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On the distribution of central values of Hecke L-functions

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by Daniel White

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Abstract

Questions regarding the behavior of the Riemann zeta function on the critical line $\frac{1}{2} + it$ can be naturally interpreted as questions regarding the family of L-functions over $\mathbb{Q}$ associated to the archimedian characters $\psi(k) = k^{-it}$ at the center point $\frac{1}{2}$. There are many families of characters besides those strictly of archimedean-type, especially as one expands their scope to proper finite extensions of $\mathbb{Q}$. Consideration of these *Hecke characters* leads immediately to analogous questions concerning their associated L-functions.

Using tools from p-adic analysis which are analogues of traditional archimedean techniques, we prove the q-aspect analogue of Heath-Brown's result on the twelfth power moment of the Riemann zeta function for Dirichlet L-functions to odd prime power moduli. In particular, our results rely on the p-adic method of stationary phase for sums of products and complement Nunes' bound for smooth square-free moduli.

We additionally prove the frequency-aspect analogue of Soundararajan's result on extreme values of the Riemann zeta function for Hecke L-functions to angular characters over imaginary quadratic number fields. This result relies on the resonance method, which is applied for the first time to this family of L-functions, where the classification and extraction of diagonal terms depends on the geometry of the associated field's complex embedding.

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Chapter 1

Preface

L-functions are meromorphic functions intrinsic to a vast number of arithmetic, algebraic and geometric structures ranging from prime numbers and Diophantine equations to number fields and automorphic representations. The study of these functions from an analytical perspective can often reveal critical information regarding their associated structures. One often defines an L-function over $\mathbb{Q}$ with respect to a multiplicative function $\psi : \mathbb{N}_m \to S^1$ known as a Hecke character, where $\mathbb{N}_m$ are the natural numbers coprime to the integer m; these Hecke characters generalize quite well over number fields and are continuous with respect to a given field's complex and p-adic embeddings. (See, for example, [26, Section 7.6] or Section 3.2.1 of this thesis for a deeper breakdown.) These characters are precisely the objects that allow an L-function to bridge so seamlessly from algebra to analysis and back.

Once such a character is specified, there is an associated Dirichlet series $L(s, \psi) = \sum \psi(n)n^{-s}$ which converges absolutely in the half-plane $\operatorname{Re} s > 1$. Analytic continuation is then used to extend the domain of $L(s, \psi)$ to the remainder of the plane with possible exception to a pole at $s = 1$. The prototypical example over $\mathbb{Q}$ with its ring of integers $\mathbb{Z}$ is the Riemann zeta function

$$\zeta(s) = \sum_{k \geqslant 1} \frac{1}{k^s}, \quad \operatorname{Re} s > 1. \tag{1.1}$$

The absence of zeros of $\zeta(s)$ on the line $s = 1 + it$ is equivalent to the prime number theorem [11, §18] which states that $\pi(X)$, the number of primes less than the given magnitude X, grows as

$$\pi(X) = \mathrm{Li}(x) + o\left(\mathrm{Li(x)}\right).$$

Further, the widely believed Riemann hypothesis (RH), which asserts that the "non-trivial" zeros of $\zeta(s)$ lie on the critical line $s = \frac{1}{2} + it$, implies that the error term above is no more than $X^{1/2+\varepsilon}$; see again [11, §18]. This t-aspect analysis of L-functions provides powerful insight into arithmetic phenomena, but is only part of a more general and far-reaching inquiry.

For $\zeta(s)$ and its generalizations over number fields, behavior on the vertical line $s = \sigma + it$ is naturally viewed as behavior of the specific family of functions $L(\sigma, \psi)$ to the archimedean characters $\psi(k) = k^{-it}$ at some real value σ often taken to be the center point $\frac{1}{2}$. Broadening our scope to include families to other character types in order to develop a more complete theory of L-functions reveals structural parallels to many classical questions and recent results on $\zeta(s)$.

The results of this thesis come from a holistic examination of L-functions over number fields and demonstrate the universality of techniques that are often developed in more classical settings.

Traditional and new tools from analytic number theory often translate in some non-trivial way to this more general environment where implementation requires the cohesion of classical analysis, p-adic analysis, and the general theory of number fields.

Many archimedean techniques have non-archimedean analogues (*e.g.* p-adic methods are useful for understanding Dirichlet L-functions) and the continuous behaviors of $\zeta(\frac{1}{2}+it)$ have discrete analogues for other families of L-functions (*e.g.* Hecke L-functions to angular characters), but the extent and depth of these types of parallels are not yet fully understood.

1.1 Organization of the thesis

The core content of this thesis comes from two research articles, the first of which is faithfully reproduced and the latter is in preparation for peer review. Specifically, Chapter 2 contains the article *Twelfth moment of Dirichlet L-functions to prime power moduli* co-authored with Djordje Milićević, which is available on the arXiv [23] in preprint form and is accepted for (and awaiting) publication in the journal *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*. Chapter 3 contains the article *Extreme values of Hecke L-functions to angular characters* which is authored solely by the author of this thesis and will be submitted to a journal shortly.

The upcoming sections in this preface chapter give brief motivation, explanation, and overview of each of the articles mentioned above to better contextualize the materials and techniques in this thesis for a general (yet expert) mathematical audience.

1.2 In the direction of high moments

A central question regarding the zeta function is understanding the distribution of the values of $\zeta(\frac{1}{2}+it)$. The Lindelöf hypothesis asserts that these values have sub-power growth in t and proof of this would be immense evidence for RH [34, §9.4]. This has motivated number theorists to dedicate significant effort to understanding the maximal growth rate of $\zeta(\frac{1}{2}+it)$. One avenue is through power moments since understanding all moments can recover a distribution completely. While no asymptotics are known for integral power moments beyond the fourth [16], high moments provide tight control on large values, and in this direction, Heath-Brown [17] proved the remarkable bound

$$\int_T^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{12} \mathrm{d}t \ll_\varepsilon T^{2+\varepsilon} \tag{1.2}$$

where the twelfth moment is the natural moment manifestation for his core theorem which bounds the measure of the set where $\zeta(\frac{1}{2}+it)$ is larger than a given magnitude. From (1.2) one may recover the pointwise Weyl bound

$$\zeta\left(\tfrac{1}{2}+it\right) \ll_\varepsilon (1+|t|)^{1/6+\varepsilon}. \tag{1.3}$$

This is an improvement on the "trivial" convexity bound (exponent $1/4+\varepsilon$) which is the threshold where improvement requires detecting cancellation among the oscillatory terms $k^{-1/2-it}$ in the short range $k \ll T^{1/2}$. Note though that (1.2) is quite a bit stronger than (1.3) as it demonstrates that large values of $\zeta(\frac{1}{2}+it)$ cannot be sustained.

The Weyl bound is a significant benchmark for results on L-functions at the center point $\frac{1}{2}$ where increasing ramification is measured instead by the conductor, such as the modulus q of a Dirichlet L-function. Sub-Weyl bounds have long been known for $\zeta(\frac{1}{2}+it)$, and much more recently, slight improvements to the Weyl bound have been established in q-aspect for Dirichlet L-functions $L(\frac{1}{2},\chi)$ to prime power moduli [24] and smooth squarefree moduli [20, 36]. In a pair of landmark papers, Petrow and Young [28, 29] have recently established the Weyl bound in general for Dirichlet L-functions.

Through an adelic lens, the pure parallel to (1.2) over $\mathbb{Q}$ is where ramification occurs at a fixed (finite) prime. In a joint paper with Djordje Milićević [23] (contained in

Chapter 2) which is to appear in *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* (2021), we prove the following.

Theorem 1. *There exists some $A > 0$, such that for every odd prime p and every $q = p^k$,*

$$\sum_{\chi \pmod q} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^{12} \ll_\varepsilon p^A q^{2+\varepsilon}.$$

The relationship between (1.2) and (1.3) holds here as well; our Theorem 1 recovers the Weyl bound (as was already known and improved upon), but, importantly, it shows that $L(\frac{1}{2}, \chi)$ is unable to sustain large size over such a family of Dirichlet characters. Arithmetic questions regarding large powers of a fixed prime are of general interest, and our work, in part, has motivated recent progress by Banks and Shparlinski [3] on the distribution of smooth numbers over residue classes.

Theorem 1 complements recent work by Nunes [27] which translates the proof of Heath-Brown to work in the context of Dirichlet L-functions to highly factorable squarefree moduli. The proof of Theorem 1 builds on both [17] and [27], but requires p-adic techniques that parallel the classical archimedean analytic techniques. The common thread among these twelfth moment results is an examination of short second moments which for Dirichlet L-functions appear as

$$S_2(\chi) = \sum_{\psi_1 \pmod{q_1}} \left|L\left(\tfrac{1}{2}, \chi\psi_1\right)\right|^2 \tag{1.4}$$

where q_1 is a proper divisor of q. Understanding these short moments on average becomes a matter of analyzing exponential sums twisted by products of Dirichlet characters. Critically though, insight by the work of Postnikov [30], which has been further formalized by Milićević [24], allows us to explicate the associated phases of these exponential sums specifically for characters to prime power moduli. The core of the problem then becomes evaluating and estimating sums of the form

$$\sum_{x \pmod{p^n}} g(x) e\left(\frac{f(x)}{p^n}\right) \tag{1.5}$$

where $f(x)$ is persistently and profitably interpreted as a p-adically analytic function and $g(x)$ is a relatively simple function invariant under translation by p. It is here that we employ the method of p-adic stationary phase. This is the proper p-adic analogue to the classical method of stationary phase for exponential integrals

$$\int_{\mathbb{R}} g(x) e(f(x)) \, \mathrm{d}x \tag{1.6}$$

which states (for $f(x)$ and $g(x)$ appropriately behaved) that vast cancellation occurs during integration except near the critical points of $f(x)$. Indeed, for (1.5) in our application, the p-adic values of x_0 for which $f'(x_0)$ is p-adically small precisely correspond to the values modulo p^n where contribution arises.

These methods are used to understand the average of $S_2(\chi\psi)$ in (1.4) over the characters ψ modulo $q_2 \mid q$ where $L(\frac{1}{2}, \chi\psi\psi_1)$ is "large" for some ψ_1 modulo q_1. From here it is a matter of parameter-tweaking and bookkeeping to Theorem 1; Section 2.1.1 within its respective article provides a deeper overview.

The problem of Dirichlet L-functions along cosets arises in other important applications; for example, our work is cited by Petrow and Young in the paper [28] where they establish a forth moment estimate along a coset of Dirichlet characters in order to generalize their landmark Weyl bound for Dirichlet L-functions [29] to general moduli.

1.3 In the direction of extreme values

The question of lower bounds on maximal behavior complements the exciting progress on bounding the maximal growth of L-functions from above, where current techniques are far from proving the expectation of the Lindelöf hypothesis. Families of L-functions are naturally oscillatory objects, where their oscillatory behavior comes from their attached (and varying) characters. The resonance method, developed by Soundararajan [31] and illustrated below, exploits these oscillations by applying to a family of L-functions another oscillatory function which is designed, through some optimization process, to "resonate" with the "louder frequencies" of the family. This pronounces the existence of large values, thereby making them easier to detect.

The generic set-up can be conveyed through the original resonator's use on the zeta function. In the range $T \leqslant t \leqslant 2T$, one can essentially treat $\zeta(\frac{1}{2}+it)$ as the Dirichlet polynomial

$$\sum_{k \leqslant T} k^{-it-1/2} \tag{1.7}$$

where it's important, for generalization, to think of $\zeta(\frac{1}{2}+it)$ instead as a family of $L(\frac{1}{2}, \psi_t)$ where ψ_t is the archimedean character $\psi_t(k) = e^{-it\log k}$ as t varies over the interval $[T, 2T]$. A resonator polynomial

$$R(t) = \sum_{n \leqslant N} r(n) n^{-it} \tag{1.8}$$

with non-negative resonator coefficients $r(n)$ and length N can then be applied by examining $\zeta(\frac{1}{2}+it)|R(t)|^2$. Integrating this quantity over $[T, 2T]$ by using (1.7) and (1.8) yields, essentially,

$$\sum_{\substack{n,m \leqslant N \\ k \leqslant T}} \frac{r(n)r(m)}{k^{1/2}} \int_T^{2T} e^{it\log(n/mk)} \, \mathrm{d}t \tag{1.9}$$

where the integral above exhibits massive cancellation outside of the diagonal case ($n = mk$) provided that the length of the resonator is $N \ll T^{1-\varepsilon}$ which ensures that n and mk can not be close enough for the integral above to avoid cancellation.

At this point, we may compare the mean of $\zeta(\frac{1}{2}+it)|R(t)|^2$ to the mean of $|R(t)|^2$, producing a lower bound on $\max_{T\leqslant t\leqslant 2T}|\zeta(\frac{1}{2}+it)|$ of the form

$$\sum_{mk\leqslant N}\frac{r(mk)r(m)}{k^{1/2}}\Big/\sum_{n\leqslant N}r(n)^2. \tag{1.10}$$

From here, it is a matter of finding the optimal choice of resonator coefficients $r(n)$ making this ratio as large as possible. Such a choice, as exhibited in [31], yields

$$\max_{T\leqslant t\leqslant 2T}\log\left|\zeta(\tfrac{1}{2}+it)\right| \geqslant (1+o(1))\sqrt{\frac{\log T}{\log\log T}}. \tag{1.11}$$

The bound (1.11) substantially improved upon the previously best known result in [1, 2] and is drastically larger than the bulk of the normal distribution (with mean value 0 and variance $(\log\log T)/2$) upon which the values $\zeta(\frac{1}{2}+it)$ essentially reside [34, §11.13]. This result is fairly close to predictions [12] from random matrix theory (and also close to the best currently available result [6, 7]) that suggest the right hand side of (3.2) may be within a factor of $(\log\log T)/\sqrt{2}$ to the true maximum.

This technique has been applied successfully across a number of different families of L-functions; see, for example, [6, 7, 9, 18, 31]. We will concern ourselves here with the application of the resonance method to Hecke L-functions over number fields. Over $\mathbb{Q}$, in addition to detecting large values of $\zeta(s)$ on the critical line $s=\frac{1}{2}+it$, this presents the question concerning extreme values of Dirichlet L-functions $L(\frac{1}{2},\chi)$. This line of inquiry involving the resonance method was first investigated by Hough [18], establishing a result of quality similar to (1.11) for extreme values lying within prescribed angular sectors for Dirichlet characters varying on prime moduli.

The situation becomes even more interesting when considering proper finite extensions of $\mathbb{Q}$. A garden of Hecke characters begins to bloom in this space, and therefore new families of Hecke L-functions arrive alongside them. Over an imaginary quadratic extension K, in addition to the usual archimedean and Dirichlet character types, angular characters make their first appearance. For principal ideals of the ring of integers $\mathcal{O}_K$ and a choice of complex embedding, these behave precisely as

$$\xi_\ell((\beta)) = e\left(\ell \arg \beta / 2\pi\right), \quad \beta \in \mathcal{O}_K,$$

for integer frequencies ℓ divisible by the number of units in $\mathcal{O}_K$. In Chapter 3, the article *Extreme values of Hecke L-functions to angular characters* contains a proof of the following.

Theorem 2. *Let K be an imaginary quadratic number field and $\mathfrak{F}_\ell$ be its associated set of angular characters with frequency ℓ. Then*

$$\max_{\substack{X \leqslant \ell \leqslant 2X \\ \xi_\ell \in \mathfrak{F}_\ell}} \log \left| L(\tfrac{1}{2}, \xi_\ell) \right| \geqslant \left(\sqrt{2} + o_K(1)\right) \sqrt{\frac{\log X}{\log \log X}}.$$

After establishing a number of foundational materials, the proof proceeds by constructing a resonator of the form

$$R(\xi_\ell) = \sum_{\mathfrak{N}\mathfrak{a} \leqslant N} r(\mathfrak{a}) \xi_\ell(\mathfrak{a})$$

which, with appropriate coefficients, will emphasize the "louder frequencies" of $L(\frac{1}{2}, \xi_\ell)$ for $X \leq \ell \leq 2X$ much in the same way that (1.8) does with (1.7). Aside from the minor appearance of the class group and some technical aspects dealing with the approximate functional equation for $L(\frac{1}{2}, \xi_\ell)$, an equation similar in spirit to (1.7), contextual differences arise when one seeks to isolate and then analyze the diagonal terms corresponding to the analogue of (1.9) which, roughly speaking, appears as

$$\sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N\\ \mathfrak{N}\mathfrak{k}\ll X\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\text{ principal}}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \sum_{k\asymp X} e(k \arg \gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}/2\pi), \tag{1.12}$$

where $\gamma_{\mathfrak{n}}$ is the generator of $\mathfrak{n}$ nearest to the positive real line with respect to rotation about the origin. It is here that one must consider the geometry of the embedding of K into $\mathbb{C}$, as the diagonal condition becomes characterized by whether the product $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}$ is principally generated by a rational element of $\mathcal{O}_K$. Indeed, this is the only way the innermost sum above can avoid cancellation when N is bounded appropriately in terms of X.

Accordingly, the length of the resonator is chosen so that our analysis of the quantity illustrated by (1.12) depends on an area of the plane where the embedding of $\mathcal{O}_K$ is "sufficiently discretized" (with respect to X) such that all $\gamma_{\mathfrak{n}}$ *near the real line* are *exactly rational*. This guarantees that the off-diagonal terms where $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}$ is not rationally generated are suppressed.

Ultimately, we arrive at a ratio analogous to (1.10), where the final steps can then occur. After factorizing the sums, it is left to examine the ratio on a logarithmic scale to delicately compute the desired bound in Theorem 2 from an optimal choice of resonator coefficients.

Chapter 2

Twelfth moment of Dirichlet L-functions to prime power moduli

2.1 Introduction

Analytic behavior of L-functions inside the critical strip encodes essential arithmetic information, and statistical information about their zeros, moments, and rate of growth along the critical line is of central importance in analytic number theory. The classical Weyl bound shows that the Riemann zeta function satisfies

$$\zeta\left(\tfrac{1}{2}+it\right) \ll_{\varepsilon} (1+|t|)^{1/6+\varepsilon} \tag{2.1}$$

where $\varepsilon > 0$ is an arbitrarily small constant that may change from one instance to another throughout this article. The widely believed Lindelöf hypothesis asserts that $\frac{1}{6}$ can be removed from the exponent above. The most recent progress in this direction is due to Bourgain [8], reducing the exponent to $\frac{13}{84}+\varepsilon$. One avenue to understanding the behavior of the Riemann zeta function along the critical line is through power moments, for which asymptotic formulas are only available up to the fourth moment [19, 25]. Higher moments provide tighter control on large values, and in this direction Heath-Brown [17] proved that, for $T \geqslant 1$,

$$\int_{T}^{2T} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{12} \mathrm{d}t \ll_{\varepsilon} T^{2+\varepsilon}. \tag{2.2}$$

This is a very elegant bound as it recovers (2.1) as a rather immediate consequence. However, (2.2) is quite a bit stronger in that it immediately implies that $\zeta\left(\frac{1}{2}+it\right)$ *cannot sustain* large values; namely that

$$\left|\left\{t \in [T, 2T] : \left|\zeta\left(\tfrac{1}{2}+it\right)\right| > V\right\}\right| \ll_{\varepsilon} T^{2+\varepsilon} V^{-12}. \tag{2.3}$$

Actually, (2.2) and (2.3) are equivalent, as is easily established via integration by parts.

Questions regarding the asymptotic behavior of $\zeta(\frac{1}{2}+it)$ as $t \to \infty$ have q-aspect analogues concerning the central values of Dirichlet L-functions $L(\frac{1}{2}, \chi)$, where χ is a primitive character modulo q and $q \to \infty$. For an account of some of the current literature on $L(\frac{1}{2}, \chi)$ and L-functions in the t-aspect, we direct the reader to the introduction of [27]. The q-analogue of (2.1), the bound $L(\frac{1}{2}, \chi) \ll_{\varepsilon} q^{1/6+\varepsilon}$, long out of reach for generic q except for real characters to odd square-free moduli [10], has been recently announced by Petrow–Young [29]. For certain families of Dirichlet L-functions, however, even small improvements are known on $q^{1/6}$; see [24] for a "sub-Weyl" bound $L(\frac{1}{2}, \chi) \ll q^{1/6-\delta}$ for prime power moduli and [20] and [36] for smooth square-free moduli.

While Dirichlet L-functions $L(\sigma+it, \chi)$ are also fruitfully used with a fixed modulus q and large $|t|$ to study arithmetic phenomena modulo q, from an adelic point of view it is more natural to consider the dependence on a large conductor q as a measurement of increasing ramification, this time at finite places, and in particular, as a pure parallel to the t-aspect, at a fixed finite place. This explains why many tools of classical "archimedean" analytic number theory have found natural p-adic analogues. The extent of this parallel is yet to be fully understood, and our aim is to explore its manifestation for high moments of L-functions. Our main theorem is a q-aspect analogue of (2.2) for Dirichlet L-functions to odd prime power moduli.

Theorem 3. *There exists a constant $A > 0$ such that, for every odd prime p and every $q = p^n$,*

$$\sum_{\chi \pmod q} \left|L\left(\tfrac{1}{2}, \chi\right)\right|^{12} \ll_\varepsilon p^A q^{2+\varepsilon}.$$

We remark that Theorem 3 complements the result of Nunes [27] where q is taken to be smooth and square-free. The structure of the proof of Theorem 3 and the main result of Nunes translate the approach taken by Heath-Brown [17] into the context of factorable and prime power moduli. For a detailed comparison between Heath-Brown's and Nunes' work, we direct the reader to the introduction of [27]. Despite the similarities, the methods of evaluation and estimation of exponential sums found throughout are quite different in the present chapter. In particular, we make extensive use of a method known as p-adic stationary phase, which we will encapsulate in Lemmata 7 and 8.

As in [17, 27], the moment estimate in Theorem 3 is a consequence of the following statement, which is reminiscent of (2.3) and its relationship to (2.2). We will establish the following.

Theorem 4. *For $V > 0$, define*

$$R(V; q) := \left\{\chi \text{ primitive of modulus } q : \left|L\left(\tfrac{1}{2}, \chi\right)\right| > V\right\}.$$

Then there exists a constant $A > 0$ such that, for every odd prime p and every $q = p^n$,

$$|R(V; q)| \ll_\varepsilon p^A q^{2+\varepsilon} V^{-12}.$$

Note that Theorem 3 follows immediately from Theorem 4 via summation by parts. From the available sharp estimates on the fourth moment of Dirichlet L-functions [15, 32], it follows that $|R(V; q)| \ll_\varepsilon q^{1+\varepsilon} V^{-4}$; see Section 2.6. Combining this and the Weyl bound for this particular class of Dirichlet L-functions [24, 30], the range of interest in Theorem 4 is $q^{1/8-\varepsilon} \leqslant V \leqslant q^{1/6+\varepsilon}$.

The p-adic methods of this chapter are very flexible. In particular, an analogue of [27, Theorem 1.2], which would sharpen Theorem 4 in the range $q^{3/20+\varepsilon} \leqslant V \leqslant q^{1/6-\varepsilon}$ (but not Theorem 3) can likely be proved with a further application of the p-adic stationary phase method to complete exponential sums with substantially more involved phases than in (2.6). It would also be of interest to investigate whether the methods of the present chapter and [27] can be unified to provide a twelfth moment bound for characters to moduli q with finitely many well-located factors as in [14] (or a hybrid moment including the archimedean average); without imposing overly onerous factorization conditions, though, this may require delicate estimates on complete sums with degenerate critical points as in [14, Lemma 7].

2.1.1 Overview

For the benefit of the reader, we present a conceptual overview of the proof, ignoring non-generic cases, coprimality conditions, q^ε factors, and so on; in particular, we use $f \preccurlyeq g$ to denote $|f| \ll_{p,\varepsilon} q^\varepsilon g$ and $f \sim g$ for $f \preccurlyeq g \preccurlyeq f$. We fix a divisor $q_1 \mid q$, and consider the short second moment

$$S_2(\chi) := \sum_{\psi_1 \pmod{q_1}} \left| L\left(\tfrac{1}{2}, \chi\psi_1\right) \right|^2. \tag{2.4}$$

We will later choose roughly $q_1 \sim V^2$, so that the expected sharp bound $S_2(\chi) \preccurlyeq q_1$ essentially matches the contribution of a single summand $|L(\frac{1}{2}, \chi)| \sim V$.

Using the approximate functional equation and executing the ψ_1-average leads to weighted dyadic sums over $n \sim N \preccurlyeq q^{1/2}$ of terms of the form $\chi(n + hq_1)\overline{\chi}(n)$, which are $Q_1 = (q/q_1)$-periodic. We apply Poisson summation, incurring the dual variable $j \preccurlyeq Q_1/N$ and the "trace function" $K_\chi(j, h; Q_1)$, which is shown in (2.19) and generically depends on $jh \preccurlyeq q/q_1^2$. The upshot of this analysis is Proposition 10, which bounds $S_2(\chi)$ roughly by

$$q_1 \Bigg(1 + Q_1^{-1/2} \sideset{}{^*}\sum_{|m| \preccurlyeq q/q_1^2} K_\chi(m; Q_1) A(m) \Bigg), \tag{2.5}$$

with somewhat messy arithmetic coefficients $A(m) \preccurlyeq 1$.

In Lemma 11, we show that the complete exponential sum $K_\chi(m; Q_1)$ exhibits square-root cancellation. This alone yields the upper bound $S_2(\chi) \preccurlyeq q_1 + (q/q_1)^{1/2}$, which is sharp for $q_1 \succcurlyeq q^{1/3}$ and recovers the Weyl subconvexity bound $L(\frac{1}{2}, \chi) \preccurlyeq q^{1/6}$ (essentially by Weyl differencing followed by completion, as in [24]).

For purposes of Theorems 3 and 4, we must consider values $q^{1/4} \preccurlyeq q_1 \preccurlyeq q^{1/3}$, in which case the weighted sum of trace functions in (2.5) is of length $Q_1^{1/2} \preccurlyeq q/q_1^2 \preccurlyeq Q_1^{2/3}$. Weights $A(m)$ make it difficult to directly estimate the sum. Instead, the key idea is sort of a *large sieve*: we argue that (roughly speaking, and as q_1 gets smaller) the vectors $(K_\chi(m; Q_1))_m$ are typically approximately orthogonal for different χ, and thus it is hard for too many of them to avoid cancellation with a single vector $(A(m))_m$. The approximate orthogonality boils down to cancellations in *incomplete sums of* *products*; since the length is over the square-root of the conductor, we apply the method of completion, incurring an additive twist. Proposition 12, our key arithmetic input, shows square-root cancellation in sums of products of rough form

$$\sum_{u \pmod{Q}}^{*} K_{\chi}^{\pm}(u; Q_1)\overline{K_{\chi'}^{\pm}(u; Q_1)}e(-uv/Q) \preccurlyeq Q^{1/2}. \tag{2.6}$$

Here, the modulus $Q \mid Q_1$ drops with the conductor of $\chi\overline{\chi'}$ (essentially the distance between χ and χ' in the dual topology), and we must first separate K_χ into two oscillatory components $K_\chi^{\pm}$ (as often happens with Bessel functions; see also [5, §9]). Lemma 11 and Proposition 12 form the heart of the chapter and are proved by a consistent application of the *p-adic method of stationary phase* to exponential sums with p-adically analytic phases, including characters to prime power moduli; see Section 2.2.

Proceeding with the large sieve idea, we estimate the the sum of $S_2(\chi\psi)$ in (2.5) over an arbitrary set Ψ of characters ψ modulo some $q_2 \mid q$ (with $q_1 \mid q_2$) by applying the Cauchy–Schwarz inequality to the m-sum and bounding sums of products of $K_{\chi\psi}^{\pm}(m; Q_1)$ using (2.6). This shows in Proposition 13 that

$$\sum_{\psi \in \Psi} S_2(\chi\psi) \preccurlyeq \left(\left(q_1 + q_1^{1/4} q_2^{1/4}\right)|\Psi| + q^{1/2}|\Psi|^{1/2} \right). \tag{2.7}$$

The bound (2.7) imposes a restriction on the size $|\Psi|$ as long as each $S_2(\chi\psi)$ is slightly bigger than $q_1 + q_1^{1/4}q_2^{1/4}$. In Section 2.6, we first fix χ and choose Ψ to be the set of characters modulo q_2 for which one of $|L(\frac{1}{2}, \chi\psi\psi_1)|$ in (2.4) exceeds V, with $q_1 = q^{-\varepsilon}V^2$ and $q_2 = q_1^3$, obtaining $|\Psi| \preccurlyeq qV^{-4}$. From here it is a matter of bookkeeping to Theorem 4 and hence Theorem 3.

2.1.2 Notation

Throughout this chapter, $\varepsilon > 0$ indicates a fixed positive number, which may be different from line to line but may at any point be taken to be as small as desired. As usual, $f \ll g$ and $f = O(g)$ indicate that $|f| \leqslant Cg$ for some effective constant $C > 0$, which may be different from line to line but does not depend on any parameters except as follows. In this introduction, all implied constants in $\ll$ and O are absolute, except that they may depend on $\varepsilon > 0$ if so indicated as in $\ll_\varepsilon$. In the rest of the chapter, we allow the implied constants (but suppress this from notation) to depend on both the odd prime p and $\varepsilon > 0$. All dependencies on p are easily seen to be polynomial, leading to the statements of Theorems 3 and 4; we do not make an effort to optimize the value of $A > 0$.

We denote the cardinality of a finite set S by $|S|$; we use the same notation for the Lebesgue measure, with the meaning clear from the context. As is customary in analytic number theory, we also write $e(z) = e^{2\pi iz}$.

2.1.3 Acknowledgments

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2.2 Preliminaries

2.2.1 Approximate functional equation

A ubiquitous tool in the analysis of L-functions inside the critical strip is the approximate functional equation (see [21, §5.2]). This equation has various manifestations depending on context and purpose. A typical form of this equation in the context of bounding central values states that one may recover the size of $L(\frac{1}{2}, \chi)$ by inserting

$s = \frac{1}{2}$ into the associated Dirichlet series which is essentially truncated at $q^{1/2}$ via a suitable smooth weight function. For our purposes, the following lemma is convenient, which follows by applying a dyadic partition of unity to [21, Theorem 5.3].

Lemma 5. *Let χ be a primitive Dirichlet character modulo q. Then,*

$$\left|L\left(\tfrac{1}{2}, \chi\right)\right|^2 \ll \log q \sum_{\substack{N \leqslant q^{1/2+\varepsilon} \\ N \text{ dyadic}}} \left| \frac{1}{\sqrt{N}} \sum_n \chi(n) V_N(n) \right|^2 + q^{-100},$$

where V_N is a smooth function depending only on N and q, whose support is contained in $[N/2, 2N]$ and whose derivatives satisfy $V_N^{(j)} \ll_j N^{-j}$ for every $j \in \mathbb{N}$.

2.2.2 p-adically analytic phases

Among the key features of our treatment of exponential sums will be: *(i)* the consistent interpretation of oscillating terms (such as characters) as exponentials with phases that are p-adically analytic functions and *(ii)* the analysis thereof. For a rigorous treatment of these concepts, we refer to [24, §2]. Recall that a p-adically analytic function f on a domain $D \subseteq \mathbb{Z}_p$ is locally expressible, around each point $a \in D$, in a p-adic ball of the form $\{x \in \mathbb{Z}_p : |x - a|_p \leqslant p^{-\varrho}\} \subseteq D$ ($\varrho \in \mathbb{Z}_{\geqslant 0}$) as the sum of its p-adically convergent Taylor power series. We let $r_p(f; a)$ denote the largest such $p^{-\varrho}$ (which is not quite the same as the p-adic radius of convergence) and $r_p(f) = \inf_{a \in D} r_p(f; a) \geqslant 0$; in all phases we will encounter, $r_p(f) \geqslant p^{-1}$ will hold. It is not hard to see that $r_p(f'; a) \geqslant r_p(f; a)$.

We will make extensive use of the p-adic logarithm, which for simplicity we define on $1 + p\mathbb{Z}_p$. Recall that, throughout this chapter, p is an odd prime.

Definition 1. The p-adic logarithm, $\log_p : 1 + p\mathbb{Z}_p \to p\mathbb{Z}_p$ is the analytic function given as

$$\log_p(1+x) := \sum_{k \geqslant 1} (-1)^{k-1} \frac{x^k}{k}.$$

Access to the above is critical due to the following lemma, with roots in Postnikov [30] and which we quote from [24, Lemma 13].

Lemma 6. *Let χ be a primitive character modulo p^n. Then there exists a p-adic unit A such that, for every p-adic integer k,*

$$\chi(1+kp) = e\left(\frac{A \log_p(1+kp)}{p^n}\right). \tag{2.8}$$

Lemma 6 allows us to explicate the phase of any exponential of the form

$$\chi(1+kp)e(f(k)/p^n)$$

when χ is a character modulo p^n.

It will be necessary to handle solutions to quadratic equations over $\mathbb{Z}_p$, which requires the use of p-adic square roots. For p an odd prime and $x \in \mathbb{Z}_p^{\times 2}$, the congruence $u^2 \equiv x \pmod{p^\kappa}$ has exactly two solutions modulo every p^κ, which reside within two p-adic towers and limit to the solutions of $u^2 = x$ as $\kappa \to \infty$. We denote these solutions $\pm x_{1/2}$. For $(\,\cdot\,)_{1/2} : \mathbb{Z}_p^{\times 2} \to \mathbb{Z}_p^\times$ to be well-defined, a choice of square root for each $y \in (\mathbb{Z}/p\mathbb{Z})^{\times 2}$ must be made. This set of choices propagates to $\mathbb{Z}_p^{\times 2}$ and represents one of the $2^{(p-1)/2}$ branches of the p-adic square root. A thorough treatment of p-adic square roots can be found in [5, §2]; we content ourselves with summarizing two properties of import to us.

Each branch $x_{1/2}$ of the square root is an analytic function expressible by a convergent power series in balls of radius $r_p \geqslant p^{-1}$. Specifically, on $1+p\mathbb{Z}_p$, the binomial expansion

$$(1 + xp)^{1/2} = \sum_{k \geqslant 0} \binom{1/2}{k} (xp)^k \tag{2.9}$$

gives the branch with values in $1 + p\mathbb{Z}_p$ (as seen by formally squaring the right-hand side), which is in fact an automorphism of $1+p\mathbb{Z}_p$. For an arbitrary $u \in \mathbb{Z}_p^{\times 2}$, a simple argument modulo p shows that

$$(u + xp)_{1/2} = u_{1/2}(1 + x\overline{u}p)^{1/2} \tag{2.10}$$

where $\overline{u}$ denotes the p-adic inverse of u. While $(\,\cdot\,)_{1/2}$ cannot in general be expected to be multiplicative, (2.10) gives it both a pseudo-morphism rule and a power expansion. Moving forward, we fix a branch to be used throughout, drop the $(\,\cdot\,)_{1/2}$ notation and simply write $(\,\cdot\,)^{1/2}$ or use a radical symbol for our chosen branch, using caution to only use (2.9), (2.10), and $\sqrt{m}^2 = m$ when exercising the usual archimedean exponent rules. For future reference, we note that, for all $u, u' \in \mathbb{Z}_p^{\times 2}$,

$$\operatorname{ord}_p(\sqrt{u} - \sqrt{u'}) = \operatorname{ord}_p(u - u'). \tag{2.11}$$

2.2.3 p-adic method of stationary phase

The following pair of lemmata establishes what is known as the p-adic method of stationary phase (see, for example, [24, §4], [5, §7]), allowing one to evaluate complete sums involving such exponentials. They are the proper p-adic analogues of the classical method of stationary phase for exponential integrals of the form $\int_{\mathbb{R}} g(x)e(f(x))\,\mathrm{d}x$ with a suitable smooth phase f and weight g, which generically proceeds in two principal steps: *(i)* showing that ranges where $|f'|$ is not suitably small are negligible, and *(ii)* close to each non-degenerate stationary point x_0 of the phase f, approximating f quadratically, with resulting Gaussian-type integrals evaluating to about

$g(x_0)e(f(x_0))/\sqrt{|f''(x_0)|}$ (see [13]).

Lemma 7. *Let p be an odd prime, $1 \leqslant \ell \leqslant n$ be integers, and $f : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ be an analytic function invariant modulo p^n under translation by $p^n\mathbb{Z}_p$. If $r_p(f) \geqslant p^{-\ell}$ and $p^{k\ell} f^{(k)}(x)/k! \equiv 0 \pmod{p^n}$ for all $x \in \mathbb{Z}_p^\times$ when $k \geqslant 2$, then*

$$\sideset{}{^*}\sum_{x \pmod{p^n}} e\left(\frac{f(x)}{p^n}\right) = \sideset{}{^*}\sum_{\substack{x_0 \pmod{p^n} \\ f'(x_0)\equiv 0 \pmod{p^{n-\ell}}}} e\left(\frac{f(x_0)}{p^n}\right).$$

Proof. Expanding $f(x)$ around x_0 gives $f(x_0 + tp^\ell) = \sum_{k\geq 0} f^{(k)}(x_0)(tp^\ell)^k/k!$. With this, observe

$$\sideset{}{^*}\sum_{x \pmod{p^n}} e\left(\frac{f(x)}{p^n}\right) = \frac{1}{p^{n-\ell}} \sideset{}{^*}\sum_{x_0 \pmod{p^n}} \sum_{t \pmod{p^{n-\ell}}} e\left(\frac{f(x_0) + f'(x_0)tp^\ell}{p^n}\right)$$

where the inner sum contributes $p^{n-\ell}e(f(x_0)/p^n)$ when $f'(x_0) \equiv 0 \pmod{p^{n-\ell}}$ and vanishes otherwise. ■

Lemma 7 reduces a complete exponential sum to p-adic neighborhoods in which $|f'(x)|_p$ is small. The following lemma is a further refined statement that explicitly evaluates these localized sums and is suited for exponential sums that we will encounter in the proof of Lemma 11.

Lemma 8. *Let p be an odd prime, $n \geqslant 2$, and $f : \mathbb{Z}_p^\times \to \mathbb{Z}_p$ be an analytic function satisfying the hypotheses in Lemma 7 for $\ell = \lceil n/2 \rceil$. Let $X \subseteq (\mathbb{Z}/p^n\mathbb{Z})^\times$ denote the solution set of $f'(x_0) \equiv 0 \pmod{p^{\lfloor n/2 \rfloor}}$, and assume that, for all $x_0 \in X$, $r_p(f; x_0) \geqslant p^{-\lfloor n/2 \rfloor}$, $f''(x_0) \in \mathbb{Z}_p$, and $p^{\lfloor n/2 \rfloor k} f^{(k)}(x_0)/k! \equiv 0 \pmod{p^n}$ for $k \geqslant 3$. Then, X is invariant under translation by $p^{\lfloor n/2 \rfloor}\mathbb{Z}$, and, for an arbitrary set of representatives $\tilde{X}$*

for $X/p^{\lfloor n/2\rfloor}\mathbb{Z}$*,*

$$\sideset{}{^*}\sum_{x \pmod{p^n}} e\left(\frac{f(x)}{p^n}\right) = p^{n/2}\sum_{x_0\in\tilde{X}} e\left(\frac{f(x_0)}{p^n}\right)\Delta_f(x_0;p^n),$$

where all summands are independent of the choice of $\tilde{X}$*, and, writing* $f'(x_0)_\circ := f'(x_0)/p^{\lfloor n/2\rfloor}$ *and* $\left(\frac{\cdot}{p}\right)$ *for the Legendre symbol,*

$$\Delta_f(x_0;p^n) = \begin{cases} 1, & 2 \mid n; \\ \epsilon(p)\left(\frac{2f''(x_0)}{p}\right)e\left(\frac{-\overline{2f''(x_0)}f'(x_0)_\circ^2}{p}\right), & 2\nmid n, p\nmid f''(x_0); \\ \sqrt{p}\mathbf{1}_{p\mid f'(x_0)_\circ}, & 2\nmid n, p \mid f''(x_0), \end{cases}$$

$$\epsilon(p) = \begin{cases} 1, & p\equiv 1 \pmod 4; \\ i, & p \equiv 3 \pmod 4). \end{cases}$$

Proof. The translational invariance of X modulo $p^{\lfloor n/2\rfloor}\mathbb{Z}$ is clear from our hypotheses and the expansion of $f'(x_0+tp^{\lfloor n/2\rfloor})$ at each $x_0\in X$. Application of Lemma 7 with $\ell = \lceil n/2\rceil$ together with an expansion of f around each $x_0 \in \tilde{X}$ gives

$$\begin{aligned} &\sideset{}{^*}\sum_{x \pmod{p^n}} e\left(\frac{f(x)}{p^n}\right) \\ &\qquad = \sum_{x_0\in\tilde{X}}\sum_{t \pmod{p^{\lceil n/2\rceil}}} e\left(\frac{f(x_0+tp^{\lfloor n/2\rfloor})}{p^n}\right) \\ &\qquad = p^{\lfloor n/2\rfloor}\sum_{x_0\in\tilde{X}} e\left(\frac{f(x_0)}{p^n}\right)\sum_{t \pmod{p^{n-2\lfloor n/2\rfloor}}} e\left(\frac{f'(x_0)_\circ t+\bar{2}f''(x_0)t^2}{p^{n-2\lfloor n/2\rfloor}}\right). \end{aligned}$$

For n even, the inner sum is trivial and the desired result follows. If n is odd, the contribution from $p \mid f''(x_0)$ is clear, while, for $p\nmid f''(x_0)$, completing the square yields for the inner sum

$$e\left(\frac{-\overline{2f''(x_0)}f'(x_0)_\circ^2}{p}\right)\sum_{t \pmod p} e\left(\frac{\bar{2}f''(x_0)t^2}{p}\right)$$
$$= \epsilon(p)\sqrt{p}\left(\frac{\bar{2}f''(x_0)}{p}\right)e\left(\frac{-\overline{2f''(x_0)}f'(x_0)_\circ^2}{p}\right),$$

by the classical evaluation of the quadratic Gauss sum. This finishes the proof. ■

Remark 1. The general (if somewhat cumbersome) conditions in Lemma 8 are easily satisfied, say, for every analytic function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ with $r_p(f) \geqslant 1/p$ and $f^{(j)}(\mathbb{Z}_p) \subseteq \mathbb{Z}_p$ for all $j \geqslant 0$. The same is true for Lemma 7 with $\ell \geqslant n/2$.

In Lemma 8, in the odd nonsingular case $2 \nmid n$, $p \nmid f''(x_0)$, we see that

$$f'(x_0 + tp^{\lfloor n/2 \rfloor}) \equiv 0 \pmod{p^{\lceil n/2 \rceil}}$$

for exactly one $t \pmod p$; picking such a representative $\tilde{x}_0 := x_0 + tp^{\lfloor n/2 \rfloor} \in \tilde{X}$, we have more simply $\Delta_f(\tilde{x}_0; p^n) = \epsilon(p)\left(\frac{2f''(\tilde{x}_0)}{p}\right)$.

Remark 2. Versions of Lemmata 7 and 8 exist for sums over other subsets of residue classes $x \pmod{p^n}$, where the phase f may have as domain a finite union of translates of $p^\lambda \mathbb{Z}_p$, with $\lambda \leqslant \ell$ in Lemma 7 and $\lambda \leqslant n/2$ in Lemma 8. The proofs of these parallel statements are the same; indeed, they only require that the sum be over a set of residues invariant under translation by a suitable $p^\ell \mathbb{Z}_p$ with $\ell \geqslant \lambda$ and $r_p(f) \geqslant p^{-\ell}$. Specifically, the proof of Proposition 12 will require Lemma 7 to be applied over quadratic residues and non-residues modulo p^n. Lemmata 7 and 8 also hold for sums of the form $\sum g(x)e(f(x)/p^n)$ where g is invariant under translation by $p^\ell \mathbb{Z}_p$.

In practice, we will apply Lemmata 7 and 8 in situations where explicitly writing the exponent of $q = p^n$ gets notationally cumbersome. To represent what are essentially square roots in these cases, we define

$$\mathrm{rt}_*(p^n) := p^{\lfloor n/2 \rfloor} \quad \text{and} \quad \mathrm{rt}^*(p^n) := p^{\lceil n/2 \rceil}.$$

2.2.4 Completion

While Lemmata 7 and 8 provide powerful tools for evaluating the types of complete exponential sums that will be found throughout, we will eventually encounter those which are incomplete. In anticipation of this, we introduce the next lemma which prepackages a technique known as completion.

Lemma 9. *Suppose f is an arithmetic function with period Q. Then*

$$\sum_{m \leqslant M} f(m) \ll \frac{1}{Q} \sum_{v \pmod{Q}} \left| \widehat{f}(v) \right| \cdot \min\left\{ M, \|v/Q\|^{-1} \right\},$$
$$\widehat{f}(v) := \sum_{u \pmod{Q}} f(u) e\left(\frac{-uv}{Q} \right),$$

where $\|x\|$ is the distance from x to the nearest integer.

Proof. Splitting the sum into residue classes modulo Q yields

$$\begin{aligned}\sum_{m \leqslant M} f(m) &= \sum_{u \pmod{Q}} f(u) \sum_{m \leqslant M} \frac{1}{Q} \sum_{v \pmod{Q}} e\left(\frac{(m-u)v}{Q} \right) \\ &= \frac{1}{Q} \sum_{v \pmod{Q}} \widehat{f}(v) \sum_{m \leqslant M} e\left(\frac{mv}{Q} \right).\end{aligned}$$

The bound

$$\sum_{m \leqslant M} e\left(\frac{mv}{Q} \right) \ll \min\{M, \|v/Q\|^{-1}\}$$

on the sum of a geometric sequence completes the proof. ∎

2.3 Short second moment

As before and throughout, p will be an odd prime and q some prime power p^n for n a positive integer. Further consider

$$p \leqslant q_1 \leqslant q_2 < q, \tag{2.12}$$

where the q_i are also powers of p. A central object to our proofs, as in [17, 27], is the short second moment. In the q-aspect, this will be a power moment which samples from a q_1-neighborhood around some fixed primitive character $\chi \pmod{q}$. This analogy is particularly natural from a p-adic point of view, as the (Pontrjagin) dual group of $\mathbb{Z}_p^*$ carries the natural dual topology, with respect to which these correspond to actual small neighborhoods of χ. We denote

$$S_2(\chi) := \sum_{\psi_1 \pmod{q_1}} \left| L\left(\tfrac{1}{2}, \chi\psi_1\right)\right|^2.$$

We will eventually analyze the size of short moments on average, but first must gather information on $S_2(\chi)$ itself.

2.3.1 Executing the short second moment

We immediately apply Lemma 5 to $S_2(\chi)$. This yields

$$\begin{aligned} S_2(\chi) &\ll q^{\varepsilon} \sum_{\psi_1 \pmod{q_1}} \sum_{\substack{N \leqslant q^{1/2+\varepsilon} \\ N \text{ dyadic}}} \left| \frac{1}{\sqrt{N}} \sum_n \chi\psi_1(n) V_N(n)\right|^2 + q^{-99} \\ &\ll \frac{q^{\varepsilon}}{N} \sum_{\psi_1 \pmod{q_1}} \left| \sum_n \chi\psi_1(n) V_N(n)\right|^2 + q^{-99} \end{aligned} \tag{2.13}$$

for some $N \leqslant q^{1/2+\varepsilon}$ by exchanging order of summation and choosing the summand which maximizes the inner sum. Denote the sum in (2.13) without the error term and q^{ε} factor as $B(N)$. Expansion and orthogonality of characters gives

$$B(N) \ll \frac{q_1}{N} \sum_{n \equiv n' \pmod{q_1}} \chi(n')\overline{\chi}(n)V_N(n')\overline{V_N(n)}.$$

We note the similarity of the resulting sum (the sum of squares of short p-adic averages, a reflection of the ψ_1-average via Parseval's identity) to those encountered with Weyl differencing in the context of factorable moduli (see, for example, [24, §5]), and we proceed similarly.

Recall that $V_N \ll 1$ with support contained in $[N/2, 2N]$. The diagonal terms corresponding to $n = n'$ contribute $O(q_1)$ to $B(N)$. The addition of this to the remaining pairs (n', n) gives

$$B(N) \ll q_1 + \frac{q_1}{N} \operatorname{Re} \sum_{h \geq 1} \sum_{n \geq 1} \chi(n + hq_1)\overline{\chi}(n)V_N(n + hq_1)\overline{V_N(n)}, \tag{2.14}$$

since each (n', n) appears above or can be accounted for by conjugation. Denote the inner sum in (2.14) as $S_{hq_1}(N; \chi)$. Since $\chi(n+hq_1)\overline{\chi}(n)$ is periodic modulo $Q_1 = q/q_1$, we may write

$$S_{hq_1}(N; \chi) = \sideset{}{^*}\sum_{r \pmod{Q_1}} \chi(r + hq_1)\overline{\chi}(r) \sum_j V_N(r + jQ_1 + hq_1)\overline{V_N(r + jQ_1)}. \tag{2.15}$$

We will apply Poisson summation to the inner sum in S_{hq_1}. Examination of

$$\begin{aligned} &\int_{\mathbb{R}} V_N(r + xQ_1 + hq_1)\overline{V_N(r + xQ_1)}e(-jx)\,\mathrm{d}x \\ &\qquad = Q_1^{-1}e(rj/Q_1) \int_{\mathbb{R}} V_N(y + hq_1)\overline{V_N(y)}e(-jy/Q_1)\,\mathrm{d}y \end{aligned} \tag{2.16}$$

shows that the Fourier transform in (2.16) is

$$Q_1^{-1} e\left(\frac{rj}{Q_1}\right) \widehat{W_{hq_1}}(j/Q_1) \quad \text{where} \quad W_{hq_1}(y) := V_N(y+hq_1)\overline{V_N(y)}. \tag{2.17}$$

Using (2.13) through (2.17) together with Poisson summation yields

$$S_2(\chi) \ll q^{\varepsilon}\left(q_1 + \frac{q_1}{N} \operatorname{Re} \sum_{h \geq 1} Q_1^{-1/2} \sum_j \widehat{W_{hq_1}}(j/Q_1) K_\chi(j,h;Q_1)\right), \tag{2.18}$$

where, for $\tilde{q}$ a proper divisor of q, we define

$$K_\chi(a,b;\tilde{q}) := \tilde{q}^{-1/2} \sideset{}{^*}\sum_{r \pmod{\tilde{q}}} \chi\big(r + b(q/\tilde{q})\big)\overline{\chi}(r) e(ar/\tilde{q}). \tag{2.19}$$

In particular, for $(m,\tilde{q}) = 1$, we also write $K_\chi(m;\tilde{q}) := K_\chi(1,m;\tilde{q}) = K_\chi(m,1;\tilde{q})$, so that

$$K_\chi(m;\tilde{q}) = \tilde{q}^{-1/2} \sideset{}{^*}\sum_{r \pmod{\tilde{q}}} \chi\big(1 + (q/\tilde{q})r\big) e(m\bar{r}/\tilde{q}). \tag{2.20}$$

This sum (which takes on the role of trace functions from the context of square-free moduli [27]) is of central importance to our arguments. We summarize some of its important properties in Lemma 11 in Section 2.4, below. In this section, we will only require the elementary reduction and vanishing claim (2.25).

By the support of V_N, we may actually take $h \ll N/q_1$ in (2.18). We will soon identify the range j that is essential to (2.18). Once this range becomes finite, we will configure our bound in a way that highlights the main object of our study.

2.3.2 Establishing the bound on $S_2(\chi)$

We first show that the contribution to (2.18) from $j = 0$ may be neglected. By Lemma 11 below,

$$K_\chi(0,h;Q_1) = \begin{cases} Q_1^{1/2}(1-p^{-1}), & Q_1 \mid h; \\ -Q_1^{1/2}/p, & Q_1/p \parallel h; \\ 0, & \text{otherwise.} \end{cases}$$

The contribution from $j = 0$ to (2.18) is then

$$q^\varepsilon q_1 Q_1^{-1/2} \operatorname{Re} \sum_{1 \leqslant h \ll N q_1^{-1}} N^{-1} \widehat{W_{hq_1}}(0) K_\chi(0,h;Q_1) \ll q^\varepsilon q_1 \sum_{\substack{1 \leqslant h \ll N q_1^{-1} \\ Q_1 \mid hp}} 1 \ll q^\varepsilon \frac{N}{Q_1}. \quad (2.21)$$

Repeated use of integration by parts shows

$$\widehat{W_{hq_1}}(y) \ll_m N \left(\frac{1}{Ny}\right)^m$$

for every positive integer m. From this bound, $h \ll N/q_1$, and the trivial bound on $K_\chi(j,h;Q_1)$, the contribution to (2.18) from $|j| > q^\varepsilon Q_1/N$ is $O(q^{-100})$. Using (2.18) and (2.21), we find

$$S_2(\chi) \ll q^\varepsilon \left(q_1 + \frac{q_1}{Q_1^{1/2}} \operatorname{Re} \sum_{0 < h \ll N/q_1} \sum_{0 < |j| \ll q^\varepsilon Q_1/N} N^{-1} \widehat{W_{hq_1}}(j/Q_1) K_\chi(j,h;Q_1) \right). \quad (2.22)$$

According to Lemma 11, noting that $(Q_1^2/p) \nmid jh \ll q^\varepsilon Q_1/q_1$, we may rewrite the double sum above as

$$\begin{aligned}
&\sum_{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1}} \sideset{}{^*}\sum_{h' \ll N(q_1 p^\eta)^{-1}} \sideset{}{^*}\sum_{0<|j'|<q^\varepsilon Q_1 (Np^\eta)^{-1}} N^{-1} \widehat{W_{h'p^\eta q_1}}(j'p^\eta/Q_1) K_\chi(j'p^\eta, h'p^\eta; Q_1) \\
&\qquad = \sum_{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1}} p^{\eta/2} \sideset{}{^*}\sum_{|m| \ll q^{1+\varepsilon}(q_1^2 p^{2\eta})^{-1}} K_\chi(m; Q_1/p^\eta) \\
&\qquad\qquad \times \sum_{\substack{h'|j'|=m \\ h' \ll N(q_1 p^\eta)^{-1} \\ |j'|<q^\varepsilon Q_1 (Np^\eta)^{-1}}} N^{-1} \widehat{W_{h'p^\eta q_1}}(j'p^\eta/Q_1). \qquad (2.23)
\end{aligned}$$

Denoting the inner sum of (2.23) as $A(m; p^\eta)$, the above becomes

$$\sum_{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1}} p^{\eta/2} \sideset{}{^*}\sum_{|m| \ll q^{1+\varepsilon}(q_1^2 p^{2\eta})^{-1}} K_\chi(m; Q_1/p^\eta) A(m; p^\eta), \qquad (2.24)$$

where $A(m; p^\eta) \ll m^\varepsilon$ by the divisor bound. The key thing is that these noisy coefficients do not depend on χ, which will allow us to remove them via an application of the Cauchy–Schwarz inequality in Section 2.5. Combining (2.22) through (2.24) we obtain Proposition 10.

Proposition 10. *Let q_1 and q be subject to the conditions in (2.12). Then there exist coefficients $A(m; p^\eta) \ll m^\varepsilon$ such that, for every primitive character χ (mod q),*

$$\begin{aligned}
&S_2(\chi) \\
&\quad \ll q^\varepsilon q_1 \Bigg(1 + Q_1^{-1/2} \operatorname{Re} \sum_{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1}} p^{\eta/2} \sideset{}{^*}\sum_{|m| \ll q^{1+\varepsilon}(q_1^2 p^{2\eta})^{-1}} K_\chi(m; Q_1/p^\eta) A(m; p^\eta) \Bigg),
\end{aligned}$$

where $K_\chi(m; Q_1/p^\eta)$ are as in (2.19).

2.4 Exponential sum estimates

In this section, we evaluate and estimate complete exponential sums to prime power moduli. Our principal tools are the p-adic stationary phase method Lemmata 7 and 8. In Lemma 11, we consider the complete exponential sum $K_\chi(j,h;\tilde{q})$ introduced in (2.19) and show that it can be expressed in terms of explicit exponentials $K_\chi^{\pm}(m;\tilde{q})$ with p-adically analytic phases. Then, in Proposition 12, we show square-root cancellation in complete sums of products of $K_\chi^{\pm}(m;\tilde{q})$ including additive twists.

2.4.1 Evaluation of $K_\chi(m;\tilde{q})$

In the following lemma, we explicitly evaluate the complete exponential sum $K_\chi(j,h;\tilde{q})$.

Lemma 11. *Let $\tilde{q}$ be a proper divisor of q and, for every $j\in\mathbb{Z}$, let $p^{\eta_j}=(j,\tilde{q})$. Then, the sum $K_\chi(j,h;\tilde{q})$ defined in (2.19) satisfies*

$$K_\chi(j,h;\tilde{q}) = \begin{cases} p^{\eta/2}K_\chi(jh/p^{2\eta};\tilde{q}/p^{\eta}), & \eta=\eta_j=\eta_h,\ \tilde{q}\neq p^{\eta};\\ \tilde{q}^{1/2}(1-p^{-1}), & \eta_j=\eta_h=\eta_{\tilde{q}};\\ -\tilde{q}^{1/2}/p, & \eta_j+\eta_h=2\eta_{\tilde{q}}-1;\\ 0, & \text{otherwise}. \end{cases} \tag{2.25}$$

Further, let A be an integer such that (2.8) holds for χ, and assume that $\tilde{q}\geqslant p^2$. Then, for $(m,\tilde{q})=1$,

$$K_\chi(m;\tilde{q}) = K_\chi^{+}(m;\tilde{q}) + K_\chi^{-}(m;\tilde{q}), \tag{2.26}$$

where

$$K_\chi^{\pm}(m;\tilde q) = \begin{cases} \Delta_\theta(s_\pm(m/A;\tilde q);\tilde q) e\left(Ag_\pm(m/A;\tilde q)/\tilde q\right), & \left(\frac{Am}{p}\right)=1; \\ 0, & \textit{otherwise}, \end{cases}$$

where, for $\left(\frac{m}{p}\right)=1$,

$$\begin{aligned} g_\pm(m;\tilde q) &= (\tilde q/q)\log_p\left(1+(q/\tilde q)s_\pm(m;\tilde q)\right) + m/s_\pm(m;\tilde q), \\ s_\pm(m;\tilde q) &= \tfrac12\left(mq/\tilde q \pm \sqrt{(mq/\tilde q)^2+4m}\right), \end{aligned} \tag{2.27}$$

θ is the phase associated to $K_\chi(m;\tilde q)$ in (2.20), and Δ_θ is as described in Lemma 8.

Proof. We first establish (2.25). Write $h \equiv h'p^{\eta_h} \pmod{\tilde q}$ and $j \equiv j'p^{\eta_j} \pmod{\tilde q}$ where $p \nmid h'j'$, and set $\eta = \min\{\eta_j,\eta_h\}$. If $p^\eta \neq \tilde q$, then by the substitution $r \mapsto \overline{j'r}$ for the variable of summation in (2.19) and a reduction to a sum over residues modulo $\tilde q/p^\eta$ we have

$$K_\chi(j,h;\tilde q) = \tilde q^{-1/2} p^\eta \sideset{}{^*}\sum_{r \pmod{\tilde q/p^\eta}} \chi\left(1+\frac{q}{\tilde q/p^\eta}p^{\eta_h-\eta} r j' h'\right) e\left(\frac{p^{\eta_j-\eta}\overline{r}}{\tilde q/p^\eta}\right). \tag{2.28}$$

In particular, this proves the first case of (2.25). The second case, when $p^\eta = \tilde q$, follows from a trivial evaluation of the definition (2.19).

We now assume $\eta_j \neq \eta_h$. For $\eta_j+\eta_h = 2\eta_{\tilde q}-1$, the situation quickly boils down, directly or with an application of (2.8), to

$$K_\chi(j,h;\tilde q) = \tilde q^{-1/2}(\tilde q/p) \sideset{}{^*}\sum_{r \pmod p} e(r/p) = -\tilde q^{1/2}/p.$$

In any other event, let ϕ be the phase associated to (2.28) where

$$\phi^{(k)}(x) = (-1)^{k-1}(k-1)!\left(A\left(\frac{p^{\eta_h-\eta}j'h'}{1+xj'hq/\tilde{q}}\right)^k\left(\frac{q}{\tilde{q}/p^\eta}\right)^{k-1} - kp^{\eta_j-\eta}x^{-(k+1)}\right) \quad (2.29)$$

for $k \geqslant 1$ and $x \in \mathbb{Z}_p^\times$ by (2.8). From (2.29), it is easy to see that $\phi^{(k)}(x)/(k-1)! \in \mathbb{Z}_p$ for $x \in \mathbb{Z}_p^\times$. Moreover

$$(\mathrm{rt}^*(\tilde{q}/p^\eta))^k/k \equiv 0 \pmod{\tilde{q}/p^\eta}$$

for all $k \geqslant 2$, so that $(\mathrm{rt}^*(\tilde{q}/p^\eta))^k\phi^{(k)}(x)/k! \equiv 0 \pmod{\tilde{q}/p^\eta}$ for all $x \in \mathbb{Z}_p^\times$ and all $k \geqslant 2$. Thus ϕ satisfies the hypotheses of Lemma 7 with $p^\ell = \mathrm{rt}^*(\tilde{q}/p^\eta)$. Since in this case $\mathrm{rt}_*(\tilde{q}/p^\eta) \geqslant p$ and exactly one of η_h and η_j equals η, we find that $K_\chi(j,h;\tilde{q})$ must vanish since no solutions to $\phi'(x) \equiv 0 \pmod p$ exist in $(\mathbb{Z}/p\mathbb{Z})^\times$. This completes the proof of (2.25).

Next, for A, $(m,\tilde{q}) = 1$, and $\tilde{q} \geqslant p^2$ as stated, the phase θ associated to $K_\chi(m;\tilde{q})$ in (2.20) agrees with the phase ϕ in (2.28) and (2.29) with $h = 1$, $j = j' = m$, and $\eta_j = \eta_h = \eta = 0$ (substituting $r \mapsto \overline{m}r$); in particular,

$$\theta'(x) = \frac{A}{1+xq/\tilde{q}} - \frac{m}{x^2} \quad \text{and} \quad \theta''(x) = -\frac{Aq/\tilde{q}}{(1+xq/\tilde{q})^2} + \frac{2m}{x^3}.$$

We will use Lemma 8. If $\left(\frac{Am}{p}\right) = -1$, there are no solutions to

$$\theta'(x_0) \equiv 0 \pmod{\mathrm{rt}_*(\tilde{q})}$$

by an obstruction modulo p, so that $K_\chi(m;\tilde{q}) = 0$ in this case. Otherwise, solving the equation $\theta'(x_0) = 0$ yields $x_0 = s_\pm(m/A;\tilde{q})$. Upon noting that $\theta''(x_0) \equiv 2mx_0^{-3} \not\equiv 0 \pmod p$, an application of Hensel's lemma gives exactly two unique solutions to the congruence above, proving (2.26). ∎

2.4.2 Sums of products

As we input Proposition 10 into estimating short second moments in aggregate over sets of characters, we will incur incomplete sums of products of trace functions $K_\chi(m;\tilde{q})$ evaluated in Lemma 11, with two different characters χ. Specifically, the inner sum in (2.32) will be estimated using the method of completion, Lemma 9. In preparation for this, in this section we prove the following proposition.

Proposition 12. *Let $\tilde{q} \geqslant p^2$ be a proper divisor of q and $K_\chi(m;\tilde{q})$ be as in (2.20). Further, let χ and χ' be two primitive Dirichlet characters modulo q with associated units A and A' as in (2.8). Denote $\delta_q(\chi,\chi') = (q/p, A-A')$ and $Q = \tilde{q}/(\tilde{q},\delta_q(\chi,\chi'))$. Then:*

(a) for $Q \geqslant p$, the expression $K_\chi^{\pm}(m;\tilde{q})\overline{K_{\chi'}^{\pm}(m;\tilde{q})}$ is Q-periodic and satisfies, for every $v \in \mathbb{Z}$,

$$\sideset{}{^*}\sum_{m \pmod Q} K_\chi^{\pm}(m;\tilde{q})\overline{K_{\chi'}^{\pm}(m;\tilde{q})}e\left(\frac{-mv}{Q}\right) \ll Q^{1/2}; \tag{2.30}$$

(b) for $Q \geqslant p^2$, the left-hand side of (2.30) vanishes unless $|v|_p = 1$;

(c) for $Q = 1$, $K_\chi^{\pm}(m;\tilde{q})\overline{K_{\chi'}^{\pm}(m;\tilde{q})} = \mathbf{1}_{(Am/p)=1}$.

Proof. Evaluation (2.26) expresses $K_\chi^{\pm}(m;\tilde{q})$ in terms of exponentials with p-adically analytic phases. Thus, as in Sections 2.2.2–2.2.3 and the proof of Lemma 11, we continue to treat the phases in complete exponential sums such as (2.30) as p-adically analytic functions on their respective domains. Since p-adic differentiation and power series will be involved, for the duration of this proof, we will use notation $K_\chi^{\pm}(u;\tilde{q})$ so as to avoid any visual suggestion that the arguments of these phases may only be rational integers.

By Lemma 11, the sum on the left-hand side of (2.30) vanishes unless $AA' \in \mathbb{Z}_p^{\times 2}$;

we assume this henceforth and restrict the sum (as we may) to primitive classes $u \pmod{Q}$ such that $(\frac{u/A}{p}) = 1$. Further, let θ_χ and $\theta_{\chi'}$ be phases associated to $K_\chi(u;\tilde{q})$ and $K_{\chi'}(u;\tilde{q})$ as in (2.20), respectively. Then, by Lemma 11, we have for $(\frac{u/A}{p}) = 1$

$$\begin{aligned} &K_\chi^{\pm}(u;\tilde{q})\overline{K_{\chi'}^{\pm}(u;\tilde{q})} \\ &\qquad = \Delta_{\theta_\chi}(s_\pm(u/A;\tilde{q});\tilde{q})\overline{\Delta_{\theta_{\chi'}}(s_\pm(u/A';\tilde{q});\tilde{q})}e\left(\frac{Ag_\pm(u/A;\tilde{q}) - A'g_\pm(u/A';\tilde{q})}{(\tilde{q},\delta_q(\chi,\chi'))Q}\right)\end{aligned}$$

where, for fixed χ and χ', the product of the Δ factors depends only on the class of $u \pmod{p}$, as is readily verified using $\theta'_\chi(x_{0,A}) = 0$ and $\theta''_\chi(x_{0,A}) \equiv 2ux_{0,A}^{-3} \pmod{p}$ with $x_{0,A} = s_\pm(u/A;\tilde{q})$ from the proof of Lemma 11. This proof also shows that $g_\pm(u;\tilde{q})$, a function analytic on its domain $\mathbb{Z}_p^{\times 2}$, is invariant modulo $\tilde{q}$ under translation by $\tilde{q}\mathbb{Z}_p$. Moreover, a moment's reflection on the definition (2.27) combined with (2.10) and (2.9) shows that for $u \in \mathbb{Z}_p^{\times 2}$ both $s_\pm(u;\tilde{q})$ and $g_\pm(u;\tilde{q})$ may be expanded into a convergent power series in $\sqrt{u}$ with coefficients in $\mathbb{Z}_p$. From this and (2.11) it follows that, for $u \in A\mathbb{Z}_p^{\times 2}$,

$$Ag_\pm(u/A;\tilde{q}) - A'g_\pm(u/A';\tilde{q}) = (\tilde{q},\delta_q(\chi,\chi'))\cdot\sigma(u)$$

is divisible by $(\tilde{q},\delta_q(\chi,\chi'))$ and invariant modulo $\tilde{q}$ under translation by $Q\mathbb{Z}_p$ (for $Q \geqslant p$). This establishes the periodicity claim and (c) follows from the definition of Δ_θ, noting that $A \equiv A' \pmod{p}$ in this case.

As for the estimate (2.30), the case $Q = p$ is trivial since we are taking $p \ll 1$, so we assume that $Q \geqslant p^2$. We will be interested in applying Lemma 7 and Remark 2 with $p^n = Q$, phase σ, and $p^\ell = \mathrm{rt}^*(Q)$. Since $s_\pm(u/A;\tilde{q})$ solves $\theta'(x_0) = 0$ in the proof of

Lemma 11, we observe

$$\frac{\mathrm{d}}{\mathrm{d}u} Ag_{\pm}(u/A;\tilde{q}) = \left\{ \frac{\partial}{\partial u} + \frac{\partial}{\partial s_{\pm}} \cdot \frac{\mathrm{d}s_{\pm}(u/A;\tilde{q})}{\mathrm{d}u} \right\} Ag_{\pm}(u/A;\tilde{q}) = \frac{1}{s_{\pm}(u/A;\tilde{q})},$$

so that, by rationalizing denominators,

$$\sigma'(u) = \frac{\pm 1/2}{\delta_q(\chi,\chi')} \left(\sqrt{(q/\tilde{q})^2 + 4A/u} - \sqrt{(q/\tilde{q})^2 + 4A'/u} \right) - v.$$

Expanding the difference of roots above according to (2.10) and (2.9) yields the quantity

$$\sum_{k\geqslant 0} \binom{1/2}{k} (q/\tilde{q})^{2k} \big((4A/u)^{1/2} (u/4A)^k - (4A'/u)^{1/2} (u/4A')^k \big),$$

which, along with (2.11), shows that the sum in (2.30) with phase σ satisfies the appropriate conditions in Lemma 7 (keeping in mind Remark 2). From here and (2.11), we see that the sum in (2.30) vanishes unless $|v|_p = 1$, as solutions to the stationary phase congruence $\sigma'(u) \equiv 0 \pmod{\mathrm{rt}_*(Q)}$ could not exist otherwise. Since $\alpha^{1/2}\beta^{1/2}/(\alpha\beta)^{1/2} \in \{\pm 1\}$ for every $\alpha, \beta \in \mathbb{Z}_p^{\times 2}$, any solutions to the stationary phase congruence must satisfy one of the four congruences

$$\delta_q(\chi,\chi')^{-1} \sum_{k\geqslant 0} \binom{1/2}{k} (q/\tilde{q})^{2k} \big(\epsilon_1 (u/4A)^k - \epsilon_2 (A'/A)^{1/2} (u/4A')^k \big) \equiv v\,(u/A)^{1/2} \pmod{\mathrm{rt}_*(Q)}$$

with $\epsilon_i \in \{\pm 1\}$, where in fact $\epsilon_1 = \epsilon_2$ unless, possibly, $\delta_q(\chi,\chi') = 1$. Each of these four congruences is polynomial in $(u/A)^{1/2}$ modulo $\mathrm{rt}_*(Q)$, satisfies the hypotheses of Hensel's lemma, and reduces to a non-degenerate linear congruence in $(u/A)^{1/2}$ modulo p. Thus there are $O(1)$ solutions modulo $\mathrm{rt}_*(Q)$ to the stationary phase congruence. The proposition then follows. ■

2.5 Short second moment estimates

Proposition 10 provides an individual bound for the short second moment $S_2(\chi)$ in terms of averages of the arithmetic function $K_\chi(m;\tilde{q})$. In this section, we leverage this result and the estimates on exponential sums from Section 2.4 to prove in Proposition 13 our penultimate result, an aggregate bound on the short second moment over a *collection* of characters modulo q_2.

Proposition 13. *Let q_1, q_2, and q be subject to the conditions in (2.12). Let χ be any primitive character modulo q, and let Ψ be any set of Dirichlet characters modulo q_2. Then*

$$\sum_{\psi_2\in\Psi} S_2(\chi\psi_2) \ll q^{\varepsilon}\left((q_1+q_1^{1/4}q_2^{1/4})|\Psi| + q^{1/2}|\Psi|^{1/2}\right).$$

Proof. We will use Proposition 10; in its notation, we may assume that $Q_1/p^\eta \gg q^{1/2-\varepsilon} \geqslant p^2$, as Proposition 13 is trivially true for $q \ll p^4$. Decomposing the quantity $K_\chi(m;Q_1/p^\eta)$ as $K_\chi^+ + K_\chi^-$ as in Lemma 11, Proposition 10 gives us

$$\sum_{\psi_2\in\Psi} S_2(\chi\psi_2) \ll q^{\varepsilon}\left(q_1|\Psi| + q_1^{3/2}q^{-1/2}\big(T^+(\Psi)+T^-(\Psi)\big)\right), \tag{2.31}$$

where

$$T^{\pm}(\Psi) := \operatorname{Re}\sum_{\psi_2\in\Psi}\ \sum_{p^\eta \ll q^{1/2+\varepsilon}q_1^{-1}} p^{\eta/2} \sideset{}{^*}\sum_{|m|\ll q^{1+\varepsilon}(q_1^2p^{2\eta})^{-1}} K_{\chi\psi_2}^{\pm}(m;Q_1/p^\eta)A(m;p^\eta)$$

and $A(m;p^\eta) \ll m^{\varepsilon}$. Application of the Cauchy–Schwarz inequality produces the bound

$$T^{\pm}(\Psi) \leqslant \frac{q^{1/2+\varepsilon}}{q_1} \times \left(\sum_{\psi_2, \psi_2' \in \Psi} \sum_{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1}} p^{-\eta} \sideset{}{^*}\sum_{|m| \ll q^{1+\varepsilon}(q_1^2 p^{2\eta})^{-1}} K^{\pm}_{\chi\psi_2}(m; Q_1/p^\eta) \overline{K^{\pm}_{\chi\psi_2'}(m; Q_1/p^\eta)} \right)^{1/2}. \tag{2.32}$$

By Proposition 12, the inner summand above is periodic modulo

$$Q_{\eta,\psi_2,\psi_2'} = \frac{Q_1/p^\eta}{(Q_1/p^\eta, (q/q_2)\delta_{q_2}(\psi_2, \psi_2'))}$$

whenever $Q_{\eta,\psi_2,\psi_2'} \geqslant p^2$. Moreover, in this case, any complete segments of the inner sum in (2.32) vanish, and an application of Lemma 9 and Proposition 12 to any remaining incomplete segment gives that

$$\sideset{}{^*}\sum_{|m| \ll q^{1+\varepsilon}(q_1^2 p^{2\eta})^{-1}} K^{\pm}_{\chi\psi_2}(m; Q_1/p^\eta) \overline{K^{\pm}_{\chi\psi_2'}(m; Q_1/p^\eta)} \ll q^\varepsilon Q^{1/2}_{\eta,\psi_2,\psi_2'}.$$

When $Q_{\eta,\psi_2,\psi_2'} \leqslant p$, we bound the inner sum of (2.32) trivially. The second to innermost sum of (2.32) is therefore asymptotically bounded above by

$$\sum_{\substack{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1} \\ Q_{\eta,\psi_2,\psi_2'} \geqslant p^2}} \frac{q^\varepsilon Q^{1/2}_{\eta,\psi_2,\psi_2'}}{p^\eta} + \sum_{\substack{p^\eta \ll q^{1/2+\varepsilon} q_1^{-1} \\ Q_{\eta,\psi_2,\psi_2'} \leqslant p}} \frac{q^{1+\varepsilon}}{q_1^2 p^{3\eta}}$$
$$\ll q^\varepsilon \left(\frac{q_2}{q_1 \delta_{q_2}(\psi_2, \psi_2')} \right)^{1/2} + q^{1+\varepsilon} \min\left(\frac{q_1 \delta_{q_2}(\psi_2, \psi_2')^3}{q_2^3}, \frac{1}{q_1^2} \right),$$

where in fact the second term only enters if $\delta_{q_2}(\psi_2, \psi_2') \gg q_2 q^{-(1/2+\varepsilon)}$. Inserting the above into (2.32), we obtain

$$T^{\pm}(\Psi) \ll \frac{q^{1/2+\varepsilon}}{q_1} \left(\frac{q_2^{1/4}}{q_1^{1/4}} |\Psi| + \frac{q^{1/2}}{q_1^{1/2}} |\Psi|^{1/2} \right).$$

Inserting this bound into (2.31), we complete the proof of Proposition 13. ■

2.6 Proof of Theorem 4

Let ε be an arbitrary, small positive real number. We will begin by defining the sets

$$R_2(V;\chi) := \{\chi_2 \pmod{q_2} : \chi\chi_2 \in R(V;q)\}$$
$$\text{and} \quad \Psi(V;\chi) := \{\psi_2 \pmod{q_2} : \psi_1\psi_2 \in R_2(V;\chi) \text{ for some } \psi_1 \pmod{q_1}\}.$$

Clearly we may assume that q is a sufficiently high power of p. Asymptotics of the fourth moment of Dirichlet L-functions due to Heath-Brown [15] and later improved by Soundararajan [32] imply that

$$\sum_{\chi \pmod{q}} \left|L\left(\tfrac{1}{2},\chi\right)\right|^4 \ll q^{1+\varepsilon},$$

from which it follows that $|R(V;q)| \ll q^{1+\varepsilon}V^{-4}$. This suffices to handle the case when $V \leqslant q^{1/8+2\varepsilon}$. On the other hand, by the Weyl bound for Dirichlet L-functions to prime power moduli due to Postnikov [30] (see also [24]), $|R_2(V;\chi)| = 0$ for $V \geqslant q^{1/6+\varepsilon}$.

Consider now the values of $q^{1/8+2\varepsilon} \leqslant V \leqslant q^{1/6+\varepsilon}$. We combine the observations

$$|R_2(V;\chi)| \leqslant \frac{1}{V^2\varphi(q_1)} \sum_{\psi_2 \in \Psi(V;\chi)} S_2(\chi\psi_2) \quad \text{and} \quad |\Psi(V;\chi)| \leqslant \varphi(q_1)R_2(V;\chi),$$

with Proposition 13 and the choices $V^2q^{-3\varepsilon} \leqslant q_1 \leqslant V^2q^{-2\varepsilon}$ and $q_2 = q_1^3$. With these choices, we obtain

$$\begin{aligned}|R_2(V;\chi)| &\ll \frac{q^{\varepsilon}}{V^2\varphi(q_1)}\Big(q_1\varphi(q_1)|R_2(V;\chi)| + q^{1/2}\varphi(q_1)^{1/2}|R_2(V;\chi)|^{1/2}\Big)\\ &\ll q^{-\varepsilon}|R_2(V;\chi)| + \frac{q^{1/2-\varepsilon}}{q_1^{3/2}}|R_2(V;\chi)|^{1/2},\end{aligned}$$

from which in turn it follows that

$$|R_2(V;\chi)| \ll \frac{q^{1+\varepsilon}}{q_1 V^4}.$$

As a consequence,

$$|R(V;q)| = \frac{1}{\varphi(q_2)} \sideset{}{^*}\sum_{\chi \pmod q} |R_2(V;\chi)| \ll q^{2+\varepsilon}V^{-12},$$

which completes the proof of Theorem 4, and hence of Theorem 3.

Chapter 3

Extreme values of Hecke *L*-functions to angular characters

3.1 Introduction

The resonance method developed by Soundararajan [31] is a flexible tool for detecting both unusually large and unusually small values in families of L-functions. In its original application to the zeta function, this method exploits the oscillatory behavior of $\zeta(\frac{1}{2}+it)$ by generating a Dirichlet polynomial

$$R(t) = \sum_{n \leqslant N} r(n) n^{-it}$$

that "resonates" with the "louder frequencies" of $\zeta(\frac{1}{2}+it)$ on $T \leqslant t \leqslant 2T$ for some later optimized choice of real-valued resonator coefficients $r(n)$. The argument proceeds by unpacking, reassembling, and analyzing what is essentially the first moment of

$$\zeta(\tfrac{1}{2}+it)|R(t)|^2 \approx \sum_{k \leqslant T} k^{-1/2} \sum_{n,m \leqslant N} r(n)r(m) \left(\frac{n}{mk}\right)^{it} \tag{3.1}$$

on this interval, where one must balance the length N of the resonator — a parameter commensurate with the strength of the resulting bound (3.2) — against the contribution of off-diagonal terms ($n \neq mk$) which are difficult to manage. Comparing the mean value of (3.1) to the mean of $|R(t)|^2$ and optimizing the resonator coefficients yields

$$\max_{T \leqslant t \leqslant 2T} \log \left|\zeta(\tfrac{1}{2} + it)\right| \geqslant (1 + o(1))\sqrt{\frac{\log T}{\log\log T}}. \tag{3.2}$$

The bound (3.2) substantially improved upon the previously best known result [1, 2] and is much larger than the bulk of the normal distribution with mean value 0 and variance $(\log\log T)/2$; see [34, §11.13]. This result is fairly close to predictions from random matrix theory [12] that suggest the right hand side of (3.2) may be within a factor of $(\log\log T)/\sqrt{2}$ to the true maximum.

Bondarenko and Seip [6, 7] have since managed to substantially extend the length of the resonator for the zeta function, thus improving (3.2), but at the cost of localization; the full strength of their result requires the entire segment $0 \leqslant t \leqslant T$ and can not recover information along dyadic ranges.

For a number field K, the pure adelic analogue for the set of values of $\zeta(\frac{1}{2}+it)$ is the set of central values of Hecke L-functions to Hecke characters that ramify only at a single place; see the upcoming Section 3.2.1 for a brief recollection of Hecke characters. Over $\mathbb{Q}$ this presents the additional problem of detecting large values of $L(\frac{1}{2}, \chi)$ for Dirichlet characters χ over prime power moduli. In a related but not identical direction, Hough [18] used the resonance method to establish a result of quality similar to (3.2) within prescribed angular sectors for Dirichlet characters varying on prime moduli, while de la Bretèche and Tenenbaum [9] have since improved the quality of this bound for such even characters (without the angular sector restriction) to the level of Bondarenko and Seip's result on the zeta function.

The complexity of Hecke characters expands as one moves to proper extensions of

$\mathbb{Q}$. For a fundamental example beyond both the archimedean and Dirichlet character types, when K is an imaginary quadratic number field, *angular characters* emerge from sole ramification at the unique complex place. On principal ideals of the ring of integers $\mathcal{O}_K$, these characters behave precisely as $\xi_\ell((\beta)) = e(\ell \arg \beta/2\pi)$ for integers ℓ divisible by $|\mathcal{O}_K^\times|$, where β is an element of a fixed complex embedding of $\mathcal{O}_K$. The integer ℓ is said to be the *frequency* of the character and there are multiple angular characters of any given permissible frequency when $\mathcal{O}_K$ is not a PID; this multiplicity is exactly equal to the size of the class group. The purpose of this chapter is to apply for the first time the resonance method to Hecke L-functions to angular characters to address for the first time the problem of detecting their large values, complementing the progress made on the archimedean and Dirichlet character types over $\mathbb{Q}$. We prove the following.

Theorem 14. *Let K be an imaginary quadratic number field and $\mathfrak{F}_\ell$ be its associated set of angular characters with frequency ℓ. Then*

$$\max_{\substack{X \leqslant \ell \leqslant 2X \\ \xi_\ell \in \mathfrak{F}_\ell}} \log \left| L(\tfrac{1}{2}, \xi_\ell) \right| \geqslant \left(\sqrt{2} + o_K(1) \right) \sqrt{\frac{\log X}{\log \log X}}.$$

The most substantial contextual difference in our application of the resonance method arises in Sections 3.4.1 and 3.4.2 when one seeks to isolate and then analyze the diagonal terms (the analogues of $n = mk$ in (3.1)) of the analogous quantity which appears, very roughly, in this chapter as

$$\sum_{\substack{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N \\ \mathfrak{N}\mathfrak{k} \ll X \\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \text{ principal}}} \frac{r(\mathfrak{a}) r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \sum_{k \asymp X} e(k \arg \gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}/2\pi),$$

where $\gamma_\mathfrak{n}$ is the generator of the principal ideal $\mathfrak{n}$ nearest to the positive real line with respect to rotation about the origin. A similar, but more relaxed, analysis precedes this in Section 3.4.1 when examining the mean of our resonator polynomial. At this point, one must consider the geometry of the embedding of K into $\mathbb{C}$. Indeed, the diagonal condition becomes characterized by whether the product $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}$ is principally generated by a rational element of $\mathcal{O}_K$. This is the only way the innermost sum above can avoid meaningful cancellation when N is bounded appropriately in terms of X.

We therefore bound length of the resonator in terms of X (as is typical) so that our analysis of the quantity illustrated above depends on an area $\mathbb{C}$ where the embedding of $\mathcal{O}_K$ is "sufficiently discretized" such that all $\gamma_{\mathfrak{n}}$ *near the real line* are *exactly rational.* As a consequence, the off-diagonal terms where the $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}$ are not rationally generated will not meaningfully contribute.

3.1.1 Notation

In this chapter, unless stated otherwise, we let ε represent a small, fixed positive number that may differ from line to line and let K be an imaginary quadratic number field. As usual, $f \ll g$ and $f = O(g)$ mean that $|f| \leqslant Cg$ for some constant $C > 0$ which may differ from line to line and depends on no parameters except for possibly ε, the number field K, or any other parameters explicitly stated. Additionally, $f = o(g)$ indicates that $f/g \to 0$ in a limit that will be clear from context. As is customary in analytic number theory, we write $e(z)$ to mean $\exp(2\pi i z)$.

3.1.2 Acknowledgments

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3.2 Preliminaries

3.2.1 Hecke characters

We now recall the basic properties of Hecke characters; the reader may reference [26, Section 7.6] for additional details. Let K be a general number field and $\mathcal{O}_K$ its ring of integers. We will denote the group of non-zero fractional ideals as $\mathfrak{I}$; it contains the subgroup $\mathfrak{P}$ of non-zero fractional principal ideals. The quotient group $\mathfrak{I}/\mathfrak{P}$ is the class group, which has finite size h_K. For a non-zero integral ideal $\mathfrak{m} \subseteq \mathcal{O}_K$, we further denote $\mathfrak{I}_\mathfrak{m}$ as the set of all fractional ideals coprime to $\mathfrak{m}$.

Suppose K has r_1 real embeddings and r_2 complex embeddings up to conjugation. A Hecke character modulo $\mathfrak{m}$ is a continuous homomorphism $\xi : \mathfrak{I}_\mathfrak{m} \to S^1$ (when interpreted as a morphism on the idele class group of K) for which there are two characters

$$\chi_f : (\mathcal{O}_K/\mathfrak{m})^\times \to S^1 \quad \text{and} \quad \chi_\infty : (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \to S^1$$

such that $\xi((\beta)) = \chi_f(\beta)\chi_\infty(\beta)$ for every $\beta \in \mathcal{O}_K$ where $((\beta), \mathfrak{m}) = 1$. A pair of characters (χ_f, χ_∞) is associated to a Hecke character if and only if $\chi_f(\epsilon)\chi_\infty(\epsilon) = 1$ for every $\epsilon \in \mathcal{O}_K^\times$. The Hecke characters associated to such a pair differ by multiplication of a character of the class group. Let $\mathcal{H}$ denote the set of all Hecke characters of K.

We now focus our attention to when K is an imaginary quadratic number field and fix an embedding into $\mathbb{C}$ so that no ambiguity may occur from conjugation. Hence we may use $\beta \in \mathcal{O}_K$ to mean both the ring element and its image under the embedding. Let $\omega_K = |\mathcal{O}_K^\times|$ and for each integer ℓ divisible by ω_K define the set

$$\mathfrak{F}_\ell := \left\{ \xi_\ell \in \mathcal{H} : \xi_\ell\big((\beta)\big) = e\left(\frac{\ell \arg \beta}{2\pi}\right) \text{ for all non-zero } \beta \in \mathcal{O}_K \right\} \tag{3.3}$$

of angular characters with frequency ℓ as in Theorem 14, where $|\mathfrak{F}_\ell| = h_K$. In the event that $\omega_K \nmid \ell$, we simply take $\mathfrak{F}_\ell$ to be empty. It will be useful to have a standard choice of generator for principal ideals that we encounter. For a principal ideal $\mathfrak{n}$, let $\gamma_\mathfrak{n}$ denote the generator of $\mathfrak{n}$ whose principal argument resides in $[-\pi\omega_K^{-1}, \pi\omega_K^{-1})$.

3.2.2 Functional equation

For any number field K, we may associate to any Hecke character its Hecke L-function

$$L(s,\xi) = \sum_{\mathfrak{k}\neq 0} \xi(\mathfrak{k})\mathfrak{N}\mathfrak{k}^{-s}$$

which converges absolutely for $\operatorname{Re} s > 1$ and has meromorphic continuation to $\mathbb{C}$. Each such L-function possesses a functional equation. In particular, for the L-functions of interest in this chapter, [22, Theorem 3.8] boils down to the following.

Theorem 15. *Let D denote the discriminant of the imaginary quadratic number field K and ξ_ℓ be an angular character of frequency ℓ. The function*

$$\Lambda(s,\xi_\ell) = |D|^{s/2}(2\pi)^{-s}\Gamma(s+|\ell|/2)L(s,\xi_\ell) \tag{3.4}$$

is entire for ξ_ℓ non-trivial and satisfies the functional equation

$$\Lambda(s,\xi_\ell) = \Lambda(1-s,\overline{\xi_\ell}). \tag{3.5}$$

When examining L-functions along the critical line, it is often beneficial to express the L-function as a smoothly weighted Dirichlet series in what is known as the approximate functional equation. Roughly speaking, and in general, this weight has approximate value one until its transition point, where it then drops rapidly to zero.

One may think of the approximate functional equation as essentially a finite sum. For Hecke L-functions, the location of this point and the decay rate are both subject to the complexity of the associated character.

Theorem 16. *For K and ξ_ℓ as in Theorem 15,*

$$L\left(\tfrac{1}{2}, \xi_\ell\right) = \sum_{\mathfrak{k} \neq (0)} \frac{\xi_\ell(\mathfrak{k}) + \overline{\xi_\ell}(\mathfrak{k})}{\sqrt{\mathfrak{N}\mathfrak{k}}} W_K\left(\mathfrak{N}\mathfrak{k}, |\ell|\right)$$

where

$$W_K\left(\frac{|D|^{1/2} y}{2\pi}, 2x\right) = V(y,x) = \frac{1}{2\pi i}\int_{(1)} \rho_s(x) y^{-s} \frac{\mathrm{d}s}{s} \tag{3.6}$$

and

$$\rho_s(x) = \frac{\Gamma(x+s+\frac{1}{2})}{\Gamma(x+\frac{1}{2})}. \tag{3.7}$$

Proof. Let $\Lambda(s, \xi_\ell)$ be as in (3.4) in Theorem 15. By a contour shift permitted by the exponential decay of the gamma function along vertical strips, we find

$$\frac{1}{2\pi i}\int_{(1)} \Lambda(s+\tfrac{1}{2}, \xi_\ell)\frac{\mathrm{d}s}{s} = \Lambda(\tfrac{1}{2}, \xi_\ell) + \frac{1}{2\pi i}\int_{(-1)} \Lambda(s+\tfrac{1}{2}, \xi_\ell)\frac{\mathrm{d}s}{s}.$$

On the right side above, applying the substitution $s \mapsto -s$ and the functional equation (3.5) from Theorem 15 before substituting (3.4) and integrating term-wise on each side, we obtain

$$L(\tfrac{1}{2}, \xi_\ell) = \frac{1}{2\pi i}\sum_{\mathfrak{k}\neq 0} \frac{\xi_\ell(\mathfrak{k}) + \overline{\xi_\ell}(\mathfrak{k})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \int_{(1)} \rho_s(|\ell|/2)\left(2\pi |D|^{-1/2}\mathfrak{N}\mathfrak{k}\right)^{-s}\frac{\mathrm{d}s}{s},$$

as desired. ∎

The following lemmata characterize the behavior of $\rho_s(x)$ and its derivatives so that we may determine where $V(y,x)$, as defined in (3.6), is "essentially supported" and how quickly it decays. The first of these demonstrates that, in suitable bounded vertical strips, $\rho_{\sigma+it}(x)$ has an approximately linear phase in t.

Lemma 17. *Let $x \geqslant 1$ and $s = \sigma + it$ where $|\sigma| \leqslant x/2$. For $\rho_s(x)$ as in (3.7), we have*

$$\rho_s(x) = (x/e)^{\sigma} (1 + \sigma/x)^{x+\sigma} \exp\left(F_{x+\sigma}(t) + i\psi(t)\right) \left(1 + O\left(x^{-1}\right)\right)$$

with

$$F_{\kappa}(t) = -\int_0^t \arctan(u/\kappa) \;\; \mathrm{d}u$$

and real-valued phase

$$\psi(t) = t \log(x + \sigma) + O\left(|t|^3 x^{-2}\right).$$

Proof. We begin by recalling Stirling's formula [33, Theorem 2.3], which states that

$$\Gamma(s) = \exp(s \log s - s) (2\pi/s)^{1/2} \left(1 + O\left(|s|^{-1}\right)\right)$$

as $|s| \to \infty$ within any sector of $\mathbb{C}$ which omits the negative real axis. Applying Stirling's formula to (3.7) and noting (3.9) shows

$$\begin{aligned} \rho_s(x) &= \frac{(x + s + \frac{1}{2})^{x+s}}{(x + \frac{1}{2})^x} e^{-s} \left(1 + O(x^{-1})\right) \\ &= (x/e)^s (1 + s/x)^{x+s} \left(1 + O\left(x^{-1}\right)\right) \end{aligned} \tag{3.8}$$

where the expansion of $\log(1+z)$ revealed

$$\begin{aligned} (x + s + \tfrac{1}{2})^{x+s} &= (x+s)^{x+s} \left(1 + \tfrac{1}{2}(x+s)^{-1}\right)^{x+s} \\ &= \sqrt{e}(x+s)^{x+s} \left(1 + O\left(x^{-1}\right)\right), \end{aligned} \tag{3.9}$$

and one notes that the error term above is uniformly bounded for $|\operatorname{Re} s| \leqslant x/2$. Toward explicating the magnitude and phase of (3.8), we may factor $(1+s/x)^{x+s}$ as the product of

$$\begin{aligned}
(1+s/x)^{x+\sigma} &= (1+\sigma/x)^{x+\sigma}\left(1+\tfrac{it}{x+\sigma}\right)^{x+\sigma} \\
&= (1+\sigma/x)^{x+\sigma}\exp\left(\tfrac{1}{2}(x+\sigma)\log\left(1+\left(\tfrac{t}{x+\sigma}\right)^2\right)+i\phi_1(t)\right),
\end{aligned}$$

where

$$\begin{aligned}
\phi_1(t) &= (x+\sigma)\arctan\left(\tfrac{t}{x+\sigma}\right) \\
&= t + O\left(|t|^3x^{-2}\right), \qquad (3.10)
\end{aligned}$$

and

$$\begin{aligned}
(1+s/x)^{it} &= (1+\sigma/x)^{it}\left(1+\tfrac{it}{x+\sigma}\right)^{it} \\
&= \exp\left(-t\arctan\left(\tfrac{t}{x+\sigma}\right)+i\phi_2(t)\right),
\end{aligned}$$

where

$$\begin{aligned}
\phi_2(t) &= t\log(1+\sigma/x)+\tfrac{t}{2}\log\left(1+\left(\tfrac{t}{x+\sigma}\right)^2\right) \\
&= t\log(1+\sigma/x)+O(|t|^3x^{-2}), \qquad (3.11)
\end{aligned}$$

noting that both ϕ_1 and ϕ_2 are real-valued. Inserting each of (3.10) and (3.11) back into (3.8) shows

$$\begin{aligned}
\rho_s(x) = &\left(1+O\left(x^{-1}\right)\right)(x/e)^\sigma(1+\sigma/x)^{x+\sigma} \\
&\times\exp\left(\tfrac{1}{2}(x+\sigma)\log\left(1+\left(\tfrac{t}{x+\sigma}\right)^2\right)-t\arctan\left(\tfrac{t}{x+\sigma}\right)\right) \\
&\times\exp\left(i\phi_1(t)+i\phi_2(t)+it(-1+\log x)\right)
\end{aligned}$$

thus yielding

$$\rho_s(x) = (x/e)^\sigma \left(1 + \sigma/x\right)^{x+\sigma} \exp\left(\tilde{F}_{x+\sigma}(t) + i\psi(t)\right)\left(1 + O\left(x^{-1}\right)\right)$$

for $\psi(t)$ as defined in the statement of Lemma 17 and

$$\tilde{F}_\kappa(t) := -t \arctan\left(\tfrac{t}{\kappa}\right) + \tfrac{1}{2}\kappa \log\left(1 + t^2/\kappa^2\right). \tag{3.12}$$

Note that $\tilde{F}_\kappa(0) = 0$ and $\tilde{F}'_\kappa(t) = -\arctan(t/\kappa)$, so that an application of the fundamental theorem of calculus shows $F_\kappa(t) = \tilde{F}_\kappa(t)$, finishing the proof. ■

Lemma 18. *For $n \geqslant 1$ and $\operatorname{Re} z \geqslant 1$, we have*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} \log \Gamma(z) = \delta_n^1 \log z + O_n\left(\operatorname{Re}(z)^{-\delta_n^1 - n + 1}\right),$$

where δ_n^m is the Kronecker delta function.

Proof. We begin with the well-known identity [35, §12.31]

$$\frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z) = \log z - \frac{1}{2z} - \int_0^\infty \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1}\right) e^{-uz}\, \mathrm{d}u.$$

The case $n = 1$ follows from the exponential decay of the integrand above, which begins at $\operatorname{Re}(z)^{-1}$. For $n \geqslant 2$, see that

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} \log \Gamma(z) = (-1)^n (n-1)! \left(\frac{1}{nz^{n-1}} + \frac{1}{2z^n} + O\left(\operatorname{Re}(z)^{-n}\right)\right),$$

which finishes the proof. ■

Lemma 19. *Let $x \geqslant 1$ and $s = \sigma + it$ with $|\sigma| \leqslant x/2$. Then, for $n \geqslant 0$,*

$$\rho_s^{(n)}(x) = \rho_s(x) \log^n(1 + s/x) \left(1 + O_n(x^{-1})\right). \tag{3.13}$$

Proof. The case $n = 0$ is clear and the case $n = 1$ follows from the logarithmic derivative

$$\frac{\mathrm{d}}{\mathrm{d}x}\rho_s(x) = \rho_s(x)\frac{\mathrm{d}}{\mathrm{d}x}\log \rho_s(x)$$

and Lemma 18. Proceeding inductively for $n \geqslant 2$, note that

$$\begin{aligned}\rho_s^{(n)}(x) &= \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}\left(\rho_s(x)\frac{\mathrm{d}}{\mathrm{d}x}\log \rho_s(x)\right)\\ &= \sum_{m=0}^{n-1}\binom{n-1}{m}\rho_s^{(m)}(x)\frac{\mathrm{d}^{n-m}}{\mathrm{d}x^{n-m}}\log \rho_s(x).\end{aligned}$$

By our inductive hypothesis (3.13) for $m < n$ and Lemma 18, we note the main contribution from $m = n - 1$ and conclude the proof. ∎

Having established Lemmata 17 and 19, we are now equipped to understand the general behavior of $V(y, x)$ and its derivatives in the approximate functional equation of $L(\frac{1}{2}, \xi_\ell)$ as introduced in Theorem 16.

Lemma 20. *Let $y \geqslant 1$, $x \geqslant 3$, $n \geqslant 0$, and $V(y, x)$ be as in (3.6). Then*

$$\frac{\partial^n}{\partial x^n}V(y, x) = \delta_n^0 + O_n\Big(x^{-n/2}(y/x)^{\sqrt{x}}\Big) \quad \text{and} \quad \frac{\partial^n}{\partial x^n}V(y, x) \ll_n x^{-n/2}(x/y)^{\sqrt{x}}, \tag{3.14}$$

where δ_n^m is the Kronecker delta function.

Proof. Let $|\sigma| = x^{1/2}$. Shifting the line of integration in (3.6) to (σ) (perhaps picking up contribution from the pole at $s = 0$), differentiating n times with respect to x, and substituting $s = \sigma + it$ yields

$$\frac{\partial^n}{\partial x^n} V(y,x) = \delta_n^0 \Delta_\sigma + \frac{y^{-\sigma}}{2\pi\sigma} \int_{-\infty}^{\infty} \rho_{\sigma+it}^{(n)}(x) y^{-it} G(\sigma,t) \, \mathrm{d}t, \tag{3.15}$$

where $\Delta_\sigma \in \{0,1\}$ indicates whether σ is negative and

$$G(\sigma,t) = \frac{\sigma}{\sigma + it}.$$

We will start by bounding the tails of the integral in (3.15), corresponding to the values $|t| \geqslant x^{1/2+\varepsilon}$. Exercising Lemmata 17 and 19 with $(1+\sigma/x)^{x+\sigma} = \exp(\sigma + O(1))$ from our choice $|\sigma| = x^{1/2}$ shows these tails are bounded by an absolute constant multiple of

$$\begin{aligned} \int_{|t| \geqslant x^{1/2+\varepsilon}} \left| \rho_{\sigma+it}^{(n)}(x) \right| \, \mathrm{d}t &\ll_n \int_{|t| \geqslant x^{1/2+\varepsilon}} \left| \rho_{\sigma+it}(x) \log^n \left(1 + \tfrac{\sigma+it}{x}\right) \right| \, \mathrm{d}t \\ &\ll x^{\sigma-n} \int_{x^{1/2+\varepsilon}}^{\infty} \exp\left(n \log t - \int_0^t \arctan\left(\tfrac{u}{x+\sigma}\right) \, \mathrm{d}u \right) \mathrm{d}t. \end{aligned}$$

Noting that $\arctan(v) \geqslant v\pi/4$ for $0 \leqslant v \leqslant 1$ and $\arctan(v) \geqslant \pi/4$ for $v \geqslant 1$ allows us to bound the right hand side above by

$$\begin{aligned} &\leqslant x^{\sigma-n} \int_{x^{1/2+\varepsilon}}^{x+\sigma} \exp\left(\tfrac{-t^2\pi}{8x} (1 + o_n(1)) \right) \mathrm{d}t \\ &\quad + x^{\sigma-n} \exp\left(\tfrac{\pi}{8}(x+\sigma) \right) \int_{x+\sigma}^{\infty} \exp\left(-\tfrac{\pi}{4} t (1 + o_n(1)) \right) \mathrm{d}t \end{aligned}$$

where utilizing the bound $\int_c^\infty e^{-u^2} \, \mathrm{d}u \ll e^{-c^2}/c$ then shows that the main contribution above comes from the left summand whose integral is bounded by

$$\ll_n \int_{x^{1/2+\varepsilon}}^{\infty} \exp\left(\frac{-t^2\pi}{16x} \right) \mathrm{d}t \ll x^{1/2-\varepsilon} \exp(-x^\varepsilon \pi/16).$$

We therefore find that the integral in (3.15) becomes

$$O_n\left(x^{\sigma-n+1/2-\varepsilon}\exp(-x^{\varepsilon}\pi/16)\right)+\int_{-x^{1/2+\varepsilon}}^{x^{1/2+\varepsilon}}\rho^{(n)}_{\sigma+it}(x)y^{-it}G(\sigma,t)\,\mathrm{d}t. \tag{3.16}$$

Another application of lemmata 17 and 19, keeping in mind $(1+\sigma/x)^{x+\sigma}$ is equal to $\exp(\sigma+O(1))$ by our choice $|\sigma|=x^{1/2}$, shows that the integral in (3.16) is

$$\ll_n x^{\sigma}\int_{-x^{1/2+\varepsilon}}^{x^{1/2+\varepsilon}}\exp\left(F_{x+\sigma}(t)\right)\left|\log^n\left(1+\tfrac{\sigma+it}{x}\right)\right|\,\mathrm{d}t \tag{3.17}$$

where, by (3.12), we may expand

$$\begin{aligned}F_{x+\sigma}(t)&=-t\arctan\left(\tfrac{t}{x+\sigma}\right)+\tfrac{1}{2}(x+\sigma)\log\left(1+\tfrac{t^2}{(x+\sigma)^2}\right)\\&=-\frac{t^2}{2x}\left(1+O(x^{-1/2}+t^2x^{-2})\right).\end{aligned} \tag{3.18}$$

Inserting (3.18) into (3.17) yields a value

$$\ll x^{\sigma}\int_{-x^{1/2+\varepsilon}}^{x^{1/2+\varepsilon}}\exp(-\tfrac{1}{4}t^2/x)\left|\log^n\left(1+\tfrac{\sigma+it}{x}\right)\right|\,\mathrm{d}t\ll x^{\sigma-n+1/2}\sigma^n$$

which can be used as an upper bound for the integral in (3.16) resulting, ultimately, in a bound for (3.15) of the form

$$\begin{aligned}\frac{\partial^n}{\partial x^n}V(y,x)&=\delta_n^0\Delta_\sigma+O_n\left(\frac{y^{-\sigma}}{2\pi\sigma}\left(x^{\sigma-n+1/2}\sigma^n\right)\right)\\&=\delta_n^0\Delta_\sigma+O_n\left(x^{-n/2}(x/y)^{\sigma}\right)\end{aligned}$$

as desired. ∎

It will be convenient to produce lower bounds on sums of products involving factors of W_K by truncating the sums, where all other factors of the summands are non-negative. To do so, we must prove that $V(y,x)$ is also non-negative for the input

(y, x) under consideration. Note that the bounds in (3.14) are insufficient here. We therefore prove the following lemma which happens to also show that $V(y, x)$ can be expressed as the ratio of an incomplete Gamma function to a complete Gamma function.

Lemma 21. *If $y > 0$ and $x > 0$, then $V(y, x) \in (0, 1)$.*

Proof. We begin by shifting the contour in the definition of $V(y, x)$ to the piece-wise contour $\gamma_\varepsilon^- + \gamma_\varepsilon + \gamma_\varepsilon^+$, where the $\gamma_\varepsilon^\pm$ sit along $\pm i[\varepsilon, \infty)$ and γ_ε forms the right half of the circle with radius ε centered at the origin. Define

$$I_\varepsilon^\pm = \frac{1}{2\pi i} \int_{\gamma_\varepsilon^\pm} \rho_s(x) y^{-s} \frac{\mathrm{d}s}{s}$$

and I_ε similarly for γ_ε, where we denote the limit as $\varepsilon \to 0^+$ of each $I_\varepsilon^\pm$ and I_ε as $I^\pm$ and I, respectively. Note that $I = 1/2$ and

$$\begin{aligned} I^- + I^+ &= \frac{1}{\pi \Gamma(x + \frac{1}{2})} \int_0^\infty \operatorname{Im} \Gamma(x + \tfrac{1}{2} + it) y^{-it} \frac{\mathrm{d}t}{t} \\ &= \frac{1}{2} \Gamma(x + \tfrac{1}{2})^{-1} \int_0^\infty u^{x-1/2} e^{-u} \operatorname{sgn} \log(u/y) \, \mathrm{d}u \end{aligned}$$

which implies that $I^- + I^+ \in \mathbb{R}$ is strictly less than $1/2$ in absolute value. The statement of the lemma follows. ■

3.3 The setup

Fix an imaginary quadratic number field K and let X be a large positive value. We begin by introducing our notion of a resonator

$$R(\xi_\ell) = \sum_{\mathfrak{N}\mathfrak{a} \leqslant N} \xi_\ell(\mathfrak{a}) r(\mathfrak{a}) \tag{3.19}$$

where $\xi_\ell \in \mathfrak{F}_\ell$ is a Hecke character as defined in (3.3) and the $r(\mathfrak{a})$ are non-negative resonator coefficients. The length of our resonator will be set by a constraint involving the frequencies of the ξ_ℓ under consideration, as is expected within the resonator method, where the goal is to build $R(\xi_\ell)$ to resonate with the oscillatory function $L(\frac{1}{2}, \xi_\ell)$ for ξ_ℓ varying among the $\mathfrak{F}_\ell$ with $X \leqslant \ell \leqslant 2X$. One then expects large central values to be pronounced when examining $L(\frac{1}{2}, \xi_\ell)|R(\xi_\ell)|^2$. To make this idea precise, let $0 \leqslant \Phi(x) \leqslant 1$ be a smooth weight function supported in $[1, 2]$ with uniformly bounded derivatives and note that

$$\max_{\substack{X \leqslant \ell \leqslant 2X \\ \xi_\ell \in \mathfrak{F}_\ell}} \left| L(\tfrac{1}{2}, \xi_\ell) \right| \geqslant \frac{\left| \sum_\ell \Phi(\ell/X) \sum_{\xi_\ell \in \mathfrak{F}_\ell} L(\frac{1}{2}, \xi_\ell) |R(\xi_\ell)|^2 \right|}{\sum_\ell \Phi(\ell/X) \sum_{\xi_\ell \in \mathfrak{F}_\ell} |R(\xi_\ell)|^2}. \tag{3.20}$$

We make provisions for the resonator coefficients before an explicit choice is later made. Let $r \geqslant 0$ be multiplicative, $r(\mathfrak{p}) = 0$ on prime ideals $\mathfrak{p}$ that are not generated by a factor of a split rational prime, and for all prime ideals $\mathfrak{p}$, $r(\mathfrak{p}) = r(\overline{\mathfrak{p}})$ and $r(\mathfrak{p}^m) = 0$ for $m \geqslant 2$. These choices do not affect the quality of our result and serve to simplify many calculations that are encountered.

3.3.1 Bounding the denominator

To start, we examine the denominator of the ratio in (3.20), upon which we wish to produce an upper bound. By orthogonality of the class group characters and the definition of $\gamma_\mathfrak{n}$, both discussed in Section 3.2.1, we have

$$\begin{aligned} \sum_{\omega_K | \ell} \Phi(\ell/X) \sum_{\xi_\ell \in \mathfrak{F}_\ell} |R(\xi_\ell)|^2 &= \sum_{\omega_K | \ell} \Phi(\ell/X) \sum_{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N} r(\mathfrak{a}) r(\mathfrak{b}) \sum_{\xi_\ell \in \mathfrak{F}_\ell} \xi_\ell(\mathfrak{a}\overline{\mathfrak{b}}) \\ &= h_K \sum_{\substack{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N \\ \mathfrak{a}\overline{\mathfrak{b}} \in P}} r(\mathfrak{a}) r(\mathfrak{b}) \sum_k \Phi(\omega_K k/X) e(\omega_K k \arg \gamma_{\mathfrak{a}\overline{\mathfrak{b}}}/2\pi), \end{aligned} \tag{3.21}$$

where P denotes the set of principal ideals of $\mathcal{O}_K$. Application of Poisson summation with a change of variable shows that the above equals

$$h_K \omega_K^{-1} X \sum_{\substack{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N \\ \mathfrak{a}\bar{\mathfrak{b}} \in P}} r(\mathfrak{a}) r(\mathfrak{b}) \sum_j \hat{\Phi}\left(X(\omega_K^{-1} j - \arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}/2\pi)\right). \tag{3.22}$$

The rapid decay of $\hat{\Phi}$ allows us to profitably split this sum into two cases according to whether $|\arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}|$ passes the threshold $X^{-1+\varepsilon}$. Note the series expansion of $\sin \arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}$ in the case $|\arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}| < X^{-1+\varepsilon}$ shows that

$$|\mathrm{Im}\, \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}| \leqslant N\left(X^{-1+\varepsilon} + O\left(X^{-2}\right)\right). \tag{3.23}$$

It is here that a constraint on the length of the resonator appears should we wish to avoid off-diagonal contribution in (3.22); later, though, we further restrict the length. Let P_0 denote the set of principal ideals generated by rational numbers and P' be the set of ideals which contain no P_0-factors. Enforcing $N \leqslant X^{1-\varepsilon}$ guarantees $|\mathrm{Im}\, \gamma_{\mathfrak{a}\bar{\mathfrak{b}}}| < 1$ by (3.23), so that $\mathfrak{a}\bar{\mathfrak{b}} \in P_0$ for X sufficiently large relative to ε.

By considering the prime factorization of $\mathfrak{a}$ and $\mathfrak{b}$, we may write

$$\mathfrak{a} = \mathfrak{a}_0 \mathfrak{a}' \quad \text{and} \quad \mathfrak{b} = \mathfrak{b}_0 \mathfrak{b}' \tag{3.24}$$

uniquely for $\mathfrak{a}_0, \mathfrak{b}_0 \in P_0$ and $\mathfrak{a}', \mathfrak{b}' \in P'$ to find that $\mathfrak{a}\bar{\mathfrak{b}} \in P_0$ if and only if $\mathfrak{a}' = \mathfrak{b}'$. By the bound $\hat{\Phi}(y) \ll_n y^{-n}$ (derived from repeated integration by parts) and the deduction above, the total contribution to the sum in (3.22) over all such pairs $(\mathfrak{a}, \mathfrak{b})$

is

$$\sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N\\ \mathfrak{a}\bar{\mathfrak{b}}\in P_0}} r(\mathfrak{a})r(\mathfrak{b})\big(\hat{\Phi}(0)+O(X^{-1})\big) \ll \hat{\Phi}(0)\sum_{\mathfrak{a}_0,\mathfrak{b}_0\in P_0} r(\mathfrak{a}_0)r(\mathfrak{b}_0)\sum_{\substack{\mathfrak{a}'\in P'\\ (\mathfrak{a}',\mathfrak{a}_0\mathfrak{b}_0)=1}} r(\mathfrak{a}')^2$$

$$= \hat{\Phi}(0)\prod_{\mathfrak{p}}{}' \left(1+4r(\mathfrak{p})^2+r(\mathfrak{p})^4\right), \tag{3.25}$$

where the primed product above is indexed over arbitrary representatives of conjugate pairs of factors of split rational primes. Again wielding the above bound on $\hat{\Phi}(y)$ and exercising the Cauchy–Schwarz inequality, we find that the off-diagonal contribution ($i.e.$ when $|\arg\gamma_{\mathfrak{a}\bar{\mathfrak{b}}}|\geqslant X^{-1+\varepsilon}$) to the sum in (3.22) is

$$\ll_m X^{-\varepsilon m}N\sum_{\mathfrak{a}\neq 0} r(\mathfrak{a})^2 \leqslant X^{1-\varepsilon m}\prod_{\mathfrak{p}}(1+r(\mathfrak{p})^2). \tag{3.26}$$

Choosing m sufficiently large relative to ε allows the quantity in (3.25) to dominate that in (3.26). Combining this observation with (3.21) and (3.22) yields

$$\sum_{\ell}\Phi(\ell/X)\sum_{\xi_\ell\in\mathfrak{F}_\ell}|R(\xi_\ell)|^2 \ll X\hat{\Phi}(0)\prod_{\mathfrak{p}}{}'\left(1+4r(\mathfrak{p})^2+r(\mathfrak{p})^4\right). \tag{3.27}$$

3.4 Resonator application

Application of Theorem 16 to the numerator in (3.20) produces

$$\sum_{\ell}\Phi(\ell/X)\sum_{\xi_\ell\in\mathfrak{F}_\ell}|R(\xi_\ell)|^2\sum_{\mathfrak{k}}\frac{\xi_\ell(\mathfrak{k})+\overline{\xi_\ell}(\mathfrak{k})}{\sqrt{\mathfrak{N}\mathfrak{k}}}W_K\big(\mathfrak{N}\mathfrak{k},|\ell|\big) \tag{3.28}$$

upon which we wish to place a lower bound. Our analysis of the sum over $\xi_\ell(\mathfrak{k})$ above will end up having built-in symmetry with the dual sum over $\overline{\xi_\ell}(\mathfrak{k})$. Further, the main contribution to the sum over $\xi_\ell(\mathfrak{k})$, hence also $\overline{\xi_\ell}(\mathfrak{k})$, will be real-valued. We thus explicitly develop our argument first (and only) for the sum in (3.28) over $\xi_\ell(\mathfrak{k})$. With this in mind, we proceed as in (3.21) and (3.22) – exchanging order of summation and using Poisson summation – to obtain

$$h_K \sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \sum_k \Phi(\omega_K k/X) W_K(\mathfrak{N}\mathfrak{k}, \omega_K k) e\left(\omega_K k \arg\gamma_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{k}}/2\pi\right)$$

$$= h_K \omega_K^{-1} X \sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \sum_j \mathcal{F}_u[\Phi(u) W_K(\mathfrak{N}\mathfrak{k}, uX)] \left(X\left(\omega_K^{-1} j - \arg\gamma_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{k}}/2\pi\right)\right) \tag{3.29}$$

where $\mathcal{F}_u[f]$ denotes the Fourier transform of $f(u)$. Over the course of this section, we will establish a lower bound on (3.28) by extracting the contribution to (3.29) from the case

$$\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \in P_0 \quad \text{and} \quad \mathfrak{N}\mathfrak{k} \ll X.$$

For $\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \in P \setminus P_0$, the rapid decay of the Fourier transform in (3.29) will be quite useful in our management, provided that $\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}$ is guaranteed to not be too small. Once focused on $\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \in P_0$ (and its $j = 0$ term in (3.29)), we will be permitted to truncate at $\mathfrak{N}\mathfrak{k} \ll X$ by the positivity of $\Phi(u)W_K(\mathfrak{N}\mathfrak{k}, uX)$ which is established in Lemma 21. Afterward, the resulting sum will undergo a somewhat lengthy factorization process enabled by Rankin's trick, as recalled in Section 3.4.3.

We begin by forming a handle on the decay of the transform of the function

$$\Phi(u)W_K(\mathfrak{N}\mathfrak{k}, uX)$$

above. It will suffice to place bounds on its derivatives. Note that

$$\frac{\mathrm{d}^n}{\mathrm{d}u^n}\Phi(u)W_K(\mathfrak{N}\mathfrak{k}, uX) = \sum_{0\leqslant m\leqslant n} \binom{n}{m} \Phi^{(n-m)}(u) \frac{\mathrm{d}^m}{\mathrm{d}u^m} W_K(\mathfrak{N}\mathfrak{k}, uX)$$

and recall Φ is supported in $[1,2]$ with uniformly bounded derivatives, as stated in

Section 3.3. Setting $c_K = 4\pi/\sqrt{D}$ and using (3.14) from Lemma 20, we find that the above is

$$\ll_n \begin{cases} X^{n/2}\left(\frac{uX}{c_K\mathfrak{N}\mathfrak{k}}\right)^{\sqrt{uX/2}}, & \text{if } u \leqslant c_K\mathfrak{N}\mathfrak{k}/X; \\ 1 + X^{n/2}\left(\frac{uX}{c_K\mathfrak{N}\mathfrak{k}}\right)^{-\sqrt{uX/2}}, & \text{if } u > c_K\mathfrak{N}\mathfrak{k}/X. \end{cases} \tag{3.30}$$

Repeating integration by parts n times on the Fourier transform of interest and exercising the bound in (3.30) shows that, for $v \neq 0$,

$$\mathcal{F}_u[\Phi(u)W_K(\mathfrak{N}\mathfrak{k}; uX)](v) \ll_n v^{-n}X^{n/2}S(\mathfrak{N}\mathfrak{k}, X) + v^{-n}\max\left(0, 2 - \tfrac{c_K\mathfrak{N}\mathfrak{k}}{X}\right) \tag{3.31}$$

where

$$S(\mathfrak{N}\mathfrak{k}, X) := \int_{\substack{u \leqslant c_K\mathfrak{N}\mathfrak{k}/X \\ u\in(1,2)}} \left(\frac{uX}{c_K\mathfrak{N}\mathfrak{k}}\right)^{\sqrt{uX/2}} \mathrm{d}u + \int_{\substack{u \geqslant c_K\mathfrak{N}\mathfrak{k}/X \\ u\in(1,2)}} \left(\frac{c_K\mathfrak{N}\mathfrak{k}}{uX}\right)^{\sqrt{uX/2}} \mathrm{d}u. \tag{3.32}$$

Using the substitution $w = uX/(c_K\mathfrak{N}\mathfrak{k})$ in (3.32) and bounding the exponents trivially based on the interval of integration reveals

$$S(\mathfrak{N}\mathfrak{k}, X) \ll_n \begin{cases} X^{-1/2}\left(c_K\mathfrak{N}\mathfrak{k}/X\right)^{\sqrt{X/2}}, & \mathfrak{N}\mathfrak{k} \leqslant X/c_K; \\ X^{-1/2}, & X/c_K < \mathfrak{N}\mathfrak{k} \leqslant 2X/c_K; \\ X^{-1/2}\left(\frac{1}{2}c_K\mathfrak{N}\mathfrak{k}/X\right)^{-\sqrt{X/2}}, & 2X/c_K < \mathfrak{N}\mathfrak{k}, \end{cases} \tag{3.33}$$

so that the insertion of (3.33) into (3.31), combined with a replacement of n with any real-valued $\eta \geqslant 1$ via interpolation (a geometric mean argument), yields

$$\begin{aligned} &\mathcal{F}_u[\Phi(u)W_K(\mathfrak{N}\mathfrak{k}; uX)](v) \\ &\ll v^{-\eta}\chi_{(0,2X/c_K]}(\mathfrak{N}\mathfrak{k}) + \frac{X^{\eta/2-1/2}}{v^{\eta}}\min\left(1, \left(c_K\mathfrak{N}\mathfrak{k}/X\right)^{\sqrt{X/2}}, \left(\tfrac{1}{2}c_K\mathfrak{N}\mathfrak{k}/X\right)^{-\sqrt{X/2}}\right). \end{aligned} \tag{3.34}$$

The next two subsections will break the outer sum of (3.29) into two cases: the *diagonal case* $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}} \in P_0$ and the *off-diagonal case* $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}} \in P \setminus P_0$.

3.4.1 Off-diagonal contribution

We first consider the *off-diagonal case* $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}} \in P \setminus P_0$, and we will do so with four subcases, the first of which is the *off-diagonal subcase I*: $\mathfrak{N}\mathfrak{k} \leqslant 2X/c_K$ and $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}| \geqslant X^{-1/2}$. It will be beneficial to subdivide the sum over $\mathfrak{k}$ in (3.29) according to the size of $\mathfrak{N}\mathfrak{k}$ along the dyadic partitions

$$D_\nu(X) = (2^{-\nu}X/c_K, 2^{-\nu+1}X/c_K]$$

and $\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}$ according to the intervals

$$I_m(X) = [mX^{-1/2}, (m+1)X^{-1/2}).$$

By the bound in (3.34), the contribution to (3.29) for $\mathfrak{N}\mathfrak{k} \leqslant 2X/c_K$ with $\eta \geqslant 1$ is

$$\ll X^{1/2-\eta/2} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{\substack{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\\ \mathfrak{N}\mathfrak{k}\leqslant 2X/c_K}} \mathfrak{N}\mathfrak{k}^{-1/2} \sum_j \left(\omega_K^{-1}j - \arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}/2\pi\right)^{-\eta} \tag{3.35}$$

where, by the definition of $\gamma_{\mathfrak{n}}$, the main contribution to the sum over j above comes from $j = 0$. Precisely,

$$\sum_j \left(\omega_K^{-1}j - \arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}/2\pi\right)^{-\eta} \ll_\eta (\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}})^{-\eta}. \tag{3.36}$$

By the restriction on the argument of $\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}$ in this subcase, we now bound the contribution to (3.35) as

$$\ll_\eta X^{1/2-\eta/2} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b})X^{\eta/2}\sum_{m\geqslant 1} m^{-\eta} \sum_{0\leqslant\nu\leqslant\log_2 X} 2^{\nu/2}X^{-1/2} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D_\nu(X)\\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|\in I_m(X)\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0}} 1$$

$$\ll_\eta \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{0\leqslant\nu\leqslant\log_2 X} 2^{\nu/2} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D_\nu(X)\\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|\in I_m(X)\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0}} 1$$

Bounding the inner sum above can take place by counting the number of $\mathcal{O}_K$ lattice points associated to each $\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}$ (translating $\mathfrak{k}$ to a generator in $\mathbb{C}$ by one of a fixed *finite* number of representatives of the class group) within the annulus defined by $D_\nu(X)$ while restricted to the appropriate angular sector. Note that the area of this annular sector is commensurate to $2^{-\nu}X^{1/2}$ and the perimeter to $2^{-\nu/2}X^{1/2}$, allowing us to estimate the number of lattice points as $\ll 2^{-\nu/2}X^{1/2}+1$.

Therefore the total contribution to (3.29) from *off-diagonal subcase I* is

$$\ll_\eta X^{1/2}\log X \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}). \tag{3.37}$$

We next consider the *off-diagonal subcase II*: $\mathfrak{N}\mathfrak{k}\leqslant 2X/c_K$ and $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|<X^{-1/2}$. By the series expansion of sine around zero, we have $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|\gg N^{-1}\mathfrak{N}\mathfrak{k}^{-1/2}$ for $\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0$, and so using (3.34) to again obtain (3.35) with its associated bound (3.36) on the sum over j shows that the contribution to (3.29) from the *off-diagonal*

subcase II is

$$\ll_\eta X^{1/2-\eta/2} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{a}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{0\leqslant\nu\leqslant\log_2 X} (X2^{-\nu})^{-1/2} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D_\nu(X)\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0\\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|<X^{-1/2}}} N^\eta (X2^{-\nu})^{\eta/2}$$

$$\ll N^\eta \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{a}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{0\leqslant\nu\leqslant\log_2 X} 2^{\nu/2-\nu\eta/2} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D_\nu(X)\\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0\\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|<X^{-1/2}}} 1.$$

Bounding the inner sum above can take place by counting lattice points within the annulus defined by $D_\nu(X)$ while restricted to the appropriate angular sector, as done in the previous subcase. This again allows us to estimate the number of lattice points as $\ll 2^{-\nu/2}X^{1/2}+1$. Substitution to achieve an upper bound then shows the contribution to (3.29) from *off-diagonal subcase II* is

$$\ll_\eta X^{1/2}N^\eta \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}). \tag{3.38}$$

In the remaining two cases, we will split $\mathfrak{N}\mathfrak{k}$ according to the intervals

$$D'_\nu(X) = (\nu 2X/c_K, (\nu+1)2X/c_K].$$

Now consider the *off-diagonal subcase III*: $\mathfrak{N}\mathfrak{k} > 2X/c_K$ and $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}| \geqslant X^{-1/2}$. With respect to (3.34), notice that we have entered the range where

$$\mathcal{F}_u[\Phi(u)W_K(\mathfrak{N}\mathfrak{k}; uX)](v)$$

decays rapidly. Again exercising (3.34), one finds that the contribution to (3.29) with the constraint $\mathfrak{N}\mathfrak{k} > 2X/c_K$ and observation (3.36) is

$$\ll_\eta X^{1/2-\eta/2} \left(\frac{2X}{c_K}\right)^{\sqrt{X/2}} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{\substack{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P\setminus P_0 \\ \mathfrak{N}\mathfrak{k}>2X/c_K}} \mathfrak{N}\mathfrak{k}^{-1/2-\sqrt{X/2}} (\arg \gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}})^{-\eta}. \quad (3.39)$$

We may bound the inner sum above

$$\sum_{\nu\geqslant 1}\sum_{m\geqslant 1} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D'_\nu(X) \\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P \\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|\in I_m(X)}} \mathfrak{N}\mathfrak{k}^{-1/2-\sqrt{X/2}} (\arg \gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}})^{-\eta}$$
$$\ll X^{\eta/2} \left(\frac{2X}{c_K}\right)^{-1/2-\sqrt{X/2}} \sum_{\nu\geqslant 1} \nu^{-1/2-\sqrt{X/2}} \sum_{m\geqslant 1} m^{-\eta} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D'_\nu(X) \\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}\in P \\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|\in I_m(X)}} 1 \quad (3.40)$$

where a lattice point counting argument, explained previously in *off-diagonal subcase I*, can be applied where the area of the angular sector is of size $X^{1/2}$ and the perimeter of size $\nu^{-1/2}X^{1/2}$. We therefore bound the inner sum of (3.40) as $\ll X^{1/2}$. Subsequent evaluation and substitution of (3.40) into (3.39) shows that the contribution from *off-diagonal subcase III* to (3.29) is

$$\ll_\eta X^{1/2} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}). \quad (3.41)$$

Finally, there is the *off-diagonal subcase IV*: $\mathfrak{N}\mathfrak{k} \geqslant 2X/c_K$ and $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}| < X^{-1/2}$. With (3.39), the bound $|\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}| \gg \mathfrak{N}\mathfrak{k}^{-1/2}N^{-1}$, and the same lattice point counting argument in previous cases, the contribution to (3.29) from the inner sum of (3.39) keeping in mind (3.36) is

$$\ll_\eta N^\eta \sum_{\nu\geqslant 1} \sum_{\substack{\mathfrak{N}\mathfrak{k}\in D'_\nu \\ |\arg\gamma_{\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}}}|<X^{-1/2}}} \mathfrak{N}\mathfrak{k}^{-1/2-\sqrt{X/2}+\eta/2} \ll N^\eta X^{\eta/2} \left(\frac{2X}{c_K}\right)^{-\sqrt{X/2}}. \quad (3.42)$$

Substitution of the above into (3.39) then bounds the contribution to (3.29) from *off-diagonal subcase IV* by

$$\ll_\eta X^{1/2} N^\eta \sum_{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N} r(\mathfrak{a}) r(\mathfrak{b}). \tag{3.43}$$

In consideration of the bounds from each of *off-diagonal subcases I-IV*, it is shown that the total contribution to (3.29) from the *off-diagonal subcases I-IV* is bounded by (3.43). Concisely, for $\eta \geqslant 1$ and $N \gg \log X$ (as will be chosen), we have

$$X \sum_{\substack{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N \\ \mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \in P \setminus P_0}} \frac{r(\mathfrak{a}) r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}} \sum_j \mathcal{F}_u[\Phi(u) W_K(\mathfrak{N}\mathfrak{k}, uX)] \left(X \left(\omega_K^{-1} j - \arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{k}} / 2\pi \right) \right)$$

$$\ll_\eta X^{1/2} N^\eta \sum_{\mathfrak{N}\mathfrak{a}, \mathfrak{N}\mathfrak{b} \leqslant N} r(\mathfrak{a}) r(\mathfrak{b}). \tag{3.44}$$

3.4.2 Diagonal contribution

With respect to the contribution from $\mathfrak{k}\mathfrak{a}\bar{\mathfrak{b}} \in P_0$ to (3.29), one may readily handle the case $j \neq 0$ of the inner sum by the approach from the previous subsection since $\omega_K^{-1} j - \arg \gamma_{\mathfrak{a}\bar{\mathfrak{b}}\mathfrak{k}} / 2\pi$ here is at least as large as its counterpart in the off-diagonal case with $j = 0$. This certainly yields no more than the quantity (3.44) found in the off-diagonal case. Further, for $j = 0$ in the inner sum of (3.29), the non-negativity of $\Phi(u) W_K(\mathfrak{N}\mathfrak{k}, uX)$ following from Lemma 21 allows one use the outer sum according to $\mathfrak{N}\mathfrak{k} \leqslant X/2c_K$ to produce a lower bound on (3.29), provided that the term resulting from truncation is $\gg$ than the quantity in (3.44) with appropriate implied constant.

Note that (3.14) from Lemma 20 demonstrates that $\mathcal{F}_u[\Phi(u) W_K(\mathfrak{N}\mathfrak{k}; uX)] (0)$ is effectively constant in this range. Write $\mathfrak{a} = \mathfrak{a}_0 \mathfrak{a}'$ and $\mathfrak{b} = \mathfrak{b}_0 \mathfrak{b}'$ as in (3.24). Further factorizing

$$\mathfrak{a}' = (\mathfrak{a}', \mathfrak{b}')\mathfrak{a}'' \quad \text{and} \quad \mathfrak{b}' = (\mathfrak{a}', \mathfrak{b}')\mathfrak{b}'', \tag{3.45}$$

we note that $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}} \in P_0$ precisely when $\mathfrak{k} = \mathfrak{k}_0\overline{\mathfrak{a}''}\mathfrak{b}''$ for $\mathfrak{k}_0 \in P_0$. The truncated sum discussed above is of magnitude

$$X \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}) \sum_{\substack{\mathfrak{N}\mathfrak{k}\leqslant X/2c_K \\ \mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}\in P_0}} \mathfrak{N}\mathfrak{k}^{-1/2} = X \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}} \sum_{\mathfrak{N}\mathfrak{k}_0\leqslant X/2c_K\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')} \mathfrak{N}\mathfrak{k}_0^{-1/2}$$
$$\gg XN^{-1} \sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N} r(\mathfrak{a})r(\mathfrak{b}).$$

when $N \ll X^{1/2+\varepsilon}$. Indeed, we now finally set $N = X^{1/4-\varepsilon}$ and $\eta = 1+\varepsilon$ so that $X^{1/2}N^{\eta} = o(X/N)$. Keeping in mind the dual sum of (3.28), we find that two times (3.29) — thus ultimately (3.28) — is bounded from below in absolute value by

$$\left(2h_K\omega_K^{-1}\hat{\Phi}(0) + o(1)\right) X \sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b}\leqslant N \\ \mathfrak{N}\mathfrak{k}\leqslant X/2c_K \\ \mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}}\in P_0}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}\mathfrak{k}}}. \tag{3.46}$$

As done in Section 3.3.1, we wish to factorize (3.46) so that we may more readily compare it to (3.27) on a subpower scale, thereby detecting large values of $|L(\frac{1}{2}, \xi_\ell)|$, which are expected to be subpower with respect to X; see again (3.20) and Theorem 14.

3.4.3 Factorization of the diagonal sum

As in (3.24) and (3.45), write $\mathfrak{a} = \mathfrak{a}_0\mathfrak{a}'$ and $\mathfrak{b} = \mathfrak{b}_0\mathfrak{b}'$ so that

$$\mathfrak{a}\overline{\mathfrak{b}} = \mathfrak{a}_0\mathfrak{b}_0(\mathfrak{a}', \mathfrak{b}')\overline{(\mathfrak{a}', \mathfrak{b}')}\mathfrak{a}''\overline{\mathfrak{b}''}$$

where $(\mathfrak{a}'', \mathfrak{b}'') = 1$ and $\mathfrak{a}''\overline{\mathfrak{b}''} \in P'$. Then for $\mathfrak{k}\mathfrak{a}\overline{\mathfrak{b}} \in P_0$, we must have

$$\mathfrak{k} = \mathfrak{k}_0\overline{\mathfrak{a}''}\mathfrak{b}''$$

for $\mathfrak{k}_0 \in P_0$. Moving forward, should context require it, we will assume in our notation that $\mathfrak{a}$ and $\mathfrak{b}$, and also $\mathfrak{a}'$ and $\mathfrak{b}'$, factorize as above in relation to one another. The sum in (3.46) is thus

$$\sum_{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b} \leqslant N} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}} \sum_{\mathfrak{N}\mathfrak{k}_0 \leqslant X/2c_k\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')} \mathfrak{N}\mathfrak{k}_0^{-1/2} \geqslant \sum_{\substack{\mathfrak{N}\mathfrak{a},\mathfrak{N}\mathfrak{b} \leqslant N \\ (\mathfrak{a}''\overline{\mathfrak{a}''}, \mathfrak{b}''\overline{\mathfrak{b}''})=1}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}}. \tag{3.47}$$

We now wish to complete the sum in this lower bound over all non-zero $\mathfrak{a}$ and $\mathfrak{b}$. We do exactly this using Rankin's trick, which is the observation that for any non-negative sequence (m_n) and any $\alpha \geqslant 0$, we have

$$\sum_{0 \leqslant n \leqslant N} m_n = \sum_{n \geqslant 0} m_n + O\left(N^{-\alpha} \sum_{n \geqslant 0} n^\alpha m_n\right).$$

Using the above, first on the sum over $\mathfrak{a}$ on the right hand side of (3.47), and then on the sum over $\mathfrak{b}$ (both times using the same value of $\alpha \geqslant 0$ to be chosen later) shows that the right side of (3.47) equals

$$\sum_{\substack{\mathfrak{a},\mathfrak{b} \neq 0 \\ (\mathfrak{a}''\overline{\mathfrak{a}''}, \mathfrak{b}''\overline{\mathfrak{b}''})=1}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}} + O\left(\Xi(\alpha, 0) + \Xi(\alpha, \alpha)\right) \tag{3.48}$$

where

$$\Xi(\alpha_1, \alpha_2) = N^{-\alpha_1-\alpha_2} \sum_{\substack{\mathfrak{a},\mathfrak{b} \neq 0 \\ (\mathfrak{a}''\overline{\mathfrak{a}''}, \mathfrak{b}''\overline{\mathfrak{b}''})=1}} \frac{r(\mathfrak{a})r(\mathfrak{b})}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}} \mathfrak{N}\mathfrak{a}^{\alpha_1}\mathfrak{N}\mathfrak{b}^{\alpha_2}. \tag{3.49}$$

We first factorize the sum in $\Xi(\alpha, \alpha)$ of (3.48), where the factorization of the expected main term in (3.48) will follow from the substitution $\alpha = 0$. Afterward, we will factorize the sum $\Xi(\alpha, 0)$ in (3.48) in a similar way. To begin, write the sum in $\Xi(\alpha, \alpha)$ as

$$\sum_{\substack{\mathfrak{a}', \mathfrak{b}' \in P' \\ (\mathfrak{a}''\overline{\mathfrak{a}''}, \mathfrak{b}''\overline{\mathfrak{b}''})=1}} \frac{r(\mathfrak{a}')r(\mathfrak{b}')}{\sqrt{\mathfrak{N}(\mathfrak{a}''\mathfrak{b}'')}} \mathfrak{N}(\mathfrak{a}'\mathfrak{b}')^{\alpha} \sum_{\substack{\mathfrak{a}_0 \in P_0 \\ (\mathfrak{a}_0, \mathfrak{a}')=1}} r(\mathfrak{a}_0)\mathfrak{N}\mathfrak{a}_0^{\alpha} \sum_{\substack{\mathfrak{b}_0 \in P_0 \\ (\mathfrak{b}_0, \mathfrak{b}')=1}} r(\mathfrak{b}_0)\mathfrak{N}\mathfrak{b}_0^{\alpha}. \tag{3.50}$$

After factorizing the two inner sums of the above expression as

$$\prod_{\mathfrak{p}}{}' A_{\alpha}(\mathfrak{p})^2 \prod_{\mathfrak{p}|\mathfrak{a}'} A_{\alpha}(\mathfrak{p})^{-1} \prod_{\mathfrak{p}|\mathfrak{b}'} A_{\alpha}(\mathfrak{p})^{-1}, \quad A_{\alpha}(\mathfrak{p}) := (1 + r(\mathfrak{p})^2 \mathfrak{N}\mathfrak{p}^{2\alpha}),$$

we write $\mathfrak{c}' = (\mathfrak{a}', \mathfrak{b}')$ so that (3.50) becomes

$$\prod_{\mathfrak{p}}{}' A_{\alpha}(\mathfrak{p})^2 \sum_{\mathfrak{c}' \in P'} r(\mathfrak{c}')^2 (\mathfrak{N}\mathfrak{c}')^{2\alpha} \prod_{\mathfrak{p}|\mathfrak{c}'} A_{\alpha}(\mathfrak{p})^{-2} \sum_{\substack{(\mathfrak{a}''\overline{\mathfrak{a}''}, \mathfrak{b}''\overline{\mathfrak{b}''})=1 \\ (\mathfrak{a}''\mathfrak{b}'', \mathfrak{c}'\overline{\mathfrak{c}'})=1}} \frac{r(\mathfrak{a}'')r(\mathfrak{b}'')}{(\mathfrak{N}\mathfrak{a}''\mathfrak{b}'')^{1/2-\alpha}} \prod_{\mathfrak{p}|\mathfrak{a}''\mathfrak{b}''} A_{\alpha}(\mathfrak{p})^{-1}. \tag{3.51}$$

With $\mathfrak{c}'' = \mathfrak{a}''\mathfrak{b}''$ (recalling that $\mathfrak{a}'', \mathfrak{b}'' \in P'$ by their definitions) and consideration of the divisor function d, we may write the innermost sum above as

$$\sum_{\substack{\mathfrak{c}'' \in P' \\ (\mathfrak{c}'', \mathfrak{c}'\overline{\mathfrak{c}'})=1}} \frac{r(\mathfrak{c}'')d(\mathfrak{c}'')}{(\mathfrak{N}\mathfrak{c}'')^{1/2-\alpha}} \prod_{\mathfrak{p}|\mathfrak{c}''} A_{\alpha}(p)^{-1} = \prod_{\mathfrak{p}\nmid\mathfrak{c}'\overline{\mathfrak{c}'}}{}' B_{\alpha}(\mathfrak{p}), \quad B_{\alpha}(\mathfrak{p}) := 1 + \frac{4r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2-\alpha}} A_{\alpha}(\mathfrak{p})^{-1}, \tag{3.52}$$

so that inserting (3.52) into the inner sum of (3.51) yields

$$\begin{aligned} &\prod_{\mathfrak{p}}{}' A_{\alpha}(\mathfrak{p})^2 B_{\alpha}(\mathfrak{p}) \sum_{\mathfrak{c}' \in P'} r(\mathfrak{c}')^2 (\mathfrak{N}\mathfrak{c}')^{2\alpha} \prod_{\mathfrak{p}|\mathfrak{c}'\overline{\mathfrak{c}'}}{}' A_{\alpha}(\mathfrak{p})^{-2} B_{\alpha}(\mathfrak{p})^{-1} \\ &\quad = \prod_{\mathfrak{p}}{}' \left(A_{\alpha}(\mathfrak{p})^2 B_{\alpha}(\mathfrak{p}) + 2r(\mathfrak{p})^2 \mathfrak{N}\mathfrak{p}^{2\alpha} \right) \\ &\quad = \prod_{\mathfrak{p}}{}' \left(1 + \frac{4r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}} \mathfrak{N}\mathfrak{p}^{\alpha} + 4r(\mathfrak{p})^2 \mathfrak{N}\mathfrak{p}^{2\alpha} + \frac{4r(\mathfrak{p})^3}{\mathfrak{N}\mathfrak{p}^{1/2}} \mathfrak{N}\mathfrak{p}^{3\alpha} + r(\mathfrak{p})^4 \mathfrak{N}\mathfrak{p}^{4\alpha} \right) \end{aligned} \tag{3.53}$$

as the factorization of the sum in $\Xi(\alpha, \alpha)$ in (3.48).

Using a similar strategy, one may also factorize the sum in $\Xi(\alpha, 0)$ of (3.48). This quantity is

$$\sum_{\substack{\mathfrak{a}',\mathfrak{b}'\in P' \\ (\mathfrak{a}''\overline{\mathfrak{a}''},\mathfrak{b}''\overline{\mathfrak{b}''})=1}} \frac{r(\mathfrak{a}')r(\mathfrak{b}')}{\sqrt{\mathfrak{N}\mathfrak{a}''\mathfrak{b}''}} (\mathfrak{N}\mathfrak{a}')^\alpha \sum_{\substack{\mathfrak{a}_0\in P_0 \\ (\mathfrak{a}_0,\mathfrak{a}')=1}} r(\mathfrak{a}_0)\mathfrak{N}\mathfrak{a}_0^\alpha \sum_{\substack{\mathfrak{b}_0\in P_0 \\ (\mathfrak{b}_0,\mathfrak{b}')=1}} r(\mathfrak{b}_0). \tag{3.54}$$

We may factorize the two inner sums of the above expression as

$$\prod_{\mathfrak{p}}{}' A_\alpha(\mathfrak{p})A_0(\mathfrak{p}) \prod_{\mathfrak{p}|\mathfrak{a}'} A_\alpha(\mathfrak{p})^{-1} \prod_{\mathfrak{p}|\mathfrak{b}'} A_0(\mathfrak{p})^{-1},$$

so that (3.54) becomes

$$\begin{aligned} \prod_{\mathfrak{p}}{}' A_\alpha(\mathfrak{p})A_0(\mathfrak{p}) \sum_{\mathfrak{c}'\in P'} r(\mathfrak{c}')^2(\mathfrak{N}\mathfrak{c}')^\alpha \prod_{\mathfrak{p}|\mathfrak{c}'} A_\alpha(\mathfrak{p})^{-1}A_0(\mathfrak{p})^{-1} \\ \times \sum_{\substack{(\mathfrak{a}''\overline{\mathfrak{a}''},\mathfrak{b}''\overline{\mathfrak{b}''})=1 \\ (\mathfrak{a}''\mathfrak{b}'',\mathfrak{c}'\overline{\mathfrak{c}'})=1}} \frac{r(\mathfrak{a}'')r(\mathfrak{b}'')}{(\mathfrak{N}\mathfrak{a}''\mathfrak{b}'')^{1/2}} \prod_{\mathfrak{p}|\mathfrak{a}''} A_\alpha(\mathfrak{p})^{-1}\mathfrak{N}\mathfrak{p}^\alpha \prod_{\mathfrak{p}|\mathfrak{b}''} A_0(\mathfrak{p})^{-1}. \end{aligned} \tag{3.55}$$

Rewriting the inner sum above as follows and factorizing allow us to obtain

$$\begin{aligned} \sum_{\substack{\mathfrak{c}''\in P' \\ (\mathfrak{c}'',\mathfrak{c}'\overline{\mathfrak{c}'})=1}} \frac{r(\mathfrak{c}'')}{(\mathfrak{N}\mathfrak{c}'')^{1/2}} \prod_{\mathfrak{p}|\mathfrak{c}''} A_0(\mathfrak{p})^{-1} \sum_{\mathfrak{a}''|\mathfrak{c}''} (\mathfrak{N}\mathfrak{a}'')^\alpha \prod_{\mathfrak{p}|\mathfrak{a}''} A_0(\mathfrak{p})A_\alpha(\mathfrak{p})^{-1} \\ = \prod_{\mathfrak{p}\nmid\mathfrak{c}'\overline{\mathfrak{c}'}}{}' \left(1 + \frac{2r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}}(A_0(\mathfrak{p})^{-1} + \mathfrak{N}\mathfrak{p}^\alpha A_\alpha(\mathfrak{p})^{-1})\right) \end{aligned}$$

so that for

$$C_\alpha(\mathfrak{p}) := A_\alpha(\mathfrak{p})A_0(\mathfrak{p}) + \frac{2r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}}(A_\alpha(\mathfrak{p}) + \mathfrak{N}\mathfrak{p}^\alpha A_0(\mathfrak{p}))$$

we may further factorize (3.55) as

$$\prod_{\mathfrak{p}}{}' C_\alpha(\mathfrak{p}) \sum_{\mathfrak{c}'\in P'} r(\mathfrak{c}')^2(\mathfrak{N}\mathfrak{c}')^\alpha \prod_{\mathfrak{p}|\mathfrak{c}'} C_\alpha(\mathfrak{p})^{-1} = \prod_{\mathfrak{p}}{}' \left(C_\alpha(\mathfrak{p}) + 2r(\mathfrak{p})^2\mathfrak{N}\mathfrak{p}^\alpha\right).$$

This ultimately yields for the sum in $\Xi(\alpha, 0)$ a factorization of

$$\prod_{\mathfrak{p}}' \Bigg(1 + \frac{2r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}}(1 + \mathfrak{N}\mathfrak{p}^{\alpha}) + r(\mathfrak{p})^2(1 + 2\mathfrak{N}\mathfrak{p}^{\alpha} + \mathfrak{N}\mathfrak{p}^{2\alpha}) \\ + \frac{2r(\mathfrak{p})^3}{\mathfrak{N}\mathfrak{p}^{1/2}}(\mathfrak{N}\mathfrak{p}^{\alpha} + \mathfrak{N}\mathfrak{p}^{2\alpha}) + r(\mathfrak{p})^4\mathfrak{N}\mathfrak{p}^{2\alpha} \Bigg). \quad (3.56)$$

Once a choice of resonator coefficients is made, it will become necessary to show that

$$\Xi(\alpha, 0) + \Xi(\alpha, \alpha) = o\left(\Xi(0, 0)\right) \quad (3.57)$$

so that the main term of (3.48) can be used as a lower bound on (3.46), and hence as a lower bound on the numerator of (3.28).

3.5 Choice of resonator coefficients

We will now choose explicit values for all remaining parameters in our application of the resonance method. Recall the properties of $r(\mathfrak{p})$ given in the beginning of Section 3.3 and let

$$L = \sqrt{\tfrac{1}{2}\log N \log\log N}; \quad r(\mathfrak{p}) = \frac{L}{\mathfrak{N}\mathfrak{p}^{1/2}\log \mathfrak{N}\mathfrak{p}}, \quad L^2 \leqslant \mathfrak{N}\mathfrak{p} \leqslant \exp(\log^2 L), \quad (3.58)$$

and $r(\mathfrak{p}) = 0$ for prime ideals $\mathfrak{p}$ outside of this range. Further, we set the Rankin's trick parameter $\alpha = 1/\log^3 L$.

3.5.1 Ratio of error terms to main term

We will need to examine each of $\Xi(\alpha, 0)$, $\Xi(\alpha, \alpha)$, and $\Xi(0, 0)$ on a logarithmic scale to demonstrate that the main term of (3.48) truly dominates the error terms. Expanding

$$\mathfrak{N}\mathfrak{p}^{\alpha} = 1 + O(\alpha \log \mathfrak{N}\mathfrak{p}), \quad \mathfrak{N}\mathfrak{p} \leqslant \exp(\log^2 L),$$

then making the crude observations

$$\sum_{\mathfrak{p}}{}' \frac{r(\mathfrak{p})^3 \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^{1/2}} \ll L^3 \sum_{\mathfrak{N}\mathfrak{p} \geqslant L^2} \mathfrak{N}\mathfrak{p}^{-2} (\log \mathfrak{N}\mathfrak{p})^{-2} \ll \frac{(\log N)^{1/2}}{(\log\log N)^{5/2}}$$

$$\text{and} \quad \sum_{\mathfrak{p}}{}' r(\mathfrak{p})^4 \log \mathfrak{N}\mathfrak{p} \ll L^4 \sum_{\mathfrak{N}\mathfrak{p} \geqslant L^2} \mathfrak{N}\mathfrak{p}^{-2} (\log \mathfrak{N}\mathfrak{p}^{-3}) \ll \frac{\log N}{(\log\log N)^3},$$

lets us deduce from (3.53) and (3.56), for $E_\alpha = \log N/(\log\log N)^3$, that

$$\begin{aligned}
&\frac{\Xi(\alpha,\alpha)}{\Xi(0,0)} + \frac{\Xi(\alpha,0)}{\Xi(0,0)} \\
&\ll \exp\left(-2\alpha \log N + \sum_{\mathfrak{p}}{}' \left((\mathfrak{N}\mathfrak{p}^\alpha - 1) \frac{4r(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}} + (\mathfrak{N}\mathfrak{p}^{2\alpha} - 1) 4 r(\mathfrak{p})^2 \right)\right) \\
&\qquad \times \exp\left(O(\alpha E_\alpha)\right) \\
&+ \exp\left(-\alpha \log N + \sum_{\mathfrak{p}}{}' \left((\mathfrak{N}\mathfrak{p}^\alpha - 1) \frac{2r(\mathfrak{p})}{\sqrt{\mathfrak{N}\mathfrak{p}}} + (2\mathfrak{N}\mathfrak{p}^{\alpha} + \mathfrak{N}\mathfrak{p}^{2\alpha} - 3) r(\mathfrak{p})^2 \right)\right) \\
&\qquad \times \exp\left(O(\alpha E_\alpha)\right). \qquad (3.59)
\end{aligned}$$

Consideration of the linear expansion

$$\mathfrak{N}\mathfrak{p}^\alpha = 1 + \alpha \log \mathfrak{N}\mathfrak{p} + O((\alpha \log \mathfrak{N}\mathfrak{p})^2), \quad \mathfrak{N}\mathfrak{p} \leqslant \exp(\log^2 L),$$

in conjunction with the definitions

$$\begin{aligned}
G_\alpha(r) &:= \sum_{\mathfrak{p}}{}' \left(\alpha \frac{r(\mathfrak{p}) \log \mathfrak{N}\mathfrak{p}}{\mathfrak{N}\mathfrak{p}^{1/2}} + \alpha^2 r(\mathfrak{p})^2 \log^2 \mathfrak{N}\mathfrak{p} \right), \\
H_\alpha(r) &:= -\alpha \log N + \alpha \sum_{\mathfrak{p}}{}' 4 r(\mathfrak{p})^2 \log \mathfrak{N}\mathfrak{p}, \qquad (3.60)
\end{aligned}$$

allows us to rewrite (3.59) as

$$\exp\left(2H_\alpha(r) + O(G_\alpha(r) + \alpha E_\alpha)\right) + \exp\left(H_\alpha(r) + O(G_\alpha(r) + \alpha E_\alpha)\right) \qquad (3.61)$$

where, by partial summation and the prime ideal theorem,

$$G_\alpha(r) \ll \alpha^2 \log N \log\log N \log\log\log N. \tag{3.62}$$

Note the right hand side of the above dominates αE_α.

3.5.2 Bounding the numerator

Bounding sums similar to the one in (3.60) is a general requirement of the resonance method and is delicately computed through partial summation and the prime ideal theorem. For example, this exact computation (after exercising the prime ideal theorem) is used in the *proof* of [4, Lemma 7.4] with the parameters $\omega(\mathfrak{p}) = a_\omega = 2$ in their notation. Outsourcing this computation, we have

$$\begin{aligned} H_\alpha(r) &\geqslant -\alpha \log N + \alpha \sum_{\mathfrak{p}} \omega(p) r(\mathfrak{p})^2 \log \mathfrak{N}\mathfrak{p} + O(1) \\ &= -\alpha \frac{\log N \log\log\log N}{\log\log N} + O\left(\alpha \frac{\log N}{\log\log N}\right). \end{aligned}$$

Applying the bounds in (3.61) and (3.62), we find that the ratio of the sum of errors to the expected main term in (3.48) is

$$\ll \exp\left(-\alpha \frac{\log N \log\log\log N}{2 \log\log N}\right).$$

Therefore, in consider of (3.48) and its relationship to (3.46), we have shown that the numerator of (3.20) is bounded from below by

$$X h_K \omega_K^{-1} \hat{\Phi}(0) \prod_{\mathfrak{p}}{}' \left(1 + \frac{4r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}} + 4r(\mathfrak{p})^2 + r(\mathfrak{p})^4\right). \tag{3.63}$$

3.6 Main result

With respect to the ratio in (3.20), we may now utilize our upper bound on its denominator (3.27) and our lower bound on its numerator (3.63). Doing so yields

$$\max_{\substack{X\leqslant \ell\leqslant 2X\\ \xi_\ell\in\mathfrak{F}_\ell}} \log\left|L(\tfrac{1}{2},\xi_\ell)\right| \geqslant o(1)+\sideset{}{'}\sum_{\mathfrak{p}} \frac{4r(\mathfrak{p})}{\mathfrak{N}\mathfrak{p}^{1/2}} = (2+o(1))\frac{L}{\log L}.$$

Recalling our exact choice of parameters in (3.58) and the value $N = X^{1/4-\varepsilon}$ set in Section 3.4.2 completes the proof of Theorem 14.

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