Australian National University

Nonlinear wave patterns in the complex KdV and nonlinear Schrödinger equations

A THESIS SUBMITTED
BY

MATTHEW CRABB

TO
THE RESEARCH SCHOOL OF PHYSICS
COLLEGE OF SCIENCE

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA, AUSTRALIAN CAPITAL TERRITORY
OCTOBER 2021

Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

Matthew Crabb
26th October 2021

Acknowledgments

This thesis was done with the supervision of Professor Nail Akhmediev. I would like to thank Nail for his assistance throughout my PhD candidature. Nail was especially helpful in helping me to write the very first manuscripts, which were in many ways the most difficult to write. Nail's assistance made it possible for me to figure out how to start putting together manuscripts that were actually presentable to a journal. The broad ideas of much of the rest of this thesis were developed and discussed in conversations with Nail. Aside from his assistance with preparing and writing papers, and discussions on the direction of research in general, I have also enjoyed many of our other conversations, so the other half of my gratitude to him goes not to specific assistance in research, but to him as a person.

I also enjoyed my time with many other members of the Department of Theoretical Physics. Peter Vouzas and Mahyar Bokaeeyan were two of the first people I met, and I have thoroughly enjoyed getting to know them both. I am also grateful to Natasha Devine, for her friendly, generous, and encouraging nature, and Peter Vassiliou, for being both a pleasure to talk to and very helpful with mathematical references and suggestions that deepened my understanding of integrable systems. The ideas of the integrable extension of the nonlinear Schrödinger equation were initially discussed with Adrian Ankiewicz. I am also glad to have met many of the other PhD candidates, and I also want to thank Karen, Luda, and Gaye for the help I received managing (and making sense of) administrative things.

People I have lived with have also played a significant part in getting this thesis finished: I would especially like to thank Mohammed Afzal Hussein and Rikiya Hagihira.

Finally, a very special thank you to Qing Huang, and to my mum. At this point, to describe the thankfulness I want to express only seems to make it seem smaller than it really is, because words can only do so much. So all that I have left to say is a simple but sincere thank you.

Publications

- (Chapter 2) M. Crabb and N. Akhmediev, Complex Korteweg-de Vries equation: A deeper theory of shallow water waves, *Physical Review E* **103**, 022216 (2021)
- (Chapter 3) M. Crabb and N. Akhmediev, Complex Korteweg-de Vries equation with a complex spatial variable in the theory of shallow water waves: Conservation laws, and relation to the split NLS equations, submitted to *Physical Review E* on October 9th, 2021, Manuscript no. EK12070
- (Chapter 4) M. Crabb and N. Akhmediev, Rogue wave multiplets in the complex Korteweg-de Vries equation *Romanian Reports in Physics*, **72**, 118 (2020)
- (Chapter 5) M. Crabb and N. Akhmediev, Doubly-periodic solutions of the class-I infinitely extended nonlinear Schrödinger equation *Physical Review E*, **99**, 052217 (2019)
- (Chapter 6) M. Crabb and N. Akhmediev, Two-breather solutions for the class-I infinitely extended nonlinear Schrödinger equation and their special cases *Nonlinear Dynamics*, **98**: 245–255 (2019)

Contents

1
Introduction

1.1 Historical origins

It is said that there were two revolutions in 20th-century physics: quantum theory, and relativity. But in a sense there was a third, less-publicised one: the revolution in nonlinear physics.

To a large extent, the roots of this subject which led to its flourishing in the last century go back to the mid to late 19th-century study of water waves. Pioneering studies in nonlinear water waves were carried out by Stokes [1], then Boussinesq [2, 3], and Korteweg and de Vries [4]. Boussinesq, as well as Korteweg and de Vries, were in part motivated by Scott Russell's observations of the solitary wave [5], which was a wave which had important features which marked it as something that could not be explained by the linear theories of the time.

The solitary wave was able to propagate long distances with neither a change in shape nor a change of its momentum. According to the predictions of the linear theory, such a wave should have decayed away, or become so steep that it became unstable and collapsed [6]. Most striking of all was the fact that these solitary waves were also able to collide with each other and emerge asymptotically unchanged, and a solitary wave with a height too large to propagate in the fluid might also break up into several smaller ones, which all shared these same characteristic properties. These properties were more akin to those of particles than those of a typical wave. These particle-like properties later motivated the name: 'soliton.'

The key difference between the linear and nonlinear descriptions of wave propagation is that the presence of nonlinear effects allows for the possibility of balancing the dispersive forces. If the nonlinear effects produce focusing forces which exactly balance the dispersive forces, then the wave will be able to preserve its shape as it moves, without any loss in momentum. When nonlinear effects are accounted for in the hydrodynamic theory, it was shown that solitons and nonlinear periodic (cnoidal) waves could collide and be reflected and emerge with the same shape [7]; that is, they were only temporarily altered in shape near an interaction.

The striking observation of the soliton makes it very easy to focus on the hydrodynamic origins of the field. We will return to this topic in a moment, but the development of nonlinear physics is really a much bigger story. In the late 19th century, the theory of nonlinear dynamics was also being developed by Henri Poincaré in the realm of celestial mechanics [8–10]. This work gave many of the techniques used in the analysis of nonlinear systems, such as the modern theory of stability for fixed points and orbits (or trajectories) in phase space. Of particular note was the notion of a homoclinic orbit, which is a trajectory connecting a saddle point to itself. This causes the orbit of the system in phase space to have a high degree of sensitivity to initial conditions, and in some cases may lead to chaotic motion.

In the early-to-mid 20th century, there was also a burgeoning interest in nonlinear oscillations. A spring is, after all, only linear so long as certain approximations remain valid; most importantly, that the displacement be small. Georg Duffing [11] investigated a model of the damped and driven oscillator with a cubic nonlinearity. Unlike the simple harmonic oscillator, the frequency of the nonlinear oscillator depends on its amplitude, and when the frequency of a driving force is varied, the Duffing oscillator also exhibits hysteresis. Nonlinear oscillations were of great interest in electrical circuits. Van der Pol considered a non-conservative nonlinear circuit which exhibited relaxation oscillations when the nonlinearity was sufficiently large [12]. These were oscillations which behave almost like sawtooth functions, where the system exhibits a sudden jump. Van der Pol and van der Mark [13] observed an interval of deterministic chaos which occurs as the circuit jumps from one frequency to another. This was an early observation of deterministic chaos decades before the famous work of Lorenz [14] in 1963. Nonlinear dynamics were also applied to the biological sciences, such as the van der Pol model of the heartbeat [15], and the Lotka-Volterra models of predator-prey populations [16].

It took until the 1960s before nonlinear physics took on its modern form. Despite all this interest in nonlinear dynamics, and despite the fact that the Boussinesq equations, and the Korteweg-de Vries (KdV) equations were respectively discovered in 1871 and 1877 by Boussinesq, the latter independently rediscovered by Korteweg and de Vries in 1895, and despite both being immediately found to provide accurate descriptions of the solitary wave in shallow water, both remained undeveloped for a long time. Their full significance was not discovered until the numerical work of Zabusky and Kruskal [17] on the problem of FPU recurrence. The FPU problem was the numerical simulation, carried out by Fermi, Pasta, and Ulam, of the oscillations of a system of point masses and springs with a nonlinear force law $F(x) = -kx(1 + \delta x)$ [18] where k is the spring constant, x the displacement between two masses, and δ the nonlinear coefficient. As initial conditions, the n masses, labelled by the index $r = 1, \ldots, n-1$, were chosen to be placed at points $\sin r\pi/n$, while their initial momenta were set to zero.

It was expected that there would be no reason that one normal mode would be preferred over any other, so the system should tend towards thermal equilibrium, with all normal modes eventually having an equal share of the total energy. Instead, the system was observed to return back to the initial state. This phenomenon became known as FPU recurrence, and the apparent contradiction with what should be physically expected known as the FPU paradox. Zabusky and Kruskal were intrigued by this phenomenon, and due to the advances in computing power, were able to do numerical experiments with continuous models rather than with a chain of a small number of nonlinear oscillators. They investigated the KdV equation, and they found that when the KdV equation was given the initial condition of a sinusoidal pulse, the wave would break up and split into a train of solitons with various amplitudes and velocities. These solitons would collide elastically, and retain their shape after interacting, so that when considered with periodic boundary conditions, the KdV system exhibited a near return to the initial state when eventually the solitons would return to the same point in space and approximately reform the initial pulse, except with the solitons having a small change in phase. This recurrence to the initial condition was qualitatively similar to the recurrence in the FPU problem. After twice the initial recurrence time, the recurrence was less close to the initial state, and the third recurrence was even further away. This gave some insight into how the interplay between dispersion and nonlinear focusing effects were able to reproduce the near-recurrence phenomenon in certain physical systems.

It should be mentioned that it is actually true that the FPU system will also eventually arrive at thermal equilibrium, given a long enough time [19], so there was really no violation of the equipartition theorem to begin with; it only seemed this way because of the computational time limits. Nevertheless, this apparent paradox marks the beginning of modern soliton research.

Shortly thereafter, Gardner, Kruskal, Greene, and Miura published a series of papers [20–26], after which the theory of the KdV equation exploded. They demonstrated a method of solution given certain initial data using the technique of inverse scattering, and showed that the KdV equation had infinitely many conservation laws. Peter Lax then proved that the KdV equation was actually an example of a class of nonlinear equations which could be treated as equivalent to the integrability of a linear system, and showed, using what is now known as a Lax pair, how the system's infinity of conservation laws emerges from its underlying structure [27].

The nonlinear Schrödinger (NLS) equation was the next major soliton equation to be studied in detail. In the mid 1960s, it appeared as an equation which described the self-focusing of wave envelopes in nonlinear media. In the context of water waves, the nonlinear Schrödinger equation was given explicitly by Zakharov [28] to answer the question of whether the Stokes wave was stable, and was shown by Zakharov and Shabat [29] to be solvable by inverse scattering, to have a Lax pair, and admit soliton solutions, like the KdV equation.

It had been shown only slightly earlier by Benjamin and Feir that the Stokes wave had a particular kind of instability, now known as modulation instability, which meant that a small, low-frequency modulation applied to the carrier wave would grow at an exponential rate [30]. This could then lead to the wavetrain breaking up. The NLS equation was also shown to describe this effect.

Modulation instability is the linear stage of a more interesting nonlinear phenomenon. Exponential growth of the initial perturbation cannot continue indefinitely. Saturation must eventually be reached. When the maximum amplitude is reached, the modulation will begin to decay. The growth and decay processes are symmetric, which is not something which is immediately obvious. As a result, the modulation will decay and the wave eventually returns to its constant-amplitude initial state, up to a change in phase. In other words, modulation instability is the first stage in the growth-decay cycle of a periodic perturbation of a wavetrain.

This means that modulation instability can manifest as what is known as a breather, which is localised in the evolution variable, but periodic in the transverse dimension. Breathers generated by modulation instability were theoretically predicted in the NLS equation by Akhmediev, Eleonskii, and Kulagin [31], and Akhmediev and Korneev [32]. These breathers, also called Akhmediev breathers, are part of a family of solutions which depend on a variable parameter determined by the frequency of the perturbation. This parameter determines the rate of the growth-decay cycle of these breathers.

Because a constant amplitude wave is unstable as a solution of the NLS equation, it represents a saddle point in the phase space of the NLS system. Modulation instability provides a linear approximation of a trajectory through this saddle point. The orbit returns to a saddle point which is equivalent to its starting point up to a phase shift. The Akhmediev breather can be thought of as giving the analytic form of this trajectory. Because the saddle point the trajectory goes to is not exactly the same as the one it started from, this is generally a heteroclinic orbit, but will be a homoclinic orbit in the limit as the modulation frequency goes to zero.

The spectral evolution of an Akhmediev breather also shows the growth-decay cycle [32]: it begins with a single fundamental frequency, which is the frequency of the background. A small modulation adds two sidebands each side of the fundamental, and further evolution then adds an infinite number of equally spaced sidebands on both sides. This spectral widening occurs due to the four-wave mixing process, which, as already mentioned, is described by the NLS equation. After maximal spreading of the spectrum, it returns back to the initial state of the original fundamental frequency. The evolution of the frequency components thus shows the same pattern as the evolution of the normal modes in the FPU problem, and the solution of the NLS equation in the form of an Akhmediev breather provides theoretical background for the explanation of FPU recurrence.

A soliton on a constant background can be considered another kind of NLS breather. These were found first by Kuznetsov [33], and studied in detail by Ma [34]. These are known as Kuznetsov-Ma solitons, also sometimes called Kuznetsov-Ma breathers [35]. Periodicity in this case is caused by the beating between the soliton and the constant background, which have different propagation constants. These breathers also form a one-parameter family of solutions, with the soliton amplitude being the free parameter.

These breathers are also particular cases of a more general two-parameter family of solutions, which are periodic in both space and time [36]. The two periods of this solution are both free parameters. When one of the periods tends to infinity, the doubly-periodic solution tends to a limit in the form of an Akhmediev breather, or a Kuznetsov-Ma soliton. On the other hand, when the periods both become infinitely large, only one peak remains. This will be a large, sudden, isolated peak which grows from a calm background.

These peaks are known as rogue waves. Their defining characteristic is that they are waves which ‘appear from nowhere and disappear without a trace’ [37]. Put another way, they are highly localised in both space and time. Rogue waves were the next major discovery in the quickly growing field of nonlinear physics. The simplest solution of the NLS equation localised in both space and time, in which the wave amplitude is given as a rational function, was given by Peregrine [38], and its relevance to rogue wave formation is now well-known [37]. The Peregrine rogue wave is also equivalent to the limiting case of the Akhmediev breather as the frequency goes to zero. This gave strong evidence that modulation instability could actually be responsible for generating a rogue wave from a plane wave background.

It is critical to note that this type of rogue wave formation is a completely distinct mechanism to that of the sudden appearance of large water waves caused by focusing effects such as obstacles or the geometry of the seabed. Even though the sudden formation of very large water waves is known to be possible in the linear theory [39], these explanations rely on additional effects also being present, rather than solely on the inherent properties of the fluid itself. Geometrical focusing effects, for instance, may produce a very large wave, apparently without warning [40]. However, it is now known that the linear theory cannot explain the appearance of rogue waves in general, because rogue waves are also readily generated from modulation instability in the ideal conditions provided by water tank experiments. This is something that cannot happen in a linear theory of water waves. It is therefore clear that these two ideas of rogue waves refer to very different physical processes. While the linear theory can explain the appearance of some large, sudden waves, it does not explain how they may be generated by a uniquely nonlinear mechanism. That said, it is worth noting that models do exist which incorporate both nonlinear focusing due to the water itself [41], as well as extrinsic factors, such as wind forcing, and these have shown that both effects may combine to enhance the effects of modulational instability [42, 43]. In this thesis, by a rogue wave we will mean only the nonlinear phenomenon.

The NLS equation was also found to be applicable to optical fibres [44, 45]. It was shown that a wave propagating through a dielectric fibre could form solitons if the quadratic dependence of the index of refraction on the electric field intensity was taken into account in a special optical frequency range. The effect of the nonlinear refractive index on a slowly-varying wave envelope can act to balance the group velocity dispersion. This enables the wave packet to form a stable coherent structure. Anomalous dispersion led to bright solitons, which form on zero background, characterised by a local increase in field intensity, whereas normal dispersion led to dark solitons, which form on non-zero background, characterised by a local decrease in field intensity.

The quadratic dependence of a medium's refractive index on the electric field, known as the optical Kerr effect, is mainly due to the medium having an optical susceptibility, or polarisation response function, which is cubic as a function of the field. However, the action of the second-order nonlinear optical susceptibility can also contribute to the quadratic term of the refractive index. This is known as cascaded nonlinearity [46].

Nonlinear optical susceptibilities lead to the generation of waves with different frequencies to the frequency of the applied field when the intensity of the field is sufficiently high. The NLS equation is one of the equations that describes these frequency transformation phenomena; in this case, it is four-wave mixing which results in the modulation instability of the wave envelope. In optics, modulation instability is also known as Bespalov-Talanov instability [47].

This was a big step forward with important practical consequences for experiments and empirical studies of nonlinear phenomena, as we will discuss.

1.2 Experiments and observations

Although this work is theoretical, it is important to recognise that the development of nonlinear physics is also significantly driven and led by empirical observations and experiment. This has been true ever since the first recorded observation of a soliton.

Nowadays, there is a large body of experimental data on the observation and reproduction of nonlinear phenomena in controlled conditions. It is now possible to generate solitons, breathers, and rogue waves easily, safely, and to put them to good use.

In water tank experiments [48, 49], rogue waves have been successfully generated, and their features have been found to agree well with theoretical predictions. In 2014, experimentalists were able to successfully generate breathers in water waves [50], and in later experiments, breathers were also successfully generated even in the wind-forced regime [51]. Water tank experiments have also successfully recreated solitons in approximately the same way in which they were first seen in the FPU simulations [52], providing additional, experimental justification for the resolution of the problem which led to the development of modern soliton theory.

Rogue waves had actually been part of nautical lore for a very long time prior to the formal observation of them striking gas platform in the Norwegian North Sea [53]. Sailors would tell stories of ships being damaged or sunk by enormous waves which appeared without any warning. A significant part of why these observations were not always taken seriously may have been the lack of place for rogue waves in the underdeveloped linear theories of the time. In fact, some physicists, such as François Arago, were so confident that they could not exist that they even ridiculed accounts of sailors who claimed to see waves over 30 metres tall [54]. By contrast, in modern times there is a wealth of empirical and theoretical knowledge on rogue waves in the ocean [55]. Seamen used to speak of rogue waves striking ships in threes, and even this has gone from ridiculed folklore to being replicated in wave tanks [56]. As the strength of the nonlinear description of water waves has become better known, there has been recognised a need to build ships to withstand a possible strike from rogue waves, and to develop rogue wave prediction mechanisms [57].

That said, rogue waves are not something which we would usually like to encounter in the ocean. More to the point, we would like some easier way of generating them than in water. This is one reason why the fact that equations like the NLS equation describing both hydrodynamic situations as well as optical ones is so important. This correspondence has enabled nonlinear phenomena to be studied far more easily with a laser than they could have been studied otherwise. To give an example of this, breathers were actually first observed not in water, but in optical fibres [58–60]. Rogue waves are now very well-studied in optical environments, where they can appear in supercontinuum generation [61–63].

One special advantage of the optical exploration of rogue waves is that while observing one rogue wave in the ocean may be difficult, the theory predicts much more intricate structures to be possible. We have mentioned sailors' stories of triplets of rogue waves, but it is predicted to also be possible for rogue waves to appear in atom-like clusters of specific numbers. To observe this spontaneously in the ocean would be a hopeless task, and to generate these even in a wave tank may also be too complicated. Lasers offer what is currently the most plausible environment for experimentally probing these structures. The study of modulation instability in optics is also useful for determining the maximum intensity of a wave which can propagate without being significantly changed as it moves through a medium [64].

All of this is still to say nothing of the large variety of more exotic nonlinear phenomena, such as spiny solitons [65], creeping solitons [66], and exploding solitons [67, 68], which have been discovered in non-integrable systems. Although the theory of integrable soliton equations is intricate and aesthetically pleasing, integrable systems are almost always built upon simplifying assumptions which can be difficult to achieve in reality. It is more common that real situations are more aptly described by non-integrable systems, about which we cannot say too much in exact mathematical terms.

1.3 The Korteweg-de Vries equation

Why did it take the better part of a century before the theory of the KdV equation was properly developed? The answer is that nonlinear partial differential equations are hard to solve, and there is no general method of exact solution which can be applied to all of them. Only late last century was it noticed that the KdV equation has an incredibly rich structure, and that this could be exploited to generate very complicated solutions in a simple way.

The first approach to solving the KdV equation in a systematic way was through the method of inverse scattering. The inverse scattering technique operates a lot like a nonlinear version of the Fourier transform [69]. The idea is to treat the solution of the KdV equation as the potential in a Sturm-Liouville equation. If, given the potential, i.e. a solution to the KdV equation, at an initial time, say $t = 0$, the asymptotic scattering data of the unknown function in the Sturm-Liouville problem are then determined at the spatial boundary. The time evolution of the scattering data must be determined by the KdV equation, since the potential is a solution of the KdV equation, so this allows the scattering data to be 'propagated forward,' so to speak, to a later time $t > 0$. Finally, the potential can be reconstructed at the time t with the Gelfand-Levitan-Marchenko equation [70]. This constructs a solution to the KdV equation at a time t from known initial conditions. The idea of the inverse scattering technique is succinctly shown in the following diagram.

$$\begin{array}{ccc}
\text{KdV solution at } t = 0 & \xrightarrow{\text{Sturm-Liouville}} & \text{scattering data at } t = 0 \\
\big\downarrow \text{time evolution} & & \big\downarrow \text{KdV equation} \\
\text{KdV solution at } t > 0 & \xleftarrow[\text{GLM equation}]{} & \text{scattering data at } t > 0
\end{array}$$

Figure 1.3.1: A rough scheme of the inverse scattering technique

Since then, the theory has grown and developed methods more elegant and easily applied.

Lax [27] showed that the KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1.1}$$

is equivalent to the compatibility condition of an overdetermined linear system:

$$-\frac{\partial^2 \psi}{\partial x^2} + u\psi = \lambda\psi \tag{1.2}$$

$$\frac{\partial \psi}{\partial t} = -4\frac{\partial^3 \psi}{\partial x^3} + 6u\frac{\partial \psi}{\partial x} + 3\frac{\partial u}{\partial x}\psi \tag{1.3}$$

where λ is a constant which should be thought of as the eigenvalue, or spectral parameter, of the system. This is often expressed in the equivalent way

$$L\psi = \lambda\psi, \ L = -\frac{\partial^2}{\partial x^2} + u \tag{1.4}$$

$$\frac{\partial \psi}{\partial t} = B\psi, \ B = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3\frac{\partial u}{\partial x} \tag{1.5}$$

so that the KdV equation is just

$$\frac{\partial L}{\partial t} = [L, B] \tag{1.6}$$

where $[\cdot, \cdot]$ denotes the commutator. The operators L and B are called the Lax pair for the KdV equation. When a Lax pair for a nonlinear partial differential equation exists, the equation will have infinitely many conserved quantities, and thus be completely integrable.

This equivalence eventually led the way for the KdV equation, a nonlinear problem, to be treated by linear methods. It was shown by Matveev [71, 72] that these equations had a certain symmetry: if (u_0, ψ_0) is already known to be a solution with a fixed eigenvalue λ_0, the system is covariant under the transformation

$$u \mapsto u - 2\frac{\partial^2}{\partial x^2} \log \psi_0 \tag{1.7}$$

$$\psi \mapsto \frac{\partial \psi}{\partial x} - \psi \frac{\partial}{\partial x} \log \psi_0 \tag{1.8}$$

This is known as a Darboux transformation (after its original discoverer [73, 74], although it was originally mentioned in a different context).

This made finding solutions far easier. Given a solution to the KdV equation, all that was now required was to solve a linear system for an associated function ψ and a certain fixed eigenvalue, and then a second solution could be generated by applying a Darboux transformation. Since this was again another solution to the KdV equation/its associated linear system, the same transformation could be applied arbitrarily many times in order to generate a family of solutions (often called a hierarchy), starting from only one, much simpler solution (often called a seed solution). Crum's theorem [75] could be used to write down the result u_n of an n-times iterated Darboux transformation as

$$u_n = u_0 - 2\frac{\partial^2}{\partial x^2} \log W(\psi_1, \ldots, \psi_n) \tag{1.9}$$

where u_0 is the seed solution, $\psi_1, \ldots, \psi_n$ are linearly independent solutions to the linear system with distinct eigenvalues, and W denotes a Wrońskian.

Multi-soliton solutions of the form (1.9) were first given explicitly by Wahlquist [76]. The Wrońskian method in general was further developed by Freeman and Nimmo [77].

When eigenvalues coincide, the solution is called degenerate. Degenerate cases are broad, and include such things as degenerate breathers and rational solutions (when they exist).

We will make use of the degenerate Darboux transformation for the KdV equation in chapter 4. In fact, we will show that the rational solutions of the KdV equation have certain properties which allow more general solutions to be constructed by adding together a particular linear combination of the Wrońskians. It is possible that this indicates the existence of a more general kind of transformation of the KdV equation which remains to be discovered.

It was also shown that the inverse scattering method for soliton equations is closely related to auto-Bäcklund transformations [78]. These are transformations which stem from geometry, essentially being transformations between equivalent surfaces [79]. Originally, they were derived as maps between pseudospherical surfaces, and shown to generate hierarchies of solutions to the sine-Gordon equation [80–82], but were later found to be widely applicable to many soliton equations, such as the KdV equation [83]. In a sense, the Bäcklund transformation is the procedure which 'adds' an additional soliton to an already-known soliton solution. From the existence of a Bäcklund transformation, it is equally possible to conclude that a given soliton equation is completely integrable.

Another very powerful method was developed by Ryogo Hirota [84–87]. Referred to as either the direct method, or the bilinear method, the idea is that with the right substitution, nonlinear equations can be written as bilinear expressions. These bilinear forms can often be written in terms of Hirota derivatives. When this is possible, the special properties of Hirota derivatives make it easy to find a solution by a method almost as simple as perturbation series.

The theory of the KdV equation quickly acquired a lot of theoretical depth. Similar ideas led to the development of the theory of the τ-function [88], which was both a powerful method for giving general solutions, and was deeply rooted in algebraic geometry, an area seemingly far from physics yet one which was nonetheless intimately connected to the theory of solitons [89]. In fact, the entire idea of inverse scattering for a large class of integrable nonlinear systems can be viewed as a Riemann-Hilbert problem, which makes this an interesting point of view to take regarding solitons and nonlinear waves in general [90]. This is also known as finite-gap integration, where the central idea is that many nonlinear integrable systems can be solved by finding certain characteristic theta functions [91–93]. This technique can also be used to describe the asymptotic behaviour of soliton equations under short-range perturbations [94, 95].

It is also common to think of the KdV equation as part of a more general family of nonlinear evolution equations. The rough idea of this is simple, and due to Lax [27]. A key observation in KdV theory, first made by Gardner, Kruskal, and Miura [20], was that the eigenvalue of the (time-independent) Schrödinger operator (1.4) is invariant as long as the evolution of u is given by the KdV equation. However, this property is not unique to the KdV equation. In fact, by choosing the operator B to be an arbitrary antisymmetric differential operator of odd order, its coefficients can be uniquely determined so as to leave the eigenvalues of L fixed. This means that the KdV equation is really part of an infinite hierarchy of nonlinear partial differential equations, all of which are completely integrable, share the same invariants, and share the same remarkable properties. This is called the KdV hierarchy.

It has been shown that the KdV hierarchy does actually have physical meaning in hydrodynamics, where each member of the hierarchy describes the evolution of long waves over an infinite sequence of long time scales [96], the higher order equations corresponding to slower changes.

As the theory grew, so too did the number of applications which the KdV equation found. What began as a theory of weakly nonlinear shallow water waves now appears in a variety of physical contexts [97]. The KdV equation has been found to accurately describe the physics of internal waves in the ocean [98], acoustics of bubbly liquids [99], and also appears in geophysics where it can be used to model magma migration [100, 101], where such solitons are sometimes called magmons. It was suggested that the KdV equation could also describe the formation of solitons in atmospheric Rossby waves [102] and the KdV equation was even famously proposed as a model for the Great Red Spot on Jupiter [103–105]. Ion-acoustic waves were predicted to be described by the KdV equation [106], and were subsequently observed in a double-plasma device [107]. The KdV equation also comes up in conformal field theory [108], where it is noted that when considered with periodic boundary conditions, its Poisson structure is equivalent to a classical limit of the Virasoro algebra [109]; its Fourier modes are a scalar multiple of the generators. It is certainly intriguing that the KdV equation appears at once in something so plain to us as shallow water waves, while also sharing an algebraic structure with theories in high-energy physics; two fields which do not seem to have much in common at first sight. Such observations as these very quickly give an impression of great depth to the study of shallow water waves.

In fact, it is possible to make a more general statement about the KdV equation. In some sense, it is a universal equation, in that the KdV equation will describe the physics of a very wide class of weakly nonlinear systems with long characteristic wavelengths [23].

1.4 The nonlinear Schrödinger equation

The nonlinear Schrödinger equation gained a lot of attention when it was discovered that it could be used to describe a variety of self-focusing phenomena in nonlinear media. When the refractive index of a dielectric depends on the square of the electric field intensity, the envelope of a slowly-varying monochromatic electric field can exhibit self-focusing or self-defocusing behaviour [110]. Equivalently, the NLS equation appears as an approximation to Maxwell's equations when the polarisation is taken to have cubic nonlinearity.

The NLS equation quickly appeared in several different areas describing wave envelopes with self-focusing properties. In optics, the NLS equation was put forth to describe light propagation in nonlinear optical fibres and waveguides, and in hydrodynamics to describe the modulation instability of the Stokes wave.

In dimensionless units, the NLS equation has the form

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} \pm |\psi|^2\psi = 0 \tag{1.10}$$

where $|\psi|$ is the amplitude of the wave envelope. The variables x and t can have different meanings, depending on the context. In hydrodynamics, t is typically time while x is the horizontal coordinate, whereas in optics t and x are generally swapped in place, and the independent variables are the transverse and longitudinal coordinates. The choice of sign is decided by whether the nonlinear effects tend to focus or defocus a wave envelope. When the sign is positive, (1.10) is called the focusing NLS equation, whereas when the sign is negative, it is called the defocusing NLS equation. In optics, this term is due to the Kerr effect [111]. In this context, the sign of the nonlinear term is determined by the ratio of the quadratic term to the leading term in the refractive index. Rays are attracted to regions which have higher refractive indices, so if this term is positive, the rays will tend to move towards the centre of the beam where the field intensity is greater and thus the refractive index higher; that is, a positive sign on the nonlinear term corresponds to a self-focusing effect. For the same reason, a negative sign on the nonlinear term corresponds to self-defocusing.

Like the KdV equation, the NLS equation was shown to have a Lax pair. Zakharov and Shabat [29] showed that the split NLS equations

$$i\frac{\partial q}{\partial t} = -\frac{1}{2}\frac{\partial^2 q}{\partial x^2} + q^2 r \tag{1.11}$$

$$i\frac{\partial r}{\partial t} = \frac{1}{2}\frac{\partial^2 r}{\partial x^2} - r^2 q \tag{1.12}$$

(which are a special case of the AKNS system, after the work of Ablowitz, Kaup, Newell, and Segur [112]), are equivalent to the compatibility condition of another linear system, which is now known as the Zakharov-Shabat system. They are, therefore, completely integrable. When $r = \pm\overline{q}$, this reduces to the NLS equation.

The AKNS system was first introduced as a means of explaining why inverse scattering was able to generate solutions to so many nonlinear evolution equations [112], and the Zakharov-Shabat system was introduced to apply the inverse scattering method to generate soliton solutions of the NLS equations. That is to say, these were introduced as little more than an abstract mathematical construct which was not directly related to anything physical at all. We will return to the split NLS equations above in chapter 3, where we give a physical meaning to them for what we believe is the first time.

Because the NLS equation is a nonlinear integrable system, it is treatable with the same family of techniques. For example, soliton solutions can be generated from the solution of the associated Riemann-Hilbert problem [90, 113], and Hirota's method [84] can also be used to generate solutions with relative ease. It was also shown that the NLS equation had a Darboux transformation [114, 115]. As in the KdV equation, this enabled the construction of whole families of solutions, such as n-soliton solutions [116], multi-breathers [117], degenerate breathers, and multi-rogue waves [118], all from simple seeds such as the trivial or plane wave solutions. Rogue wave multiplets were especially interesting. It may seem reasonable to suspect that rogue waves should appear in arbitrary numbers. This turns out to be false. For example, the NLS equation has a rogue wave triplet solution [119], but not a doublet. Rogue waves of the NLS equation appear only in very particular multiplet structures, which, when plotted in the xt-plane, are striking in their resemblance to atomic shells [120, 121].

There are also solutions to the NLS equation which do not seem to be easily obtainable by Darboux transformations. A two-parameter family of doubly-periodic solutions have been derived by assuming a t-varying linear relationship between the real and imaginary parts (up to a phase term) for both the focusing [36] and defocusing [122–124] cases. These are known to reduce to the first-order solutions generated from Darboux transformations in various limits. These doubly-periodic solutions have been themselves used in the Darboux transformation in order to generate rogue waves which form on a doubly-periodic background [125], rather than growing from the plane wave, as they are usually considered.

The NLS equation is now widely applied in physics. We have already mentioned the original work wherein the NLS equation was shown to describe the stability of the Stokes wave. In optics, largely because of how generally the NLS equation can be derived from Maxwell's equations, it has a very broad range of applicability, such as pulses in optical fibres and nonlinear guided waves [126].

It was even shown that while the KdV equation could describe ion-acoustic waves with long wavelengths, ion waves with short wavelengths were described by the NLS equation [127]. This led to the experimental observation of the Peregrine rogue wave in plasmas [128]. The Hashimoto transformation establishes an equivalence between the NLS equation and the motion of vortex filaments [129, 130], for which breathers have been theoretically predicted [131]. The Heisenberg ferromagnet equation is equivalent to the NLS equation, making it applicable to the description of spin chains [132]. The NLS equation can also be seen as a particular case of the Gross-Pitaevskii equation, which means that it also describes the physics of Bose-Einstein condensates [133].

These may seem disconnected. However, similar to the KdV equation, the NLS equation can also be said to be a kind of universal equation for the equation of a weakly nonlinear monochromatic wavetrain [134, 135]. This justifies the natural appearance of the NLS equation describing the self-focusing effects of weakly nonlinear wave envelopes in diverse areas of physics, even though they may not seem to be connected in an immediately obvious way.

1.5 Thesis overview

This thesis is on the development of the theory of nonlinear waves. We explore new theoretical foundations for the theory of shallow water waves, by revisiting the Korteweg-de Vries equation. We develop a new theory based on treating the complex KdV equation, by which we mean the KdV equation where both the spatial variable and the independent variable are complex functions, as the fundamental equation of motion for the whole body of fluid. We build on this theory by reinterpreting its conservation laws, showing consistency, and linking it to the split NLS equations. We also give several new solutions to both the KdV equation, and an integrable extension of the nonlinear Schrödinger equation.

Chapter 2 is based on the paper [136], and builds the theory of the complex KdV equation from first principles. We consider an ideal fluid in a flat channel, and from Levi-Cività's equation, show that the first-order approximation of its complex velocity is given by the complex KdV equation. We also show that this is consistent with the more familiar real KdV equation describing the amplitude. This is done with a direct proof that the complex KdV equation describing velocity throughout the channel implies that the surface elevation satisfies the real KdV equation. Unlike the real KdV equation, the complex KdV equation allows for the direct calculation of particle motion at all points within the fluid. We give an example, and show that this is qualitatively almost identical to numerical simulations by other authors.

Chapter 3 is based on the manuscript [137], and continues the theory of the complex KdV equation. We give examples of conservation laws of the complex KdV equation, and show that they have especially natural physical interpretations, some of which are 'hidden' when considering only the real KdV equation as a theory for shallow water waves. We also point out that the complex KdV equation describing the velocity field of the fluid implies that the split NLS equations in fact have a direct physical meaning – that is, the split NLS equations directly give the fundamental modes of the components of velocity. This also reduces to the defocusing nonlinear Schrödinger equation when talking about the wave amplitude.

Chapter 4 gives examples of a previously unnoticed symmetry in the rational solutions of the KdV equation. It is found that when writing the rational solutions of the KdV equation in terms of Wrońskians, a linear combination of the Wrońskians of orders n and $n+2$ also gives a solution of the KdV equation. When considering complex solutions, there are regions off the real axis for which the solutions may be nonsingular. By varying the constant which gives relative 'weight' to the two Wrońskians, it is shown that the peak of the rational solution can actually split into multiple peaks. Chapter 4 is based on the paper [138].

Chapters 5 and 6 deal with solutions to the class I integrable extension of the NLS equation. These handle cases in which we want to incorporate higher order nonlinear and dispersive effects as part of an extension of the same integrable system. The solutions are found by taking the known solution for the NLS equation, and solving for frequency-like and velocity-like parameters which are given in terms of the strengths of the higher-order terms. Chapter 5 is based on the work [139], and gives the one-parameter doubly-periodic solution to the infinite extension. This is the most general known one-parameter family of solutions to the extended NLS equation. The solutions found include special cases, such as the Akhmediev breather.

Chapter 6 is based on the paper [140], and is on two-breather solutions to the class I integrable extension to the NLS equation. This requires the introduction of additional free parameters corresponding to the second breather component. We give exact solutions, and discuss several special cases. In particular, we examine degenerate breathers, breather-to-soliton transformations, as well as rogue wave triplets. These solutions also generalise some earlier work on solutions to extended NLS equations which incorporated fourth-order nonlinearities and dispersions. We also point out that the form of the general solutions is related to the single breather solution in a very simple way.

Chapter 7 is the concluding chapter. We discuss some possibilities for future research, and the ideas of this thesis as a whole.

2

The Complex Korteweg-de Vries Equation

Using Levi-Civitá's theory of ideal fluids, we derive the complex Korteweg-de Vries (KdV) equation, in which both the spatial variable and the dependent variable are complex, and show that it describes the complex velocity of a shallow fluid up to first order. We use perturbation theory, and the long wave, slowly varying velocity approximations for shallow water. The complex KdV equation describes the nontrivial dynamics of all water particles from the surface to the bottom of the water layer. A crucial new step made in our work is the proof that a natural consequence of the complex KdV theory is that the wave elevation is described by the real KdV equation. The complex KdV approach in the theory of shallow fluids is thus more fundamental than the one based on the real KdV equation. We demonstrate how it allows direct calculation of the particle trajectories at any point of the fluid, and that these results agree well with numerical simulations of other authors.

2.1 Introduction

Water waves can be classified into many types [141]. One shared feature is, as Feynman put it, that they have all the possible complications that a wave can have. For instance, when treated with full generality, water waves are commonly considered to be nonlinear phenomena [142]. Consequently, to precisely describe water waves is infamously difficult. One convenient approximate method of describing water waves is to give an evolution equation for the elevation of the water surface [143, 144]; in the lowest order nonlinear approximation, this leads to the nonlinear Schrödinger equation for the complex envelope of waves in deep water [145], and the real Korteweg-de Vries (KdV) equation for the elevation of a nonlinear shallow water wave [3, 4, 146]. Higher-order physical effects can be incorporated in each of these equations for more descriptive power [144, 147–149]. However, these approaches still restrict us to only the motion of the surface.

When a fluid is ideal, we can also describe its state of motion with the potential. As is often detailed in many standard texts on hydrodynamics [6], a well-behaved potential can be treated as the independent variable in the description of the fluid, rather than a spatial coordinate. This approach was pioneered by Levi-Cività in 1907, who derived an equation satisfied by all fluid motions without singularities in a channel of arbitrary depth [150]. The theory was developed in the early twentieth century, but due to the difficult nature of the resulting equation it has fallen by the wayside. Modern interest in this approach has been partly revived by Levi [151].

In this work, we give a thorough derivation of a complex KdV equation describing first order perturbations in the complex velocity around steady flow. Importantly, not only the dependent function in the KdV equation but the spatial variable is considered to be complex as well. This approach provides a complete description of the flow in the fluid up to first order, and is not limited to describing only the water elevation. Thus, the main advantage of the complex KdV equation in hydrodynamics is that it describes the dynamics of water particles not only at the surface but also throughout the entire body of the fluid.

A new step forward in our work is that if the complex KdV equation describes a first-order perturbation of the complex velocity for a shallow fluid, then the fluid's elevation is naturally determined by a solution of the standard real KdV equation.

As an example application of this theory, we give a simple demonstration of how the motion of the entire fluid may be described with a basic periodic solution to the complex KdV equation. We illustrate how the particle displacement from a given point may be determined, along with their trajectories, and show that in the limit as the period becomes infinite, the familiar soliton solution may be correctly described.

2.2 The Levi-Cività theory of flow in ideal fluids

Water of depth h occupies a channel with a flat bottom, and waves of a height $\eta = \eta(x,t)$ above the mean level propagate along the surface, the axis of x being taken along the bottom, and in the direction of propagation, while y represents a vertical coordinate. A diagram is shown in Fig.2.2.1. At any point x at a time t the

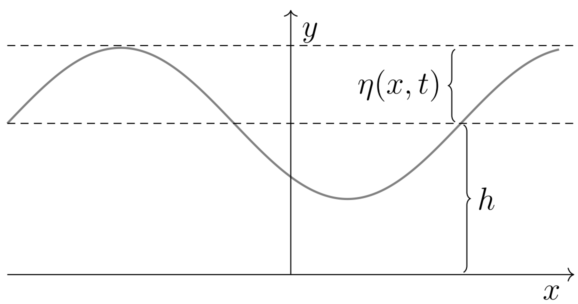

Figure 2.2.1: *Schematic of water layer with average depth h. The function $\eta(x,t)$ describes the water elevation above the average level.*

water surface is described by the equation $y = h + \eta(x,t)$. Thus,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v,$$

where u and v are the horizontal and vertical components of the fluid velocity, respectively.

When the ratio of wave height to wavelength is very small, we have the linearised theory

$$\frac{\partial \eta}{\partial t} = \frac{\partial \psi}{\partial x}$$

where ψ is the stream function, with the convention $d\psi = vdx - udy$. If we additionally suppose that the motion is irrotational, and that the wave is long enough that surface tension can be neglected, then the pressure just inside the fluid's surface must be very nearly equal to the pressure just outside the fluid's surface, and this implies that, approximately,

$$g\eta \simeq \frac{\partial \phi}{\partial t},$$

where ϕ is the velocity potential, with $d\phi = -udx - vdy$.

Since there is no flow through the bottom of the channel, the stream function is constant on the bottom of the channel, and we can choose $\psi = 0$ when $y = 0$. The complex potential $w = \phi + i\psi$ is then real when $y = 0$, and w can be analytically continued into the region $-h \leqslant y < 0$. Doing so leads to Cisotti's equation,

$$\frac{\partial^2}{\partial t^2}\{w(z+ih,t) + w(z-ih,t)\} + ig\frac{\partial}{\partial z}\{w(z+ih,t) - w(z-ih,t)\} = 0, \quad (2.1)$$

where $z = x + iy$, and w is a holomorphic function of z on its domain.

More details can be found in the well-known book by Milne-Thomson [6]. The equation (2.1) is complex but linear. However, we are also interested in nonlinear waves.

Beginning from just Bernoulli's principle, if we let $q = \sqrt{u^2 + v^2}$ be the total speed of the fluid at a given position and time, then

$$-\frac{\partial \phi}{\partial t} + \tfrac{1}{2}q^2 + gy + \frac{p}{\rho} = f(t)$$

where ρ is the density of the fluid, p is the pressure, and the time-derivative of the velocity potential is the energy due to acceleration within the fluid.

Now we suppose that the fluid surface is free to move along a variable curve given by $y = h + \eta(x,t)$. Along the free surface, the pressure is constant, so we have

$$-\frac{\partial \phi}{\partial t} + \tfrac{1}{2}q^2 + g\eta = 0. \tag{2.2}$$

We can define the velocity potential ϕ such that any constant or function of time on the right hand side can be set to zero. Note also that for surface waves, $0 \leqslant |\psi| \leqslant |\psi_0|$, where $\psi = \psi_0$ on the free surface $y = h + \eta(x,t)$.

Let us denote $\beta = u - iv$. We reiterate

$$\beta = -\frac{\partial w}{\partial z}. \tag{2.3}$$

Since w and β are both real along the bottom of the channel, and certainly assumed to be holomorphic throughout the body of fluid, we can analytically continue both functions to the region where $-h \leqslant y < 0$. Also, since w is holomorphic at every point of the flow, we can invert the relationship to write z, and the complex velocity β, as functions of the potential w rather than position. The velocity potential and stream function satisfy, by (2.3), the differential equation

$$dx + idy = -\frac{d\phi + id\psi}{u - iv}.$$

Streamlines are defined by $d\psi = 0$, so, with

$$ds^2 = dx^2 + dy^2$$

defining the line element on the free surface, and after collecting real and imaginary parts, which are respectively

$$dx = -\frac{ud\phi - vd\psi}{q^2},$$
$$dy = -\frac{vd\phi + ud\psi}{q^2},$$

we have

$$\frac{\partial y}{\partial s} = -\frac{v}{q^2}\frac{\partial \phi}{\partial s}$$

on a streamline. We also have, by definition,

$$-d\phi = udx + vdy,$$

so it is easy to see that

$$\frac{\partial \phi}{\partial s} = -q,$$

and consequently

$$\frac{\partial \eta}{\partial s} = \frac{v}{q}$$

on a streamline. Bernoulli's equation (2.2) can now be differentiated with respect to an arc length s along the free surface $y = h + \eta(x,t)$ to give

$$\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial s} + \frac{gv}{q} = 0,$$

or

$$\frac{\partial q^2}{\partial t} + q\frac{\partial q^2}{\partial s} + 2gv = 0. \tag{2.4}$$

As stated only shortly before, we can recast Bernoulli's principle, now in the form (2.4), in terms of the complex velocity β and the complex potential w. We obtain through a simple change of variable $d\phi = -qds$ the equation

$$\frac{\partial}{\partial t}|\beta|^2 - |\beta|^2\frac{\partial}{\partial \phi}|\beta|^2 - ig(\overline{\beta} - \beta) = 0. \tag{2.5}$$

Here, instead of the condition that the free surface be defined by $y = h + \eta(x,t)$, we have instead the free surface defined by the streamline $\psi = \psi_0$, and the complex velocity should be understood as a function $\beta = \beta(\phi + i\psi_0, t)$, with the conjugate velocity $\overline{\beta}$ given by $\overline{\beta} = \beta(\phi - i\psi_0, t)$. Lastly, since $\psi = 0$ and therefore w is real along the bottom of the channel, $y = 0$, we can extend the differential equation for β to all w, rather than just for $w = \phi$. The differential equation (2.5) can therefore be presented in the form

$$\begin{aligned}
- &\quad \frac{\partial}{\partial t}\log\{\beta(w+i\psi_0,t)\beta(w-i\psi_0,t)\} + \frac{\partial}{\partial w}\{\beta(w+i\psi_0,t)\beta(w-i\psi_0,t)\} + \\
+ &\quad ig\left\{\frac{1}{\beta(w+i\psi_0,t)} - \frac{1}{\beta(w-i\psi_0,t)}\right\} = 0. \qquad (2.6)
\end{aligned}$$

This equation was first derived by Levi-Cività [150] over a century ago, albeit in a more limited form, dealing only with steady flow. However, due to the fact that a differential equation of this type is difficult to solve, there has been only relatively minor attention given to this particular equation. Having said that, even though this equation is challenging to solve in the general case, it is nonetheless possible to gain insight into possible fluid dynamics by applying certain techniques, such as perturbation series.

2.3 Linear perturbations around steady uniform flow

Suppose that the complex velocity of the wave has the form of a small perturbation around an otherwise constant flow parallel to the bottom of the channel, so

$$\beta(w,t) = c\{1 + \varepsilon\alpha(w,t)\},$$

where ε is a small parameter, α is a complex function of the potential and c is a real constant. Since we are restricting ourselves to considering only small perturbations around a steady horizontal flow, most of the flow across the surface at any given point will be due to the horizontal movement, and therefore we can take the flux over the free surface to be approximately $\psi_0 = -ch$. This will clearly be true for periodic functions, but also for the general problem of the solitary wave [6], since the fluid must return to steady motion infinitely far away from the travelling pulse.

By hypothesis, $\varepsilon^2 \simeq 0$, so we can disregard those terms of all but first order. To first order, the perturbation satisfies

$$\begin{aligned} &-\frac{\partial}{\partial t}\log[1+\varepsilon\{\alpha(w+ich,t)+\alpha(w-ich,t)\}]+ \\ &+c^2\varepsilon\frac{\partial}{\partial w}\{\alpha(w+ich,t)+\alpha(w-ich,t)\}+ \\ &+\frac{ig\varepsilon}{c}\{\alpha(w+ich,t)-\alpha(w-ich,t)\}=0. \end{aligned} \tag{2.7}$$

This is the equation which will determine the stability of a small perturbation in the velocity.

For simplicity, first, we will consider motion which can be reduced to steady flow, such as, for example, a solitary wave or periodic motion. When the motion is steady, all the time dependence will be implicit, and contained in the potential, so that (2.7) becomes

$$\left\{\frac{d}{dw}\cos\left(ch\frac{d}{dw}\right)-\frac{g}{c^3}\sin\left(ch\frac{d}{dw}\right)\right\}\alpha(w)=0. \tag{2.8}$$

Here, the sine and cosine terms should be understood, like the exponential, in the sense of formal power series of an operator. That is,

$$\cos\left(\frac{d}{dw}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{d^{2n}}{dw^{2n}}, \tag{2.9}$$

and similar for the sine operator.

The ease of manipulation which results from the form (2.8) immediately suggests a simple solution for steady flow in the form

$$\alpha(w)=\alpha_0 e^{i\mu w}, \tag{2.10}$$

where μ is a solution of the transcendental equation

$$\tanh ch\mu=\frac{c^3\mu}{g}, \tag{2.11}$$

and α_0 a constant. Integration gives the equation

$$z = -\frac{1}{c}\left\{w + \frac{i\varepsilon\alpha_0}{\mu}e^{i\mu w} + O(\varepsilon^2)\right\}$$

and thus

$$\eta = \frac{\varepsilon\alpha_0}{\mu}e^{ch\mu}\cos\mu cx + O(\varepsilon^2) \tag{2.12}$$

for the stationary form of the free surface, up to first order. We see that $\mu = k/c$, approximately, where k is the wavenumber of the flow, and that (2.11) is just the dispersion relation for waves in an inviscid fluid of depth h.

Now suppose the motion is steady. We can write the potential purely as a function of z; $w = w(z)$, and to lowest order in ε, we have

$$w = -cz + O(\varepsilon).$$

To lowest order in the z-plane:

$$-\frac{d}{dz}\{\beta(z+ih) + \beta(z-ih)\} = \frac{ig}{c^2}\{\beta(z+ih) - \beta(z-ih)\}. \tag{2.13}$$

This is effectively equivalent to Cisotti's equation (2.1).

For the case in which the channel is shallow, with a flat bottom, we can obtain an approximate equation of motion by expanding in powers of h, which will be a convergent series when h is sufficiently small.

2.4 The nonlinear theory

We can more generally consider a perturbation series in powers of the small parameter ε of the form

$$\beta = c(1 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdots + \varepsilon^n\beta_n + \cdots), \tag{2.14}$$

where $\beta_n = \beta_n(z,t)$. We can expand (2.6) to each order of ε, and solve the equations obtained at each order sequentially. The constant flow velocity c will have distinct values for the shallow and deep regimes, and these must be treated as two separate limiting cases to make the problem manageable.

2.4.1 The complex KdV equation

For shallow fluids, we consider perturbations around a constant flow of uniform speed $c = \sqrt{gh}$. We can expand $\beta(w \pm ich, t)$ in powers of h, since h is small. We then apply the change of variable $\beta(w,t) \mapsto \beta(z,t)$, under which the derivative transforms as

$$\frac{\partial \beta}{\partial w} \mapsto -\frac{\partial}{\partial z} \log \beta(z,t). \tag{2.15}$$

Higher order derivatives transform as

$$\begin{aligned}
\frac{\partial^2 \beta}{\partial w^2} &\mapsto \frac{1}{\beta} \frac{\partial^2}{\partial z^2} \log \beta, \\
\frac{\partial^3 \beta}{\partial w^3} &\mapsto -\frac{1}{\beta^2} \frac{\partial^3}{\partial z^3} \log \beta + \frac{1}{\beta^3} \frac{\partial \beta}{\partial z} \frac{\partial^2}{\partial z^2} \log \beta,
\end{aligned}$$

and so on. Truncating the series at third order gives the approximate equation

$$\begin{gathered}
-\beta^2 \frac{\partial \beta}{\partial z} + (gh)^{\frac{3}{2}} \frac{\partial}{\partial z} \log \beta + \\
+ \frac{gh^3}{2} \left(\frac{\partial^3 \beta}{\partial z^3} - 3 \frac{1}{\beta} \frac{\partial \beta}{\partial z} \frac{\partial^2 \beta}{\partial z^2} \right) + \\
+ \frac{g^{\frac{5}{2}} h^{\frac{9}{2}}}{6} \left(\frac{1}{\beta^3} \frac{\partial \beta}{\partial z} \frac{\partial^2}{\partial z^2} \log \beta - \frac{1}{\beta^2} \frac{\partial^3}{\partial z^3} \log \beta \right) \simeq 2\beta \frac{\partial \beta}{\partial t}
\end{gathered}$$

after a change of variable to the z-plane. We then assume a perturbation series solution of the form

$$\frac{\beta(z,t)}{\sqrt{gh}} = \sum_{n=0}^{\infty} \varepsilon^n \beta_n(z,t).$$

The lowest order term is identically zero with this substitution, as we have seen. The only surviving coefficient of ε gives the first order approximation

$$\frac{2(gh)^{\frac{3}{2}}}{3}\left(h^2\frac{\partial^3\beta_1}{\partial z^3}-\frac{3}{\sqrt{gh}}\frac{\partial\beta_1}{\partial t}\right)\varepsilon,$$

while the second-order terms read

$$\bigg\{-6\beta_1\frac{\partial\beta_1}{\partial z}-\frac{5h^2}{3}\frac{\partial\beta_1}{\partial z}\frac{\partial^2\beta_1}{\partial z^2}+h^2\beta_1\frac{\partial^3\beta_1}{\partial z^3}- \\ -\frac{2}{\sqrt{gh}}\beta_1\frac{\partial\beta_1}{\partial t}+\frac{2h^2}{3}\frac{\partial^3\beta_2}{\partial z^3}-\frac{6}{\sqrt{gh}}\frac{\partial\beta_2}{\partial t}\bigg\}(gh)^{\frac{3}{2}}\varepsilon^2.$$

Since the same stray terms that appear as the coefficients of ε appear as coefficients of ε^2 with β_2 instead of β_1, it is reasonable to suspect that these terms actually belong to a higher order of smallness.

If we let a typical wave in the fluid have a characteristic length scale l, and typical amplitude a, then to say that the wave is long is only the statement that the fraction h/l is much less than 1, and to say that the fluid is shallow is to say that the fraction a/h is small. When the complex potential exists and is holomorphic, the fluid must be irrotational, and this requires that the wave amplitude be sufficiently small compared to its depth. This suggests taking the small parameter

$$\frac{a}{h}=\varepsilon.$$

For a wave which is typically long, the wave will normally appear to change only slowly, and over a long horizontal distance of the same order as l. We can thus put

$$\left(\frac{h}{l}\right)^2=\delta,$$

where δ is small. This suggests defining the holomorphic dimensionless coordinate

$$Z=X+iY=\frac{z}{l}.$$

Assuming that these orders of magnitude are both equally small, $\varepsilon = \delta$, the condition that the fluid is shallow is now that the vertical coordinate Y is much smaller than 1 everywhere in the fluid; more specifically, $Y = O(\varepsilon^{\frac{1}{2}})$.

Writing equations in terms of Z, we see that the variation of the first order perturbation β_1 must be infinitesimal with respect to t. This is physically intuitive since the characteristic length scale of the wave is large. However, upon the introduction of a longer time scale, $t' = \varepsilon t$, β_1 can be seen to have a nontrivial slow variation; we have at first order a complex Korteweg-de Vries (KdV) equation:

$$\frac{l}{\sqrt{gh}} \frac{\partial \beta_1}{\partial t'} = -3\beta_1 \frac{\partial \beta_1}{\partial Z} + \frac{1}{3} \frac{\partial^3 \beta_1}{\partial Z^3}.$$

If we take the ratio of t' to the characteristic time $l/\sqrt{gh}$, we can define a dimensionless long time scale T by

$$\frac{t'}{T} = \frac{l}{\sqrt{gh}}.$$

In terms of Z and T, we have the fully dimensionless form of the complex KdV equation,

$$\frac{\partial \beta_1}{\partial T} = -3\beta_1 \frac{\partial \beta_1}{\partial Z} + \frac{1}{3} \frac{\partial^3 \beta_1}{\partial Z^3}. \tag{2.16}$$

The conditions of validity are that the wave is generally long, and the perturbation varies slowly with time.

The idea of reviving the theory of the complex KdV equation comes from the work of Levi [151]. However, some fundamental errors made in previous work [151, 152] did not allow this theory to be used in practice. Namely, in [151], the water elevation h was directly replaced by the expression $h = h_0 \varepsilon^{\frac{1}{3}}$ with h_0 being an unspecified parameter. The smallness parameter introduced this way is incompatible with approximations required for establishing correspondence with the real KdV equation. In the work [152], an intended small parameter ε was not dimensionless, making the idea of smallness in a perturbation series poorly defined. Here, by instead scaling the complex variable by a characteristic length scale l, the typical length of the wave enters through the derivative terms, providing a physically clear interpretation of how small each of the terms are. The scales of the flow, a, h, and l, then naturally form the dimensionless parameter ε, which is also the same order of smallness as in the real KdV equation [153], up to a possible constant factor.

Unlike the better-known real KdV equation [4], the complex KdV equation describes a perturbation of the complex velocity around an otherwise steady, uniform flow. It thus describes the entire fluid motion, not just the elevation of the free surface. Consequently, our approach allows us to describe the trajectories of the fluid particles across the whole layer of liquid. This has not actually been carried out at all up until now, although this is one of the important advantages of the complex KdV equation. Below, we provide the first demonstration of how this is done, as well as point out its excellent qualitative agreement with numerical results of other authors.

But before we do, we will show, for the first time, how the complex KdV theory leads naturally to the standard theory of the real KdV equation for the surface elevation of shallow water.

2.4.2 Relation to the real KdV equation

We can construct a similarly dimensionless potential $\tilde{w}$ by the scaling

$$\tilde{w} = \frac{w}{\sqrt{ghl}} = -Z + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots \tag{2.17}$$

The corresponding first order perturbation of the potential in dimensionless form will give β_1 through

$$-\frac{\partial\gamma_1}{\partial Z} = \beta_1(Z, T). \tag{2.18}$$

Substituting this into (2.16) gives the potential KdV equation for the function γ_1:

$$\frac{\partial\gamma_1}{\partial T} = \frac{3}{2}\left(\frac{\partial\gamma_1}{\partial Z}\right)^2 + \frac{1}{3}\frac{\partial^3\gamma_1}{\partial Z^3}. \tag{2.19}$$

The complex potential also gives one of the simplest methods for obtaining the elevation η. We have assumed that, whatever the form of the free surface, it corresponds to the constant value of the stream function $\psi = -ch$. The imaginary part of the dimensionless potential $\tilde{w} = \tilde{\phi} + i\tilde{\psi}$ on the surface is then just

$$\tilde{\psi} = -\frac{h}{l}.$$

Suppose that a holomorphic solution for the first order perturbation of the potential $\gamma_1 = \gamma_1(Z, T)$ is obtained. If γ_1 has imaginary part $\tilde{\psi}_1$, then we must have on the free surface $y = h + \eta$ the equation

$$\frac{\eta}{l} \simeq \varepsilon \tilde{\psi}_1|_{Y=(h+\eta)/l} \tag{2.20}$$

in terms of the dimensionless variables, accurate to first order in ε. In general, this will give an implicit equation for η.

In fact, since Y has already been required to be small, of order $\sqrt{\varepsilon}$, we can avoid the complications of implicit equations for the free surface by approximating the first order perturbation of the potential as

$$\gamma_1(Z, T) \simeq \gamma_1(X, T) - iY\beta_1(X, T).$$

Recalling that the stream function vanishes on the bottom of the channel, it follows that $\gamma_1(X, T)$ is real, so the first order perturbation in the stream function must be approximately

$$\tilde{\psi}_1 \simeq -Y\beta_1(X, T).$$

From (2.20), the surface elevation η can be given in terms of $\beta_1(X, T)$ as

$$\eta(X, T) \simeq -\varepsilon\{h + \eta(X, T)\}\beta_1(X, T),$$

or

$$\eta(X, T) \simeq -h\varepsilon\beta_1(X, T) \tag{2.21}$$

to first order in ε.

The vanishing of the stream function along the bottom is equivalent to the vanishing of the vertical component of the velocity. We see then that $\beta_1(X, T)$ must be real.

Given that $\beta_1(X,T)$ is a real solution to (2.16) with the real variable X instead of the complex variable Z, we have obtained in (2.21) the result that the surface elevation η of a shallow fluid is described by a real solution of the KdV equation in X and T, up to a constant of proportionality, and accurate to first order in ε.

We therefore make the claim that the complex KdV equation is more fundamental than the real KdV equation, since we have shown that the description of the surface elevation by the real KdV equation naturally emerges as a consequence of a perturbation of the complex velocity being described by the complex KdV equation. The complex KdV equation, however, also allows for a full picture of the motion of the entire fluid up to first order, not just the surface elevation to which we are limited by considering only its real solutions. Not only mathematically, then, but also physically, we have thus justified the claim that it is the complex KdV equation which takes the most fundamental place in the hydrodynamic theory of shallow fluids.

It is also worth mentioning that the τ-function for the complex KdV equation, which provides maybe the simplest mathematical lens through which to view the KdV equation [84], can be simply related to the first order perturbation of the potential, since

$$\log \tau(Z,T) \propto \int^{Z} \gamma_1(Z',T)dZ'. \tag{2.22}$$

In the complex KdV equation, the τ-function is thus a logarithmic measure of flux in the fluid due to perturbations around steady flow. To develop any more sophisticated physical connection between the complex potential and the τ-function further would be beyond the scope of this work, so we leave this as what we believe to be an interesting comment.

Terms of the third order of smallness give an equation for β_2. Naturally, the calculations are more complicated for higher orders of the perturbation series, and are less immediately enlightening, so we relegate these to the appendix. However, we stress that higher-order corrections to the wave motion can be naturally calculated in an extended version of this theory.

2.5 Particle motion in periodic waves

Ignoring the constant background flow, from a fixed point Z a fluid particle will move to the point $Z + dZ'$ in a time dT under the influence of the perturbation β_1. In this section, we will drop the subscript without confusion, since we are only working to first order in ε, in the KdV regime.

The rate of change of the particle's position will be given by the differential equation

$$\frac{d\overline{Z}'}{dT} = \beta, \tag{2.23}$$

where, again, the bar denotes complex conjugation.

A periodic wave will have the form $\beta = \beta(mZ - nT)$. Integrating with respect to T gives

$$\overline{Z}' = \frac{\gamma(mZ - nT)}{n}. \tag{2.24}$$

As before, we can make the lowest order approximation

$$\gamma(mZ - nT) \simeq \gamma(mX - nT) - imY\beta(mX - nT),$$

so that the new position of the particle is given by the horizontal and vertical coordinates

$$\begin{aligned} X' &= \frac{1}{n}\gamma(mX - nT) + C_1, \\ Y' &= \frac{m}{n}Y\{\beta(mX - nT) + C_2\}, \end{aligned}$$

where C_1 and C_2 are constants of integration (when seeing that a term of the form C_2Y can appear, it must be remembered that the initial position (X, Y) is fixed).

To be definite, we will consider a particular periodic solution of the complex KdV equation. From (2.16), we see that we have a periodic solution of the form

$$\beta(Z, T) = -\tfrac{4}{3}k^2m^2 \operatorname{cn}^2(mZ + nT, k), \tag{2.25}$$

where k is the modulus while the frequency is

$$n = -\tfrac{4}{3}(2k^2 - 1)m^3.$$

We also have one of the form

$$\beta(Z,T) = -\tfrac{4}{3}m^2 \operatorname{dn}^2(mZ + nT, k), \tag{2.26}$$

where

$$n = -\tfrac{4}{3}(k^2 - 2)m^3,$$

and another of the form $\operatorname{cn}^2 + k \operatorname{cn} \operatorname{dn}$ [154]. As $k \to 1$, the first two tend to the same sech^2-type solution, while the third tends to 0. Also, because of the identity

$$\operatorname{dn}^2(u,k) = k'^2 + k^2 \operatorname{cn}^2(u,k), \tag{2.27}$$

where k' is the complementary modulus, the cn^2 and dn^2-type solutions can be simply transformed into one another through Galilean symmetry. We will therefore use the term cnoidal wave to refer to both of these solutions.

Corresponding to the dn^2-solution, the perturbation in the potential is

$$\gamma(Z,T) = -\tfrac{4}{3}mE(\operatorname{am}\zeta, k), \tag{2.28}$$

where $E(u,k)$ is the elliptic integral of the second kind, and $\zeta = mZ - nT$. If we look at the cn^2-solution instead, γ just picks up an extra term which is just a multiple of ζ from the relation (2.27). Again, because Z refers to a constant position in the fluid, not a coordinate which follows the particle, this will not lead to qualitatively different behaviour up to Galilean symmetry.

The coordinates of the fluid particle are given by

$$X' = -\frac{4m}{3n}E(\operatorname{am}\xi, k) + C_1, \tag{2.29}$$

$$Y' = -\frac{4m^3Y}{3n}\{\operatorname{dn}^2(\xi,k) + C_2\}, \tag{2.30}$$

where $\xi = mX - nT$. By use of the identity

$$E(\operatorname{am} u, k) = Z(u, k) + \frac{Eu}{K},$$

where $K = K(k)$, $E = E(k)$ are the complete elliptic integrals of the first and second kind, respectively, and $Z(u, k)$ is the Jacobi zeta function with modulus k, we can write

$$\xi' - \xi'_0 = -\frac{4m^2}{3n} Z(\xi, k). \tag{2.31}$$

Here ξ' represents horizontal position in the comoving frame and ξ'_0 is a fixed constant. The fluid motion is now easily seen to be periodic, with identical motion at points separated by a horizontal distance $2K$ in the comoving frame, i.e. $\xi \sim \xi + 2K$.

The trajectory of any particle in the fluid must be a closed curve (up to the constant horizontal flow). However, it cannot be the usual ellipse, which is characteristic of linear flow. It is easy to see this by physical principles alone. Because cnoidal waves are characterised by long, relatively flat regions between narrow peaks, the trajectory of a particle cannot be symmetric in two axes. Instead, it will travel along a nearly flat line, and then its vertical velocity will increase and decrease quickly. This means that its trajectory must be a curve which has a broad, flat base. The width of this curve will also increase with the modulus k, and be constant with respect to Y, since ξ' is independent of Y. The height of the trajectory will also decrease until it is completely flat at the bottom of the channel.

We give an illustration of the periodic motion of a cnoidal wave in the complex KdV equation in Fig.2.5.1, taking k large enough that the flat regions between peaks becomes clearly defined. The surface has the form of a cnoidal wave, while the curves on and within the fluid display the trajectories of fluid particles as we descend to the bottom. The trajectories in the cnoidal wave are most similar to bean curves, rather than ellipses or circles, which are familiar from the linear theory. However, when k is small enough, the trajectories may be well-approximated by ellipses. The particle trajectories are in close agreement with earlier numerical approximations [155], but in the framework of the complex KdV theory, they appear naturally with very little

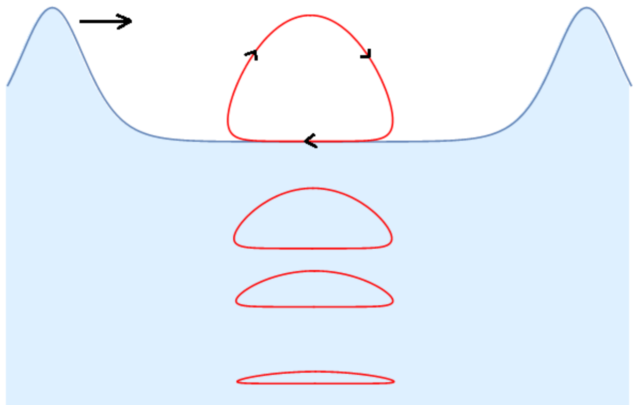

Figure 2.5.1: *An illustration of the paths of the particles in a rightward-propagating cnoidal wave (not to scale). The surface elevation is described by the real KdV equation, but the motion of the particles within the fluid is described by the complex KdV equation (2.16). The motion of the particle at a given point is determined by (2.30, 2.31). The trajectory depends on the wave modulus, frequency, and depth in the fluid.*

work.

As $k \to 1$, we have also $K \to \infty$, so the motion is no longer periodic. Instead, we have the well-known sech^2-shaped soliton solution. In this case, the particle trajectories become parabolic arcs [156].

Lastly, we will give an example calculation of the elevation from the complex KdV equation. The simplest case to work with is that of a first order perturbation in the potential of the form

$$\gamma = \tfrac{4}{3} a \tanh a(Z + \tfrac{4}{3} a^2 T). \tag{2.32}$$

This leads to the implicit equation for the surface

$$\frac{\eta}{l} \simeq \frac{\frac{4}{3}\varepsilon a \sin 2a(h+\eta)/l}{\cos 2a(h+\eta)/l + \cosh 2a(X + \frac{4}{3}a^2 T)}.$$

An initial measurement taken at $T = 0$ will have a maximum of η_0 located at $X = 0$, roughly given by the transcendental equation

$$\frac{\eta_0}{l} \simeq \tfrac{4}{3}\varepsilon a \tan\left(a \frac{h + \eta_0}{l}\right).$$

When h and l are known and the dimensionless parameter a is fixed, this allows an approximation of η_0. Assuming that $h + \eta_0$ is much smaller than l, we can take $\tan\theta \sim \theta$, and estimate the maximal elevation as

$$\eta_0 \simeq \tfrac{4}{3}a^2 h\varepsilon, \tag{2.33}$$

which agrees with the result (2.21). It follows that η can be put in terms of η_0 as

$$\eta \simeq \eta_0 \operatorname{sech}^2 a(X + \tfrac{4}{3}a^2 T) \tag{2.34}$$

up to first order. This characterises the form of the free surface traced out by a solitary wave. This was first derived from the tanh-potential (2.32) by McCowan [156], but this connection is more quickly and easily obtained by the simple relation (2.21), given a solution to (2.16) or (2.19).

2.6 Conclusion

In this chapter, we derived the complex KdV equation for shallow water waves. We showed that it describes a small perturbation of the complex velocity around steady flow, for the slowly-varying propagation of long waves in a shallow, incompressible, inviscid fluid. An important conclusion of our derivation is that if the complex KdV equation describes a complex velocity for a shallow fluid, then the elevation must be given by a real solution of the KdV equation. In other words, the theory of the real KdV equation is just a particular case of what we have derived here.

We reproduced the periodic waves in terms of elliptic functions and the well known result for the soliton solution. However, the complex KdV equation allowed us to calculate not only the surface wave profile, but also the trajectories of particles within the fluid, and show that particles in cnoidal waves move along bean curve-like trajectories. Our results were in close agreement with numerical work based on the real KdV equation [155], but had the bonus of being a natural and easy consequence of the theoretical framework. This gives even more evidence for the validity of the link between the two approaches established in the present work.

Having complete information about the dynamics of all fluid particles, not only at the surface but also throughout the whole body of fluid, may make it possible to solve more complicated problems. For example, this may include the class of problems related to internal waves when the fluid is stratified, or may allow the techniques of complex analysis [157] to be applied, such as conformal transformations, which may help to solve problems involving special bottom profiles or underwater barriers and obstacles.

For simplicity, we restricted ourselves to only one type of boundary condition at the bottom, with zero friction. We have no doubt this can be generalised to other conditions that will result in more general solutions. Generalisation to higher-order corrections is also straightforward in principle. The choices here have no limits.

We should also mention that the complex KdV equation has a greater variety of solutions than the real one. This includes regular [158–160], blow-up [161, 162] or complexiton [163] solutions. This expands the range of phenomena that the complex KdV equation may describe. On the other hand, we stress that exact solutions, even if they are highly involved, can be found using well known techniques such as the Darboux transformation.

Applications of the complex KdV equation are not limited to water waves. This equation can be also used in the seemingly unrelated problem of unidirectional crystal growth [164]. Last but not least, the theory of shallow water waves is spreading into the optical domain [165] which could be another area of future application.

Appendix

For completeness, we give the results of the calculations for second order perturbations. The second order terms β_2 in a shallow fluid satisfy the equation

$$\begin{aligned} \frac{\partial \beta_2}{\partial T} = {\beta_1}^2 \frac{\partial \beta_1}{\partial Z} + \frac{\beta_1}{18} \frac{\partial^3 \beta_1}{\partial Z^3} - \frac{5}{18} \frac{\partial \beta_1}{\partial Z} \frac{\partial^2 \beta_1}{\partial Z^2} + \\ + \frac{1}{9} \frac{\partial^3 \beta_2}{\partial Z^3} - 3 \frac{\partial (\beta_1 \beta_2)}{\partial Z}. \end{aligned} \tag{2.35}$$

where $\beta_1 = \beta_1(Z,T)$ is a solution of (2.16). Having β_1 in explicit form provides the possibility for higher-order extensions of the present theory. That is, working to higher order in perturbation theory will result in equations which include nonlinear and dispersive effects for shallow fluids, accurate to higher powers of the small parameter $\varepsilon = (h/l)^2$. We note here that taking terms which are of higher order in h/l will never result in a theory which is valid for anything other than shallow fluids, which are defined by h/l not being small in the first place.

3

Complex KdV equation with complex spatial variable: Conservation laws, and relation to the split NLS equations

The complex KdV equation, with a complex spatial variable, describes the complex velocity of a unidirectional weakly nonlinear shallow water wave. This provides much more information about the wave motion than the real KdV equation. This more general equation requires rethinking of the whole theory. Firstly, the complex conservation laws have to be derived and their relation to the conservation laws of the real KdV equation has to be found; we show they are consistent with the hydrodynamic interpretation of both equations. Secondly, the defocusing nonlinear Schrödinger equation is known to be obtained from the real KdV equation in the quasi-monochromatic approximation. In the complex KdV equation, we show that this leads to the split system of nonlinear Schrödinger equations, and that this describes the fundamental modes of velocity.

3.1 Introduction

The real KdV equation [4] is well known to describe the water elevation of a long, small amplitude wave in a shallow, horizontal channel. However, the more general complex KdV equation gives much more information about the whole motion of the fluid than the real one. Based on the work of Levi-Cività [150], we have shown earlier [136] that the complex KdV equation describes the two-dimensional wave motion of an ideal fluid in a canal of small depth. In contrast to the real KdV equation, the complex KdV equation describes not only the elevation in the canal, but is actually an equation for the evolution of the complex velocity at each point, thus being an equation of motion for the whole layer of fluid. When looking at only the surface, it also follows from this more general equation for the whole fluid that the water elevation does satisfy the real KdV equation. On the other hand, it was also shown [4, 136] that the water elevation can be approximated by the velocity along the bottom of the channel. Thus, the two equations are tightly linked to each other and describe the same physical phenomenon. However, the advantage of placing the complex KdV equation in the more fundamental position is that in this case we work with a function which defines the entire flow. This opens up clear advantages that do not exist when considering only the surface elevation obtained using the real KdV equation. One of them is the possibility of applying conformal transformations, which may help to solve for flows in channels with nontrivial geometries.

We write the complex KdV equation in the standard dimensionless form as

$$\frac{\partial\beta}{\partial t} - 6\beta\frac{\partial\beta}{\partial z} + \frac{\partial^3\beta}{\partial z^3} = 0. \tag{3.1}$$

where $\beta = u - iv$ is the complex velocity of a water particle located at the fixed spatial point (x, y), and $z = x + iy$ is the complex spatial variable. Here, the velocity vector $\mathbf{u}$ has horizontal and vertical components u and v. The constant coefficients in

Eq.(3.1) can be rescaled to other values if needed. As we are dealing with the shallow water, only the values of variable y small compared to 1 are considered. The only remaining real variable in Eq.(3.1) is the time t, which represents the long timescale over which the wave evolves.

The complex velocity β is a holomorphic function of z for all t. It is taken to be real along the real axis, $y = 0$. The latter is a boundary condition in this problem: the vertical component of velocity vanishes on the bottom of the channel. Not every imaginable solution of Eq.(3.1) satisfies this boundary condition. Only holomorphic functions β which are real on the line $y = 0$ describe shallow water waves. This is the physical constraint necessary to describe the motion of an ideal fluid in a horizontal channel.

The horizontal velocity on the bottom of the channel, $\beta(x, t)$, is proportional to the water elevation

$$\eta(x, t) \simeq -h\varepsilon\beta(x, t) \tag{3.2}$$

where ε is a small parameter (see Eq.(21) of [136]). Moreover, it was shown that $\eta(x, t)$ satisfies the real KdV equation. This means that in the hydrodynamic context and with proper scaling, the solution of the complex KdV equation can be considered as an analytic continuation of the corresponding solution of the real KdV equation. For example, a soliton solution of the complex KdV equation can be considered an analytic continuation of the soliton solution of the real KdV equation, up to a constant factor.

An illustration of the relevant physical quantities is given in Fig.3.1.1. Thus, Eq. (3.1) describes velocities of the water particles in a horizontal layer of shallow water with the average height h [136] as shown in Fig.3.1.1. The blue line in Fig.3.1.1 corresponds to the water's surface, and on the surface, the velocity vector $\mathbf{u}$ is a tangent to this curve. The water elevation $\eta(x, t)$ evolves in time according to the real KdV equation, while the velocity of particles at any fixed point evolves in time according to the complex KdV equation (3.1).

The fact that we can describe the physics of waves in a shallow ideal fluid with analytic solutions of the complex KdV equation means that this equation should also

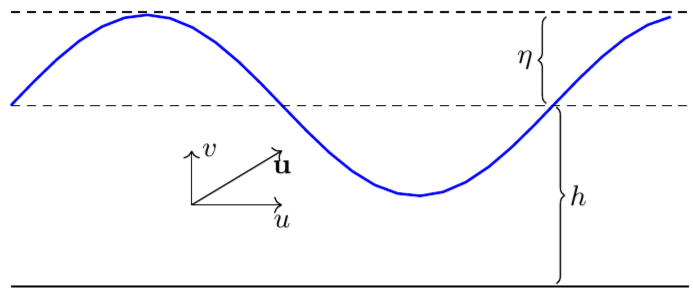

Figure 3.1.1: *Water waves in a channel with average depth h. The vector $\mathbf{u} = (u, v)$ is the water particle velocity at a fixed point and can be given as the complex velocity $\beta = u - iv$. η gives water elevation around the average level. In the KdV regime they are related by (3.2).*

have conservation laws. Some of the lowest order ones are derived in this work. More generally, we will show that when there is a conservation law in the real KdV equation, it is possible to define a similar integral over the whole area occupied by the fluid.

We will also demonstrate one interesting difference in these two equations. When a periodic wave is expanded via the quasi-monochromatic approximation in the real KdV equation, it can be shown that the wave envelope is described by the nonlinear Schrödinger (NLS) equation [135, 166]. We will demonstrate that there is a key difference in the complex KdV equation. Instead, the result is the split NLS equations [167], which is a particular case of the AKNS system [112]. This does not describe the wave envelope directly, but rather describes the fundamental modes of the velocity field. We will also show that this is not inconsistent with the usual theory, but rather complementary. This is also what we believe to be the first directly physical origin of the split NLS equations to be given so far, showing that this system of equations should be considered as more than an abstract construct.

3.2 Conserved quantities

It is immediately clear from the physics of the problem that the complex KdV theory must incorporate some conservation laws. For example, in the derivation of the governing equation, it was implicitly assumed that there was no significant dissipation, so we should have some notion of energy conservation. We already know that when looking at only the motion of the surface of the fluid, we will return to real KdV theory, in which there are infinitely many conserved quantities which can be put in terms of the surface elevation, but because we have a theory of the two-dimensional fluid motion, we must have some way of linking these concepts.

3.2.1 Density, vorticity, and centre of mass velocity

The very simplest conserved quantities are in fact assumed in the derivation of the complex KdV theory. The vorticity is assumed to vanish, and the fluid is assumed incompressible. From these requirements we are able to give a description of the fluid in terms of a holomorphic potential function. This is equivalent to the vanishing of the integral

$$\oint \beta dz$$

around any closed contour in the fluid, and this follows by definition. Because this is a local statement; i.e. it is true for any contour around any point in the fluid except the particles on very surface, this is technically a stronger statement than in real KdV theory, but we can recover the more familiar conservation laws as a particular case of this.

If we suppose that at a fixed moment in time, we can stretch the contour out to follow the surface of the fluid, the bottom of the channel, together with two horizontal parts at infinity closing the contour, then we can return to a familiar conserved quantity from the real KdV theory. Since $\beta \to 0$ as $x \to \pm\infty$ at all times and all values of y within the fluid, the horizontal parts of the contours will give an arbitrarily small contribution to the integral and can be taken to be zero in the limit. Over the surface, say $y = H(x,t)$ in dimensionless coordinates, the integral is just

$$\int_{-\infty}^{\infty} \beta(x + iH, t)dx = \Delta\phi$$

where $\Delta\phi$ is the change in the velocity potential from $x = -\infty$ to $x = \infty$ on the surface. By assuming that the fluid returns to steady flow at infinity, it follows that we can always take $\Delta\phi$ to be constant.

There must be an equal and opposite contribution to the integral from the bottom of the channel, and because the horizontal velocity $u_0(x,t)$ along the bottom is directly proportional to the surface elevation $\eta(x,t)$ above the mean, this implies

$$\frac{d}{dt}\int_{-\infty}^{\infty}\eta(x,t)dx = 0 \tag{3.3}$$

This is the trivial conservation law of the real KdV equation. Viewed this way, it is easily seen to be really just a statement of mass density conservation for an irrotational fluid.

We can see something similar by considering the contour integral

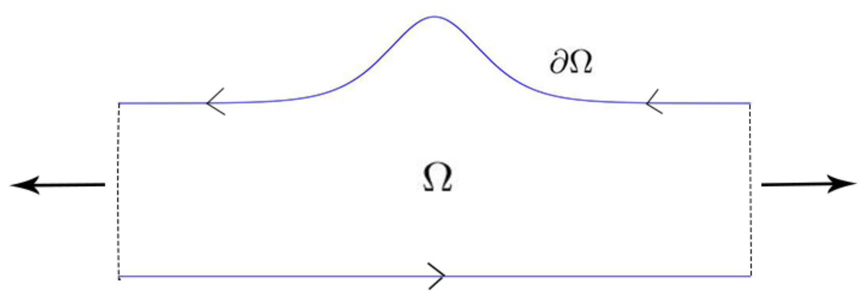

Figure 3.2.1: *The contour of integration over which we integrate to evaluate a conserved quantity. The vertical segments of the contour can be pushed out to infinity, and give only zero contribution to the total integral in the limit.*

$$\oint_C \beta z dz = 0 \tag{3.4}$$

for any closed contour C. If we let position and velocity be given in rectangular coordinates by the vectors $\mathbf{x} = (x, y)$ and $\mathbf{u} = (u, v)$, then the real and imaginary parts of βz can be written

$$\beta z = \mathbf{u}\cdot\mathbf{x} + i\det(\mathbf{u},\mathbf{x}) \tag{3.5}$$

The imaginary part is proportional to the angular momentum of the particle at $\mathbf{x}$

about the origin. This has no direct analogue in terms of the conserved quantities of the real KdV equation.

The real part of (3.5) gives

$$\oint_C \mathbf{u} \cdot \mathbf{x} ds = 0 \tag{3.6}$$

where ds is a line element on the contour.

When $\mathbf{u} \cdot \mathbf{x} > 0$, the particle at $\mathbf{x}$ with velocity $\mathbf{u}$ will be moving away from the origin, whereas when $\mathbf{u} \cdot \mathbf{x} < 0$, it will be moving nearer to the origin. When summing up $\mathbf{u} \cdot \mathbf{x}$ over the free surface, like that shown in Fig. (3.2.1), we will generally be left with a function of time, because the integral will depend on the shape of the surface at a given moment. Oscillatory motion will not give a net contribution to the integral, because the contributions from the rising parts (increasing potential energy) and falling parts (increasing kinetic energy) will cancel out, but translational motion will give a net contribution. A solitary wave, for example, moves a particle along a parabolic path [156].

We can therefore say that the nonzero parts of the integral (3.6) over the free surface will be due to a net translational motion of the horizontal coordinate of the centre of mass of the fluid. This is cancelled by the integral over the bottom of the channel, and both of these will generally be a function of time. Any number of peaks can be written either as a series of solitons or a periodic wave, and these progress in the same direction, without a change in speed when they are not overtaking. This means that the horizontal coordinate of the centre of mass must also move with a constant speed. Equivalently, each half of the integral (3.6) is only a linear function of time. Cancelling out the surface integral with the integral over the bottom justifies

$$\int_{-\infty}^{\infty} \frac{\partial u_0}{\partial t} x dx \propto \int_{-\infty}^{\infty} \frac{\partial \eta}{\partial t} x dx$$

as a conserved quantity. Toda [153] gives this the interpretation of momentum. In fact, we can see from the above argument that this more specifically represents the constant momentum of the fluid's centre of mass.

3.2.2 Kinetic and potential energy

For the non-trivial conservation laws, we will require some idea of area integrals. We therefore want to emphasise thinking of the fluid as occupying a thin region of the complex plane rather than requiring everything to be transferred back to the two-dimensional real plane. To do this, recall that every smooth function on the plane can be written in terms of z and $\overline{z}$, and that areas can be defined in terms of complex 2-forms; i.e. oriented two-dimensional area elements $dz \wedge d\overline{z}$, where $\wedge$ denotes the exterior product. For shorthand we will write

$$d^2z = dz \wedge d\overline{z}$$

which is equivalent to $d^2z = -2idxdy$.

The (dimensionless) kinetic energy at a point z will be proportional to $|\beta|^2$, so we expect that

$$\int_\Omega |\beta|^2 d^2z \tag{3.7}$$

where the integration is carried out over the entire body Ω of fluid at a given time, will be a constant of the motion. If we take

$$\beta = -\frac{\partial w}{\partial z} \tag{3.8}$$

then the energy integral transforms under (3.8) as a conformal transformation $z \mapsto w$ with Jacobian determinant $|\beta|^2$, i.e.

$$\int_\Omega \frac{\partial w}{\partial z}\frac{\partial \overline{w}}{\partial \overline{z}} d^2z = \int_\Omega d^2w$$

The range of values the potential w takes was assumed to be completely independent of time, so the total area represented by the integral must be a constant of the motion. The imaginary part of w was also required to be small, and we can always arrange that the real part of w is bounded by applying a Galilean transformation if necessary, so the integral can always be given a finite value. It follows that (3.7) is conserved, and can be given some finite value, say

$$T = \int_{\Omega} |\beta|^2 d^2 z \tag{3.9}$$

with

$$\dot{T} = 0$$

where the dot denotes a total derivative with respect to time.

If we look at only the horizontal component of velocity along the bottom, $u_0(x,t)$, we get the continuity equation

$$\frac{\partial}{\partial t}(u_0^2) = \frac{\partial}{\partial x}\left\{4u_0^3 + \left(\frac{\partial u_0}{\partial x}\right)^2 - 2u_0\frac{\partial^2 u_0}{\partial x^2}\right\} \tag{3.10}$$

This is more than just a rote check; this actually gives another more intuitive way to see that T should be constant. Since $u_0 \propto \eta$, we have

$$\frac{d}{dt}\int_{-\infty}^{\infty} \eta^2 dx = 0 \tag{3.11}$$

Because η represents displacement from equilibrium, we can identify the constant

$$V = \int_{-\infty}^{\infty} \eta^2 dx \tag{3.12}$$

with the total potential energy of the fluid, up to some unimportant scalar factor.

Again, in deriving the KdV equation, we have assumed that there be no dissipative effects on the fluid, such as frictional forces between the fluid and the channel itself, so the total energy has already implicitly been assumed constant. Since the potential energy V of the motion is constant and the atmospheric pressure has also been assumed constant, the kinetic energy T must also be constant. T is clearly proportional to the integral of the square of the speed at any point, taken over the whole fluid, so we can identify T with the integral (3.7), which gives us another proof. Alternatively, we could start from $\dot{T} = 0$ to obtain the result $\dot{V} = 0$.

3.2.3 General conservation laws

The complex KdV equation may lead to equations of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_z}{\partial z} + \frac{\partial J_{\overline{z}}}{\partial \overline{z}} = 0 \tag{3.13}$$

but this is not enough to entail any conservation law. The reason for this is because the region Ω which the fluid occupies is only a thin strip in the plane, and it is not necessary for the relevant physical quantities to vanish at the boundaries, since there is still nontrivial motion on the surface.

If we define

$$Q = \int_\Omega \rho(z, \overline{z}, t) i d^2 z \tag{3.14}$$

then Q evolves in time as

$$\dot{Q} = -\int_\Omega \left(\frac{\partial J_z}{\partial z} + \frac{\partial J_{\overline{z}}}{\partial \overline{z}} \right) i d^2 z \tag{3.15}$$

$$= i \oint_{\partial\Omega} J_{\overline{z}} dz - J_z d\overline{z} \tag{3.16}$$

by Stokes' theorem. Here $\partial\Omega$ denotes the boundary of the region Ω. It is obvious that $\dot{Q} = 0$ if its density ρ is a holomorphic function. But this is somewhat trivial. More generally, we will require the contribution to the integrals from the real axis to cancel with the surface terms when the functions J_z and $J_{\overline{z}}$ are only smooth, not necessarily holomorphic.

There is no loss in generality in considering our conserved quantity Q to be real, since (3.13) can be separated into real and imaginary parts if not. When Q is real, then we must have $J_{\overline{z}} = \overline{J}_z = J$. Then

$$\dot{Q} = -2\Im \left(\oint_{\partial\Omega} \overline{J} dz \right) \tag{3.17}$$

where $\Im$ denotes the imaginary part. There is a clear geometric interpretation here: with the same idea as in (3.5) can write

$$\oint_{\partial\Omega} \overline{J} dz = \oint_{\partial\Omega} J_{\parallel} ds + i \oint_{\partial\Omega} J_{\perp} ds \tag{3.18}$$

where $J_{\parallel}$ and $J_{\perp}$ are the parts of the corresponding current vector $\mathbf{J} = (J_x, J_y)$ which are respectively tangential and normal to the boundary curve $\partial\Omega$. This makes it clear that (3.18) is just the physical statement that Q is constant if and only if the amount of the current $\mathbf{J}$ entering the fluid at any given moment in time is balanced by the amount leaving.

However, it is not immediately clear when (3.18) holds. This makes for some difficulties, because (3.13) only implies the existence of a conserved quantity when the integral over the boundaries vanishes. This means that a transformation which converts one equation of the continuity type to another, like Gardner's transformation, is not necessarily enough.

We can gain some clarity by approximating these terms with a first-order series expansion in y. There is no real loss in information by doing so, since we have already done this repeatedly when dealing with the derivation of the complex KdV equation and its relation to the real KdV equation [136]. There we applied an expansion in perturbation series with a small parameter ε, and the shallow-fluid condition was equivalent to the estimate that $y = O(\varepsilon^{\frac{1}{2}})$ at most. Without loss of generality, each term in the perturbation series was taken to be $O(1)$. In particular, $\beta = O(1)$, so in any approximation of β, $O(\varepsilon)$ terms can be neglected, since they should appear at the next order of smallness. With this in mind, if we take the series expansion

$$J(z, \overline{z}, t) = J(x, x, t) + iy \left[\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y} \right]_{y=0} + O(y^2) \tag{3.19}$$

then the real and imaginary parts of the current

$$J = J_x + iJ_y$$

can be identified with

$$J_x = J(x,x,t), \;\; J_y = y\left[\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}\right]_{y=0} \tag{3.20}$$

Return to (3.13), and notice that it can also be written in terms of real and imaginary parts of J as the more familiar

$$\frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = 0 \tag{3.21}$$

which can also be written to first order as

$$\frac{\partial \rho}{\partial t} + 2\frac{\partial J_x}{\partial x} + \left.\frac{\partial J_y}{\partial y}\right|_{y=0} \simeq 0 \tag{3.22}$$

Because J_x, the real part of J, is taken as the value of J on the real axis, this provides a very simple way of seeing where these area integrals (3.17) will be constant: If (3.13) reduces to one of the members of the well-known infinite family of conservation laws of the real KdV equation when taken solely on the real axis, then (3.13) would reduce to

$$\frac{\partial \rho}{\partial t} + 2\frac{\partial J_x}{\partial x} = 0$$

and we must therefore have, for all of these forms of conservation laws, that the y-component must change an order of magnitude more slowly than the other terms, so that it is justified to neglect the term

$$\left.\frac{\partial J_y}{\partial y}\right|_{y=0} \simeq 0$$

in (3.22).

These are both equations of motion for ρ. It follows that a conserved area integral of the form (3.14) can be given when there is a corresponding conserved quantity

$$\int_{-\infty}^{\infty} \rho(x,x,t)dx$$

in the real KdV equation. This establishes an equivalence between the conserved quantities of the complex KdV equation, taken as integrals over the area occupied by the fluid, and the real KdV equation, taken as integrals over the real axis.

We have already seen an example of this with kinetic energy and potential energy. For the kinetic energy, we can take the current

$$J = -3|\beta|^2\beta - \overline{\beta}\frac{\partial^2\beta}{\partial z^2} \tag{3.23}$$

which reduces to the conservation law (3.10) when $y = 0$, remembering that $\partial_{\overline{z}}\beta = \partial_z\overline{\beta} = 0$.

The next simplest example is

$$J = 6\beta\left|\frac{\partial\beta}{\partial z}\right|^2 + \frac{\partial\overline{\beta}}{\partial\overline{z}}\frac{\partial^3\beta}{\partial z^3} + 4|\beta|^2\beta^2 + 3|\beta|^4 + \overline{\beta}^2\frac{\partial^2\beta}{\partial z^2} + \\ + 2|\beta|^2\frac{\partial^2\beta}{\partial z^2} - \overline{\beta}\left(\frac{\partial\beta}{\partial z}\right)^2$$

and this is associated with the conserved quantity

$$\int_\Omega \left\{\left|\frac{\partial\beta}{\partial z}\right|^2 - |\beta|^2(\beta+\overline{\beta})\right\} id^2z \tag{3.24}$$

These two can be simply motivated by considering the already-known conserved quantities, and then seeking to generalise them to an area integral over Ω. For these two cases, there are only so many choices which even make sense. For example, in generalising (3.10), an integral of β^2 does not give any new information, since we already know that β is holomorphic, so this could be trivially put in the form (3.13) with the term $J_{\overline{z}} = 0$. Similar holds for $\overline{\beta}^2$. This leaves the only possible choice which holds non-trivial information as $|\beta|^2$. In a similar way, the integral (3.24) is also the only non-trivial choice.

It is easy to see that this becomes more complicated quite quickly. This naturally raises the question of how these may be generated in some recursive process. As already pointed out, Gardner's transformation is not sufficient here, because the boundary of the fluid, $\partial\Omega$, is not pushed out to infinity. This remains a problem which may be answered in future research.

Lastly, we have dealt only with simple analogues of the conservation laws of the real KdV equation. This argument does not preclude the existence of more conserved quantities of the form (3.14) which do not have simple analogues, or even any analogues, in the real KdV equation at all.

3.3 Physical origin of the split NLS equations

Next, we will point out an interesting physical connection between the complex KdV theory and the split NLS equations.

It is well known that the nonlinear Schrödinger equation can be derived from the KdV equation in the quasi-monochromatic approximation [135, 166, 168]. However, this is typically done when the KdV equation is assumed to be a strictly real-variable entity. When the KdV equation is considered with a complex spatial variable, things are slightly different. This is because when considering the Fourier series of a complex solution to the KdV equation, the real and imaginary parts are generally independent, so there is no need to assume that each term also comes with its complex conjugate in the sum.

We will show that this has consequences.

Let ε be a small real parameter, let $\zeta = \varepsilon z$, $\tau = \varepsilon t$, and let a solution to the KdV equation (3.1) be formally given by the Fourier series

$$\beta(z,t) = \varepsilon^2 A_0(\zeta,\tau) + \sum_{n=1}^{\infty} \varepsilon^n \{A_n(\zeta,\tau)e^{ni\theta} + A_{-n}(\zeta,\tau)e^{-ni\theta}\} \tag{3.25}$$

where $\theta = kz - \omega t$ and ω is a function of the wavenumber k. In order that the lowest order terms vanish in (3.1), the frequency and wavenumber must be related by the dispersion relation

$$\omega = -k^3 \tag{3.26}$$

The Fourier series (3.25) represents the decomposition of a KdV-type wave into a superposition of slowly-modulated harmonics.

The linear terms are easily given in series form. The first few terms in the expansion of the nonlinear part of the KdV equation are

$$\beta\frac{\partial\beta}{\partial z} = ikA_1^2e^{2i\theta}\varepsilon^2 + \left(A_1\frac{\partial A_{-1}}{\partial\zeta} + ikA_0A_1e^{i\theta} + \right. \\ \left. + ikA_0A_1e^{2i\theta}\right)\varepsilon^3 + [+ \leftrightarrow -] + O(\varepsilon^4)$$

For (3.25) to satisfy the KdV equation (3.1), we find that the following series must vanish:

$$\left\{\left(\frac{\partial A_1}{\partial\tau} - 3k^2\frac{\partial A_1}{\partial\zeta}\right)e^{i\theta} - ik(k^2A_2 + A_1^2)e^{2i\theta}\right\}\varepsilon^2 + \\ + \left\{\frac{\partial A_0}{\partial\tau} + 3ik\left(\frac{\partial^2 A_1}{\partial\zeta^2} - 2A_0A_1\right)e^{i\theta} + \cdots\right\}\varepsilon^3 + \\ + [+ \leftrightarrow -] + \cdots = 0$$

We use the notation $[+ \leftrightarrow -]$ to mean all the rest of the terms obtained by replacing $A_ne^{ni\theta} \leftrightarrow A_{-n}e^{-ni\theta}$. In the real-variable case, this would represent the complex conjugate terms. Again, here they do not, since there is no requirement that the terms in the Fourier series be real.

The coefficients of $e^{ni\theta}$ in (3.1) must vanish, since these terms are linearly independent. This leads to the immediate conditions

$$A_{\pm 2} = -\left(\frac{A_{\pm 1}}{k}\right)^2 \tag{3.27}$$

and

$$\frac{\partial A_0}{\partial\tau} = 6\frac{\partial(A_{-1}A_1)}{\partial\zeta} \tag{3.28}$$

In a frame moving with the group velocity $\omega'(k) = -3k^2$, say

$$\zeta \mapsto \zeta + 3k^2\tau \tag{3.29}$$

it is easy to see that (3.28) implies

$$A_0 \simeq \frac{2}{k^2} A_{-1} A_1$$

to leading order. This change of variable also eliminates the first order ζ-derivative at $O(\varepsilon^2)$.

Over a higher order time scale, i.e. if we make the additional scaling

$$\tau \mapsto \varepsilon\tau, \tag{3.30}$$

the vanishing of the coefficients of the fundamental mode give the equation

$$\frac{\partial A_1}{\partial \tau} + 3ik\frac{\partial^2 A_1}{\partial \zeta^2} - \frac{6i}{k} A_{-1} A_1^2 = 0 \tag{3.31}$$

Following through the same procedure for the coefficients of $e^{-i\theta}$ gives a similar equation with A_{-1} in place of A_1, and we are left with the coupled pair of equations

$$\begin{aligned} i\frac{\partial A_1}{\partial \tau} - 3k\frac{\partial^2 A_1}{\partial \zeta^2} + \frac{6}{k} A_{-1} A_1^2 = 0 \\ i\frac{\partial A_{-1}}{\partial \tau} + 3k\frac{\partial^2 A_{-1}}{\partial \zeta^2} - \frac{6}{k} A_1 A_{-1}^2 = 0 \end{aligned} \tag{3.32}$$

which are the split nonlinear Schrödinger equations [167] for a complex variable ζ.

The split NLS system (3.32) therefore physically represents the evolution of the slowly-modulated fundamental mode of a periodic solution to the complex KdV equation. Because the complex KdV equation physically represents the first order perturbations of a velocity field [136], the split NLS equations can be physically interpreted as determining the evolution of the velocity of the fundamental mode.

If we take the real and imaginary parts of the solutions to the split NLS equations as

$$A_{\pm 1} = P_{\pm 1} + iQ_{\pm 1} \tag{3.33}$$

and set $\beta = u - iv$, then we can see that the horizontal and vertical components of velocity have fundamental modes

$$u = \{(P_1 + P_{-1})\cos\theta - (Q_1 - Q_{-1})\sin\theta\}\varepsilon + O(\varepsilon^2)$$
$$v = -\{(Q_1 + Q_{-1})\cos\theta + (P_1 - P_{-1})\sin\theta\}\varepsilon + O(\varepsilon^2)$$

i.e. the sum and difference of A_1 and A_{-1} directly give the coefficients of the fundamental.

This way of looking at the split NLS equations is also completely consistent with the usual interpretation of the usual NLS equation in hydrodynamics.

In keeping with our physical interpretation, when the spatial variable z is real, the function $\beta(z,t)$ also becomes real, because it represents a complex velocity in a channel with a horizontal bottom, and there is only horizontal flow at the very bottom. Recall from (3.2) that the horizontal velocity on the bottom, i.e. $\beta(x,t)$, is directly proportional to the elevation of the surface, which satisfies a real-variable KdV equation [136]. So in the context of the hydrodynamic problem, when z is reduced to a real variable x, and ζ to a real variable ξ, we are forced to take $A_{-n} = \overline{A}_n$ so that $\beta(x,t)$ is real. The coefficients A_n then represent the modes in the expansion of the surface elevation, up to a constant of proportionality. The split NLS system (3.32) reduces to the familiar defocusing nonlinear Schrödinger equation

$$i\frac{\partial A_1}{\partial \tau} + 3k\frac{\partial^2 A_1}{\partial \xi^2} - \frac{6}{k}|A_1|^2 A_1 = 0 \tag{3.34}$$

together with its usual interpretation: describing the evolution of the amplitude $|A_1|$ of a slowly modulated wavetrain [28].

Another less formal way to see that the split NLS equations come out of the complex KdV equation as a description of shallow water waves is the following. It is known [4] that the cnoidal solution of the real KdV equation, representing the wave amplitude, approaches the Stokes approximation for values of the wave modulus near 1. This wave has a Fourier series the same as (3.25) except that $A_{-n} = \overline{A}_n$ for all n. As just mentioned, the NLS equation is known to physically describe the evolution of the fundamental mode in a Stokes wave. In shallow water, the Stokes wave suffers a defocusing effect, so it is the defocusing NLS equation [28] which must govern the evolution of the fundamental term A_1.

If we now look at a solution of the complex KdV equation, and replace the complex conjugate terms in the Fourier series, then we must replace the nonlinear term $|A_1|^2 A_1$ in the NLS equation with $A_1^2 A_{-1}$. The defocusing NLS equation would then look like

$$i\frac{\partial A_1}{\partial \tau} + \frac{\partial^2 A_1}{\partial \xi^2} - \nu |A_1|^2 A_1 = 0$$
$$\downarrow$$
$$i\frac{\partial A_1}{\partial \tau} + \frac{\partial^2 A_1}{\partial \zeta^2} - \nu A_1^2 A_{-1} = 0$$

where $A_{\pm 1}$ and ζ are complex and $\nu > 0$ is a positive constant, so that the dispersive and nonlinear terms have opposite sign. Because of the KdV dispersion relation (3.26) making the frequency ω, and thus also the phase θ, an odd function of k, the Fourier series is unchanged by making the replacements $A_n \leftrightarrow A_{-n}$, $k \mapsto -k$. So we must also have

$$-i\frac{\partial A_{-1}}{\partial \tau} + \frac{\partial^2 A_{-1}}{\partial \zeta^2} - \nu A_{-1}^2 A_1 = 0$$

as the evolution equation for A_{-1}. It follows that A_1 and A_{-1} are given by the split NLS equations, which, again, are now the slowly-evolving fundamental modes in the expansion of the complex velocity.

We have now shown that the split NLS equations both physically stem from the complex KdV equation, and have an interpretation as the equation of a velocity field, which is entirely consistent with the hydrodynamic theory of the nonlinear Schrödinger equation.

3.4 Conclusion

In this work, we have discussed some of the consequences of considering the complex KdV equation as the fundamental evolution equation of a weakly nonlinear shallow water wave. We discussed its conservation laws, and showed that considering solutions to the complex KdV equation to define a complex velocity leads to physically natural conservation laws defined over the whole body of fluid in the simplest few cases. In particular, while the simplest nontrivial conservation law of the real KdV equation is associated with potential energy, the analogous conserved quantity in the complex KdV equation is associated with a conserved kinetic energy. We showed that even in the more complicated, less physically obvious cases, it is still possible to define conserved quantities which correspond to the known conserved quantities of the real KdV equation.

By considering a periodic solution of the complex KdV equation, we showed that in the quasi-monochromatic approximation, it is possible to derive the split NLS equations. The split NLS equations were shown to directly describe the fundamental modes of the velocity of a periodic wave in the KdV equation. This gives possibly the first directly physical interpretation of the system of split NLS equations, thus showing that it is not merely a mathematical tool, as it was originally introduced, but rather that it emerges on its own in hydrodynamics. When considering the special case in which both spatial variables are real, the relation (3.2) between the horizontal velocity on the bottom of the channel and the elevation of the surface reduced the split NLS equations to the defocusing nonlinear Schrödinger equation, and gave the correct interpretation: an equation of the amplitude of the wave envelope.

The real KdV equation is one of the foundational keystones of modern physics [17, 20, 22, 97]. The knowledge gained while studying this equation provided several crucial contributions to mathematical physics [169–171]. Its direct extension to the equation with a complex valued function of real temporal and spatial variables is also known [172], and these solutions can also be treated in analogy with the solutions of the real KdV equation [158, 173–175]. However, the real complications start when one of the independent variables (spatial one) is also taken to be complex. It is especially beneficial that this complication occurred to be not a mathematical artefact, but to describe the real physical problem of shallow water waves and to include much more detail in their description [136]. Our present work sheds more light on this technique adding more tools that can be used in using this approach in practice. We believe these new steps may lead to a whole new strategy of problem solving in the theory of shallow water waves and, potentially, some other fields based on the complex KdV equation [164, 165, 176].

4

Rogue Wave Mutliplets in the Complex Korteweg-de Vries Equation

We present a multi-parameter family of rational solutions to the complex Korteweg–de Vries (KdV) equations. This family of solutions includes particular cases with high-amplitude peaks at the centre, as well as a multitude of cases in which high-order rogue waves are partially split into lower-order fundamental components. We present an empirically-found symmetry which introduces a parameter controlling the splitting of the rogue wave components into multi-peak solutions, and allows for nonsingular solutions at higher order.

4.1 Introduction

Rogue waves are known to exist on deep ocean surfaces [55, 145], within water in the form of internal rogue waves [177, 178], in optical fibres in supercontinuum generation [59, 61], in the vacuum in the form of quantum fluctuations [179] and even in the theory of gravitational waves [180]. Their universality has been confirmed by water tank experiments [181], in quadratic nonlinear crystals [182], and, most strikingly, in our encounters with extreme natural phenomena [183]. The most common approach to the description of rogue waves is using the exact solutions of integrable evolution systems such as three-wave interaction [184], nonlinear Schrödinger (NLS) [148, 185], Kadomtsev-Petviashvili [186, 187], and Davey-Stewartson [188] equations, among others [189]. Rational Peregrine-like solutions of these equations provide a good approximation to describing rogue wave formation in a variety of physical situations [190, 191].

The real Korteweg-de Vries (KdV) equation [4] is the basis of the most common tool for the (1+1)-dimensional modelling of shallow water waves, which has been in use since the work of Boussinesq [3]. Numerical modelling done by Zabusky and Kruskal revealed the presence of soliton solutions of this equation [17], and the inverse scattering technique developed for the KdV equation enabled the derivation of analytic solutions for given initial conditions with zeros at infinity [20]. Being the historic first among the integrable nonlinear evolution equations, the KdV equation attracted significant attention from both physicists and mathematicians [169, 171, 192, 193].

Despite such extensive interest, until very recently, the KdV equation was thought to lack rogue wave solutions. This is true, but only if the wave described by the KdV equation is purely real. If we consider complex-valued solutions to the KdV equation, it is possible to derive rogue wave solutions [194]. Being the first work on this subject, the paper [194], however, presented only selected (fixed-parameter) rogue wave solutions and did not reveal the large variety of possible features of this important class of solutions. In this work, we provide a more detailed mathematical treatment and derive families of rogue wave solutions with free parameters that determine a range of features. The presence of several parameters in our equations makes our approach much more powerful than in previous work [194]. Here, by use of a simple symmetry of the n-fold Darboux transformation, we show that rational solutions to the KdV equation can be substantially generalised to describe a much larger variety of rogue waves. In principle, depending on the choice of parameters involved, these rational solutions may be either singular or nonsingular. We also show that higher order rogue waves in the complex KdV equation can appear in multi-peak formations, in a similar way to the rogue waves of the NLS equation [119, 121, 195].

4.2 The n-th order rational solution for the complex KdV equation

We will consider the complex KdV equation in the form

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial z} + \frac{\partial^3 u}{\partial z^3} = 0. \tag{4.1}$$

where $z = x+iy$ is a complex variable, and $u = u(z,t)$ a complex function. Regardless of whether u is real or complex, (4.1) is also the condition of compatibility of the system

$$\frac{\partial \psi}{\partial t} = -4\frac{\partial^3 \psi}{\partial z^3} + 6u\frac{\partial \psi}{\partial z} + 3\frac{\partial u}{\partial z}\psi, \tag{4.2}$$

$$-\frac{\partial^2 \psi}{\partial z^2} + u\psi = \lambda\psi. \tag{4.3}$$

This equivalence has several consequences. One of the most important for our purposes is that the system (4.2, 4.3) is Darboux covariant, giving us a dressing method to construct nontrivial solutions to (4.1) from simple ones. Given an initial (seed) solution $u = u_0$ to the KdV equation, and n linearly independent solutions $\psi_1, \ldots, \psi_n$ to the associated linear system (4.2, 4.3), with corresponding spectral parameters

$\lambda = \lambda_1, \ldots, \lambda = \lambda_n$, the n-fold Darboux transformation of u_0 is given by [71]

$$u_n = u_0 - 2\frac{\partial^2}{\partial z^2} \log W_{n+1}, \tag{4.4}$$

where W_n is the Wrońskian determinant of the functions $\psi_1, \ldots, \psi_n$ with respect to z:

$$W_n = \begin{vmatrix} \psi_1 & \psi_2 & \ldots & \psi_n \\ \partial_z\psi_1 & \partial_z\psi_2 & \ldots & \partial_z\psi_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial_z^{n-1}\psi_1 & \partial_z^{n-1}\psi_2 & \ldots & \partial_z^{n-1}\psi_n \end{vmatrix}, \tag{4.5}$$

and $u = u_n$ will be another solution to the KdV equation (4.1). To be more concise we omit writing explicitly the dependence of W_n on the functions $\psi_1, \ldots, \psi_n$.

In order that the transformation (4.4) be non-trivial, the parameters λ_k must be distinct. In order to obtain a Darboux transformation in the degenerate case $\lambda_k \to \lambda$ for all $k = 1, 2, \ldots, n$, we define $\psi_1, \ldots, \psi_n$ such that $\psi(z, t; \lambda_k) = \psi_k(z, t)$. Then, expanding the matrix element $\partial_z^{\,i}\psi_j$ as a Taylor series with respect to λ_k, we have

$$\lim_{\substack{\lambda_k \to \lambda \\ 1 \leqslant k \leqslant n}} W_n = \begin{vmatrix} \psi & \partial_\lambda\psi & \ldots & \partial_\lambda^{n-1}\psi \\ \partial_z\psi & \partial_\lambda\partial_z\psi & \ldots & \partial_\lambda^{n-1}\partial_z\psi \\ \vdots & \vdots & \ddots & \vdots \\ \partial_z^{n-1}\psi & \partial_\lambda\partial_z^{n-1}\psi & \ldots & \partial_\lambda^{n-1}\partial_z^{n-1}\psi \end{vmatrix}, \tag{4.6}$$

i.e. in the degenerate limit W_n becomes the Wrońskian of the functions $\psi, \partial_\lambda\psi, \ldots, \partial_\lambda^{n-1}\psi$.

So if we take, for example, the simple constant seed solution $u_0(z, t) = c$, we can take as linearly independent solutions to the system (4.2, 4.3) the functions

$$\psi_k(z, t) = \cosh \omega_k\{z + 2(3c - 2\omega_k^{\,2})t\} \tag{4.7}$$

where $\omega_k = \sqrt{c - \lambda_k}$ for $\lambda_k \neq c$. In (4.7), each eigenvalue λ_k is distinct.

We will also note here that due to the translational invariance $u(z, t) \mapsto u(z - z_0, t - t_0)$ of the KdV equation (4.1), we can introduce a second constant, via $t \mapsto t - t_0$. This will not affect the choice of ψ in any substantial way, but this will be relevant later, so we allow for arbitrary shifts in z and t. For simplicity's sake, we set a background $c = 0$, and the function ψ in (4.7) becomes

$$\psi(z, t; \lambda_k) = \cos \sqrt{\lambda_k}\{z + 4\lambda_k(t - it_0)\}, \tag{4.8}$$

from which we get

$$W_2(z, t) = -\tfrac{1}{2}z - 6\lambda(t - it_0) - \frac{\sin 2\sqrt{\lambda}\{z + 4\lambda(t - it_0)\}}{4\sqrt{\lambda}},$$

with $W_2(z, t) \to -z$ as $\lambda \to 0$.

The imaginary part of z ensures that the Wrońskian $W_2(z, t)$ has no zeros in z and t except for the origin, and in the limit as $\lambda \to c$, in this case as $\lambda \to 0$, the Wrońskian becomes a polynomial in x and t. The corresponding degenerate solution u_1 to the complex KdV equation will thus always be a non-singular rational function as $\lambda \to 0$ if y is chosen appropriately. In the simplest case, $u_1(x, t)$ becomes

$$u_1(z, t) = \frac{2}{z^2} = \frac{2}{(x + iy)^2}.$$

The 3×3 Wrońskian becomes in the limit

$$\lim_{\lambda \to 0} W_3(x, t) = -\tfrac{2}{3}(x + iy_0)^3 - 8(t - it_0).$$

For better clarity, we will write K_n for the limit of the Wrońskian W_{n+1} as $\lambda \to 0$, i.e.

$$K_n(x + iy_0, t - it_0) = \lim_{\lambda \to 0} W_{n+1}, \tag{4.9}$$

so that in general, the n-th order rational solution of the KdV equation is given by

$$u_n(x, t) = -2\frac{\partial^2}{\partial x^2} \log K_n(x, t). \tag{4.10}$$

Singularities do not appear in all parts of the complex plane. If we choose the parameters of the solution such that the system

$$\Re\{K_n(x+iy,t-it_0)\}=0,\ \Im\{K_n(x+iy,t-it_0)\}=0$$

has no solution in real values of x and t. To find a nonsingular second-order solution, we observe that if t_0 is strictly real, then

$$\Im\{K_2(x+iy,t-it_0)\}=2(x^2y-\tfrac{1}{3}y^{\,2}-4t_0),$$

and this is a quadratic in x with no real zero in x in the region

$$\frac{y^{\,3}+12t_0}{y}<0.$$

In this region of the complex plane, the solutions will not have any singularities for any values of x.

The constant t_0 allows us to avoid any real zeros in the denominator of the second-order solution, since if $t_0=0$, the equation $K_2(x+iy,t-it_0)=0$ always has roots in real values of x and t for any choice of y.

The second-order rational solution of the complex KdV equation is

$$u_2(z,t)=6z\frac{z^3-24(t-it_0)}{\{z^3+12(t-it_0)\}^2}. \tag{4.11}$$

The plot of this function for fixed parameters y and t_0 is shown in Fig. 4.2.1. With these restrictions on y, this has the form of a rogue wave with the maximal amplitude at the origin, being given in terms of y and t_0 by

$$|u_2(0,0)|=6\frac{|y^{\,4}+24yt_0|}{(y^{\,3}-12t_0)^2}.$$

It is a straightforward exercise to write rational solutions of any order n. To give a few more examples, the Wrońskians K_n as $\lambda\to 0$ for the third to fifth-order solutions

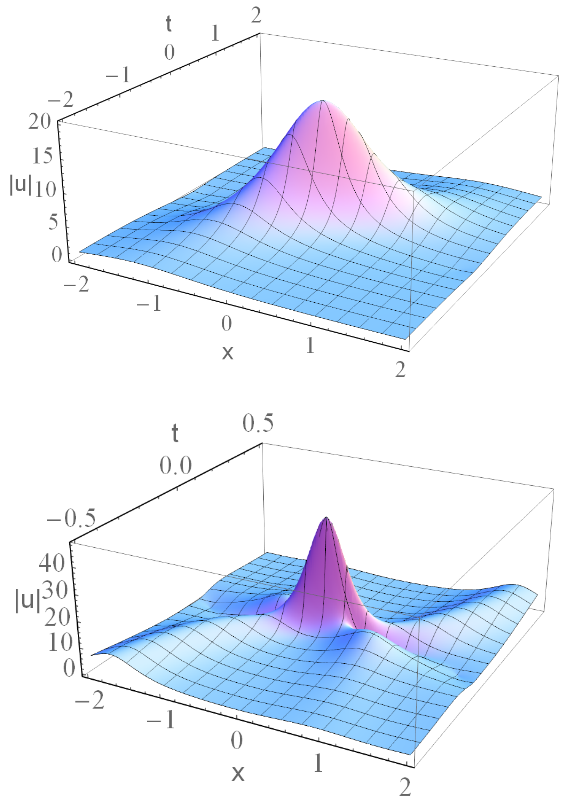

Figure 4.2.1: *(a) The plot of the second-order rogue wave (4.11) with $y = \frac{3}{2}$ and $t_0 = \frac{1}{2}$. The maximal amplitude is $|u_2(0,0)| = 20.0816$. (b) The plot of the third-order rogue wave, defined by (4.13). Parameters are $y = \frac{1}{2}$, $t_0 = -\frac{1}{24}$, and $g = -\frac{6}{5}i$. The maximal amplitude is $|u_3(0,0)| = 50$. The background for both cases is $c = 0$.*

are

$$
\begin{aligned}
K_3(z,t) &= \frac{4}{15}(z^6 + 60z^3 t - 720t^2), \\
K_4(z,t) &= \frac{32}{515}(z^{10} + 180z^7 t + 302400zt^3), \\
K_5(z,t) &= \frac{256}{33075}(z^{15} + 420z^{12}t + 25200z^9t^2 + \\
&+ 2116800z^6t^3 - 254016000z^3t^4 - \\
&- 1524096000t^5).
\end{aligned}
$$

up to translations $z \mapsto z + z_0,\ t \mapsto t - it_0$.

Since $K_n(z,t)$ is an $(n+1)\times(n+1)$ determinant, the explicit formulae quickly become more cumbersome with increasing n, although the description in terms of the determinant holds for all n. The second-order rogue wave in Fig. 4.2.1(a) is the simplest one among the hierarchy of KdV rogue waves. It has a single maximum, and smoothly growing and decaying fronts.

The symmetry

$$u(z,t) \mapsto c + u(z + 6ct, t) \tag{4.12}$$

of the KdV equation allows us to chose the background c arbitrarily, but at the expense of a travelling velocity of $6c$. The two lowest order solutions with this adjustment become

$$\begin{aligned} u_1(z,t) &= c - \frac{2}{(z+6ct)^2}, \\ u_2(z,t) &= c - 2\frac{\partial^2}{\partial z^2}\log\{(z+6ct)^3 + 12(t - it_0)\}. \end{aligned}$$

These are generalisations of previously obtained rogue wave solutions. Namely, when $c = -1$, $y = -\frac{1}{2}$ and $t_0 = \frac{1}{24}$, the above solutions coincide with those derived in [194] by the complex Miura transformation.

Multi-peak solutions

The higher-order solutions of the hierarchy are more complicated. Moreover, the usual expressions for them are always singular and may not describe physical situations. In order to obtain solutions which can be nonsingular, we have to go beyond the standard dressing technique.

Solutions of order $n \geqslant 3$ can be further generalised by making use of an empirically found symmetry which allows us to replace the functions K_n with linear combinations of K_n and K_{n-2} : If

$$V_n(z,t;g) = K_n(z,t) - gK_{n-2}(z,t),\ n = 3, 4, \ldots, \tag{4.13}$$

where g is an arbitrary constant, then

$$u = u_n(z,t;g) = -2\frac{\partial^2}{\partial z^2}\log V_n(z,t;g)\ (n \geqslant 3) \tag{4.14}$$

is also a solution of the KdV equation (4.1). This symmetry can be verified by direct substitution as we have done for all cases found here, i.e. up to $n = 5$, and we conjecture it holds in general for all n.

If we were to extend this symmetry to the $n = 2$ case, then it would make sense to identify K_0 as simply a constant, since this would generate the solution $u = 0$. Then (4.13) for $n = 2$ would just be introducing a complex additive constant. If $\Re(g) = 0$, it is identical to the substitution $t \mapsto t - it_0$.

When $n = 3$, we recover a third-order rational solution,

$$u_3(z,t;g) = \frac{P_3(z,t-it_0;g)}{\{V_3(z,t-it_0;g)\}^2}, \tag{4.15}$$

$$\begin{aligned} P_3(z,t;g) = {} & 450g^2 + 2160gz^5 + 8294400zt^3 + \\ & + 1036800z^4t^2 + 192z^{10}, \end{aligned} \tag{4.16}$$

up to the symmetry (4.12). The parameters t_0, g and c here allow us to control the shape of the rogue wave. One example of this solution for background $c = 0$, with $y = \frac{1}{2}$, $t_0 = -\frac{1}{24}$ and $g = -\frac{6}{5}i$ is shown in Fig. 4.2.1(b). This same choice of parameters with background $c = -1$ again reduces to the particular third-order rogue wave solution given in [194]. This, like (4.11), has a single central peak at the origin, but now there are two sets of tails, resembling a two-soliton collision.

When the parameter g is purely imaginary, the rogue wave is dominated by its central peak. The imaginary part of g affects the relative heights of the peaks and the tails. When $\Im(g)$ becomes large, the peaks reduce in size relative to the tails, and for sufficiently large values, the peaks may be even smaller than the tails. On the other hand, the real part of g causes splitting of the lower order components, so that they do not directly collide at the origin. Instead, with $\Re(g) \neq 0$, we see growth of multiple peaks. In Fig. 4.2.2(a), the central peak splits into two smaller ones, their locations and amplitudes depending on g and t_0. Another example is shown in Fig. 4.2.2(b), in which we see that the effect of the parameter t_0, and position y on the imaginary axis, can be to transfer amplitude from one peak to another.

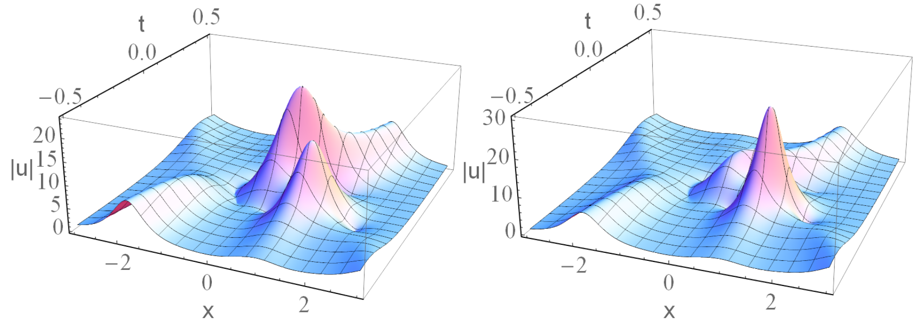

Figure 4.2.2: *Two plots of the third-order rogue wave. (a) Parameters are the same as in Fig. 4.2.1(b) except now $g = -5 - \frac{6}{5}i$. A second peak has grown and the larger peak has decreased in total amplitude and moved out from the origin. (b) Parameters are the same as in (a) except now $t_0 = -\frac{1}{40}$. One of the peaks has faded and the other peak has gained more amplitude.*

The general fourth-order rational solution in explicit form is given by

$$u_4(z,t;g) = \frac{P_4(z,t-it_0;g)}{\{K_4(z,t) - gK_2(z,t)\}^2}, \tag{4.17}$$

$$\begin{aligned} P_4(z,t;g) = {} & 50\{32z^9 + 4032z^6t + 967680t^3 + 105gz^2\}^2 - \\ & - 60z\{35g + 48z^4(z^3+84t)\}\{16z^{10} + \\ & + 2880z^7t + 4838400zt^3 + 175g(z^3+12t)\}. \end{aligned} \tag{4.18}$$

The explicit form of the fifth-order rational solution is

$$u_5(z,t;g) = \frac{P_5(z,t-it_0;g)}{\{K_5(z,t) - gK_3(z,t)\}^2}, \tag{4.19}$$

$$\begin{aligned} P_5(z,t;g) = {} & 60z\{972405g^2(43200t^3 + 5400t^2z^3 + z^9)+ \\ & + 2048(-154857907814400000t^9 + \\ & + 19357238476800000z^3t^8 + 184354652160000z^6t^7 + \\ & + 17667320832000z^9t^6 + 512096256000z^{12}t^5 + \\ & + 1447891200z^{15}t^4 + 120960z^{21}t^2 + 504z^{24}t + z^{27})\}- \\ & - 282240g(9144576000t^6 - 21772800z^6t^4 - \\ & - 483840z^9t^3 + 22680z^{12}t^2 + 252z^{15}t + z^{18}). \end{aligned} \tag{4.20}$$

Higher order rational solutions can also be written in similar form, but quickly exceed reasonable limits of presentability.

Two examples of the fourth-order solution for given sets of parameters and zero background c are shown in Fig. 4.2.3. The profile of this solution can again take a multiplicity of forms. When the parameter g is purely imaginary, as in Fig. 4.2.3(a), most of the rogue wave amplitude is concentrated in the central peak, although two small, symmetrically located side peaks are also present. The maximal amplitude of the central peak here is 82.

An example of the fourth order rogue wave for g with nonzero real part is shown in Fig. 4.2.3(b). Again, the real part of g causes multiple peaks to grow. There we have three distinct large peaks but of smaller amplitudes, each roughly 20 to 30. Their relative locations and values of velocity are again determined by g and t_0.

This extreme localisation shown in all graphs is the characteristic feature of rogue waves [37].

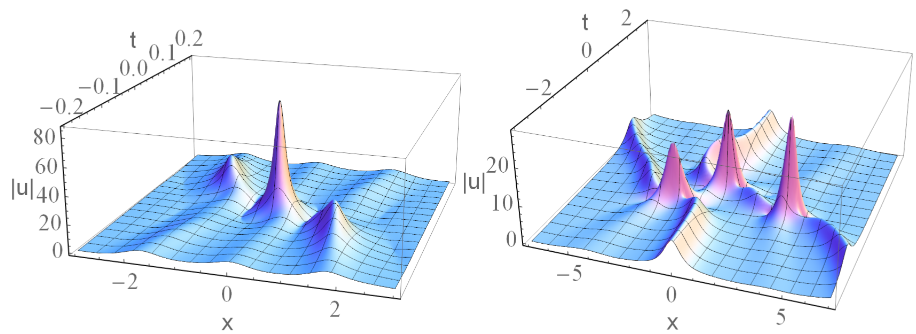

Figure 4.2.3: *Two plots of the fourth-order rogue wave. (a) Parameters are* $y = -\frac{1}{2}$, $t_0 = \frac{1}{24}$, $g = -6i$ *and* $c = 0$*. The maximal velocity is* $|u_4(0,0)| = 82$*. (b) Parameters are the same except now* $g = 1888 - 12i$*. With* g *having nonzero real part, the main peak has decreased in amplitude and the smaller peaks have grown and moved away from the origin.*

4.3 Conclusion

The crucial step taken in our work is the generalisation (4.13). Despite being as simple as a linear superposition law for the Wrońskians, it has important nontrivial consequences for the whole family of rational solutions, allowing them to be non-singular i.e. physically relevant rogue waves. It also adds the complex parameter g that is essential in the higher order rational solutions for removing singularities and for splitting the higher-order rogue wave into its fundamental components. When the real part of g is zero, then the rogue wave has the highest peak at the origin and smaller local maxima around it, as shown in Fig. 4.2.3(a). On the other hand, when g has nonzero real part, the central large peak decreases and smaller side peaks grow while separating from each other. This type of splitting of higher order rogue waves into multiplet structures has also been observed in the case of NLS rogue waves [119, 121, 195] and their extensions [140]. However, the splitting of the higher-order rogue waves of the complex KdV equation is more complicated. The complete classification of all forms of rogue waves here remains open for investigation.

Lastly, we point out that complex solutions of the KdV equation are also applicable to unidirectional crystal growth [164] and complex KdV-like equations serve to model dust-acoustic waves in magnetoplasmas [176]. The KdV hierarchy itself also finds applications in modern theories of quantum gravity [196].

As such, these new solutions presented in this work, previously not thought to exist, may find much wider use in various areas of physics.

5

Doubly-Periodic Solutions of the Class I Infinitely Extended Nonlinear Schrödinger Equation

We present doubly-periodic solutions of the infinitely extended nonlinear Schrödinger equation with an arbitrary number of higher-order terms and corresponding free real parameters. Solutions have one additional free variable parameter that allows to vary periods along the two axes. The presence of infinitely many free parameters provides many possibilities in applying the solutions to nonlinear wave evolution. Being general, this solution admits several particular cases which are also given in this chapter. We give a thorough introduction to the key ideas which will be used for both the current chapter and the next.

5.1 Introduction

Evolution equations are a powerful tool for describing a great variety of physical effects. Using evolution equations, one can explain phenomena that would otherwise be difficult to interpret. Examples of such phenomena include solitons [17, 197], modulation instability [30, 198, 199], supercontinuum generation [59], Fermi-Pasta-Ulam recurrence [200], rogue waves [55, 184, 188, 191, 201], etc. It is especially helpful when the evolution equations under study are integrable [202]. Unfortunately, this is not always the case. Not all evolution equations are integrable [143, 144, 203]. Finding new integrable equations [149, 201, 204], and extending the existing ones to allow for incorporating new, physically relevant terms [205–209], is therefore an important direction of research in nonlinear dynamics.

The nonlinear Schrödinger (NLS) equation is one of the fundamental examples of a completely integrable equation [29, 167] which finds application in such areas as the description of water waves [28, 55, 145, 191], pulses in optical fibres [44, 210, 211], Bose-Einstein condensates [212–215], waves in the atmosphere [216], as well as plasma [217] and many other physical systems [218–222]. However, in none of these fields is the NLS equation absolutely accurate. Extensions of the NLS equation that have physical relevance are therefore essential.

Various extensions of the NLS equation have been considered [223–225] that increase the accuracy of the description of nonlinear wave phenomena in these systems by incorporating higher-order effects [184, 206, 207, 226]. Higher-order terms in these extensions are responsible for linear dispersion, as well as nonlinear effects such as self-phase modulation, pulse self-steepening, the Raman effect, and so on [208, 210]. These higher-order terms are important in nonlinear optics [227, 228], ocean wave dynamics [143, 144, 148] and especially in modelling high-amplitude rogue wave phenomena [188, 191, 201].

In general, these extensions lift the integrability for most particular physical problems. However, in special cases, we can obtain extensions which remain integrable, and, in addition, we can add infinitely many higher-order terms with variable coefficients representing the strength of these effects, adding substantial flexibility to the evolution equation.

There are two types of extensions of the NLS equation [229–232]. For clarity, we call them here the class I and class II extensions. The next higher-order terms in the class I extension correspond to the Hirota equation [229–231], while the next higher-order terms in the class II extension correspond to the Sasa-Satsuma equation [232, 233].

Both of these extensions take into account higher order dispersive effects, without any restriction on the magnitude of these effects, i.e. without requiring them to be small perturbations. In practice, waves are affected by more than just second-order dispersion, so solutions to the infinite equations are an important development in that they allow a generalisation of the fundamental structures which appear in the 'basic' nonlinear Schrödinger equation to account for these effects. When the number of higher-order terms is limited to the third order, integrability can be achieved with variation of two free parameters [125]. For infinitely extended equations, the number of free parameters is also infinite.

The presence of two classes of integrable extensions thus widens the range of problems that can be solved analytically. Remarkably, solutions to both classes can be found in general form, even for the case of an infinite number of terms, and an infinite number of corresponding parameters. In order to find these solutions, we can start with the known solutions of the NLS equation and extend them, recalculating the parameters of the solution. This can be done for soliton, breather and rogue wave solutions [230, 231]. In the present work, we further expand this approach to doubly-periodic solutions. They include as particular cases solitons and breathers.

5.2 The class I infinitely extended NLS equation

First, we give a brief exposition of the class I infinitely extended nonlinear Schrödinger equation. It is the integrable equation written in general form [229, 230]

$$i\psi_x + F(\psi, \psi^*) = 0, \tag{5.1}$$

where the operator $F(\psi, \psi^*)$ is defined through

$$F = \sum_{n=1}^{\infty}(\alpha_{2n}K_{2n} - i\alpha_{2n+1}K_{2n+1}), \tag{5.2}$$

with the operators K_n defined recursively by the integrals of the nonlinear Schrödinger equation [229], and where each coefficient α_n is an arbitrary real number; that is,

$$K_n(\psi) = (-1)^n \frac{\delta}{\delta\psi^*}\int p_{n+1}dt,$$

where p_n is the n-th integral of the basic nonlinear Schrödinger equation, and p_{n+1} can be defined recursively as

$$p_{n+1} = \psi\frac{\partial}{\partial t}\left(\frac{p_n}{\psi}\right) + \sum_{r=1}^{n} p_{n-r}p_r,\ p_1 = |\psi|^2.$$

Exact forms for the class I form of the operators $K_n(\psi)$ are given in [230]. The four lowest order operators K_n are presented below.

$$\begin{aligned}
K_2(\psi) &= \psi_{tt} + 2|\psi|^2\psi, \\
K_3(\psi) &= \psi_{ttt} + 6|\psi|^2\psi_t, \\
K_4(\psi) &= \psi_{tttt} + 8|\psi|^2\psi_{tt} + 6\psi|\psi|^4 + \\
&\quad +4\psi|\psi_t|^2 + 6\psi_t^2\psi^* + 2\psi^2\psi_{tt}^*. \\
K_5(\psi) &= \psi_{ttttt} + 10|\psi|^2\psi_{ttt} + 10(\psi\,|\psi_t|^2)_t + \\
&\quad +20\psi^*\psi_t\psi_{tt} + 30|\psi|^4\psi_t.
\end{aligned} \tag{5.3}$$

The coefficients α_n determine the strength of the dispersive effects of order n, as well as higher-order nonlinear effects. The whole infinite equation (5.1) is integrable for arbitrary values of α_n.

To be specific, we start with the standard focusing NLS equation:

$$i\frac{\partial\psi}{\partial x} + \alpha_2\left(\frac{\partial^2\psi}{\partial t^2} + 2|\psi|^2\psi\right) = 0 \tag{5.4}$$

where $\psi = \psi(x,t)$ is the wave envelope, x is the distance along the fibre or along the water surface, while t is the retarded time in the frame moving with the group velocity of wave packets. The coefficient α_2 scales the dispersion and nonlinear terms in a way convenient for the extensions.

The equation with the terms up to the third order is the Hirota equation

$$i\frac{\partial\psi}{\partial x} + \alpha_2\left(\frac{\partial^2\psi}{\partial t^2} + |\psi|^2\psi\right) - i\alpha_3\left(\frac{\partial^3\psi}{\partial t^3} + 6|\psi|^2\frac{\partial\psi}{\partial t}\right) = 0 \tag{5.5}$$

while including up to fourth order terms gives the Lakshmanan-Porsezian-Daniel (LPD) equation [205], and so on. Particular solutions of the first-order to the equation (5.1) have been given in [230, 231]. Solutions of the mKdV equation, which is another particular case of (5.1), are provided in [234]. Any of the extensions of (5.1) with only a few nonzero terms can be considered individually.

As mentioned, the numbers α_n can take any values whatsoever, and do not need to be viewed as representing small perturbations for the equation (5.1) to be completely integrable. This allows us to find solutions for which any order of dispersion can be taken into account without the need for approximation or numerical techniques. This extension substantially widens the range of applicability of the NLS equation for solving nonlinear wave evolution problems. However, physical applications, in general, do require dispersive effects to decrease rapidly in strength with increasing order n. Convergence will thus not be an issue in practice for series involving α_n, and we will therefore be comfortable leaving the operator F for the whole equation (5.1), as well as any other associated parameters, in the form of an infinite series when necessary.

Among the more general families of solutions to the nonlinear Schrödinger equation (5.4) are the doubly-periodic solutions [36]. The two periods of this family can be varied, thus providing particular cases in the form of solitons, breathers, cnoidal, dnoidal and Peregrine waves when one or two of these periods tend to infinity or zero [36]. Unlike breather solutions, however, these doubly-periodic solutions do not decay in either space or time, and instead have the special property of being periodic in both the x and t variables.

In this work, we show that doubly-periodic solutions can be found for the class I equation. This solution includes infinitely many parameters α_n in full generality. We also show that particular limiting cases of this family include the Akhmediev breather and soliton solutions.

5.3 Doubly-periodic solutions

There are two types of doubly periodic solutions to the nonlinear Schrödinger equation, which can be classified as type-A and type-B [235]. Each of them is expressed in terms of Jacobi elliptic functions, with the modulus k as the free parameter of the family. First, we consider the type-A solutions.

5.3.1 Type-A Solutions

Type-A solutions of Eq. (5.1) are of the form

$$\psi(x,t) = \frac{k\,\mathrm{sn}\,(Bx/k,k) - iC(t+vx)\,\mathrm{dn}\,(Bx/k,k)}{k - kC(t+vx)\,\mathrm{cn}\,(Bx/k,k)} e^{i\phi x} \tag{5.6}$$

where

$$C(t) = \sqrt{\frac{k}{1+k}}\,\mathrm{cn}\left(\sqrt{\frac{2}{k}}t, \sqrt{\frac{1-k}{2}}\right).$$

The constants B, v and ϕ in the solution (5.6) are given in terms of the coefficients α_n of the equation (5.1). Taking into account the lowest order terms, step by step, we find:

$$\begin{aligned} B &= 2\alpha_2 + 8\alpha_4 + \left(32 - \frac{4}{k^2}\right)\alpha_6 + \\ &+ \left(128 - \frac{32}{k^2}\right)\alpha_8 + \left(512 - \frac{192}{k^2} + \frac{12}{k^4}\right)\alpha_{10} + \\ &+ \left(2048 - \frac{1024}{k^2} + \frac{144}{k^4}\right)\alpha_{12} + \cdots \end{aligned} \tag{5.7}$$

$$\begin{aligned} \phi &= 2\alpha_2 + \left(8 - \frac{2}{k^2}\right)\alpha_4 + \left(32 - \frac{12}{k^2}\right)\alpha_6 + \\ &+ \left(128 - \frac{64}{k^2} + \frac{6}{k^4}\right)\alpha_8 + \left(512 - \frac{320}{k^2} + \frac{60}{k^4}\right)\alpha_{10} + \\ &+ \left(2048 - \frac{1536}{k^2} + \frac{432}{k^4} - \frac{20}{k^6}\right)\alpha_{12} + \cdots \end{aligned} \tag{5.8}$$

$$\begin{aligned} v &= 4\alpha_3 + \left(16 - \frac{2}{k^2}\right)\alpha_5 + \left(64 - \frac{16}{k^2}\right)\alpha_7 + \\ &+ \left(256 - \frac{96}{k^2} + \frac{6}{k^4}\right)\alpha_9 + \left(1024 - \frac{512}{k^2} + \frac{72}{k^4}\right)\alpha_{11} + \\ &+ \left(4096 - \frac{2560}{k^2} + \frac{576}{k^4} - \frac{20}{k^6}\right)\alpha_{13} + \cdots \end{aligned} \tag{5.9}$$

An important observation here is that the expression for v which is responsible for the 'tilt' in (x, t)-plane discussed below includes only odd-order coefficients α_n. If these are zero, v is also zero.

In order to determine the general forms, with all α_n included, we note that only one of these sets of polynomial coefficients is algebraically independent. For $n \geqslant 1$, the coefficient of α_{2n-1} in v is half the coefficient of α_{2n} in B, and is also the sum of the coefficients of α_{2n} in B and ϕ. It is therefore sufficient to determine the coefficients of B, since, if we let

$$B = \sum_{n=1}^{\infty} B_n \alpha_{2n},$$

then

$$v = \sum_{n=1}^{\infty} \tfrac{1}{2} B_{n+1} \alpha_{2n+1}, \;\; \phi = \sum_{n=1}^{\infty} (\tfrac{1}{2} B_{n+1} - B_n) \alpha_{2n}.$$

Calculating further the other terms of B_n, we find that they are the polynomials

$$B_n = 2^{2n-1} \sum_{r=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^r (2r)!(n-r-1)!}{2^{4r}(r!)^3(n-2r-1)!} \frac{1}{k^{2r}},$$

$\lfloor \cdot \rfloor$ being the floor function, i.e. $\lfloor m \rfloor$ is the largest integer which is no greater than m. Now the full expression for B is given explicitly by the series formula

$$B = \sum_{n=1}^{\infty} 2^{2n-1} \sum_{r=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r-1}{r} \frac{1}{k^{2r}} \alpha_{2n}. \tag{5.10}$$

It also follows that

$$v = \sum_{n=1}^{\infty} 2^{2n} \sum_{r=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r}{r} \frac{1}{k^{2r}} \alpha_{2n+1}, \tag{5.11}$$

and the phase factor ϕ is

$$\phi = \sum_{n=1}^{\infty} 2^{2n-1} \Bigg\{ 2 \sum_{r=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r}{r} \frac{1}{k^{2r}} - \\ - \sum_{r=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r-1}{r} \frac{1}{k^{2r}} \Bigg\} \alpha_{2n} \tag{5.12}$$

We plot an example of the type-A solution for Eq. (5.1) in Fig. 5.3.1. For this example, we take the modulus $k = 0.7$, and the coefficients $\alpha_n = 1/n!$ up to $n = 10$, restricting ourselves with the case when all terms higher than $n = 10$ are zero. We can see from Fig. 5.3.1 that v introduces a tilt to the solutions and appears to operate similarly to a velocity parameter in a boost transformation. However, note that v cannot be interpreted exactly as a velocity, as there is no function f such that we could write $\psi(x,t) = f(t + vx)$ as we could do with a travelling wave, except in the case that $\phi = B = 0$.

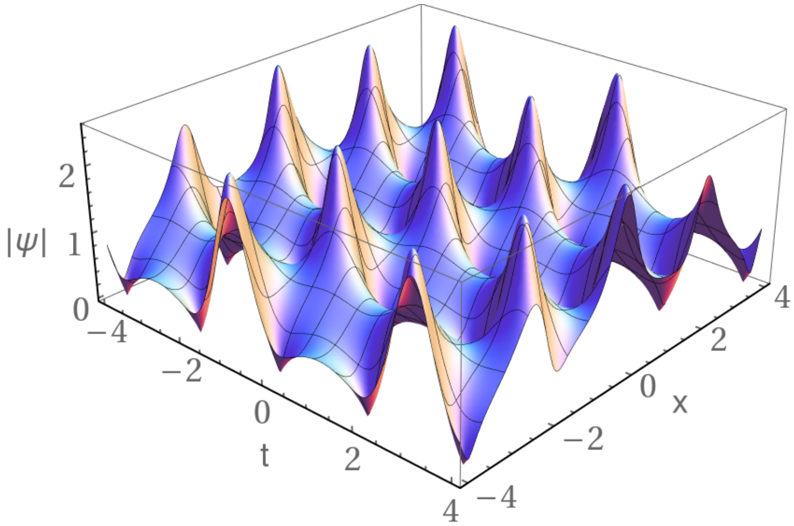

Figure 5.3.1: *Type-A solution for Eq. (5.1), with $k = 0.7$, $\alpha_n = 1/n!$ up to $n = 10$ and all other $\alpha_n = 0$. Notice that v is nonzero.*

From equation (5.6), we can see that the parameter B/k can be associated with a frequency of the modulation along the x axis when $v = 0$. On the other hand, the real quarter-period along the t axis is:

$$\sqrt{\frac{k}{2}} K \left(\sqrt{\frac{1-k}{2}} \right),$$

where $K(k)$ denotes the complete elliptic integral of the first kind with modulus k. However, just as v cannot be precisely interpreted as a velocity, neither can B/k be thought of as a modulation frequency exactly, except when $v = 0$ and the solution is periodic along the x-axis.

5.3.2 The Akhmediev Breather Limit

In the limit as modulus $k \to 1$, we have

$$\lim_{k\to 1} B = \sum_{n=1}^{\infty} \binom{2n}{n} n F(1-n, 1; \tfrac{3}{2}; \tfrac{1}{2}) \alpha_{2n}, \tag{5.13}$$

$$\lim_{k\to 1} \phi = \sum_{n=1}^{\infty} \binom{2n}{n} \alpha_{2n} \tag{5.14}$$

$$\lim_{k\to 1} v = \sum_{n=1}^{\infty} \binom{2n}{n} (2n+1) F(-n, 1; \tfrac{3}{2}; \tfrac{1}{2}) \alpha_{2n+1} \tag{5.15}$$

where $F(a, b; c; z)$ is Gauss' hypergeometric function. The type-A solution then reduces to the Akhmediev breather with modulation parameter $\sqrt{2}$ [230]; i.e. the solution becomes

$$\lim_{k\to 1} \psi(x,t) = \frac{\sqrt{2}\sinh Bx - i\cos\sqrt{2}(t+vx)}{\sqrt{2}\cosh Bx - \cos\sqrt{2}(t+vx)} e^{i\phi x} \tag{5.16}$$

with B, ϕ, and v given by (5.13), (5.14), and (5.15), respectively. An example of this limiting case is plotted in Fig. 5.3.2.

5.3.3 Type-B Solutions

Type-B solutions can be considered as the analytic continuation of the type-A solutions for values of the modulus $k > 1$. Using the corresponding transformations of the elliptic functions with modulus $\kappa = 1/k$, these solutions take the form

$$\psi(x,t) = \kappa e^{i\phi x} \frac{\sqrt{1+\kappa}\,\mathrm{sn}(Bx, \kappa) - iA(t+vx)\,\mathrm{cn}(Bx, \kappa)}{\sqrt{1+\kappa} - A(t+vx)\,\mathrm{dn}(Bx, \kappa)} \tag{5.17}$$

where the function $A(t)$ is given by

$$A(t) = \mathrm{cd}\left(\sqrt{1+\kappa}t, \sqrt{\frac{1-\kappa}{1+\kappa}}\right),$$

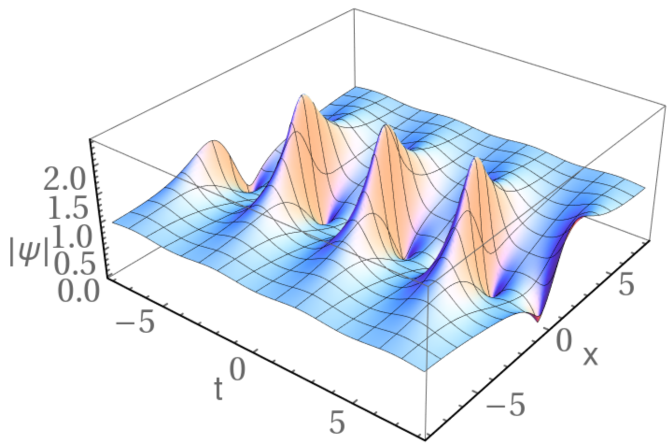

Figure 5.3.2: *The limiting case $k \to 1$ of the type-A solutions, with $\alpha_n = (n!)^2/(2n)!$ up to $n = 12$, all other $\alpha_n = 0$. This is the Akhmediev breather solution of Eq. (5.1), periodic in t and with the growth-decay cycle in x.*

and κ is in the range $0 < \kappa < 1$. In this case, we find:

$$B = \sum_{n=1}^{\infty} 2^{2n-1} \sum_{r=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r-1}{r} \kappa^{2r} \alpha_{2n}, \tag{5.18}$$

$$\phi = \sum_{n=1}^{\infty} 2^{2n-1} \Bigg\{ 2 \sum_{r=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r}{r} \kappa^{2r} - \\ - \sum_{r=0}^{\lfloor \frac{1}{2}n \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r-1}{r} \kappa^{2r} \Bigg\} \alpha_{2n} \tag{5.19}$$

$$v = \sum_{n=1}^{\infty} 2^{2n} \sum_{r=0}^{\lfloor \frac{1}{2}(n+1) \rfloor} \frac{(-1)^r}{2^{4r}} \binom{2r}{r} \binom{n-r}{r} \kappa^{2r} \alpha_{2n+1}. \tag{5.20}$$

These are just the same polynomials as previously given for the type-A solutions, but with reciprocal argument $\kappa = 1/k$.

We plot an example of type-B solutions in Fig. 5.3.3. The solution is qualitatively

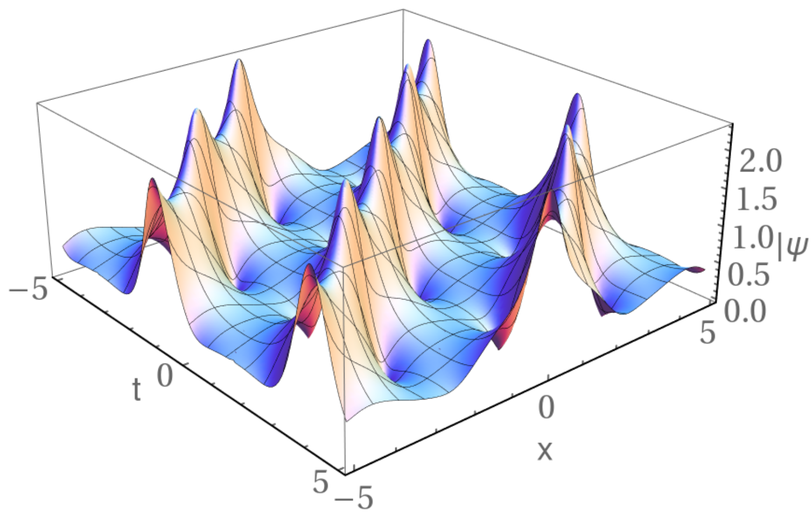

Figure 5.3.3: *The type-B solution (5.17), where* $\kappa = 0.7$, $\alpha_n = 1/n!$ *up to* $n = 8$, *with all other* $\alpha_n = 0$. *The peaks of this solution are aligned along lines of constant* $\tau = t + vx$.

different from the type-A solution as the location of maxima are now different.

The limit as $\kappa \to 1$ is identical to the limit as $k \to 1$ for the reasons just stated previously, and we again recover the separatrix breather solution (5.16). However, by changing the parameter κ, we can vary the periods of the type-B solutions while always keeping the functions analytic, so that in the limit $\kappa \to 0$ we recover the soliton solution

$$\lim_{\kappa \to 0} \psi(x,t) = 2e^{i\phi x} \operatorname{sech}(2t + vx) \tag{5.21}$$

with

$$v = \sum_{n=1}^{\infty} 2^{2n-1} \alpha_{2n+1}$$

and

$$\phi = \sum_{n=1}^{\infty} 2^{2n} \alpha_{2n},$$

which is the general soliton solution for the equation (5.1), up to scaling [230].

5.4 Phase portrait of solutions

The transformation of the two periodic solutions into the Akhmediev breather (AB) when $k \to 1$ and $\kappa \to 1$ can be illustrated by the phase portrait of these solutions which is shown in Fig. 5.4.1. Although the dynamical system that we are dealing with is infinite-dimensional, the dynamics of the solutions still can be presented on a two-dimensional plane which can be considered as a projection of the infinite-dimensional phase space onto a plane. The Akhmediev breather solution (5.16) on this plane is represented by the heteroclinic orbits connecting two saddle-points. It is a separatrix between the type-A and type-B solutions, represented respectively by the periodic orbits A and B.

The projection of infinite-dimensional phase space onto a plane requires certain tricks, as the solutions involve drift. In order to avoid the corresponding shifts, we make the change of variable $\xi = Bx$, $\tau = t + vx$, and define the new function

$$u(\xi,\tau) = \psi(x,t)e^{-i\phi x}. \tag{5.22}$$

The counterbalancing exponential factor allows us to stop the rotation of ψ around the origin in the complex plane. Then it is easy to see that the trajectory corresponding to the AB solution satisfies the equation

$$\{\mathrm{Re}\, u(\xi,T)\}^2 + \{\mathrm{Im}\, u(\xi,T) - 1\}^2 = 2,\ T = \frac{n\pi}{\sqrt{2}}, \tag{5.23}$$

where $\mathrm{Re}\, u$ and $\mathrm{Im}\, u$ are the real and imaginary parts of u, respectively, and n is any integer. The trajectories defined by Eq. (5.23) are circular so long as we trace the evolution in ξ along these lines of constant τ. They are shown as black curves in Fig. 5.4.1. Similar precautions should be taken for the doubly-periodic orbits.

The difference between the type-A and type-B solutions can be seen clearly from Fig. 5.4.1. Trajectories for the type-A solutions never cross the real axis, whereas trajectories for the type-B solutions do. Therefore, each time they complete one full path, the phase change is either zero or 2π. The periodicity of solutions along the

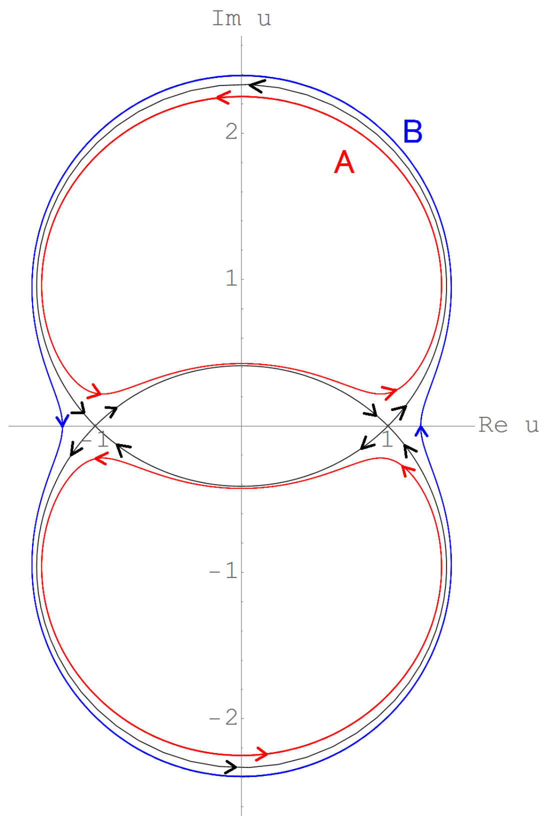

Figure 5.4.1: *The phase portrait of the periodic dynamics around the Akhmediev breather shown by the black curves. The type-A solution is shown by the red curve while the type-B solution is shown by the blue curve. Here u is defined by (5.22), and the two saddle points are at $u = \pm 1$. The trajectories are drawn along the lines $\xi = Bx$ and $\tau = t + vx$. Thus, evolution is in ξ along lines of constant τ.*

x-axis depends on the strength of the dispersion, or the values of the coefficients α_n, through ϕ, B, and v.

5.5 The case with zero even-order terms

In the absence of any even order terms in Eq. (5.1), it becomes real:

$$\frac{\partial \psi}{\partial x} - \sum_{n=1}^{\infty} \alpha_{2n+1} K_{2n+1}[\psi] = 0, \tag{5.24}$$

Then we have $\phi = 0$ and $B = 0$, and the type-B solution takes the real-valued form

$$\psi(x,t) = u(\tau) = \frac{\kappa A(\tau)}{\sqrt{1+\kappa} - A(\tau)}, \tag{5.25}$$

where $\tau = t + vx$. Note that here v can be interpreted as a velocity since u has the form of a travelling wave.

The real quarter-period in τ is equal to the real quarter-period in t of the usual type-B solutions, which is

$$\frac{1}{\sqrt{1+\kappa}} K\left(\sqrt{\frac{1-\kappa}{1+\kappa}}\right).$$

The real quarter-period in x, for fixed t, is $\sqrt{\kappa}K(\kappa)/v$ since $\phi = B = 0$.

In particular, with the normalisation $\alpha_3 = -1$ and all other $\alpha_n = 0$, this becomes the periodic solution to the mKdV equation

$$\psi_x + \psi_{ttt} + 6\psi^2\psi_t = 0,$$

given by

$$\psi(x,t) = \frac{\kappa \operatorname{cd}\left(\sqrt{1+\kappa}(t-4x), \sqrt{\frac{1-\kappa}{1+\kappa}}\right)}{\sqrt{1+\kappa} - \operatorname{cd}\left(\sqrt{1+\kappa}(t-4x), \sqrt{\frac{1-\kappa}{1+\kappa}}\right)}. \tag{5.26}$$

We plot an example of this solution in Fig. 5.5.1.

The degree of generality of our solutions allows one to consider many other particular cases. For example, some of the polynomial coefficients have real zeros for certain values of n. Taking $k^2 = \frac{1}{4}$ causes the influence of fourth-order dispersion on ϕ to vanish, as well as the effect of the sixth-order dispersion on the modulation frequency B, and similar for eighth-order dispersion in ϕ when $k^2 = \frac{1}{2}$. Considering all these cases can be useful for practical application of these solutions.

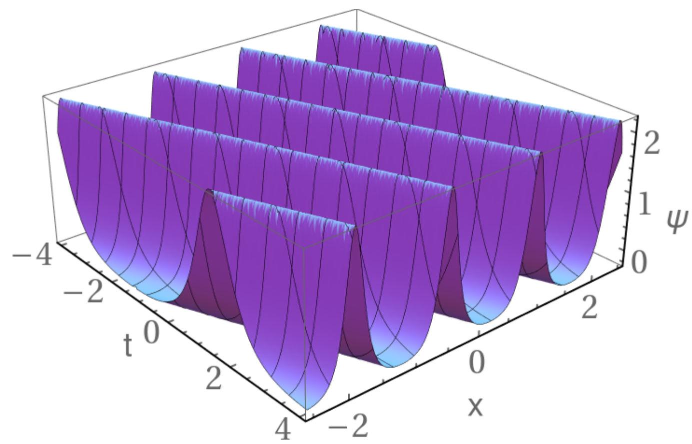

Figure 5.5.1: *Plot of real-valued mKdV equation solution ψ given by (5.26), where $k = \frac{1}{2}$. Here the type-B solution (5.17) reduces to a periodic solution propagating with speed* $v = 4$.

5.6 Conclusion

We have presented doubly-periodic solutions of type-A and type-B for the class I infinitely extended nonlinear Schrödinger equation. These solutions are expressed in terms of Jacobi elliptic functions, and have two variable periods along the two axes of the system. Being rather general, they include important cases of solutions: among them, the Akhmediev breather and the soliton solution are the limiting cases when the modulus of the elliptic functions is one or zero. As another particular case, we give a periodic solution of the mKdV equation.

As the equation under consideration has an infinite number of free parameters, this can be useful in modelling various physical problems of nonlinear wave evolution with a large degree of flexibility in choosing the parameters. The integrability of this equation allows one to write all solutions in explicit form, adding significantly more power into the analysis.

6

Two-breather solutions for the class I infinitely extended nonlinear Schrödinger equation and their special cases

We derive the two-breather solution of the class I infinitely extended nonlinear Schrödinger equation. We present a general form of this multi-parameter solution that includes infinitely many free parameters of the equation and free parameters of the two-breather components. Particular cases of this solution include rogue wave triplets, and special cases of 'breather-to-soliton' and 'rogue wave-to-soliton' transformations. The presence of many parameters in the solution allows one to describe wave propagation problems with higher accuracy than with the use of the basic NLS equation.

6.1 Introduction

In this chapter, we build upon the work introduced in the previous chapter, and consider 2-breather solutions of the class I infinitely extended NLS equation equation. These are multi-parameter solutions that involve both the free parameters of the equation, and free parameters of the solution, which together control the features of the two breather components, such as their localisation, propagation, and their relative position and frequencies. The presence of an infinite number of free parameters allows us to consider many particular cases, such as breather-to-soliton conversion, which is exclusive to higher-order extensions of the basic equation.

We also derive several limiting cases, the most important one of which is the general second-order rogue wave solution, a particular case of the 2-breather collision. However, only a limited number of special cases can be given in the frame of the present thesis. We leave others for future work in this direction.

As mentioned in the previous chapter, even though the operators K_n in (5.3) rapidly become more complicated and the resulting differential equation of order n becomes much harder to solve, exact solutions can be found explicitly by using already known solutions to the NLS equation as a guide, and a large class of breather and soliton solutions are already known [230, 231]. In previous works [230, 231], it has been seen that the effect of nonzero odd order operators is to transform t as $t \mapsto t + vx$ with v being a function of all coefficients α_{2n+1}. The effect of the nonzero even order operators is to transform x as $x \mapsto Bx$, with B being a function of the parameters α_{2n}.

However, it is easily seen that it is not quite so simple in the case of higher-order solutions involving more parameters, such as 2-breather solutions, because in this case, the solutions can be thought of as asymptotically separating into several components.

In this chapter we extend the approach from chapter 5 to a general family of second-order solutions, so we introduce parameters B_1 and B_2, and v_1 and v_2, to play an analogous role for the two distinct breather components. This enables us to generalise the 2-breather to the infinite extension of the NLS equation, and we now proceed to the analysis of these solutions.

6.2 The 2-breather solution

Higher analogues of the Akhmediev breathers can be obtained through iterations of the Darboux transformation [116, 120]. After transforming the plane wave solution e^{ix} with a Darboux transformation, with an eigenvalue λ such that $\lambda^2 \neq -1$, and repeating this transformation twice, we get the 2-breather solution to the basic NLS equation. This can then be generalised to the 2-breather solution of the extended equation. The general 2-breather solution is of the form

$$\psi(x,t) = \left\{1 + \frac{G(x,t) + iH(x,t)}{D(x,t)}\right\} e^{i\phi x}, \tag{6.1}$$

where

$$\begin{aligned}
G(x,t) &= -({\kappa_1}^2 - {\kappa_2}^2)\left\{\frac{{\kappa_1}^2\delta_2}{\kappa_2}\cosh\delta_1 B_1 x\cos\kappa_2 t_2 - \right. \\
&- \frac{{\kappa_2}^2\delta_1}{\kappa_1}\cosh\delta_2 B_2 x\cos\kappa_1 t_1 - \\
&- \left. ({\kappa_1}^2 - {\kappa_2}^2)\cosh\delta_1 B_1 x\cosh\delta_2 B_2 x\right\},
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
H(x,t) &= -2({\kappa_1}^2 - {\kappa_2}^2)\left\{\frac{\delta_1\delta_2}{\kappa_2}\sinh\delta_1 B_1 x\cos\kappa_2 t_2 - \right. \\
&- \frac{\delta_1\delta_2}{\kappa_1}\sinh\delta_2 B_2 x\cos\kappa_1 t_1 - \\
&- \delta_1\sinh\delta_1 B_1 x\cosh\delta_2 B_2 x + \\
&+ \left. \delta_2\cosh\delta_1 B_1 x\sinh\delta_2 B_2 x\right\},
\end{aligned} \tag{6.3}$$

$$\begin{aligned}
D(x,t) &= 2({\kappa_1}^2 + {\kappa_2}^2)\frac{\delta_1\delta_2}{\kappa_1\kappa_2}\cos\kappa_1 t_1\cos\kappa_2 t_2 + 4\delta_1\delta_2 \times \\
&\times (\sinh\delta_1 B_1 x\sinh\delta_2 B_2 x + \sin\kappa_1 t_1\sin\kappa_2 t_2) - \\
&- (2{\kappa_1}^2 - {\kappa_1}^2{\kappa_2}^2 + {\kappa_2}^2)\cosh\delta_1 B_1 x\cosh\delta_2 B_2 x - \\
&- 2({\kappa_1}^2 - {\kappa_2}^2)\left\{\frac{\delta_1}{\kappa_1}\cosh\delta_2 B_2 x\cos\kappa_1 t_1 - \right. \\
&- \left. \frac{\delta_2}{\kappa_2}\cosh\delta_1 B_1 x\cos\kappa_2 t_2\right\},
\end{aligned} \tag{6.4}$$

Here κ_1 and κ_2 are the modulation parameters,

$$\delta_m = \tfrac{1}{2}\kappa_m\sqrt{4-\kappa_m^2}$$

is the growth rate of the modulational instability for each breather component, and the shorthand notation t_m indicates $t_m = t + v_m x$ for $m = 1, 2$. Note that whenever t_m appears, we have ignored a constant of integration, and we have also done the same whenever $\delta_m B_m x$ appears. The most general solution allows for the replacements $t_m \mapsto t_m - T_m$, and $\delta_m B_m x \mapsto \delta_m B_m(x - X_m)$, where T_m and X_m are real constants which determine relative positions along the axes of t and x, respectively, which we might include to incorporate a time delay in one breather component, for instance. For the time being, we set these constants to be both zero without substantial loss, to address their significance later.

The phase factor ϕ is independent of the modulation, since this part has no physical effect on the modulation when it is real, and here it takes the same real value as it does for the plane wave solution, i.e.

$$\phi = \sum_{n=1}^{\infty}\binom{2n}{n}\alpha_{2n}. \tag{6.5}$$

The values B_m determine the modulation frequency of each component, and the parameters v_m, although they cannot be considered velocities in the usual sense, introduce a tilt to $|\psi|$ relative to the axes of x and t. They are given explicitly by

$$B_m = \sum_{n=1}^{\infty}\binom{2n}{n} nF(1-n, 1; \tfrac{3}{2}; \tfrac{1}{4}\kappa_m^2)\alpha_{2n}, \tag{6.6}$$

$$v_m = \sum_{n=1}^{\infty}\binom{2n}{n}(2n+1)F(-n, 1; \tfrac{3}{2}; \tfrac{1}{4}\kappa_m^2)\alpha_{2n+1}, \tag{6.7}$$

with $m = 1, 2$, where $F(a, b; c; z)$ is the Gaussian hypergeometric function. Note that there is a simple relationship between v_m and B_m: the coefficient of α_{2n} in B_m is twice the coefficient of α_{2n-1} in v_m. The first two terms of B_m for the two-breather solution have been previously given in [236]. Our new solution extends these coefficients to arbitrary orders of dispersion and nonlinearity.

Also notice that the parameters are the same functions of κ_m, for both $m = 1, 2$. This is at least suggested by symmetry. If two successive Darboux transformations generate a 2-breather solution, then there must be two independent eigenvalues, corresponding to two independent modulation parameters. Physically, we could reason that there should be no way of knowing which breather component is which, so the order in which each component was generated by Darboux transformation should be equally irrelevant. If so, it should then follow that B_1 is the same function of κ_1 as B_2 is of κ_2, and similar for v_1 and v_2, and this would also imply that B_m and v_m are the same functions as for the single-breather solution, which are already known [231]. It is worth considering whether this property extends to the general n-breather solution: i.e. whether, in general, we can find $B_1, \ldots, B_n$ and $v_1, \ldots, v_n$ which are the same functions of their respective modulation parameters $\kappa_1, \ldots, \kappa_n$, but we do not answer this question here.

The growth rate δ_m in both components will be real when κ_m is real, but the eigenvalues of the Darboux transformation are free to take any complex value at all, although the transformations are trivial when the eigenvalues are real, and thus so are the modulation parameters. Real-valued modulation parameters correspond to Akhmediev breathers, whereas imaginary-valued modulation parameters correspond to Kuznetsov-Ma solitons, the functional form of the breathers being otherwise equivalent. An example of the 2-breather solution for a fixed ratio of two real modulation parameters is shown in Fig. (**??**), and an example which shows the difference between real and imaginary modulation parameters is given in Fig. (6.2.2). In Fig. (6.2.3) we give an example of the effects of altering the ratio of the modulation of the two components.

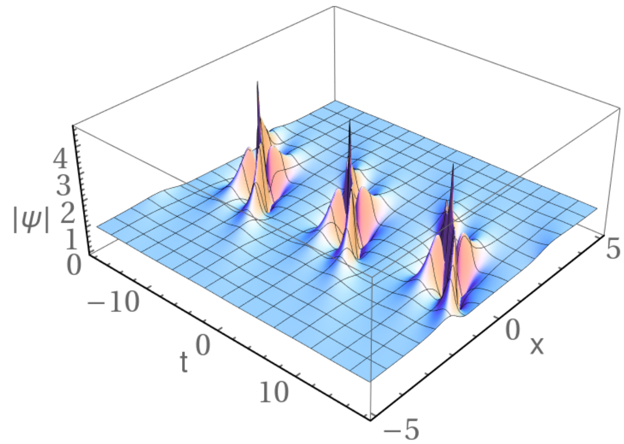

Figure 6.2.1: *The 2-breather solution (6.1) of Eq.(??). The modulation parameters are at a ratio $\kappa_1 : \kappa_2 = 1 : 2$. Parameters of the equation are: $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{6}$, $\alpha_4 = \frac{1}{24}$, $\alpha_5 = \frac{1}{30}$, $\alpha_6 = \frac{1}{144}$, with all higher $\alpha_n = 0$. The wave profile is tilted in the (x, t)-plane due to the nonzero v_m.*

6.3 Breather-to-soliton conversion

If we choose parameters α_n such that $B_m = 0$, the 2-breather solution may then behave in a way which is unique to the extension of the nonlinear Schrödinger equation [236], in the sense that it is only when higher orders of dispersion and nonlinearity are accounted for that it is possible to take $B_m = 0$ without obtaining a trivial or otherwise degenerate solution.

For example, if we choose α_2 such that $B_2 = 0$ for all κ_2, then writing $B_1 = B$, it is easy to show that B must take the value

$$B = \sum_{n=1}^{\infty} \binom{2n+2}{n+1} (n+1) E\left(-n, 1; \tfrac{3}{2}; \tfrac{1}{4}{\kappa_1}^2 | \tfrac{1}{4}{\kappa_2}^2\right) \alpha_{2n+2},$$

where we define the function E as the difference of hypergeometric functions:

$$E(a, b; c; z_1|z_2) = F(a, b; c; z_1) - F(a, b; c; z_2).$$

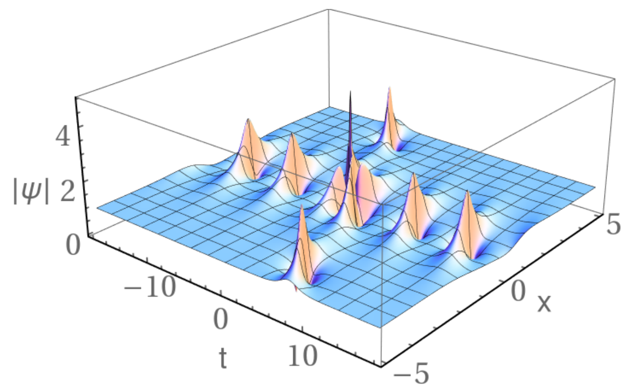

Figure 6.2.2: *A collision between an Akhmediev breather and Kuznetsov-Ma soliton, with* $\kappa_1 = 1$, *and* $\kappa_2 = i$. *Here* $\alpha_n = 1/n!$ *up to* $n = 8$, *with all higher* $\alpha_n = 0$.

We can then simplify the general 2-breather solution considerably. We obtain

$$G(x,t) = ({\kappa_1}^2 - {\kappa_2}^2)\Bigg\{\frac{{\kappa_1}^2\delta_2}{\kappa_2}\cosh\delta_1 Bx\cos\kappa_2 t_2 - \\ - \frac{{\kappa_2}^2\delta_1}{\kappa_1}\cos\kappa_1 t_1 - ({\kappa_1}^2 - {\kappa_2}^2)\cosh\delta_1 Bx\Bigg\},$$

$$H(x,t) = 2\delta_1({\kappa_1}^2 - {\kappa_2}^2)\sinh\delta_1 Bx\left(1 - \frac{\delta_2}{\kappa_2}\cos\kappa_2 t_2\right),$$

$$D(x,t) = 2({\kappa_1}^2 + {\kappa_2}^2)\frac{\delta_1\delta_2}{\kappa_1\kappa_2}\cos\kappa_1 t_1\cos\kappa_2 t_2 + \\ + 4\delta_1\delta_2\sin\kappa_1 t_1\sin\kappa_2 t_2 - \\ - (2{\kappa_1}^2 - {\kappa_1}^2{\kappa_2}^2 + {\kappa_2}^2)\cosh\delta_1 Bx - \\ - 2({\kappa_1}^2 - {\kappa_2}^2)\left(\frac{\delta_1}{\kappa_1}\cos\kappa_1 t_1 - \right. \\ \left. - \frac{\delta_2}{\kappa_2}\cosh\delta_1 Bx\cos\kappa_2 t_2\right). \quad (6.8)$$

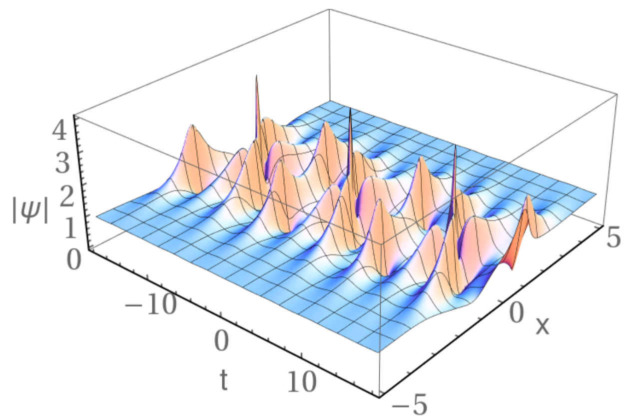

Figure 6.2.3: *The 2-breather solution with $\alpha_n = 1/n!$ up to $n = 8$, with all higher $\alpha_n = 0$, but now with $\kappa_1 = \frac{3}{2}$, and $\kappa_2 = 1$.*

An example of this solution is given in Fig. 6.3.1. The difference of this solution from the one shown in Fig. 6.2.1 is that the wave profiles at $x \to \pm\infty$ are not plane waves. The periodic set of tails from each breather maximum extends to infinity, reminiscent of periodically repeating solitons. This is the phenomenon that is known as breather-to-soliton conversion [236]. Clearly, these 'solitons' do not have a separate spectral parameter related to them.

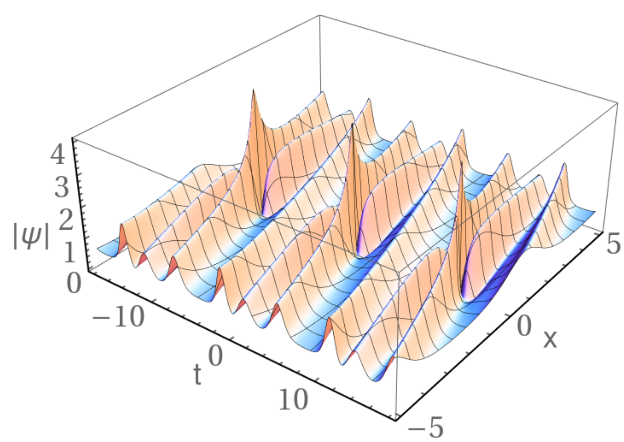

Figure 6.3.1: *A wave profile of a 'breather-to-soliton conversion'. We use the same set of parameters as in Fig. 6.2.3, except α_2 is now chosen such that $B_2 = 0$. This choice extends to infinity the tails of the breathers that would otherwise decay.*

6.4 The 2-breather solution in the semirational limit

When one of the modulation parameters, say κ_2, tends to zero, we obtain the semirational limit, i.e. a solution obtained as a combination of polynomials and circular or hyperbolic functions. Then, writing κ for κ_1, and δ for δ_1, the functions G, H, and D become

$$\begin{aligned} G(x,t) &= \tfrac{1}{8}\kappa^2\{\kappa^2(1+4t_2^2+4B_2^2x^2)-1\}\cosh\delta B_1 x + \\ &+ \kappa\delta\cos\kappa t_1, \end{aligned}$$

$$\begin{aligned} H(x,t) &= 2\kappa B_2 x(\delta\cos\kappa t_1 - \kappa\cosh\delta B_1 x) + \\ &+ \tfrac{1}{4}\delta\kappa^2(1+4t_2^2+4B_2^2x^2)\sinh\delta B_1 x \end{aligned}$$

$$\begin{aligned} D(x,t) &= \frac{\delta}{\kappa}\{4-\tfrac{1}{4}\kappa^2(1+4t_2^2+4B_2^2x^2)\}\cos\kappa t_1 + \\ &+ 4\delta B_2 x\sinh\delta B_1 x + \delta t_2\sin\kappa t_1 - \\ &- \{4+\tfrac{1}{4}\kappa^2(1+4t_2^2+4B_2^2x^2)\}\cosh\delta B_1 x, \end{aligned} \tag{6.9}$$

and the parameters B_2 and v_2 are reduced to

$$B_2 = \sum_{n=1}^{\infty}\binom{2n}{n} n\alpha_{2n}, \tag{6.10}$$

and

$$v_2 = \sum_{n=1}^{\infty}\binom{2n}{n}(2n+1)\alpha_{2n+1}. \tag{6.11}$$

This semirational 2-breather solution is a superposition of a Peregrine solution with the Akhmediev breather, since taking the limit $\kappa_2 \to 0$ reduces the frequency of one of the breathers to zero, meaning that it is transformed to a Peregrine solution. A plot of this solution is shown in Fig. 6.4.1. Here, the central feature is roughly the second order rogue wave while the peaks away from the origin belong to the remaining first-order breather.

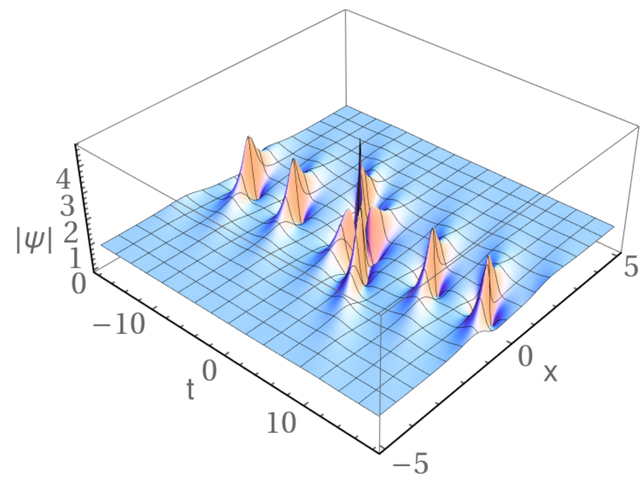

Figure 6.4.1: *The 2-breather solution in the semirational limit. Here the nonzero modulation parameter is $\kappa = 1$, with α_n the same as in Fig. (6.2.1). It can be considered as a superposition of the Akhmediev breather with the Peregrine solution.*

6.5 The degenerate two-breather limit

If both eigenvalues of the Darboux transformation are taken to be equal, so that both modulation parameters κ_m are also equal, we obtain the case of degenerate breathers. Direct calculations provide no solution. In this case, one modulation parameter should instead be taken as a small perturbation from the other, say $|\kappa_1 - \kappa_2| = \varepsilon$. Then, we take the limit as the perturbation ε becomes arbitrarily small, so that the solution remains well-defined at all times. Namely, if we put $\kappa_1 = \kappa$, and $\kappa_2 = \kappa + \varepsilon$, we have

$$B_2 = \sum_{n=1}^{\infty} \binom{2n}{n} nF(1-n, 1; \tfrac{3}{2}; \tfrac{1}{4}(\kappa+\varepsilon)^2)\alpha_{2n},$$

and

$$v_2 = \sum_{n=1}^{\infty} \binom{2n}{n} (2n+1)F(-n, 1; \tfrac{3}{2}; \tfrac{1}{4}(\kappa+\varepsilon)^2)\alpha_{2n+1}.$$

Next, recalling the identity

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z),$$

take the Maclaurin series of the $G(x,t)$, $H(x,t)$, and $D(x,t)$ with respect to ε. In the limit as $\varepsilon \to 0$, the ratio of these series will be a well-defined solution with equal eigenvalues; it is thus sufficient to consider only the lowest-order non-vanishing terms in the series expansion for $D(x,t)$ in ε, which in this case happen to be the coefficients of ε^2. By this method we obtain the degenerate 2-breather solution in the form (6.1) with

$$\begin{aligned} G(x,t) &= 2\kappa^2\Bigg[1+\cosh 2\delta Bx+\Bigg\{\Bigg(\kappa B-\frac{2\delta^2}{\kappa}B- \\ &- \delta^2 B'\Bigg)x\sinh\delta Bx- \\ &- \frac{\kappa}{\delta}\left(1-\frac{\delta^2}{\kappa^2}\right)\cosh\delta Bx\Bigg\}\cos\kappa(t+vx)- \\ &- \{t+(v+\kappa v')x\}\delta\cosh\delta Bx\sin\kappa(t+vx)\Bigg], \end{aligned}$$

$$\begin{aligned} H(x,t) &= 2\kappa\left[\left\{\left(\frac{2\delta^2}{\kappa^2}-1\right)\kappa B+2\delta^2 B'\right\}x+\right. \\ &+ \tfrac{1}{2}\delta\left\{\tfrac{1}{2}\left(\frac{2\delta^2}{\kappa^2}-1\right)Bx-\frac{\delta^2}{\kappa}B'\right\}x\cosh\delta Bx\times \\ &\times \cos\kappa(t+vx)+\frac{\kappa}{\delta}\left(\frac{2\delta^2}{\kappa^2}-1\right)\sinh 2\delta Bx- \\ &- \delta^2\sinh\delta Bx\{\cos\kappa(t+vx)+ \\ &+ \left.\kappa\sin\kappa(t+vx)\}\{t+(v+\kappa v')x\}\right], \end{aligned}$$

$$\begin{aligned}
D(x,t) &= \frac{\kappa^2}{32\delta^2}\Bigg[-8\kappa^2\left(1+\frac{\delta^2}{\kappa^2}\right)-\frac{64\delta^4}{\kappa^2}(t+vx)^2- \\
&- 64\delta^2\left(1-\frac{2\delta^2}{\kappa^2}\right)^2 B^2x^2 - 32\cosh 2\delta Bx - \\
&- \frac{128\delta^2}{\kappa}\left\{\left(2-\frac{4\delta^2}{\kappa^2}\right)B-\frac{\delta^2}{\kappa}B'\right\}x\sinh\delta Bx\times \\
&\times \cos\kappa(t+vx) - 32\delta\Bigg\{\kappa\cos\kappa(t+vx)+ \\
&+ \frac{4\delta^2}{\kappa^2}\{t+(v+\kappa v')x\}\sin\kappa(t+vx)\Bigg\}\cosh\delta Bx + \\
&+ \frac{32\delta^2}{\kappa^2}\Bigg\{\cos 2\kappa(t+vx)+\Bigg(4\left(1-\frac{2\delta^2}{\kappa^2}\right)\kappa BB'x - \\
&- 2\delta^2B'^2\Bigg)\kappa^2x-2v'\{2t+(v+\kappa v')x\}\Bigg)\Bigg\}\Bigg],
\end{aligned} \tag{6.12}$$

where $B = B_1$, and we use B' and v' to denote the partial derivatives of B_2 and v_2 with respect to ε evaluated at the point $\varepsilon = 0$, i.e. when $\kappa_2 \to \kappa$. That is,

$$B' = \tfrac{1}{3}\kappa\sum_{n=1}^{\infty}\binom{2n}{n}n(1-n)F(2-n,2;\tfrac{5}{2};\tfrac{1}{4}\kappa^2)\alpha_{2n},$$

and

$$v' = -\tfrac{1}{3}\kappa\sum_{n=1}^{\infty}\binom{2n}{n}(2n+1)nF(1-n,2;\tfrac{5}{2};\tfrac{1}{4}\kappa^2)\alpha_{2n+1},$$

etc. We drop the subscripts due to the fact that as $\varepsilon \to 0$ both modulation parameters take equal values anyway. A plot of this solution is given in Fig.6.5.1.

The degenerate breather solution is a one-parameter family of solutions which represents the collision of two breathers with the same modulation parameter κ, or, equivalently, with equal frequencies. It can be considered a generalisation of the known 2-soliton solution for the class I extension of the nonlinear Schrödinger equation [232].

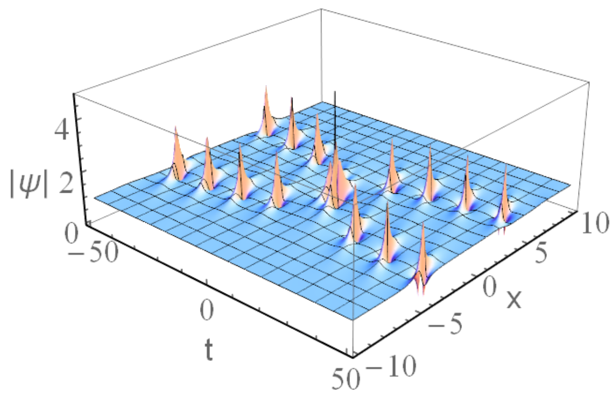

Figure 6.5.1: *The degenerate 2-breather solution. We take the set of α_n the same as in Fig. 6.2.1, and the modulation parameters $\kappa_1 = \kappa_2 = \frac{1}{2}$. The two breathers collide with the high peak at the origin due to the synchronised phases.*

6.6 Second-order rogue wave solution

When the frequency of the degenerate breather tends to zero, the spacing between the successive peaks in Fig. 6.5.1 becomes infinitely large, pushing them out to infinity. What remains at the origin is the second-order rogue wave. In order to derive this solution, we take the limit $\kappa \to 0$ in the expressions (6.12). However, calculations show that this limit cannot be found directly. In order to find it, we repeatedly apply l'Hôpital's rule to the degenerate breather solution as $\kappa \to 0$. The derivatives of G, H, and D with respect to κ at the point $\kappa = 0$ vanish up to $O(\varepsilon^6)$. The resulting functions G, H, and D become polynomials:

$$\begin{aligned}
G(x,t) = {} & 12\{-3 + 24(3B^2 - BB'' - vv'')x^2 + \\
& + 80B^4x^4 - 192v''xt + 96B^2x^2(t+vx)^2 + \\
& + 24(t+vx)^2 + 16(t+vx)^4\},
\end{aligned} \tag{6.13}$$

$$\begin{aligned}
H(x,t) = {} & 576B''x + 2304B''x(t+vx)^2 - 24Bx\{15 - \\
& - 8(B+16B'')Bx^2 - 16B^4x^4 + \\
& + 192v''x(t+vx) - 32B^2x^2(t+vx)^2 + \\
& + 24(t+vx)^2 - 16(t+vx)^4\},
\end{aligned} \tag{6.14}$$

$$\begin{aligned}
D(x,t) = {} & -9 - 36\{11B^2 - 48BB'' + 64B''^2 - \\
& - 16(v - v'')v''\}x^2 - 48\{9B^4 - 6B^2v^2 + \\
& + 16(3v^2 - B^2)BB'' + 16(v^2 - 3B^2)vv''\}x^4 - \\
& - 64(B^4 + 3B^2v^2 + 3v^4)B^2x^6 + 576v''xt - \\
& - 768v''xt^3 + 288(B^2 - 8BB'' - 8vv'')x^2t^2 - \\
& - 192B^2x^2t^4 + 576\{(B - 8B'')Bv + \\
& + 4(B^2 - v^2)v''\}x^3t - 768B^2vx^3t^3 - \\
& - 192(B^2 + 6v^2)B^2x^4t^2 - 384(B^2 + \\
& + 2v^2)B^2vx^5t - 108(t+vx)^2 - \\
& - 48(t+vx)^4 - 64(t+vx)^6,
\end{aligned} \tag{6.15}$$

where, in the same limit as $\kappa \to 0$,

$$\begin{aligned}
B &= \sum_{n=1}^{\infty} \binom{2n}{n} n\alpha_{2n}, \\
v &= \sum_{n=1}^{\infty} \binom{2n}{n} (2n+1)\alpha_{2n+1}.
\end{aligned}$$

The first-order derivatives B' and v' vanish as $\kappa \to 0$, but the second-order derivatives still remain, and in the limit as $\kappa \to 0$ are

$$B'' = -\frac{1}{3}\sum_{n=1}^{\infty}\binom{2n}{n}n(n-1)\alpha_{2n},$$
$$v'' = -\frac{1}{3}\sum_{n=1}^{\infty}\binom{2n}{n}(2n+1)n\alpha_{2n+1}.$$

This solution is shown in Fig. 6.6.1. It is, naturally, the second-order rogue wave, but slanted and rescaled in the (x, t)-plane relative to the second-order rogue wave of the NLS equation [31, 37].

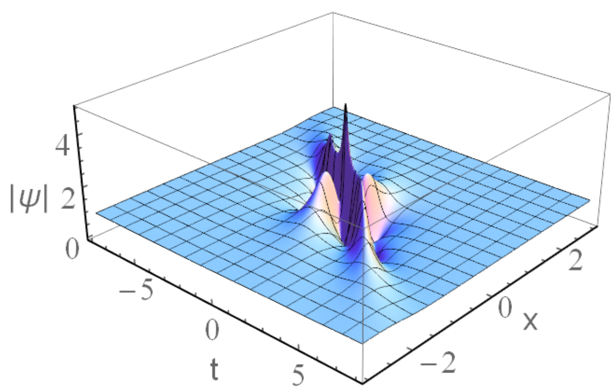

Figure 6.6.1: *The second-order rogue wave, Eqs.(6.13),(6.14),(6.15) obtained from the degenerate 2-breather solution shown in Fig. 6.5.1 in the limit* $\kappa \to 0$, *with some stretching due to higher-order effects.*

6.7 Rogue wave triplets

It is well known that the general n-th order rogue wave has the remarkable property of being able to split into $\frac{1}{2}n(n+1)$ first-order components [195]. The second-order rogue wave discussed above is only a particular case of a more general rogue wave structure, where all three first-order components are located at the origin, and have merged into one single peak. In order to obtain the more general solution where the three components are not merged together, known as the rogue wave triplet [119], we re-introduce the constants of integration into the general 2-breather solution, i.e.

$$\delta_m B_m x \mapsto \delta_m (B_m x - \varepsilon X_m),$$
$$t + v_m x \mapsto t - T_m \varepsilon + v_m x,$$

where X_m and T_m are arbitrary, and the parameter ε is introduced to make sure that the Taylor series in the degenerate limit still vanishes up to $O(\varepsilon^2)$. The values of X_m and T_m determine the location of the components of the breather components. They add additional free parameters to the solution which we have previously given for the restricted case in which $X_m = T_m = 0$. Notice also that we do not make the replacement $x \mapsto x - X_m \varepsilon$ directly, but, for simplicity's sake, instead define X_m to account for the higher-order terms in B_m.

In order to further simplify parametrisation, we assume that X_m and T_m are functions of the modulation parameter κ, and are of the order $O(\kappa)$. Then, defining free parameters ξ and η independent of κ, such that

$$\kappa \xi = 48(X_1 - X_2),$$
$$\kappa \eta = 48(T_1 - T_2),$$

we have in the limit as $\kappa \to 0$ the rogue wave triplet solution in the form

$$\psi(x,t) = \left\{ 1 + \frac{\hat{G}(x,t) + i\hat{H}(x,t)}{\hat{D}(x,t)} \right\} e^{i\phi x}, \tag{6.16}$$

with

$$\begin{aligned}
\hat{G}(x,t) &= G(x,t) - 48\xi Bx - 48\eta(t+vx), && (6.17)\\
\hat{H}(x,t) &= H(x,t) + 12\xi - 48\xi B^2x^2 - \\
&- 96\eta Bx(t+vx) + 48\xi(t+vx)^2, && (6.18)\\
\hat{D}(x,t) &= D(x,t) - (\xi^2+\eta^2) + 12\{\xi(3B-4B'') - \\
&- 4\eta v''\}x + 16\xi B^3x^3 + 12\eta(1+4B^2x^2)\times \\
&\times (t+vx) - 48\xi Bx(t+vx)^2 \\
&- 16\eta(t+vx)^3, && (6.19)
\end{aligned}$$

where $\hat{G}$, $\hat{H}$ and $\hat{D}$ now contain two new free parameters, ξ and η, which determine the separation of the fundamental rogue wave components in the triplet [119], and where G, H, and D are as given in Eqs. (6.13)-(6.15), for the particular case in which $\xi = \eta = 0$. An example of the formation of rogue wave triplets, corresponding to nonzero ξ and η, is shown in Fig. 6.7.1. When both $\xi = 0$ and $\eta = 0$, all three components merge at the origin, as in Fig. 6.6.1.

Generally, the coefficient B in Eq. (6.16) determines the degree of localisation along the x-axis. Larger values of B will correspond to narrower peaks, whereas smaller values of B will correspond to broader peaks, and $B = 0$ to minimal localisation. A point of interest here is that it is again possible to choose a parametrisation for which B is any fixed constant. If we choose, for instance,

$$\alpha_2 = c - \frac{1}{2}\sum_{n=1}^{\infty}\binom{2n+2}{n+1}(n+1)\alpha_{2n+2},$$

we end up with $B = c$, where c is a free parameter. However, B'' is entirely independent of the choice of c, since the coefficient of α_2 in B'' is zero. As the simplest example, we consider the completely de-localised case, $B = 0$, with B'' remaining

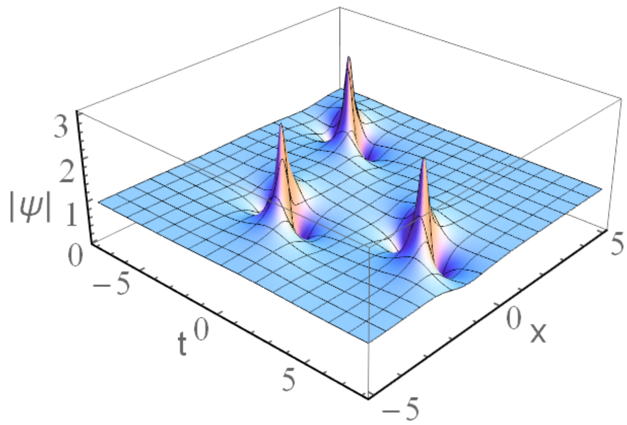

Figure 6.7.1: *The second-order rogue wave triplet (6.16), with separation parameters* $\xi = -\eta = 480$, *and the extended equation parameters given by* $\alpha_2 = \frac{1}{2}$, $\alpha_3 = \frac{1}{27}$, $\alpha_4 = \frac{1}{50}$, $\alpha_5 = \frac{1}{81}$, $\alpha_6 = \frac{1}{200}$, $\alpha_7 = \frac{1}{343}$. *For this choice of parameters, we have* $B = \frac{77}{50}$, $v = \frac{1324}{1323}$, $B'' = -\frac{7}{25}$, *and* $v'' = -\frac{2894}{3969}$.

arbitrary. The rogue wave solution then reduces to (6.16) with

$$\begin{aligned}
\hat{G}(x,t) &= G_0(x,t) - 48\eta(t+vx),\\
\hat{H}(x,t) &= H_0(x,t) + 12\xi + 48\xi(t+vx)^2,\\
\hat{D}(x,t) &= D_0(x,t) - (\xi^2+\eta^2) - 48(\xi B'' + \eta v'')x+\\
&\quad + 12\eta(t+vx) - 16\eta(t+vx)^3.
\end{aligned}$$

where

$$\begin{aligned}
G_0(x,t) &= 12\{-3 - 192v''x(t+vx) + 24(t+vx)^2 + \\
&\quad + 16(t+vx)^4\}, \\
H_0(x,t) &= 576B''x\{1 + 4(t+vx)^2\}, \\
D_0(x,t) &= -9 - 576\{4B''^2 + v''^2\}x^2 + 576v''x(t+vx) - \\
&\quad - 768v''x(t+vx)^3 - 108(t+vx)^2 - \\
&\quad - 48(t+vx)^4 - 64(t+vx)^6.
\end{aligned}$$

Here, G_0, H_0, and D_0 are as given for the case where the components are merged and $B = 0$, and $\hat{G}$, $\hat{H}$, $\hat{D}$ incorporate the shifting of the first-order components through ξ and η.

When $B = 0$, the second-order rogue wave acquires soliton-like tails similar to those in Fig. 6.3.1. When, additionally, $\xi = \eta = 0$, rogue waves merge at the origin

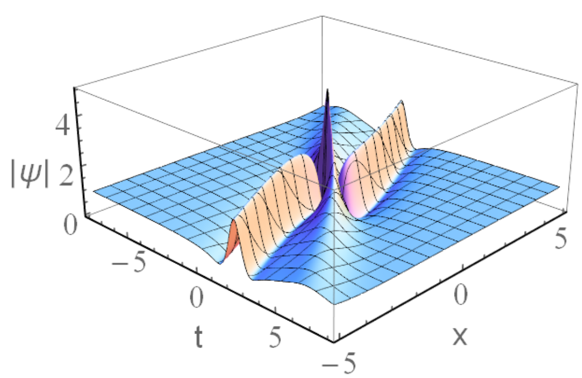

Figure 6.7.2: *The second-order rogue wave solution with 'soliton'-like tails when α_2 chosen such that $B = 0$, and $\xi = \eta = 0$. Other parameters are the same as in Fig. 6.7.1.*

to form a second-order rogue wave with extended tails. This case is shown in Fig. 6.7.2. When ξ or η is not zero, the components split, resulting in the disappearance of the central peak. This case is shown in Fig. 6.7.3. Here, the central peak is absent but the long tails remain, consisting of delocalised first-order components.

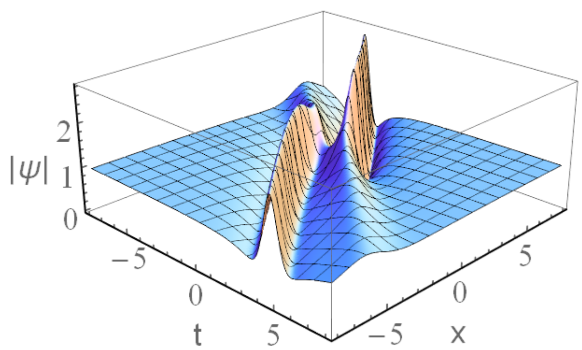

Figure 6.7.3: *The second-order rogue wave solution with 'soliton'-like tails when* $B = 0$, *but now* $\xi = \eta = 48$.

Conclusions

We have derived the general 2-breather solution for the class I infinitely extended nonlinear Schrödinger equation, and given many limiting cases; namely, breather-to-soliton conversions, the semirational limit, the degenerate 2-breather, and, probably most importantly, the general second-order rogue wave solution. These solutions completely describe a large family of second-order solutions to the class I extension of the NLS equation, and exhibit rich behaviour.

These results provide a more detailed analysis of the formation of nonlinear wave structures such as breathers and rogue waves when higher-order effects come into play, and leaves a large range of future related work wide open.

7

Final Remarks

This thesis has been on the study of nonlinear waves; both the analysis of solutions to nonlinear partial differential equations and the physics they describe, and a derivation of new physical models for describing nonlinear behaviour in hydrodynamics in particular.

We presented a derivation of the Korteweg-de Vries equation as a description of the complex velocity of a unidirectional, weakly nonlinear wave in a shallow, ideal fluid, and showed that it also reduces to the real KdV equation when considering only the surface elevation. We showed that there is a physically consistent interpretation of the conserved quantities in complex KdV theory, and we also showed that in the quasi-monochromatic approximation, Fourier modes of complex KdV solutions are described by the split NLS system.

We also showed that a more general kind of rational solution to the KdV equation exists than was previously known. These solutions were found by a process nearly the same as linear superposition, except that only one specific combination was found to generate new solutions. Recently, it has been suggested that other nonlinear PDEs may have similar symmetries which would lead to solutions being generated by the same process [237].

Finally, we obtained solutions for the class I infinite extension of the NLS equation. We obtained both the general one-parameter family of first-order solutions, and a large class of solutions of the second order. These solutions describe the effects of higher order nonlinearity and dispersion.

Because the derivation of the complex KdV equation as a new way of looking at the dynamics of a weakly nonlinear shallow water wave is quite closely connected to fundamentals, this work is open-ended in a sense, and provides a lot of fertile ground for further growth. One of the strongest advantages of putting the complex KdV equation first is the fact that it allows us to speak in terms of complex potentials. This offers far greater possibilities than discussing the elevation alone. Not only does the potential immediately provide complete information on the motion of the entire body of fluid, but it offers a flexibility that is entirely absent when focusing on the elevation. When we think of the KdV regime in terms of its complex variable description first, then it becomes clear that the typical KdV equation can be related to a much broader variety of fluid flows via conformal transformations. In particular, in chapter 2 we considered only those flows for which the complex potential is real precisely on the real axis. This restriction is completely artificial, and was made only to make the simplification of Levi-Civitàʼs equation as easy as possible. In principle, though, the stream function can take constant values on any boundary we like. By applying a conformal transformation, the streamlines can be mapped onto other conformally equivalent but generally nontrivial geometries. These would then give weakly nonlinear flows over geometries other than a channel with a flat, horizontal bottom, such as channels with an obstacle introduced into them. Problems such as these can often be nearly intractable with real methods alone.

At the very beginning of this thesis, the field of nonlinear physics was described as something of a revolutionary development akin to that of quantum mechanics and relativity of the past century. It seems fitting to end by returning to this original statement.

Major leaps forward in physics have, historically, also gone together with a new way of describing, and talking about, the new physics. This began with Newton and Leibniz and the development of calculus to describe the problems of classical mechanics in a mathematically precise way. Quantum mechanics is often put in the language of functional analysis, and general relativity is most effectively discussed in the language of differential geometry. In the field of integrable nonlinear systems, it seems that we are persistently led towards the field of algebraic geometry.

This connection is in fact immediately suggested by the form of the KdV equation. Ignoring the time derivative, the nonlinear and dispersive terms form the differential equation of an elliptic curve, in this case parameterised by the Weierstrass $\wp$-function. This makes Hirota's method, often associated with the substitution (4.4) (often with $u_0 = 0$) quite suggestive, because this substitution takes us from speaking about elliptic functions to speaking directly about a theta function. Similar is true of the sine-Gordon equation; in fact, the pendulum equation is one of the simplest nonlinear systems conceivable, and leads very directly to the theory of elliptic functions. The Kadomtsev-Petviashvili (KP) equation also emerges naturally in the field of algebraic geometry in the resolution of Novikov's conjecture [238–240]. The fact that a very physical system emerges from an apparently unrelated and very abstract area of mathematics is surprising, and perhaps ought to suggest to us that there may be a more natural language for nonlinear systems.

None of this is to advocate for unnecessary abstraction in physics, but rather, in the spirit of Freeman Dyson's famous lecture Missed Opportunities [241], to say that greater cooperation between pure mathematics and physics is likely to bring about an easier way of looking at nonlinear systems, and make what were once fearsome problems simple, and clear.

References

[1] G. Stokes. On the theory of oscillatory waves. *Transactions of the Cambridge Philosophical Society*, 8, 1847.

[2] J. V. Boussinesq. Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. *Journal de Mathématiques Pures et Appliquées*, 1871.

[3] J. V. Boussinesq. *Essai sur la theórie des eaux courantes*. Imprimerie Nationale, 1877.

[4] D. J. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philosophical Magazine*, 39:422–443, 1895.

[5] J. Scott Russell. Report on waves. *Fourteenth meeting of the British Association for the Advancement of Science*, 1844.

[6] L. M. Milne-Thomson. *Theoretical Hydrodynamics*. Dover New York, fifth edition, 1968.

[7] D. J. Benney and J. C. Luke. On the interactions of permanent waves of finite amplitude. *Journal of Mathematics and Physics*, 43:309–313, 1964.

[8] H. Poincaré and R. Magini. *Les méthodes nouvelles de la mécanique céleste*, volume 1. Gauthiers-Villars, Paris, 1892.

[9] H. Poincaré and R. Magini. *Les méthodes nouvelles de la mécanique céleste*, volume 2. Gauthiers-Villars, Paris, 1893.

[10] H. Poincaré and R. Magini. *Les méthodes nouvelles de la mécanique céleste*, volume 3. Gauthiers-Villars, Paris, 1899.

[11] Georg Duffing. *Erzwungene Schwingungen Bei veraänderlicher Eigenfrequenz und ihre technische Bedeutung*. Braunschweig, F. Vieweg & Sohn, 1918.

[12] B. van der Pol. On relaxation-oscillations. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 2:978–992, 1926.

[13] B. van der Pol and J. van der Mark. Frequency demultiplication. *Nature*, 120:363–364, 1927.

[14] Edward N. Lorenz. Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*, 20:130–141, 1963.

[15] B. van der Pol and J. van der Mark. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 6(38):763–775, 1928.

[16] Alfred J. Lotka. Analytical note on certain rhythmic relations in organic systems. *Proceedings of the National Academy of Sciences*, 6(7):410–415, 1920.

[17] N.J. Zabusky and M. D. Kruskal. Interactions of solitons in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, 15:240–243, 1965.

[18] E. Fermi and J. Pasta and S. Ulam. Studies of Nonlinear Problems. Los Alamos National Laboratory, 1955.

[19] Miguel Onorato, Lara Vozella, Davide Proment, and Yuri V. Lvov. Route to thermalization in the α-Fermi–Pasta–Ulam system. *Proceedings of the National Academy of Sciences*, 112:4208–4213, 2015.

[20] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Method for solving the Korteweg-de Vries equation. *Physical Review Letters*, 19:1095–1097, 1967.

[21] Robert M. Miura. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *Journal of Mathematical Physics*, 9:1202–1204, 1968.

[22] Robert M. Miura, Clifford S. Gardner, and Martin D. Kruskal. Korteweg-de Vries equation and generalizations II: Existence of conservation laws and constants of motion. *Journal of Mathematical Physics*, 9:1204–1209, 1968.

[23] C. H. Su and C. S. Gardner. Korteweg-de Vries equation and generalizations. III. Derivation of the Korteweg-de Vries equation and Burgers equation. *Journal of Mathematical Physics*, 10(3):536, 1969.

[24] C. S. Gardner. Korteweg-de Vries equation and generalizations IV: The Korteweg-de Vries equation as a Hamiltonian system. *Journal of Mathematical Physics*, 12:1548–1551, 1971.

[25] Martin D. Kruskal, Robert M. Miura, Clifford S. Gardner, and Norman J. Zabusky. Korteweg-de Vries equation and generalizations V: Uniqueness and nonexistence of polynomial conservation laws. *Journal of Mathematical Physics*, 11:952–960, 1970.

[26] Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura. Korteweg-de Vries equation and generalizations VI: Methods for exact solution. *Communications on Pure and Applied Mathematics*, 27:97–133, 1974.

[27] Peter D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Communications on Pure and Applied Mathematics*, 21:467–490, 1968.

[28] V. E. Zakharov. Stability of periodic waves of finite amplitude on the surface of a deep fluid. *Journal of Applied Mechanics and Technical Physics*, 9:190–194, 1972.

[29] V. E. Zakharov and A. B. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Soviet Journal of Experimental and Theoretical Physics*, 34:62, 1972.

[30] T. Brooke Benjamin and J. E. Feir. The disintegration of wave trains on deep water Part 1: Theory. *Journal of Fluid Mechanics*, 27:417–430, 1967.

[31] N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin. Generation of periodic trains of picosecond pulses in an optical fiber: exact solutions. *Soviet Journal of Experimental and Theoretical Physics*, 62:894, 1985.

[32] N. N. Akhmediev and V. I. Korneev. Modulation instability and periodic solutions of the nonlinear Schrödinger equation. *Theoretical and Mathematical Physics*, 69:1089–1093, 1986.

[33] E. A. Kuznetsov. Solitons in a parametrically unstable plasma. *Doklady Akademii nauk SSSR*, 1977.

[34] Yan-Chow Ma. The perturbed plane-wave solutions of the cubic Schrödinger equation. *Studies in Applied Mathematics*, 60:43–58, 1979.

[35] Li-Chen Zhao, Liming Ling, and Zhan-Ying Yang. Mechanism of Kuznetsov-Ma breathers. *Physical Review E*, 97:022218, 2018.

[36] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin. Exact first-order solutions of the nonlinear Schrödinger equation. *Theoretical and Mathematical Physics*, 72:809–818, 1987.

[37] N. Akhmediev, A. Ankiewicz, and M. Taki. Waves that appear from nowhere and disappear without a trace. *Physics Letters A*, 373:675–678, 2009.

[38] D. H. Peregrine. Water waves, nonlinear Schrödinger equations and their solutions. *The Journal of the Australian Mathematical Society. Series B. Applied Mathematics*, 25(1):16–43, 1983.

[39] Amaury Mathis, Luc Froehly, Shanti Toenger, Frédéric Dias, Goëry Genty, and John M. Dudley. Caustics and rogue waves in an optical sea. *Scientific Reports*, 5, 2015.

[40] Christian Kharif and Efim Pelinovsky. Physical mechanisms of the rogue wave phenomenon. *European Journal of Mechanics B*, 22:603–634, 2003.

[41] Maura Brunetti. Formation of rogue waves under forcing fields. *Swiss Physical Society Communications*, 47(50):26–28, 2015.

[42] Maura Brunetti and Jérôme Kasparian. Modulational instability in wind-forced waves. *Physics Letters A*, 378:3626–3630, 2014.

[43] Maura Brunetti, Nadège Marchiando, Nicolas Berti, and Jérôme Kasparian. Nonlinear fast growth of water waves under wind forcing. *Physics Letters A*, 378:1025–1030, 2014.

[44] Akira Hasegawa and Frederick Tappert. Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers I: Anomalous dispersion. *Applied Physics Letters*, 23:142–144, 1973.

[45] Akira Hasegawa and Frederick Tappert. Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers II: Normal dispersion. *Applied Physics Letters*, 23:171–172, 1973.

[46] Roland Schiek. Nonlinear refraction caused by cascaded second-order nonlinearity in optical waveguide structures. *Journal of the Optical Society of America B*, 10:1848, 1993.

[47] V. I. Bespalov and V. I. Talanov. Filamentary structure of light beams in nonlinear liquids. *JETP Letters*, 3:307, 1966.

[48] A. Chabchoub, N. Hoffmann, M. Onorato, and N. Akhmediev. Super rogue waves: Observation of a higher-order breather in water waves. *Phys. Rev. X*, 2:011015, Mar 2012.

[49] A. Chabchoub, N. Hoffmann, M. Onorato, A. Slunyaev, A. Sergeeva, E. Pelinovsky, and N. Akhmediev. Observation of a hierarchy of up to fifth-order rogue waves in a water tank. *Phys. Rev. E*, 86:056601, Nov 2012.

[50] A. Chabchoub, B. Kibler, J. M. Dudley, and N. Akhmediev. Hydrodynamics of periodic breathers. *Philosophical Transactions of the Royal Society A*, 372:20140005, 2014.

[51] D. Eeltink, A. Lemoine, H. Branger, O. Kimmoun, C. Kharif, J. D. Carter, A. Chabchoub, M. Brunetti, and J. Kasparian. Spectral up- and downshifting of Akhmediev breathers under wind forcing. *Physics of Fluids*, 29:107103, 2017.

[52] S. Trillo, G. Deng, G. Biondini, M. Klein, G. F. Clauss, A. Chabchoub, and M. Onorato. Experimental observation and theoretical description of multisoliton fission in shallow water. *Physical Review Letters*, 117:144102, Sep 2016.

[53] Sverre Haver. A possible freak wave event measured at the Draupner Jacket, January 1 1995. In *Proceedings of the 2004 Rogue Waves*, 01 2004.

[54] Ian S. F. Jones and Joyce E. Jones. *Oceanography in the Days of Sail.* Hale & Iremonger, Sydney, NSW, Second edition, 2008.

[55] C. Kharif, E. Pelinovsky, and A. Slunyaev. *Rogue Waves in the Ocean.* Springer, 2009.

[56] A. Chabchoub and N. Akhmediev. Observation of rogue wave triplets in water waves. *Physics Letters A*, 377:2590–2593, 2013.

[57] A. V. Slunyaev. Predicting rogue waves. *Moscow University Physics Bulletin*, 72:236–249, 2017.

[58] G. Van Simaeys, Ph. Emplit, and M. Haelterman. Experimental demonstration of the Fermi-Pasta-Ulam recurrence in a modulationally unstable optical wave. *Physical Review Letters*, 87, 2001.

[59] J. M. Dudley, G. Genty, F. Dias, B. Kibler, and N. Akhmediev. Modulation instability, Akhmediev breathers and continuous wave supercontinuum generation. *Optics Express*, 17:21497, 2009.

[60] B. Kibler, J. Fatome, C. Finot, G. Millot, G. Genty, B. Wetzel, N. Akhmediev, F. Dias, and J. M. Dudley. Observation of Kuznetsov-Ma soliton dynamics in optical fibre. *Scientific Reports*, 2, 2012.

[61] D. R. Solli, C. Ropers, P. Koonath, and B. Jalali. Optical rogue waves. *Nature*, 450:1054–1057, 2007.

[62] Daniel R. Solli, Claus Ropers, and Bahram Jalali. Optical rogue waves and stimulated supercontinuum generation. *Nonlinear Optics and Applications IV*, 2010.

[63] N. Akhmediev, J. M. Dudley, D. R. Solli, and S. K. Turitsyn. Recent progress in investigating optical rogue waves. *Journal of Optics*, 15:060201, 2013.

[64] R. Schiek, H. Fang, R. Malendevich, and G. I. Stegeman. Measurement of modulational instability gain of second-order nonlinear optical eigenmodes in a one-dimensional system. *Physical Review Letters*, 86:4528–4531, 1993.

[65] Wonkeun Chang, Jose M. Soto-Crespo, Peter Vouzas, and Nail Akhmediev. Spiny solitons and noise-like pulses. *J. Opt. Soc. Am. B*, 32(7):1377–1383, Jul 2015.

[66] Wonkeun Chang, Adrian Ankiewicz, Nail Akhmediev, and J. M. Soto-Crespo. Creeping solitons in dissipative systems and their bifurcations. *Physical Review E*, 76, 2007.

[67] N. Akhmediev and J. Soto-Crespo. Exploding solitons and Shil'nikov's theorem. *Physics Letters A*, 317:287–292, 10 2003.

[68] Wonkeun Chang and Nail Akhmediev. Exploding solitons vs rogue waves in laser cavities. In *Advanced Photonics*, page NM3A.6. Optical Society of America, 2014.

[69] Mark Ablowitz and Harvey Segur. *Solitons and the Inverse Scattering Transform*. Society for Industrial Mathematics, 1980.

[70] I. M. Ge'lfand and B. M. Levitan. On the determination of a differential equation from its spectral function. *American Mathematical Society Translations*, 1:253–304, 1955.

[71] Vladimir B. Matveev and Mikhail A. Salle. *Darboux Transformations and Solitons*. Springer-Verlag, 1991.

[72] V. B. Matveev. Generalized Wrońskian formula for solutions of the KdV equations: First applications. *Physics Letters A*, 166(3):205–208, 1992.

[73] G. Darboux. Sur une proposition relative aux équations linéaires. *Comptes Rendus de l'Acadmie des Sciences*, 94:1456–1459, 1882.

[74] G. Darboux. *Leçons sur la Theorie des Surfaces*. Gauthier-Villars, Paris, Second edition, 1912.

[75] M. Crum. Associated Sturm-Liouville systems. *Quarterly Journal of Math*, 6:121–127, 1955.

[76] Hugo D. Wahlquist. Bäcklund transformation of potentials of the Korteweg-de Vries equation and the interaction of solitons with cnoidal waves. In Robert Miura, editor, *Bäcklund Transformations, the Inverse Scattering Method, Solitons, and Their Applications: Proceedings of the NSF Research Workshop on Contact Transformations, held in Nashville, Tennessee, 1974*, pages 162–183, Berlin Heidelberg, 1976. Springer.

[77] N. C. Freeman and J. J. C. Nimmo. Soliton solutions of the Korteweg-de Vries and the Kadomtsev-Petviashvili equations: The Wroǹskian technique. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 389(1797):319–329, 1983.

[78] Miki Wadati, Heiji Sanuki, and Kimiaki Konno. Relationships among Inverse Method, Bäcklund Transformation and an Infinite Number of Conservation Laws. *Progress of Theoretical Physics*, 53(2):419–436, 02 1975.

[79] C. Rogers and W. K. Schief. *Bäcklund and Darboux Transformations*. Cambridge University Press, 2002.

[80] Edmond Bour. Théorie de la déformation des surfaces. *Journal de l'École Impériale Polytechnique*, 19:1–48, 1862.

[81] A. V. Bäcklund. Om ytor med konstant negativ krökning. *Lunds Universitets Årsskrift*, 19:1–48, 1883.

[82] L. Bianchi. Sulla trasformazione di bäcklund per le superficie pseudosferiche. *Rendiconti Lincei*, 5:3–12, 1892.

[83] Hugo D. Wahlquist and Frank B. Estabrook. Bäcklund transformation for solutions of the Korteweg-de Vries equation. *Physical Review Letters*, 31:1386–1390, 1973.

[84] Ryogo Hirota. *The Direct Method in Soliton Theory*. Cambridge University Press, 2004.

[85] Ryogo Hirota. Direct method of finding exact solutions of nonlinear evolution equations. *Journal of the Physical Society of Japan*, pages 40–68, 1976.

[86] Ryogo Hirota. Reduction of soliton equations in bilinear form. *Physica D*, 18:161–170, 1986.

[87] Ryogo Hirota. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Physical Review Letters*, 27:1192–1194, 1971.

[88] T. Miwa, M. Jimbo, and E. Date. *Solitons: Differential Equations, Symmetries and Infinite Dimensional Algebras*. Cambridge Tracts in Mathematics. Cambridge University Press, 2000.

[89] Mikio Sato. Soliton equations as dynamical systems on infinite dimensional Grassmann manifold. In *Nonlinear Partial Differential Equations in Applied Science; Proceedings of The U.S.-Japan Seminar, Tokyo, 1982*, volume 81 of *North-Holland Mathematics Studies*, pages 259–271. North-Holland, 1983.

[90] Eugene D. Belokolos, Alexander I. Bobenko, Viktor Z. Enol'skii, Alexander R. Its, and Vladimir B. Matveev. *Algebro-geometric Approach to Nonlinear Evolution Equations*. Springer Series in Nonlinear Dynamics. Springer-Verlag, 1994.

[91] I. M. Krichever and S. P. Novikov. Holomorphic bundles over algebraic curves and non-linear equations. *Russian Mathematical Surveys*, 35:53–79, 1980.

[92] B. A. Dubrovin, V. B. Matveer, and S. P. Novikov. Non-linear equations of Korteweg-de Vries type, finite-zone linear operators and Abelian varieties. *Russian Mathematical Surveys*, pages 53–140, 1981.

[93] B. A. Dubrovin. Theta functions and non-linear equations. *Russian Mathematical Surveys*, 36:11–92, 1981.

[94] Spyridon Kamvissis and Gerald Teschl. Stability of periodic soliton equations under short range perturbations. *Physics Letters A*, 364:480–483, 2007.

[95] Spyridon Kamvissis and Gerald Teschl. Long-time asymptotics of the periodic Toda lattice under short-range perturbations. *Journal of Mathematical Physics*, 53:073706, 2012.

[96] R. A. Kraenkel, M. A. Manna, and J. G. Pereira. The Korteweg–de Vries hierarchy and long water-waves. *Journal of Mathematical Physics*, 36:307–320, 1995.

[97] D. G. Crighton. Applications of KdV. *Acta Applicandae Mathematicae*, pages 39–67, 1995.

[98] A. R. Osborne and T. L. Burch. Internal solitons in the Andaman sea. *Science*, 208:451–460, 1980.

[99] Leen Van Wijngaarden. Evolving solitons in bubbly flows. *Acta Applicandae Mathematica*, 39:507–516, 1995.

[100] John A. Whitehead and Karl R. Helfrich. The Korteweg-de Vries equation from laboratory conduit and magma migration equations. *Geophysical Research Letters*, 13:545–546, 1986.

[101] Peter Olson and Ulrich Christensen. Solitary wave propagation in a fluid conduit within a viscous matrix. *Journal of Geophysical Research*, 91:6367, 1986.

[102] L. G. Redekopp. On the theory of solitary Rossby waves. *Journal of Fluid Mechanics*, 82(4):725–745, 1977.

[103] T. Maxworthy and L. G. Redekopp. New theory of the Great Red Spot from solitary waves in the Jovian atmosphere. *Nature*, 260:509–511, 1976.

[104] T. Maxworthy and L. G. Redekopp. A solitary wave theory of the great red spot and other observed features in the Jovian atmosphere. *Icarus*, 29:261–271, 1976.

[105] T. Maxworthy, L. G. Redekopp, and P. D. Weidman. On the production and interaction of planetary solitary waves: Applications to the Jovian atmosphere. *Icarus*, 33:388–409, 1978.

[106] Haruichi Washimi and Toshiya Taniuchi. Propagation of ion-acoustic solitary waves of small amplitude. *Physical Review Letters*, 17:996–998, 1966.

[107] H. Ikezi, R. J. Taylor, and D. R. Baker. Formation and interaction of ion-acoustic solitions. *Physical Review Letters*, 25:11–14, 1970.

[108] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. Springer, 1997.

[109] J.-L. Gervais. Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets. *Physics Letters B*, 160:277–278, 1985.

[110] V. I. Talanov. Self focusing of wave beams in nonlinear media. *Soviet Journal of Experimental and Theoretical Physics Letters*, 1965.

[111] Gadi Fibich. *The Nonlinear Schrödinger Equation: Singular Solutions and Optical Collapse*. Springer, 2015.

[112] Mark J. Ablowitz, David J. Kaup, Alan C. Newell, and Harvey Segur. The inverse scattering transform – Fourier analysis for nonlinear problems. *Studies in Applied Mathematics*, 53:249–315, 1974.

[113] Spyridon Kamvissis. Focusing nonlinear Schrödinger equation with infinitely many solitons. *Journal of Mathematical Physics*, 36:4175–4180, 1995.

[114] M. A. Salle. Darboux transformations for non-Abelian and nonlocal equations of the Toda chain type. *Theoretical and Mathematical Physics*, 53:1092–1099, 1982.

[115] V. Matveev, M. Salle, and A. Rybin. Darboux transformations and coherent interaction of the light pulse with two-level media. *Inverse Problems*, 4:173, 01 1999.

[116] N. N. Akhmediev and N. V. Mitzkevich. Extremely high degree of n-soliton pulse compression in an optical fiber. *IEEE Journal of Quantum Electronics*, 27:849–857, 1991.

[117] Akhmediev N., Korneev V. I., and Mitskevich N. V. N-modulation signals in a single-mode optical waveguide under nonlinear conditions. *Soviet Journal of Experimental and Theoretical Physics*, 67(1):89–95, 1988.

[118] David J. Kedziora, Adrian Ankiewicz, and Nail Akhmediev. Second-order nonlinear Schrödinger equation breather solutions in the degenerate and rogue wave limits. *Physical Review E*, 85:066601, 2012.

[119] Adrian Ankiewicz, David J. Kedziora, and Nail Akhmediev. Rogue wave triplets. *Physics Letters A*, 375:2782–2785, 2011.

[120] David J. Kedziora, Adrian Ankiewicz, and Nail Akhmediev. Triangular rogue wave cascades. *Physical Review E*, 86:056602, 2012.

[121] David J. Kedziora, Adrian Ankiewicz, and Nail Akhmediev. Classifying the hierarchy of nonlinear Schrödinger-equation rogue wave solutions. *Physical Review E*, 88(013207), 2013.

[122] D. Mihalache and N. C. Panoiu. Exact solutions of the nonlinear Schrödinger equation for the normal-dispersion regime in optical fibers. *Physical Review A*, 45(9):6730–6734, 1992.

[123] D. Mihalache and N. C. Panoiu. Exact solutions of nonlinear Schrödinger equation for positive group velocity dispersion. *Journal of Mathematical Physics*, 33(6):2323–2328, 1992.

[124] Nail Akhmediev and Adrian Ankiewicz. First-order exact solutions of the nonlinear Schrödinger equation in the normal-dispersion regime. *Physical Review A*, 47:3213–3221, 1993.

[125] Jinbing Chen, Dmitry E. Pelinovsky, and Robert E. White. Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation. *Physical Review E*, 100:052219, 2019.

[126] D. Mihalache, M. Bertolotti, and C. Sibilia. IV. Nonlinear wave propagation in planar structures. volume 27 of *Progress in Optics*, pages 227–313. Elsevier, 1989.

[127] Maki Saito, Shinsuke Watanabe, and Hiroshi Tanaka. Modulational instability of ion wave in plasma with negative ion. *Journal of the Physical Society of Japan*, 53:2304–2310, 1984.

[128] H. Bailung, S. K. Sharma, and Y. Nakamura. Observation of Peregrine solitons in a multicomponent plasma with negative ions. *Physical Review Letters*, 107, 2011.

[129] L. S. da Rios. Sul moto d'un liquido indefinito con un filetto vorticoso di forma qualunque. *Rendiconti del Circolo Matematico di Palermo*, 22:117–135, 1906.

[130] Hidenori Hasimoto. A soliton on a vortex filament. *Journal of Fluid Mechanics*, 51:477–485, 1972.

[131] Hayder Salman. Breathers on quantized superfluid vortices. *Physical Review Letters*, 111, 2013.

[132] Ludwig D. Faddeev and Leon A. Takhtajan. *Hamiltonian Methods in the Theory of Solitons*. Springer, 1987.

[133] Lev Pitaevskii and Sandro Stringari. *Bose-Einstein Condensation and Superfluidity*. Oxford University Press, 2016.

[134] D. J. Benney and A. C. Newell. The propagation of nonlinear wave envelopes. *Journal of Mathematics and Physics*, 46:133–139, 1967.

[135] Mark J. Ablowitz. *Nonlinear Dispersive Waves – Asymptotic Analysis and Solitons*. Cambridge University Press, 2011.

[136] M. Crabb and N. Akhmediev. Complex Korteweg–de Vries equation: A deeper theory of shallow water waves. *Physical Review E*, 103:022216, 2021.

[137] M. Crabb and N. Akhmediev. Complex Korteweg-de Vries equation in the theory of shallow water waves: Conservation laws, and relation to the AKNS and defocusing NLS equations. Submitted to Physical Review E, manuscript EK12070.

[138] M. Crabb and N. Akhmediev. Rogue wave multiplets in the complex Korteweg-de Vries equation. *Romanian Reports in Physics*, 72(118), 2020.

[139] M. Crabb and N. Akhmediev. Doubly periodic solutions of the class-I infinitely extended nonlinear Schrödinger equation. *Physical Review E*, 99:052217, 2019.

[140] M. Crabb and N. Akhmediev. Two-breather solutions for the class I infinitely extended nonlinear Schrödinger equation and their special cases. *Nonlinear Dynamics*, 98:245–255, 2019.

[141] Alessandro Toffoli and Elzbieta M. Bitner-Gregersen. *Types of Ocean Surface Waves, Wave Classification, In: Encyclopedia of Maritime and Offshore Engineering, Basic Hydrodynamics and Wave Dynamics*, pages 1–8. John Wiley & Sons, 2017.

[142] Adrian Constantin. Nonlinear water waves: Introduction and overview. *Philosophical Transactions of the Royal Society A*, 2017.

[143] Yu. V. Sedletsky. The fourth-order nonlinear Schrödinger equation for the envelope of Stokes waves on the surface of a finite-depth fluid. *Journal of Experimental and Theoretical Physics*, 97:180–193, 2003.

[144] A. V. Slunyaev. A high-order nonlinear envelope equation for gravity waves in finite-depth water. *Journal of Experimental and Theoretical Physics*, 101:926–941, 2005.

[145] A. Osborne. *Nonlinear Ocean Waves and the Inverse Scattering Transform.* Elsevier, 2010.

[146] Roger Grimshaw and Zihua Liu. Nonlinear Periodic and Solitary Water Waves on Currents in Shallow Water. *Studies in Applied Mathematics*, 139:60–77, 2017.

[147] Kristian B. Dysthe. Note on a modification to the nonlinear Schrödinger equation for application to deep water waves. *Proceedings of the Royal Society A: Mathematical, Physical, and Engineering Sciences*, 369:105–114, 1979.

[148] Karsten Trulsen and Kristian B. Dysthe. A modified nonlinear Schrödinger equation for broader bandwidth gravity waves on deep water. *Wave Motion*, 24:281–289, 1996.

[149] T. R. Marchant. High-order interaction of solitary waves on shallow water. *Studies in Applied Mathematics*, 109:1–17, 2002.

[150] T. Levi-Cività. Sulle onde progressive di tipo permanente. *Rendiconti Accademia dei Lincei*, 1907.

[151] D. Levi. Levi-Cività theory for irrotational water waves in a one-dimensional channel and the complex Korteweg-de Vries equation. *Theoretical and Mathematical Physics*, 99:705–709, 1994.

[152] D. Levi and M. Sanielevici. Irrotational water waves and the complex Korteweg-de Vries Equation. *Physica D*, 98:510–514, 1996.

[153] Morikazu Toda. *Nonlinear Waves and Solitons*. KTK Scientific Publishers, Tokyo, 1989.

[154] Avinash Khare and Avadh Saxena. Linear superposition for a class of nonlinear equations. *Physics Letters A*, 377:2761–2765, 2013.

[155] Handan Borluk and Henrik Kalisch. Particle dynamics in the KdV approximation. *Wave Motion*, 49:691–709, 2012.

[156] J. M. McCowan. On the solitary wave. *The Philosophical Magazine: Series 5*, 32, 1891.

[157] N. E. Joukowsky. Über die Konturen der Tragflächen der Drachenflieger. *Zeitschrift für Flugtechnique und Motorluftschiffahrt*, 1:281–284, 1910.

[158] Yi Zhang, Yi neng Lv, Ling ya Ye, and Hai qiong Zhao. The exact solutions to the complex KdV equation. *Physics Letters A*, 367:465–472, 2007.

[159] Juan-Ming Yuan and Jiahong Wu. The complex KdV equation with or without dissipation. *Discrete and Continuous Dynamical Systems*, 5:489–512, 2005.

[160] Judith R. Miller. Stability properties of solitary waves in a complex modified KdV system. *Mathematics and Computers in Simulation (MATCOM)*, 55:557–565, 2001.

[161] Y. Charles Li. Simple explicit formulae for finite time blow up solutions to the complex KdV equation. *Chaos, Solitons, and Fractals*, 39:369–372, 2009.

[162] Björn Birnir. An example of blow-up, for the complex KdV equation and existence beyond the blow-up. *SIAM Journal of Applied Mathematics*, 47:710–725, 1987.

[163] Hong-Li An and Yong Chen. Numerical complexiton solutions for the complex KdV equation by the homotopy perturbation method. *Applied Mathematics and Computation*, 203:125–133, 2008.

[164] Michel Kerszberg. A simple model for unidirectional crystal growth. *Physics Letters A*, 105:241–244, 1984.

[165] Stefan Wabnitz, Christophe Finot, Julien Fatome, and Guy Millot. Shallow water rogue wavetrains in nonlinear optical fibers. *Physics Letters A*, 377:932–939, 2013.

[166] Roger Grimshaw, Dmitry Pelinovsky, Efim Pelinovsky, and Tatiana Talipova. Wave group dynamics in weakly nonlinear long-wave models. *Physica D*, 159:35–57, 2001.

[167] Nail Akhmediev and Adrian Ankiewicz. *Solitons, Nonlinear Pulses and Beams*. Chapman and Hall, London, 1997.

[168] R. S. Johnson. *A Modern Introduction to the Mathematical Theory of Water Waves*. Cambridge University Press, 1997.

[169] Robert M. Miura. The Korteweg – de Vries Equation: A Survey of Results. *SIAM Review*, 18:412–459, 1976.

[170] A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin. The soliton: A new concept in applied science. *Proceedings of the IEEE*, 61:1443–1483, 1973.

[171] John W. Miles. The Korteweg-de Vries equation: A historical essay. *Journal of Fluid Mechanics*, 106:131, 1981.

[172] Ying-Ying Sun, Juan-Ming Yuan, and Da-Jun Zhang. Solutions to the Complex Korteweg-de Vries Equation: Blow-up Solutions and Non-Singular Solutions. *Communications in Theoretical Physics*, 61:415–422, 2014.

[173] H. I. Abdel-Gawad, M. Tantawy, and R. E. Abo Elkhair. On the extension of solutions of the real to complex KdV equation and a mechanism for the construction of rogue waves. *Waves in Random and Complex Media*, 26:397–406, 2016.

[174] Subhrajit Modak, Akhil Pratap Singh, and Prasanta Kumar Panigrahi. Complex solitary waves and soliton trains in KdV and mKdV equations. *European Physical Journal B*, 89, 2016.

[175] Julia Cen and Andreas Fring. Complex solitons with real energies. *Journal of Physics A: Mathematical and Theoretical*, 49:365202, 2016.

[176] A. P. Misra and Yunliang Wang. Dust-acoustic solitary waves in a magnetized dusty plasma with nonthermal electrons and trapped ions. *Communications in Nonlinear Science and Numerical Simulation*, 22:1360–1369, 2015.

[177] R. Grimshaw, E. Pelinovsky, T. Taipova, and A. Sergeeva. Rogue internal waves in the ocean: Long wave model. *The European Physical Journal Special Topics*, 185:195–208, 2010.

[178] Matthew H. Alford, Thomas Peacock, and et al MacKinnon. The formation and fate of internal waves in the South China Sea. *Nature*, 521(7550):65–69, 2015.

[179] Mathieu Manceau, Kirill Yu. Spasibko, Gerd Leuchs, Radim Filip, and Maria V. Chekhova. Indefinite-mean Pareto photon distribution from amplified quantum noise. *Physical Review Letters*, 123, 2019.

[180] C. Bayindir and M. Arik. Rogue quantum gravitational waves. *arxiv.org/abs/1908.02601*, 2019.

[181] A. Chabchoub, N. P. Hoffmann, and N. Akhmediev. Rogue wave observation in a water wave tank. *Physical Review Letters*, 106, 2011.

[182] Roland Schiek and Fabio Baronio. Spatial Akhmediev breathers in slab waveguides. *Physical Review Research*, 2019.

[183] I. Nikolkina and I. Didenkulova. Rogue waves in 2006–2010. *Natural Hazards and Earth System Sciences*, 11:2913–2924, 2011.

[184] Fabio Baronio, Antonio Degasperis, Matteo Conforti, and Stefan Wabnitz. Solutions of the vector nonlinear Schrödinger equations: Evidence for deterministic rogue waves. *Physical Review Letters*, 109(044102), 2012.

[185] Pierre Gaillard. Families of quasi-rational solutions of the NLS equation and multi-rogue waves. *Journal of Physics A*, 44:435204, 2011.

[186] P. Dubard and V. B. Matveev. Multi-rogue waves solutions to the focusing NLS equation and the KP-I equation. *Natural Hazards and Earth Systems Science*, 11:667–672, 2011.

[187] Yuji Kodama. KP solitons in shallow water. *Journal of Physics A: Mathematical and Theoretical*, 43:434004, 2010.

[188] Yasuhiro Ohta and Jianke Yang. Rogue waves in the Davey-Stewartson I equation. *Physical Review E*, 86, 2012.

[189] Yongshuai Zhang, Deqin Qiu, Dumitru Mihalache, and Jingsong He. The loop rogue wave solutions for the Wadati–Konno–Ichikawa equation. *Chaos*, 28:103108, 2018.

[190] Victor I. Shrira and Vladimir V. Geogjaev. What makes the Peregrine soliton so special as a prototype of freak waves? *Journal of Engineering Mathematics*, 67:11–22, 2010.

[191] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, and F. T. Arecchi. Rogue waves and their generating mechanisms in different physical contexts. *Scientific Reports*, 528:47–89, 2013.

[192] R. K. Bullough and P. J. Caudrey. Solitons and the Korteweg—de Vries Equation: Integrable Systems in 1834–1995. *Acta Applicandae Mathematicae*, pages 193–228, 1995.

[193] M. Lakshmanan and S. Rajasekar. *Nonlinear Dynamics*. Advanced Texts in Physics. Springer, 2003.

[194] A. Ankiewicz, Mahyar Bokaeeyan, and N. Akhmediev. Shallow-water rogue waves: An approach based on complex solutions of the Korteweg–de Vries equation. *Physical Review E*, 99(050201(R)), 2019.

[195] Adrian Ankiewicz and Nail Akhmediev. Multi-rogue waves and triangular numbers. *Romanian Reports in Physics*, 2017.

[196] Ramakrishnan Iyer, Clifford V. Johnson, and Jeffrey S. Pennington. Nonperturbative string theory from water waves. *Journal of Physics A: Mathematical and Theoretical*, 44:375401, 2011.

[197] V. S. Gerdjikov, I. M. Uzunov, E. G. Evstatiev, and G. L. Diankov. Nonlinear Schrödinger equation and N-soliton interactions. *Physical Review E*, 1997.

[198] S. Trillo and S. Wabnitz. Dynamics of the nonlinear modulational instability in optical fibers. *Optics Letters*, 16:986, 1991.

[199] A. Mussot and A. Kudlinski. Modulation instability, loaded dice. *Nature Photonics*, 6:415–416, 2012.

[200] Arnaud Mussot, Alexandre Kudlinski, Maxime Droques, Pascal Szriftgiser, and Nail Akhmediev. Fermi-Pasta-Ulam Recurrence in Nonlinear Fiber Optics: The Role of Reversible and Irreversible Losses. *Physical Review X*, 4, 2014.

[201] F. Baronio, M. Conforti, S. Wabnitz, and A. Degasperis. Rogue Waves of the Vector Nonlinear Schrödinger Equations. In *IEEE Proceedings of 2013 Conference on lasers and electro-optics Europe and International quantum electronics conference*, 2013.

[202] M. J. Ablowitz and P. A. Clarkson. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Number 149 in London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge, 1991.

[203] Sergei Sakovich. Integrability of a generalized short pulse equation revisited. *Applied Mathematics Research*, 2, 2018.

[204] K. Porsezian. Completely integrable nonlinear Schrödinger type equations on moving space curves. *Physical Review E*, 55:3785–3788, 1997.

[205] K. Porsezian, M. Daniel, and M. Lakshmanan. On the integrability aspects of the one-dimensional classical continuum isotropic biquadratic Heisenberg spin chain. *Journal of Mathematical Physics*, 33:1807–1816, 1992.

[206] D. Mihalache, N. C. Panoiu, F. Moldoveanu, and D. M. Baboiu. The Riemann problem method for solving a perturbed nonlinear Schrödinger equation describing pulse propagation in optic fibres. *Journal of Physics A: Mathematical and General*, 27:6177–6189, 1994.

[207] D. Mihalache, L. Torner, F. Moldoveanu, N.-C. Panoiu, and N. Truta. Inverse-scattering approach to femtosecond solitons in monomode optical fibers. *Physical Review E*, 48:4699–4709, 1993.

[208] Marek Trippenbach and Y. B. Band. Effects of self-steepening and self-frequency shifting on short-pulse splitting in dispersive nonlinear media. *Physical Review A*, 57:4791–4803, 1991.

[209] Yoshimasa Matsuno. Bilinearization of Nonlinear Evolution Equations II. Higher-Order Modified Korteweg-de Vries Equations. *Journal of the Physical Society of Japan*, 49:787–794, 1980.

[210] Govind Agrawal. *Nonlinear Fiber Optics*. Elsevier, 4 edition, 2006.

[211] John Dudley, Goëry Genty, and Stéphane Coen. Supercontinuum generation in photonic crystal fiber. *Reviews of Modern Physics*, 78:1135–1184, 2006.

[212] G. Theocharis, Z. Rapti, P. G. Kevrekidis, D. J. Frantzeskakis, and V. V. Konotop. Modulational instability of Gross-Pitaevskii-type equations in1+1dimensions. *Physical Review A*, 67, 2003.

[213] V. B. Bobrov and S. A. Trigger. Bose–Einstein condensate wave function and nonlinear Schrödinger equation. *Bulletin of the Lebedev Physics Institute*, 43:266–269, 2016.

[214] Luigi Galati and Shijun Zheng. Nonlinear Schrödinger equations for Bose-Einstein condensates. In *AIP Conference Proceedings*, 2013.

[215] Thudiyangal Mithun and Kenichi Kasamatsu. Modulation instability associated nonlinear dynamics of spin–orbit coupled Bose–Einstein condensates. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 52:045301, 2019.

[216] Lennart Stenflo and Mattias Marklund. Rogue waves in the atmosphere. *Journal of Plasma Physics*, 76:293–295, 2010.

[217] Eugene B. Kolomeisky. Nonlinear plasma waves in an electron gas. *Journal of Physics A: Mathematical and Theoretical*, 51:35LT02, 2018.

[218] Catherine Sulem and Pierre-Louis Sulem. *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*. Number 139 in Applied Mathematical Sciences. Springer, 1999.

[219] Yakup Yıldırım, Nisa Çelik, and Emrullah Yaşar. Nonlinear Schrödinger equations with spatio-temporal dispersion in Kerr, parabolic, power and dual power law media: A novel extended Kudryashov's algorithm and soliton solutions. *Results in Physics*, 7:3116–3123, 2017.

[220] C. Castro and J. Mahecha. On Nonlinear Quantum Mechanics, Brownian Motion, Weyl Geometryand Fisher Information. *Progress in Physics*, 1(38), 2006.

[221] Marek Czachor. Nonlinear Schrödinger equation and two-level atoms. *Physical Review A*, 53:1310–1315, 1996.

[222] A. G. Vladimirov, S. V. Gurevich, and M. Tlidi. Effect of Cherenkov radiation on localized-state interaction. *Physical Review A*, 97, 2018.

[223] G. H. M. Roelofs and P. H. M. Kersten. Supersymmetric extensions of the nonlinear Schrödinger equation: Symmetries and coverings. *Journal of Mathematical Physics*, 33:2185–2206, 1992.

[224] K. Ohkuma, Yoshi H. Ichikawa, and Y. Abe. Soliton propagation along optical fibers. *Optics Letters*, 12:516, 1987.

[225] Maysaa Mohamed Al Qurashi, Abdullahi Yusuf, Aliyu Isa Aliyu, and Mustafa Inc. Optical and other solitons for the fourth-order dispersive nonlinear Schrödinger equation with dual-power law nonlinearity. *Superlattices and Microstructures*, 105:183–197, 2017.

[226] P. A. Andreev. First principles derivation of NLS equation for BEC with cubic and quintic nonlinearities at nonzero temperature: Dispersion of linear waves. *International Journal of Modern Physics B*, 27:1350017, 2013.

[227] M. J. Potasek and M. Tabor. Exact solutions for an extended nonlinear Schrödinger equation. *Physics Letters A*, 154:449–452, 1991.

[228] Solange B. Cavalcanti, José C. Cressoni, Heber R. da Cruz, and Artur S. Gouveia-Neto. Modulation instability in the region of minimum group-velocity dispersion of single-mode optical fibers via an extended nonlinear Schrödinger equation. *Physical Review A*, 43:6162–6165, 1991.

[229] D. J. Kedziora, A. Ankiewicz, A. Chowdury, and N. Akhmediev. Integrable equations of the infinite nonlinear Schrödinger equation hierarchy with time variable coefficients. *Chaos*, 25:103114, 2015.

[230] A. Ankiewicz, D. J. Kedziora, A. Chowdury, U. Bandelow, and N. Akhmediev. Infinite hierarchy of nonlinear Schrödinger equations and their solutions. *Physical Review E*, 2016.

[231] A. Ankiewicz and N. Akhmediev. Rogue wave solutions for the infinite integrable nonlinear Schrödinger equation hierarchy. *Physical Review E*, 96, 2017.

[232] A. Ankiewicz, U. Bandelow, and N. Akhmediev. Generalised Sasa–Satsuma Equation: Densities Approach to New Infinite Hierarchy of Integrable Evolution Equations. *Zeitschrift für Naturforschung, Section A, Journal of Physical Sciences*, 73:1121–1128, 2018.

[233] U. Bandelow, A. Ankiewicz, Sh. Amiranashvili, and N. Akhmediev. Sasa-Satsuma hierarchy of integrable evolution equations. *Chaos*, 28:053108, 2018.

[234] A. Ankiewicz and N. Akhmediev. Rogue wave-type solutions of the mKdV equation and their relation to known NLSE rogue wave solutions. *Nonlinear Dynamics*, 91:1931–1938, 2018.

[235] J. M. Soto-Crespo, N. Devine, and N. Akhmediev. Adiabatic transformation of continuous waves into trains of pulses. *Physical Review A*, 96, 2017.

[236] Amdad Chowdury, Wieslaw Krolikowski, and N. Akhmediev. Breather solutions of a fourth-order nonlinear Schrödinger equation in the degenerate, soliton, and rogue wave limits. *Physical Review E*, 96, 2017.

[237] Wen-Xiu Ma. A polynomial conjecture connected with rogue waves in the KdV equation. *PDEs in Applied Mathematics*, 3:100023, 2021.

[238] Takahiro Shiota. Characterization of Jacobian varieties in terms of soliton equations. *Inventiones Mathematicae*, 83:333–382, 1986.

[239] Motohico Mulase. KP Equations, Strings, and the Schottky Problem. In *Papers Dedicated to Professor Mikio Sato on the Occasion of his Sixtieth Birthday*, volume 2, pages 473–492. 1988.

[240] Igor Krichever and Takahiro Shiota. Soliton equations and the Riemann-Schottky problem. *arXiV.org/abs/1111.0164v1 [math.AG]*, 2011.

[241] Freeman J. Dyson. Missed opportunities. *Bulletin of the American Mathematical Society*, 78(5):635–653, 1972.