

Modification of Social Dominance in Social Networks by Selective Adjustment of Interpersonal Weights

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Abstract—According to the DeGroot-Friedkin model of a social network, an individual’s social power evolves as the network discusses individual opinions over a sequence of issues. Under mild assumptions on the connectivity of the network, the social power of every individual converges to a constant nonnegative value as the number of issues discussed increases. If the network has a special topology, namely the “star topology”, then all social power accumulates with the individual at the centre of the star. This paper studies the strategic introduction of new individuals and/or interpersonal relationships into a social network with the star topology so as to *reduce the social power of the centre individual*. In fact, several strategies are proposed. For each strategy, we derive necessary and sufficient conditions on the strength of the new interpersonal relationships, based on local information, which *ensures that the centre individual no longer has the greatest social power within the social network*. Interpretations of these conditions reveal that the strategies are remarkably intuitive and that certain strategies are favourable compared to others, all of which is sociologically expected.

I. INTRODUCTION

In recent years, the systems and control community has turned to study of networked systems and multi-agent systems in the context of social sciences. Of particular interest are social networks, where groups of people interact with acquaintances through *interpersonal relationships*.

One problem of “opinion dynamics” has been of particular interest: how do the opinions of individuals for a given issue evolve as they discuss this issue in a social network? A recent survey on opinion dynamics is presented in [1]. An important aspect is *social power*, which in one sense can be considered as the weight/power/influence an individual has on the opinion discussion, relative to the weight/power/influence of the other individuals in the social network. This relativity arises due to interpersonal relationships and their strengths (which may be unidirectional). This concept has been studied in the seminal works [2], [3]. The dynamical evolution of social power has been studied in [4]. Selecting the most

influential individual in social diffusion models has been studied in [5]. The centralised DeGroot-Friedkin model for the evolution of social power has been proposed and analysed in [6]. Distributed discrete- and continuous-time DeGroot-Friedkin models have been proposed and studied in [7] and [8], respectively.

According to French Jr. and Snyder in [9], “*leadership* is the potential social influence of one part of the group over another.” From the perspective of opinion dynamics, a leader can therefore be seen as an individual or a group of individuals that has a disproportionate amount of control over the opinion discussion process. In the context of social power, one can therefore refer to a leader/leader group as the *socially dominant* individual/group of individuals. The fact that social power tends to accumulate with one individual or a subgroup of individuals in a social network has been reported empirically in [4] and theoretically in [6]. This individual or subgroup is defined explicitly by the interpersonal relationships in the social network. Motivated by this concept of social dominance/leadership, and using the centralised DeGroot-Friedkin model to describe the social network, we begin in this paper with network topologies that have a single socially dominant individual, and seek to study *strategies, including introduction of new individuals into the network and/or establishment of new interpersonal relationships, that will cause the social dominance to shift to another individual*. The DeGroot-Friedkin model will be introduced formally in the sequel, but we describe it here at a high level so the reader is better able to understand the contributions of the paper. The DeGroot-Friedkin model considers a social network which discusses opinions of individuals on a sequence of issues/topics. At the end of the discussion of any given issue (which occurs when consensus on opinions is reached), each individual reflects on its influence, i.e. social power, on the overall discussion. For the following issue, each individual sets its self-weight (the weight the individual places on its own opinion relative to its neighbours’ opinions) to be equal to its social power from the previous issue. The DeGroot-Friedkin model describes the dynamical evolution of the social power of all individuals over the sequence of issues.

It was shown in [6] that if the social network has a constant topology, each individual’s social power converges to a constant (equilibrium) value as issues were sequentially discussed. For the special star topology, all social power at equilibrium accumulated with a single individual k , in an “autocratic configuration”. In this paper, we show that by *strategic* introduction of new individuals and/or new interpersonal relationships into the social network, not only is

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the autocratic configuration broken but *if the new relationship is sufficiently strong, other identifiable individual(s) will have social power at equilibrium greater than individual k* . Specifically, we derive necessary and sufficient conditions *based on local information* for the relationship strength. This is in contrast to many control strategies on networked systems which rely on global information [5], [10]. In fact, a number of different strategies are considered.

While the results are initially presented mathematically as inequalities, we provide detailed analysis and interpretation. In doing so, we show that the strategies are remarkably intuitive and precisely what one would expect when considered within a sociological context. While the strategies are intuitive, the novel contributions of this paper are the necessary and sufficient conditions for guaranteeing that the strategy succeeds, expressed as analytic functions of key interpersonal relationship strengths. The fact that the strategies affect the *social power of individuals* in a social network which is sequentially discussing issues implies that we have developed strategies for *affecting/influencing the opinion dynamics process*.

The rest of the paper is structured as follows. In Section II, we provide notations, an introduction to graph theory and the DeGroot-Friedkin model. At the same time, a formal problem statement is given. The main results are presented in Section III and conclusions are drawn in Section IV.

II. BACKGROUND AND FORMAL PROBLEM STATEMENT

We begin by introducing some mathematical notations used in the paper. Let $\mathbf{1}_n$ and $\mathbf{0}_n$ denote, respectively, the $n \times 1$ column vectors of all ones and all zeros. For a vector $\mathbf{x} \in \mathbb{R}^n$, $0 \preceq \mathbf{x}$ and $0 \prec \mathbf{x}$ indicate component-wise inequalities, i.e., for all $i \in \{1, 2, \dots, n\}$, $0 \leq x_i$ and $0 < x_i$, respectively. Let Δ_n denote the n -simplex, the set given by $\{\mathbf{x} \in \mathbb{R}^n : 0 \preceq \mathbf{x}, \mathbf{1}_n^\top \mathbf{x} = 1\}$. The canonical basis of \mathbb{R}^n is given by $\mathbf{e}_1, \dots, \mathbf{e}_n$. Define $\tilde{\Delta}_n = \Delta_n \setminus \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\text{int}(\Delta_n) = \{\mathbf{x} \in \mathbb{R}^n : 0 \prec \mathbf{x}, \mathbf{1}_n^\top \mathbf{x} = 1\}$. Note that the terms “node”, and “individual” are used interchangeably.

A. Graph Theory

The interaction between individuals in a social network is modelled using a weighted directed graph, denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each individual is a node in the finite, nonempty set of nodes $V = \{v_1, \dots, v_n\}$. The set of ordered edges is $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We denote an ordered edge as $e_{ij} = (v_i, v_j) \in \mathcal{E}$, and because the graph is directed, in general e_{ij} and e_{ji} may not both exist. An edge e_{ij} is said to be outgoing with respect to v_i and incoming with respect to v_j . The presence of an edge e_{ij} connotes that individual j learns of, and takes into account, the opinion value of individual i when updating its own opinion. The relative interaction matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$ associated with \mathcal{G} has nonnegative entries c_{ij} , termed “relative interpersonal weights” in [6]. The entries of \mathbf{C} have properties such that $0 < c_{ij} \leq 1 \Leftrightarrow e_{ji} \in \mathcal{E}$ and $c_{ij} = 0$ otherwise. It is assumed that $c_{ii} = 0$ (i.e. with no self-loops), and we impose the restriction that $\sum_{j=1}^n c_{ij} = 1$ (i.e. that \mathbf{C} is a row-stochastic matrix).

A directed path is a sequence of edges of the form $(v_{p_1}, v_{p_2}), (v_{p_2}, v_{p_3}), \dots$ where $v_{p_i} \in \mathcal{V}, e_{ij} \in \mathcal{E}$. Node i is reachable from node j if there exists a directed path from v_j to v_i . A graph is said to be strongly connected if every node is reachable from every other node. The relative interaction matrix \mathbf{C} is irreducible if and only if the associated graph \mathcal{G} is strongly connected. If \mathbf{C} is irreducible then it has a unique left eigenvector $\boldsymbol{\gamma}^\top$ satisfying $\boldsymbol{\gamma}^\top \mathbf{1}_n = 1$, with all entries strictly positive, associated with the eigenvalue 1 (Perron-Frobenius Theorem, see [11]). Henceforth, we shall call $\boldsymbol{\gamma}^\top$ the *dominant left eigenvector of \mathbf{C}* , with i^{th} entry γ_i .

B. The DeGroot-Friedkin Model

We define $\mathcal{S} = \{0, 1, 2, \dots\}$ to be the set of indices of sequential issues which are being discussed by the social network. For a given issue $s \in \mathcal{S}$, the social network’s opinion dynamics process is described using the discrete-time DeGroot opinion pooling algorithm (with constant weights throughout the discussion of the issue). At the end of the discussion (i.e. when the DeGroot model has effectively reached steady state), each individual reflects upon, and judges its impact on the discussion. This mechanism is termed reflected self-appraisal, with “reflection” referring to the fact that adjustments to weights are made following the completion of discussion on the particular issue s . The individual then updates its own self-weight (the weight an individual applies to its own opinion value in the DeGroot algorithm) and discussion begins on the next issue $s + 1$ (using the same DeGroot algorithm but now with adjusted weights). We now explain the mathematical modelling of the opinion dynamics for an issue, and the updating of self-weights from one issue to the next.

For each issue $s \in \mathcal{S}$, individual i updates its opinion $y_i(s, \cdot) \in \mathbb{R}$ at time $t + 1$ as

$$y_i(s, t + 1) = w_{ii}(s)y_i(s, t) + \sum_{j \neq i}^n w_{ij}(s)y_j(s, t) \quad (1)$$

where $w_{ii}(s)$ is the self-weight individual i places on its own opinion and $w_{ij}(s)$ is the weight given by individual i to the opinion of its neighbour individual j . Note that $w_{ij}(s), \forall i, j$ is constant for any given s . The element $w_{ij}, j \neq i$ is defined as $w_{ij} = (1 - w_{ii})c_{ij}$ which ensures that $\sum_{j=1}^n w_{ij} = 1$ for any w_{ii} , because we have assumed above that $\sum_{j=1}^n c_{ij} = 1$. In other words, $\sum_{j=1}^n w_{ij}(s) = 1$ continues to hold as $w_{ii}(s)$ is updated (the precise mechanism for updating $w_{ii}(s)$ will be detailed shortly). It was shown in [6] that if \mathbf{C} is irreducible, and $\exists i : w_{ii} > 0$ then consensus of opinions is reached: $y_k(\infty) = y_j(\infty), \forall k, j$.

We now detail the updating of the self-weight (also referred to as individual social power, self-confidence or self-esteem) of individual i , which is denoted by $x_i(s) = w_{ii}(s) \in [0, 1]$ for convenience [6]. Define the vector $\mathbf{x}(s) = [x_1(s) \ \dots \ x_n(s)]^\top$ as the vector of self-weights for the individuals of the social network, with starting self-weight $0 \leq x_i(0) \leq 1$ satisfying $\sum_i x_i(0) = 1$. The DeGroot-Friedkin model states that the self-weight vector $\mathbf{x}(s)$ is

updated as

$$\mathbf{x}(s+1) = \mathbf{F}(\mathbf{x}(s)) \quad (2)$$

where $\mathbf{F}(\mathbf{x}(s))$ is defined as

$$\mathbf{F}(\mathbf{x}(s)) = \begin{cases} \mathbf{e}_i & \text{if } x_i(s) = \mathbf{e}_i, \text{ for any } i \\ \alpha(\mathbf{x}(s)) \begin{bmatrix} \frac{\gamma_1}{1-x_1(s)} \\ \vdots \\ \frac{\gamma_n}{1-x_n(s)} \end{bmatrix} & \text{otherwise} \end{cases}$$

with $\alpha(\mathbf{x}(s)) = 1 / \sum_{i=1}^n \frac{\gamma_i}{1-x_i(s)}$. We omit the details on the precise formulation of the model, and the derivation of the map \mathbf{F} due to space limitations; we refer the reader to the original work [6] which introduces the model.

Much of this paper will deal with scenarios where the underlying graph has a star topology or its variants, the definition of which is now given.

Definition 1 (Star topology). *A strongly connected graph¹ \mathcal{G} is said to have star topology if there exists a node i , which is called the centre node, such that every edge of \mathcal{G} is either to or from node i*

Note that the irreducibility of \mathbf{C} implies that the star topology must include edges in both directions between the centre node v_i and every other node $v_j, j \neq i$. We now provide two results regarding the convergence of $\mathbf{F}(\mathbf{x}(s))$ as $s \rightarrow \infty$, with an interpretation given in Remark 1.

Lemma 1 (Lemma 3.2, [6]). *Suppose that $n \geq 3$, and suppose further that \mathcal{G} has star topology, which without loss of generality has centre node v_1 . Let \mathbf{C} be the row-stochastic and irreducible relative interaction matrix, with zero diagonal entries, associated with \mathcal{G} . Then for all initial conditions $\mathbf{x}(0) \in \tilde{\Delta}_n$, $\lim_{s \rightarrow \infty} \mathbf{x}(s) = \mathbf{e}_1$.*

Theorem 1 (Theorem 4.1, [6]). *For $n \geq 3$, consider the DeGroot-Friedkin dynamical system (2) with a relative interaction matrix \mathbf{C} that is row-stochastic, irreducible, and has zero diagonal entries. Assume that the digraph \mathcal{G} associated with \mathbf{C} does not have star topology and define $\boldsymbol{\gamma}^\top$ as the dominant left eigenvector of \mathbf{C} . Then,*

- (i) *For all initial conditions $\mathbf{x}(0) \in \tilde{\Delta}_n$, the self-weights $x_i(s)$ converge to x_i^* as $s \rightarrow \infty$. Here, $\mathbf{x}^* \in \text{int}(\Delta_n)$ is the unique fixed point satisfying $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$.*
- (ii) *There holds $x_i^* < x_j^*$ if and only if $\gamma_i < \gamma_j$, for any i, j , where γ_i is the i^{th} entry of the dominant left eigenvector $\boldsymbol{\gamma}$. There holds $x_i^* = x_j^*$ if and only if $\gamma_i = \gamma_j$.*
- (iii) *The unique fixed point \mathbf{x}^* is determined only by $\boldsymbol{\gamma}^\top$, and is independent of the initial conditions.*

C. Formal Problem Statement

In this paper, we investigate how additional nodes and/or edges strategically connected to a star topology can change the social power at equilibrium, \mathbf{x}^* . To that end, we begin

¹While it is possible to have a star graph that is not strongly connected, this paper, similarly to [6], deals only with strongly connected graphs.

first by providing definitions which will aid in describing our problem and discussing the results obtained. Moreover, we are interested in comparing the social power of individuals within the network at equilibrium, i.e. when $s \rightarrow \infty$. We will therefore refer to the equilibrium value x_i^* as the social power of individual i when there is no ambiguity (as opposed to the evolving $x_i(s)$ when $s < \infty$). To simplify the problem, we do not study the evolution of the opinions $y_i(s, t)$, which as observed below (1), will always reach a consensus if \mathbf{C} is irreducible and $\mathbf{x}(0) \in \Delta_n$.

Definition 2 (Autocratic Network). *A social network is said to be an autocratic configuration, with node v_i being the autocrat, if $\mathbf{x}(s) = \mathbf{e}_i$.*

Definition 3 (Social dominance/leadership). *Node v_i is said to be the socially dominant/leader node in the network if $x_i^* > x_j^*$ for all $j \neq i$. In other words, at equilibrium, the social power of individual i is greater than the social power of any other individual in the social network.*

Remark 1 (Autocratic tendency). *Lemma 1 has an important social connotation. One can consider $x_i(0)$ as individual i 's estimate of its social power when the social network is first formed, before any discussions occur. For any initial estimate $\mathbf{x}(0) \in \tilde{\Delta}_n$ (that is, no individual i believes $x_i(0) = 1$), the star topology network tends to an autocratic configuration at equilibrium, $\mathbf{x}^* = \mathbf{e}_1$. This implies that, for initial issues, opinion discussion will occur with everyone contributing to the final consensus value. However, the centre individual increasingly guides the outcome of discussions until, for $s = \infty$, only the centre individual's opinion value matters.*

Remark 2. *In [6], the constant entries c_{ij} of \mathbf{C} are termed "relative interpersonal weights", and we will stay with this terminology. However, one can also consider c_{ij} as the amount of relative "trust" individual i allocates to individual j . In other words, c_{ij} captures the strength of a unidirectional interpersonal relationship (unidirectional since $c_{ij} \neq c_{ji}$ in general).*

For a given graph \mathcal{G} with star topology, with centre node v_1 , let us call the other nodes *subject nodes* in the sense that they are subjects to the autocrat centre node. We are going to study how the autocracy can be disrupted by introduction of a perturbation to the star graph. This leads us to define a new type of node. An *attacker node* is a node v_j which forms edges e_{ji}, e_{ij} with a subject node. In doing so, we modify the graph \mathcal{G} to become $\bar{\mathcal{G}}$ which is no longer a star. In Fig. 1, v_1 is the centre node, while nodes $v_i, i = 2, \dots, 7$ are subject nodes, and node v_8 is the attacker node, forming edges with node v_7 . We call node v_j an attacker node because, as it will become apparent in the sequel, the weights c_{ji} and c_{ij} can modify/reduce the social power x_1^* of the centre node v_1 . In other words, v_j attacks the social dominance of v_1 . Note that two edges, e_{ji}, e_{ij} must be formed to ensure that $\bar{\mathcal{G}}$ remains strongly connected. Actually, there are a number of interesting ways to attack the social dominance of v_1 , and we list below some of the most important/fundamental methods.

An example of each is given in Figures 1-4.

Topology Variation 1 (Single Attack). *Suppose that $n \geq 4$. Suppose further that \mathcal{G} has star topology, with v_1 being the centre node, and with $n-2$ subject nodes, $v_i, i = 2, \dots, n-1$. A single attacker node v_n attaches to subject node v_{n-1} by forming edges $e_{n-1,n}, e_{n,n-1}$, forming the graph $\bar{\mathcal{G}}$.*

Topology Variation 2 (Coordinated Double Attack). *Suppose that $n \geq 5$. Suppose further that \mathcal{G} has star topology, with v_1 being the centre node, and with $n-3$ subject nodes, $v_i, i = 2, \dots, n-2$. Two attacker nodes v_{n-1} and v_n attach to subject node v_{n-2} by forming the set of edges $\{e_{n-2,n-1}, e_{n-1,n-2}, e_{n-2,n}, e_{n,n-2}\}$. This forms $\bar{\mathcal{G}}$.*

Topology Variation 3 (Uncoordinated Double Attack). *Suppose that $n \geq 5$. Suppose further that \mathcal{G} has star topology, with v_1 being the centre node, and with $n-3$ subject nodes, $v_i, i = 2, \dots, n-2$. One attacker node v_{n-1} attaches to subject node v_{n-3} with edges $e_{n-3,n-1}, e_{n-1,n-3}$. A second attacker node v_n attaches to subject node v_{n-2} with edges $e_{n-2,n}, e_{n,n-2}$. This forms $\bar{\mathcal{G}}$.*

Topology Variation 4 (Two Dissenting Subjects). *Suppose that $n \geq 4$. Suppose further that \mathcal{G} has star topology, with v_1 being the centre node, and with $n-1$ subject nodes, $v_i, i = 2, \dots, n$. There are no attacker nodes. Subject nodes v_{n-1} and v_n form edges $e_{n,n-1}, e_{n-1,n}$, forming $\bar{\mathcal{G}}$.*

In the next section, we investigate the above topological variations of the star graph. Note that Topology Variations 1-4 have modified graphs $\bar{\mathcal{G}}$ which *do not have star topology*. From Theorem 1, it immediately follows that $x_1^* < 1$ for all Topology Variations. In other words, v_1 is no longer the autocrat but if the perturbation from the star topology (caused by the new edges) is small, v_1 remains *socially dominant*. What we will show is that if the interpersonal weights associated with these new edges exceed a given threshold, the socially dominant node changes from v_1 to some other node. Of particular note is that the thresholds depend only on local information. It is worth emphasising at this stage that, in Variations 1-3, it is useless for an attacker node v_n to attach to the centre node v_1 instead of a subject node; the topology remains a strongly connected star, and thus v_1 remains the socially dominant autocrat.

Note that when new edges are introduced, we assume each individual i adjusts its weights c_{ij} to ensure that the new \bar{C} is row-stochastic. Take Topology Variation 4 as an example. The relative interaction matrix \bar{C} , associated with star graph \mathcal{G} (without edges $e_{n,n-1}$ and $e_{n-1,n}$), is assumed to be row-stochastic. The relative interaction matrix \bar{C} associated with $\bar{\mathcal{G}}$ is also implicitly assumed to be row-stochastic with zero diagonal. That is, after the addition of edges $e_{n,n-1}, e_{n-1,n}$, adjustments are made to the original weights $c_{n,j}, c_{n-1,k}$ to ensure that \bar{C} is row-stochastic.

Remark 3 (Ordering of Social Power). *Although Theorem 1 states that x^* is uniquely determined by γ^\top , there are no results available which allow one to analytically compute the value of x^* given γ^\top . What is available is Statement (ii) of*

Theorem 1, which states that the ordering of x_i^ is consistent with the ordering of γ_i . A recent paper [12] computed an upper bound on x_i^* which is dependent on γ_i . However, we are in this paper interested in the ordering of individual social power, as opposed to the precise values of social power. This is reflected in Definition 3.*

III. MAIN RESULTS

In order to place the focus on discussion of the *social connotations* of each result, and due to space limitations, the proofs of the results presented in this section are not included in this paper. The proofs for each theorem and corollary, which all follow the same format, can be found in the extended version of this paper on arXiv [13]. Simulations demonstrating our main results are also provide in [13].

Topology Variation 1 is in fact a special case of Topology Variation 2, obtained by setting $c_{n,n-2} = 0$ and by removing node v_n . As such, the first result we present will be on Variation 2, the discussions on Variation 2 are all applicable to the fundamental strategy Variation 1 with minor adjustments.

A. Topology Variation 2: Coordinated Double Attack

Consider now Topology Variation 2. Firstly, define $\beta_1 = c_{n-2,n-1} \in (0, 1)$ and $\beta_2 = c_{n-2,n} \in (0, 1)$ as the two adjustable interpersonal weights associated with the two attackers. Note that because \bar{C} is assumed to be row-stochastic, it is implied that $\beta_1 + \beta_2 + c_{n-2,1} = 1 \Rightarrow \beta_1 + \beta_2 < 1$ because $c_{n-2,1} > 0$. It is straightforward to observe that

$$\bar{C}(\beta_1, \beta_2) = \begin{bmatrix} 0 & c_{12} & \dots & c_{1,n-1} & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 - (\beta_1 + \beta_2) & 0 & \dots & 0 & \beta_1 & \beta_2 \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 \end{bmatrix}$$

We do not display the exact form of \bar{C} for other Topology Variations in this paper due to space limitations, but they are provided in the Appendix of [13].

Theorem 2 (Coordinated Double Attack). *For a social network with Topology Variation 2, with initial conditions $x(0) \in \tilde{\Delta}_n$, and described by the DeGroot-Friedkin model, the following statements are true:*

- (i) *For all $\beta_1, \beta_2 \in (0, 1)$, there holds $x_i^* < x_1^*$ for all $i \neq 1, n-2, n-1, n$, and $x_n^*, x_{n-1}^* < x_{n-2}^*$.*
- (ii) *There holds 1) $x_1^* > x_i^*, \forall, i \neq 1$ if and only if $\beta_1 + \beta_2 < 1 - c_{1,n-2} = \sum_{i=2}^{n-3} c_{1,i}$, or 2) $x_{n-2}^* > x_i^*, \forall, i \neq n-2$ if and only if $\beta_1 + \beta_2 > 1 - c_{1,n-2}$. There holds $x_1^* = x_{n-2}^*$ if and only if $\beta_1 + \beta_2 = 1 - c_{1,n-2}$.*
- (iii) *There holds $x_n^* > x_1^*$ (respectively $x_{n-1}^* > x_1^*$) if and only if $\beta_2 > (1 - \beta_1)/(1 + c_{1,n-2})$ (respectively $\beta_1 > (1 - \beta_2)/(1 + c_{1,n-2})$).*
- (iv) *There holds $x_{n-1}^* < x_n^*$ or $x_{n-1}^* > x_n^*$ if and only if $\beta_1 < \beta_2$ or $\beta_1 > \beta_2$ respectively. If $\beta_1 = \beta_2$, then $x_{n-1}^* = x_n^*$.*

Corollary 1 (Generalised Placement of Coordinated Double Attack). *Suppose that instead of attaching to subject node*

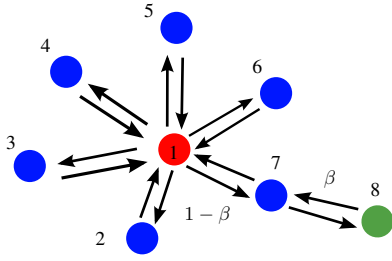


Fig. 1. Topology Variation 1 (Single Attacker) with $n = 8$, attacker is green.

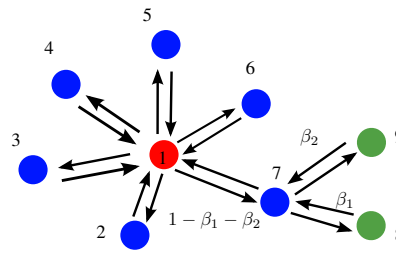


Fig. 2. Topology Variation 2 (Coordinated Double Attacker) with $n = 9$, attackers are green.

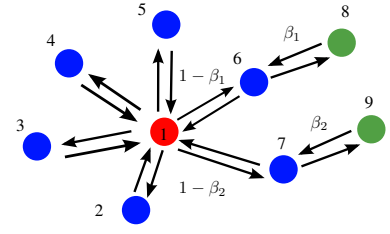


Fig. 3. Topology Variation 3 (Uncoordinated Double Attacker) with $n = 9$, green attackers.

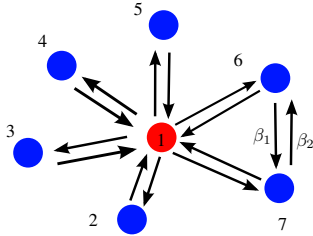


Fig. 4. Topology Variation 4 (Two Dissenting Subjects) with $n = 7$.

v_{n-2} , attacker nodes v_{n-1}, v_n can attach to any subject node $v_i, i \in \{2, \dots, n-2\}$ by forming the set of edges $\{e_{n-1,i}, e_{i,n-1}, e_{n,i}, e_{i,n}\}$. The lower bound on $\beta_1 + \beta_2 = c_{n-1,i} + c_{n,i}$ required to have $x_{n-2}^* > x_1^*$ is minimised if v_{n-1} and v_n attach to v_k where $k = \operatorname{argmax}_{j \in \{2, \dots, n-2\}} c_{1,j}$.

The above mathematical results can be interpreted in the following social context. From Statement (i), we conclude that subject individual i , for $i \in \{2, \dots, n-3\}$, will never have greater social power at equilibrium than the centre individual v_1 , $x_i^* < x_1^*$, regardless of how β_1, β_2 changes. Moreover, the attacker nodes v_{n-1}, v_n will never have greater social power than the subject node v_{n-2} which it is attached to.

Remark 2 states that c_{ij} can be considered the trust level accorded to individual j by individual i . The key result is Statement (ii), which indicates that the *combined trust* given to attackers v_{n-1} and v_n by subject node v_{n-2} must exceed the *combined trust* given to subjects v_2, \dots, v_{n-3} by centre node v_1 , in order for centre node v_1 to lose social dominance (and thus subject v_{n-2} becomes socially dominant). It is most interesting to note that *it is only the sum* of the trust/influence $\beta_1 + \beta_2$ that is relevant, and *there is no requirement on the individual magnitudes* of β_1, β_2 .

Corollary 1 delivers an intuitive, powerful, and socially relevant result. It states that the attackers v_{n-1}, v_n should seek to form an interpersonal relationship with *the subject individual* v_k that individual v_1 trusts the most. This will minimise the required amount of trust subject v_k accords attacker v_n before centre individual v_1 loses social dominance.

Statement (iii) reveals that the attackers can also obtain social power greater than the centre individual v_1 if β_1 and β_2 exceed some lower bounding inequalities. Specifically, we observe that the inequality, which if satisfied ensures that attacker v_n has social power greater than centre v_1 ,

simply involves the variables β_1, β_2 and $c_{1,n-2}$. Moreover, there always exists a pair β_1, β_2 , satisfying $\beta_1 + \beta_2 < 1$ (and thus maintaining $\sum_{j=1}^n c_{n-2,j} = 1$), which ensures both attacker individuals v_{n-1}, v_n have social power greater than the centre v_1 . This is detailed in the proof [13, Theorem 2] and is desirable from the view point of the two attacker individuals.

B. Topology Variation 3: Uncoordinated Double Attack

Define $\beta_1 = c_{n-3,n-1} \in (0, 1)$ and $\beta_2 = c_{n-2,n} \in (0, 1)$.

Theorem 3 (Uncoordinated Double Attack). *For a social network with Topology Variation 3, with initial conditions $\mathbf{x}(0) \in \tilde{\Delta}_n$, and described by the DeGroot-Friedkin model, the following statements are true:*

- (i) *For all $\beta_1, \beta_2 \in (0, 1)$, there holds $x_i^* < x_1^*$ for all $i \in \{2, \dots, n-4\}$, and $x_{n-1}^* < x_{n-3}^*$ and $x_n^* < x_{n-2}^*$.*
- (ii) *There holds $x_1^* > x_i^*$ for all $i \neq 1$ if and only if $\beta_1 < 1 - c_{1,n-3}$ and $\beta_2 < 1 - c_{1,n-2}$. If $\beta_1 > 1 - c_{1,n-3}$ (respectively $\beta_2 > 1 - c_{1,n-2}$), then $x_{n-3}^* > x_1^*$ (respectively $x_{n-2}^* > x_1^*$).*
- (iii) *For $i \in \{1, 2\}$, there holds $x_{n-2+i}^* > x_1^*$ if and only if $\beta_i > 1/(1 + c_{1,n-4+i})$.*
- (iv) *There holds $x_{n-3}^* > x_{n-2}^*$ if and only if $\frac{1-\beta_2}{1-\beta_1} > \frac{c_{1,n-2}}{c_{1,n-3}} \Leftrightarrow \frac{c_{1,n-3}}{c_{n-3,1}} > \frac{c_{1,n-2}}{c_{n-2,1}}$.*

Due to space limitations, we discuss social implications of Theorem 3 only if the conclusions differ significantly from the discussion in the previous subsection.

The most interesting conclusion drawn from Theorem 3 arises when we compare to Theorem 2 which concerns Topology Variation 2. With Topology Variation 2, for the centre individual v_1 to lose its social dominance we require *the sum of the trust values* $\beta_1 + \beta_2$ to exceed a lower bound, and there are no separate lower bounding inequalities for β_1 or β_2 . With Topology Variation 3, centre individual v_1 loses social dominance if and only if either β_1 or β_2 exceed their respective lower bounding inequalities. Importantly, these two lower bounding inequalities *are independent of each other*. This clearly points to the fact that a **coordinated attack** on the social dominance of the centre node is more desirable, an idea which is socially intuitive.

From Statement (iii), both attacker nodes have larger social power than the centre node if and only if $\beta_1 > 1/(1 + c_{1,n-3})$ and $\beta_2 > 1/(1 + c_{1,n-2})$, which implies that $\beta_1 + \beta_2 > 1/(1 + c_{1,n-3}) + 1/(1 + c_{1,n-2})$. From Statement (iii)

Theorem 2, with Topology Variation 2, both attacker nodes have larger social power than the centre node if and only if $\beta_2 > (1 - \beta_1)/(1 + c_{1,n-2})$ and $\beta_1 > (1 - \beta_2)/(1 + c_{1,n-2})$, which implies that $\beta_1 + \beta_2 > 2/(2 + c_{1,n-2})$. Since both $1 + c_{1,n-2}$ and $1 + c_{1,n-3}$ are smaller than $2 + c_{1,n-2}$, it follows that $1/(1 + c_{1,n-3}) + 1/(1 + c_{1,n-2}) > 2/(2 + c_{1,n-2})$, which implies that a **coordinated attack on the social dominance of the centre node is also more efficient for the attackers.**

C. Topology Variation 4: Two Dissenting Subjects

Topology Variation 4 is different from the ones studied above in the sense that there are no attacker nodes. Instead, one can consider this variation as one where two subjects form a relationship in *dissent* from the leader. Letting $\beta_1 = c_{n-1,n} \in (0, 1)$ and $\beta_2 = c_{n,n-1} \in (0, 1)$, analysis yields:

Theorem 4 (Two Dissenting Subjects). *For a social network with Topology Variation 4, with initial conditions $\mathbf{x}(0) \in \Delta_n$, and described by the DeGroot-Friedkin model, the following statements are true:*

- (i) For all $\beta_1, \beta_2 \in (0, 1)$, there holds $x_i^* < x_1^*$ for all $i \neq 1, n-1, n$.
- (ii) There holds $x_n^* > x_1^*$ if and only if $\beta_1 > (1 - c_{1,n})/(c_{1,n-1} + \beta_2)$ with $\beta_1 \in (0, 1)$. There exists such a $\beta_1 \in (0, 1)$ only if $\beta_2 > \sum_{i=2}^{n-2} c_{1,i}$.
- (iii) There holds $x_{n-1}^* > x_1^*$ if and only if $\beta_2 > (1 - c_{1,n-1})/(c_{1,n} + \beta_1)$ with $\beta_2 \in (0, 1)$. There exists such a $\beta_2 \in (0, 1)$ only if $\beta_1 > \sum_{i=2}^{n-2} c_{1,i}$.
- (iv) There holds $x_n^* < x_{n-1}^*$ if and only if $\beta_2 > \beta_1 c_{1,n} + c_{1,n-1}(c_{1,n} - 1)$ or equivalently $\beta_1 < (\beta_2 + c_{n-1}(1 - c_{1,n}))$

Note that the inequality in statement (ii) can be rewritten as $\beta_2 > (1 - c_{1,n} - \beta_1 c_{1,n-1})/\beta_1$ with $\beta_2 \in (0, 1)$ which is satisfiable only if $\beta_1 > (1 - c_{1,n})/(1 + c_{1,n-1})$. Similarly, the inequality in statement (iii) is equivalent to $\beta_1 > (1 - c_{1,n-1} - \beta_2 c_{1,n})/\beta_2$ with $\beta_1 \in (0, 1)$ which is satisfiable only if $\beta_2 > (1 - c_{1,n-1})/(1 + c_{1,n})$.

We now interpret Statement (ii), which we view, along with Statement (iii), as the key result of the theorem. A similar conclusion can be drawn for Statement (iii) but we do not include it due to space limitations. In order to make centre node v_1 lose social dominance, the dissent subject nodes v_{n-1} and v_n must adopt a cooperative strategy. From their definitions, we can interpret β_1 as the trust given by v_{n-1} to v_n while β_2 is the trust given by v_n to v_{n-1} . A necessary condition for individual v_n to have social power greater than centre node v_1 is that $\beta_2 > \sum_{i=2}^{n-2} c_{1,i}$. This means that not only must v_{n-1} trust v_n sufficiently (as given by the inequality $\beta_1 > (1 - c_{1,n})/(c_{1,n-1} + \beta_2)$), but individual v_n must reciprocate by ensuring that it trusts v_{n-1} sufficiently ($\beta_2 > \sum_{i=2}^{n-2} c_{1,i}$). Unless the two dissenting nodes build a cooperative and sufficiently strong bilateral relationship, centre node v_1 will remain socially dominant.

Remark 4. *It should be noted that we have analysed the strategies and discussed results and implications from the perspective of attacker and subject nodes. It would be of*

great interest in future work to analyse from the leader's perspective and consider how to ensure robustness and protection against external influence. This also opens up future work using a game theoretic approach.

IV. CONCLUSIONS

Social networks with the star topology converge to an autocratic configuration, with the centre individual holding all the social power. This paper proposed a number of different strategies, involving introduction of new individuals and/or new interpersonal relationships into the social network, in order to move social dominance from the centre individual to a subject individual. Necessary and sufficient conditions have been developed, and based on these conditions it has been argued that the strategies are sociologically intuitive. Numerous future directions exist. Firstly, we wish to generalise the results on uncoordinated attack and coordinated attack to arbitrary numbers of attacker nodes. Different leadership groups, and dissent topologies will also be explored. We also wish to investigate whether such straightforward strategies exist for more general topologies, and lastly, study strategies concerning social power for a subgroup of individuals.

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