * R^2 - 2\pi R^2$

area $(\Delta ABC) = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

or in degrees

area $(\Delta ABC) = (\angle BAC + \angle ABC + \angle ACB - 180)\frac{\pi}{180} * R^2$

We can call $(\angle BAC + \angle ABC + \angle ABC - \pi)$ the angle excess of the spherical triangle.

Therefore, the area of any spherical triangle is __________________

All spherical triangles have an angle sum greater than π (180˚).

Since every triangle on the sphere has ________, every triangle on the sphere has an interior angle sum _________ than ___ or _________.

Exit Ticket

Find the area of a triangle on a sphere with radius 20 meters, whose angles are $\frac{4\pi}{9}$, $\frac{5\pi}{9}$ and $\frac{2\pi}{3}$.

Find the angle excess of a spherical triangles area ia 200 ft^2 on a sphere whose radius is 10 ft.

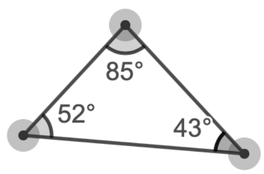
The total surface area of a sphere is $4\pi R^2$. Given this information, what is the largest sum a triangles area could be without overlap.

ANNOTATED STUDENT HANDOUT

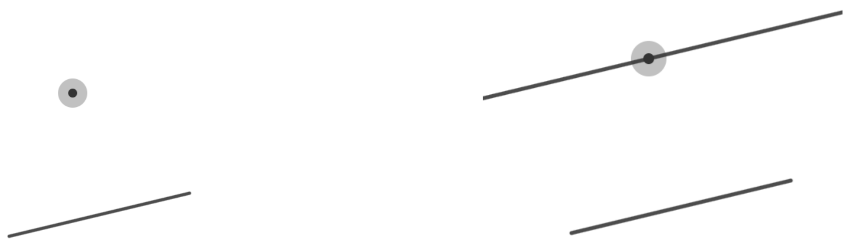
Do Now

In the space below, draw a triangle in the Euclidean plane. Then using a protractor, find the sum of its angles.

Triangles will vary. Below is an example of one triangle. All angle sums will total 180 or π.



Given a line and a point not on that line, draw a line parallel to the given line and on the given point.



Get out your lacrosse ball:

Draw a triangle that has three right angles. What is the angle sum of this triangle?

Triangle will be an eighth of the ball and its angle sum will be 270 or $\frac{3\pi}{2}$.

Now draw a great circle and a point not on the great circle. Is it possible to draw a line parallel to the great circle that goes through the point?

This is a trick question for students to explore. It is impossible to draw parallel lines in spherical geometry.

Notes

We just revealed some very important postulates in both Euclidean and spherical geometry. Let's make sure we summarize our findings.

In Euclidean Geometry: The sum of the angles in a triangle is always equal to 180˚ or π.

- Given a line and a point not on the line, there is one unique line that contains the point and is parallel to the given line.

In Spherical Geometry:

- The sum of the angles in a triangle is always greater than 180˚ or π.

- Given a line and a point not on the line, there is no line that contains the point and is parallel to the given line.

We have now seen at least one triangle whose angle is greater than 180˚ or π. Let's take a deeper look at what triangles look like in spherical geometry.

Proofs

The area of spherical triangles = (angle excess)* R^2

Assume the triangle lies on one hemisphere. If not you can break the triangle up into smaller triangles that do lie in one hemisphere. A triangle is the intersection of three lunes. The lune given by $\overline{AB} \cap \overline{AC}$ (color red), the lune given by $\overline{AB} \cap \overline{BC}$ (color yellow), and the lune given by $\overline{BC} \cap \overline{AC}$(color blue). Each of these three lunes have an antipodal lune and the intersection of these three lunes is an antipodal triangle $\Delta A'B'C'$. Consider the following depiction of the sphere, from Notes on Lines and Triangles in Spherical Geometry by Doug Norris [2]. We will denote the lunes by the angle that includes them, BAC(red),ABC(yellow) and ACB(blue). Starting with ΔABC add the lune ABC, add to them the lune ACB. This covers ΔABC three times. Now add in the antipodal lune to CAB and the antipodal triangle $\Delta A'B'C'$. This covers the hemisphere determined by $\overline{BC}$, counting the triangle three times.

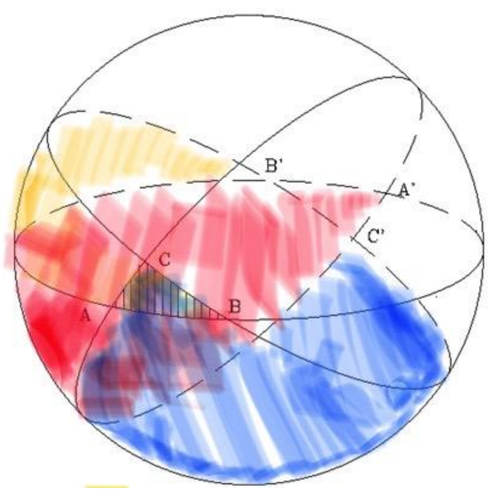
We know the following areas are:

Hemisphere = $2\pi R^2$

Lune BAC= $2\,\theta_{BAC}\,\pi\,R^2$

LuneABC= $2\,\theta_{ABC}\pi\,R^2$

Lune ACB= $2\,\theta_{ACB}\pi\,R^2$



Thus:

$2\pi R^2 = 2\,\theta_{BAC} * R^2 + 2\,\theta_{ABC} * R^2 + 2\,\theta_{ACB} * R^2 - 3 * area(\Delta ABC) + area\Delta ABC$

$2\pi R^2 = 2(\theta_{BAC} + \theta_{ABC} + \theta_{ACB}) * R^2 - 2 * area\,(\Delta ABC)$

$2 * area\,(\Delta ABC = 2(\theta_{BAC} + \theta_{ABC} + \theta_{ACB}) * R^2 - 2\pi R^2$

area $(\Delta ABC) = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

or in degrees

area $(\Delta ABC) = (\angle BAC + \angle ABC + \angle ACB - 180)\frac{\pi}{180} * R^2$

We can call $(\angle BAC + \angle ABC + \angle ABC - \pi)$ the angle excess of the spherical triangle.

Therefore, the area of any spherical triangle is (angle excess)* R^2

All spherical triangles have an angle sum greater than π (180˚).
Since every triangle on the sphere has area, every triangle on the sphere has an interior angle sum greater than π or 180˚.

Exit Ticket

Find the area of a triangle on a sphere with radius 20 meters, whose angles are $\frac{4\pi}{9}$, $\frac{5\pi}{9}$ and $\frac{2\pi}{3}$.

$area\ (\Delta ABC) = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

$= (\frac{4\pi}{9} + \frac{5\pi}{9} + \frac{2\pi}{3} - \pi) * 20^2$

$\approx 837.76\ m^2$

Find the angle excess of a spherical triangles area ia 200 ft^2 on a sphere whose radius is 10 ft.

$area\ (\Delta ABC) = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

$200 = (x)\ 10^2$

x =2 radians

x ≈114.59

The total surface area of a sphere is $4\pi R^2$. Given this information, what is the largest sum a triangles area could be without overlap.

$area\ (\Delta ABC) = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

$4\pi R^2 = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi) * R^2$

$4\pi = (\theta_{BAC} + \theta_{ABC} + \theta_{ACB} - \pi)$

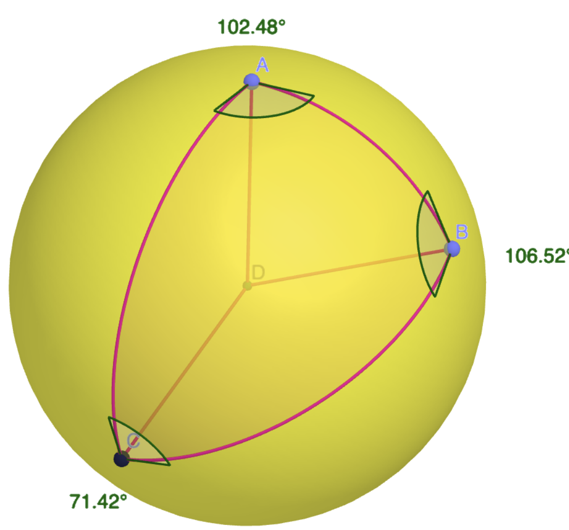
$5\pi = \theta_{BAC} + \theta_{ABC} + \theta_{ACB}$ or $900 = \angle BAC + \angle ABC + \angle ACB$

<u>**Spherical Geometry Homework**</u>

1) Given the moon has a radius of 1737.4 km, find the area of a triangle whose angles are all 90˚. What fraction of the whole sphere is the area of this triangle?

2) The Bermuda triangle is a famous triangle with points at Florida, Puerto Rico and Bermuda. The angles of each vertex on the triangle with corresponding locations are 52.8˚, 54.8˚, and 74.1˚.
   a) What is the angle sum of this triangle? Explain if your answer makes sense for context of the problem
   b) Find the area of the Bermuda triangle.

3) Given the following sphere has a radius of 100 m. Find the area of the given triangle.



** Spherical diagrams are made with Geogebra [3].*

<u>**Spherical Geometry Homework Answer Key**</u>

1) Given the moon has a radius of 1737.4 km, find the area of a triangle whose angles are all 90˚. What fraction of the whole sphere is the area of this triangle?

area $=(angle\ excess) * R^2$

$=\frac{\pi}{2} * 1737.4^2$

≈4741,541.01 km^2

2) The Bermuda triangle is a famous triangle with points at Florida, Puerto Rico and Bermuda. The angles of each vertex on the triangle with corresponding locations are 52.8˚, 54.8˚, and 74.1˚.

a) What is the angle sum of this triangle? Explain if your answer makes sense for context of the problem

angle sum= 52.8+54.8+ 74.1

= 181.7˚

This makes sense because this is a spherical triangle whose angle sum must be more than 180˚.

b) Find the area of the Bermuda triangle. (radius of earth = 6378 km)

In degrees

area $=(angle\ excess) * \frac{2\pi}{360} * R^2$

$= (1.7) * \frac{2\pi}{360} * 6378^2$

≈ 1206966.79 km^2

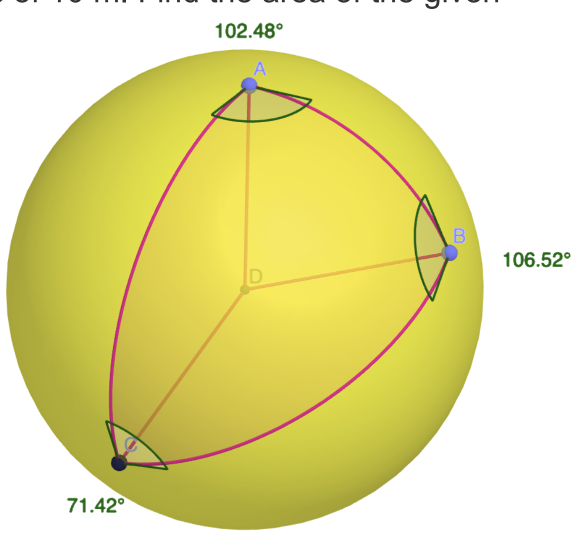
3) Given the following sphere has a radius of 10 m. Find the area of the given triangle.

In degrees

area $=(angle\ excess) * \frac{2\pi}{360} * R^2$

$=(100.42) * \frac{2\pi}{360} * 100^2$

≈ 175.27 m^2



Lesson 8: Hyperbolic Geometry and Triangles

Teacher Overview**:**

This lesson is a brief introduction into hyperbolic planes and triangles. Students start off by reviewing properties of Euclidean and spherical geometries. Students then get introduced to hyperbolic geometry through the Poincaré half plane. After giving an example of measuring length in the half plane, students define geodesics in the Poincaré half plane. After that students are introduced to hyperbolic triangles and their angle sum. Students finish the introduction into hyperbolic space by defining angle defect and finding a hyperbolic triangle's area.

Geometric Overview:

By introducing hyperbolic geometry, students are shown yet another space that is different from planar geometry. In this space we take a different approach to the fifth postulate where there are an infinite number of parallel lines that go through a point not on the original line. Before going through this,we make sure the axioms of absolute geometry hold. After looking at the fifth postulate, we introduce the hyperbolic length of The Poincaré' Half Plane. We then look at the area of a triangle in the Poincaré' Half Plane. This helps prove that the angle sum of a triangle in hyperbolic space is less than π or 180. In summation we look at a chart that compares and contrasts plane, spherical and hyperbolic geometries.

Learning Targets:

- 85% of students will be able to describe hyperbolic space.
- 85% of students will be able to identify and construct hyperbolic geodesics.
- 85% of students will be able to find the angle sum of a hyperbolic triangle.
- 85% of students will be able to find the area of a hyperbolic triangle.
- 85% of students will be able to compare and contrast Euclidean, spherical and hyperbolic geometries.

Prerequisite Knowledge:

- Lessons 1-7

Allotted Time:

- 1 Class period (70 minutes)

Materials:

- For students
    - Student note handout
    - Student homework assignment
    - Compass

    - Straightedge
    - Computer
- For teachers
    - Annotated student note sheet
    - Answer key for homework assignment:

<u>Lesson Overview</u>

- Do Now: (5 minutes)
    - Euclidean and spherical geometries chart
- Notes: (50 minutes)
    - Define Poincaré half plane
    - Look at the first four postulates from Euclidean geometry in the Poincaré half plane
    - Look at parallel lines in the Poincaré half plane
    - Develop the triangle angle sum in hyperbolic space
    - Define the area of triangle in Poincaré half plane
- Exit Ticket: (15 minutes)
    - Find the area of a triangles in a Poincaré half plane
    - Find the angle defect of triangle in Poincaré half plane

STUDENT HANDOUT

Do Now

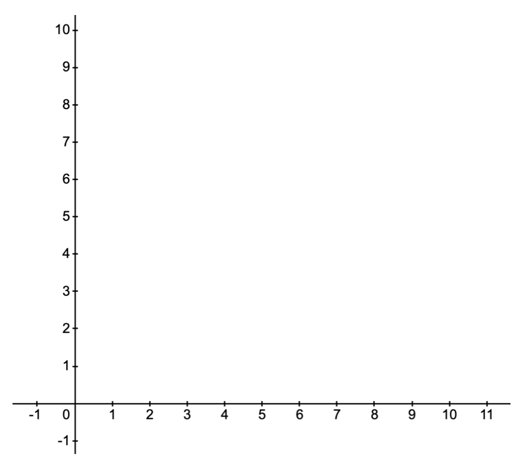
Fill out the following chart:

Geometry	What is the Geodesic	Do Postulates 1-4 Hold?	How does the 5th postulate work	Triangle Angle Sum
Euclidean				
Spherical				

Notes

Hyperbolic Space

Let's take a look at a new space called the Poincaré half plane. In the Poincaré half plane the metric = _______________. In other words the hyperbolic length of the horizontal lines from (0,1) to (1,1) is the same as (0,2) to (2,2), and (1,5) to (6,5). Let's take a look at this on a graph. Notice that all horizontal lines drawn have a hyperbolic length of 1.Even though the hyperbolic length of each horizontal line is 1, there is a shorter line found by using the geodesic of the Poincaré Half Plane. These geodesics are half circles with a center on the x-axis. We will draw them on this diagram later on.



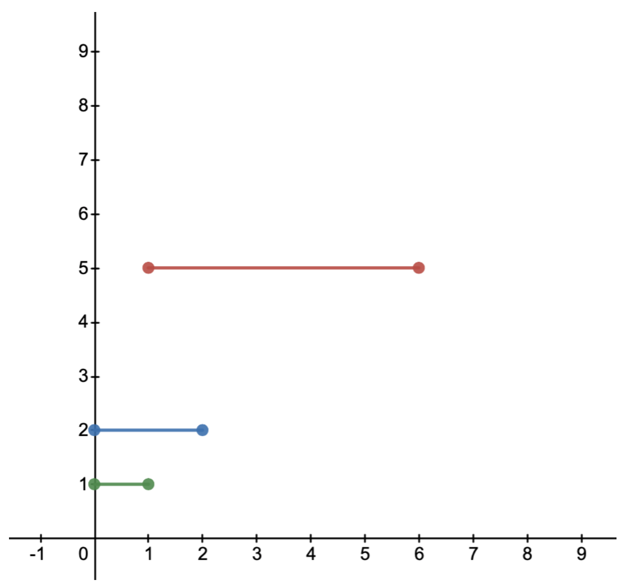
Postulate 1:To draw a line from any point to any point.
In hyperbolic space the geodesic has two types of lines.

1)____________ ____________ If the two points are on the same x coordinate, the shortest distance between them is the ____________ ____________ line that connects them.

2)_______-_____________ centered on the ___-_________ .- Because distance is divided when moving upwards, a ____________ line can be used to get from one point to another in the shortest hyperbolic distance.

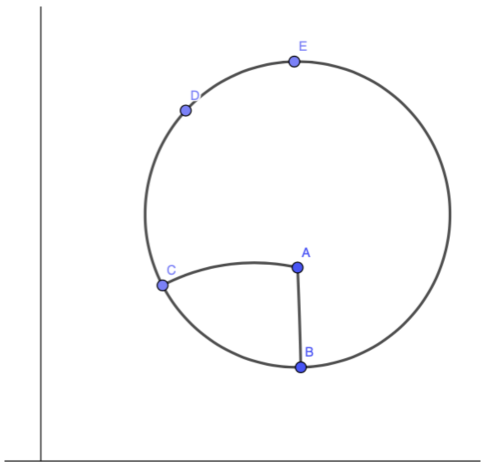
Draw a few examples of each below:

Draw the geodesics connecting points with the same y-coordinate:
To draw the geodesics, start by finding the perpendicular bisector of each horizontal line and extending the bisector to the x-axis. The center of the half circle for each geodesic will be at the intersection of the perpendicular bisector and x-axis. Use your compass to draw the half circle centered at the geodesic intersection point with a radius that goes through each endpoint.



Postulate 2: To produce a finite straight line continuously in a straight line.
When drawing a complete semicircle, the distance as we get closer to the x axis gets larger and larger. As it gets ____________ close to the x axis, the line is ____________ long.

Postulate 3: To describe a circle with any center and radius.
Notice that in the Poincaré half plane, the figure of a circle is the same as a circle in ____________ ____________. The difference is the placement of the center.

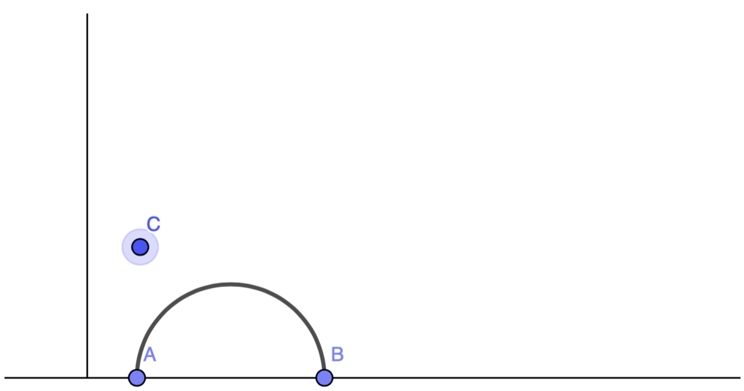


Segments connecting points that aren't vertical are _________ _________. These line segments are a part of an infinite hyperbolic line that takes the form of a planar semicircle centered on the x axis. In the diagram, the geodesics AB, and AC are drawn. **Draw hyperbolic geodesis AD and AE.**

All segments from the center of the circle to any point on the circle are equal. This means the vertical geodesic lengths BA and AE are equal in hyperbolic length. Although AB and AE aren't congruent in planar length, they are equal in hyperbolic length because every length in the hyperbolic geometry is reversely proportional to its distance to the x-axis. In hyperbolic geometry ___=___=___=___.

Postulate 4: All right angles are congruent.
To construct a right angle, start by constructing a _______ __________ whose center is on the x axis. Then draw a __________ ______ through the __________ of the half circle. You can create infinitely more congruent right angles by drawing more half circle geodesics with different radii all centered at the intersection of the vertical line and the x axis.

Postulate 5 (Playfair's Axiom): Given a straight line m and a point P not on m, there are infinitely many straight lines that are parallel to m and contain P.
To construct a line _________ to AB that goes through C, Use your compass and draw a half circle centered at ___ with radius _____. This will create a half circle that does not intersect with AB. We can continue to create infinitely more parallel hyperbolic lines by moving the center to either side on the x axis making sure that line does not intersect with the original line AB.



Hyperbolic triangles:

1. Go to https://www.geogebra.org/m/Z4SWyEnC
2. Create three points(A,B,C) on a horizontal line above the x-axis.
3. Using the hyperbolic line tool, connect points A and B, then B and C, and finally C and A. This is a hyperbolic triangle!
4. Using the angle measure tool, find the angle of all three angles.
   The angle sum of your triangle is ________.
5. Move the three points A, B, and C around and find a pattern with the angle sum of the triangle. State the result that you find below.

How does your finding relate to triangles in Euclidean and spherical geometries?

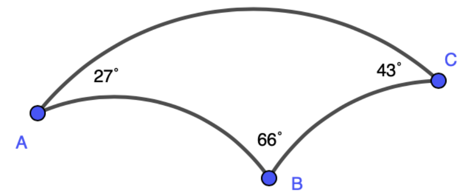
The __________ of a hyperbolic triangle with angles A,B,C is ___________________.

Triangle area

The area of a triangle in hyperbolic space is equal to its ____________.

Exit Ticket

Find the area of hyperbolic triangle ΔABC.



Find the angle sum of a triangle whose sum is 1.05

ANNOTATED STUDENT HANDOUT

Do Now

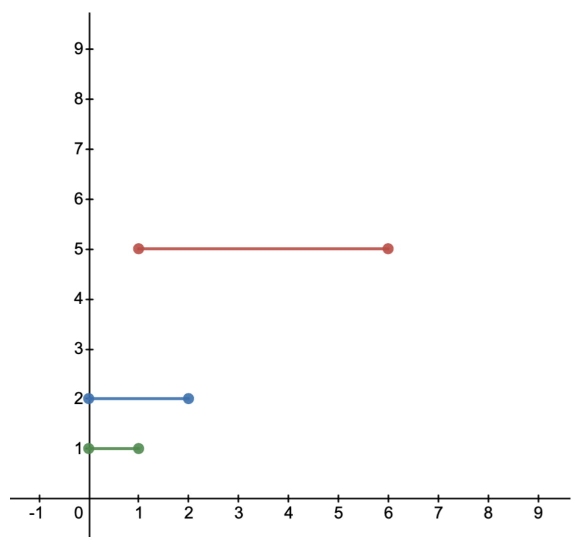
Fill out the following chart:

Geometry	What is the Geodesic	Do postulates 1-4 Hold?	How does the 5th postulate work	Triangle Angle Sum
Euclidean	The shortest distance is a straight line between two two points.	All 4 postulates hold	Given a line and a point not on the line, there is a one unique line that contains the point and is parallel to the given line.	The angle sum of every triangle is equal to 180˚.
Spherical	The shortest distance is a great circle.	All 4 postulates hold	Given a line and a point not on the line, there is no line that contains the point and is parallel to the given line.	The angle sum of every triangle is greater than 180.

Notes

Hyperbolic Space

Let's take a look at a new space called the Poincaré half plane.In the Poincaré half plane the metric = $\frac{euclidean\ distance}{y}$. In other words the hyperbolic length of the horizontal lines from (0,1) to (1,1) is the same as (0,2) to (2,2), and (1,5) to (6,5). Let's take a look at this on a graph. Notice that all horizontal lines drawn have a hyperbolic length of 1. Even though the hyperbolic length of each horizontal line is 1, there is a shorter line found by using the geodesic of the Poincaré Half Plane. These geodesics are half circles with a center on the x-axis. We will draw them on this diagram later on.



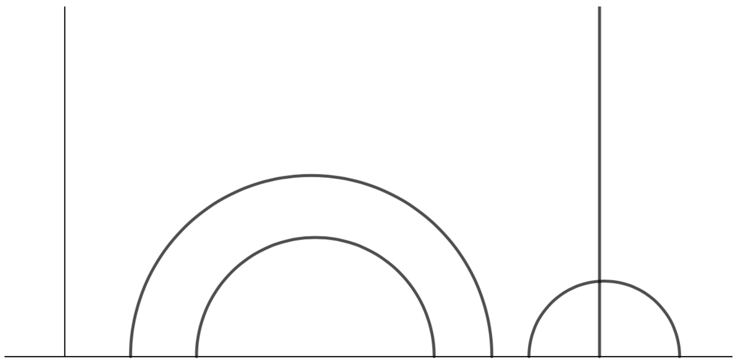
Let’s take a look at Euclid’s first four postulates in hyperbolic space.

Postulate 1:To draw a line from any point to any point.

In hyperbolic space the geodesic has two types of lines.

1) Vertical Lines If the two points are on the same x coordinate, the shortest distance between them is the vertical straight line that connects them.
2) Semi-circles centered on the X axis.- Because distance is divided when moving upwards, a semi-circle line can be used to get from one point to another in the shortest hyperbolic distance.

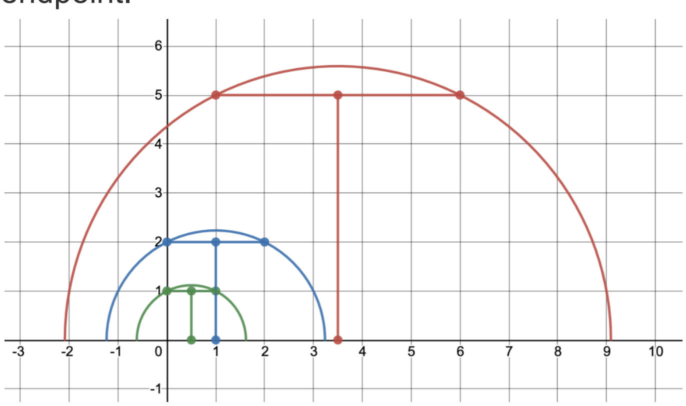
Draw a few examples of each below:



**Poincaré half plane drawings are made with Geogebra [4].*

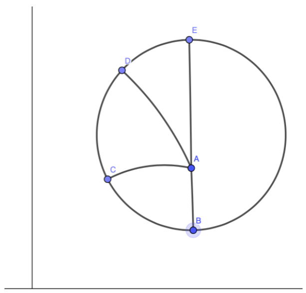
Let’s also go back and draw the geodesics for the horizontal lines we drew earlier.

To draw the geodesics, start by finding the perpendicular bisector of each horizontal line and extending it to the x-axis. The center of the half circle for each geodesic will be at the intersection of the perpendicular bisector and x-axis. Use your compass to draw the half circle centered at the intersection point with a radius that goes through each endpoint.



Postulate 2: To produce a finite straight line continuously in a straight line.
When drawing a complete semicircle, the distance as we get closer to the x axis gets larger and larger. As it gets infinitely close to the x axis, the line is infinitely long.

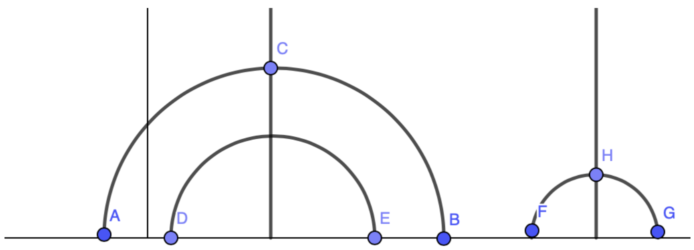
Postulate 3: To describe a circle with any center and radius.
Notice that in the Poincaré half plane, the figure of a circle is the same as a circle in Euclidean geometry. The difference is the placement of the center. In the diagram, connect lines AB AC AD and AE. In hyperbolic geometry AB=AC=AD=AE.



Segments connecting points that aren't vertical are bowed geodesics. These line segments are a part of an infinite hyperbolic line that takes the form of a planar semicircle centered on the x axis. In the diagram, the geodesics AB, and AC are drawn. **Draw hyperbolic geodesis AD and AE.**

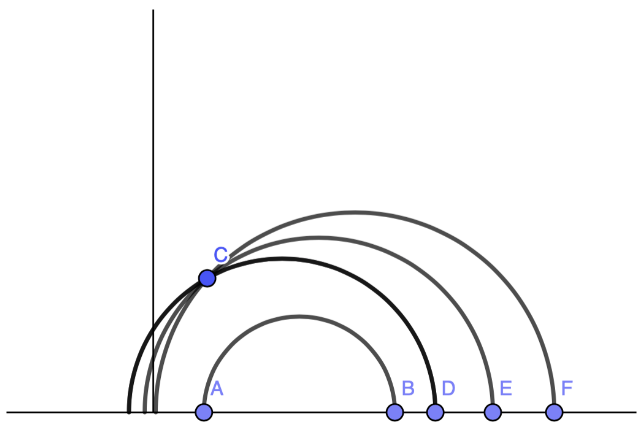
All segments from the center of the circle to any point on the circle are equal. This means the vertical geodesic lengths BA and AE are equal in hyperbolic length. Although AB and AE aren't congruent in planar length, they are equal in hyperbolic length because every length in the hyperbolic geometry is reversely proportional to its distance to the x-axis. In hyperbolic geometry AB=AC=AD=AE.

Postulate 4: All right angles are congruent.
To construct a right angle, start by constructing a half circle whose center is on the x axis. Then draw a vertical line through the center of the half circle. You can create infinitely more congruent right angles by drawing more half circle geodesics with different radii all centered at the intersection of the vertical line and the x axis.



Postulate 5 (Playfair's Axiom):Given a straight line m and a point P not on m, there are infinitely many straight lines that are parallel to m and contain P.

To construct a line parallel to AB that goes through C, Use your compass and draw a half circle centered at C with radius BC. This will create a half circle that does not intersect with AB. We can continue to create infinitely more parallel hyperbolic lines by moving the center to either side on the x-axis making sure that line does not intersect with the original line AB.



Hyperbolic triangles:

6. Go to https://www.geogebra.org/m/Z4SWyEnC
7. Create three points(A,B,C) on a horizontal line above the x-axis.
8. Using the hyperbolic line tool, connect points A and B, then B and C, and finally C and A. This is a hyperbolic triangle!
9. Using the angle measure tool, find the angle of all three angles.
   The angle sum of your triangle is ________.(Answers will vary)
10. Move the three points A, B, and C around and find a pattern with the angle sum of the triangle. State the result that you find below.
    The angle sum of any triangle in the hyperbolic geometry is less than 180˚.

How does your finding relate to triangles in Euclidean and spherical geometries?

The angle sum of any triangle in the Euclidean geometry is equal to 180˚.
The angle sum of any triangle in the spherical geometry is greater than 180˚.
The angle sum of any triangle in the hyperbolic geometry is less than 180˚.

The defect of a hyperbolic triangle with angles A,B,C is the quantity $\pi * \frac{180-A-B-C}{180}$

Triangle area

The area of a triangle in hyperbolic space is equal to its defect.

Exit Ticket

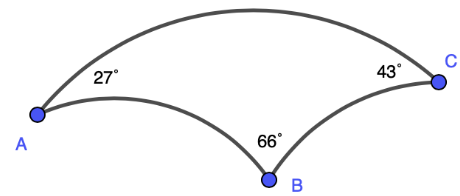
Find the area of hyperbolic triangle ΔABC.

Area = $\pi * \frac{180-A-B-C}{180}$

$= \pi * \frac{180-27-66-43}{180}$

$= \pi * \frac{44}{180}$

≈ 0.77



Find the angle sum of a triangle whose sum is 1.05

Area = $\pi * \frac{180-A-B-C}{180}$

$1.05 = \pi * \frac{180-x}{180}$

$60 = 180 - x$

$x = 120$

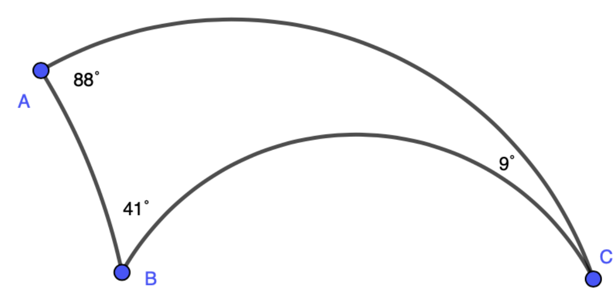
Hyperbolic Geometry Homework

Fill out the following chart:

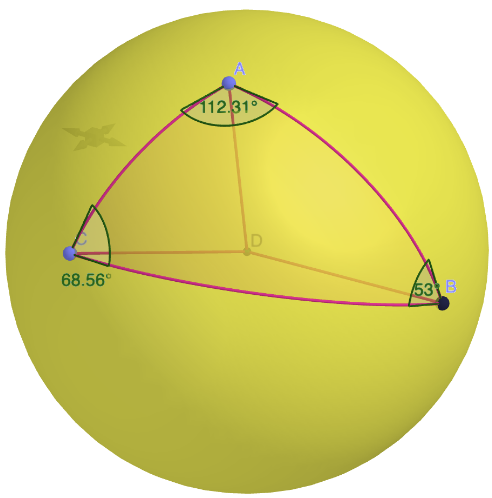
Geometry	What is the Geodesic	Do Postulates 1-4 Hold?	How does the 5th postulate work	Triangle Angle Sum
Euclidean				
Spherical				
Hyperbolic				

Find the angle sum and area of each triangle.

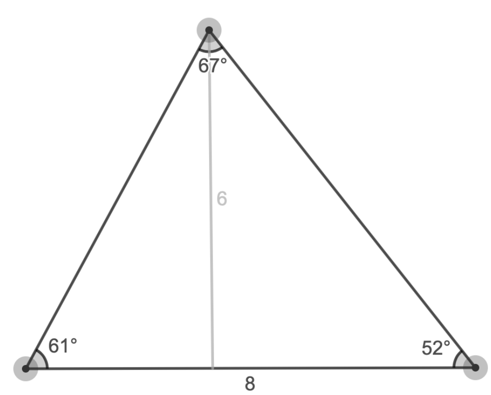
1) Hyperbolic triangle



2) Spherical triangle with R =10.



3) Euclidean triangle:



Hyperbolic Geometry Homework Answer Key

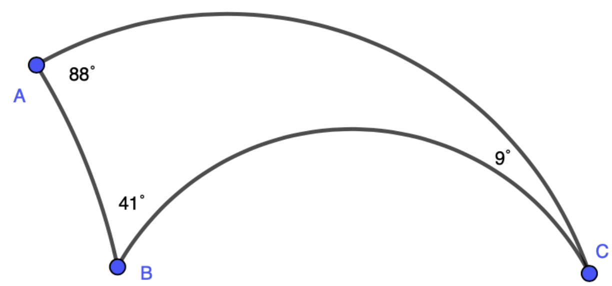
Fill out the following chart:

Geometry	What is the Geodesic	Do Postulates 1-4 Hold?	How does the 5th postulate work	Triangle Angle Sum
Euclidean	The shortest distance is a straight line between two two points.	All 4 postulates hold	Given a line and a point not on the line, there is a one unique line that contains the point and is parallel to the given line.	The angle sum of every triangle is equal to 180˚.
Spherical	The shortest distance is a great circle.	All 4 postulates hold	Given a line and a point not on the line, there is no line that contains the point and is parallel to the given line.	The angle sum of every triangle is greater than 180.
Hyperbolic	Half circles centered on the x axis and vertical lines	All 4 postulates hold	Given a line and point not on the line, there are infinitely many lines that contain the point and are parallel to the given line.	The angle sum of every triangle is less than 180.

Find the angle sum and area of each triangle.

1) **Hyperbolic triangle**

Angle sum= 88+41+9
=138˚



area= $\pi * \frac{180-angle\ sum}{180}$

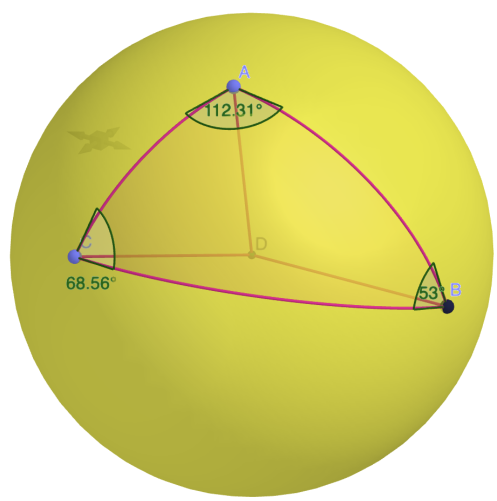
$= \pi * \frac{180-138}{180}$

$= \pi * \frac{42}{180}$

≈ 0.73

2) **Spherical triangle with R = 10 m.**

Angle sum= 112.31+68.56+53
=233.87˚



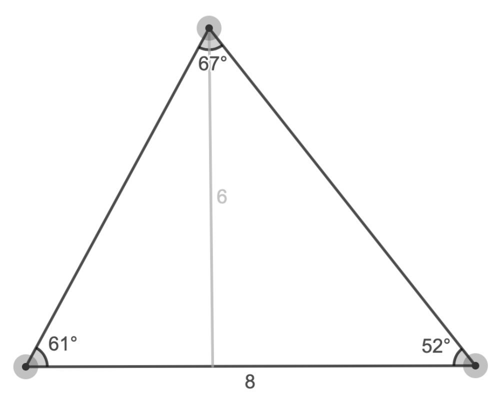
Area = $(Angle\ sum\ -\ 180)\frac{\pi}{180}\ R^2$

$= (233.87\ -\ 180)\frac{\pi}{180}\ 10^2$

≈ 94.02 m^2

3) **Euclidean triangle**

Angle sum= 67+52+61
=180˚



Area = ½ (base* height)
= ½ (6*8)
≈ 24

References

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