Synchronisation of micro-mechanical oscillators inside one cavity using feedback control

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Abstract— **The purpose of this work is to develop a systematic approach towards synchronisation of two micro-mechanical oscillators inside one optical cavity using feedback control. We first obtain the linear quantum stochastic state space model for the optomechanical system considered in this paper. Then we design a measurement-based optimal controller, aimed at achieving complete quantum synchronisation of the two mechanical oscillators with different natural frequencies, in the linear quadratic Gaussian setting. In addition, simulation results are provided, which show how system parameters impact on the control effect. These findings shed light on the synchronised network of oscillators that can be used for memory and quantum state transfer.**

I. INTRODUCTION

Optomechanical systems, in which optical resonators interact with mechanical oscillators, offer a platform for studying a wide range of nonlinear and quantum effects. These systems have been studied in the context of quantum-limited detection of forces and displacements, the production of nonclassical states of light, synchronisation and chaotic dynamics; see [2], [14]. In this project, we consider an optomechanical system which consists of multiple optical and mechanical modes. In such a system, the motion of a given mechanical mode will modulate the intracavity optical field, which will in turn drive other mechanical modes. This can be thought of as an optically mediated coupling between the mechanical modes; see [8].

As the number of mechanical oscillators increases, the interactions between different modes become more complicated. In this situation, quantum network theory and the (S, L, H) representation of cascade quantum systems is of much help to obtain the corresponding linear stochastic state space model, which is widely used in control engineering; see [5], [9]. Once we acquire linear quantum stochastic system models given by a set of quantum stochastic differential equations (QSDEs), some existing control techniques turn out to be applicable, which may significantly reduce the workload of designing controllers and help pilot experiments. To begin with, we are concerned with a membrane whose mechanical motion is coupled to another membrane via the light field; see [7], [8]. Specifically, in this system setup, there are two micro-mechanical oscillators coupled to the fields of an optical cavity, and the cavity fields can induce an effective coupling between the mechanical oscillators via radiation pressure force.

The control goals of such optomechanical systems comprise synchronisation of each mechanical oscillator, superposition of different mechanical modes, multimode entanglement and phonon-phonon coupling. In this paper, the aim is to synchronise two micro-mechanical oscillators inside one optical cavity using feedback control. Synchronisation, which prevalently occurs in nature, is of great technological interest since it can contribute to signal processing and novel memory concepts; see [3], [11]. Moreover, at the nanoscale, synchronisation mechanisms are likely to be integrated with current nanofabrication capabilities and to facilitate scaling up to network sizes; see [16].

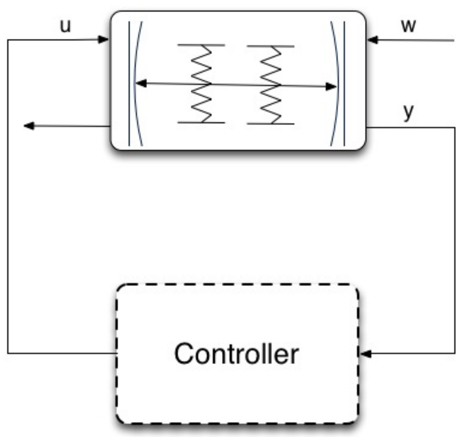

Fig. 1. The closed-loop feedback plant-controller system.

We first reinterpret the optomechanical plant in the form of QSDEs from the (S, L, H) description. And then we formulate an optimal control problem, aimed at synchronising the two mechanical modes inside the cavity. Here we mention another paper [16], in which two micro-mechanical oscillators are synchronised by tuning the optical coupling strength without feedback control involved. By contrast, the focus of our research is the application of feedback that takes advantage of acquired information about quantum plants. As figure. 1 shows, the output of the controller, which tunes the input optical power, is fed back to the optomehanical

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plant. Also, the two mechanical oscillators in [16] can be further synchronised by minimising the difference of their amplitudes using the feedback control scheme proposed in this paper. The linear quantum state space model we apply in this paper makes it easier for us to sort out how system parameters affect the behaviour of the quantum plant, and assists in adjustments to system parameters in order to observe salient synchronisation effects via feedback control.

The paper is organised as follows. We begin in Section II by presenting the linear quantum state space model for the optomechanical plant. In Section III, we design a measurement-based optimal controller in the linear quadratic Gaussian (LQG) setting, whose output modulates the amplitude, phase and frequency of the laser. Simulation results are studied as well. Section IV provides some concluding remarks and future research directions.

Notations. In this paper the asterisk is used to indicate the Hilbert space adjoint $X^\star$ of an operator X, as well as the complex conjugate $z^\star = x - iy$ of a complex number $z = x + iy$ (here, $i = \sqrt{-1}$ and x, y are real). Real and imaginary parts are denoted by $\Re(z) = \frac{z+z^\star}{2}$ and $\Im(z) = \frac{z-z^\star}{2i}$ respectively. The conjugate transpose $A^\dagger$ of a matrix $A = \{a_{ij}\}$ is defined by $A^\dagger = \{a^\star_{ji}\}$. Also defined are the conjugate $A^\sharp = \{a^\star_{ij}\}$ and the transpose $A^T = \{a_{ji}\}$ matrices, so that $A^\dagger = \left(A^T\right)^\sharp = \left(A^\sharp\right)^T$. Det$(A)$ denotes the determinant of a matrix A, and Tr(A) represents the trace of A. The mean value (quantum expectation) of an operator X is denoted by $\langle X \rangle$. The commutator of two operators X, Y is defined by $[X, Y] = XY - YX$. The anticommutator of two operators X, Y is defined by $\{X, Y\} = XY + YX$. The tensor product of operators X, Y defined on Hilbert spaces $\mathbb{H}, \mathbb{G}$ is denoted $X \otimes Y$, and is defined on the tensor product Hilbert space $\mathbb{H} \otimes \mathbb{G}$.

II. LINEAR QUANTUM STOCHASTIC SYSTEM MODEL

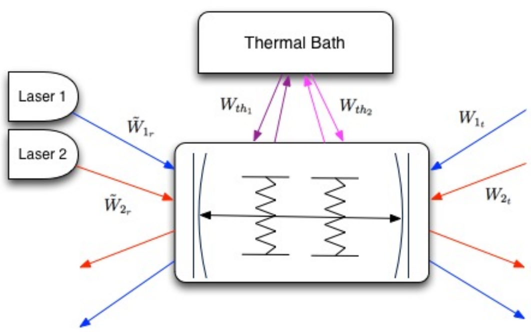

Fig. 2. Two mechanical oscillators are connected to one thermal bath inside an optical cavity. The cavity is driven by two continuous wave (cw) lasers.

As shown in Fig. 2, we consider setups that are composed of two micro-mechanical oscillators inside one optical cavity. Following the notations in [9], we first define

$$\begin{aligned} a_{m_i} &= \frac{q_{m_i} + ip_{m_i}}{2}, \\ a^\star_{m_i} &= \frac{q_{m_i} - ip_{m_i}}{2}, \\ a_{o_j} &= \frac{q_{o_j} + ip_{o_j}}{2}, \\ a^\star_{o_j} &= \frac{q_{o_j} - ip_{o_j}}{2} \end{aligned}$$

where $i, j = 1, 2$. $a_{\{m_1, m_2\}}$ and $a_{\{o_1, o_2\}}$ are the annihilation operators of the mechanical modes and optical modes respectively, with $q_{\{m_1, m_2, o_1, o_2\}}$ being the corresponding position operators and $p_{\{m_1, m_2, o_1, o_2\}}$ being the momentum operators of each individual mode.

The operator X of an open quantum system evolves in the Heisenberg picture as (see [5])

$$dX = \mathcal{L}(X)\, dt + [X, L]\, dW^\star + [L^\star, X]\, dW$$

where the Lindblad operator $\mathcal{L}(X)$ is given by

$$\mathcal{L}(X) = -i[X, H] + \frac{1}{2}\left(L^\star [X, L] + [L^\star, X] L\right).$$

The effective Hamiltonian of this optomechanical system is (see [7], [8])

$$H = \frac{\hbar}{4}\sum_{i=1,2} \omega_i \left(q^2_{m_i} + p^2_{m_i}\right) - \frac{\hbar}{4}\sum_{j=1,2} \Delta_j \left(q^2_{o_j} + p^2_{o_j}\right) + \hbar \sum_{i,j=1,2} g_{ij} q_{m_i} q_{o_j}$$

where the cavity fields are transformed into the frame rotating at the driving frequency $\omega_{L,j}$, and $\Delta_j = \omega_{L,j} - \omega_{0,j}$ denotes the detuning parameter with respect to the resonance frequency of the corresponding cavity mode, $\omega_{0,j}$. The mechanical oscillation frequencies are given by ω_i. The optomechanical interaction is induced by the radiation pressure force that is proportional to the light field intensities, and leads to a coupling rate between the jth optical and the ith mechanical mode. Please note the interaction term in this model is linearised around the optical steady state, and $g_{ij} (i, j = 1, 2)$ denote the optomechanical coupling rates. $\hbar$ is the reduced Planck constant.

In order to apply the (S, L, H) description of a quantum system, we first extract the system Hamiltonian, which is commonly described by the following quadratic form

$$H = \frac{1}{2} x^T N x$$

where

$$N = \begin{bmatrix} \frac{\omega_1}{2} & 0 & 0 & 0 & g_{11} & 0 & g_{12} & 0 \\ 0 & \frac{\omega_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\omega_2}{2} & 0 & g_{21} & 0 & g_{22} & 0 \\ 0 & 0 & 0 & \frac{\omega_2}{2} & 0 & 0 & 0 & 0 \\ g_{11} & 0 & g_{21} & 0 & -\frac{\Delta_1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{\Delta_1}{2} & 0 & 0 \\ g_{12} & 0 & g_{22} & 0 & 0 & 0 & -\frac{\Delta_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\Delta_2}{2} \end{bmatrix},$$

and the vector x comprises the position and momentum operators of both optical and mechanical modes,

$$x = \begin{bmatrix} q_{m_1} & p_{m_1} & q_{m_2} & p_{m_2} & q_{o_1} & p_{o_1} & q_{o_2} & p_{o_2} \end{bmatrix}^T.$$

The coupling operators of this optomechanical system are given by

$$\begin{aligned} L_{m_i} &= \sqrt{\gamma_{m_i}} a_{m_i}, \\ L_{o_j} &= \begin{bmatrix} \sqrt{\gamma_{o_{r_j}}} a_{o_j} \\ \sqrt{\gamma_{o_{t_j}}} a_{o_j} \end{bmatrix} \end{aligned}$$

where γ_{m_i} $(i = 1, 2)$ denotes the mechanical dissipation term. $\gamma_{o_{r_j}}$ and $\gamma_{o_{t_j}}$ $(j = 1, 2)$ are related to the cavity linewidth; see [15].

We collect all the coupling operators in a vector as

$$L = \begin{bmatrix} L_{m_1} & L_{m_2} & L_{o_1}^T & L_{o_2}^T \end{bmatrix}^T$$

which is equivalent to

$$L = \Lambda x$$

where

$$\Lambda = \begin{bmatrix} \frac{\sqrt{\gamma_{m_1}}}{2} & \frac{i\sqrt{\gamma_{m_1}}}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{\gamma_{m_2}}}{2} & \frac{i\sqrt{\gamma_{m_2}}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\gamma_{o_{1_r}}}}{2} & \frac{i\sqrt{\gamma_{o_{1_r}}}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{\gamma_{o_{1_t}}}}{2} & \frac{i\sqrt{\gamma_{o_{1_t}}}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\gamma_{o_{2_r}}}}{2} & \frac{i\sqrt{\gamma_{o_{2_r}}}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{\gamma_{o_{2_t}}}}{2} & \frac{i\sqrt{\gamma_{o_{2_t}}}}{2} \end{bmatrix}.$$

Now we are going to find the linear quantum state space model of this optomechanical system.

First we define $(n \in \mathbb{N})$

$$\begin{aligned} \Theta_{2n} &= I_n \otimes J, \\ \Gamma_{2n} &= P_{2n} I_n \otimes M, \end{aligned}$$

with

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and

$$M = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

I_n denotes the n dimensional identity matrix, and P_{2n} is a $2n \times 2n$ permutation matrix so that if we consider a column vector $a = \begin{bmatrix} a_1 & a_2 & \cdots & a_{2n} \end{bmatrix}^T$,

$$P_{2n} a = \begin{bmatrix} a_1 & a_3 & \cdots & a_{2n-1} & a_2 & a_4 & \cdots & a_{2n} \end{bmatrix}^T$$

with $P_{2n} P_{2n}^T = P_{2n}^T P_{2n} = I_{2n}$.

Given the Hamiltonian and coupling operators of the optomechanical system considered in this paper, we can obtain the system coefficient matrices as follows (see [9]):

$$\begin{aligned} A &= 2\Theta_8 \left(N + \Im \left(\Lambda^\dagger \Lambda \right) \right), \\ B &= 2i\Theta_{12} \begin{bmatrix} -\Lambda^\dagger & \Lambda^T \end{bmatrix} \Gamma_8. \end{aligned}$$

By plugging Λ and N in, we have

$$A = \begin{bmatrix} -\frac{\gamma_{m_1}}{2} & \omega_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\omega_1 & -\frac{\gamma_{m_1}}{2} & 0 & 0 & -2g_{11} & 0 & -2g_{12} & 0 \\ 0 & 0 & -\frac{\gamma_{m_2}}{2} & \omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_2 & -\frac{\gamma_{m_2}}{2} & -2g_{21} & 0 & -2g_{22} & 0 \\ 0 & 0 & 0 & 0 & -\frac{\gamma_{o_{1_r}}+\gamma_{o_{1_t}}}{2} & -\Delta_1 & 0 & 0 \\ -2g_{11} & 0 & -2g_{21} & 0 & \Delta_1 & -\frac{\gamma_{o_{1_r}}+\gamma_{o_{1_t}}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\gamma_{o_{2_r}}+\gamma_{o_{2_t}}}{2} & -\Delta_2 \\ -2g_{12} & 0 & -2g_{22} & 0 & 0 & 0 & \Delta_2 & -\frac{\gamma_{o_{2_r}}+\gamma_{o_{2_t}}}{2} \end{bmatrix},$$

$$B = \begin{bmatrix} -\sqrt{\gamma_{m_1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{\gamma_{m_1}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{\gamma_{m_2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\gamma_{m_2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\gamma_{o_{1_r}}} & 0 & -\sqrt{\gamma_{o_{1_r}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{o_{1_r}}} & 0 & -\sqrt{\gamma_{o_{1_r}}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{o_{2_r}}} & 0 & -\sqrt{\gamma_{o_{2_t}}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\gamma_{o_{2_r}}} & 0 & -\sqrt{\gamma_{o_{2_t}}} \end{bmatrix}.$$

And if we choose $D = I_{12}$, then

$$C = -B^T.$$

Therefore, the dynamics of two mechanical oscillators inside a two-mode optical cavity can be captured by linear

QSDEs as follows

$$\begin{aligned} dx &= Axdt + Bd\tilde{w} \\ dy &= Cxdt + Dd\tilde{w}. \end{aligned} \tag{1}$$

with A, B, C, D given above. The quantum noise input is

$$\tilde{w} = \left[\begin{array}{ccc} W_{th_1} + W_{th_1}^\star & \frac{W_{th_1} - W_{th_1}^\star}{i} & W_{th_2} + W_{th_2}^\star \\ \frac{W_{th_2} - W_{th_2}^\star}{i} & \tilde{W}_{1_r} + \tilde{W}_{1_r}^\star & \frac{\tilde{W}_{1_r} - \tilde{W}_{1_r}^\star}{i} \\ W_{1_t} + W_{1_t}^\star & \frac{W_{1_t} - W_{1_t}^\star}{i} & \tilde{W}_{2_r} + \tilde{W}_{2_r}^\star \\ \frac{\tilde{W}_{2_r} - \tilde{W}_{2_r}^\star}{i} & W_{2_t} + W_{2_t}^\star & \frac{W_{2_t} - W_{2_t}^\star}{i} \end{array} \right]^T.$$

The two mechanical oscillators are connected to a thermal bath, and W_{th_1} and W_{th_2} denote the thermal noise inputs to each oscillators respectively. Similarly, $\tilde{W}_{\{1_r,2_r\}}$ and $W_{\{1_t,2_t\}}$ represent the quantum noise inputs to the optical cavity from electromagnetic fields, coupled to each optical mode respectively. Note that $\tilde{W}_{1_r}$ and $\tilde{W}_{2_r}$ are coherent inputs, that is,

$$\begin{aligned} d\tilde{W}_{1_r} &= |\alpha_1| \exp(i\theta_1)\, dt + dW_{1_r} \\ d\tilde{W}_{2_r} &= |\alpha_2| \exp(i\theta_2)\, dt + dW_{2_r} \end{aligned}$$

where W_{1_r}, W_{1_t}, W_{2_r} and W_{2_t} denote vacuum inputs. $|\alpha_1| \exp(i\theta_1)$ and $|\alpha_2| \exp(i\theta_2)$ are the complex amplitudes corresponding to two different coherent fields, with $\left|\alpha_{\{1,2\}}\right|$ and $\theta_{\{1,2\}}$ being the classical amplitudes and phases respectively.

We can rewrite (1) as

$$\begin{aligned} dx &= (Ax + G_0 u_0)\, dt + Bdw \\ dy &= (Cx + T_0 u_0)\, dt + dw. \end{aligned} \tag{2}$$

where

$$u_0 = \left[\begin{array}{cc} 2|\alpha_1| \cos\theta_1 & 2|\alpha_1| \sin\theta_1 \\ 2|\alpha_2| \cos\theta_2 & 2|\alpha_2| \sin\theta_2 \end{array} \right]^T,$$

and

$$w = \left[\begin{array}{ccc} W_{th_1} + W_{th_1}^\star & \frac{W_{th_1} - W_{th_1}^\star}{i} & W_{th_2} + W_{th_2}^\star \\ \frac{W_{th_2} - W_{th_2}^\star}{i} & W_{1_r} + W_{1_r}^\star & \frac{W_{1_r} - W_{1_r}^\star}{i} \\ W_{1_t} + W_{1_t}^\star & \frac{W_{1_t} - W_{1_t}^\star}{i} & W_{2_r} + W_{2_r}^\star \\ \frac{W_{2_r} - W_{2_r}^\star}{i} & W_{2_t} + W_{2_t}^\star & \frac{W_{2_t} - W_{2_t}^\star}{i} \end{array} \right]^T.$$

Also, here

$$G_0 = BT_0$$

with

$$T_0 = \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right]^T.$$

Table II shows quantum Itô terms for vacuum noise and thermal noise. k_n is the mean occupation number of the thermal phonons which is associated with the temperature of the system.

TABLE I

QUANTUM ITÔ TABLE FOR NOISE INPUTS.

dX/dY	$dW_{\{1_r,1_t,2_r,2_t\}}$	$dW^\star_{\{1_r,1_t,2_r,2_t\}}$
$dW_{\{1_r,1_t,2_r,2_t\}}$	0	dt
$dW^\star_{\{1_r,1_t,2_r,2_t\}}$	0	0

dX/dY	$dW_{th_{\{1,2\}}}$	$dW^\star_{th_{\{1,2\}}}$
$dW_{th_{\{1,2\}}}$	0	$(1+k_n)\, dt$
$dW^\star_{th_{\{1,2\}}}$	$k_n dt$	0

III. SYNCHRONISATION OF MECHANICAL OSCILLATORS

Now, we formulate the LQG control problem for the purpose of synchronising the two mechanical oscillators using feedback control.

A metric which gauges the level of quantum complete synchronisation is given in [12] as follows

$$S_c(t) = \frac{4}{\left\langle q_-^2(t) + p_-^2(t) \right\rangle}.$$

Here

$$\begin{aligned} q_-(t) &= q_{m_1}(t) - q_{m_2}(t) \\ p_-(t) &= p_{m_1}(t) - p_{m_2}(t). \end{aligned}$$

Note that

$$\begin{aligned} \left\langle q_-^2(t) \right\rangle \left\langle p_-^2(t) \right\rangle &\geqslant \left| \left\langle q_-(t), p_-(t) \right\rangle \right|^2 \\ &\geqslant \left| \mathrm{Tr}\left(\rho \frac{[q_-(t), p_-(t)]}{2i} \right) \right|^2 \\ &= 4, \end{aligned}$$

and therefore, we have

$$\frac{4}{S_c(t)} \geqslant 2\sqrt{\left\langle q_-^2(t) \right\rangle \left\langle p_-^2(t) \right\rangle} \geqslant 4,$$

namely

$$S_c(t) \in (0, 1]. \tag{3}$$

The performance index in this LQG control problem is

$$S_c = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} S_c(t)\, dt \tag{4}$$

which defines the level of synchronisation of the two mechanical modes at steady states. In addition, the more closer S_c is to 1, the more synchronised the two mechanical oscillators are.

A. CONTROLLER DESIGN

As shown in Fig. 3, now we include the control input to the optomechanical plant, with making a homodyne measurement of the $dW_{1_t} + dW^\star_{1_t}$ field (transmissive light), then the plant model becomes

$$\begin{aligned} dx &= (Ax + G_0 u_0)\, dt + Bdw + Gudt, \\ dy_m &= C_m x dt + D_m dw \end{aligned} \tag{5}$$

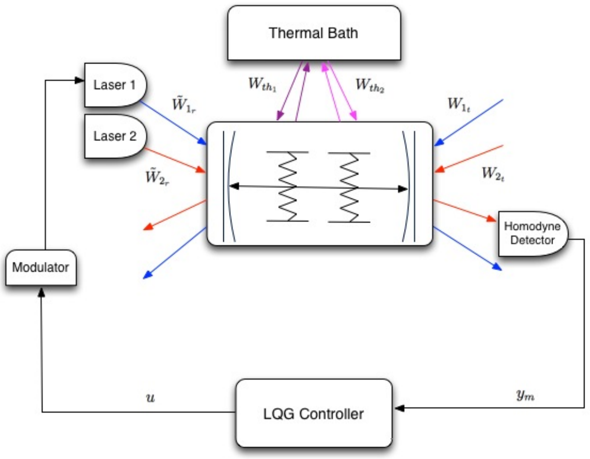

Fig. 3. The composite system comprise the optomechanical plant and a measurement-based LQG controller.

where

$$D_m = \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

and

$$C_m = D_m C.$$

The measurement-based LQG controller is given by

$$\begin{aligned} d\hat{x} &= A\hat{x}dt + K\left(dy_m - C_m\hat{x}dt\right) + \left(G_0u_0 + Gu\right)dt, \\ u &= L_1\hat{x} + L_2u_0 \end{aligned} \tag{6}$$

where

$$G = BT_u$$

with

$$T_u = \left[\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]^T.$$

K is the steady-state Kalman gain given by (see [4])

$$K = P_eC_m^T + BD_m^T$$

where P_e is the steady-state solution to the following Riccati equation

$$\begin{aligned} &\left(A - BD_m^TC_m\right)P_e + P_e\left(A - BD_m^TC_m\right)^T \\ &- P_eC_m^TC_mP_e + B\left(S_{\tilde{w}} - D_m^TD_m\right)B^T = 0. \end{aligned}$$

Here

$$S_w = \frac{dwdw^T + \left(dwdw^T\right)^T}{2dt} = \left(1 + 2k_n\right)I_4 \oplus I_8.$$

In this LQG control problem, the quadratic cost is defined as

$$J\left(x,t\right) = \left\langle \lim_{t_f\to\infty}\frac{1}{t_f}\int_0^{t_f}\left(x^TQx + u^TRu\right)dt\right\rangle,$$

where $Q = C_z^TC_z$ with

$$C_z = \left[\begin{array}{cccccccc} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right],$$

and $R > 0$.

The optimal feedback gains, L_1 and L_2, are given by (see [1])

$$\begin{aligned} L_1 &= -R^{-1}G^TS_1, \\ L_2 &= -R^{-1}G^TS_2 \end{aligned}$$

where S_1 satisfies the following Riccati equation

$$S_1A + A^TS_1 + Q - S_1GR^{-1}G^TS_1 = 0,$$

and S_2 is determined by

$$S_2 = -\left(A^T - S_1GR^{-1}G^T\right)^{-1}S_1G_0.$$

By turning on measurement-based LQG feedback control, we get the closed-loop system model as follows

$$d\eta\left(t\right) = \mathcal{A}\eta\left(t\right)dt + \mathcal{B}dw\left(t\right)$$

where

$$\eta\left(t\right) = \left[\begin{array}{c} x\left(t\right) \\ \hat{x}\left(t\right) \\ u_0 \end{array}\right],$$

and

$$\mathcal{A} = \left[\begin{array}{cc} A_{11} & A_{12} \\ 0_{4\times 8} & 0_{4\times 4} \end{array}\right]$$

$$\mathcal{B} = \left[\begin{array}{c} B \\ KD_m \\ 0_{4\times 12} \end{array}\right].$$

Here

$$A_{11} = \left[\begin{array}{cc} A & GL_1 \\ KC_m & A - KC_m + GL_1 \end{array}\right]$$

$$A_{12} = \left[\begin{array}{c} G_0 + GL_2 \\ G_0 + GL_2 \end{array}\right].$$

The covariance matrix $\tilde{P}\left(t\right)$ is defined as

$$\tilde{P}\left(t\right) = \frac{1}{2}\left\langle \eta\left(t\right)\eta^T\left(t\right) + \left(\eta\left(t\right)\eta^T\left(t\right)\right)^T\right\rangle$$

which satisfies the following differential equation

$$\dot{\tilde{P}}\left(t\right) = \mathcal{A}\tilde{P}\left(t\right) + \tilde{P}\left(t\right)\mathcal{A}^T + \mathcal{B}S_w\mathcal{B}^T,$$

with $\tilde{P} = \lim_{t\to\infty}\tilde{P}\left(t\right)$ satisfying

$$\mathcal{A}\tilde{P} + \tilde{P}\mathcal{A}^T + \mathcal{B}S_w\mathcal{B}^T = 0.$$

If we define P as

$$P = \frac{1}{2}\left\langle\left(\left[\begin{array}{c} x \\ \hat{x} \end{array}\right]\left[\begin{array}{c} x \\ \hat{x} \end{array}\right]^T\right) + \left(\left[\begin{array}{c} x \\ \hat{x} \end{array}\right]\left[\begin{array}{c} x \\ \hat{x} \end{array}\right]^T\right)^T\right\rangle,$$

then $\tilde{P}$ can be partitioned into 4 blocks as follows

$$\tilde{P} = \left[\begin{array}{cc} P & P_{12} \\ P_{12}^T & P_{22} \end{array}\right],$$

where

$$\begin{aligned} &A_{11}P + PA_{11}^T + A_{12}P_{12}^T + P_{12}A_{12}^T \\ &+ \left(\begin{array}{c} B \\ KD_m \end{array}\right)S_w\left(\begin{array}{c} B \\ KD_m \end{array}\right)^T = 0, \end{aligned}$$

TABLE II

PARAMETERS FOR THE OPTOMECHANICAL PLANT.

Parameter	Value
Natural frequency of oscillator 1 ω_1	πMHz
Natural frequency of oscillator 2 ω_1	2πMHz
Mean number of the thermal phonons k_n	$0.01 \sim 10$
Mechanical dissipation terms $\gamma_{m_1} = \gamma_{m_2} = \gamma_m$	$0.001 \sim 0.1$MHz
Optomechanical coupling terms $g_{11} = g_{12} = g_1$	0.1MHz
Optomechanical coupling terms $g_{21} = g_{22} = g_2$	$0.01 \sim 1$MHz
Laser detunings $\triangle_1 = -\triangle_2 = \triangle$	1.5πMHz
Cavity decay rates $\gamma_{o_{1_r}} = \gamma_{o_{2_r}} = \gamma_{o_r}$	1MHz
Cavity decay rates $\gamma_{o_{1_t}} = \gamma_{o_{2_t}} = \gamma_{o_t}$	1MHz
Initial complex amplitude of laser 1 $\|\alpha_2\| e^{i\theta_2}$	$10^2 \sim 10^4 e^{i\frac{\pi}{4}}$
Initial complex amplitude of laser 2 $\|\alpha_2\| e^{i\theta_2}$	$10^2 \sim 10^4 e^{i\frac{\pi}{4}}$

$$P_{12} = -A_{11}^{-1} A_{12} P_{22},$$
$$P_{22} = u_0 u_0^T.$$

Therefore the performance index is given by

$$S_c = \frac{4}{\mathrm{Tr}\left(\mathcal{C}_z P \mathcal{C}_z^T\right)}$$

where

$$\mathcal{C}_z = \left[\begin{array}{cc} C_z & 0_{2\times 8} \end{array}\right].$$

With the control signal $u = \left[\begin{array}{cc} u_1 & u_2 \end{array}\right]^T$ fed back to the plant, the amplitude of the coherent field corresponding to the first laser becomes

$$\left|\alpha_1^{'}\right| = \frac{\sqrt{\left(u_1 + 2\left|\alpha_1\right|\cos\theta_1\right)^2 + \left(u_2 + 2\left|\alpha_1\right|\sin\theta_1\right)^2}}{2},$$

and the phase becomes

$$\theta_1^{'} = \arctan\left(\frac{u_2 + 2\left|\alpha_1\right|\sin\theta_1}{u_1 + 2\left|\alpha_1\right|\cos\theta_1}\right).$$

Therefore, in order to implement this control method, we need to adjust the coherent field complex amplitude accordingly, which is equivalent to tuning the amplitude and phase of the first laser.

B. SIMULATION RESULTS

In the following simulation, we refer to the physical values in [7], [13], and solve LQG control problems with the assistance of Matlab; see [6], [10].

Example 1: Assume the experiment is conducted in the unresolved sideband regime where the mechanical frequency is comparable or smaller than the optical cavity linewidth. The interaction between the motion of the two mechanical oscillators is mediated by a quantised light field in a laser driven high-finesse cavity. Also, S_c^{on} denotes the performance index (see (4)) with optimal feedback control, while S_c^{off} is used when feedback control is turned off at steady states. Table II shows the values of system parameters we use.

(i) $k_n = 0.01 \sim 10; \gamma_m = 0.01MHz; g_1 = g_2 = 0.1$MHz.

As shown in Figure. 4, both S_c^{on} (solid line) and S_c^{off} (dashed line) become smaller as the temperature of the system increases (k_n is proportional to the temperature of the optomechanical system). When the temperature grows

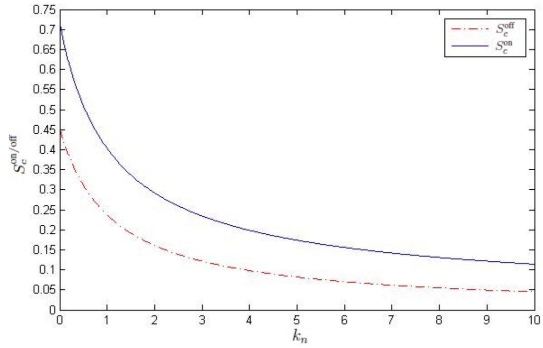

Fig. 4. Plot of the quantum complete synchronisation metric $S_c^{\mathrm{on/off}}$ as a function of k_n.

higher, the thermal force gets stronger, and the influence of thermal and quantum noise renders S_c^{on} (S_c^{off}) smaller. By turning on the optical feedback control, we can see that the quantum complete synchronisation metric turns out to be closer to 1 at steady states. This indicates that the controller works well at synchronising the two mechanical oscillators regardless of the temperature.

(ii) $k_n = 0.01, 1; \gamma_m = 0.001 \sim 0.1MHz; g_1 = g_2 =$ 0.1MHz.

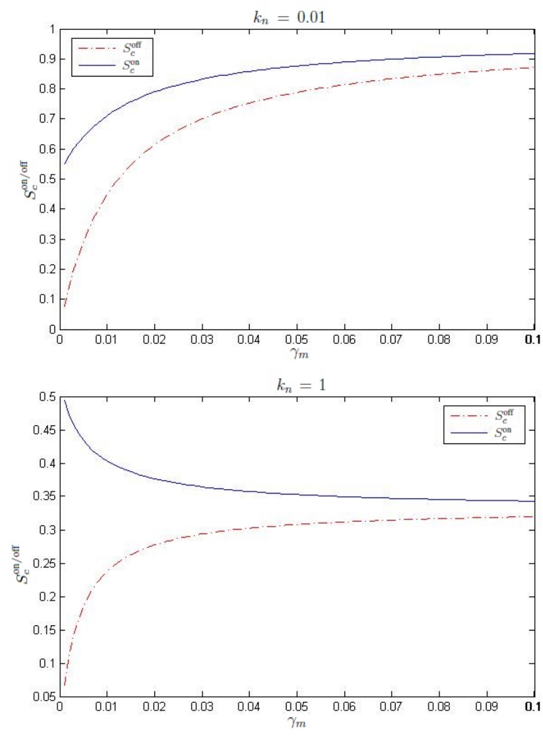

Fig. 5. Plot of the quantum complete synchronisation metric $S_c^{\mathrm{on/off}}$ as a function of γ_m (MHz).

Fig. 5 depicts how mechanical dissipation terms influence the synchronisation effect in quantum and classical regimes. As γ_m increases, S_c^{off} (dashed line) grows gradually in both quantum and classical regimes, while S_c^{on} (solid line) goes up when k_n is very small but goes down when k_n is big.

Furthermore, in the classical regime, the synchronisation effect with optimal feedback control becomes less salient as γ_m varies from 0.001MHz to 0.1MHz. This is because though we tune the intensity of the laser guided by the feedback control scheme and the radiation pressure force changes accordingly, the thermal force appears to impair the mutual interaction between the two micro-mechanical oscillators when k_n is big.

(iii) $k_n = 0.01, 1; \gamma_m = 0.01\text{MHz}; g_1 = 0.1\text{MHz}; g_2 = 0.01 \sim 1\text{MHz}$.

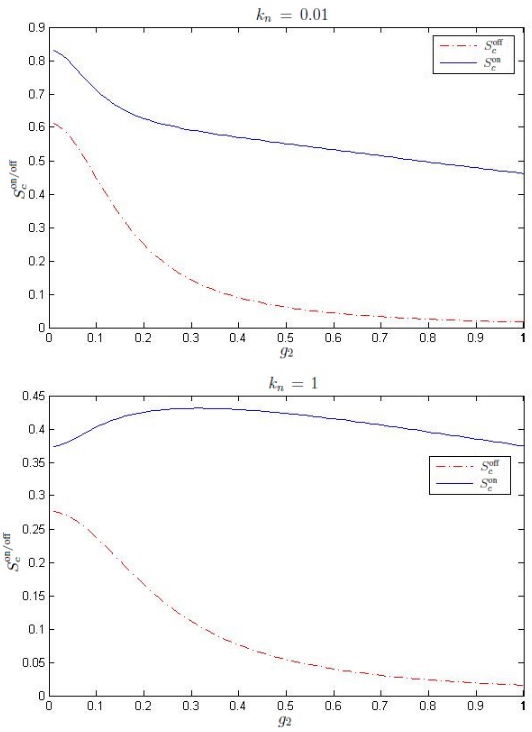

Fig. 6. Plot of the quantum complete synchronisation metric $S_c^{\text{on/off}}$ as a function of g_2 (MHz).

Fig. 6 shows how optomechanical coupling strength affects the synchronisation effect in quantum and classical regimes. In the quantum regime ($k_n = 0.01$), it can be seen that the level of quantum complete synchronisation goes down as g_2 increases attributed to the powerful radiation pressure force. We feed the optimal control input back to the quantum plant by tuning the first laser, and we can observe that as g_2 rises the control effect on synchronisation becomes more significant. However, in the classical regime ($k_n = 1$), S_c^{on} (solid line) achieves a local maximum when g_2 is around ≈ 0.3MHz, that is, the measurement-based controller works best if $g_2 \approx 0.3$MHz in this case.

IV. CONCLUSIONS

We have obtained the linear stochastic state space model of the quantum plant, based on which a measurement-based LQG controller is designed to synchronise the two mechanical oscillators inside a cavity using feedback control. The purpose is to implement the proposed control scheme to this practical optomechanical system. Furthermore, simulation results illustrate how to adjust parameters in the original system setup in order to achieve prominent synchronisation effects. This research allows us to think about controlled (synchronised) network of oscillators that can be used for memory and quantum state transfer. Future work includes designing coherent controllers and conducting experiments accordingly. Results of the experiments will be reported later.

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