ANALYSIS OF WATER WAVES IN THE PRESENCE OF GEOMETRY AND DAMPING

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ABSTRACT

Gary Moon: Analysis of Water Waves in the Presence of Geometry and Damping
(Under the direction of Jeremy Marzuola)

The evolution of waves on the surface of a body of water (or another approximately inviscid liquid) is governed by the free-surface Euler equations; that is, the incompressible Euler equations coupled with a kinematic and a dynamic boundary condition on the free surface. We assume that the flow has zero vorticity in the bulk of the fluid domain and so consider the irrotational free-surface Euler equations (the water waves system). Two major themes are present in our study of the water waves system. The first is the consideration of flows in the presence of substantial geometric features. The second theme is the consideration of the effects of damping, which is an essential tool in the numerical study of water waves. In both contexts, our objective is to consider the local-in-time well-posedness of the water waves system and to study the lifespan of solutions (i.e., the timescales on which solutions to the water waves system persist).

To my family and Σοφια.

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CHAPTER 1

Laying the Foundation

1.1 Introduction

The (irrotational) water waves problem concerns the evolution of the interface $\mathcal{S}_t$ separating an inviscid, incompressible, irrotational fluid from a vacuum region. One may consider the effects of gravity (gravity water waves), surface tension (capillary water waves) or gravity and surface tension (gravity-capillary water waves). The water waves problem can be considered in arbitrary d-dimensional spaces (e.g., $\mathbb{R}^d$ or $\mathbb{T}^d$) with the physically relevant dimensions being $d = 2$ or $d = 3$. We shall restrict ourselves to consideration of the $2d$ problem.

We shall take the fluid domain Ω_t to be a subset of $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. The dynamics of the flow are governed by the incompressible, irrotational Euler equations, coupled with two boundary conditions (BCs) on the interface (the so-called kinematic and dynamic boundary conditions):

$$\begin{cases} \mathbf{D}_t\mathbf{v} := \partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\dfrac{p}{\rho_0} - \mathbf{g} & \text{in } \Omega_t \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \Omega_t \\ \operatorname{curl}\mathbf{v} = 0 & \text{in } \Omega_t\ . \\ \mathbf{D}_t \text{ is tangent to } \displaystyle\bigcup_t (\mathcal{S}_t \times \{t\}) \subset \mathbb{T}_x \times \mathbb{R}^2_{y,t} & \\ p = -\tau H_{\mathcal{S}_t} + p_{\text{ext}} & \text{on } \mathcal{S}_t \end{cases} \tag{1.1.1}$$

The system (1.1.1) is known as the free-surface Euler equations or the water waves system. The first equation in (1.1.1) is the momentum equation; this is just $\mathbf{F} = m\mathbf{a}$. The second equation is the continuity equation representing the conservation of mass. In the context of incompressible flows considered here, this equation reduces to insisting that the velocity field be divergence-free. The third equation imposes the assumption of irrotationality via requiring that the velocity field be curl-free. Moving on, the fourth line is the kinematic BC, which requires that the free surface move with the fluid. The kinematic BC implies that fluid particles may not cross the free boundary and that any fluid particle on the free surface at the initial time

will remain on the free surface for all time. Finally, the last equation is the dynamic BC. The dynamic BC is a stress balance which governs the pressure jump across the interface. When surface tension is considered, the dynamic BC insists that the pressure jump is proportional to the mean curvature of the free surface (we will explain the additional "external pressure" term p_{ext} momentarily). On the other hand, if surface tension is neglected, we require that the pressure at the interface be equal to the atmospheric pressure p_{atm}, the pressure in the region above the fluid. In either case, we assume that p_{atm} is constant and, without loss of generality, we assume that this constant is zero. As stated in (1.1.1), the dynamic BC accounts for surface tension. However, by taking $\tau = 0$, we recover the dynamic BC for gravity waves.

In (1.1.1), $\mathbf{v}$ is the flow velocity, $\mathbf{D}_t$ is the hydrodynamic derivative, ρ_0 is the (constant) density, p denotes the pressure, $\mathbf{g} := (0, g)$ with g being acceleration due to gravity, τ is the coefficient of surface tension and $H_{\mathcal{S}_t}$ is the mean curvature of the free surface. The term p_{ext} is an external pressure which serves to effect the damping and, in the standard (undamped) free-surface Euler equations, we just have $p_{\text{ext}} \equiv 0$. Given that we assume constant density ρ_0, we may, without loss of generality, assume unit density $\rho_0 = 1$. We shall hereafter make this assumption. We will impose free-slip boundary conditions on any remaining portions of $\partial\Omega_t$:

$$\mathbf{v} \cdot \hat{\mathbf{n}} = 0 \text{ on } \partial\Omega_t \backslash \mathcal{S}_t, \tag{1.1.2}$$

where $\hat{\mathbf{n}}$ is the outward unit normal vector field on $\mathcal{S}_t$. The boundary condition (1.1.2) is also called a no-penetration or solid-wall boundary condition.

It is often desirable to not work directly with equation (1.1.1), but instead to reformulate the problem (e.g., by reducing it to a system on the interface). Beginning from (1.1.1), there are many ways to reformulate the water waves problem. We have the vortex sheet formulation (e.g., [AmMa1, BMO2, CCG1, Düll2, AmbEtAl]), the Zakharov-Craig-Sulem formulation (e.g., [Zak, CrSu, ABZ1, Lan1, GMS3]), holomorphic coordinates and the conformal method (e.g., [Nal, Wu1, DKSZ, HIT, H-GIT]), other Lagrangian formulations (e.g., [Yos2, Cra, ChLi, Lin, CoSh1]), a coordinate-free geometric formulation (e.g., [Arn, EbMa, BeGu2, deP, ShZe1]), other variational formulations (e.g., [Pet, Luke, Balk, ClDu, KBEW]), and more (e.g., [AFM, AbHa, FoNa, AsFo, VaDe]). See Chapter 1 of [Lan2] for an overview of many of the aforementioned formulations of the water waves problem. We are primarily concerned with vortex sheet formulations, however the Zakharov-Craig-Sulem formulation will also make an appearance. Vortex sheet formulations are a popular choice for numerical

modeling of water waves [BMO2, HLS1, HLS2, BHL2, Bea3]. For example, the explicit representation of the Dirichlet-Neumann map via layer potentials is well adapted to the needs of numerical computation [WiVa].

Though we present analytical results here, this thesis is substantially motivated by numerical work. The formulation which we utilize in Chapter 2 is a vortex sheet model for water waves in the presence of geometry proposed by Ambrose, Camassa, Marzuola, McLaughlin, Robinson and Wilkening in [AmbEtAl]. The objective of the authors in [AmbEtAl] was to obtain accurate and efficient algorithms for numerically solving the two-dimensional, free-surface Euler equations in a geometric setting. The model allows for variable topography, smooth obstacles in the fluid flow and a (constant) background current.

Another important concept in the numerical simulation of water waves is that of damping. It is often of interest to model water waves on an (effectively) unbounded domain, such as on the open ocean. However, when carrying out numerical experiments one is forced to work on a bounded domain and, depending on the boundary conditions imposed, this can create problems. In particular, waves can reflect off of the boundary and propagate back into the domain, which creates interference. One way to counteract this problem is to add a damping term to the equation. The damping term is designed to dissipate the energy causing outgoing waves to decay.

The form of damping we shall consider, which we call Clamond damping, was first introduced in the remarkable work of Clamond, et. al. in the setting of $3d$ water waves [CFGK2]. Clamond damping is a type of modified sponge-layer, which is effected via the application of an external pressure at a portion of the interface:

$$p_{\text{ext}} \coloneqq \partial_x^{-1}(\chi_\omega \partial_x \varphi) \text{ (modulo a Bernoulli constant).} \tag{1.1.3}$$

In the above, $\omega \subset [0, 2\pi)$ is the connected interval on which we damp the fluid, χ_ω is a smooth, non-negative cut-off function, which is positive on ω, and φ is the velocity potential. Equation (1.1.3) is simply the $2d$ analogue of the $3d$ damper given in [CFGK2]:

$$p_{\text{ext},3d} \coloneqq \nabla^{-1} \cdot (\chi_\omega \nabla \varphi) \text{ (modulo a Bernoulli constant).} \tag{1.1.4}$$

We do not explicitly deal with the Bernoulli constant as it is a function of time alone and so will not have any impact on the energy estimates with which we are concerned. On the other hand, the Bernoulli constant can

be quite important computationally. Though in some contexts it can be incorporated into the velocity potential (particularly, when χ_ω is constant) and thus taken to vanish (in fact, this gives a sort of gauge condition enforcing uniqueness of the velocity potential), it cannot generically be set to zero. Generally, the treatment of the Bernoulli constant will depend upon the method one uses to resolve the equations. See [CFGK2] for further details.

Numerical experiments have shown Clamond damping to be extraordinarily effective [CFGK2]. However, Clamond damping is a linear phenomenon and the question of why it performs so well for the full (nonlinear) water waves system is still open. For example, there is no proof that Clamond damping dissipates energy. Given that Clamond damping is so highly effective numerically, it is our belief that a more thorough understanding of this damping mechanism is important and we hope to initiate this process of better understanding Clamond damping from an analytical viewpoint.

Our objective is to study the local well-posedness of the water waves system. As we shall discuss shortly, the water waves system is known to be well-posed in a variety of settings. What is new in this work is that we study the water waves system in a more geometric setting and/or subject to Clamond damping. By "geometric setting", we mean variable topography, smooth obstacles in the flow and a (constant) background current. In addition to well-posedness, we are concerned with determining the timescales on which solutions persist.

1.2 A Brief History of the Water Waves Problem, Vortex Sheets and Damped Water Waves

This thesis sits at the intersection of a number of fascinating topics in fluid dynamics. First and foremost, there is, of course, the study of the water waves system. Additionally, there is the theory of vortex sheets and vortex methods for modeling fluids. Finally, we have the issue of damping water waves, which is closely related to the control theory. Before proceeding further, we give an overview of these three areas of research. Given the breadth and depth of the literature on these highly active research questions, any overview is bound to contain but a proper subset of the existing results. This will certainly be true of ours.

1.2.1 A Brief History of the Water Waves Problem

The water waves problem belongs to the class of problems known as free boundary problems, which are notoriously challenging to analyze. The earliest well-posedness results made strong assumptions on the data and the geometry of the domain. Namely, they considered data/geometry that was either analytic or perturbative. For example, Kano-Nishida proved well-posedness of the gravity water waves problem with analytic Cauchy data and a flat bottom in [KanNis]. Shinbrot and Sinbrot-Reeder also studied gravity water

waves with analytic data and flat geometry [Shi, ReSh]. Another collection of early results studied the well-posedness of the water waves system in Sobolev spaces under the assumption that the initial configuration of the interface be a small perturbation of still water and the bottom, if present, be a small perturbation of flat. Included in this group is the earliest local well-posedness result for the (full) water waves system of which the author is aware due to Nalimov, who introduced the Lagrangian approach of holomorphic coordinates [Nal]. Other works in this regime include those of Craig [Cra] and Yosihara [Yos1, Yos2]. One important motivation for the smallness assumption in the case of gravity waves is that it has long been known that such an assumption implies that the Taylor sign condition holds:

$$-\partial_{\hat{\mathbf{n}}} p \geqslant c > 0 \text{ on } \mathcal{S}_t. \tag{1.2.1}$$

The condition (1.2.1) is critical for the well-posedness of the gravity water-waves problem. In fact, it is known that the gravity water waves problem may be ill-posed if (1.2.1) fails [Ebin1].

The need for a smallness assumption was first overcome for $2d$ infinite-depth water waves. In her seminal work, Wu utilized Lagrangian coordinates and the conformal method to show that the gravity water waves problem is well-posed by proving that (1.2.1) always holds for gravity waves over infinite depth as long as the free surface is non-self-intersecting [Wu1] (see [Wu2] for a similar treatment of the corresponding $3d$ problem using Clifford analysis in place of complex analysis). An alternative proof, utilizing a vortex sheet framework, is given by Ambrose-Masmoudi in [AmMa1, AmMa3]. On the other hand, Beyer-Günther showed well-posedness of the Cauchy problem for a capillary drop noting that their methods extend to the well-posedness of capillary waves over an infinite-depth fluid [BeGu1]. Iguchi and Ambrose independently provided proofs, via distinct approaches, of the well-posedness of the two-dimensional gravity-capillary water waves problem [Amb1, Igu]. Ambrose-Masmoudi prove a similar result in the case $d = 3$ [AmMa2]. These results have been extended to allow for vorticity, rough Cauchy data and to provide simpler proofs [Lin, CoSh1, CoSh2, HIT, AIT1, ZhZh, ShZe1, ShZe2, ShZe3]. The well-posedness of the free-surface Euler equations was considered via the inviscid limit of the free-surface Navier-Stokes equations by Schweizer [Schw] and Masmoudi-Rousset [MaRo], who considered gravity-capillary and gravity waves, respectively.

The aforementioned work of Iguchi actually proved that the two-dimensional gravity-capillary water waves problem is well-posed in the presence of bathymetric variations [Igu]. Also, Ogawa-Tani prove the

local well-posedness of the $2d$ gravity-capillary water waves problem over variable topography and, additionally, they show that, as surface tension tends to zero, solutions converge to solutions of the gravity system [OgTa]. Well-posedness of the gravity water waves problem in the presence of topography was shown by Lannes in [Lan1], utilizing Eulerian coordinates. The work of Lannes was extended by Ming-Zhang to account for the effects of surface tension [MiZh]. The work of Alazard-Burq-Zuily extended this work by allowing for rough initial data and virtually arbitrary topography (the only restriction on the geometry is a non-cavitation assumption) [ABZ1, ABZ3]. For further extensions of the aforementioned results on water waves in the finite-depth setting, including non-zero vorticity, emerging bottom, rougher Cauchy Data, non-localized Cauchy data and Coriolis forcing, the interested reader can consult [ABZ4, CaLa, H-GIT, Mel, MiWa1, MiWa2, MiWa3, Schw, WZZZ].

For roughly the past ten years, a particularly active area of research has been low regularity well-posedness of the water waves problem. This work was initiated by Alazard-Burq-Zuily in their beautiful works [ABZ1, ABZ3], where they showed local well-posedness of the water waves system as soon as the initial velocity field is Lipschitz regular. A key component of their study is detailed paradifferential analysis, following earlier work of Alazard-Métivier [AlMe]. The good unknown of Alinhac [Ali1, Ali2] plays a crucial role in the paradifferential analysis of the water waves system. The Lipschitz regularity of the initial velocity field is a natural well-posedness threshold for differential equations and is the limit of the energy methods employed by Alazard-Burq-Zuily. However, by exploiting certain features of the equation and utilizing various analytical tools, one can prove well-posedness below this threshold for some quasilinear equations.

One rather powerful approach to low-regularity well-posedness is the low-regularity Strichartz paradigm, which originated in the work of Bahouri-Chemin and Tataru (e.g., see [BaCh1, BaCh2, Tat]; also see [SmTa]). For an expository presentation of this strategy applied to quasilinear wave equations, the interested reader may consult Chapter 9 of [BCD]. Christianson-Hur-Staffilani were the first to prove Strichartz estimates for the water waves system, however they were not working in the low regularity setting [CHS] and so did not achieve a gain in the regularity threshold. They proved Strichartz estimates, in a semiclassical regime, for the gravity-capillary (infinite depth) water wave system and were able to deduce a local smoothing effect. The low regularity Strichartz paradigm was first successfully applied to the water waves system by Alazard-Burq-Zuily [ABZ2, ABZ5], who coupled it with their paradifferential approach. Some further example of low-regularity Strichartz estimates applied to the water waves system include

[Ai1, Ai2, dePNg2, dePNg1, Ngu1, Ngu2].

To our knowledge, the most substantial reduction of the well-posedness threshold comes from the work of Ai-Ifrim-Tataru [AIT1], in the case of gravity waves, and Nguyen [Ngu1], in the case of gravity-capillary waves. Both of these results regard the two-dimensional system. The approach of Ai-Ifrim-Tataru in [AIT1] is the application of "balanced cubic energy estimates" utilizing holomorphic coordinates and Nguyen utilizes the low-regularity Strichartz paradigm [Ngu1]. This comparison is, however, a bit biased as Ai-Ifrim-Tataru essentially just utilize energy estimates and so Strichartz estimates can still be applied in their framework to obtain further gains. Some other relavent work is that of Kinsey-Wu [KiWu] and Wu [Wu6] on a priori estimates for and local well-posedness of the water waves system where the interface fails to be C^1.

An important related question regards the lifespan of solutions to the water waves problem, usually in the small-data setting. Here some interesting results are provided by Hunter-Ifrim-Tataru and their collaborators, who have applied their "modified energy method" to the water waves system. The modified energy method was introduced in [HITW] to study the lifespan of a Bürgers-Hilbert equation. The main idea of the modified energy method, as applied to a quadratically nonlinear equation, is to use a normal form transformation to construct a modified energy functional which satisfies cubically nonlinear estimates. As such, when considering a quadratically nonlinear, quasilinear equation, the modified energy estimates can be used to prove local well-posedness with a cubic lifespan. The modified energy method has been applied to gravity waves over infinite depth [HIT, IfTa4], gravity waves over finite depth [H-GIT] and capillary waves over infinite depth [IfTa3]. Of course, normal form methods can also be applied more directly to obtain long-time existence of solutions to the water waves system. For example, in [BFF], Berti-Feola-Franzoi consider periodic, two-dimensional gravity-capillary water waves over a flat bottom, reduce the system to Birkhoff normal form up to degree three and thereby obtain a cubic lifespan. On the other hand, Berti-Feola-Pusateri consider periodic, two-dimensional gravity water waves over infinite depth and prove a quartic lifespan, again via Birkhoff normal form methods [BFP].

Alazard-Burq-Zuily studied the gravity water waves system in certain analytic function spaces in [ABZ6]. There the authors showed that the system is locally-in-time well-posed in these analytic function spaces and that, for Cauchy data of size ε, solutions persist on a $O(\frac{1}{\varepsilon})$ timescale. The primary tools used to obtain this result were energy estimates and a careful study of the Dirichlet-Neumann map in the analytic function spaces under consideration, which followed in the tradition of [Lan1, AlMe, ABZ1, ABZ3]. Ionescu-Pusateri proved a lifespan between quadratic and cubic (specifically, $O(\varepsilon^{-\frac{5}{3}+})$) for $3d$

gravity-capillary water waves, which is valid for almost all values of the gravitation and surface tension coefficients [IoPu4]. To prove this lifespan, Ionescu-Pusateri use, among other tools, a combination of paradifferential analysis, partial normal form transformations and sharp estimates for small divisors.

While we are primarily concerned with lifespan as a function of the size of the initial data, in the small data regime, there is another collection of interesting long-time existence results. These results measure the lifespan in terms of various dimensionless parameters used to characterize the flow. Some commonly used dimensionless constants include

$$\epsilon := \frac{a}{H},\ \mu := \frac{H^2}{L^2},\ \beta := \frac{b}{H},\ \mathrm{Bo} := \frac{\rho g L^2}{\tau} \tag{1.2.2}$$

where a is the order of amplitude of the free surface waves, H is the characteristic water depth, L is the characteristic wavelength in the longitudinal direction and b is the order of amplitude of the bathymetric variations. It is common to call ϵ the nonlinearity parameter, μ the shallowness parameter and β the topography parameter. Of course, Bo is the Bond number. The general objective of characterizing the lifespan in terms of such dimensionless constants is to rigorously justify various simplified models (e.g., KdV, Green-Naghdi, NLS and so forth) in asymptotic regimes (e.g., the shallow water regime corresponds to $\mu \ll 1$).

In [AlLa], it was shown that solutions to the water waves system persist on timescales of order $\mathcal{O}(\frac{1}{\beta \vee \epsilon})$. The main tools included a detailed study of the Dirichlet-Neumann map and a Nash-Moser iteration scheme. Given that this lifespan is uniform in μ, this large-time well-posedness result can be used to help justify asymptotic models in the shallow water regime. This was improved by Mésognon-Gireau, who proved in [Més] that solutions have a lifespan of order $\mathcal{O}(\frac{1}{\epsilon})$. The analysis of [Més] is of particular interest to us due to the use of commuting vector fields.

The next group of large-time well-posedness results concern the question of global or almost-global existence of solutions, under the assumption of small, localized, smooth initial data. Additionally, to the best of the author's knowledge, all almost-global and global well-posedness results obtained so far require the assumption of vanishing vorticity in the bulk of the fluid domain. Most of these results are in the setting of infinite depth, however global regularity in the finite-depth setting has been considered very recently (assuming flat geometry). Further, such results tend to be easier to obtain in $3d$ as opposed to $2d$ due to better rates of decay in higher dimension. It also tends to be easier in this setting to work with gravity waves due to

differences in the dispersion relation and resonant interactions. Intimately related to the question of global regularity, there is the question of whether solutions scatter to a linear solution as $t \to +\infty$ and, more generally, the long-time asymptotic behavior of solutions. For a good survey on the global regularity problem for water waves, see [IoPu2].

The earliest global well-posedness results were proved in dimension $d = 3$ in the infinite-depth setting. In [GMS2], Germain-Masmouid-Shatah used their celebrated "method of space-time resonances", introduced in [GMS1] to study the quadratic NLS, to prove the existence of global solutions to the gravity water waves system, also showing that these solutions scatter. Wu provided an alternative proof, but did not consider scattering [Wu5]. The existence of global solutions to the three-dimensional capillary water waves problem was proved via the method of space-time resonances by Germain-Masmoudi-Shatah in [GMS3], where they again show that solutions scatter. The global regularity and scattering of solutions to the $3d$ gravity-capillary water waves system was shown by Deng-Ionescu-Pausader-Pusateri in [DIPP]. Global existence and scattering results were extended to the finite-depth setting, in the case of a flat bottom, by Wang for gravity waves [Wang2] and capillary waves [Wang3].

The first result of this sort for the $2d$ water waves system was obtained by Wu, who showed the existence of almost-global solutions to the gravity water waves system [Wu4]. An alternative proof of Wu's almost-global existence result was obtained by Hunter-Ifrim-Tataru in [HIT]. Wu's result was upgraded to global well-posedness by Alazard-Delort in [AlDe1, AlDe2] and Ionescu-Pusateri [IoPu1] independently, where a modified scattering is proved in both. Ifrim-Tataru simplified the proof of this result in [IfTa2] by using their method of "testing by wave packets" which was introduced in [IfTa1] for studying the cubic NLS. The global regularity problem for the $2d$ capillary water waves system was independently solved by Ionescu-Pusateri [IoPu3] and Ifrim-Tataru [IfTa3], where modified scattering is demonstrated in both cases. The question of the existence of global solutions to the $2d$ gravity-capillary water waves system is still open. The best result to date, to the author's knowledge, is that of Berti-Delort, who proved that almost-global periodic solutions exist for almost all choices of g and τ [BeDe]. Finally, we note that Wang extended the global existence and modified scattering results of Alazard-Delort and Ionescu-Pusateri to the case of a flat bottom in [Wang4], where moreover solutions were allowed to have infinite energy.

We noted above that all global well-posedness results for the water waves system require a smallness assumption on the Cauchy data. In fact, we know that this restriction on the size of the data is necessary. This is because a solution to the water waves system with smooth, but large, data can form a singularity in finite

time. Two types of singularities which are known to form for solutions to the $2d$ or $3d$ water waves system with or without surface tension are splash and splat singularities [CasEtAl3, CasEtAl2, CasEtAl1, CasEtAl4, CoSh3]. A splash singularity occurs when the free boundary self-intersects at a point, while a splat singularity occurs when the free boundary self-intersects along an arc. There are other possible singularities studied in the context of water waves (e.g., squirt singularities in which a smaller volume of fluid is ejected from a larger volume [CFdlL]), however, to the author's knowledge, these are not definitively known to occur. An interesting notion related to singularity formation is that of breakdown criteria. That is, to determine necessary and sufficient conditions for singularity formation. To the author's knowledge, the first breakdown criterion proved for equations in fluid dynamics is the celebrated result of Beale-Kato-Majda [BKM], which says that a smooth solution $\mathbf{u}$ to the $3d$ Euler equations on $[0, T]$ can be continued to some time $T^* > T$ if and only if

$$\int_0^{T_*} \|\operatorname{curl} \mathbf{u}\|_{L^\infty(\mathbb{R}^3)}\, dt < +\infty. \tag{1.2.3}$$

Beale-Kato-Majda-type breakdown criteria for the irrotational and rotational gravity water waves system were obtained in [WaZh] and [WZZZ], respectively.

The water waves problem is a highly active area of research and the above outlined questions are far from the only questions which one can ask about the problem. We will give just a small handful of examples of other results obtained for the water waves system to try to demonstrate just how many fascinating questions one can pose about this problem. In [Zhu2], Zhu proved a propagation of singularities result for the gravity-capillary water waves system, which, to our knowledge, represented the first propagation result for a quasilinear dispersive equation. To prove this result, the author had to define a new notion of wavefront set, the quasi-homogeneous wavefront set, that is well-adapted to studying singularities of the gravity-capillary water waves system. This built off of the earlier work of Nakamura [Nak], who defined the homogeneous wavefront set to study propagation for the Schrödinger equation, and others. Alazard-Ifrim-Tataru initiated the study of Morawetz estimates for water waves. The authors proved local energy decay (or a Morawetz estimate) for gravity waves in [AlIT1] and for gravity-capillary waves for large Bond number in [AlIT2] (note that their convention for defining the Bond number is the opposite of the one we use, so they refer to low Bond number). Symmetries and conservation laws for gravity water waves are studied in [BeOl]. On the other hand, in [Olv], Olver proved that the $2d$ gravity water waves problem has exactly eight nontrivial

conservation laws.

As alluded to above, there is the question of providing rigorous mathematical justifications for the various models used to describe the dynamics of water waves in different asymptotic regimes (e.g., see Duchêne's memoir [Duc] or the book of Lannes [Lan2]). There is also the question of special solutions of the water waves system. For example, we have the issue of the existence of solitary waves [Bea1, FrHy] and the conjecture of Stokes regarding the angle formed by the crest of a steady wave of maximal amplitude which was proved in [AFT]. An interesting result on the regularity, or lack thereof, of the flow map for the gravity-capillary water waves system in $3d$ was obtained in [CMSW]. As just these few examples show, there are seemingly countless questions one can ask about the water waves system.

1.2.2 Previous Results on Vortex Sheets and the Vortex Sheet Formulation of the Water Waves Problem

As discussed above, there are numerous ways to formulate the water waves problem (various coordinate systems, parameterizations of the interface and so on). The model with which we are primarily concerned utilizes the vortex sheet formulation. The classical vortex sheet problem (also called the Kelvin-Helmholtz problem) considers the interface between two incompressible, inviscid, irrotational, density-matched fluids moving past each other in two dimensions, neglecting the effects of surface tension. In such a scenario the vorticity is concentrated entirely along the interface due to the jump in tangential velocity (while the normal velocity is continuous).

It has long been known that the Kelvin-Helmholtz problem is ill-posed in the usual sense due to the well-known Kelvin-Helmholtz instability (see, e.g., [CaOr2, Moo]). Recall that the Kelvin-Helmholtz instability is a linear phenomenon: linearizing the vortex sheet equations about the equilibrium solution, high-frequency Fourier modes become unbounded causing small disturbances to grow exponentially. Nevertheless, the Kelvin-Helmholtz problem is well-posed in analytic function spaces [CaOr1, SSBF]. Importantly, these ill-posedness results neglect the effects of surface tension, which exhibits a smoothing/restoring effect. When surface tension is incorporated, high-frequency Fourier modes remain bounded in the linearization. This led Birkhoff to conjecture that the vortex sheet problem with surface tension is well-posed [Bir]. Building off of this, Beale-Hou-Lowengrub showed that the linearized two-dimensional vortex sheet problem with surface tension is well-posed, even far from equilibrium [BHL1] (see [HTZ] for the corresponding result in three dimensions). It was proven by Iguchi-Tanaka-Tani in [ITT] that the (nonlinear) vortex sheet problem with surface tension is well-posed when the initial configuration of

the free surface is a small perturbation of a flat interface. This smallness assumption was removed by Ambrose who showed that the vortex sheet problem with surface tension is well-posed, at least in the infinite-depth setting [Amb1]. This local well-posedness result also holds in dimension $d = 3$ [AmMa2].

In spite of the classical vortex sheet problem assuming that the upper and lower fluids are density-matched, this assumption is not necessary and vortex sheet formulations have been widely used to study water waves and other phenomena in fluid dynamics. This approach (i.e., using the vortex sheet formulation to model phenomena in fluid dynamics) belongs to the broader class of tools known as vortex methods. The seminal work on vortex sheet formulations is that of Baker-Meiron-Orszag, which considered two-dimensional water waves [BMO2]. Vortex sheet formulations have also been applied to study other phenomena in fluid dynamics (e.g., gas bubbles in liquids [BaMo, Hua, Yang]).

A particularly useful framework for studying vortex sheets (and vortex sheet formulations more broadly), particularly in the presence of surface tension, was developed by Hou-Lowengrub-Shelley (HLS) in their beautiful paper [HLS1] (see also [HLS2]). This framework was developed from a numerical perspective to create a non-stiff algorithm for modeling $2d$ interfacial flow under the influence of surface tension. The HLS framework rests on three key ideas. The first, influenced by earlier work of Mullins on "curve shortening" in the context of grain boundaries [Mul], is to select a special frame of reference by choosing particular geometric coordinates (as opposed to Cartesian coordinates). The second is to pick a favorable, renormalized arclength parameterization of the interface. The third, primarily relevant for numerical work, is the use of a small-scale decomposition (SSD); that is, terms which are unstable at small spatial scales are identified so that they can be computed implicitly, whereas the remaining terms are computed explicitly. It is worth noting that the terms showing up in the SSD also tend to require care when studying the equations analytically, however there are additional terms that require similar care that do not appear in the HLS SSD (see [Amb1] for further discussion). We shall discuss the HLS framework further in the sequel, but one particular benefit, following from the first key idea, is that one obtains a highly simplified expression for the curvature of the interface $H_{\mathcal{S}_t}$, which is relevant when considering surface tension due to the Laplace-Young condition at the interface.

The HLS framework is powerful and, in addition to classical vortex sheets, has been used to study water waves [AmMa1, CHS, CCG1, Düll1, Düll2], Darcy flows [Amb2, Amb4, CCG2], hydroelastic waves [AmSi, LiAm] and flame fronts [AkAm]. Moreover, although the HLS framework is necessarily two-dimensional, the main insights have been extended to study $3d$ flows. In the case of three-dimensional

flows, isothermal coordinates take the place of the arclength parameterization. Examples of numerical and analytical work using this framework can be found in [Amb3, AmMa2, AmMa3, CCG3, HouZh].

The well-posedness theory of the vortex sheet formulation of the water waves problem has been developed by several authors. All of the results of which we are aware deal with the infinite-depth setting. Ambrose proved in [Amb1] that the vortex sheet formulation of the two-dimensional gravity-capillary water waves problem is well-posed and this model was shown to be well-posed in the zero surface tension limit by Ambrose-Masmoudi in [AmMa1]. Ambrose-Masmoudi prove analogous results in three dimensions in [AmMa2, AmMa3]. Cheng-Coutand-Shkoller extended the work of Ambrose and Ambrose-Masmoudi to allow for vorticity in the bulk of the fluid [CCS1] and then further considered the limit as the density ratio goes to zero to obtain solutions to the water waves system with vorticity [CCS2]. Christianson-Hur-Staffilani utilized a vortex sheet formulation to prove semiclassical Strichartz estimates and local smoothing for the two-dimensional water waves system with surface tension over infinite depth [CHS].

There are many more important results on vortex sheets. We must mention the celebrated work of Delort [Del] in which the global existence of a weak solution to the Euler equations with vortex sheet Cauchy data is demonstrated, under the assumption that the vortex sheet strength is the sum of a Radon measure with distinguished sign and an (arbitrary) L^p function with $p > 1$. The work of DiPerna and Majda on the Euler equations and "concentration-cancellation" provided critical tools for Delort's result [DiMa1, DiMa2]. The proof of Delort was simplified by Majda in [Maj1] where it was also shown that solutions to the Navier-Stokes equations with vortex sheet initial data with distinguished sign converge to weak solutions of the $2d$ Euler equations in the high-Reynolds-number limit. Another simplified proof was given by Schochet who also obtained Majda's convergence result and extended to $p = 1$ [Sch]. Delort's result was further extended to remove the distinguished sign assumption in the presence of particular symmetries [LNX1] and then further to exterior domains [LNX2]. The impossibility of splash singularities for vortex sheets was shown in [CoSh4, FIL]. For some further interesting results on vortex sheets and vortex methods, see [ADL, Cou, LNS, Wu3]. Good introductions to vortex sheets and vortex dynamics include [Saf, SaBa]. The survey article [BaLa] by Bardos-Lannes is well worth reading and covers the Kelvin-Helmholtz problem, the Rayleigh-Taylor problem and the vortex sheet formulation of the water waves problem. A good overview of vortex methods can be found in [Set].

1.2.3 Previous Results on Damped Water Waves

When we refer to damping water waves, we are referring to the application of a sponge layer or numerical beach; that is, an artificial, dissipative term supported near the boundary that removes energy from the system. However, in the literature, there are other systems which are known as damped water waves, (free-surface) damped Euler equations and so on. For the sake of completeness, we will briefly discuss some results obtained for these systems and ultimately explain how they differ from the damping which we consider.

First, there are several interesting results on the so-called damped Euler equations, including the free-surface damped Euler equations. In this context, the damped Euler equations are given by

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + a\mathbf{v} = -\dfrac{1}{\rho}\nabla p + \mathbf{f} \text{ in } \Omega_t \\ \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega_t \end{cases}, \tag{1.2.4}$$

where $a > 0$ is the damping coefficient and $\mathbf{f}$ represents any body forces to be considered. If one is considering the free-surface equations, the dynamic and kinematic boundary conditions are the same. The free boundary problem for (1.2.4) is globally-in-time well-posed for gravity waves (i.e., $\mathbf{f} = -\mathbf{g}$) [Lian2], gravity-capillary waves [Lian3] and in the case of two fluids [Lian4]. Further, as the surface tension coefficient tends to zero, solutions to the gravity-capillary system converge to solutions of the gravity system [Lian1]. In the free boundary case, the damper $a\mathbf{v}$ indeed dissipates energy and solutions converge to the equilibrium almost exponentially (resp. exponentially) without (resp. with) surface tension [Lian2] (resp. [Lian3]). In [ChSa], infinite-energy solutions to (1.2.4), with no free boundary and $\mathbf{f} \equiv 0$, are studied. Finally, we note that Saut studies the Charney-Stommel model of the Gulf-Stream which reduces to a damped version of the stationary Euler equations in $2d$ [Saut].

Comparing the Euler and Navier-Stokes equations, one sees that the presence of viscosity has a dissipative effect which makes the analysis of these systems substantially different (e.g., the Navier-Stokes equations have a parabolic character, whereas the Euler equations have a hyperbolic character). Thus, it is reasonable to think that incorporating viscosity into the water waves system would yield a form of damping. Of course, one could simply study the free-surface Navier-Stokes equations directly (e.g., [Bea2, CasEtAl5, GuoTi1, GuoTi2, Tani]), however in doing so one loses some important benefits of studying the water waves system: generally speaking, viscosity is not compatible with potential flow and the

existence of a velocity potential is critical for many of the formulations of the water waves system (this is also the reason why the study of water waves with vorticity has a rather different character). Nevertheless, there has been a good deal of work considering various ways to add "artificial viscosity" to potential flows. A particularly popular model of this form is the Dias-Dyachenko-Zakharov (DDZ) model, which, for two-dimensional gravity water waves, is given by

$$\begin{cases} \Delta\varphi = 0 & \text{in } \Omega_t \\ \partial_t\eta = \partial_y\varphi + 2\nu\partial_x^2\varphi - \partial_x\eta\partial_x\varphi & \text{on } \mathcal{S}_t \\ \partial_t\varphi = -\dfrac{1}{2}|\nabla\varphi|^2 - 2\nu\partial_y^2\varphi - g\eta & \text{on } \mathcal{S}_t \\ \partial_y\varphi \to 0 & \text{as } y \to -\infty \end{cases}, \tag{1.2.5}$$

where η is a function describing the location of the free surface and ν is the coefficient of viscosity. Surface tension can be incorporated into the DDZ model in the usual way (i.e., by modifying the dynamic boundary condition via the Laplace-Young jump condition). Dias-Dyachenko-Zakharov obtained (1.2.5) by adding dissipative (viscous) terms to the dynamic and the kinematic boundary condition (causing the free surface to experience dissipative effects) based upon an analysis of the linearized Navier-Stokes equations and further showed that the NLS derived from this model is the classical damped NLS [DDZ]. This model was extended to water waves over finite depth by Dutykh-Dias [DuDi]. Ambrose-Bona-Nicholls study a truncated-series DDZ model, which they show to be well-posed locally in time [ABN]. Note that by truncated series, we mean that they expanded the given operators (via the method of operator expansions) and then truncated the resulting series at a given order (in this case second order). The full DDZ model is considered in [NgNi], where Ngom-Nicholls show that the model is locally well-posed (with a substantially simplified well-posedness theory) and gives a substantially more stable numerical scheme for studying water waves. Asymptotic models for damped water waves were derived from the DDZ model in [GrSc]. Again, it makes sense to refer to these as models for damped water waves as solutions persist globally in time and decay to equilibrium exponentially in time [NgNi]. The DDZ model was investigated from a computational perspective in [KaNi], where the exponential decay of solutions was observed numerically. As we noted above, the DDZ model is not the only model incorporating "artificial viscosity" into the water waves system. Many of the references discussed above contain references to other such models and the interested reader can consult them for more information on this interesting perspective on damped water waves (e.g., [GrSc] has a

good discussion of some other models as do [DuDi, DDZ]).

Though the above systems are indeed models for damped water waves, they are different from the damping with which we are concerned in at least one critical way. Namely, in the above models, the damping is effected on the entirety of the domain. As noted above, we are concerned with damping as it can be applied to the numerical simulation of water waves. More particularly, we want the waves to propagate freely in the majority of the domain and only be damped in a localized neighborhood of the boundary to avoid spurious reflections. In that respect, above forms of damping are not appropriate. Now, that is not to say that these models could not be adapted to that purpose (e.g., by localizing the effect of the viscosity to a small neighborhood of the boundary), however investigating that possibility, while a fascinating question for future research, is beyond the scope of this work.

Having briefly detoured to discuss some different perspectives on damped water waves, we return to considering damping in the sense in which we wish to study it. That is in the form of an artificial dissipative term localized near the boundary (e.g., a sponge layer). There is a vast literature regarding numerical aspects of the damped water waves problem (e.g., numerically evaluating the performance of various dampers). For further information on numerical aspects of damping water waves, the interested reader may consult [BMO3, Bonn, CBS, CFGK2, Clem1, Clem2, Ducr, GrHo, IsOr, JKR, JenEtAl, Rom, Wes] as well as the references therein. Closely related to damping are tools such as absorbing boundary conditions, radiation boundary conditions, artificial boundary conditions, perfectly matched layers and so on; just a few references on these methods include [BaTu, Ber, Col, EnMa, Gil1, Gil2, Giv1, Giv2, Hu, Tsy, TuYe]. In spite of the vast numerical literature on the subject, the analytical study of damped (in our sense of the word) nonlinear water waves is rather more sparse.

An important exception would be Alazard's wonderful papers on the stabilization of the water waves system [Ala3, Ala4]. In [Ala3], utilizing the Zakharov-Craig-Sulem formulation, the popular damper

$$p_{\text{ext},1} := \chi_1 G(\eta)\psi \tag{1.2.6}$$

is considered, χ_1 is a non-negative cut-off function which serves to localize the damping, η is a function giving the displacement of the free surface, $G(\eta)$ is the normalized Dirichlet-Neumann map and ψ is the trace of the velocity potential along the free surface. Here, $p_{\text{ext},1}$ represents an external pressure applied at the free surface via modifying the dynamic boundary condition. The damper (1.2.6) is a natural choice from the

Hamiltonian perspective. If $\mathcal{S}_t$ is the graph of a function η, then the water waves system can be written as a Hamiltonian system with Hamiltonian energy

$$\mathscr{H} = \frac{g}{2}\int_0^{2\pi} \eta^2\, dx + \tau \int_0^{2\pi} \sqrt{1+\eta_x^2} - 1\, dx + \frac{1}{2}\int_0^{2\pi}\int_{-h}^{\eta(x,t)} |\nabla\varphi|^2\, dydx, \tag{1.2.7}$$

where $\{y = -h\}$ is the (flat) bottom of the fluid domain. Then, one has the Hamiltonian equations

$$\frac{\partial\eta}{\partial t} = \frac{\delta\mathscr{H}}{\delta\psi},\ \frac{\partial\psi}{\partial t} = -\frac{\delta\mathscr{H}}{\delta\eta} - p_{\text{ext},1} \tag{1.2.8}$$

and, using (1.2.8) and noticing that $p_{\text{ext},1} = \chi_1\partial_t\eta$, one can deduce that

$$\frac{d\mathscr{H}}{dt} = \int_0^{2\pi} \frac{\delta\mathscr{H}}{\delta\eta}\partial_t\eta + \frac{\delta\mathscr{H}}{\delta\psi}\partial_t\psi\, dx = -\int_0^{2\pi} \partial_t\eta p_{\text{ext},1}\, dx = -\int_0^{2\pi} \chi_1(\partial_t\eta)^2\, dx \leqslant 0. \tag{1.2.9}$$

Thus, it is easily seen that $p_{\text{ext},1}$ induces dissipation of the energy. The real achievement of [Ala3] is to show that $p_{\text{ext},1}$ stabilizes the water waves system with the rate of convergence being exponential in time.

An analogous result is obtained in [Ala4] for the $2d$ gravity water waves system. In the gravity case, the pneumatic damper is taken to satisfy

$$p_{\text{ext},2}(x,t) = \partial_x^{-1}\left(\chi_2(x)\int_{-h}^{\eta(x,t)} \varphi_x(x,y,t)\, dy\right). \tag{1.2.10}$$

The reason that the Hamiltonian damper (1.2.6) is not considered is due to difficulties in showing that the Cauchy problem is well-posed. A similar, though slightly more involved, argument shows that (1.2.10) causes the Hamiltonian energy to decay. The main result of [Ala4] is that $p_{\text{ext},2}$, which satisfies (1.2.10), stabilizes the water waves system with the energy decaying to zero exponentially in time.

The question of stabilizability of the water waves system belongs to the broader field of control theory for water waves. Within control theory, the problems of stabilizability, controllability and observability are closely related. These questions are likewise important for the numerical simulation of water waves. For example, the question of controllability relates to the generation of waves via a wave maker. As was the case for the problem of stabilizability, the literature on the analytic study of control of the (full) water waves system is rather sparse. The earliest results on the control of the water waves system applied to the linearized system and were obtained by Reid-Russell [ReRu], as well as Reid [Reid1, Reid2]. Control theory for linear

water waves is still an active area of research (e.g., see [Mot, CGS]).

The first results on the controllability of the full (nonlinear) water waves system were obtained in the beautiful paper [ABH] by Alazard, Baldi and Han-Kwan, which considered control via an external pressure (i.e., a pneumatic wave maker). In this paper, the authors prove that the $2d$ gravity-capillary water waves system is locally exactly controllable in arbitrarily short time: given a control domain ω and a control time $T^* > 0$, the water waves system is controllable in time T^* for sufficiently small initial data. There is also a smallness assumption on the desired final configuration. In other words, one can, in arbitrarily small time, generate arbitrary (sufficiently small amplitude) periodic (gravity-capillary) surface waves by applying an external pressure to a localized portion of the free surface of a fluid. The smallness assumptions of [ABH] are rather restrictive, but the stabilization result of [Ala3], which imposed a milder smallness assumption, can be combined with the small-data control result of [ABH] to yield a larger-data control result via a strategy of Dehman-Lebeau-Zuazua [DLZ] which exploits the time-reversibility of the water waves system. The controllability result of [ABH] was extended to higher dimensions in [Zhu1] subject to the requirement that the control domain ω (a subset of $\mathbb{T}^d$) satisfies a geometric control condition (GCC). Given a Riemannian manifold (M, g), compact and without boundary for simplicity, and a damping region $\omega \subset M$, the GCC is as follows:

$$\text{Every geodesic of } M \text{ must eventually enter the damping region } \omega. \tag{1.2.11}$$

In the case of control problems, the GCC is the same with control domain replacing damping region. We note that our formulation of the GCC in (1.2.11) is not unique; namely, the proper way to formulate a GCC will depend on the geometry of the problem (e.g., compact vs. non-compact or with boundary vs. without boundary). For more on the GCC, see the seminal works [RaTa, BLR]. The GCC is a rather natural requirement for control/stabilization/observation problems (at least when dealing with hyperbolic or other non-dissipative equations) and appears frequently in the control theory literature (e.g., [BoRo, BuGe, DGL, JoLa, Leb, LLTT]). One might ask why the GCC was absent from the control result of [ABH]. However, that would be a bit misleading as the GCC was not absent, but rather implicit: the GCC is always satisfied on $\mathbb{T}$!

Thus far, we have discussed control and stabilization results for the water waves system. This leaves the final piece of the control theory triad: observability. We know from the work of Lions that observability is intimately connected to stabilizability and controllability (in fact, observability is dual to null controlability)

[Lio]. So, given that we have control and stabilization results for the water waves system, we might expect to have observation results. This is indeed the case as there are observability results in [ABH, Zhu1]. Moreover, Alazard proves boundary observability of the gravity water waves system in $2d$ and $3d$, where the fluid domain is taken to be a rectangular tank bounded by a flat bottom, vertical walls and a free surface [Ala5]. Boundary observability implies that one can control the energy of the system via measurements at the boundary (i.e., where the free surface meets the vertical walls).

Finally, there are a number of results in the control theory literature regarding equations arising in fluid dynamics, many of which are closely related to the water waves system. Namely, control theory results exist for the Euler equations, various asymptotic models for water waves (e.g., KdV and Saint-Venant), the Benjamin-Ono equation (used to model internal waves in deep water) and the Navier-Stokes equations. Though related, these results are nevertheless substantially different and, generally, tend to be easier to obtain. A crucial difference is that the water waves problem is a free boundary problem, often formulated nonlocally, whereas the results on the aforementioned equations are generally posed on a fixed domain. The interested reader may consult [Cor1, Cor2, Gla] for some control theory results on the Euler equations and [LiRo] for a controllability and stabilizability result for the Benjamin-Ono equation. For an expository presentation on control of nonlinear equations, including the equations from fluid dynamics given above (i.e., Euler, Navier-Stokes, KdV and Saint-Venant), one may consult [Cor3].

1.3 Preliminary Results

In this section, we will discuss some preliminary results which will be utilized frequently throughout this thesis. Many of these results will be stated without proof, however, for the sake of completeness, we provide references where proofs can be found.

1.3.1 Fourier and Microlocal Analysis on $\mathbb{T}$

We begin by discussing some basic Fourier analysis, and a hint of microlocal analysis, on the circle $\mathbb{T}$. We will first define the Fourier transform and Fourier series on $\mathbb{T}$. Then, we will state some of the most important properties of the Fourier transform which will be important for our analysis. Next, we will discuss the definition of pseudodifferential operators (ΨDO) on $\mathbb{T}$. We end by investigating how differentiation interacts with band-limited functions.

A good reference for Fourier analysis on $\mathbb{T}$ is Chapter 3 of [Tay1] and we will partly follow this

presentation of the material. If $f \in L^1 = L^1(\mathbb{T})$, we define its Fourier transform by

$$\mathcal{F}(u)(k) = \hat{u}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} e^{-ikx} u(x)\, dx. \tag{1.3.1}$$

Here we have introduced a notational convention we will use throughout this thesis to reduce notational clutter. When the underlying domain of a function space is suppressed, it should be assumed to be $\mathbb{T}$. Thus, we denote $H^r = H^r(\mathbb{T})$, $L^p = L^p(\mathbb{T})$ and so on. We have the corresponding mapping property:

$$\mathcal{F} : L^1 \to \ell^\infty(\mathbb{Z}) \text{ is a continuous linear map.} \tag{1.3.2}$$

Let $s(\mathbb{Z})$ denote the space of rapidly decreasing functions on $\mathbb{Z}$, where, by rapidly decreasing, we mean that

$$\sup_{k \in \mathbb{Z}} \langle k \rangle^N \left| u(k) \right| < +\infty \ \forall N. \tag{1.3.3}$$

In the above equation, we have introduced the Japanese bracket $\langle \cdot \rangle$ which is defined by $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$. Letting $\mathcal{D}$ denote the space of test functions (i.e., $\mathcal{D} = C^\infty$), we then have the further mapping property

$$\mathcal{F} : \mathcal{D} \to s(\mathbb{Z}). \tag{1.3.4}$$

Moreover, for $u \in \mathcal{D}$ and $v \in s(\mathbb{Z})$, we have

$$(\mathcal{F}\, u, v)_{\ell^2} = (u, \mathcal{F}^*\, v)_{L^2}, \tag{1.3.5}$$

where

$$\mathcal{F}^*(u)(x) = \sum_{k \in \mathbb{Z}} u(k) e^{ikx}. \tag{1.3.6}$$

The following mapping properties for $\mathcal{F}^*$ holds:

$$\mathcal{F}^* : s(\mathbb{Z}) \to \mathcal{D}, \tag{1.3.7}$$

$$\mathcal{F}^* : \ell^1(\mathbb{Z}) \to L^\infty. \tag{1.3.8}$$

We make note of the Fourier inversion formula on $\mathcal{D}$:

$$u(x) = \sum_{k\in\mathbb{Z}} \hat{u}(k)e^{ikx} \qquad (u \in \mathcal{D}). \tag{1.3.9}$$

We have

$$\mathcal{F}^*\mathcal{F} = \mathrm{id} \text{ on } \mathcal{D}, \tag{1.3.10}$$

$$\mathcal{F}\mathcal{F}^* = \mathrm{id} \text{ on } s(\mathbb{Z}). \tag{1.3.11}$$

For $u, v \in \mathcal{D}$, we have the Parseval identity and the Plancherel identity:

$$(u, v)_{L^2} = (\hat{u}, \hat{v})_{\ell^2}, \tag{1.3.12}$$

$$\|u\|_{L^2}^2 = \|\hat{u}\|_{\ell^2}^2. \tag{1.3.13}$$

By continuity, $\mathcal{F}$ extends from $\mathcal{D}$ to a unitary map

$$\mathcal{F} : L^2 \to \ell^2. \tag{1.3.14}$$

In addition, $\mathcal{F}^*$ has a unique continuous extension to $\ell^2(\mathbb{Z})$ with

$$\mathcal{F}^* : \ell^2(\mathbb{Z}) \to L^2 \text{ unitarily.} \tag{1.3.15}$$

We can extend the definition of the Fourier transform to the class of Schwartz distributions by duality. Given $u \in \mathcal{D}'$, we set

$$\mathcal{F}\, u(k) = \hat{u}(k) := \frac{1}{2\pi}\langle u, e^{-ikx}\rangle. \tag{1.3.16}$$

We then have

$$\mathcal{F} : \mathcal{D}' \to s'(\mathbb{Z}), \tag{1.3.17}$$

where $s'(\mathbb{Z})$ denotes those functions $a : \mathbb{Z} \to \mathbb{C}$ with at-most-polynomial growth:

$$a \in s'(\mathbb{Z}) \iff \big|a(k)\big| \lesssim \langle k\rangle^N \text{ for some } N. \tag{1.3.18}$$

It further holds that

$$\mathcal{F}^* : s'(\mathbb{Z}) \to \mathcal{D}', \tag{1.3.19}$$

where

$$\mathcal{F}^*(a)(x) = \sum_{k\in\mathbb{Z}} a(k)e^{ikx}. \tag{1.3.20}$$

The Fourier inversion formulae extend by duality:

$$\mathcal{F}^*\mathcal{F} = \mathrm{id} \text{ on } \mathcal{D}', \tag{1.3.21}$$

$$\mathcal{F}\mathcal{F}^* = \mathrm{id} \text{ on } s'(\mathbb{Z}). \tag{1.3.22}$$

Remark 1.3.1. *Here is a good place to introduce some notational conventions which we shall utilize.*

1. *We use $A \lesssim B$ to denote $A \leqslant cB$ for some constant $c > 0$.*
2. *We take $A \lesssim_{a_1,\dots,a_k} B$ to mean $A \leqslant c(a_1,\dots,a_k)B$.*
3. *By $A \sim B$ we mean $B \lesssim A \lesssim B$.*
4. *Finally, for $r \in \mathbb{R}$, $r+$ denotes $r + h$ for an arbitrary small, positive parameter h. For example, Lemma 1.3.9 implies that*

$$\|uv\|_{L^2} \lesssim \|u\|_{L^2}\|v\|_{H^{1/2+}}.$$

We can then use the Fourier transform to define Sobolev spaces. Namely, for $r \in \mathbb{R}$, we have

$$H^r := \left\{u \in \mathcal{D}' : \langle D\rangle^r u \in L^2\right\}, \tag{1.3.23}$$

where $D := \frac{1}{i}\partial$. In addition, for $r \in \mathbb{R}$, we also have the homogeneous Sobolev space

$$\dot{H}^r := \left\{u \in \mathcal{D}' : |D|^{\frac{r}{2}} u \in L^2\right\}. \tag{1.3.24}$$

We have the following elementary Sobolev embedding result:

Lemma 1.3.2. *If $r > \frac{1}{2}$, then $H^r \hookrightarrow L^\infty$ with the estimate*

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^r}. \tag{1.3.25}$$

Further, if $r > \frac{3}{2}$, then $H^r \hookrightarrow \mathrm{Lip}$ with

$$\|f\|_{\mathrm{Lip}} \lesssim \|f\|_{H^r}. \tag{1.3.26}$$

Proof. See Proposition 3.3 in Chapter 4 of [Tay1]. The proof is based on the proof of the corresponding embeddings for $H^r(\mathbb{R})$ which are the content of Proposition 1.3 and Corollary 1.4 (also in Chapter 4). □

We will need to be able to work with Fourier multipliers and ΨDO on $\mathbb{T}$. For example, we just made use of Fourier multipliers on $\mathbb{T}$ when we defined Sobolev spaces above. Thus, we will need to be able to define such operators on the circle. It turns out that we have some choice in how to go about doing this. Suppose we consider $\mathbb{T}$ as a locally compact abelian group (the circle group) with Pontryagin dual $\widehat{\mathbb{T}} = \mathbb{Z}$. From this perspective, it would be natural to consider symbols $a(x,\xi)$ which are functions on $\mathbb{T} \times \widehat{\mathbb{T}} = \mathbb{T} \times \mathbb{Z}$. On the other hand, suppose we think $\mathbb{T}$ as a one-dimensional compact manifold (without boundary). In this case, it would make the most sense to consider symbols $a(x,\xi)$ which are functions on the cotangent bundle $T^*\mathbb{T}$. We know that the circle has trivial tangent bundle and therefore the cotangent bundle is trivial. That is to say that $T^*\mathbb{T}$ is isomorphic (diffeomorphic) to $\mathbb{T} \times \mathbb{R}$: $T^*\mathbb{T} \cong \mathbb{T} \times \mathbb{R}$. We think of the former perspective as giving rise to a global description and the latter to a local description. Fortunately, we know that these perspectives are entirely equivalent (see [McL]). We shall choose to utilize the local perspective, thinking of $\mathbb{T}$ as a manifold and working (implicitly) with local coordinates and partitions of unity.

So, when dealing with ΨDO, we will be working with symbols defined on $X \times \mathbb{R}$ for an open set X, where we think of X as being a component of an atlas of charts covering $\mathbb{T}$. We will not fully explore the details of defining ΨDO on a manifold, ensuring that the operators are invariant under coordinate transformations (i.e., diffeomorphisms $\mathbb{R} \to \mathbb{R}$) and so on. However, rest assured that everything indeed works as it should. For all of the details, see [Hör] or [Tay2]. Given $m \in \mathbb{R}$ and $0 \leqslant \rho, \delta \leqslant 1$, we further define the Hörmander symbol class $S^m_{\rho,\delta}(X)$ for an open set $X \subset \mathbb{R}$ to consist of those $p \in C^\infty(X \times \mathbb{R})$ such that, for any $k, \ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and every compact $K \subset X$, we have

$$\left|\partial_x^\ell \partial_\xi^k p(x,\xi)\right| \lesssim (1+|\xi|)^{m-\rho k+\delta \ell} \qquad (x \in K,\ \xi \in \mathbb{R}). \tag{1.3.27}$$

The most important symbol class for us will be $S^m_{1,0}(X)$, however the more exotic symbol class $S^m_{1,1}(X)$ will also make an appearance, at least implicitly.

We conclude this subsection with a Bernstein-type inequality regarding the effect of differentiating

band-limited functions:

Lemma 1.3.3. *Define* $A := \left\{\frac{1}{2} \leqslant |\xi| \leqslant 2\right\}$ *and* $B = \{|\xi| \leqslant 1\}$. *Take* $k, p \in \mathbb{N}_0$ *and* $u \in L^2$. *Then, it holds that*

$$\operatorname{supp} \hat{u} \subset 2^p B \implies \left\|\partial_x^k u\right\|_{L^2} \lesssim 2^{pk} \|u\|_{L^2}, \tag{1.3.28}$$

$$\operatorname{supp} \hat{u} \subset 2^p A \implies \left\|\partial_x^k u\right\|_{L^2} \sim 2^{pk} \|u\|_{L^2}. \tag{1.3.29}$$

Proof. See Lemma 2.1 in [BCD]. Also, see Lemma 1.1.2 in [AlGe]. □

1.3.2 Multilinear and Nonlinear Estimates

Given that we are going to be dealing with nonlinear equations, we will frequently need to confront multilinear and nonlinear estimates. More specifically, we will want to be able to estimate the product of functions and the composition of functions. Bony's paradifferential calculus provides an incredibly powerful tool for achieving such estimates via the paraproduct and the paralinearization formula.

In order to construct the paraproduct, we will need to first build up frequency filtering operators of Littlewood-Paley theory. To that end, let $\phi \in C_c^\infty(\mathbb{R})$, $0 \leqslant \phi \leqslant 1$, be a radial function such that

$$\phi(\xi) = 1 \text{ for } |\xi| \leqslant \frac{1}{2} \text{ and } \phi(\xi) = 0 \text{ for } |\xi| \geqslant 1. \tag{1.3.30}$$

Now, set $\Phi(\xi) = \phi(2^{-1}\xi) - \phi(\xi)$ and observe that Φ is supported in a dyadic shell:

$$\operatorname{supp} \Phi \subset \left\{2^{-1} \leqslant |\xi| \leqslant 2\right\}. \tag{1.3.31}$$

It further holds that

$$1 = \phi(\xi) + \sum_{p \geqslant 0} \Phi(2^{-p}\xi) \qquad (\forall \xi). \tag{1.3.32}$$

Notice that there are never more than two non-zero terms in the sum (1.3.32).

For $p \in \mathbb{N}_0$, we define the frequency filtering operator S_p acting on $\mathcal{D}'$ by

$$S_p u := \phi_p(D)u, \text{ where } \phi_p(\xi) := \phi(2^{-p}\xi). \tag{1.3.33}$$

For $p \in \mathbb{Z}$, we further define the dyadic blocks Δ_p, again acting on $\mathcal{D}'$, by

$$\Delta_p u \coloneqq S_{p+1}u - S_p u \text{ for } p \in \mathbb{N}_0,\ \Delta_{-1}u \coloneqq S_0 u,\ \Delta_p u \coloneqq 0 \text{ for } p \leqslant -2. \tag{1.3.34}$$

If $p \in \mathbb{N}_0$, the dyadic blocks satisfy

$$\Delta_p u = \Phi_p(D)u, \text{ where } \Phi_p(\xi) \coloneqq \Phi(2^{-p}\xi). \tag{1.3.35}$$

In fact, equation (1.3.35) can be, and frequently is, used as the definition of the dyadic blocks. Observe that the above definitions imply that

$$S_p u = \sum_{q \leqslant p-1} \Delta_q u, \tag{1.3.36}$$

which explains why the operators S_p are also called partial sum operators.

We now state the important properties of the frequency filtering operators and the dyadic blocks. First, the spectrum of the frequency filtering operators is contained in a dyadic ball:

$$\operatorname{supp} \mathcal{F}(S_p u) \subset \left\{|\xi| \leqslant 2^{p-1}\right\}. \tag{1.3.37}$$

Similarly, the spectrum of dyadic blocks is contained in a dyadic shell:

$$\operatorname{supp} \mathcal{F}(\Delta_p u) \subset \left\{2^{p-1} \leqslant |\xi| \leqslant 2^{p+1}\right\}. \tag{1.3.38}$$

We additionally have the following dyadic partition of unity:

$$\mathrm{id} = S_0 + \sum_{p \geqslant 0} \Delta_p = \sum_{p \geqslant -1} \Delta_p. \tag{1.3.39}$$

In particular, for $u \in \mathcal{D}'$, we have

$$\sum_{p=-1}^{P} \Delta_p u \xrightarrow{\mathcal{D}'} u \text{ as } P \to +\infty. \tag{1.3.40}$$

We are now prepared to define the paraproduct, at least formally. Given $u, v \in \mathcal{D}'$, we define

$$T_u v \coloneqq \sum_{p \geqslant 2} S_{p-2} u \Delta_p v. \tag{1.3.41}$$

The regularity of the paraproduct $T_u v$ is controlled by v and, in particular, $T_u v$ cannot be more regular than v. We then have Bony's paraproduct decomposition:

$$uv = T_u v + T_v u + R(u, v), \tag{1.3.42}$$

where the remainder $R(\cdot, \cdot)$ is given by

$$R(u, v) := \sum_{|p-q| \leqslant 2} \Delta_p u \Delta_q v. \tag{1.3.43}$$

The term remainder is justified as $R(u, v)$ is smoother than both u and v as soon as it is defined.

We now want to make things a bit more precise and not just rely on formal computations. We begin with the following result.

Lemma 1.3.4. *Let $a \in L^\infty$ and $u \in H^r$ for some $r \in \mathbb{R}$. Then, the paraproduct $T_a u$ is well-defined and we further have*

$$\|T_a u\|_{H^r} \lesssim \|a\|_{L^\infty} \|u\|_{H^r}. \tag{1.3.44}$$

Proof. This is a well-known result and can be found in many references. For example, see Proposition C.8 in [B-GS]. Also, see [Mét] where this result is the content of Proposition 5.2.1. □

The above result is by no means sharp and we can define $T_a u$ for much less regular a. However, we will not need to use any more specialized results and so do not bother stating them. Now, we need to deal with the remainder. We want to determine a sufficient condition for the remainder to be well-defined, quantify in what sense the remainder is smoother and estimate $R(u, v)$ in terms of u and v. The following result will take care of all of this:

Lemma 1.3.5. *Let $r, t \in \mathbb{R}$ be such that $s + t > 0$. Then, for $u \in H^r$ and $v \in H^t$, the remainder $R(u, v)$ is well-defined (by* (1.3.43)*) and satisfies*

$$\|R(u, v)\|_{H^{s+t-1/2}} \lesssim \|u\|_{H^r} \|v\|_{H^t}. \tag{1.3.45}$$

Proof. See Theorem C.9 in [B-GS]. □

Finally, we want to record the paralinearization formula:

Lemma 1.3.6. *Let $r > \frac{1}{2}$. Then, for any $u \in H^r$ and any function $F \in C^\infty(\mathbb{R})$, we have*

$$F(u) = T_{F'(u)}u + R(u), \tag{1.3.46}$$

where $R(u) \in H^{2r-\frac{1}{2}}$.

Proof. This result goes back to Bony's seminal work [Bony]. A good proof of this result can be found in Theorem 2 of Section 16.3 in [CoMe]. A wonderful discussion of a variant formulation can be found in Chapter 3 of [Tay4]. Some other good resources include [Mét, BCD, AlGe]. □

Littlewood-Paley theory and paraproducts (and paradifferential operators more broadly) have wide applicability throughout analysis. Of course, the primary benefit of Littlewood-Paley analysis is that it allows for a detailed study of frequency interactions. For some interesting examples applied to the quasilinear Schrödinger and Camassa-Choi equations, respectively, see [MMT, HaMa]. The paradifferential calculus was originally introduced to study the propagation of singularities for nonlinear equations [Bony]. Since its introduction, paradifferential analysis has been fruitfully applied to many problems. For some examples of its application to the system under study in this thesis, the water waves system, see [AlMe, ABZ1, ABZ3, ABH]. Some other interesting applications include the Euler-Maxwell system of plasma dynamics [GeMa], the MHD system [LXZ] and the hydrodynamic flow of liquid crystals [XuZh]. For further discussion of applications, see [Mét, Tay4, Tay5].

Now, we arrive at the main results of this subsection. First, we have a Moser-type estimate for the composition $F \circ u$ of a smooth function F and a Sobolev function u:

Lemma 1.3.7. *If $F : \mathbb{R} \to \mathbb{C}$ is C^∞ and $u \in H^r \cap L^\infty$ with $r \geqslant 0$, then*

$$\|F(u)\|_{H^r} \lesssim 1 + \|u\|_{H^r}; \tag{1.3.47}$$

the implied constant is of the form $C\|F'\|_{C^K}(1 + \|u\|_{L^\infty}^K)$ for $0 < r < K$.

Proof. A particularly efficient approach to proving this result is via the paralinearization formula of Lemma 1.3.6. See Section 3.1 in [Tay4]. □

Remark 1.3.8. *Lemma 1.3.2 implies that Lemma 1.3.7 shall apply to any $u \in H^r$ whenever $r > \frac{1}{2}$.*

With the Moser-type estimate in hand, we next want to obtain a good product estimate. There is, of course, the well-known Sobolev algebra property, which says that, for $r > \frac{1}{2}$, H^r is a Banach algebra and $\|uv\|_{H^r} \lesssim \|u\|_{H^r}\|v\|_{H^r}$. However, we will need a more powerful result as we want to be able to deal with multiplication by rougher functions (or even distributions). To that end, we have the following:

Lemma 1.3.9. *Suppose that $u \in H^r$ and $v \in H^t$ with $r + t > 0$. Then, for all ρ satisfying $\rho \leqslant \min(r, t)$ and $\rho < r + t - \frac{1}{2}$, we have $uv \in H^\rho$ with the following estimate:*

$$\|uv\|_{H^\rho} \lesssim \|u\|_{H^r}\|v\|_{H^t}. \tag{1.3.48}$$

Proof. As noted above, this is yet another result that follows readily from Bony's paradifferential theory. Namely, we can apply Bony's paraproduct decomposition (1.3.42). See Theorem C.10 of [B-GS]. □

1.3.3 The Hilbert Transform

The Hilbert transform $\mathcal{H}$ will play a prominent role in our analysis, particularly in Chapter 2. As such, it shall be helpful to establish some of its mapping properties. We begin by examining the action of $\mathcal{H}$ on L^2:

Lemma 1.3.10. *The Hilbert transform $\mathcal{H}$ is an L^2-isometry.*

Proof. This is a consequence of Plancherel's theorem, combined with the fact that $\mathcal{H} := -i\,\mathrm{sgn}(D)$.

More specifically, we have the following. Begin with the Hilbert transform $\mathcal{H} : \mathcal{D} \to \mathcal{D}$, defined by $\mathcal{H}\,u := -i\,\mathrm{sgn}(D)u$. By Plancherel's theorem, $\mathcal{H} : \mathcal{D} \to \mathcal{D}$ is an isometry. Since $\mathcal{D}$ is dense in L^2, $\mathcal{H}$ has a unique, densely-defined extension to L^2 and, abusing notation mildly, we also denote this extension by $\mathcal{H}$. Using Plancherel to justify taking the necessary limits, $\mathcal{H}$ is then an isometry on L^2. □

Lemma 1.3.11. *For $r \in \mathbb{R}$, the Hilbert transform $\mathcal{H}$ is a continuous (bounded) linear operator on H^r; in fact, $\mathcal{H}$ is an isometry of H^r:*

$$\|\mathcal{H}\,u\|_{H^r} = \|u\|_{H^r}.$$

Proof. Since Fourier multipliers commute, Lemma 1.3.10 implies

$$\|\mathcal{H}\,u\|_{H^r} = \|\langle D\rangle^r\,\mathcal{H}\,u\|_{L^2} = \|\mathcal{H}\langle D\rangle^r u\|_{L^2} = \|\langle D\rangle^r u\|_{L^2} = \|u\|_{H^r}.$$

□

We have the following useful commutator estimates for commutators involving the Hilbert transform:

Lemma 1.3.12. *Let $f \in H^r$ for $r \in \mathbb{R}$. Then, the operator $[\mathcal{H}, f]$ is bounded $L^2 \to H^{r-1}$ and $H^{-1} \to H^{r-2}$. Further, for $j = -1, 0$, we have*

$$\|[\mathcal{H}, f](u)\|_{H^{r-1+j}} \lesssim \|f\|_{H^r} \|u\|_{H^j}. \tag{1.3.49}$$

Proof. See Lemma 3.7 in [Amb1]. □

Lemma 1.3.13. *If $f \in H^r$ for $r \geqslant 3$, then $[\mathcal{H}, f]$ is a bounded operator $H^{r-2} \to H^r$. If $f \in H^{r-1/2}$ for $r \geqslant 4$, then $[\mathcal{H}, f]$ is a bounded operator mapping $H^{r-2} \to H^{r-1/2}$. In addition, for $j \in \left\{0, -\frac{1}{2}\right\}$, we have the estimate*

$$\|[\mathcal{H}, f](u)\|_{H^{r+j}} \lesssim \|f\|_{H^{r+j}} \|u\|_{H^{r-2}}.$$

Proof. See Lemma 3.8 in [Amb1]. □

1.4 Overview of the Thesis

In Chapter 2, we consider a vortex sheet formulation for $2d$ gravity-capillary water waves in a geometric setting proposed by Ambrose, et. al. in [AmbEtAl]. This model allows for bathymetric variations, (smooth) obstacles in the flow and a background flow. We start by introducing the model and writing down the evolution equations. Then, by proving energy estimates, we show that this model is locally well-posed. We then turn to study the lifespan of solutions with the main result being that, for small Cauchy data of size $\varepsilon \ll 1$ and zero background current, the lifespan satisfies

$$T(\varepsilon) \gtrsim \log \frac{1}{\varepsilon}. \tag{1.4.1}$$

We also obtain lifespans for large data and for non-zero background current. We next seek to incorporate Clamond damping into this model. We begin by deriving new evolution equations for the damped system. Then, again by proving energy estimates, we show that the model remains well-posed and that all of the lifespan results, obtained for the undamped problem, apply to the damped problem.

The lifespan given in (1.4.1) is by no means optimal, rather this lifespan is what is attainable by using basic energy estimates. In particular, given that the vortex sheet equations are quadratically nonlinear, the classical local well-posedness theory for quasilinear hyperbolic equations suggests a quadratic (i.e., $\mathcal{O}(\frac{1}{\varepsilon})$) lifespan [Kato2, Kato1, Maj2]. However, actually obtaining this lifespan will require some rather technical

analysis. Our next objective is then to begin working towards obtaining this quadratic lifespan. To this end, in Chapter 3, we will introduce a toy model for water waves subject to Clamond damping. Our toy model has a parameter and we primarily focus on two values of this parameter, which correspond to capillary waves and gravity waves. For technical reasons, these two cases require quite distinct analysis, however, in both cases, we are able to prove that solutions exhibit a quadratic lifespan. We include some discussion of how this toy model relates to the vortex sheet equations and why we believe that the methods employed to study the toy model will extend to the vortex sheet formulation of the gravity-capillary water waves system. As such, this chapter is somewhere between a trial run and a proof of concept.

CHAPTER 2

Local Well-Posedness of the Gravity-Capillary Water Waves System in the Presence of Geometry and Damping

2.1 Plan of the Chapter

We consider a vortex sheet model for two-dimensional gravity-capillary water waves with a (constant) background current over obstacles and topography proposed by Ambrose, et. al. in [AmbEtAl]. For simplicity of presentation, we limit ourselves to the case of a single obstacle, however our techniques apply to the case of any finite number of obstacles. The velocity is given by the gradient of a scalar potential φ, which is represented via layer potentials on the different components of the boundary. The variables which we evolve are θ, the tangent angle formed by the interface with the horizontal; γ, the vortex sheet strength; ω, the density of the layer potential on the bottom and β, the density of the layer potential on the obstacle. We note that $\gamma := \mu_\alpha$, where μ is the density of the layer potential on the free surface.

The system of equations which we consider is nonlocal and, in particular, is of the form

$$\begin{cases} (\mathrm{id} + \mathscr{K}[\Theta])\partial_t \Theta = \mathfrak{F}(\Theta) \\ \Theta(t=0) = \Theta_0 \end{cases}, \tag{2.1.1}$$

where $\Theta := (\theta, \gamma, \omega, \beta)^t$ and $\mathscr{K}[\cdot]$ is a compact operator. We introduce the parameter B to denote the size of the initial data:

$$B := \|\Theta_0\|_X, \tag{2.1.2}$$

where X is the energy space. We will obtain our main lifespan results in the context of small data and, in this setting, we take

$$B = \varepsilon \ll 1. \tag{2.1.3}$$

Our first main objective will be to show that the model proposed in [AmbEtAl] is well-posed and that solutions persist on a timescale of order $O(\log \frac{1}{\varepsilon})$ (resp. $O(1)$) in the presence of zero (resp. non-zero)

background current. Our approach will be to first consider the model problem

$$\begin{cases} \partial_t \Theta = \mathfrak{F}(\Theta) \\ \Theta(t=0) = \Theta_0 \end{cases}, \tag{2.1.4}$$

beginning by proving the desired results for this model problem via energy estimates. Then, we will deduce mapping properties of $(\mathrm{id} + \mathscr{K})^{-1}$ that imply that the results proved for the model problem (2.1.4) are also true of the water waves system (2.1.1).

Our next primary objective will be to modify the system (2.1.1) to incorporate the Clamond damper and show that the same results hold for the damped system. We do so by following the same approach as for the non-damped system (i.e., first consider the model problem for damped water waves and then use mapping properties of $(\mathrm{id} + \mathscr{K})^{-1}$ to obtain the desired result). As noted above, we primarily utilize energy estimates and, in particular, we largely follow the approach of [Amb1].

The $\mathcal{O}(\log \frac{1}{\varepsilon})$ existence time obtained here is certainly not sharp, particularly being less than the $\mathcal{O}(\frac{1}{\varepsilon})$ existence time suggested by the nonlinearity (this follows from the classical local well-posedness theory for quasilinear hyperbolic equations; e.g., see [Kato2, Kato1, Maj2]). However, obtaining the sharper existence time requires a more detailed study of the system and, as such, we have decided to leave this to future work and here simply focus on results obtainable by energy methods.

The plan of this chapter is as follows. In Section 2, we give an overview of the main results. We then proceed to give a brief overview of the model which we utilize in Section 3. Next, in Section 4, we begin the process of proving our first main result and prove a bound on the growth of the energy of solutions to the model problem (2.1.4), then, in Section 5, we prove the existence of solutions to the system (2.1.4). In Sections 6 and 7, we complete the proof of the local-in-time well-posedness of the (undamped) model system. Section 8 contains a study of the lifespan of solutions to the model system. Section 9 considers the damped model system and here we show that the results of Sections 4-8 all extend to the damped model problem. Section 10 is concerned with the needed mapping properties of $(\mathrm{id} + \mathscr{K})^{-1}$ and extending the results of Sections 4-9 to the full water waves system (2.1.1). In the final section, Section 11, we prove the solvability of the integral equations arising in the system, which gives an alternative approach to the one given in [AmbEtAl]. One of the reasons we include this proof is that it can be more readily extended to $3d$ than the proof given in [AmbEtAl].

2.2 Main Results

Here we will state the main results of this chapter. As outlined above, our first main result is to show that that this system is well-posed locally in time and to obtain a lower bound on the lifespan of solutions. Next, we consider a damped version of the system and show that all of the results obtained for the non-damped system apply to the damped system.

Let $\mathbf{V}_0 := (V_0, 0)$ denote the background current and introduce the scale of spaces

$$X_r := H^r \times H^{r-1/2} \times H^1 \times H^1 \qquad (r \in \mathbb{R}). \tag{2.2.1}$$

Notice that we clearly have $X_t \subset X_r$ for $t > r$. Utilizing this notation, our first main result is then the following:

Theorem 2.2.1. *Let s be sufficiently large. The system* (2.1.1) *is locally well-posed (in the sense of Hadamard). Namely, there exists a unique solution $\Theta \in C([0, T(B, |V_0|)]; X_s)$ to the system* (2.1.1) *and the flow map is Lipschitz continuous from X_1 into $C([0, T]; X_1)$. In the case of small Cauchy data and zero background current (i.e., $V_0 = 0$), we have*

$$T(\varepsilon) \gtrsim \log \frac{1}{\varepsilon}. \tag{2.2.2}$$

On the other hand, for large Cauchy data, we have

$$T(B, |V_0|) \gtrsim \begin{cases} B^{1-N} & V_0 = 0 \\ \min\big((1 + |V_0|)^{-2}, B^{1-N}\big) & V_0 \neq 0 \end{cases}, \tag{2.2.3}$$

where N is a parameter given in equation (2.5.58).

Remark 2.2.2. *We note that the solution is not guaranteed to remain of size $O(\varepsilon)$ on the given lifespans. Rather, all that is assured is that the energy remains bounded, and thus the solutions persist, on the stated timescales. The $O(\log \frac{1}{\varepsilon})$ lifespan when $V_0 = 0$ is certainly not sharp. In fact, as noted above, the quadratic nonlinearity exhibited by the system* (2.1.1) *suggests an $O(\frac{1}{\varepsilon})$ lifespan. However, actually proving that solutions exist on an $O(\frac{1}{\varepsilon})$ timescale is not a trivial matter and will require more delicate analysis [AlLa, Més]. On the other hand, proving that solutions persist on an $O(\log \frac{1}{\varepsilon})$ timescale can be done using*

only energy estimates and a Grönwall argument. As such, in this paper, which is largely based on energy methods, we simply prove the $O(\log \frac{1}{\varepsilon})$ lifespan. We are presently working on a follow-up result with Thomas Alazard and Jeremy Marzuola in which we prove the $O(\frac{1}{\varepsilon})$ lifespan.

Remark 2.2.3. *The existence time of $O((1+|V_0|)^{-2} \wedge B^{1-N})$ when $V_0 \neq 0$ may not be sharp, however substantial improvements are not possible. In fact, when $V_0 \neq 0$, numerical simulations have shown splash singularities to occur in $O(1)$ time, even starting from a flat interface [AmbEtAl].*

We next consider a damped version of the system. As noted above, we implement a modified sponge layer damper, which we call Clamond damping, first introduced in [CFGK2]. Recall that Clamond damping utilizes a pneumatic damper with the external pressure given by (1.1.3) (i.e., $p_{\text{ext}} := \partial_x^{-1}(\chi_\omega \partial_x \varphi)$). Though we use the same notation ω for the damping region and the density of the single layer potential on the bottom, this will cause no confusion as context will always make clear what ω represents.

We derive evolution equations which account for the Clamond damping and we denote the new right-hand side by $\mathfrak{F}_D$. We then arrive at the damped water waves system:

$$\begin{cases} (\mathrm{id} + \mathscr{K}[\Theta])\partial_t \Theta = \mathfrak{F}_D(\Theta) \\ \Theta|_{t=0} = \Theta_0 \end{cases} . \tag{2.2.4}$$

Our second main result is as follows:

Theorem 2.2.4. *All of the results of Theorem 2.2.1 apply to the damped system. In particular, take s to be sufficiently large. Then,* (2.2.4) *is locally-in-time well-posed with the flow map being Lipschitz regular from X_1 into $C([0,T];X_1)$ and the solution Θ belonging to $C([0,T(B,|V_0|)];X_s)$. For small Cauchy data and zero background flow, the lifespan $T(\varepsilon)$ satisfies* (2.2.2) *and, in the case of large Cauchy data, we again have* (2.2.3).

Remark 2.2.5. *In [AmbEtAl], Ambrose, et. al. actually present two formulations of the water waves problem. Namely, in addition to the vortex sheet formulation we consider here, they propose a dual formulation via Cauchy integrals. The energy methods employed here would yield results analogous to those of Theorem 2.2.1 for the Cauchy integral formulation. Further, Clamond damping can be implemented in the Cauchy integral formulation and the results of Theorem 2.2.4 can similarly be obtained for the Cauchy integral formulation via the energy arguments utilized here.*

2.3 A Brief Overview of the Model

Our objective here is to give a brief overview of the model which we utilize. We will discuss the domain as well the relevant variables and parameters with which we work. Finally, we will write down the evolution equations which govern the system. For full details on the model, the reader should consult [AmbEtAl].

2.3.1 The Domain

At time t, the fluid is contained in a domain $\Omega_t \subset \mathbb{T} \times \mathbb{R}$ of finite vertical extent. The fluid domain is bounded above by a free surface $\mathcal{S}_t$ and below by a fixed, solid boundary $\mathcal{B}$. We assume Ω_t is multiconnected and $\partial\Omega_t \backslash (\mathcal{S}_t \cup \mathcal{B})$ is composed of smooth Jordan curves. We describe the location of the free boundary via a parameterized curve, $\mathcal{S}_t : (\xi(\alpha,t), \eta(\alpha,t))$, where t denotes time and α is the parameter along $\mathcal{S}_t$. Here, $\xi(\alpha) - \alpha$ and η are both periodic with period 2π.

The bottom is fixed (i.e., time-independent) and also described by a parameterized curve $\mathcal{B} : (\xi_1(\alpha), \eta_1(\alpha))$ with the same periodicity. Additionally, the multiconnectedness of Ω_t corresponds to one or more obstacles in the flow. For simplicity of notation and presentation we utilize a single obstacle $\mathcal{O}$ (i.e., $\complement\Omega_t = \mathcal{O} \cup \mathcal{U}_t$, where $\mathcal{U}_t$ is unbounded). However, we note that the extension to an arbitrary, finite number of obstacles is immediate and all of our results apply to this case. Of course, due to periodicity this one obstacle will recur periodically and so, in fact, corresponds to a periodic array of obstacles. We denote $C := \partial\mathcal{O} = \partial\Omega_t \backslash (\mathcal{S}_t \cup \mathcal{B})$. We assume that the obstacle is fixed and that its boundary is given by a closed parameterized curve $C : (\xi_2(\alpha), \eta_2(\alpha))$ with ξ_2 and η_2 being 2π-periodic.

It will frequently be beneficial to utilize a complexified description of the domain and to this end define

$$\zeta := \xi + i\eta \text{ and } \zeta_j := \xi_j + i\eta_j. \tag{2.3.1}$$

Regarding orientation, we parameterize the boundary of the fluid domain so that the normal on $\mathcal{S}_t$ points into the vacuum region, the normal on $\mathcal{B}$ points into the fluid region and the normal on C points into the fluid region. We denote the length of one period of the free surface by $L = L(t)$, the length of one period of $\mathcal{B}$ by L_1 and the length of C by L_2.

For technical reasons, we shall want the interface to be free of self-intersections. In order to ensure that this is so, we impose the chord-arc condition on ζ:

$$\exists \mathfrak{c} > 0 : \left| \frac{\zeta(\alpha) - \zeta(\alpha')}{\alpha - \alpha'} \right| > \mathfrak{c} \qquad (\forall \alpha \neq \alpha'). \tag{2.3.2}$$

This condition will rule out self-intersections (e.g., splash and splat singularities) as well as cusps.

We shall also assume that the depth of the water is bounded away from zero, as is the distance from the free surface to the boundary of the obstacle. Namely, there exist positive constants $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ so that

$$\eta - \eta_1 \geqslant \mathfrak{h}, \tag{2.3.3}$$

$$\eta - \eta_2 \geqslant \tilde{\mathfrak{h}}. \tag{2.3.4}$$

These are non-cavitation assumptions and mean that neither the bottom nor the obstacle go dry. These non-cavitation assumptions are critical for our analysis. The question of removing these assumptions is quite fascinating and much work is yet to be done. Some progress has been made in considering the water waves system in a simply connected domain in the absence of assumption (2.3.3). For more on this fascinating problem, the interested reader can consult [deP, MiWa1, MiWa2, MiWa3]. Asymptotic models for water waves are studied in this context in, for example, [CamEtAl, LanMét].

Finally, we introduce the notation ζ_d, which we define by

$$\zeta_d(\alpha, t) \coloneqq \zeta(\alpha, t) - \zeta(0, t). \tag{2.3.5}$$

The value of $\zeta(0, t)$ is, in general, unimportant. It is worth noting that $\partial_\alpha \zeta_d = \partial_\alpha \zeta$.

2.3.2 The Dynamics of the Free Surface

We will now briefly discuss how the evolution of the free surface is described in this model. At each point on $\mathcal{S}_t$, there is a unit tangent vector $\hat{\mathbf{t}} = \left|(\xi_\alpha, \eta_\alpha)\right|^{-1}(\xi_\alpha, \eta_\alpha)$ and a unit normal vector $\hat{\mathbf{n}} = \left|(\xi_\alpha, \eta_\alpha)\right|^{-1}(-\eta_\alpha, \xi_\alpha)$, where the subscript α denotes differentiation with respect to the parameter α. We let U denote the normal velocity and V the tangential velocity:

$$\partial_t(\xi, \eta) = U\hat{\mathbf{n}} + V\hat{\mathbf{t}}. \tag{2.3.6}$$

A key observation underlying the HLS framework is that the shape of the free surface $\mathcal{S}_t$ is solely determined by the normal velocity U, while changes in the tangential velocity V serve only to reparameterize the interface [HLS1]. The tangential velocity V will be chosen so as to enforce a renormalized arclength parameterization of $\mathcal{S}_t$.

The HLS framework utilizes a geometric frame of reference to describe the location of the free surface,

as opposed to the usual Cartesian coordinates (ξ, η) [HLS1]. The first of the geometric coordinates is $\theta = \theta(\alpha, t)$, which denotes the tangent angle formed by $\mathcal{S}_t$ with the horizontal:

$$\theta := \arctan \frac{\eta_\alpha}{\xi_\alpha}. \tag{2.3.7}$$

Using this new variable, we can write $\hat{\mathbf{t}} = (\cos\theta, \sin\theta)$ and $\hat{\mathbf{n}} = (-\sin\theta, \cos\theta)$. In addition, we have

$$\xi(\alpha) = \alpha + \partial_\alpha^{-1}(s_\alpha \cos\theta(\alpha) - 1), \tag{2.3.8}$$

where, in this case, ∂_α^{-1} denotes the mean-zero antiderivative (for more details, see section 2.2 of [AmbEtAl]).

The other geometric coordinate is the arclength element $s_\alpha = s_\alpha(\alpha, t)$ given by $s_\alpha := \sqrt{\xi_\alpha^2 + \eta_\alpha^2}$. It is straightforward to see that

$$\partial_t s_\alpha = V_\alpha - \theta_\alpha U. \tag{2.3.9}$$

We further note that L is given by $L(t) = \int_0^{2\pi} s_\alpha(\alpha, t)\, d\alpha$, where the integral is over one period. We may again differentiate with respect to time and use equation (2.3.6) to infer the evolution equation for L:

$$\partial_t L = -\int_0^{2\pi} \theta_\alpha U \, d\alpha. \tag{2.3.10}$$

In fact, one can either take s_α or L to be the other independent variable describing $\mathcal{S}_t$. Here, one sees one of the major strengths of the HLS framework. Recall that the dynamic boundary condition in (1.1.1) requires that the pressure jump across the free boundary be proportional to the curvature of the free boundary. In this setting, where the free boundary is given by a parameterized curve, the curvature is given in terms of the Cartesian coordinates (ξ, η) by

$$H_{\mathcal{S}_t} = \kappa(\zeta) = \frac{\xi_\alpha \eta_{\alpha\alpha} - \eta_\alpha \xi_{\alpha\alpha}}{(\xi_\alpha^2 + \eta_\alpha^2)^{\frac{3}{2}}}. \tag{2.3.11}$$

However, we can express the curvature in the geometric coordinates (θ, s_α) as

$$\kappa(\zeta) = \frac{\theta_\alpha}{s_\alpha}. \tag{2.3.12}$$

The tangential velocity V is selected to enforce that s_α be independent of the spatial variable, which

yields a renormalized arclength parameterization of $\mathcal{S}_t$. Considering the equations for $\partial_t s_\alpha$ and $\partial_t L$ leads to the choice

$$V := \partial_\alpha^{-1}\left(\theta_\alpha U - \frac{1}{2\pi}\int_0^{2\pi} \theta_\alpha U \, d\alpha\right). \tag{2.3.13}$$

Implicit in (2.3.13) is a constant of integration, which we are free to choose. Reasonable choices include taking the constant of integration so as to force (i) V to have mean zero, (ii) $V(0,t) = 0$ or (iii) $\xi(0,t) = 0$. This constant of integration will not be terribly important for us, but we generally take $V(0,t) = 0$ as it will simplify a later computation. It is straightforward to check that such a choice of V leads to $L = 2\pi s_\alpha$ for all time (also see [AmbEtAl]).

Our next objective is to give a definition of the normal velocity U along the free surface. We recall that the fluid velocity satisfies the (irrotational) free-surface Euler equations (1.1.1). Notice that the assumptions of irrotationality and incompressibility imply that the velocity field is given by the gradient of a harmonic scalar potential φ. With this in mind, we shall write $\mathbf{v} = \nabla\varphi$ with $\varphi = \varphi_0 + \varphi_1 + \varphi_2 + \chi(a_0\nabla\varphi_{\text{cyl}} + \mathbf{V}_0)$, noting that each of the φ_j's corresponds to a different part of the boundary of the fluid region – the interface $\mathcal{S}_t$, the bottom $\mathcal{B}$ and the boundary of the obstacle $\mathcal{C}$. The constant a_0 is a circulation parameter and φ_{cyl} is given by

$$\varphi_{\text{cyl}}(z) := \mathfrak{Re}\left\{\frac{1}{2}z - i\log\sin\frac{1}{2}(z - z_c)\right\}, \tag{2.3.14}$$

where $z_c \in \mathcal{O}$. We note that it is only necessary to introduce φ_{cyl} in the case of a nonzero background flow, which is why we have introduced the coefficient

$$\chi := \begin{cases} 1 & V_0 \neq 0 \\ 0 & V_0 = 0 \end{cases}. \tag{2.3.15}$$

As previously noted, U must be determined by the physics and we have

$$U := \partial_{\hat{\mathbf{n}}}\varphi. \tag{2.3.16}$$

We take the φ_j's to be given by layer potentials (a double layer potential on the free surface and single layer potentials on the bottom as well as on the boundary of the obstacle).

We utilize a double layer potential at the free boundary. At a point (x, y) in the fluid region, we have

$$\varphi_0(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mu(\alpha')\frac{(x-\xi(\alpha'), y-\eta(\alpha'))}{\left|(x-\xi(\alpha'), y-\eta(\alpha'))\right|^2}\cdot \hat{\mathbf{n}}(\alpha')\, d\sigma(\alpha').$$

It is of benefit for us to rewrite the above integral as an integral over a single period as opposed to an integral over the real line. We achieve this by summing over periodic images and applying a formula of Mittag-Lefler [AbFo] to obtain a complex cotangent kernel:

$$\varphi_0(x,y) = \Re\mathfrak{e}\left\{\frac{1}{4\pi i}\int_0^{2\pi} \mu(\alpha')\zeta_\alpha(\alpha')\cot\frac{1}{2}((x+iy)-\zeta(\alpha'))\, d\alpha'\right\}, \tag{2.3.17}$$

where (x, y) is in the fluid region. It is the gradient of (2.3.17) which interests us and so we apply $(\partial_x - i\partial_y)$, integrate by parts to retain the cotangent kernel and set $\gamma := \mu_\alpha$. We obtain

$$(\partial_x - i\partial_y)\varphi_0(x,y) = \frac{1}{4\pi i}\int_0^{2\pi} \gamma(\alpha')\cot\frac{1}{2}((x+iy)-\zeta(\alpha'))\, d\alpha'.$$

Of course, this gradient will be singular on $\mathcal{S}_t$, however we can take the limit as we approach the free boundary by using the Plemelj formulae. This process yields

$$\lim_{(x,y)\to(\xi(\alpha),\eta(\alpha))}(\partial_x - i\partial_y)\varphi_0(x,y) = \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \gamma(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\, d\alpha' + \frac{\gamma(\alpha)\zeta_\alpha^*(\alpha)}{2s_\alpha^2}, \tag{2.3.18}$$

where the pv denotes a principal value integral, $\gamma := \mu_\alpha$ is the (non-normalized) vortex sheet strength and $(\cdot)^*$ denotes complex conjugation. Note that the integral in (2.3.18) is the (complex conjugate of the) complexified Birkhoff-Rott integral. We denote the real Birkhoff-Rott integral as $\mathbf{BR} = (BR_1, BR_2)$ and so

$$\mathfrak{C}(\mathbf{BR})^*(\alpha) = \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \gamma(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\, d\alpha', \tag{2.3.19}$$

where $\mathfrak{C} : (a, b) \mapsto a + ib$. Following (2.3.18), we can write

$$\lim_{(x,y)\to(\xi,\eta)} \nabla\varphi_0(x,y) = \mathbf{BR} + \frac{\gamma}{2s_\alpha}\hat{\mathbf{t}}. \tag{2.3.20}$$

The quantity $\frac{\gamma}{2s_\alpha}$ is known as the true vortex sheet strength as it gives the jump in the tangential component of

the velocity across the free boundary:

$$[\mathbf{v}]|_{\mathcal{S}_t} \cdot \hat{\mathbf{t}} = \frac{\gamma}{2s_\alpha}. \tag{2.3.21}$$

A key aspect of our methods that restricts us to considering the $2d$ case involves the simplification of the Birkhoff-Rott integral in (2.3.19), namely summing over periodic images to obtain a complex cotangent kernel. It is also worthwhile to reinforce that the integral defining **BR** is a singular integral as this fact shall be important in the analysis to come.

We shall use a single layer potential on $\mathcal{B}$, where we impose Neumann boundary conditions. That is, for φ_1, we have

$$\varphi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega(\alpha')s_{1,\alpha}(\alpha')\log\big|(x-\xi_1(\alpha'), y-\eta_1(\alpha'))\big|\, d\alpha',$$

where (x,y) is a point in the fluid region and $s_{1,\alpha}$ is the arclength parameter on the bottom. Notice that in this case the arclength parameter $s_{1,\alpha}$ depends upon α. We again take the gradient and express it as a complex quantity:

$$(\partial_x - i\partial_y)\varphi_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega(\alpha')s_{1,\alpha}(\alpha')\frac{(x-\xi_1(\alpha')) - i(y-\eta_1(\alpha'))}{\big|(x-\xi_1(\alpha')) + i(y-\eta_1(\alpha'))\big|^2}\, d\alpha'.$$

As we did with the double layer potential, we can sum over periodic images and obtain a cotangent kernel. This process results in

$$(\partial_x - i\partial_y)\varphi_1(x,y) = \frac{1}{4\pi}\int_0^{2\pi} \omega(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}((x+iy)-\zeta_1(\alpha'))\, d\alpha'.$$

This integral is not singular on $\mathcal{S}_t$ and so we can evaluate it along the free boundary in a straightforward way. We define $\mathbf{Y} := \nabla\varphi_1(\zeta)$ and so have

$$\mathfrak{C}(\mathbf{Y})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi} \omega(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))\, d\alpha'. \tag{2.3.22}$$

We again use a single layer potential on the boundary of the obstacle O:

$$\varphi_2(x,y) = \frac{1}{2\pi}\int_0^{2\pi} \beta(\alpha')s_{2,\alpha}(\alpha')\log\big|(x-\xi_2(\alpha'), y-\eta_2(\alpha'))\big|\, d\alpha',$$

where, once more, (x,y) is a point in the fluid region and $s_{2,\alpha}$ is the arclength parameter on C. Again, $s_{2,\alpha}$ is

dependent upon α. We can write the gradient as an integral over a single period like in the previous cases. Given that the computations are basically identical to the ones we just did for the bottom, we shall omit them. Ultimately, we set $\mathbf{Z} \coloneqq \nabla\varphi_2(\zeta)$ and have

$$\mathfrak{C}(\mathbf{Z})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi} \beta(\alpha')s_{2,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta_2(\alpha'))\,d\alpha'. \tag{2.3.23}$$

Once more, we notice that the integral defining $\mathbf{Z}$ is not singular.

It shall be convenient to introduce the notation $\mathbf{W} \coloneqq \mathbf{BR} + \mathbf{Y} + \mathbf{Z} + \chi\big(\nabla\varphi_{\text{cyl}}(\zeta) + \mathbf{V}_0\big)$. With this notation in place, utilizing (2.3.16), we can write U along the interface as

$$U(\alpha) = \mathbf{W}(\alpha)\cdot\hat{\mathbf{n}}(\alpha). \tag{2.3.24}$$

We shall write $U = U_0 + U_1 + U_2 + \chi U_3$, where

$$U_0 \coloneqq \mathbf{BR}\cdot\hat{\mathbf{n}},\ U_1 \coloneqq \mathbf{Y}\cdot\hat{\mathbf{n}},\ U_2 \coloneqq \mathbf{Z}\cdot\hat{\mathbf{n}},\ U_3 \coloneqq \nabla\varphi_{\text{cyl}}(\zeta)\cdot\hat{\mathbf{n}}\mathbf{V}_0\cdot\hat{\mathbf{n}}.$$

Given the singular nature of $\mathbf{BR}$, it will be useful to decompose it, as well as $\mathbf{BR}_\alpha$, into a singular term and a smooth remainder. To this end, we shall utilize the following decompositions, proofs of which can be found in [Amb1]:

$$\mathfrak{C}(\mathbf{BR})^* = \frac{1}{2i}\mathcal{H}\left(\frac{\gamma}{\zeta_\alpha}\right) + K[\zeta]\gamma, \tag{2.3.25}$$

$$\mathbf{BR}_\alpha = \frac{1}{2s_\alpha}\mathcal{H}(\gamma_\alpha)\hat{\mathbf{n}} - \frac{1}{2s_\alpha}\mathcal{H}(\gamma\theta_\alpha)\hat{\mathbf{t}} + \mathbf{m}, \tag{2.3.26}$$

where $K[\cdot]$ is a smoothing operator (see Lemma 3.5 in [Amb1] or Lemma 2.4.5 below) given by

$$K[\zeta]f(\alpha) \coloneqq \frac{1}{4\pi i}\int_{b-\pi}^{b+\pi} f(\alpha')\left[\cot\frac{1}{2}(\zeta_d(\alpha)-\zeta_d(\alpha')) - \frac{1}{\zeta_\alpha(\alpha')}\cot\frac{1}{2}(\alpha-\alpha')\right]d\alpha', \tag{2.3.27}$$

where b can be any real number. On the other hand, $\mathbf{m}$ is given by

$$\mathfrak{C}(\mathbf{m})^* \coloneqq \frac{\zeta_\alpha}{2i}\left[\mathcal{H},\zeta_\alpha^{-2}\right]\left(\gamma_\alpha - \frac{\gamma\zeta_{\alpha\alpha}}{\zeta_\alpha}\right) + \zeta_\alpha K[\zeta]\left(\frac{\gamma_\alpha}{\zeta_\alpha} - \frac{\gamma\zeta_{\alpha\alpha}}{\zeta_\alpha^2}\right). \tag{2.3.28}$$

So, the singular parts of $\mathbf{BR}$ and $\mathbf{BR}_\alpha$ are given by Hilbert transforms, while the smooth part of $\mathfrak{C}(\mathbf{BR})^*$ is

given by $K[\zeta]\gamma$ and the smooth part of $\mathbf{BR}_\alpha$ is given by $\mathbf{m}$. We shall occasionally write $\mathbf{m} = \mathbf{B} + \mathbf{R}$, where $\mathbf{B}$ is the commutator term and $\mathbf{R}$ is the term involving the operator $K[\zeta]$.

The terms in $\mathbf{m}$ arise upon approximating $\zeta(\alpha) - \zeta(\alpha')$ to first order via Taylor expansion and then rewriting the remainder. The reader may turn to [Amb1] for all of the details. The singular nature of $\mathbf{BR}$, as opposed to the other terms in $\mathbf{W}$, means that it will, at times, be useful to distinguish it from the remaining terms. To do so, we write $\mathbf{W} = \mathbf{BR} + \widetilde{\mathbf{W}}$.

A quantity which shall appear frequently in the work to come is $\partial_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}})$, which results from the choice of $V \neq \mathbf{W}\cdot\hat{\mathbf{t}}$. Using the geometric identity $\hat{\mathbf{t}}_\alpha = \theta_\alpha\hat{\mathbf{n}}$, we can formulate a convenient expression for $\partial_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}})$:

$$\partial_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}}) = \theta_\alpha U + s_{\alpha t} - \mathbf{W}_\alpha\cdot\hat{\mathbf{t}} - \mathbf{W}\cdot(\theta_\alpha\hat{\mathbf{n}}) = s_{\alpha t} - \mathbf{W}_\alpha\cdot\hat{\mathbf{t}}. \tag{2.3.29}$$

We can obtain a useful representation of $\mathbf{BR}_\alpha\cdot\hat{\mathbf{t}}$ via equation (2.3.26). The remaining terms in $\mathbf{W}_\alpha$ are regular and so are quite a bit simpler to grasp. We can simply compute them as follows, integrating by parts to retain the cotangent kernel:

$$\partial_\alpha\mathfrak{C}(\mathbf{Y})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi}\partial_{\alpha'}\left(\frac{\omega(\alpha')s_{1,\alpha}(\alpha')\zeta_\alpha(\alpha)}{\zeta_{1,\alpha}(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha) - \zeta_1(\alpha'))\,d\alpha', \tag{2.3.30}$$

$$\partial_\alpha\mathfrak{C}(\mathbf{Z})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi}\partial_{\alpha'}\left(\frac{\beta(\alpha')s_{2,\alpha}(\alpha')\zeta_\alpha(\alpha)}{\zeta_{2,\alpha}(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha) - \zeta_2(\alpha'))\,d\alpha', \tag{2.3.31}$$

$$\partial_\alpha(\nabla\varphi_{\text{cyl}}(\zeta(\alpha))) = \nabla^2\varphi_{\text{cyl}}(\zeta(\alpha))\zeta_\alpha(\alpha), \tag{2.3.32}$$

where $\nabla^2\varphi_{\text{cyl}}$ denotes the Hessian of φ_{cyl}.

2.3.3 Evolution Equations

Recall that, following the approach in [AmbEtAl], the variables which we shall evolve are θ, γ, ω and β. Notice that we do not explicitly evolve s_α or L. This will cause no trouble as after solving for the given variables, we can obtain U and then easily solve for s_α and/or L. Here we wish to write out the system of evolution equations for θ, γ, ω and β. Derivations of the evolution equations can be found in [AmbEtAl].

Utilizing the definition of θ, we can easily see that

$$\theta_t = \frac{U_\alpha + \theta_\alpha V}{s_\alpha}. \tag{2.3.33}$$

Using (2.3.26), we can rewrite (2.3.33) as

$$\theta_t = \frac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \frac{\theta_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \frac{1}{s_\alpha}\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}} + \frac{\mathbf{m}\cdot\hat{\mathbf{n}}}{s_\alpha}. \tag{2.3.34}$$

Recall that γ is the vortex sheet strength and related to the velocity potential at the free surface by $\gamma := \mu_\alpha$, where μ, on the other hand, is the density of the double layer potential at the free surface. Hence, via standard layer potential theory (e.g., see [Fol]), we know that μ represents the jump in φ_0 across the free boundary. The derivation of the evolution equation for γ is substantially more involved than that for θ. Roughly, one begins from (2.3.20) and rearranges to obtain an expression for γ, which is then differentiated with respect to time. One rewrites the resulting expression using the Bernoulli equation and then uses the Laplace-Young condition on the pressure at the interface. This is where we can exploit the highly simplified expression for the curvature of the free surface in (2.3.12). This process yields the following evolution equation for γ:

$$\gamma_t = \partial_\alpha\left(\frac{2\tau}{s_\alpha}\theta_\alpha + \frac{1}{s_\alpha}(V - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma - \frac{\gamma^2}{4s_\alpha^2} - 2g\eta\right) - 2s_\alpha\mathbf{W}_t\cdot\hat{\mathbf{t}} + 2(V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}). \tag{2.3.35}$$

We can rewrite equation (2.3.35) by expanding the derivative and applying (2.3.29). We then have

$$\begin{aligned}\gamma_t = {} & \frac{2\tau}{s_\alpha}\theta_{\alpha\alpha} + \frac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) + \frac{\gamma_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \frac{\gamma}{s_\alpha}\Big(s_{\alpha t} - \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}} - \mathbf{m}\cdot\hat{\mathbf{t}}\Big) \\ & - 2s_\alpha\mathbf{W}_t\cdot\hat{\mathbf{t}} - \frac{\gamma\gamma_\alpha}{2s_\alpha^2} - 2g\eta_\alpha + 2\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big)\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}.\end{aligned} \tag{2.3.36}$$

Observe that the γ_t equation is nonlocal; in particular, it is an integro-differential equation due to the presence of $\mathbf{W}_t\cdot\hat{\mathbf{t}}$.

Finally, we turn our attention to the evolution equations for ω and β. Recall that ω is the density of the layer potential on the bottom and β is the density of the layer potential on the obstacle. In order to write the evolution equations (and later equations) more compactly, we introduce some notation for the integral kernels. These integral kernels, as well as the evolution equations for ω and β, arise from enforcing the

homogeneous Neumann boundary conditions on the solid boundaries. On the free surface, we have

$$
\begin{aligned}
k^1_{\mathcal{S}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{1}{2s_{1,\alpha}(\alpha)}\zeta_{1,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_1(\alpha)-\zeta(\alpha'))\right\},\\
k^2_{\mathcal{S}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{1}{2s_{2,\alpha}(\alpha)}\zeta_{2,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_2(\alpha)-\zeta(\alpha'))\right\}.
\end{aligned}
\tag{2.3.37}
$$

Notice that the integral kernels in (2.3.37) are time-dependent. The kernels on the bottom are

$$
\begin{aligned}
k^1_{\mathcal{B}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{is_{1,\alpha}(\alpha')}{2s_{1,\alpha}(\alpha)}\zeta_{1,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_1(\alpha)-\zeta_1(\alpha'))\right\},\\
k^2_{\mathcal{B}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{is_{1,\alpha}(\alpha')}{2s_{2,\alpha}(\alpha)}\zeta_{2,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_2(\alpha)-\zeta_1(\alpha'))\right\}.
\end{aligned}
\tag{2.3.38}
$$

Finally, the kernels on the boundary of the obstacle are given by

$$
\begin{aligned}
k^1_{\mathcal{C}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{is_{2,\alpha}(\alpha')}{2s_{1,\alpha}(\alpha)}\zeta_{1,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_1(\alpha)-\zeta_2(\alpha'))\right\},\\
k^2_{\mathcal{C}}(\alpha,\alpha') &= \Re\mathrm{e}\left\{\frac{is_{2,\alpha}(\alpha')}{2s_{2,\alpha}(\alpha)}\zeta_{2,\alpha}(\alpha)\cot\frac{1}{2}(\zeta_2(\alpha)-\zeta_2(\alpha'))\right\}.
\end{aligned}
\tag{2.3.39}
$$

Notice that, at first appearance, it seems that the kernels $k^1_{\mathcal{B}}$ and $k^2_{\mathcal{C}}$ are also singular. However, they are in fact not singular (see [AmbEtAl] for details). We also note that the kernels in (2.3.38) and (2.3.39) are independent of time.

Utilizing this notation, the evolution equations for ω and β are given by

$$
\begin{aligned}
\left(\frac{1}{2}\omega_t(\alpha)+\frac{1}{2\pi}\int_0^{2\pi}\omega_t(\alpha')k^1_{\mathcal{B}}(\alpha,\alpha')\,d\alpha'\right) &= -\frac{1}{2\pi}\int_0^{2\pi}\gamma(\alpha')k^1_{\mathcal{S},t}(\alpha,\alpha')\,d\alpha' - \frac{1}{2\pi}\int_0^{2\pi}\gamma_t(\alpha')k^1_{\mathcal{S}}(\alpha,\alpha')\,d\alpha'\\
&\quad - \frac{1}{2\pi}\int_0^{2\pi}\beta_t(\alpha')k^1_{\mathcal{C}}(\alpha,\alpha')\,d\alpha'
\end{aligned}
\tag{2.3.40}
$$

and

$$
\begin{aligned}
\left(\frac{1}{2}\beta_t(\alpha)+\frac{1}{2\pi}\int_0^{2\pi}\beta_t(\alpha')k^2_{\mathcal{C}}(\alpha,\alpha')\,d\alpha'\right) &= -\frac{1}{2\pi}\int_0^{2\pi}\gamma(\alpha')k^2_{\mathcal{S},t}(\alpha,\alpha')\,d\alpha' - \frac{1}{2\pi}\int_0^{2\pi}\gamma_t(\alpha')k^2_{\mathcal{S}}(\alpha,\alpha')\,d\alpha'\\
&\quad - \frac{1}{2\pi}\int_0^{2\pi}\omega_t(\alpha')k^2_{\mathcal{B}}(\alpha,\alpha')\,d\alpha'.
\end{aligned}
\tag{2.3.41}
$$

The equations for ω_t and β_t are integral equations and so, like the evolution equation for γ, are nonlocal.

Combining (2.3.34), (2.3.36), (2.3.40) and (2.3.41), we have the full water waves system which we shall study:

$$\begin{cases} \theta_t = \dfrac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \dfrac{\theta_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \dfrac{1}{s_\alpha}\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}} + \dfrac{\mathbf{m}\cdot\hat{\mathbf{n}}}{s_\alpha} \\ \gamma_t = \dfrac{2\tau}{s_\alpha}\theta_{\alpha\alpha} + \dfrac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) + \dfrac{\gamma_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \dfrac{\gamma}{s_\alpha}\Big(s_{\alpha t} - \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}} - \mathbf{m}\cdot\hat{\mathbf{t}}\Big) \\ \qquad -2s_\alpha\mathbf{W}_t\cdot\hat{\mathbf{t}} - \dfrac{\gamma\gamma_\alpha}{2s_\alpha^2} - 2g\eta_\alpha + 2\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big)\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} \\ \omega_t = -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\omega_t(\alpha')k^1_{\mathcal{B}}(\cdot,\alpha')\,d\alpha' - \dfrac{1}{\pi}\int_0^{2\pi}\gamma(\alpha')k^1_{\mathcal{S},t}(\cdot,\alpha')\,d\alpha' \\ \qquad -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\gamma_t(\alpha')k^1_{\mathcal{S}}(\cdot,\alpha')\,d\alpha' - \dfrac{1}{\pi}\int_0^{2\pi}\beta_t(\alpha')k^1_{\mathcal{C}}(\cdot,\alpha')\,d\alpha' \\ \beta_t = -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\beta_t(\alpha')k^2_{\mathcal{C}}(\cdot,\alpha')\,d\alpha' - \dfrac{1}{\pi}\int_0^{2\pi}\gamma(\alpha')k^2_{\mathcal{S},t}(\cdot,\alpha')\,d\alpha' \\ \qquad -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\gamma_t(\alpha')k^2_{\mathcal{S}}(\cdot,\alpha')\,d\alpha' - \dfrac{1}{\pi}\int_0^{2\pi}\omega_t(\alpha')k^2_{\mathcal{B}}(\cdot,\alpha')\,d\alpha' \\ \theta(t=0) = \theta_0,\ \gamma(t=0) = \gamma_0,\ \omega(t=0) = \omega_0,\ \beta(t=0) = \beta_0 \end{cases} . \tag{2.3.42}$$

Remark 2.3.1. *1. Compare the integral kernels given above in equations* (2.3.37)-(2.3.39) *with the K_{kj} and G_{kj} in Table 1 in [AmbEtAl]. Note that there are superficial differences between the kernels we use and the kernels in [AmbEtAl] due to a minor difference of how the arclength terms $s_{k,\alpha}$ are handled, but they are otherwise the same.*

2. The equations in (2.3.42) *correspond to the the first equation in (2.10), equation (4.14) and the system (4.17) with $N = 2$ in [AmbEtAl]. The equation we utilize for γ_t in* (2.3.42) *more closely corresponds to the evolution equation obtained in Appendix D of [AmbEtAl].*

Remark 2.3.2. *As noted above, the evolution equations for γ, ω and β are nonlocal. In fact, we can now clearly see that the system* (2.3.42) *is of the form* (2.1.1)*. We shall refer to $\mathfrak{F}(\Theta)$ as the right-hand side of the system and write $\mathfrak{F} = (\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4)^t$. Since $(\mathrm{id} + \mathscr{K})$ is invertible (see [AmbEtAl] or Section 10 below), we have*

$$\partial_t\Theta = (\mathrm{id} + \mathscr{K}[\Theta])^{-1}\mathfrak{F}(\Theta).$$

This motivates the plan of attack outlined earlier:

1. Obtain energy estimates for the model problem (2.1.4).

2. *Use mapping properties of* $(\mathrm{id} + \mathscr{K}[\cdot])^{-1}$ *to conclude that the estimates still hold for the full system* (2.3.42).

2.4 The Right-Hand Side $\mathfrak{F}$

In order to carry out the strategy outlined in Remark 2.3.2, we will need to determine the correct model problem, which necessitates determining which terms belong to the right-hand side $\mathfrak{F}(\Theta)$, and write down the model problem (2.1.4) in a way that is amenable to carrying out the needed energy estimates. This will involve exploiting some subtle cancellation. We begin by decomposing the system (2.3.42) into terms that belong to $\mathscr{K}[\Theta]\Theta_t$ (i.e., those that involve a nonlocal operator acting on γ_t, ω_t or β_t; no equation involves nonlocal operators acting on θ_t) and those that belong in the right-hand side $\mathfrak{F}(\Theta)$ (all other terms). Noting that the evolution equation for θ contains no nonlocal terms, we write

$$\gamma_t = F_\gamma + N_\gamma,$$
$$\omega_t = F_\omega + N_\omega,$$
$$\beta_t = F_\beta + N_\beta,$$

where the F terms belong to the right-hand side and the N terms arise from $\mathscr{K}[\Theta]$ being applied to Θ_t. This can be done immediately in the case of the ω_t equation and the β_t equation. In particular, we have

$$F_\omega = -\frac{1}{\pi}\int_0^{2\pi} \gamma(\alpha')k^1_{S,t}(\alpha,\alpha')\,d\alpha', \tag{2.4.1}$$
$$F_\beta = -\frac{1}{\pi}\int_0^{2\pi} \gamma(\alpha')k^2_{S,t}(\alpha,\alpha')\,d\alpha'. \tag{2.4.2}$$

Then, N_ω contains the remaining integrals in (2.3.40), multiplied by 2 to clear the factor of $\frac{1}{2}$ in front of ω_t, with N_β defined analogously from equation (2.3.41).

For the γ_t equation, we begin by noticing that the only terms in N_γ will arise from $\mathbf{W}_t$; in particular, only $\mathbf{BR}_t$, $\mathbf{Y}_t$ and $\mathbf{Z}_t$ will contribute terms to N_γ. As such, we will write $\mathbf{BR}_t = F_{\mathbf{BR}} + N_{\mathbf{BR}}$, $\mathbf{Y}_t = F_{\mathbf{Y}} + N_{\mathbf{Y}}$ and $\mathbf{Z}_t = F_{\mathbf{Z}} + N_{\mathbf{Z}}$. We now compute the relevant pieces of $\mathbf{W}_t$, integrating by parts to retain the cotangent

kernel:

$$\partial_t \mathfrak{C}(\mathbf{BR})^*(\alpha) = \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \gamma_t(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\,d\alpha' + \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \partial_{\alpha'}\left(\frac{\gamma(\alpha')(\zeta_t(\alpha)-\zeta_t(\alpha'))}{\zeta_\alpha(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\,d\alpha', \tag{2.4.3}$$

$$\partial_t \mathfrak{C}(\mathbf{Y})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi} \omega_t(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))\,d\alpha' + \frac{1}{4\pi}\int_0^{2\pi} \partial_{\alpha'}\left(\frac{\omega(\alpha')s_{1,\alpha}(\alpha')\zeta_t(\alpha)}{\zeta_{1,\alpha}(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))\,d\alpha', \tag{2.4.4}$$

$$\partial_t \mathfrak{C}(\mathbf{Z})^*(\alpha) = \frac{1}{4\pi}\int_0^{2\pi} \beta_t(\alpha')s_{2,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta_2(\alpha'))\,d\alpha' + \frac{1}{4\pi}\int_0^{2\pi} \partial_{\alpha'}\left(\frac{\beta(\alpha')s_{2,\alpha}(\alpha')\zeta_t(\alpha)}{\zeta_{2,\alpha}(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha)-\zeta_2(\alpha'))\,d\alpha'. \tag{2.4.5}$$

Now, we can clearly see that $\mathfrak{C}(F_{\mathbf{BR}})^*$ is the second integral in equation (2.4.3) and $\mathfrak{C}(N_{\mathbf{BR}})^*$ is the first integral. It is the same for $F_{\mathbf{Y}}$, $F_{\mathbf{Z}}$, $N_{\mathbf{Y}}$ and $N_{\mathbf{Z}}$.

2.4.1 Rewriting $F_{\mathbf{BR}}$

Given that $F_{\mathbf{BR}}$ is given by a singular integral, it will be beneficial to decompose it into smaller pieces. This decomposition will additionally give rise to the previously mentioned cancellation. We begin by using the Leibniz rule to rewrite $F_{\mathbf{BR}}$:

$$\mathfrak{C}(F_{\mathbf{BR}})^* = \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \partial_{\alpha'}\left(\frac{\gamma(\alpha')}{\zeta_\alpha(\alpha')}\right)(\zeta_t(\alpha)-\zeta_t(\alpha'))\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\,d\alpha' - \frac{1}{4\pi i}\,\mathrm{pv}\int_0^{2\pi} \frac{\gamma(\alpha')}{\zeta_\alpha(\alpha')}\zeta_{t\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta(\alpha'))\,d\alpha'.$$

We want to rewrite $\zeta_{t\alpha}$. Utilizing the identity $\zeta_\alpha = s_\alpha e^{i\theta}$ gives

$$\partial_t\zeta_\alpha = \partial_t(s_\alpha e^{i\theta}) = s_{\alpha t}e^{i\theta} + s_\alpha(i\theta_t e^{i\theta}) = \frac{s_{\alpha t}}{s_\alpha}\zeta_\alpha + i\theta_t\zeta_\alpha.$$

We now substitute equation (2.3.34) for θ_t to obtain

$$\zeta_{t\alpha} = \frac{s_{\alpha t}}{s_\alpha}\zeta_\alpha + i\zeta_\alpha\left(\frac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \frac{\theta_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \frac{1}{s_\alpha}\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}} + \frac{\mathbf{m}\cdot\hat{\mathbf{n}}}{s_\alpha}\right). \tag{2.4.6}$$

We can now decompose $F_{\mathbf{BR}}$ into a singular term involving the Hilbert transform and a remainder term involving a smoothing operator K. To carry this out, we make use of a similar decomposition of the

Birkhoff-Rott integral given above in (2.3.25). Decomposing $F_{\mathbf{BR}}$ similarly yields

$$\begin{aligned}\mathfrak{C}(F_{\mathbf{BR}})^* = {}& [\zeta_t, \mathcal{H}]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) + [\zeta_t, K[\zeta]]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) \\ & - \frac{1}{2i}\mathcal{H}\left(\frac{\zeta_{t\alpha}}{\zeta_\alpha}\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta]\left(\zeta_{t\alpha}\left(\frac{\gamma}{\zeta_\alpha}\right)\right). \end{aligned} \tag{2.4.7}$$

We will then substitute in equation (2.4.6). After substituting, we will factor some of the terms out of the Hilbert transform, thus picking up some commutators, exploit the identity $\mathcal{H}^2 = -\,\mathrm{id}$ and do a bit of rearranging. The result of these operations is

$$\begin{aligned}\mathfrak{C}(F_{\mathbf{BR}})^* = {}& [\zeta_t, \mathcal{H}]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) + [\zeta_t, K[\zeta]]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - \frac{s_{\alpha t}}{2is_\alpha}\mathcal{H}\left(\frac{\gamma}{\zeta_\alpha}\right) \\ & - \frac{s_{\alpha t}}{s_\alpha}K[\zeta]\gamma + \frac{\gamma\gamma_\alpha}{4s_\alpha^2\zeta_\alpha} - \frac{1}{4s_\alpha^2}\left[\mathcal{H}, \frac{\gamma}{\zeta_\alpha}\right](\mathcal{H}(\gamma_\alpha)) - \frac{i}{2s_\alpha^2}K[\zeta](\gamma\mathcal{H}(\gamma_\alpha)) \\ & - \frac{1}{2s_\alpha}\mathcal{H}\left(\frac{\gamma\mathbf{m}\cdot\hat{\mathbf{n}}}{\zeta_\alpha}\right) - \frac{i}{s_\alpha}K[\zeta](\gamma\mathbf{m}\cdot\hat{\mathbf{n}}) - \frac{V - \mathbf{W}\cdot\hat{\mathbf{t}}}{2s_\alpha\zeta_\alpha}\mathcal{H}(\gamma\theta_\alpha) \\ & - \frac{1}{2s_\alpha}\left[\mathcal{H}, \frac{V - \mathbf{W}\cdot\hat{\mathbf{t}}}{\zeta_\alpha}\right](\gamma\theta_\alpha) - \frac{i}{s_\alpha}K[\zeta](\gamma\theta_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}})) - \frac{1}{2s_\alpha}\mathcal{H}\left(\frac{\gamma\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}}{\zeta_\alpha}\right) \\ & - \frac{i}{s_\alpha}K[\zeta](\gamma\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}). \end{aligned} \tag{2.4.8}$$

This is the decomposed version of $F_{\mathbf{BR}}$ which we shall use. We can now see the cancellation that will occur between $F_{\mathbf{BR}}$ and $(V - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha\cdot\hat{\mathbf{t}}$.

2.4.2 Obtaining the Cancellation

To obtain the desired cancellation, we begin by considering

$$\begin{aligned}(V - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} &= (V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{BR}_\alpha\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}) \\ &= (V - \mathbf{W}\cdot\hat{\mathbf{t}})\left(-\frac{1}{2s_\alpha}\mathcal{H}(\gamma\theta_\alpha) + \mathbf{m}\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}\right). \end{aligned}$$

We therefore have

$$\begin{aligned}2(V - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} - 2s_\alpha F_{\mathbf{BR}}\cdot\hat{\mathbf{t}} &= -\frac{V - \mathbf{W}\cdot\hat{\mathbf{t}}}{s_\alpha}\mathcal{H}(\gamma\theta_\alpha) + \frac{V - \mathbf{W}\cdot\hat{\mathbf{t}}}{s_\alpha}\mathcal{H}(\gamma\theta_\alpha) \\ &\quad + 2(V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{m}\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}) - 2s_\alpha\mathbf{br}_0\cdot\hat{\mathbf{t}} \\ &= 2(V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{m}\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}) - 2s_\alpha\mathbf{br}_0\cdot\hat{\mathbf{t}}, \end{aligned}$$

where

$$\mathfrak{C}(\mathbf{br}_0)^* := \mathfrak{C}(F_{\mathbf{BR}})^* + \frac{V - \mathbf{W}\cdot\hat{\mathbf{t}}}{2s_\alpha\zeta_\alpha}\mathcal{H}(\gamma\theta_\alpha).$$

Most of the terms in $\mathbf{br}_0$ shall be routine to estimate, however we do have one transport term which we wish to isolate. As such, we write

$$\mathfrak{C}(\mathbf{br}_0)^* = \frac{\gamma\gamma_\alpha}{4s_\alpha^2\zeta_\alpha} + \mathfrak{C}(\mathbf{br}_1)^*,$$

which implies that

$$2s_\alpha\mathbf{br}_0\cdot\hat{\mathbf{t}} = \frac{\gamma\gamma_\alpha}{2s_\alpha^2} + 2\,\mathfrak{Re}\left\{\mathfrak{C}(\mathbf{br}_1)^*\zeta_\alpha\right\}.$$

This prepares us to write down the right-hand side of the γ_t equation (those terms belonging to $\mathfrak{F}_2$):

$$\begin{aligned}\mathfrak{F}_2(\Theta) = {} & \frac{2\tau}{s_\alpha}\theta_{\alpha\alpha} + \frac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) + \frac{\gamma_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) - \frac{\gamma\gamma_\alpha}{s_\alpha^2} \\ & + \frac{\gamma}{s_\alpha}\Big(s_{\alpha t} - \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}} - \mathbf{m}\cdot\hat{\mathbf{t}}\Big) - 2g\eta_\alpha + 2\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big)(m\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}) \\ & - 2s_\alpha\big[\mathbf{br}_1 + F_{\mathbf{Y}} + F_{\mathbf{Z}} + \chi\partial_t(\nabla\varphi_{\text{cyl}}(\zeta))\big]\cdot\hat{\mathbf{t}}. \end{aligned} \tag{2.4.9}$$

2.4.3 Writing Down the System $\Theta_t = \mathfrak{F}(\Theta)$

As previously noted, we will first consider the model problem (2.1.4). In (2.1.4), the right-hand side $\mathfrak{F}(\Theta)$ is given by

$$\begin{aligned}\mathfrak{F}_1(\Theta) &= \frac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \frac{\theta_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) + \frac{1}{s_\alpha}\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}} + \frac{\mathbf{m}\cdot\hat{\mathbf{n}}}{s_\alpha} \\ \mathfrak{F}_2(\Theta) &= \frac{2\tau}{s_\alpha}\theta_{\alpha\alpha} + \frac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) + \frac{\gamma_\alpha}{s_\alpha}\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big) - \frac{\gamma\gamma_\alpha}{s_\alpha^2} \\ &\quad + \frac{\gamma}{s_\alpha}\Big(s_{\alpha t} - \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}} - \mathbf{m}\cdot\hat{\mathbf{t}}\Big) - 2g\eta_\alpha + 2\big(V - \mathbf{W}\cdot\hat{\mathbf{t}}\big)(\mathbf{m}\cdot\hat{\mathbf{t}} + \widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}) \\ &\quad - 2s_\alpha\big[\mathbf{br}_1 + F_{\mathbf{Y}} + F_{\mathbf{Z}} + \chi\partial_t(\nabla\varphi_{\text{cyl}}(\zeta))\big]\cdot\hat{\mathbf{t}} \\ \mathfrak{F}_3(\Theta) &= -\frac{1}{\pi}\int_0^{2\pi}\gamma(\alpha')k^1_{\mathcal{S},t}(\alpha,\alpha')\,d\alpha' \\ \mathfrak{F}_4(\Theta) &= -\frac{1}{\pi}\int_0^{2\pi}\gamma(\alpha')k^2_{\mathcal{S},t}(\alpha,\alpha')\,d\alpha'. \end{aligned} \tag{2.4.10}$$

Remark 2.4.1. *Though simpler than* (2.3.42)*, the system* (2.1.4) *is still a rather complicated, quasilinear system. In order to handle this, we will utilize an approach which is quite common in the study of quasilinear*

hyperbolic equations. Namely, we will first work with a regularized version of our system and then pass to the limit as the regularization parameter $\delta \to 0^+$ to solve the non-regularized system. The regularization scheme that we shall use is much like the one used in [Amb1] and the interested reader can consult this paper for further details. Also, Section 16.1 of [Tay3] has a good presentation of this approach to quasilinear hyperbolic equations, including a detailed example of its application to a quasilinear, symmetric hyperbolic system. (see also [MaBe] for applications to equations arising in fluid dynamics).

2.4.4 The Regularized Evolution Equations for the System (2.1.4)

Now, we want to obtain an appropriately regularized version of the system (2.1.4). We begin by simply writing down the regularized evolution equations, and then we will go back to briefly discuss how the regularized terms are constructed. Beginning with θ, we have

$$\theta_t^\delta = \frac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}(\mathcal{J}_\delta\,\gamma_\alpha^\delta) + \frac{1}{s_\alpha^\delta}\,\mathcal{J}_\delta\big(\big(V^\delta - \mathbf{W}^\delta\cdot\hat{\mathbf{t}}^\delta\big)\,\mathcal{J}_\delta\,\theta_\alpha^\delta\big) + \frac{1}{s_\alpha^\delta}\widetilde{\mathbf{W}}_\alpha^\delta\cdot\hat{\mathbf{n}}^\delta + \frac{\mathbf{m}^\delta\cdot\hat{\mathbf{n}}^\delta}{s_\alpha^\delta} + \mu^\delta. \tag{2.4.11}$$

Notice that there is no term corresponding to μ^δ in the non-regularized equation. Its purpose is to enforce the condition that $\zeta^\delta(\alpha) - \alpha$ be 2π-periodic and it is given by

$$\mu^\delta(t) := -\frac{\displaystyle\int_0^{2\pi} s_{\alpha t}^\delta\zeta_\alpha^\delta + iU_\alpha^\delta\zeta_\alpha^\delta + V^\delta\zeta_{\alpha\alpha}^\delta\,d\alpha}{\displaystyle is_\alpha^\delta\int_0^{2\pi}\zeta_\alpha^\delta\,d\alpha}. \tag{2.4.12}$$

See [Amb1] for the derivation of μ^δ and the proof that it enforces the aforementioned periodicity condition. The same calculations and arguments work in the present setting with the only difference being the terms contained in U.

We now turn to the γ_t equation:

$$\gamma_t^\delta = \frac{2\tau}{s_\alpha^\delta}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta + \frac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}((\gamma^\delta)^2\,\mathcal{J}_\delta\,\theta_\alpha^\delta) + \frac{1}{s_\alpha^\delta}\,\mathcal{J}_\delta\big(\big(V^\delta - \mathbf{W}^\delta\cdot\hat{\mathbf{t}}^\delta\big)\,\mathcal{J}_\delta\,\gamma_\alpha^\delta\big) - \frac{\mathcal{J}_\delta(\gamma^\delta\,\mathcal{J}_\delta\,\gamma_\alpha^\delta)}{(s_\alpha^\delta)^2} + m_\gamma^\delta. \tag{2.4.13}$$

The term m_γ^δ is primarily a remainder term, but it does contain one term not appearing in the non-regularized system. Notice that in the regularized evolution equation for γ we have pulled a factor of γ^δ through the

Hilbert transform. The cost of doing so is a (smooth) commutator which we also include in m_γ^δ. We thus have

$$m_\gamma^\delta = \frac{\gamma^\delta}{s_\alpha^\delta}\Big(s_{\alpha t}^\delta - \widetilde{\mathbf{W}}_\alpha^\delta \cdot \hat{\mathbf{t}}^\delta - \mathbf{m}^\delta \cdot \hat{\mathbf{t}}^\delta\Big) - 2g\eta_\alpha^\delta + 2\,\mathcal{J}_\delta\Big(\big(V^\delta - \mathbf{W}^\delta \cdot \hat{\mathbf{t}}^\delta\big)\,\mathcal{J}_\delta(\mathbf{m}^\delta \cdot \hat{\mathbf{t}}^\delta + \widetilde{\mathbf{W}}_\alpha^\delta \cdot \hat{\mathbf{t}}^\delta)\Big)$$
$$- 2s_\alpha^\delta\,\mathcal{J}_\delta\big(\big[\mathbf{br}_1^\delta + F_{\mathbf{Y}}^\delta + F_{\mathbf{Z}}^\delta + \chi\partial_t(\nabla\varphi_{\text{cyl}}(\zeta^\delta))\big]\cdot\hat{\mathbf{t}}^\delta\big) - \big[\,\mathcal{H}, \gamma^\delta\big]\Big(\frac{\gamma^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta}{2(s_\alpha^\delta)^2}\Big). \tag{2.4.14}$$

For ω and β, we have

$$\omega_t^\delta = -\frac{1}{\pi}\int_0^{2\pi} \gamma^\delta(\alpha')k_{\mathcal{S},t}^{1,\delta}(\alpha,\alpha')\,d\alpha' \tag{2.4.15}$$

and

$$\beta_t^\delta = -\frac{1}{\pi}\int_0^{2\pi} \gamma^\delta(\alpha')k_{\mathcal{S},t}^{2,\delta}(\alpha,\alpha')\,d\alpha'. \tag{2.4.16}$$

The regularized system we consider is then

$$\begin{cases} \theta_t^\delta = \dfrac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}(\mathcal{J}_\delta\,\gamma_\alpha^\delta) + \dfrac{1}{s_\alpha^\delta}\,\mathcal{J}_\delta\big(\big(V^\delta - \mathbf{W}^\delta\cdot\hat{\mathbf{t}}^\delta\big)\,\mathcal{J}_\delta\,\theta_\alpha^\delta\big) + \dfrac{1}{s_\alpha^\delta}\widetilde{\mathbf{W}}_\alpha^\delta\cdot\hat{\mathbf{n}}^\delta + \dfrac{\mathbf{m}^\delta\cdot\hat{\mathbf{n}}^\delta}{s_\alpha^\delta} + \mu^\delta \\ \gamma_t^\delta = \dfrac{2\tau}{s_\alpha^\delta}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta + \dfrac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}((\gamma^\delta)^2\,\mathcal{J}_\delta\,\theta_\alpha^\delta) + \dfrac{1}{s_\alpha^\delta}\,\mathcal{J}_\delta\big(\big(V^\delta - \mathbf{W}^\delta\cdot\hat{\mathbf{t}}^\delta\big)\,\mathcal{J}_\delta\,\gamma_\alpha^\delta\big) - \dfrac{\mathcal{J}_\delta(\gamma^\delta\,\mathcal{J}_\delta\,\gamma_\alpha^\delta)}{(s_\alpha^\delta)^2} + m_\gamma^\delta \\ \omega_t^\delta = -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\gamma^\delta(\alpha')k_{\mathcal{S},t}^{1,\delta}(\alpha,\alpha')\,d\alpha' \\ \beta_t^\delta = -\dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\gamma^\delta(\alpha')k_{\mathcal{S},t}^{2,\delta}(\alpha,\alpha')\,d\alpha' \\ \theta^\delta(t=0) = \theta_0,\ \gamma^\delta(t=0) = \gamma_0,\ \omega^\delta(t=0) = \omega_0,\ \beta^\delta(t=0) = \beta_0 \end{cases}. \tag{2.4.17}$$

We shall now succinctly describe the various terms appearing in the regularized equations, beginning with the family of mollifiers $\mathcal{J}_\delta$. For each $\delta \in (0,1]$, we have a corresponding operator $\mathcal{J}_\delta$, which is an approximation of the identity. We conceptualize the operator $\mathcal{J}_\delta$ as truncating the Fourier series via zeroing out modes with wavenumber greater than δ^{-1}. There are other ways to conceptualize these mollifiers, for example one might also conceptualize $\mathcal{J}_\delta$ as convolution with an approximation of the Dirac mass depending on the parameter δ. Notice that, given our conceptualization of $\mathcal{J}_\delta$, each $\mathcal{J}_\delta$ will be a Fourier multiplier and so will commute with other Fourier multipliers. In addition, $\mathcal{J}_\delta$ will be self-adjoint. We shall now state two results regarding the action of $\mathcal{J}_\delta$ on Sobolev spaces H^r. The first is

Lemma 2.4.2. *Let $\delta \in (0,1]$ and $u \in H^r$ for some $r \in \mathbb{R}$. Then, for any $k \in \mathbb{N}_0$, we have $\mathcal{J}_\delta u \in H^{r+k}$ with*

$$\|\mathcal{J}_\delta u\|_{H^{r+k}} \lesssim \delta^{-k}\|u\|_{H^r}.$$

Proof. This is part of the content of Lemma 3.5 in [MaBe], which is proved in the appendix for Chapter 3 in the same. □

Lemma 2.4.2 tells us a couple of interesting properties of the mollifiers $\mathcal{J}_\delta$. First, by taking $k = 0$, we see that $\mathcal{J}_\delta$ is a bounded (and hence continuous) linear operator on H^r for any $r \in \mathbb{R}$. The second property is that we can exchange derivatives of $\mathcal{J}_\delta u$ for powers of δ^{-1}. This is, in fact, a Bernstein-type result in disguise. Indeed, recall that we are conceptualizing the action of $\mathcal{J}_\delta$ as truncating the Fourier series by zeroing out modes with frequencies greater than δ^{-1}. Put another way, we have

$$\operatorname{supp}\mathcal{F}(\mathcal{J}_\delta u) \subset \left\{|\xi| \leqslant \delta^{-1}\right\}.$$

A Bernstein-type lemma (similar to Lemma 1.3.3, but not focused on dyadic frequencies) will then give

$$\left\|\partial_\alpha^k \mathcal{J}_\delta u\right\|_{L^2} \lesssim \delta^{-k}\|\mathcal{J}_\delta u\|_{L^2} \lesssim \delta^{-k}\|u\|_{L^2},$$

where the last inequality follows from the first property we noted (specifically, $\mathcal{J}_\delta$ is a bounded operator on L^2 which we identify with H^0).

The plan of attack outlined in Remark 2.4.1 necessitates taking the limit as $\delta \to 0^+$ of the sequence of regularized solutions. Thus, we will need to ensure that the sequence of solutions converges and proving this will require estimating terms of the form $\mathcal{J}_\delta u - \mathcal{J}_{\tilde{\delta}} u$ in norm. The next result allows us to do this:

Lemma 2.4.3. *For $u \in H^1$ and $\delta,\ \tilde{\delta} \in (0,1]$,*

$$\|\mathcal{J}_\delta u - \mathcal{J}_{\tilde{\delta}} u\|_{L^2} \leqslant \max(\delta, \tilde{\delta})\|u\|_{H^1}.$$

Proof. This is Lemma 2.2 of [Amb1]. See Lemma 3.5 (and its proof in the appendix) in [MaBe]. □

Consider a sequence $\delta_k \to 0^+$. Notice that Lemma 2.4.3 implies that $\{\mathcal{J}_{\delta_k} u\}_{k=1}^{+\infty}$ is a Cauchy sequence (in L^2) as soon as $u \in H^1$.

Most of the nuance in defining the regularized terms lies in constructing ζ^δ and $\mathbf{BR}^\delta$. We shall define ζ^δ and $\mathbf{BR}^\delta$ exactly as in [Amb1] and the interested reader can find all of the details in that paper. The remaining regularized terms are defined in the same way as the non-regularized ones with ζ, $\mathbf{BR}$, γ, etc. replaced with ζ^δ, $\mathbf{BR}^\delta$, γ^δ, etc. For example, $\mathfrak{C}(\hat{\mathbf{n}}^\delta) := \frac{i\zeta^\delta_\alpha}{s^\delta_\alpha}$, where $s^\delta_\alpha := \left|\zeta^\delta_\alpha\right|$, Θ^δ solves (2.4.17) and

$$\mathfrak{C}(\mathbf{Y}^\delta)^*(\alpha) := \frac{1}{4\pi}\int_0^{2\pi} \omega^\delta(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta^\delta(\alpha)-\zeta_1(\alpha'))\,d\alpha'.$$

We now state some useful results regarding the term ζ_d and the operator K used in the decomposition (2.3.25).

Lemma 2.4.4. *Let $r \geqslant 0$. If $\theta \in H^r$, then $\zeta_d \in H^{r+1}$ with the estimate*

$$\|\zeta_d\|_{H^{r+1}} \lesssim 1 + \|\theta\|_{H^r}. \tag{2.4.18}$$

Proof. We define ζ exactly the same as z in [Amb1]. Ergo, the desired estimate follows directly from Lemma 3.2 in [Amb1] . □

We include the following two results regarding mapping properties of K which will be of use to us.

Lemma 2.4.5. *If $\zeta_d \in H^{r+1}$, $r \in \mathbb{Z}$ with $r \geqslant 3$, then $K[\zeta] : H^j \to H^{r+j-1}$, for $j \in \{1, 0, -1\}$, with the estimate*

$$\|K[\zeta]f\|_{H^{r+j-1}} \lesssim \|f\|_{H^j}(1+\|\theta\|_{H^r})^3. \tag{2.4.19}$$

Proof. We shall show that $K[\zeta] : H^{-1} \to H^{r-2}$ with the corresponding estimate; the proofs of the other claims are contained in Lemma 3.5 of [Amb1]. In proving this mapping property, we follow the proof given in [Amb1]. We begin by writing $K = K_1 + K_2$, where

$$K_1[\zeta]f(\alpha) = \frac{1}{2\pi i}\int_0^{2\pi} f(\alpha')\left[\frac{1}{\zeta_d(\alpha)-\zeta_d(\alpha')} - \frac{1}{\zeta_\alpha(\alpha')(\alpha-\alpha')}\right]d\alpha', \tag{2.4.20}$$

$$K_2[\zeta]f(\alpha) = \frac{1}{4\pi i}\int_{\alpha-\pi}^{\alpha+\pi} f(\alpha')\left[g\left(\frac{1}{2}(\zeta_d(\alpha)-\zeta_d(\alpha'))\right) - \frac{1}{\zeta_\alpha(\alpha')}g\left(\frac{1}{2}(\alpha-\alpha')\right)\right]d\alpha'. \tag{2.4.21}$$

In the above definition, g is a function, holomorphic at the origin, such that

$$\cot z = \frac{1}{z} + g(z).$$

Notice that the choice of limits of integration in the definition of K_2 allows us to integrate over one period while avoiding the poles of g, which by definition must be the non-zero integer multiples of 2π – this choice of limits of integration will force $\left|\alpha - \alpha'\right| \leqslant \pi$.

First, consider

$$\partial_\alpha^{r-2} K_1[\zeta] f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} f(\alpha') \partial_\alpha^{r-2} \left[\frac{1}{\zeta_d(\alpha) - \zeta_d(\alpha')} - \frac{1}{\zeta_\alpha(\alpha')(\alpha - \alpha')} \right] d\alpha'.$$

We then apply one of the $r-2$ derivatives to the quantity inside the brackets:

$$\partial_\alpha^{r-2} K_1[\zeta] f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} f(\alpha') \partial_\alpha^{r-3} \left[-\frac{\zeta_\alpha(\alpha)}{(\zeta_d(\alpha) - \zeta_d(\alpha'))^2} + \frac{1}{\zeta_\alpha(\alpha')(\alpha - \alpha')^2} \right] d\alpha'.$$

By rearranging the factors of ζ_α, we can write the quantity in brackets as a derivative with respect to α':

$$\partial_\alpha^{r-2} K_1[\zeta] f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\alpha')}{\zeta_\alpha(\alpha')} \partial_\alpha^{r-3} \partial_{\alpha'} \left[\frac{1}{\alpha - \alpha'} - \frac{\zeta_\alpha(\alpha)}{\zeta_d(\alpha) - \zeta_d(\alpha')} \right] d\alpha'.$$

Then, by integrating by parts and recognizing the quantity in brackets as a ratio of divided differences, we can rewrite this expression to obtain

$$\partial_\alpha^{r-2} K_1[\zeta] f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} \partial_{\alpha'}^{-1} \left(\frac{f(\alpha')}{\zeta_\alpha(\alpha')} \right) \partial_\alpha^{r-3} \partial_{\alpha'}^2 \left[\frac{q_2(\alpha, \alpha')}{q_1(\alpha, \alpha')} \right] d\alpha'.$$

We introduced above some notation used in [Amb1]:

$$q_1(\alpha, \alpha') := \frac{\zeta_d(\alpha) - \zeta_d(\alpha')}{\alpha - \alpha'}, \; q_2(\alpha, \alpha') := \frac{\zeta_d(\alpha) - \zeta_d(\alpha') - \zeta_\alpha(\alpha)(\alpha - \alpha')}{(\alpha - \alpha')^2}. \tag{2.4.22}$$

From here, we deduce the immediate bound

$$\left| \partial_\alpha^{r-2} K_1[\zeta] f(\alpha) \right| \lesssim \left\| \frac{f}{\zeta_\alpha} \right\|_{H^{-1}} \left\| \frac{q_2}{q_1} \right\|_{H^{r-1}}.$$

In particular, notice that since $\frac{q_2}{q_1}$ is in H^{r-1}, in both variables (see Lemma 3.4 in [Amb1]), we know that $\frac{q_2}{q_1}$ will be in $W_\alpha^{r-3,\infty}$ and $H_{\alpha'}^2$. Lemma 1.3.9 and the Sobolev algebra property then imply that

$$\left| \partial_\alpha^{r-2} K_1[\zeta] f(\alpha) \right| \lesssim \|f\|_{H^{-1}} \left\| \frac{1}{\zeta_\alpha} \right\|_{H^{1+}} \|q_2\|_{H^{r-1}} \left\| \frac{1}{q_1} \right\|_{H^{r-1}}.$$

Finally, we can apply Lemma 1.3.7 in conjunction with Lemma 3.4 from [Amb1] (an estimate on the H^r norms of the divided differences q_1 and q_2) to deduce that

$$\|K_1[\zeta]f\|_{H^{r-2}} \lesssim \|f\|_{H^{-1}}(1 + \|\theta\|_{H^r})^3. \tag{2.4.23}$$

A similar modification of the argument in [Amb1] implies that

$$\|K_2[\zeta]f\|_{H^{r-2}} \lesssim \|f\|_{H^{-1}}(1 + \|\theta\|_{H^r})^2. \tag{2.4.24}$$

Combining (2.4.23) and (2.4.24) gives the desired result. □

Lemma 2.4.6. *If $\theta, \tilde{\theta} \in H^1$, and the associated $\zeta, \tilde{\zeta}$ satisfy equations* (2.3.2)*,* (2.5.6) *and* (2.5.7)*, then we have the following Lipschitz estimate for K:*

$$\left\|K[\zeta]f - K[\tilde{\zeta}]f\right\|_{H^1} \lesssim \|f\|_{H^1}\left\|\theta - \tilde{\theta}\right\|_{H^1}. \tag{2.4.25}$$

Proof. See Lemma 3.6 in [Amb1]. □

As noted earlier, the above regularization scheme is common in studying quasilinear PDE. The usual plan of attack in using such a scheme is to prove that solutions to the regularized equations exist and that those solutions satisfy an appropriate uniform (in δ) energy estimate. The energy estimate allows one to deduce a common existence time (independent of δ) for the regularized solutions. Then, one can show that the limit as $\delta \to 0^+$ of the regularized solutions exists and satisfies the non-regularized system. Carrying out the above plan will be the focus of the next two sections. We will begin by defining a suitable energy and then establishing the uniform energy estimate.

2.5 The Energy Estimate

Now that we have the appropriate evolution equations, as well as the above preliminary remarks and results under our belts, we shall begin the process of proving the first main result. The results in the next two sections are, unless otherwise noted, all concerning the regularized equations. For the sake of the reader, we shall, for the most part, drop the δ notation in the regularized equations. The reader should presume all quantities are regularized in the manner discussed above unless and until otherwise stated.

A quantity which shall be of fundamental importance to the analysis in the sequel is the energy for a

solution $(\theta, \gamma, \omega, \beta)$.

Definition 2.5.1. *Inspired by [Amb1], we define the energy of a solution to the regularized system as follows*

$$\mathcal{E}(t) = \sum_{j=0}^{s+1} \mathcal{E}^j(t), \tag{2.5.1}$$

where

$$\mathcal{E}^0 = \frac{1}{2}\Big(\|\theta\|_{L^2}^2 + \|\gamma\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\beta\|_{L^2}^2\Big), \tag{2.5.2}$$

$$\mathcal{E}^1 = \frac{1}{2}\Big(\|\partial_\alpha\omega\|_{L^2}^2 + \|\partial_\alpha\beta\|_{L^2}^2\Big), \tag{2.5.3}$$

$$\mathcal{E}^j = \frac{1}{2}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)^2 + \frac{1}{4\tau s_\alpha}(\partial_\alpha^{j-2}\gamma)\Lambda(\partial_\alpha^{j-2}\gamma) + \frac{\gamma^2}{16\tau^2 s_\alpha^2}(\partial_\alpha^{j-2}\gamma)^2\, d\alpha \qquad (2 \leqslant j \leqslant s+1). \tag{2.5.4}$$

We define $\Lambda \coloneqq \mathcal{H}\,\partial_\alpha$ and note that Λ is a Fourier multiplier: $\Lambda = |D|$. We will write $\mathcal{E}^j = \mathcal{E}_1^j + \mathcal{E}_2^j + \mathcal{E}_3^j$.

We note that in [Amb1], the coefficient of surface tension appeared in the energy implicitly via the Weber number:

$$\mathrm{We} = \frac{\rho_1 + \rho_2}{2\tau}.$$

In our case (i.e., the case of water waves where we have normalized to have unit density in the fluid), we have $\mathrm{We} = \frac{1}{2\tau}$.

Definition 2.5.2. *For $\mathcal{E}$ as above, we have*

$$\mathcal{E}(t) \sim \|\theta(t)\|_{H^s}^2 + \|\gamma(t)\|_{H^{s-1/2}}^2 + \|\omega(t)\|_{H^1}^2 + \|\beta(t)\|_{H^1}^2 = \|\Theta(t)\|_{H^s\times H^{s-1/2}\times H^1\times H^1}^2. \tag{2.5.5}$$

We therefore define the energy space to be $X \coloneqq H^s \times H^{s-1/2} \times H^1 \times H^1$. We shall let $\mathfrak{X}$ denote the subset of X where five conditions are satisfied:

- *the chord-arc condition* (2.3.2) *holds:*

$$\exists \mathfrak{c} > 0 : \left|\frac{\zeta(\alpha) - \zeta(\alpha')}{\alpha - \alpha'}\right| > \mathfrak{c} \qquad (\forall \alpha \neq \alpha');$$

- *the non-cavitation assumptions* (2.3.3) *and* (2.3.4) *hold:*

$$\eta - \eta_1 \geqslant \mathfrak{h} \text{ and } \eta - \eta_2 \geqslant \tilde{\mathfrak{h}};$$

- *we have*

$$s_\alpha \geqslant 1, \tag{2.5.6}$$

  with equality holding in the case $\theta = 0$*;*

- *and we have the following uniform-in-time bound on the energy:*

$$\exists 0 < \mathfrak{e} < +\infty : \ \mathcal{E} < \mathfrak{e}. \tag{2.5.7}$$

  Notice that $\mathfrak{X}$ *will depend upon the constants* $\mathfrak{c}$, $\mathfrak{h}$, $\tilde{\mathfrak{h}}$ *and* $\mathfrak{e}$ *chosen.*

Henceforth, we shall for the most part restrict our attention to $\mathfrak{X}$ *as this is where we shall seek solutions.*

Remark 2.5.3. *We shall assume throughout that* s *is sufficiently large for all computations to make sense; we are not seeking sharp regularity results. Here we simply remark that we shall at least require that* $s > \frac{3}{2}$*. Notice then that, by Lemma 1.3.2,* $H^{s-1/2} \hookrightarrow L^\infty$*, and therefore* $\Theta \in (L^\infty)^4$*. Lemma 2.4.4 implies that* $\zeta_d \in H^{s+1}$*. We will further have* $\psi := \varphi|_{\mathcal{S}_t} \in H^{s+1/2}$*, therefore* $\varphi \in H^{s+1}$ *and* $\mathbf{v} = \nabla\varphi \in H^s$*. It follows, again from Lemma 1.3.2, that* $\zeta, \mathbf{v} \in \mathrm{Lip}$*. This is in line with the standard regularity requirements for proving local well-posedness by energy methods (see, e.g., [ABZ1]). Further, the definition of* s_α*, the definition of the energy and the bound on the energy in* (2.5.7) *imply that* $s_\alpha \in L^\infty$*. Of course, this implies that* $L \in L^\infty$ *as well.*

Definition 2.5.2 implies that, for $\Theta \in \mathfrak{X}$, we have $\|\Theta\|_X \lesssim 1$. Further, by Remark 2.5.3, we also have $\|s_\alpha\|_{L^\infty}, \|L\|_{L^\infty} \lesssim 1$. Before proceeding to the main energy estimate, we begin by obtaining some a priori estimates for some important quantities appearing in our evolution equations. These estimates will be used repeatedly in the sequel when proving the main energy estimate.

Lemma 2.5.4. *The following estimates hold for s sufficiently large:*

$$\|\mathbf{BR}\|_{L^2} \lesssim \sqrt{\mathcal{E}} + \mathcal{E}^2, \tag{2.5.8}$$

$$\|\mathbf{Y}\|_{L^2} \lesssim \sqrt{\mathcal{E}}, \tag{2.5.9}$$

$$\|\mathbf{Z}\|_{L^2} \lesssim \sqrt{\mathcal{E}}, \tag{2.5.10}$$

$$\|\mathbf{Y}\|_{H^{s+1}} \lesssim \sqrt{\mathcal{E}} + \mathcal{E}, \tag{2.5.11}$$

$$\|\mathbf{Z}\|_{H^{s+1}} \lesssim \sqrt{\mathcal{E}} + \mathcal{E}, \tag{2.5.12}$$

$$\left\|\nabla\varphi_{\text{cyl}}(\zeta)\right\|_{H^{s+1}} \lesssim 1 + \sqrt{\mathcal{E}}. \tag{2.5.13}$$

These estimates hold for both the regularized and non-regularized terms.

Proof. We use the representation (2.3.25) and Lemma 2.4.5 to estimate

$$\|\mathbf{BR}\|_{L^2} \lesssim \|\gamma\|_{L^2}\left\|\frac{1}{\zeta_\alpha}\right\|_{L^\infty} + \|\gamma\|_1(1 + \|\theta\|_{H^s})^3.$$

It then follows that

$$\|\mathbf{BR}\|_{L^2} \lesssim \|\gamma\|_{H^{s-1/2}}(1 + \|\theta\|_{H^s})^3. \tag{2.5.14}$$

To estimate the norm of $\mathbf{Y}$, consider

$$\left|\mathfrak{C}(\mathbf{Y})^*(\alpha)\right| \lesssim \int_0^{2\pi}\left|\omega(\alpha')\right|\left|s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha) - \zeta_1(\alpha'))\right| d\alpha' \lesssim \|\omega\|_{L^2}.$$

This implies the estimate (2.5.9). Next, we consider

$$\begin{aligned}\left|\partial_\alpha^{s+1}\mathfrak{C}(\mathbf{Y})^*(\alpha)\right| &\leqslant \frac{1}{4\pi}\int_0^{2\pi}\left|\omega(\alpha')s_{1,\alpha}(\alpha')\right|\left|\partial_\alpha^{s+1}\cot\frac{1}{2}(\zeta(\alpha) - \zeta_1(\alpha'))\right| d\alpha' \\ &\lesssim \|\omega\|_{L^2}\|\zeta_d\|_{H^{s+1}}.\end{aligned}$$

It then follows from Lemma 2.4.4 that

$$\|\mathbf{Y}\|_{H^{s+1}} \lesssim \|\omega\|_{L^2} + \|\omega\|_{L^2}\|\zeta_d\|_{H^{s+1}} \lesssim \|\omega\|_{H^1}(1 + \|\theta\|_{H^s}).$$

The proofs of (2.5.10) and (2.5.12) are nearly identical to that of (2.5.9) and (2.5.11). Next, recalling

the definition of φ_{cyl} in (2.3.14), it is easy to see that $\nabla\varphi_{\text{cyl}}$ is a smooth function and so we can apply Lemma 1.3.7 to obtain

$$\left\|\nabla\varphi_{\text{cyl}}(\zeta)\right\|_{H^{s+1}} \lesssim 1 + \|\zeta_d\|_{H^{s+1}} \lesssim 1 + \|\theta\|_{H^s}. \tag{2.5.15}$$

□

Lemma 2.5.5. *For s sufficiently large, we can control the H^s norms of the unit vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ (both regularized and non-regularized) in $\mathfrak{X}$ where we have the following estimates:*

$$\|\hat{\mathbf{n}}\|_{H^s} \lesssim 1 + \sqrt{\mathcal{E}}. \tag{2.5.16}$$

$$\left\|\hat{\mathbf{t}}\right\|_{H^s} \lesssim 1 + \sqrt{\mathcal{E}}, \tag{2.5.17}$$

Proof. We shall only prove the estimate for $\hat{\mathbf{n}}$ as the argument for $\hat{\mathbf{t}}$ is totally analogous. Upon writing $\mathfrak{C}(\hat{\mathbf{n}}) = \frac{i\zeta_\alpha}{s_\alpha}$, Lemma 2.4.4 gives

$$\|\hat{\mathbf{n}}\|_{H^s} \lesssim \|\zeta_\alpha\|_{H^s} \leqslant \|\zeta_d\|_{H^{s+1}} \lesssim 1 + \|\theta\|_{H^s}.$$

□

Lemma 2.5.6. *Let $s \in \mathbb{R}$ be sufficiently large. Then, on $\mathfrak{X}$, we can bound s_α above and below by*

$$1 \leqslant s_\alpha \lesssim 1 + \sqrt{\mathfrak{e}}. \tag{2.5.18}$$

This estimate holds for the non-regularized s_α and the regularized s_α^δ.

Proof. The lower bound is simply equation (2.5.6) in the definition of $\mathfrak{X}$. To obtain the upper bound, we can apply the definition of s_α, Lemma 2.4.4 and Lemma 1.3.2. In particular, these results together imply that

$$s_\alpha \leqslant \|\zeta_\alpha\|_{L^\infty} \lesssim \|\zeta_\alpha\|_{H^{1/2+}} \leqslant \|\zeta_d\|_{H^{s+1}} \lesssim 1 + \|\theta\|_{H^s} \lesssim 1 + \sqrt{\mathcal{E}} < 1 + \sqrt{\mathfrak{e}}.$$

□

Lemma 2.5.7. *For s sufficiently large and $(\theta,\gamma,\omega,\beta)\in\mathfrak{X}$, the following estimates hold:*

$$|s_{\alpha t}| \lesssim \mathcal{E}+\mathcal{E}^3+\chi(1+|V_0|)(\sqrt{\mathcal{E}}+\mathcal{E}^{\frac{3}{2}}), \tag{2.5.19}$$

$$\|\mathbf{m}\cdot\hat{\mathbf{t}}\|_{H^s} \lesssim \sqrt{\mathcal{E}}+\mathcal{E}^{\frac{9}{2}}, \tag{2.5.20}$$

$$\|V\|_{L^2} \lesssim \mathcal{E}+\mathcal{E}^3+\chi(1+|V_0|)(\sqrt{\mathcal{E}}+\mathcal{E}^{\frac{3}{2}}), \tag{2.5.21}$$

$$\|\partial_\alpha(V-\mathbf{W}\cdot\hat{\mathbf{t}})\|_{H^{s-1}} \lesssim \sqrt{\mathcal{E}}+\mathcal{E}^{\frac{9}{2}}+\chi(1+|V_0|)(1+\mathcal{E}^{\frac{3}{2}}), \tag{2.5.22}$$

$$|\mu| \lesssim \sqrt{\mathcal{E}}+\mathcal{E}^{\frac{9}{2}}+\chi(1+|V_0|)(1+\mathcal{E}^2). \tag{2.5.23}$$

The estimate for $\|\mathbf{m}\cdot\hat{\mathbf{n}}\|_{H^s}$ is the same as the estimate given above for $\|\mathbf{m}\cdot\hat{\mathbf{t}}\|_{H^s}$. Finally, we remark that all of these estimates hold for the regularized and non-regularized terms.

Proof. We have $|L_t|\leqslant\|\theta\|_{H^1}\|U\|_{L^2}$. An application of Lemma 2.5.4 yields the desired result.

We recall that $\mathbf{m}$ is composed of two types of terms, a commutator and an integral remainder (see (2.3.28)). Beginning with the commutator, we use Lemma 1.3.13 to control the H^s norm:

$$\|\mathbf{B}\cdot\hat{\mathbf{t}}\|_{H^s} \lesssim \|\zeta_\alpha\|_{H^s}^2\|\zeta_\alpha^{-2}\|_{H^s}\left\|\gamma_\alpha-\frac{\gamma\zeta_{\alpha\alpha}}{\zeta_\alpha}\right\|_{H^{s-2}}$$

Observing that, $\zeta_{\alpha\alpha}=\partial_\alpha(s_\alpha e^{i\theta})=\theta_\alpha\zeta_\alpha$, we use the Sobolev algebra property and Lemma 1.3.7 to deduce that

$$\|\mathbf{B}\cdot\hat{\mathbf{t}}\|_{H^s} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^6. \tag{2.5.24}$$

On the other hand, we can use Lemma 2.4.5 to estimate the H^s norm of $\mathbf{R}\cdot\hat{\mathbf{t}}$:

$$\|\mathbf{R}\cdot\hat{\mathbf{t}}\|_{H^s} \lesssim \|\zeta_\alpha\|_{H^s}^2\left(\left\|\frac{\gamma_\alpha}{\zeta_\alpha}\right\|_{H^1}+\left\|\frac{\gamma\zeta_{\alpha\alpha}}{\zeta_\alpha^2}\right\|_{H^1}\right)(1+\|\theta\|_{H^s})^3.$$

The Sobolev algebra property and the identity $\zeta_{\alpha\alpha}=\theta_\alpha\zeta_\alpha$ imply that

$$\|\mathbf{R}\cdot\hat{\mathbf{t}}\|_{H^s} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^8. \tag{2.5.25}$$

Adding (2.5.24) and (2.5.25) gives the desired estimate for $\|\mathbf{m}\cdot\hat{\mathbf{t}}\|_{H^s}$.

Moving on, we immediately see that

$$\|V\|_{L^2} = \left\|\partial_\alpha^{-1}(\theta_\alpha U + s_{\alpha t})\right\|_{L^2} \sim \|\theta_\alpha U + s_{\alpha t}\|_{\dot{H}^{-1}} \leqslant \|\theta_\alpha U\|_{L^2} + |s_{\alpha t}|.$$

Recalling that $|s_{\alpha t}| \leqslant \|\theta\|_{H^1}\|U\|_{L^2}$, we deduce from Lemma 1.3.9 that

$$\|V\|_{L^2} \lesssim \|\theta\|_{H^{3/2+}}\|U\|_{L^2}.$$

From here, Lemma 2.5.4 gives the stated estimate for $\|V\|_{L^2}$. Next, recalling equations (2.3.29) and (2.3.26), we have

$$\left\|\partial_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}})\right\|_{H^{s-1}} \lesssim |s_{\alpha t}| + \|\mathcal{H}(\gamma\theta_\alpha)\|_{H^{s-1}} + \left\|\mathbf{m}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1}} + \left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}\right\|_{H^{s-1}}.$$

Lemma 2.5.4 allows us to estimate the final term. We can dispose of the Hilbert transform term by applying Lemma 1.3.11 and the Sobolev algebra property. Controlling $|s_{\alpha t}|$ and $\left\|\mathbf{m}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1}}$ as in equations (2.5.19) and (2.5.20) then gives (2.5.22).

Now, all that is left is to control $|\mu|$. Just as in [Amb1], we can use the chord-arc condition (2.3.2) to bound the denominator from below:

$$\left|is_\alpha \int_0^{2\pi} \zeta_\alpha \, d\alpha\right| \geqslant |s_\alpha|\mathfrak{c} \geqslant \mathfrak{c}. \tag{2.5.26}$$

The estimate on the first term in the numerator is likewise straightforward:

$$\left|\int_0^{2\pi} s_{\alpha t}\zeta_\alpha \, d\alpha\right| \leqslant 2\pi|s_\alpha||s_{\alpha t}|. \tag{2.5.27}$$

The second term in the numerator will be a bit different. We have

$$\left|\int_0^{2\pi} iU_\alpha\zeta_\alpha \, d\alpha\right| \leqslant 2\pi|s_\alpha|\|U_\alpha\|_{L^2}. \tag{2.5.28}$$

We begin by computing U_α:

$$U_\alpha = \mathbf{BR}_\alpha \cdot \hat{\mathbf{n}} - \theta_\alpha \mathbf{BR} \cdot \hat{\mathbf{t}} + \mathbf{Y}_\alpha \cdot \hat{\mathbf{n}} - \theta_\alpha \mathbf{Y} \cdot \hat{\mathbf{t}} + \mathbf{Z}_\alpha \cdot \hat{\mathbf{n}} - \theta_\alpha \mathbf{Z} \cdot \hat{\mathbf{t}}$$
$$+ \chi(-\theta_\alpha \mathbf{V}_0 \cdot \hat{\mathbf{t}} + \partial_\alpha(\nabla\varphi_{\text{cyl}}(\zeta)) \cdot \hat{\mathbf{n}} - \theta_\alpha \nabla\varphi_{\text{cyl}}(\zeta) \cdot \hat{\mathbf{t}}). \tag{2.5.29}$$

Therefore, applying Lemma 1.3.9, we estimate

$$\|U_\alpha\|_{L^2} \leqslant \|\mathbf{BR}_\alpha \cdot \hat{\mathbf{n}}\|_{L^2} + \|\theta\|_{H^s}\|\mathbf{BR} \cdot \hat{\mathbf{t}}\|_{L^2} + \|\mathbf{Y}_\alpha \cdot \hat{\mathbf{n}}\|_{L^2} + \|\theta\|_{H^s}\|\mathbf{Y} \cdot \hat{\mathbf{t}}\|_{L^2} + \|\mathbf{Z}_\alpha \cdot \hat{\mathbf{n}}\|_{L^2} + \|\theta\|_{H^s}\|\mathbf{Z} \cdot \hat{\mathbf{t}}\|_{L^2}$$
$$+ \chi(|V_0|\|\theta\|_{H^s}\|\hat{\mathbf{t}}\|_{L^2} + \|\partial_\alpha(\nabla\varphi_{\text{cyl}}(\zeta)) \cdot \hat{\mathbf{n}}\|_{L^2} + \|\theta\|_{H^s}\|\nabla\varphi_{\text{cyl}}(\zeta) \cdot \hat{\mathbf{t}}\|_{L^2}).$$

We can control the L^2 norm of $\hat{\mathbf{t}}$ using Lemma 2.5.5. Then, we can apply 2.5.4 and equation (2.3.26) yielding

$$\|U_\alpha\|_{L^2} \lesssim \|\mathcal{H}(\gamma_\alpha)\|_{L^2} + \|\mathbf{m} \cdot \hat{\mathbf{n}}\|_{L^2} + \|\theta\|_{H^s}\|\gamma\|_{H^{s-1/2}}(1 + \|\theta\|_{H^s})^4 + \|\omega\|_{H^1}(1 + \|\theta\|_{H^s})^2$$
$$+ \|\theta\|_{H^s}\|\omega\|_{H^1}(1 + \|\theta\|_{H^s}) + \|\beta\|_{H^1}(1 + \|\theta\|_{H^s})^2 + \|\theta\|_{H^s}\|\beta\|_{H^1}(1 + \|\theta\|_{H^s})$$
$$+ \chi(|V_0|\|\theta\|_{H^s}(1 + \|\theta\|_{H^s}) + (1 + \|\theta\|_{H^s})^2 + \|\theta\|_{H^s}(1 + \|\theta\|_{H^s})^2).$$

Using Lemma 1.3.10 as well as the bound on the H^s norm of $\mathbf{m} \cdot \hat{\mathbf{n}}$ and rearranging a bit gives

$$\|U_\alpha\|_{L^2} \lesssim (1 + \|\theta\|_{H^s})^2\big[\|\gamma\|_{H^{s-1/2}}(1 + \|\theta\|_{H^s})^6 + \|\omega\|_{H^1} + \|\beta\|_{H^1} + \chi(1 + |V_0|)(1 + \|\theta\|_{H^s})\big]. \tag{2.5.30}$$

At this point, we need only control the final part of the numerator. By writing

$$\zeta_\alpha = s_\alpha e^{i\theta}, \tag{2.5.31}$$

we can rewrite this term and proceed estimating:

$$\left|\int_0^{2\pi} V\theta_\alpha\zeta_\alpha \, d\alpha\right| \leqslant |s_\alpha|\|V\|_{L^2}\|\theta_\alpha\|_{L^2}. \tag{2.5.32}$$

Noting our estimate for the L^2 norm of V above completes the proof. □

Lemma 2.5.8. *The $H^{s-1/2}$ norm of $\mathbf{br}_1$ is controlled by the energy. In particular, we have*

$$\|\mathbf{br}_1\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{13} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^8. \tag{2.5.33}$$

Proof. We begin by recalling that

$$\begin{aligned}
\mathfrak{C}(\mathbf{br}_1)^* = {} & [\zeta_t, \mathcal{H}]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) + [\zeta_t, K[\zeta]]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - \frac{s_{\alpha t}}{2is_\alpha}\mathcal{H}\left(\frac{\gamma}{\zeta_\alpha}\right) \\
& - \frac{s_{\alpha t}}{s_\alpha}K[\zeta]\gamma - \frac{1}{4s_\alpha^2}\left[\mathcal{H}, \frac{\gamma}{\zeta_\alpha}\right](\mathcal{H}(\gamma_\alpha)) - \frac{i}{2s_\alpha^2}K[\zeta](\gamma\,\mathcal{H}(\gamma_\alpha)) \\
& - \frac{1}{2s_\alpha}\mathcal{H}\left(\frac{\gamma\mathbf{m}\cdot\hat{\mathbf{n}}}{\zeta_\alpha}\right) - \frac{i}{s_\alpha}K[\zeta](\gamma\mathbf{m}\cdot\hat{\mathbf{n}}) - \frac{1}{2s_\alpha}\left[\mathcal{H}, \frac{V-\mathbf{W}\cdot\hat{\mathbf{t}}}{\zeta_\alpha}\right](\gamma\theta_\alpha) \\
& - \frac{i}{s_\alpha}K[\zeta](\gamma\theta_\alpha(V-\mathbf{W}\cdot\hat{\mathbf{t}})) - \frac{1}{2s_\alpha}\mathcal{H}\left(\frac{\gamma\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}}{\zeta_\alpha}\right) - \frac{i}{s_\alpha}K[\zeta](\gamma\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}).
\end{aligned}$$

We will proceed term by term and as such write $\mathbf{br}_1 = \sum_{j=1}^{12}\mathbf{br}_{1,j}$. We begin by using Lemma 1.3.13 to obtain

$$\left\|[\mathcal{H}, \zeta_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{H^{s-1/2}} \lesssim \|\zeta_t\|_{H^{s-1/2}}\left\|\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right\|_{H^{s-2}}.$$

We observe that

$$\partial_\alpha\zeta_t = \partial_t(s_\alpha e^{i\theta}) = s_{\alpha t}e^{i\theta} + is_\alpha\theta_t e^{i\theta} = \frac{s_{\alpha t}}{s_\alpha}\zeta_\alpha + i\theta_t\zeta_\alpha. \tag{2.5.34}$$

Hence, we estimate

$$\begin{aligned}
\|\zeta_t\|_{H^{s-1/2}} &\sim \|\zeta_t\|_{L^2} + \|\partial_\alpha\zeta_t\|_{H^{s-3/2}} \\
&\lesssim \|\zeta_t\|_{L^2} + |s_{\alpha t}|\|\zeta_\alpha\|_{H^{s-3/2}} + \|\theta_t\|_{H^{s-3/2}}\|\zeta_\alpha\|_{H^{s-3/2}}.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\|\zeta_t\|_{H^{s-1/2}} \lesssim {} & \|U\|_{L^2} + \|V\|_{L^2} + |s_{\alpha t}|\|\zeta_\alpha\|_{H^{s-3/2}} \\
& + (1+\|\theta\|_{H^s})\Big(\|\mathcal{H}(\gamma_\alpha)\|_{H^{s-3/2}} + \|\mathbf{m}\cdot\hat{\mathbf{n}}\|_{H^{s-3/2}} + \|\theta_\alpha\|_{H^{s-3/2}}\|V-\mathbf{W}\cdot\hat{\mathbf{t}}\|_{H^{s-3/2}} + \left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}\right\|_{H^{s-3/2}}\Big).
\end{aligned}$$

We can now invoke Lemma 2.5.4 and Lemma 2.5.7 to conclude that

$$\|\zeta_t\|_{H^{s-1/2}} \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^9 + \chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^4. \tag{2.5.35}$$

Then for $\mathbf{br}_{1,1}$, we have

$$\|\mathbf{br}_{1,1}\|_{H^{s-1/2}} \lesssim \|\zeta_t\|_{H^{s-1/2}} \left\|\frac{1}{\zeta_\alpha}\right\|_{H^{s-2}} \left\|\frac{1}{\zeta_\alpha}\right\|_{H^{s-1}} \|\gamma\|_{H^{s-1}} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^2\|\zeta_t\|_{H^{s-1/2}},$$

which implies that

$$\|\mathbf{br}_{1,1}\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{11} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^6. \tag{2.5.36}$$

For $\mathbf{br}_{1,2}$, we begin by writing

$$\|\mathbf{br}_{1,2}\|_{H^{s-1/2}} \lesssim \left\|K[\zeta]\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{H^{s-1/2}} + \|\zeta_t\|_{H^{s-1/2}}\left\|K[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{H^{s-1/2}}.$$

We can then apply Lemmas 2.4.5 and 1.3.7 along with the Sobolev algebra property to obtain

$$\|\mathbf{br}_{1,2}\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^4\|\zeta_t\|_{H^{s-1/2}} + \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^4\|\zeta_t\|_{H^{s-1/2}}.$$

It then follows that

$$\|\mathbf{br}_{1,2}\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{13} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^8. \tag{2.5.37}$$

The Sobolev algebra property in conjunction with Lemmas 2.5.4, 2.5.7, 2.4.5, 1.3.7 imply that

$$\|\mathbf{br}_{1,3}\|_{H^{s-1/2}} \lesssim |s_{\alpha t}|\left\|\mathcal{H}\left(\frac{\gamma}{\zeta_\alpha}\right)\right\|_{H^{s-1/2}} \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^5 + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^3, \tag{2.5.38}$$

$$\|\mathbf{br}_{1,4}\|_{H^{s-1/2}} \lesssim |s_{\alpha t}|\|\gamma\|_{H^1}(1+\|\theta\|_{H^s})^3 \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^7 + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^5, \tag{2.5.39}$$

$$\|\mathbf{br}_{1,6}\|_{H^{s-1/2}} \lesssim \|\gamma\,\mathcal{H}(\gamma_\alpha)\|_{H^1}(1+\|\theta\|_{H^s})^3 \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^3, \tag{2.5.40}$$

$$\|\mathbf{br}_{1,8}\|_{H^{s-1/2}} \lesssim \|\gamma\mathbf{m}\cdot\hat{\mathbf{n}}\|_{H^1}(1+\|\theta\|_{H^s})^3 \lesssim \|\gamma\|^2_{H^{s-1/2}}(1+\|\theta\|_{H^s}) \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{11}, \tag{2.5.41}$$

$$\|\mathbf{br}_{1,10}\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^1}\|\theta_\alpha\|_{H^1}\|(V-\mathbf{W}\cdot\hat{\mathbf{t}})\|_{H^1}(1+\|\theta\|_{H^s})^3 \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^{11} + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^6, \tag{2.5.42}$$

$$\|\mathbf{br}_{1,12}\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^1}\left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}\right\|_{H^1}(1+\|\theta\|_{H^s})^3 \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^5 + \chi\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^5. \tag{2.5.43}$$

On the other hand, we can use Lemma 1.3.13 with Lemmas 1.3.7, 2.5.4 and 2.5.7 to obtain

$$\|\mathbf{br}_{1,5}\|_{H^{s-1/2}} \lesssim \left\|\frac{\gamma}{\zeta_\alpha}\right\|_{H^{s-1/2}} \|\mathcal{H}(\gamma_\alpha)\|_{H^{s-2}} \lesssim \|\gamma\|^2_{H^{s-1/2}}(1+\|\theta\|_{H^s}) \lesssim \mathcal{E}+\mathcal{E}^{\frac{3}{2}}, \tag{2.5.44}$$

$$\|\mathbf{br}_{1,9}\|_{H^{s-1/2}} \lesssim (1+\|\theta\|_{H^s})\|\theta\|_{H^s}\|\gamma\|_{H^{s-1/2}}\left\|V-\mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^9+\chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^4. \tag{2.5.45}$$

The final two estimates are rather routine. By Lemmas 2.5.4 and 2.5.7, we have

$$\|\mathbf{br}_{1,7}\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})\|\mathbf{m}\cdot\hat{\mathbf{n}}\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^9, \tag{2.5.46}$$

$$\|\mathbf{br}_{1,11}\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})\left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}\right\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^3+\chi\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^3. \tag{2.5.47}$$

Putting together the estimates (2.5.36)-(2.5.47), we deduce that (2.5.33) holds.

□

Lemma 2.5.9. *We have the estimate*

$$\|m_\gamma\|_{H^{s-1/2}} \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{17}+\chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^{12}. \tag{2.5.48}$$

Proof. We begin by breaking m_γ into smaller parts:

$$m_\gamma = m_\gamma^1+m_\gamma^2+m_\gamma^3+m_\gamma^4, \tag{2.5.49}$$

where

$$m_\gamma^1 := \frac{\gamma}{s_\alpha}\left(s_{\alpha t}-\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}-\mathbf{m}\cdot\hat{\mathbf{t}}\right),\ m_\gamma^2 := -2g\eta_\alpha+2\,\mathcal{J}_\delta((V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta(\mathbf{m}\cdot\hat{\mathbf{t}}+\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}})), \tag{2.5.50}$$

$$m_\gamma^3 := -2s_\alpha\,\mathcal{J}_\delta([\mathbf{br}_1+F_{\mathbf{Y}}+F_{\mathbf{Z}}+\chi\partial_t(\nabla\varphi_{\mathrm{cyl}}(\zeta))]\cdot\hat{\mathbf{t}}),\ m_\gamma^4 := -[\mathcal{H},\gamma]\left(\frac{\gamma\,\mathcal{J}_\delta\,\theta_\alpha}{2s_\alpha^2}\right). \tag{2.5.51}$$

$$\tag{2.5.52}$$

Beginning with m_γ^1, we have, by Lemma 1.3.9,

$$\|m_\gamma^1\|_{H^{s-1/2}} \lesssim |s_{\alpha t}|\|\gamma\|_{H^{s-1/2}}+\left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}}\|\gamma\|_{H^{s-1/2}}+\|\mathbf{m}\cdot\hat{\mathbf{t}}\|_{H^{s-1/2}}\|\gamma\|_{H^{s-1/2}}.$$

We can apply Lemma 2.5.4 and Lemma 2.5.7:

$$\left\|m_\gamma^1\right\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^8 + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^3. \tag{2.5.53}$$

Next, we consider

$$\left\|m_\gamma^2\right\|_{H^{s-1/2}} \lesssim \|\eta_\alpha\|_{H^{s-1/2}} + \left\|\mathcal{J}_\delta((V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta(\mathbf{m}\cdot\hat{\mathbf{t}}+\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}))\right\|_{H^{s-1/2}}$$

Using the fact that $\eta_\alpha = s_\alpha \sin\theta$ and the Sobolev algebra property, we obtain

$$\left\|m_\gamma^2\right\|_{H^{s-1/2}} \lesssim \|\theta\|_{H^s} + \left\|V-\mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{H^s}(\left\|\mathbf{m}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} + \left\|\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}}).$$

It then follows from Lemma 2.5.4 and Lemma 2.5.7 that

$$\left\|m_\gamma^2\right\|_{H^{s-1/2}} \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{17} + \chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^{12}. \tag{2.5.54}$$

Moving on, we next consider m_γ^3:

$$\left\|m_\gamma^3\right\|_{H^{s-1/2}} \lesssim \|\mathbf{br}_1\|_{H^{s-1/2}} + \left\|F_{\mathbf{Y}}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} + \left\|F_{\mathbf{Z}}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} + \chi\left\|\partial_t(\nabla\varphi_{\mathrm{cyl}}(\zeta))\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}}.$$

Lemma 2.5.8 gives control of the first term on the right-hand side. We recall that

$$(F_{\mathbf{Y}}\cdot\hat{\mathbf{t}})(\alpha) = \Re\mathrm{e}\left\{\frac{\zeta_\alpha(\alpha)}{4\pi s_\alpha}\int_0^{2\pi}\partial_{\alpha'}\left(\frac{\omega(\alpha')s_{1,\alpha}(\alpha')\zeta_t(\alpha)}{\zeta_{1,\alpha}(\alpha')}\right)\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))\,d\alpha'\right\}.$$

We therefore have

$$\left|(F_{\mathbf{Y}}\cdot\hat{\mathbf{t}})(\alpha)\right| \lesssim \left|\zeta_\alpha(\alpha)\right|\left|\zeta_t(\alpha)\right|\|\omega\|_{H^1}\left\|\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\cdot))\right\|_{L^2}.$$

Hence,

$$\left\|F_{\mathbf{Y}}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} \lesssim \|\zeta_\alpha\|_{H^{s-1/2}}\|\zeta_t\|_{H^{s-1/2}}\|\omega\|_{H^1}(1+\|\zeta\|_{H^{s-1/2}}) \lesssim \|\omega\|_{H^1}(1+\|\theta\|_{H^s})^2\|\zeta_t\|_{H^{s-1/2}}.$$

We can use (2.5.35) to obtain

$$\left\|F_{\mathbf{Y}} \cdot \hat{\mathbf{t}}\right\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{11} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{6}.$$

We can similarly estimate

$$\left\|F_{\mathbf{Z}} \cdot \hat{\mathbf{t}}\right\|_{H^{s-1/2}} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{11} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{6}.$$

Finally, we estimate

$$\left\|\partial_t(\nabla\varphi_{\mathrm{cyl}}(\zeta))\right\|_{H^{s-1/2}} \lesssim \|\zeta_t\|_{H^{s-1/2}}(1+\|\zeta_0\|_{H^{s-1/2}}) \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{10} + \chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^{5}.$$

We thus conclude that

$$\left\|m_\gamma^3\right\|_{H^{s-1/2}} \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{14} + \chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^{9}. \tag{2.5.55}$$

For m_γ^4, we use Lemma 1.3.13 and the Sobolev algebra property to estimate

$$\left\|m_\gamma^4\right\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}\|\gamma\,\mathcal{J}_\delta\,\theta_\alpha\|_{H^{s-2}} \lesssim \|\theta\|_{H^s}\|\gamma\|_{H^{s-1/2}}^2 \lesssim \mathcal{E}^{\frac{3}{2}}. \tag{2.5.56}$$

Upon combining estimates (2.5.53)-(2.5.56), it follows that

$$\left\|m_\gamma\right\|_{H^{s-1/2}} \lesssim \sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^{17} + \chi(1+|V_0|)(1+\sqrt{\mathcal{E}})^{12}. \tag{2.5.57}$$

□

We now arrive at the main energy estimate. Our objective shall be to show that the time derivative of $\mathcal{E}$ is controlled by a suitable polynomial in $\sqrt{\mathcal{E}}$. What will be most important is the lowest order term as this term will control the small-data lifespan. We define

$$\mathfrak{P}(\mathcal{E}) := \mathcal{E} + \mathcal{E}^N + \chi(1+|V_0|)(\sqrt{\mathcal{E}} + \mathcal{E}^M), \tag{2.5.58}$$

where $N, M \in 2^{-1}\mathbb{Z}$, $N > M$, are taken to be sufficiently large ($M, N \geqslant 11$ will work).

Theorem 2.5.10. *For s sufficiently large and for $\mathfrak{P}(\mathcal{E})$ given as above, it holds that*

$$\frac{d\mathcal{E}}{dt} \lesssim \mathfrak{P}(\mathcal{E}).$$

Proof. We begin with the $\mathcal{E}^j$'s. We first compute

$$\frac{d\mathcal{E}_1^j}{dt} = \int_0^{2\pi} (\partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-1}\theta_t)\, d\alpha.$$

Substituting the RHS of equation (2.4.11) for θ_t above, we write

$$\begin{aligned}
\frac{d\mathcal{E}_1^j}{dt} &= \frac{1}{2s_\alpha^2}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-1}\mathcal{H}(\mathcal{J}_\delta\,\gamma_\alpha))\, d\alpha + \frac{1}{s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-1}(\mathbf{m}\cdot\hat{\mathbf{n}}))\, d\alpha \\
&\quad + \frac{1}{s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)\Big(\partial_\alpha^{j-1}\,\mathcal{J}_\delta\big((V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta\,\theta_\alpha\big)\Big)\, d\alpha + \frac{1}{s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-1}(\widetilde{\mathbf{W}}_\alpha\cdot\hat{\mathbf{n}}))\, d\alpha \\
&= A_1^j + I + II + III,
\end{aligned}$$

where we have used the fact that $\partial_\alpha\mu = 0$.

In II, we want to separate out the term where all of the derivatives land on θ_α as it will require more care in analysis. To do this, we rewrite II using the Leibniz rule as follows:

$$\begin{aligned}
II &= \frac{1}{s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)\,\mathcal{J}_\delta\Big((V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta\,\partial_\alpha^j\theta\Big)\, d\alpha + \frac{1}{s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-1}\theta)\left(\sum_{\ell=1}^{j-1}\binom{j-1}{\ell}\mathcal{J}_\delta\Big(\partial_\alpha^\ell(V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta\,\partial_\alpha^{j-\ell}\theta\Big)\right) d\alpha \\
&= Z_1^j + R_1^j.
\end{aligned}$$

We have singled out two terms, namely A_1^j and Z_1^j. Consideration of A_1^j will be temporarily deferred to exploit some cancellation with terms arising in the sequel, while Z_1^j is a transport term which we will consider in short order. Before examining the transport term, we will estimate terms I, III and R_1^j.

We begin by considering an arbitrary individual summand from R_1^j, which by Hölder's inequality is bounded above by

$$\left\|\partial_\alpha^{j-1}\theta\right\|_{L^2}\left\|\mathcal{J}_\delta\Big(\partial_\alpha^\ell(V-\mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta\,\partial_\alpha^{j-\ell}\theta\Big)\right\|_{L^2}.$$

Clearly, $\left\|\partial_\alpha^{j-1}\theta\right\|_{L^2}$ is bounded by the H^s norm of θ as $j \leqslant s+1$ and so we focus on bounding the other term. We can use Lemma 2.4.2 to dispense with the outermost instance of $\mathcal{J}_\delta$, and then the Sobolev lemma in

conjunction with the Sobolev algebra property imply that

$$\left\|\partial_\alpha^\ell\big(V-\mathbf{W}\cdot\hat{\mathbf{t}}\big)\,\mathcal{J}_\delta\,\partial_\alpha^{j-\ell}\theta\right\|_{L^2}\lesssim\left\|\partial_\alpha\big(V-\mathbf{W}\cdot\hat{\mathbf{t}}\big)\right\|_{s-1}\|\mathcal{J}_\delta\,\theta\|_s$$

as $\ell\leqslant j-1\leqslant s$. Then, another application of Lemmas 2.4.2 and 2.5.7 imply that

$$R_1^j\lesssim\mathcal{E}(1+\sqrt{\mathcal{E}})^8+\chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^3\lesssim\mathfrak{P}(\mathcal{E}). \tag{2.5.59}$$

Moving on, we can utilize Hölder's inequality and Lemma 2.5.7 to estimate I, while III can be controlled using Lemma 2.5.4:

$$I+III\lesssim\mathfrak{P}(\mathcal{E}). \tag{2.5.60}$$

We now proceed to consider the transport term Z_1^j. If we rewrite Z_1^j exploiting the self-adjointness of $\mathcal{J}_\delta$, we can recognize a perfect derivative in the factors of θ and integrate by parts to obtain

$$Z_1^j=-\frac{1}{2s_\alpha}\int_0^{2\pi}(\mathcal{J}_\delta\,\partial_\alpha^{j-1}\theta)^2\partial_\alpha\big(V-\mathbf{W}\cdot\hat{\mathbf{t}}\big)\,d\alpha.$$

Then, application of Lemmas 1.3.9, 2.4.2 and 2.5.7 readily give us control of Z_1^j:

$$Z_1^j\lesssim\mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^8+\chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^3\lesssim\mathfrak{P}(\mathcal{E}). \tag{2.5.61}$$

As noted earlier, we delay estimating A_1^j and so now move on to $\mathcal{E}_2^j$. We begin by computing

$$\frac{d\mathcal{E}_2^j}{dt}=\frac{1}{4\tau s_\alpha}\int_0^{2\pi}(\partial_\alpha^{j-2}\gamma_t)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha-\frac{s_{\alpha t}}{4\tau s_\alpha^2}\int_0^{2\pi}(\partial_\alpha^{j-2}\gamma)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha.$$

As with the estimate for $\frac{d\mathcal{E}_1^j}{dt}$, we substitute the regularized evolution equation (2.4.13) for γ_t, which yields

$$\begin{aligned}
\frac{d\mathcal{E}_2^j}{dt} &= \frac{1}{2s_\alpha^2}\int_0^{2\pi}(\mathcal{J}_\delta\,\partial_\alpha^j\theta)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha + \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi}(\mathcal{H}(\gamma^2\,\mathcal{J}_\delta\,\partial_\alpha^{j-1}\theta)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&\quad + \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi}\sum_{\ell=1}^{j-2}\binom{j-2}{\ell}\mathcal{H}(\partial_\alpha^\ell(\gamma^2)\,\mathcal{J}_\delta\,\partial_\alpha^{j-\ell-1}\theta)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&\quad + \frac{1}{4\tau s_\alpha^2}\int_0^{2\pi}\mathcal{J}_\delta\Big(\big(V-\mathbf{W}\cdot\hat{\mathbf{t}}\big)\,\mathcal{J}_\delta\,\partial_\alpha^{j-1}\gamma\Big)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&\quad + \frac{1}{4\tau s_\alpha^2}\int_0^{2\pi}\sum_{\ell=1}^{j-2}\binom{j-2}{\ell}\mathcal{J}_\delta\Big(\partial_\alpha^\ell\big(V-\mathbf{W}\cdot\hat{\mathbf{t}}\big)\,\mathcal{J}_\delta\,\partial_\alpha^{j-\ell-1}\gamma\Big)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&\quad - \frac{1}{4\tau s_\alpha^3}\int_0^{2\pi}(\partial_\alpha^{j-2}\,\mathcal{J}_\delta(\gamma\,\mathcal{J}_\delta\,\gamma_\alpha))\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha + \frac{1}{4\tau s_\alpha}\int_0^{2\pi}(\partial_\alpha^{j-2}m_\gamma)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&\quad - \frac{s_{\alpha t}}{4\tau s_\alpha^2}\int_0^{2\pi}(\partial_\alpha^{j-2}\gamma)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \\
&= A_2^j + S_1^j + I + Z_2^j + II + III + IV + V.
\end{aligned}$$

First, we shall exploit the primary cancellation which we mentioned earlier. In particular, recalling that $\Lambda \coloneqq H\partial_\alpha$, we consider

$$A_1^j + A_2^j = \frac{1}{2s_\alpha^2}\int_0^{2\pi}(\partial_\alpha^{j-1}\theta)\,\mathcal{H}(\mathcal{J}_\delta\,\partial_\alpha^j\gamma)\,d\alpha + \frac{1}{2s_\alpha^2}\int_0^{2\pi}(\mathcal{J}_\delta\,\partial_\alpha^j\theta)\,\mathcal{H}(\partial_\alpha^{j-1}\gamma)\,d\alpha.$$

Noting that $\mathcal{J}_\delta$ is a self-adjoint operator which commutes with spatial differentiation and integrating by parts in the second integral, we obtain

$$A_1^j + A_2^j = \frac{1}{2s_\alpha^2}\int_0^{2\pi}(\partial_\alpha^{j-1}\theta)\,\mathcal{H}(\mathcal{J}_\delta\,\partial_\alpha^j\gamma)\,d\alpha - \frac{1}{2s_\alpha^2}\int_0^{2\pi}(\partial_\alpha^{j-1}\theta)\,\mathcal{H}(\mathcal{J}_\delta\,\partial_\alpha^j\gamma)\,d\alpha = 0. \tag{2.5.62}$$

Much like the A's, consideration of S_1^j will be delayed to exploit some secondary cancellation. We will first estimate $I - V$ and then consider the second transport term Z_2^j. In estimating these terms, we shall repeatedly encounter terms of the form $\int(\partial_\alpha^j f)\Lambda(\partial_\alpha^\ell g)\,d\alpha$. As such, it will be of use to obtain a preliminary estimate for such terms. By applying Plancherel's theorem and recalling that Λ is a Fourier multiplier, we can write

$$\int_0^{2\pi}(\partial_\alpha^j f)\Lambda(\partial_\alpha^\ell g)\,d\alpha = \sum_{k\in\mathbb{Z}}\mathcal{F}(\partial_\alpha^j f)|k|\,\mathcal{F}(\partial_\alpha^\ell g) = \sum_{k\in\mathbb{Z}}|k|^{\frac{1}{2}}\,\mathcal{F}(\partial_\alpha^j f)\cdot|k|^{\frac{1}{2}}\,\mathcal{F}(\partial_\alpha^\ell g).$$

This immediately implies the estimate

$$\int_0^{2\pi} (\partial_\alpha^k f)\Lambda(\partial_\alpha^\ell g)\, d\alpha \lesssim \left\|\partial_\alpha^j f\right\|_{H^{1/2}} \left\|\partial_\alpha^\ell g\right\|_{H^{1/2}} \leqslant \|f\|_{H^{j+1/2}} \|g\|_{H^{\ell+1/2}}. \tag{2.5.63}$$

Utilizing the estimate (2.5.63), it is straightforward to estimate

$$I + II + IV + V \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.64}$$

For III, we want to first use the Leibniz rule to isolate the term where all of the derivatives land on γ_α:

$$\begin{aligned} III &= -\frac{1}{4\tau s_\alpha^3} \int_0^{2\pi} \mathcal{J}_\delta(\gamma(\partial_\alpha^{j-2}\, \mathcal{J}_\delta\, \gamma_\alpha))\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha \\ &\quad - \frac{1}{4\tau s_\alpha^3} \sum_{\ell=1}^{j-2} \binom{j-2}{\ell} \int_0^{2\pi} \mathcal{J}_\delta((\partial_\alpha^\ell \gamma)\, \mathcal{J}_\delta(\partial_\alpha^{j-2-\ell}\gamma_\alpha))\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha \\ &= Z_3^j + R_2^j. \end{aligned} \tag{2.5.65}$$

Z_3^j is a transport term and we shall consider it alongside the other transport term Z_2^j as we treat them in very similar ways. For R_2^j, we begin by applying (2.5.63) and Lemma 2.4.2, to eliminate the outermost instance of $\mathcal{J}_\delta$, to an arbitrary summand:

$$\int_0^{2\pi} \mathcal{J}_\delta((\partial_\alpha^\ell \gamma)\, \mathcal{J}_\delta(\partial_\alpha^{j-1-\ell}\gamma))\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha \lesssim \left\|(\partial_\alpha^\ell \gamma)\, \mathcal{J}_\delta(\partial_\alpha^{j-1-\ell}\gamma)\right\|_{H^{1/2}} \left\|\partial_\alpha^{j-2}\gamma\right\|_{H^{1/2}}.$$

We want to apply Lemma 1.3.9, but we will need to be careful about which factor we place in the higher regularity space. First, recall that $1 \leqslant \ell \leqslant j-2 \leqslant s-1$. If $\ell = j-2$, then $j-1-\ell = 1$ and, upon applying Lemma 2.4.2 again, we have the estimate

$$\left\|(\partial_\alpha^\ell \gamma)\, \mathcal{J}_\delta(\partial_\alpha^{j-1-\ell}\gamma)\right\|_{H^{1/2}} \left\|\partial_\alpha^{j-2}\gamma\right\|_{H^{1/2}} \lesssim \left\|\partial_\alpha^{j-2}\gamma\right\|_{H^{1/2}} \|\partial_\alpha \gamma\|_{H^{1/2+}} \|\gamma\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}^3.$$

On the other hand, if $\ell \leqslant j-3$, we can put $\partial_\alpha^\ell \gamma$ in the higher regularity space (again we apply Lemma 2.4.2 twice):

$$\left\|(\partial_\alpha^\ell \gamma)\, \mathcal{J}_\delta(\partial_\alpha^{j-1-\ell}\gamma)\right\|_{H^{1/2}} \left\|\partial_\alpha^{j-2}\gamma\right\|_{H^{1/2}} \lesssim \|\partial_\alpha^\ell \gamma\|_{H^{1/2+}} \left\|\partial_\alpha^{j-1-\ell}\gamma\right\|_{H^{1/2}} \|\gamma\|_{H^{s-1/2}} \lesssim \|\gamma\|_{H^{s-1/2}}^3,$$

where we used the fact that $j-1-\ell \leqslant j-2 \leqslant s-1$. In either case, we have the estimate

$$R_2^j \lesssim \|\gamma\|_{H^{s-1/2}}^3 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.66}$$

We now arrive at the Z^j transport terms. We begin by considering a general integral of the form $\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha$. Recalling that $\Lambda = \mathcal{H}\,\partial_\alpha$ and that the Hilbert transform is anti-self-adjoint, we can write

$$\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha = -\int_0^{2\pi} f_\alpha\, \mathcal{H}(g f_\alpha)\, d\alpha.$$

Now, we will pull g out of the Hilbert transform and pick up a commutator:

$$\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha = -\int_0^{2\pi} g f_\alpha\, \mathcal{H}(f_\alpha)\, d\alpha - \int_0^{2\pi} f_\alpha [\mathcal{H}, g](f_\alpha).$$

Of course, $\mathcal{H}(f_\alpha) = \Lambda(f)$, so we can move the first integral over to the left-hand side and we are left with

$$2\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha = -\int_0^{2\pi} f_\alpha [\mathcal{H}, g](f_\alpha)\, d\alpha.$$

Finally, upon integrating the right-hand side by parts and dividing through by the factor of 2, we obtain

$$\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha = \frac{1}{2}\int_0^{2\pi} \partial_\alpha([\mathcal{H}, g](f_\alpha)) f\, d\alpha. \tag{2.5.67}$$

Then, Hölder's inequality and Lemma 1.3.12 imply that

$$\int_0^{2\pi} g f_\alpha \Lambda(f)\, d\alpha \leqslant \|[\mathcal{H}, g](f_\alpha)\|_{H^1} \|f\|_{L^2} \lesssim \|f_\alpha\|_{H^{-1}} \|g\|_{H^3} \|f\|_{L^2} \lesssim \|f\|_{L^2}^2 \|g\|_{H^3}. \tag{2.5.68}$$

After exploiting the symmetry of $\mathcal{J}_\delta$, Z_2^j is of this form and so we have:

$$\int_0^{2\pi} (V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathcal{J}_\delta\, \partial_\alpha^{j-2}\gamma)_\alpha \Lambda(\mathcal{J}_\delta\, \partial_\alpha^{j-2}\gamma)\, d\alpha \lesssim \left\|\partial_\alpha^{j-2}\gamma\right\|_{L^2}^2 \left\|V - \mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{H^3}.$$

Then, Lemmas 2.5.4 and 2.5.7 give

$$Z_2^j \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^8 + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^3 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.69}$$

Next, we consider Z_3^j, after rewriting by again exploiting the symmetry of $\mathcal{J}_\delta$. We again apply the estimate of equation (2.5.68) in conjunction with the fact that $\mathcal{J}_\delta$ commutes with ∂_α, as well as $\mathcal{H}$, and Lemma 2.4.2 to obtain

$$Z_3^j = \int_0^{2\pi} \gamma(\mathcal{J}_\delta\, \partial_\alpha^{j-2}\gamma)_\alpha \Lambda(\mathcal{J}_\delta\, \partial_\alpha^{j-2}\gamma)\, d\alpha \lesssim \left\|\partial_\alpha^{j-2}\gamma\right\|_{L^2}^2 \|\gamma\|_{H^3} \lesssim \|\gamma\|_{H^{s-1/2}}^3 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.70}$$

.

We continue and now compute

$$\begin{aligned}\frac{d\mathcal{E}_3^j}{dt} &= -\frac{s_{\alpha t}}{8\tau^2 s_\alpha^3}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)^2\, d\alpha + \frac{1}{16\tau^2 s_\alpha^2}\int_0^{2\pi} \gamma\gamma_t(\partial_\alpha^{j-2}\gamma)^2\, d\alpha + \frac{1}{16\tau^2 s_\alpha^2}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\gamma_t)\, d\alpha \\ &= I + II + III.\end{aligned}$$

It is obvious that

$$I + II \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.71}$$

To estimate III, we substitute in the RHS of the evolution equation for γ:

$$\begin{aligned}III &= \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha})\, d\alpha + \frac{1}{32\tau^2 s_\alpha^4}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}(\mathcal{H}(\gamma^2\,\mathcal{J}_\delta\,\theta_\alpha))\, d\alpha \\ &\quad + \frac{1}{16\tau^2 s_\alpha^3}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\,\mathcal{J}_\delta((V - \mathbf{W}\cdot\hat{\mathbf{t}})\,\mathcal{J}_\delta\,\gamma_\alpha))\, d\alpha \\ &\quad - \frac{1}{16\tau^2 s_\alpha^4}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)\partial_\alpha^{j-2}\,\mathcal{J}_\delta(\gamma\,\mathcal{J}_\delta\,\gamma_\alpha)\, d\alpha + \frac{1}{16\tau^2 s_\alpha^2}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}m_\gamma)\, d\alpha \\ &= S_2^j + C_1^j + C_2^j + C_3^j + C_4^j.\end{aligned}$$

We first examine the sum of S_1^j and S_2^j:

$$S_1^j + S_2^j = \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi} \mathcal{H}(\gamma^2\,\mathcal{J}_\delta\,\partial_\alpha^{j-1}\theta)\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha + \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha})\, d\alpha.$$

We exploit the fact that Λ is self-adjoint and that $\Lambda\,\mathcal{H} = -\partial_\alpha$ to rewrite this as

$$\frac{1}{8\tau s_\alpha^3}\int_0^{2\pi} -\partial_\alpha(\gamma^2\,\mathcal{J}_\delta\,\partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-2}\gamma)\, d\alpha + \frac{1}{8\tau s_\alpha^3}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\mathcal{J}_\delta\,\partial_\alpha^{j}\theta)\, d\alpha.$$

When we expand the derivative in the first integral, we obtain the cancellation (when the derivative lands on $\mathcal{J}_\delta\, \partial^{j-1}\theta$) and are left with

$$S_1^j + S_2^j = -\frac{1}{4\tau s_\alpha^3} \int_0^{2\pi} \gamma\gamma_\alpha (\mathcal{J}_\delta\, \partial_\alpha^{j-1}\theta)(\partial_\alpha^{j-2}\gamma)\, d\alpha. \tag{2.5.72}$$

We can then use Hölder's inequality and Lemma 1.3.9 to obtain

$$S_1^j + S_2^j \lesssim \left\| \gamma\gamma_\alpha (\partial_\alpha^{j-1}\theta) \right\|_{L^2} \left\| \partial_\alpha^{j-2}\gamma \right\|_{L^2} \lesssim \|\theta\|_{H^s} \|\gamma\|_{H^{s-1/2}}^3 \lesssim \mathcal{E}^2 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.73}$$

There are no surprises in the C^j's; we have

$$C_1^j + C_2^j + C_3^j + C_4^j \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.74}$$

Collecting these estimates, we now deduce that

$$\frac{d\mathcal{E}^j}{dt} \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.75}$$

We now proceed to examine $\mathcal{E}^1$ and begin by computing

$$\begin{aligned} \frac{d\mathcal{E}^1}{dt} &= \frac{d}{dt}\left\{ \frac{1}{2}\int_0^{2\pi} (\partial_\alpha \omega)^2\, d\alpha + \frac{1}{2}\int_0^{2\pi} (\partial_\alpha \beta)^2\, d\alpha \right\} \\ &= \int_0^{2\pi} (\partial_\alpha \omega)(\partial_\alpha \omega_t)\, d\alpha + \int_0^{2\pi} (\partial_\alpha \beta)(\partial_\alpha \beta_t)\, d\alpha \\ &= I + II. \end{aligned} \tag{2.5.76}$$

Via Hölder's inequality, we have $I \leqslant \|\omega\|_{H^1} \|\partial_\alpha \omega_t\|_{L^2}$. Given that

$$\partial_\alpha \omega_t(\alpha) = -\frac{1}{\pi}\int_0^{2\pi} \gamma(\alpha')\partial_\alpha k_{S,t}^1(\alpha,\alpha')\, d\alpha',$$

we compute

$$\partial_\alpha k^1_{\mathcal{S},t}(\alpha,\alpha') = \Re\mathrm{e}\left\{\frac{s_{1,\alpha\alpha}(\alpha)\zeta_t(\alpha')}{2s^2_{1,\alpha}(\alpha)}\partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha)-\zeta(\alpha'))\right\}$$
$$-\Re\mathrm{e}\left\{\frac{\zeta_t(\alpha')}{2s_{1,\alpha}(\alpha)}\partial^2_\alpha \cot\frac{1}{2}(\zeta_1(\alpha)-\zeta(\alpha'))\right\}.$$

Notice that $s_{1,\alpha\alpha} = \partial_\alpha\left|\zeta_{1,\alpha}\right| = \frac{1}{2s_{1,\alpha}}(\zeta_{1,\alpha\alpha}\zeta^*_{1,\alpha} + \zeta_{1,\alpha}\zeta^*_{1,\alpha\alpha})$ and so $\left|s_{1,\alpha\alpha}\right| \lesssim 1$. Again applying Hölder's inequality, we deduce that

$$\left|\partial_\alpha\omega_t(\alpha)\right| \lesssim \|\gamma\|_{L^2}\left\|\partial_\alpha k^1_{\mathcal{S},t}(\alpha,\cdot)\right\|_{L^2},$$

so the only task at hand is to control the L^2 norm of the derivative of $k^1_{\mathcal{S},t}$. From the above computation, we use Lemma 1.3.9 to estimate

$$\left\|\partial_\alpha k^1_{\mathcal{S},t}(\alpha,\cdot)\right\|_{L^2} \lesssim \|\zeta_t\|_{L^2}\left(\left\|\cot\frac{1}{2}(\zeta_1(\alpha)-\zeta(\cdot))\right\|_{H^{3/2+}} + \left\|\cot\frac{1}{2}(\zeta_1(\alpha)-\zeta(\cdot))\right\|_{H^{5/2+}}\right).$$

Lemma 1.3.7 and (2.5.35) then imply that

$$\|\partial_\alpha\omega_t\|_{L^2} \lesssim \mathcal{E}(1+\sqrt{\mathcal{E}})^{12} + \chi(1+|V_0|)\sqrt{\mathcal{E}}(1+\sqrt{\mathcal{E}})^7.$$

This implies that we have the following estimate for I:

$$I \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^{12} + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^7 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.77}$$

For the second term, we may once more apply Hölder's inequality to obtain $II \leqslant \|\beta\|_{H^1}\|\partial_\alpha\beta_t\|_{L^2}$. The estimate for $\|\partial_\alpha\beta_t\|_{L^2}$ is very similar to the estimate for $\|\partial_\alpha\omega_t\|_{L^2}$. We omit the calculations, but note that we have

$$II \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^{12} + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^7 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.78}$$

Putting together equations (2.5.77) and (2.5.78), we have the following estimate in terms of the energy for the time derivative of $\mathcal{E}^1$:

$$\frac{d\mathcal{E}^1}{dt} \lesssim \mathcal{E}^{\frac{3}{2}}(1+\sqrt{\mathcal{E}})^{12} + \chi(1+|V_0|)\mathcal{E}(1+\sqrt{\mathcal{E}})^7 \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.79}$$

We can similarly estimate

$$\frac{d\mathcal{E}^0}{dt} \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.80}$$

At last, upon combining (2.5.80), (2.5.79) and (2.5.75), we have now shown that

$$\frac{d\mathcal{E}}{dt} \lesssim \mathfrak{P}(\mathcal{E}). \tag{2.5.81}$$

□

2.6 Existence of Solutions

We continue in this section to carry out the plan sketched earlier for obtaining solutions to the non-regularized system. Having established the uniform energy estimate in the previous section, our next goal will be to show that solutions to the regularized system exist, at least for a short time.

Theorem 2.6.1. *Given initial data $\Theta_0 \in \mathfrak{X}$, there exists, for any $\delta \in (0,1]$, a unique solution $\Theta^\delta \in \mathfrak{X}$ which solves the regularized system* (2.4.17)*. Further, there exists a time $T^\delta > 0$ such that $\Theta^\delta \in C^1([0,T^\delta];\mathfrak{X})$. A priori, T^δ may depend upon the regularization parameter δ. In addition, T^δ may depend on ε, $|V_0|$, s and $\mathfrak{X}$. Notice that the solution belonging to $\mathfrak{X}$ implies that the chord-arc condition* (2.3.2)*, the non-cavitation assumptions* (2.3.3) *and* (2.3.4)*, the lower bound on the arclength metric* (2.5.6) *and the uniform energy bound* (2.5.7) *are all satisfied on $[0,T^\delta]$.*

Remark 2.6.2. *Though the existence time T^δ obtained from Theorem 2.6.1 is allowed to depend upon δ, we will prove a result in the sequel showing that there is a uniform (in δ) time interval $[0,T]$ on which solutions to the regularized system exist for any $\delta \in (0,1]$. This existence time T will, of course, still depend on ε, $|V_0|$, s and $\mathfrak{X}$.*

Proof of Theorem 2.6.1. We define $\mathfrak{F}^\delta : \mathbb{R}^4 \to \mathbb{R}^4$, $\mathfrak{F}^\delta = (\mathfrak{F}_1^\delta, \mathfrak{F}_2^\delta, \mathfrak{F}_3^\delta, \mathfrak{F}_4^\delta)$, by letting $\mathfrak{F}_1^\delta$ denote the right-hand side of (2.4.11), $\mathfrak{F}_2^\delta$ the RHS of (2.4.13), $\mathfrak{F}_3^\delta$ the RHS of (2.4.15) and $\mathfrak{F}_4^\delta$ the RHS of (2.4.16). We shall use the Picard theorem to establish the existence of solutions to the regularized equations. As such, we wish to show that $\mathfrak{F}$ satisfies a particular Lipschitz bound on $\mathfrak{X}$. In particular, given $\Theta, \Theta' \in \mathfrak{X}$, we claim that

$$\left\|\mathfrak{F}^\delta(\Theta) - \mathfrak{F}^\delta(\Theta')\right\|_X \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.1}$$

Notice that in (2.6.1) the implied constant can depend on the regularization parameter δ. This dependence

will generally be in the form of negative powers of δ (see Lemma 2.4.2). We use the triangle inequality to break the left-hand side of (2.6.1) up into smaller, more manageable, pieces. We observe that many of the terms will be the same as those in [Amb1] and so satisfy the desired estimate. In these cases, we will not reprove the estimates, but refer the reader to that work for details.

We begin with $\mathfrak{F}_1^\delta = \mathfrak{F}_{1,1}^\delta + \mathfrak{F}_{1,2}^\delta + \mathfrak{F}_{1,3}^\delta + \mathfrak{F}_{1,4}^\delta + \mathfrak{F}_{1,5}^\delta$. From Theorem 5.1 of [Amb1], we have

$$\left\|\mathfrak{F}_{1,1}^\delta(\Theta) - \mathfrak{F}_{1,1}^\delta(\Theta')\right\|_{H^s} \lesssim_\delta \|\Theta - \Theta'\|_X. \tag{2.6.2}$$

By applying Lemma 2.4.2, adding and subtracting, and utilizing the Sobolev algebra property, we can bound the $\mathfrak{F}_{1,2}^\delta$ difference by

$$C\big(\|(V - \mathbf{W}\cdot\hat{\mathbf{t}}) - (V' - \mathbf{W}'\cdot\hat{\mathbf{t}}')\|_{H^s}\|\mathcal{J}_\delta\,\theta_\alpha\|_{H^s} + \|V' - \mathbf{W}'\cdot\hat{\mathbf{t}}'\|_{H^s}\|\mathcal{J}_\delta(\theta_\alpha - \theta'_\alpha)\|_{H^s}\big).$$

The second term is straightforward; in particular, we apply Lemma 2.4.2 and the uniform energy estimates:

$$\|V' - \mathbf{W}'\cdot\hat{\mathbf{t}}'\|_{H^s}\|\mathcal{J}_\delta(\theta_\alpha - \theta'_\alpha)\|_{H^s} \lesssim \delta^{-1}\|\theta - \theta'\|_{H^s} \lesssim_\delta \|\Theta - \Theta'\|_X. \tag{2.6.3}$$

We can use the energy estimates to easily bound the first term by a constant multiple of

$$\|(V - \mathbf{W}\cdot\hat{\mathbf{t}}) - (V' - \mathbf{W}'\cdot\hat{\mathbf{t}}')\|_{L^2} + \|\partial_\alpha(V - \mathbf{W}\cdot\hat{\mathbf{t}}) - \partial_\alpha(V' - \mathbf{W}'\cdot\hat{\mathbf{t}}')\|_{H^{s-1}}.$$

For the first piece, we must estimate $\|V - V'\|_{L^2}$ and $\|\mathbf{W}\cdot\hat{\mathbf{t}} - \mathbf{W}'\cdot\hat{\mathbf{t}}'\|_{L^2}$. First, it is straightforward to see that

$$\|V - V'\|_{L^2} \lesssim \|U\|_{L^2}\|\theta_\alpha - \theta'_\alpha\|_{H^{1/2+}} + \|\theta'_\alpha\|_{H^{1/2+}}\|U - U'\|_{L^2}.$$

It is clear that the first term is controlled by $C\|\theta - \theta'\|_{H^s} \lesssim \|\Theta - \Theta'\|_X$. Hence, we need only control $\|U - U'\|_{L^2}$ by a constant multiple of $\|\Theta - \Theta'\|_X$.

By definition, we have

$$\begin{aligned}\|U - U'\|_{L^2} \leqslant{}& \|\mathbf{BR}\cdot\hat{\mathbf{n}} - \mathbf{BR}'\cdot\hat{\mathbf{n}}'\|_{L^2} + \|\mathbf{Y}\cdot\hat{\mathbf{n}} - \mathbf{Y}'\cdot\hat{\mathbf{n}}'\|_{L^2} + \|\mathbf{Z}\cdot\hat{\mathbf{n}} - \mathbf{Z}'\cdot\hat{\mathbf{n}}'\|_{L^2}\\ &+ \chi(|V_0|\|n_1 - n_1'\|_{L^2} + \|\nabla\varphi_{\text{cyl}}(\zeta)\cdot\hat{\mathbf{n}} - \nabla\varphi_{\text{cyl}}(\zeta')\cdot\hat{\mathbf{n}}'\|_{L^2}).\end{aligned}$$

That $\|\mathbf{BR}\cdot\hat{\mathbf{n}}-\mathbf{BR}'\cdot\hat{\mathbf{n}}'\|_{L^2}\lesssim_\delta\|\Theta-\Theta'\|_X$ follows from Theorem 5.1 of [Amb1]. Observe that, by adding and subtracting, we have for the second term

$$\left\|\mathbf{Y}\cdot\hat{\mathbf{n}}-\mathbf{Y}'\cdot\hat{\mathbf{n}}'\right\|_{L^2}\leqslant\left\|(\mathbf{Y}-\mathbf{Y}')\cdot\hat{\mathbf{n}}\right\|_{L^2}+\left\|\mathbf{Y}'\cdot(\hat{\mathbf{n}}-\hat{\mathbf{n}}')\right\|_{L^2}.$$

The second term is easily bounded:

$$\left\|Y'\cdot(\hat{\mathbf{n}}-\hat{\mathbf{n}}')\right\|_{L^2}\lesssim\left\|\zeta-\zeta'\right\|_{L^2}\lesssim\left\|\theta-\theta'\right\|_{L^2}\leqslant\left\|\Theta-\Theta'\right\|_X. \tag{2.6.4}$$

For the first term, we begin by considering

$$\begin{aligned}&\left|(\mathfrak{C}(\mathbf{Y})^*(\alpha)-\mathfrak{C}(\mathbf{Y}')^*(\alpha))\mathfrak{C}(\hat{\mathbf{n}})(\alpha)\right|\\&=\frac{\left|i\zeta_\alpha(\alpha)\right|}{4\pi s_\alpha}\left|\int_0^{2\pi}\omega(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))\,d\alpha'-\int_0^{2\pi}\omega'(\alpha')s_{1,\alpha}(\alpha')\cot\frac{1}{2}(\zeta'(\alpha)-\zeta_1(\alpha'))\,d\alpha'\right|.\end{aligned} \tag{2.6.5}$$

This is bounded above by a constant multiple of

$$\int_0^{2\pi}\left|\omega(\alpha')-\omega'(\alpha')\right|d\alpha'+\int_0^{2\pi}\left|\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))-\cot\frac{1}{2}(\zeta'(\alpha)-\zeta_1(\alpha'))\right|d\alpha'. \tag{2.6.6}$$

By Hölder's inequality,

$$\int_0^{2\pi}\left|\omega(\alpha')-\omega'(\alpha')\right|d\alpha'\lesssim\left\|\omega-\omega'\right\|_{L^2}. \tag{2.6.7}$$

On the other hand,

$$\int_0^{2\pi}\left|\cot\frac{1}{2}(\zeta(\alpha)-\zeta_1(\alpha'))-\cot\frac{1}{2}(\zeta'(\alpha)-\zeta_1(\alpha'))\right|d\alpha'\lesssim\left|\zeta(\alpha)-\zeta'(\alpha)\right|, \tag{2.6.8}$$

given that $|\zeta-\zeta_1|$ is bounded away from zero - in fact, recall that we require $\eta-\eta_1\geqslant\mathfrak{h}>0$ - and thus the map $\zeta\mapsto\cot\frac{1}{2}(\zeta-\zeta_1)$ is Lipschitz continuous with the Lipschitz constant depending upon the water depth $\mathfrak{h}$. It then follows that

$$\left\|(\mathbf{Y}-\mathbf{Y}')\cdot\hat{\mathbf{n}}\right\|_{L^2}\lesssim\left\|\omega-\omega'\right\|_{L^2}+\left\|\zeta-\zeta'\right\|_{L^2}\lesssim\left\|\theta-\theta'\right\|_{L^2}+\left\|\omega-\omega'\right\|_{L^2}\leqslant\left\|\Theta-\Theta'\right\|_X. \tag{2.6.9}$$

Therefore, from (2.6.4) and (2.6.9), we conclude that

$$\left\|\mathbf{Y}\cdot\hat{\mathbf{n}}-\mathbf{Y}'\cdot\hat{\mathbf{n}}'\right\|_{L^2}\lesssim\left\|\Theta-\Theta'\right\|_X. \tag{2.6.10}$$

The estimate for the third terms is entirely analogous:

$$\left\|\mathbf{Z}\cdot\hat{\mathbf{n}}-\mathbf{Z}'\cdot\hat{\mathbf{n}}'\right\|_{L^2}\lesssim\left\|\Theta-\Theta'\right\|_X. \tag{2.6.11}$$

The remaining terms contain no surprises and upon carrying out these computations we obtain

$$\left\|U-U'\right\|_{L^2}\lesssim_\delta\left\|\Theta-\Theta'\right\|_X. \tag{2.6.12}$$

From here, we deduce that

$$\left\|V-V'\right\|_{L^2}\lesssim_\delta\left\|\Theta-\Theta'\right\|_X \text{ and } \left|s_{\alpha t}-s'_{\alpha t}\right|\lesssim_\delta\left\|\Theta-\Theta'\right\|_X.$$

Next, we have

$$\begin{aligned}\left\|\mathbf{W}\cdot\hat{\mathbf{t}}-\mathbf{W}'\cdot\hat{\mathbf{t}}'\right\|_{L^2}\leqslant&\left\|\mathbf{BR}\cdot\hat{\mathbf{t}}-\mathbf{BR}'\cdot\hat{\mathbf{t}}'\right\|_{L^2}+\left\|\mathbf{Y}\cdot\hat{\mathbf{t}}-\mathbf{Y}'\cdot\hat{\mathbf{t}}'\right\|_{L^2}+\left\|\mathbf{Z}\cdot\hat{\mathbf{t}}-\mathbf{Z}'\cdot\hat{\mathbf{t}}'\right\|_{L^2}\\&+\chi(|V_0|\left\|t_1-t_1'\right\|_{L^2}+\left\|\nabla\varphi_{\mathrm{cyl}}(\zeta)\cdot\hat{\mathbf{t}}-\nabla\varphi_{\mathrm{cyl}}(\zeta')\cdot\hat{\mathbf{t}}'\right\|_{L^2}).\end{aligned}$$

It is easily observable that this will satisfy the same estimate as $\left\|U-U'\right\|_{L^2}$ and thus

$$\left\|\mathbf{W}\cdot\hat{\mathbf{t}}-\mathbf{W}'\cdot\hat{\mathbf{t}}'\right\|_{L^2}\lesssim_\delta\left\|\Theta-\Theta'\right\|_X. \tag{2.6.13}$$

It therefore follows that

$$\left\|(V-\mathbf{W}\cdot\hat{\mathbf{t}})-(V'-\mathbf{W}'\cdot\hat{\mathbf{t}}')\right\|_{L^2}\lesssim_\delta\left\|\Theta-\Theta'\right\|_X.$$

Continuing to estimate term-by-term as we have been leads us to conclude that

$$\left\|\mathfrak{F}^\delta_{1,2}(\Theta)-\mathfrak{F}^\delta_{1,2}(\Theta')\right\|_{H^s}\lesssim_\delta\left\|\Theta-\Theta'\right\|_X. \tag{2.6.14}$$

Proceeding in this fashion, we arrive at the estimate

$$\left\|\mathfrak{F}_1^\delta(\Theta) - \mathfrak{F}_1^\delta(\Theta')\right\|_{H^s} \lesssim \left\|\Theta - \Theta'\right\|_X. \tag{2.6.15}$$

Moving on to $\mathfrak{F}_2^\delta$, Theorem 5.1 of [Amb1] implies that

$$\left\|\mathfrak{F}_{2,1}^\delta(\Theta) - \mathfrak{F}_{2,1}^\delta(\Theta')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X, \tag{2.6.16}$$

$$\left\|\mathfrak{F}_{2,2}^\delta(\Theta) - \mathfrak{F}_{2,2}^\delta(\Theta')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.17}$$

Further, using the above estimates derived in estimating $\mathfrak{F}_1^\delta$, it is easy to obtain the bounds

$$\left\|\mathfrak{F}_{2,3}^\delta(\Theta) - \mathfrak{F}_{2,3}^\delta(\Theta')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X, \tag{2.6.18}$$

$$\left\|\mathfrak{F}_{2,4}^\delta(\Theta) - \mathfrak{F}_{2,4}^\delta(\Theta')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.19}$$

For $\mathfrak{F}_{2,5}^\delta$, we shall utilize the decomposition of $m_\gamma = m_\gamma^1 + m_\gamma^2 + m_\gamma^3 + m_\gamma^4$ from Lemma 2.5.9. The following estimates are rather simple:

$$\left\|m_\gamma^1 - (m_\gamma^1)'\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X, \tag{2.6.20}$$

$$\left\|m_\gamma^2 - (m_\gamma^2)'\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.21}$$

For m_γ^3, we have

$$\begin{aligned}\left\|m_\gamma^3 - (m_\gamma^3)'\right\|_{H^{s-1/2}} \lesssim{}& \left\|\mathcal{J}_\delta(\mathbf{br}_1 \cdot \hat{\mathbf{t}} - \mathbf{br}_1' \cdot \hat{\mathbf{t}}')\right\|_{H^{s-1/2}} + \left\|\mathcal{J}_\delta(F_{\mathbf{Y}} \cdot \hat{\mathbf{t}} - F_{\mathbf{Y}}' \cdot \hat{\mathbf{t}}')\right\|_{H^{s-1/2}} \\ &+ \left\|\mathcal{J}_\delta(F_{\mathbf{Z}} \cdot \hat{\mathbf{t}} - F_{\mathbf{Z}}' \cdot \hat{\mathbf{t}}')\right\|_{H^{s-1/2}} \\ &+ \left\|\chi\, \mathcal{J}_\delta(\partial_t(\nabla\varphi_{\text{cyl}}(\zeta)) \cdot \hat{\mathbf{t}} - \partial_t(\nabla\varphi_{\text{cyl}}(\zeta')) \cdot \hat{\mathbf{t}}')\right\|_{H^{s-1/2}}.\end{aligned}$$

For the first term, we add and subtract, and use Lemma 2.4.2, to obtain the bound

$$\left\|\mathcal{J}_\delta(\mathbf{br}_1 \cdot \hat{\mathbf{t}} - \mathbf{br}_1' \cdot \hat{\mathbf{t}}')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|(\mathbf{br}_1 - \mathbf{br}_1') \cdot \hat{\mathbf{t}}\right\|_{L^2} + \left\|\mathbf{br}_1' \cdot (\hat{\mathbf{t}} - \hat{\mathbf{t}}')\right\|_{L^2}.$$

Given that $\mathbf{br}_1$ is bounded in L^2, the second term is easily bounded by $C\|\Theta - \Theta'\|_X$, as we have seen many

times before. For the first term, we shall begin by writing $\mathbf{br}_1 = \sum \mathbf{br}_{1,j}$. Beginning with $\mathbf{br}_{1,1}$, we have

$$\|(\mathbf{br}_{1,1} - \mathbf{br}'_{1,1}) \cdot \hat{\mathbf{t}}\|_{L^2} \lesssim \left\| [\mathcal{H}, \zeta_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta'_\alpha}\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}.$$

We now add and subtract:

$$\begin{aligned}\|(\mathbf{br}_{1,1} - \mathbf{br}'_{1,1}) \cdot \hat{\mathbf{t}}\|_{L^2} \lesssim &\left\| [\mathcal{H}, \zeta_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2} \\ &+ \left\| [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \frac{1}{\zeta'_\alpha}\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}.\end{aligned}$$

We begin by considering the first term:

$$\begin{aligned}&\left\| [\mathcal{H}, \zeta_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2} \\ &\leqslant \left\| \mathcal{H}\left((\zeta_t - \zeta'_t)\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right)\right\|_{L^2} + \left\| \mathcal{H}\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)(\zeta_t - \zeta'_t)\right\|_{L^2} \\ &\lesssim \|\zeta_t - \zeta'_t\|_{L^2}.\end{aligned}$$

Recalling that $\zeta_t = Un_1 + Vt_1 + i(Un_2 + Vt_2)$, it follows that

$$\begin{aligned}&\left\| [\mathcal{H}, \zeta_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2} \\ &\lesssim \|U - U'\|_{L^2} + \|V - V'\|_{L^2} \\ &\lesssim \|\Theta - \Theta'\|_X.\end{aligned}$$

We use Lemma 1.3.12, and the fact that ζ'_t is bounded in H^1, for the second term:

$$\left\| [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \frac{1}{\zeta'_\alpha}\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2} \lesssim \left\| \frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \frac{1}{\zeta'_\alpha}\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right\|_{L^2}.$$

We next add and subtract to obtain

$$\begin{aligned}\left\| [\mathcal{H}, \zeta'_t]\left(\frac{1}{\zeta_\alpha}\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \frac{1}{\zeta'_\alpha}\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2} \lesssim &\left\| \partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\left(\frac{1}{\zeta_\alpha} - \frac{1}{\zeta'_\alpha}\right)\right\|_{L^2} \\ &+ \left\| \frac{1}{\zeta'_\alpha}\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}\end{aligned}$$

We have the following bound for the first term:

$$\left\|\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\left(\frac{1}{\zeta_\alpha}-\frac{1}{\zeta'_\alpha}\right)\right\|_{L^2} \lesssim \|\zeta_\alpha-\zeta'_\alpha\|_{L^2} \lesssim \|\theta-\theta'\|_{L^2} \leqslant \|\Theta-\Theta'\|_X.$$

On the other hand, for the second term, we add and subtract:

$$\left\|\frac{1}{\zeta'_\alpha}\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)-\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2} \lesssim \left\|\frac{\gamma}{\zeta_\alpha}-\frac{\gamma'}{\zeta'_\alpha}\right\|_{H^1} \lesssim \|\theta-\theta'\|_{H^1}+\|\gamma-\gamma'\|_{H^{\leqslant}}\|\Theta-\Theta'\|_X.$$

We have shown that

$$\|(\mathbf{br}_{1,1}-\mathbf{br}'_{1,1})\cdot\hat{\mathbf{t}}\|_{L^2} \lesssim \|\Theta-\Theta'\|_X. \tag{2.6.22}$$

We again add and subtract in $\mathbf{br}_{1,2}$:

$$\begin{aligned}
&\left\|[K[\zeta],\zeta_t]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-\left[K[\zeta'],\zeta'_t\right]\left(\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}\\
&\leqslant \left\|K[\zeta]\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta'_t\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}\\
&\quad+\left\|\zeta_tK[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-\zeta'_tK[\zeta']\left(\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}.
\end{aligned}$$

We begin by adding and subtracting in the first term:

$$\begin{aligned}
&\left\|K[\zeta]\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta'_t\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}\\
&\leqslant \left\|K[\zeta]\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2}\\
&\quad+\left\|K[\zeta']\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta'_t\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2}.
\end{aligned}$$

We use Lemma 2.4.6 to estimate the first term

$$\left\|K[\zeta]\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2} \lesssim \|\theta-\theta'\|_{H^1} \leqslant \|\Theta-\Theta'\|_X.$$

To estimate the second term, we apply Lemma 2.4.5:

$$\left\|K[\zeta']\left(\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)-K[\zeta']\left(\zeta'_t\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right)\right\|_{L^2} \lesssim \left\|\zeta_t\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)-\zeta'_t\partial_\alpha\left(\frac{\gamma'}{\zeta'_\alpha}\right)\right\|_{L^2}.$$

By adding and subtracting, we obtain

$$\left\| K[\zeta']\left(\zeta_t \partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta']\left(\zeta_t' \partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2} \lesssim \|\zeta_t - \zeta_t'\|_{L^2} + \left\|\frac{\gamma}{\zeta_\alpha} - \frac{\gamma'}{\zeta_\alpha'}\right\|_{H^1}.$$

The right-hand side is then easily bounded by $C\|\Theta - \Theta'\|_X$. We therefore have

$$\left\| K[\zeta]\left(\zeta_t \partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta']\left(\zeta_t' \partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2} \lesssim \|\Theta - \Theta'\|_X.$$

As usual, we can add and subtract to obtain the bound

$$\begin{aligned}
&\left\| \zeta_t K[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - \zeta_t' K[\zeta']\left(\partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2} \\
&\lesssim \|\zeta_t - \zeta_t'\|_{L^2} + \left\| K[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta']\left(\partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2}.
\end{aligned}$$

We know that the first term is bounded by $C\|\Theta - \Theta'\|_X$. For the second term, we add and subtract again:

$$\begin{aligned}
&\left\| K[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta']\left(\partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2} \\
&\lesssim \left\| K[\zeta]\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right) - K[\zeta']\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right)\right)\right\|_{L^2} + \left\| K[\zeta']\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2}.
\end{aligned}$$

Lemma 2.4.6 implies that the first term is bounded by $C\|\Theta - \Theta'\|_X$. On the other hand, we can control the second term via Lemma 2.4.5

$$\left\| K[\zeta']\left(\partial_\alpha\left(\frac{\gamma}{\zeta_\alpha}\right) - \partial_\alpha\left(\frac{\gamma'}{\zeta_\alpha'}\right)\right)\right\|_{L^2} \lesssim \left\|\frac{\gamma}{\zeta_\alpha} - \frac{\gamma'}{\zeta_\alpha'}\right\|_{H^1}.$$

By adding and subtracting again, we can control the right-hand side by $C\|\Theta - \Theta'\|_X$. We have now shown that

$$\|(\mathbf{br}_{1,2} - \mathbf{br}_{1,2}') \cdot \hat{\mathbf{t}}\|_{L^2} \lesssim \|\Theta - \Theta'\|_X. \tag{2.6.23}$$

The estimates for the remaining $\mathbf{br}_{1,j}$ terms follow in a similar fashion. We have now shown that

$$\|(\mathbf{br}_1 - \mathbf{br}_1') \cdot \hat{\mathbf{t}}\|_{L^2} \lesssim_\delta \|\Theta - \Theta'\|_X \tag{2.6.24}$$

and therefore

$$\left\|\mathbf{br}_1 \cdot \hat{\mathbf{t}} - \mathbf{br}_1' \cdot \hat{\mathbf{t}}'\right\|_{L^2} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.25}$$

The remaining terms are estimated much like those we have already seen. Ultimately, we obtain

$$\left\|\mathfrak{F}_2^\delta(\Theta) - \mathfrak{F}_2^\delta(\Theta')\right\|_{H^{s-1/2}} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.26}$$

We now consider $\mathfrak{F}_3^\delta$:

$$\left|\mathfrak{F}_3^\delta(\Theta(\alpha)) - \mathfrak{F}_3^\delta(\Theta'(\alpha))\right| = \frac{1}{\pi}\left|\int_0^{2\pi} \gamma(\alpha')k^1_{\mathcal{S},t}(\alpha,\alpha')\,d\alpha' - \int_0^{2\pi} \gamma'(\alpha')(k^1_{\mathcal{S},t})'(\alpha,\alpha')\,d\alpha'\right|,$$

where $k^1_{\mathcal{S}}$ is given in (2.3.37). It thus follows that

$$k^1_{\mathcal{S},t}(\alpha,\alpha') = -\,\Re\mathrm{e}\left\{\frac{\zeta_t(\alpha')}{2s_{1,\alpha}(\alpha)}\partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta(\alpha'))\right\}.$$

Upon adding and subtracting, we have

$$\begin{aligned}&\left|\mathfrak{F}_3^\delta(\Theta(\alpha)) - \mathfrak{F}_3^\delta(\Theta'(\alpha))\right| \\ &\lesssim \int_0^{2\pi}\left|k^1_{\mathcal{S},t}(\alpha,\alpha')\right|\left|\gamma(\alpha') - \gamma'(\alpha')\right|\,d\alpha' + \int_0^{2\pi}\left|\gamma'(\alpha')\right|\left|k^1_{\mathcal{S},t}(\alpha,\alpha') - (k^1_{\mathcal{S},t})'(\alpha,\alpha')\right|\,d\alpha'.\end{aligned}$$

Hölder's inequality then implies

$$\left|\mathfrak{F}_3^\delta(\Theta(\alpha)) - \mathfrak{F}_3^\delta(\Theta'(\alpha))\right| \lesssim \left\|\gamma - \gamma'\right\|_{L^2} + \left\|k^1_{\mathcal{S},t}(\alpha,\cdot) - (k^1_{\mathcal{S},t})'(\alpha,\cdot)\right\|_{L^2}.$$

We are thus left to estimate the second term and we begin by adding and subtracting:

$$\left\|k^1_{\mathcal{S},t}(\alpha,\cdot) - (k^1_{\mathcal{S},t})'(\alpha,\cdot)\right\|_{L^2} \lesssim \left\|\partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta(\cdot)) - \partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta'(\cdot))\right\|_{L^2} + \left\|\zeta_t - \zeta_t'\right\|_{L^2}.$$

Via Lipschitz continuity, we can estimate

$$\left\|\partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta(\cdot)) - \partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta'(\cdot))\right\|_{L^2} \lesssim \left\|\zeta - \zeta'\right\|_{L^2} \lesssim \left\|\Theta - \Theta'\right\|_X.$$

Further, as we have seen already,

$$\|\zeta_t - \zeta_t'\|_{L^2} \lesssim \|U - U'\|_{L^2} + \|V - V'\|_{L^2} \lesssim_\delta \|\Theta - \Theta'\|_X.$$

We have thus shown that

$$\|\mathfrak{F}_3^\delta(\Theta) - \mathfrak{F}_3^\delta(\Theta')\|_{L^2} \lesssim_\delta \|\Theta - \Theta'\|_X.$$

Similarly, we have

$$\begin{aligned}\left|\partial_\alpha \mathfrak{F}_3^\delta(\Theta(\alpha)) - \partial_\alpha \mathfrak{F}_3^\delta(\Theta'(\alpha))\right| \lesssim &\int_0^{2\pi} \left|\partial_\alpha k_{\mathcal{S},t}^1(\alpha,\alpha')\right|\left|\gamma(\alpha') - \gamma'(\alpha')\right| d\alpha' \\ &+ \int_0^{2\pi} \left|\gamma'(\alpha')\right|\left|\partial_\alpha k_{\mathcal{S},t}^1(\alpha,\alpha') - \partial_\alpha (k_{\mathcal{S},t}^1)'(\alpha,\alpha')\right| d\alpha'.\end{aligned}$$

Recall that

$$\begin{aligned}\partial_\alpha k_{\mathcal{S},t}^1(\alpha,\alpha') = &\,\Re\mathrm{e}\left\{\frac{s_{1,\alpha\alpha}(\alpha)\zeta_t(\alpha')}{2s_{1,\alpha}^2(\alpha)}\partial_\alpha \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta(\alpha'))\right\} \\ &- \Re\mathrm{e}\left\{\frac{\zeta_t(\alpha')}{2s_{1,\alpha}(\alpha)}\partial_\alpha^2 \cot\frac{1}{2}(\zeta_1(\alpha) - \zeta(\alpha'))\right\}.\end{aligned}$$

Then, applying Hölder's inequality, we estimate

$$\left|\partial_\alpha \mathfrak{F}_3^\delta(\Theta(\alpha)) - \partial_\alpha \mathfrak{F}_3^\delta(\Theta'(\alpha))\right| \lesssim \|\gamma - \gamma'\|_{L^2} + \left\|\partial_\alpha k_{\mathcal{S},t}^1(\alpha,\cdot) - \partial_\alpha (k_{\mathcal{S},t}^1)'(\alpha,\cdot)\right\|_{L^2}.$$

By adding and subtracting, then using Lipschitz estimates, we obtain

$$\left\|\partial_\alpha k_{\mathcal{S},t}^1(\alpha,\cdot) - \partial_\alpha (k_{\mathcal{S},t}^1)'(\alpha,\cdot)\right\|_{L^2} \lesssim \|\zeta - \zeta'\|_{L^2} + \|\zeta_t - \zeta_t'\|_{L^2}.$$

As we have seen, the right-hand side is controlled by $C(\delta)\|\Theta - \Theta'\|_X$. It then follows that

$$\|\partial_\alpha \mathfrak{F}_3^\delta(\Theta) - \partial_\alpha \mathfrak{F}_3^\delta(\Theta')\|_{L^2} \lesssim_\delta \|\Theta - \Theta'\|_X.$$

We therefore conclude that

$$\|\mathfrak{F}_3^\delta(\Theta) - \mathfrak{F}_3^\delta(\Theta')\|_{H^1} \lesssim_\delta \|\Theta - \Theta'\|_X. \tag{2.6.27}$$

Finally, we move on to $\mathfrak{F}_4^\delta$, where we begin by recalling that

$$\left|\mathfrak{F}_4^\delta(\Theta) - \mathfrak{F}_4^\delta(\Theta')\right| = \frac{1}{\pi}\left|\int_0^{2\pi} \gamma(\alpha')k_{\mathcal{S},t}^2(\alpha,\alpha')\,d\alpha' - \int_0^{2\pi} \gamma'(\alpha')(k_{\mathcal{S},t}^2)'(\alpha,\alpha')\,d\alpha'\right|;$$

note that $k_{\mathcal{S}}^2$ is given in (2.3.37). Virtually the same arguments used to derive the Lipschitz estimate for $\mathfrak{F}_3^\delta$ then imply that

$$\left\|\mathfrak{F}_4^\delta(\Theta) - \mathfrak{F}_4^\delta(\Theta')\right\|_{H^1} \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.28}$$

Combining the estimates (2.6.15), (2.6.26), (2.6.27) and (2.6.28) leads us to deduce the Lipschitz continuity of $\mathfrak{F}$:

$$\left\|\mathfrak{F}^\delta(\Theta) - \mathfrak{F}^\delta(\Theta')\right\|_X \lesssim_\delta \left\|\Theta - \Theta'\right\|_X. \tag{2.6.29}$$

Therefore, the Picard theorem for ODE on Banach spaces implies that solutions to the regularized system exist, at least for a short time, and have the desired regularity. □

Now that we have proven the existence of solutions to the regularized system, we want to take a limit of the solutions $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ as $\delta \to 0^+$. In order to do that, we will, as previously mentioned, need to prove that the solutions exist on a common time interval and show that $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ converges. We begin by obtaining an existence time independent of δ. To that end, we have the following corollary of the uniform energy estimate Theorem 2.5.10 (and the existence result of Theorem 2.6.1):

Corollary 2.6.3. *If the regularity index s of the energy space X is sufficiently large, then there exists a positive $T = T(\varepsilon, |V_0|, s, \mathfrak{X})$ such that, for any $\delta \in (0,1]$, the solution Θ^δ of the regularized initial value problem is in $C^1([0,T];\mathfrak{X})$. In particular, notice that T is independent of the regularization parameter δ.*

Proof. We will follow the proof in [Amb1]. The main difference between our proof and the proof of [Amb1] is that we have the non-cavitation assumptions in our definition of $\mathfrak{X}$ given the added geometry in our setting.

We want to apply the continuation theorem for ODE on a Banach space. Given a solution Θ^δ to the regularized system obtained from Theorem 2.6.1, we will be able to continue this solution so long as it remains in $\mathfrak{X}$. Ergo, we want to show that Θ^δ cannot leave $\mathfrak{X}$ until some time which is independent of δ.

We will treat each of the five conditions defining $\mathfrak{X}$ individually and begin with the uniform energy bound $\mathcal{E} < \mathfrak{e}$, which we have imposed on the initial data. Then, the uniform energy estimate of Theorem 2.5.10 gives a time $T_1 > 0$, independent of δ, such that we will have $\mathcal{E} < \mathfrak{e}$ for $0 \leqslant t \leqslant T_1$. The periodicity

implies that the lower bound on the arclength element $s_\alpha \geqslant 1$ will automatically be verified.

We next want to consider our (regularized) non-cavitation assumptions:

$$\eta^\delta - \eta_1 \geqslant \mathfrak{h} \text{ and } \eta^\delta - \eta_2 \geqslant \tilde{\mathfrak{h}}.$$

These assumptions are imposed on our Cauchy data and we seek to prove that we can propagate them forward in time, for at least some small time, independent of δ. Notice that, as our bottom and obstacle are independent of time,

$$\partial_t \eta^\delta = \partial_t(\eta^\delta - \eta_1) = \partial_t(\eta^\delta - \eta_2).$$

Then, if we can control $\left\|\partial_t \eta^\delta\right\|_{L^\infty}$ independent of δ, we can then ensure that both non-cavitation conditions will be satisfied on a uniform-in-δ time interval. For this, we can use

$$\eta^\delta(\alpha, t) := \eta^\delta(0, t) + \int_0^\alpha s_\alpha(t) \sin\theta^\delta(\alpha', t)\, d\alpha'.$$

So, propagating the non-cavitation conditions forward in time comes down to our ability to control $\|\partial_t s_\alpha\|_{L^\infty}$ and $\|\partial_t \theta\|_{L^\infty} \lesssim \|\partial_t \theta\|_{H^{1/2+}}$ uniformly in δ. We can do this by Lemma 2.5.7 and Theorem 2.5.10. There is also technically the issue of $\partial_t \eta^\delta(0, t)$. By choosing the constant of integration implicit in the definition of V, we can set $V(0, t) = 0$. We then have $\partial_t(\xi^\delta, \eta^\delta)(0, t) = (\mathbf{W}^\delta \cdot \hat{\mathbf{n}}^\delta)(0, t)\hat{\mathbf{n}}^\delta(0, t)$. We can control this term in L^∞ via Lemmas 2.5.4 and 2.5.5, if we can control $\mathbf{BR}^\delta \cdot \hat{\mathbf{n}}^\delta$ in, say, $H^{1/2+}$. We can do so by writing $\mathfrak{C}(\mathbf{BR}^\delta)^* = \frac{1}{2i}\mathcal{H}(\frac{\gamma^\delta}{\zeta^\delta_\alpha}) + K[\zeta^\delta]\gamma^\delta$ and applying Lemma 2.4.5. We thereby obtain times $T_2, T_3 > 0$, both independent of δ, such that the first non-cavitation assumption (2.3.3) holds on $[0, T_2]$ and the second non-cavitation assumption (2.3.4) holds on $[0, T_3]$.

Finally, we need to consider the chord-arc condition. Recall the divided difference q_1, which we introduced in (2.4.22). We can write the chord-arc condition in terms of q_1:

$$\left|q_1(\alpha, \alpha')\right| > \mathfrak{c}. \tag{2.6.30}$$

We can handle the chord-arc condition much like we handled the uniform energy bound: since the chord-arc condition is imposed on the Cauchy data, bounding $|\partial_t q_1|$ uniformly in δ will ensure the existence of some (perhaps small, but independent of δ) $T_4 > 0$ such that the chord-arc condition is satisfied on $[0, T_4]$. Recall

from Lemma 3.4 in [Amb1] that the following estimate holds:

$$\|q_1\|_{H^{r-1}} \lesssim \|\zeta_d\|_{H^r}. \tag{2.6.31}$$

If we first apply Sobolev embedding (Lemma 1.3.2) and then apply (2.6.31), we obtain

$$\|\partial_t q_1\|_{L^\infty} \lesssim \|\partial_t q_1\|_{H^{1/2+}} \lesssim \left\|\partial_t \zeta_d^\delta\right\|_{H^{3/2+}}. \tag{2.6.32}$$

From here, we can utilize the definition of $\zeta_d^\delta(\alpha,t) := \int_0^\alpha s_\alpha(t) e^{i\theta^\delta(\alpha',t)}\, d\alpha'$. Specifically, we use this definition to compute the time derivative. We know from Lemma 2.5.7 and Theorem 2.5.10 that we can control $|\partial_t s_\alpha|$ and $\|\partial_t \theta\|_{H^r}$ uniformly in δ, at least for small enough r ($r = \frac{3}{2}+$ is certainly small enough to make this work). Taking $T = \min\{T_1, T_2, T_3, T_4\}$ will give the desired uniform time. □

Having obtained a common time interval on which regularized solutions exist, we now move on to establish that we can take a limit as $\delta \to 0^+$. That is, we want to show that the sequence $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ converges. To achieve this, we will demonstrate that $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ is a Cauchy sequence in $C([0,T];Y)$, where $Y \supset X$. Here it will be helpful to recall some notation we introduced earlier in equation (2.2.1): $X_r := H^r \times H^{r-1/2} \times H^1 \times H^1$. Our choice will thus be to take $Y = X_1$. We have the following:

Theorem 2.6.4. *If s is sufficiently large, then the sequence of solutions $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ of the regularized IVP* (2.4.17)*, indexed by the regularization parameter δ, is a Cauchy sequence in $C([0,T];X_1)$.*

Proof. Here we will want to estimate the difference of regularized solutions with different regularization parameters. In particular, what we would like to obtain is some estimate of the form

$$\sup_{t\in[0,T]} \left\|(\theta^\delta - \theta^{\tilde\delta}, \gamma^\delta - \gamma^{\tilde\delta}, \omega^\delta - \omega^{\tilde\delta}, \beta^\delta - \beta^{\tilde\delta})\right\|_{X_1} \lesssim f(\delta,\tilde\delta), \tag{2.6.33}$$

where $f(\delta,\tilde\delta) \to 0$ as $(\delta,\tilde\delta) \to (0,0)$.

Following [Amb1], we introduce an energy for the difference of two regularized solutions with different values of the regularization parameter, which will control $\left\|(\theta^\delta - \theta^{\tilde\delta}, \gamma^\delta - \gamma^{\tilde\delta}, \omega^\delta - \omega^{\tilde\delta}, \beta^\delta - \beta^{\tilde\delta})\right\|_{X_1}^2$. Define $\mathcal{E}_d$ to be given by

$$\mathcal{E}_d := \mathcal{E}_d^1 + \mathcal{E}_d^0 + \frac{1}{2}\left\|\omega^\delta - \omega^{\tilde\delta}\right\|_{H^1}^2 + \frac{1}{2}\left\|\beta^\delta - \beta^{\tilde\delta}\right\|_{H^1}^2, \tag{2.6.34}$$

where

$$\mathcal{E}_d^1 := \frac{1}{2}\int_0^{2\pi} (\partial_\alpha(\theta^\delta - \theta^{\tilde{\delta}}))^2 + \frac{1}{4\tau s_\alpha^\delta}(\gamma^\delta - \gamma^{\tilde{\delta}})\Lambda(\gamma^\delta - \gamma^{\tilde{\delta}}) + \frac{(\gamma^\delta)^2}{16\tau^2(s_\alpha^\delta)^2}(\gamma^\delta - \gamma^{\tilde{\delta}})^2 \, d\alpha, \tag{2.6.35}$$

$$\mathcal{E}_d^0 := \frac{1}{2}\int_0^{2\pi} (\theta^\delta - \theta^{\tilde{\delta}})\Lambda(\theta^\delta - \theta^{\tilde{\delta}}) + \frac{1}{4\tau s_\alpha^\delta}(\gamma^\delta - \gamma^{\tilde{\delta}})^2 + (\theta^\delta - \theta^{\tilde{\delta}})^2 \, d\alpha. \tag{2.6.36}$$

Noting that the regularized solutions all satisfy the same initial condition, regardless of the value of the regularization parameter δ, so we have $\mathcal{E}_d(0) = 0$. Our goal will then be to come up with a suitable bound on the growth of $\mathcal{E}_d$ over time. We begin by computing

$$\begin{aligned}
\frac{d\mathcal{E}_d^1}{dt} &= \int_0^{2\pi} \partial_\alpha(\theta_t^\delta - \theta_t^{\tilde{\delta}})\partial_\alpha(\theta^\delta - \theta^{\tilde{\delta}}) \, d\alpha + \int_0^{2\pi} \frac{1}{4\tau s_\alpha^\delta}(\gamma_t^\delta - \gamma_t^{\tilde{\delta}})\Lambda(\gamma^\delta - \gamma^{\tilde{\delta}}) \, d\alpha \\
&\quad + \int_0^{2\pi} \frac{(\gamma^\delta)^2}{16\tau^2(s_\alpha^\delta)^2}(\gamma_t^\delta - \gamma_t^{\tilde{\delta}})(\gamma^\delta - \gamma^{\tilde{\delta}}) \, d\alpha \\
&\quad + \int_0^{2\pi} \partial_t\left(\frac{1}{4\tau s_\alpha^\delta}\right)(\gamma^\delta - \gamma^{\tilde{\delta}})\Lambda(\gamma^\delta - \gamma^{\tilde{\delta}}) + \partial_t\left(\frac{(\gamma^\delta)^2}{16\tau^2(s_\alpha^\delta)^2}\right)(\gamma^\delta - \gamma^{\tilde{\delta}})^2 \, d\alpha. \\
&= d_1 + d_2 + d_3 + d_4.
\end{aligned}$$

We begin with d_1 and plug in for θ_t^δ and $\theta_t^{\tilde{\delta}}$ from (2.4.11):

$$d_1 = \int_0^{2\pi} \left(\frac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}(\mathcal{J}_\delta\,\gamma_{\alpha\alpha}^\delta) - \frac{1}{2(s_\alpha^{\tilde{\delta}})^2}\,\mathcal{H}(\mathcal{J}_{\tilde{\delta}}\,\gamma_{\alpha\alpha}^{\tilde{\delta}})\right)(\theta_\alpha^\delta - \theta_\alpha^{\tilde{\delta}}) \, d\alpha + e_1, \tag{2.6.37}$$

where e_1 is the remainder. We now examine d_2, again plugging in for γ_t^δ and $\gamma_t^{\tilde{\delta}}$ from (2.4.13). These substitutions yield

$$\begin{aligned}
d_2 &= \int_0^{2\pi} \frac{1}{4\tau s_\alpha^\delta}\left(\frac{2\tau}{s_\alpha^\delta}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \frac{2\tau}{s_\alpha^{\tilde{\delta}}}\,\mathcal{J}_{\tilde{\delta}}\,\theta_{\alpha\alpha}^{\tilde{\delta}}\right)\Lambda(\gamma^\delta - \gamma^{\tilde{\delta}}) \, d\alpha \\
&\quad + \int_0^{2\pi} \frac{1}{4\tau s_\alpha^\delta}\left(\frac{1}{2(s_\alpha^\delta)^2}\,\mathcal{H}((\gamma^\delta)^2\,\mathcal{J}_\delta\,\theta_\alpha^\delta) - \frac{1}{2(s_\alpha^{\tilde{\delta}})^2}\,\mathcal{H}((\gamma^{\tilde{\delta}})^2\,\mathcal{J}_{\tilde{\delta}}\,\theta_\alpha^{\tilde{\delta}})\right)\Lambda(\gamma^\delta - \gamma^{\tilde{\delta}}) \, d\alpha + e_2,
\end{aligned} \tag{2.6.38}$$

with e_2 again being the remainder. By adding together d_1 and d_2, we will obtain a cancellation.

Recall that s_α^δ is bounded above and away from zero below, independent of δ, by Lemma 2.5.6. Let w_1 denote the sum of the integral in (2.6.37) and the first integral in (2.6.38). Upon an integration by parts and

noting the bounds on s_α^δ, we have

$$
\begin{aligned}
w_1 = &-\int_0^{2\pi}\left(\frac{1}{2(s_\alpha^\delta)^2}\Lambda(\mathcal{J}_\delta\,\gamma^\delta) - \frac{1}{2(s_\alpha^{\tilde\delta})^2}\Lambda(\mathcal{J}_{\tilde\delta}\,\gamma^{\tilde\delta})\right)(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\,d\alpha \\
&+\int_0^{2\pi}\left(\frac{1}{2(s_\alpha^\delta)^2}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \frac{1}{2(s_\alpha^{\tilde\delta})^2}\,\mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta}\right)\Lambda(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha \\
\sim &-\int_0^{2\pi}\Big(\Lambda(\mathcal{J}_\delta\,\gamma^\delta) - \Lambda(\mathcal{J}_{\tilde\delta}\,\gamma^{\tilde\delta})\Big)(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\,d\alpha \\
&+\int_0^{2\pi}\Big(\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta}\Big)\Lambda(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha.
\end{aligned}
\tag{2.6.39}
$$

Recall from Remark 1.3.1 that we use $A \sim B$ to denote $B \lesssim A \lesssim B$. We now add and subtract in each of the two integrals in (2.6.39) to obtain

$$
\begin{aligned}
w_1 \sim &-\int_0^{2\pi}\Lambda(\mathcal{J}_\delta\,\gamma^\delta - \mathcal{J}_{\tilde\delta}\,\gamma^\delta)(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi}\Lambda(\mathcal{J}_{\tilde\delta}(\gamma^\delta - \gamma^{\tilde\delta}))(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\,d\alpha \\
&+\int_0^{2\pi}\big(\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^\delta\big)\Lambda(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha \\
&+\int_0^{2\pi}\Big(\mathcal{J}_{\tilde\delta}(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\Big)\Lambda(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha.
\end{aligned}
$$

The second and fourth integrals above will cancel. In order to obtain this cancellation, we will need to use the fact that $\mathcal{J}_\delta$ is self-adjoint, and commutes with differentiation and the Hilbert transform. The commutation properties follow from the fact that we are conceptualizing $\mathcal{J}_\delta$ as a Fourier multiplier and so will commute with other Fourier multipliers such as differentiation and the Hilbert transform (Lemma 3.5 in [MaBe] has an alternative proof that $\mathcal{J}_\delta$ commutes with ∂_α). Turning now to the first integral, we integrate by parts and apply Hölder's inequality:

$$
-\int_0^{2\pi}\Lambda(\mathcal{J}_\delta\,\gamma^\delta - \mathcal{J}_{\tilde\delta}\,\gamma^\delta)(\theta_{\alpha\alpha}^\delta - \theta_{\alpha\alpha}^{\tilde\delta})\,d\alpha \leqslant \big\|\mathcal{H}(\mathcal{J}_\delta\,\gamma_{\alpha\alpha}^\delta - \mathcal{J}_{\tilde\delta}\,\gamma_{\alpha\alpha}^\delta)\big\|_{L^2}\Big\|\theta_\alpha^\delta - \theta_\alpha^{\tilde\delta}\Big\|_{L^2}.
$$

Using Lemma 1.3.10 and Lemma 2.4.3 as well as the uniform energy estimate of Theorem 2.5.10, we can control the first norm by

$$
\big\|\mathcal{H}(\mathcal{J}_\delta\,\gamma_{\alpha\alpha}^\delta - \mathcal{J}_{\tilde\delta}\,\gamma_{\alpha\alpha}^\delta)\big\|_{L^2} \leqslant \max(\delta,\tilde\delta)\big\|\gamma_{\alpha\alpha}^\delta\big\|_{H^1} \lesssim \max(\delta,\tilde\delta).
$$

The second norm above is clearly controlled by $\sqrt{\mathcal{E}_d}$. Finally, turning to the third integral, we use the fact that Λ is self-adjoint to rewrite it as

$$\int_0^{2\pi} \mathcal{H}(\mathcal{J}_\delta\,\theta^\delta_{\alpha\alpha\alpha} - \mathcal{J}_{\tilde\delta}\,\theta^\delta_{\alpha\alpha\alpha})(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha \leqslant \left\|\mathcal{H}(\mathcal{J}_\delta\,\theta^\delta_{\alpha\alpha\alpha} - \mathcal{J}_{\tilde\delta}\,\theta^\delta_{\alpha\alpha\alpha})\right\|_{L^2}\left\|\gamma^\delta - \gamma^{\tilde\delta}\right\|_{L^2}.$$

The second norm is again easily controlled by $\sqrt{\mathcal{E}_d}$, while for the first norm we apply Lemmas 1.3.10 and 2.4.3 as well as Theorem 2.5.10:

$$\left\|\mathcal{H}(\mathcal{J}_\delta\,\theta^\delta_{\alpha\alpha\alpha} - \mathcal{J}_{\tilde\delta}\,\theta^\delta_{\alpha\alpha\alpha})\right\|_{L^2} \leqslant \max(\delta,\tilde\delta)\left\|\theta^\delta_{\alpha\alpha\alpha}\right\|_{H^1} \lesssim \max(\delta,\tilde\delta).$$

We have now shown that

$$w_1 \lesssim \max(\delta,\tilde\delta)\,\sqrt{\mathcal{E}_d}. \tag{2.6.40}$$

The cancellation we saw in the sum of d_1 and d_2 corresponds to the primary cancellation from Theorem 2.5.10. So, we should expect some further cancellation which corresponds to the secondary cancellation of Theorem 2.5.10. Consider d_3 and plug in from (2.4.13):

$$d_3 = \int_0^{2\pi} \frac{(\gamma^\delta)^2}{16\tau^2(s^\delta_\alpha)^2}\left(\frac{2\tau}{s^\delta_\alpha}\,\mathcal{J}_\delta\,\theta^\delta_{\alpha\alpha} - \frac{2\tau}{s^{\tilde\delta}_\alpha}\,\mathcal{J}_{\tilde\delta}\,\theta^{\tilde\delta}_{\alpha\alpha}\right)(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha + e_3, \tag{2.6.41}$$

where e_3 once again denotes the remaining terms. To obtain the analogue of the secondary cancellation of Theorem 2.5.10, we let w_2 denote the sum of the second integral in (2.6.38) and the integral in (2.6.41). Utilizing the self-adjointness of Λ as well as the identity $\mathcal{H}^2 = -\,\mathrm{id}$, which implies that $\Lambda\,\mathcal{H} = -\partial_\alpha$, we get

$$\begin{aligned} w_2 = &-\int_0^{2\pi} \frac{1}{4\tau s^\delta_\alpha}\left(\frac{1}{2(s^\delta_\alpha)^2}\partial_\alpha((\gamma^\delta)^2\,\mathcal{J}_\delta\,\theta^\delta_\alpha) - \frac{1}{2(s^{\tilde\delta}_\alpha)^2}\partial_\alpha((\gamma^{\tilde\delta})^2\,\mathcal{J}_{\tilde\delta}\,\theta^{\tilde\delta}_\alpha)\right)(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha \\ &+ \int_0^{2\pi} \frac{(\gamma^\delta)^2}{16\tau^2(s^\delta_\alpha)^2}\left(\frac{2\tau}{s^\delta_\alpha}\,\mathcal{J}_\delta\,\theta^\delta_{\alpha\alpha} - \frac{2\tau}{s^{\tilde\delta}_\alpha}\,\mathcal{J}_{\tilde\delta}\,\theta^{\tilde\delta}_{\alpha\alpha}\right)(\gamma^\delta - \gamma^{\tilde\delta})\,d\alpha. \end{aligned} \tag{2.6.42}$$

We now use the Leibniz rule to expand the derivative in the first integral above then add and subtract in the

appropriate integral. This process yields

$$
\begin{aligned}
w_2 = &-\int_0^{2\pi} \frac{1}{4\tau s_\alpha^\delta}\left(\frac{1}{(s_\alpha^\delta)^2}\gamma^\delta\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta - \frac{1}{(s_\alpha^{\tilde\delta})^2}\gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_{\tilde\delta}\,\theta_\alpha^{\tilde\delta}\right)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi} \frac{1}{8\tau s_\alpha^\delta}\left(\frac{1}{(s_\alpha^\delta)^2}(\gamma^\delta)^2\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \frac{1}{s_\alpha^\delta s_\alpha^{\tilde\delta}}(\gamma^\delta)^2\,\mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta}\right)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi} \frac{1}{8\tau s_\alpha^\delta}\left(\frac{1}{s_\alpha^\delta s_\alpha^{\tilde\delta}}(\gamma^\delta)^2\,\mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta} - \frac{1}{(s_\alpha^{\tilde\delta})^2}(\gamma^{\tilde\delta})^2\,\mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta}\right)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&+\int_0^{2\pi} \frac{(\gamma^\delta)^2}{8\tau(s_\alpha^\delta)^2}\left(\frac{1}{s_\alpha^\delta}\,\mathcal{J}_\delta\,\theta_{\alpha\alpha}^\delta - \frac{1}{s_\alpha^{\tilde\delta}}\,\mathcal{J}_{\tilde\delta}\,\theta_{\alpha\alpha}^{\tilde\delta}\right)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha.
\end{aligned}
$$

Observe that the second and fourth integrals above cancel.

We turn now to controlling the remaining integrals which do not cancel. Let $w_{2,1}$ and $w_{2,3}$ denote the first integral above and the third integral above, respectively; these are the remaining integrals which need to be controlled. We will again make use of Lemma 2.5.6 to bound s_α^δ below (away from zero) and above for any $\delta \in (0,1]$. We begin with $w_{2,1}$, where, after adding and subtracting, we have

$$
\begin{aligned}
w_{2,1} \sim &-\int_0^{2\pi} (\gamma^\delta\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta - \gamma^{\tilde\delta}\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi} (\gamma^{\tilde\delta}\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta - \gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_\delta\,\theta_\alpha^\delta)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi} (\gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_\delta\,\theta_\alpha^\delta - \gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_{\tilde\delta}\,\theta_\alpha^\delta)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \\
&-\int_0^{2\pi} (\gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_{\tilde\delta}\,\theta_\alpha^\delta - \gamma^{\tilde\delta}\gamma_\alpha^{\tilde\delta}\,\mathcal{J}_{\tilde\delta}\,\theta_\alpha^{\tilde\delta})(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha.
\end{aligned}
$$

Utilizing Hölder's inequality, Lemma 1.3.9 and the uniform energy estimate of Theorem 2.5.10, we estimate the first integral in $w_{2,1}$:

$$
-\int_0^{2\pi} (\gamma^\delta\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta - \gamma^{\tilde\delta}\gamma_\alpha^\delta\,\mathcal{J}_\delta\,\theta_\alpha^\delta)(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha \lesssim \left\|\gamma^\delta-\gamma^{\tilde\delta}\right\|_{L^2}^2 \lesssim \mathcal{E}_d.
$$

We recognize a perfect derivative and integrate by parts to rewrite the second integral of $w_{2,1}$:

$$
-\int_0^{2\pi} \gamma^{\tilde\delta}\,\mathcal{J}_\delta\,\theta_\alpha^\delta(\gamma_\alpha^\delta-\gamma_\alpha^{\tilde\delta})(\gamma^\delta-\gamma^{\tilde\delta})\,d\alpha = \frac{1}{2}\int \partial_\alpha(\gamma^{\tilde\delta}\,\mathcal{J}_\delta\,\theta_\alpha^\delta)(\gamma^\delta-\gamma^{\tilde\delta})^2\,d\alpha.
$$

Then, Hölder's inequality, Lemma 1.3.9 and the uniform energy estimate imply that

$$\frac{1}{2}\int_0^{2\pi} \partial_\alpha(\gamma^{\tilde{\delta}}\,\mathcal{J}_\delta\,\theta^\delta_\alpha)(\gamma^\delta-\gamma^{\tilde{\delta}})^2\,d\alpha \leqslant \left\|\partial_\alpha(\gamma^{\tilde{\delta}}\,\mathcal{J}_\delta\,\theta^\delta_\alpha)(\gamma^\delta-\gamma^{\tilde{\delta}})\right\|_{L^2}\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2} \lesssim \left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|^2_{L^2} \lesssim \mathcal{E}_d.$$

For the third integral of $w_{2,1}$, we use Hölder's inequality to obtain the bound:

$$-\int_0^{2\pi}(\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha\,\mathcal{J}_\delta\,\theta^\delta_\alpha-\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha\,\mathcal{J}_{\tilde{\delta}}\,\theta^\delta_\alpha)(\gamma^\delta-\gamma^{\tilde{\delta}})\,d\alpha \leqslant \left\|\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha(\mathcal{J}_\delta\,\theta^\delta_\alpha-\mathcal{J}_{\tilde{\delta}}\,\theta^\delta_\alpha)\right\|_{L^2}\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2}.$$

Then, by Lemma 1.3.9, Lemma 2.4.3 and Theorem 2.5.10, we get

$$\left\|\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha(\mathcal{J}_\delta\,\theta^\delta_\alpha-\mathcal{J}_{\tilde{\delta}}\,\theta^\delta_\alpha)\right\|_{L^2}\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2} \lesssim \max(\delta,\tilde{\delta})\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2} \lesssim \max(\delta,\tilde{\delta})\,\sqrt{\mathcal{E}_d}.$$

Considering the final integral in $w_{2,1}$, Hölder's inequality, Lemma 1.3.9, the uniform energy estimate of Theorem 2.5.10 and 2.4.2 yield

$$-\int_0^{2\pi}(\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha\,\mathcal{J}_{\tilde{\delta}}\,\theta^\delta_\alpha-\gamma^{\tilde{\delta}}\gamma^{\tilde{\delta}}_\alpha\,\mathcal{J}_{\tilde{\delta}}\,\theta^{\tilde{\delta}}_\alpha)(\gamma^\delta-\gamma^{\tilde{\delta}})\,d\alpha \lesssim \left\|\theta^\delta-\theta^{\tilde{\delta}}\right\|_{H^1}\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2} \lesssim \mathcal{E}_d.$$

It then follows that

$$w_{2,1} \lesssim \mathcal{E}_d + \max(\delta,\tilde{\delta})\,\sqrt{\mathcal{E}_d}. \tag{2.6.43}$$

We now proceed to examine $w_{2,3}$, which we can rewrite as

$$w_{2,3} \sim -\int_0^{2\pi}\mathcal{J}_{\tilde{\delta}}\,\theta^{\tilde{\delta}}_{\alpha\alpha}(\gamma^\delta+\gamma^{\tilde{\delta}})(\gamma^\delta-\gamma^{\tilde{\delta}})^2\,d\alpha.$$

Then, by Hölder's inequality, we have

$$w_{2,3} \lesssim \left\|\mathcal{J}_{\tilde{\delta}}\,\theta^{\tilde{\delta}}_{\alpha\alpha}(\gamma^\delta+\gamma^{\tilde{\delta}})(\gamma^\delta-\gamma^{\tilde{\delta}})\right\|_{L^2}\left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|_{L^2}.$$

We conclude by applying Lemma 1.3.9, Lemma 2.4.2 and Theorem 2.5.10 to control the right-hand side above:

$$w_{2,3} \lesssim \left\|\gamma^\delta-\gamma^{\tilde{\delta}}\right\|^2_{L^2} \lesssim \mathcal{E}_d. \tag{2.6.44}$$

Combining (2.6.43) with (2.6.44) and recalling the secondary cancellation, it therefore holds that

$$w_2 \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}. \tag{2.6.45}$$

We are now left to estimate d_4 as well as the remainder terms: e_1, e_2 and e_3. These terms are all rather straightforward. We have

$$d_4 \lesssim \mathcal{E}_d, \tag{2.6.46}$$

$$e_1 \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}, \tag{2.6.47}$$

$$e_2 \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}, \tag{2.6.48}$$

$$e_3 \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}. \tag{2.6.49}$$

We have found that

$$\frac{d\mathcal{E}_d^1}{dt} \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}. \tag{2.6.50}$$

We can estimate the remaining terms in a similar way. Doing so gives

$$\frac{d\mathcal{E}_d}{dt} \lesssim \mathcal{E}_d + \max(\delta, \tilde{\delta}) \sqrt{\mathcal{E}_d}, \tag{2.6.51}$$

which we can rewrite as

$$\frac{d\sqrt{\mathcal{E}_d}}{dt} \lesssim \sqrt{\mathcal{E}_d} + \max(\delta, \tilde{\delta}). \tag{2.6.52}$$

Upon solving the differential inequality in (2.6.52), recalling that $\mathcal{E}_d(0) = 0$, we find that

$$\sqrt{\mathcal{E}_d(t)} \leqslant \max(\delta, \tilde{\delta})(e^{ct} - 1), \tag{2.6.53}$$

where c is the implied constant in (2.6.52). Now, we recall that, by the definition of $\mathcal{E}_d$, we have

$$\left\| (\theta^\delta - \theta^{\tilde{\delta}}, \gamma^\delta - \gamma^{\tilde{\delta}}, \omega^\delta - \omega^{\tilde{\delta}}, \beta^\delta - \beta^{\tilde{\delta}}) \right\|_{X_1} \lesssim \sqrt{\mathcal{E}_d}. \tag{2.6.54}$$

Finally, we take the supremum and utilize (2.6.53):

$$\sup_{t\in[0,T]}\left\|(\theta^\delta-\theta^{\tilde\delta},\gamma^\delta-\gamma^{\tilde\delta},\omega^\delta-\omega^{\tilde\delta},\beta^\delta-\beta^{\tilde\delta})\right\|_{X_1}\lesssim\sup_{t\in[0,T]}\sqrt{\mathcal{E}_d(t)}\lesssim\max(\delta,\tilde\delta)(e^{cT}-1). \tag{2.6.55}$$

This is of the desired form (2.6.33) and so we see that $\left\{(\theta^\delta,\gamma^\delta,\omega^\delta,\beta^\delta)\right\}_{\delta\in(0,1]}$ is indeed a Cauchy sequence in $C([0,T];X_1)$.

□

We are now able to take a limit of the regularized solutions as $\delta\to 0^+$. The next step is, of course, to show that this limit solves the non-regularized system. However, in order to do this we will need to ensure that the limit has enough regularity to make sense of it as a solution. We will thus reserve this proof for the next section, where we consider the regularity.

2.7 Regularity of Solutions

At this point we know that the sequence of solutions $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$ to the regularized system converges to a limit as $\delta\to 0^+$. In this section, we will show that this limit solves the non-regularized system ((2.1.4) with right-hand side given by (2.4.10)), that this solution is unique and that it lies in $C([0,T];\mathfrak{X})$. We shall begin by first showing that the limit solves the non-regularized system and is continuous, with respect to time, in the weak topology.

Theorem 2.7.1. *Let $(\theta,\gamma,\omega,\beta)$ denote the limit as $\delta\to 0^+$ of the sequence of solutions $\left\{(\theta^\delta,\gamma^\delta,\omega^\delta,\beta^\delta)\right\}_{\delta\in(0,1]}$ to the regularized system* (2.4.17)*. Then, $(\theta,\gamma,\omega,\beta)$ solves the non-regularized system* (2.1.4) *with right-hand side given by* (2.4.10)*. Additionally, $(\theta,\gamma,\omega,\beta)\in C_{\mathrm{W}}([0,T];\mathfrak{X})$, where $C_{\mathrm{W}}([0,T];\mathfrak{X})$ denotes the space of weakly continuous function from $[0,T]$ into $\mathfrak{X}$. Finally, $(\theta,\gamma,\omega,\beta)$ is additionally in $C([0,T];X_r)$ for $1\leqslant r<s$, where X_r is as defined in* (2.2.1)*.*

Proof. We follow the proof of Theorem 5.4 in [Amb1] as it is nearly identical.

We have $\Theta\in C([0,T];X_1)$, which is the limit of the sequence of solutions of the regularized system, $\left\{\Theta^\delta\right\}_{\delta\in(0,1]}$. We know from the uniform energy estimate of Theorem 2.5.10 that, for any $\delta\in(0,1]$, $\left\|\Theta^\delta\right\|_X\lesssim 1$ and, noting that the unit ball $B_1\subset X$ is compact in the weak topology (indeed, X is a Hilbert space), it follows that Θ^δ converges weakly in X. Given that $X\subset X_1$, we must have $\Theta^\delta\rightharpoonup\Theta$ with $\Theta\in X$. Further, given that Θ is defined as the limit of Θ^δ as $\delta\to 0^+$, where Θ^δ satisfies the chord-arc condition (2.3.2), the non-cavitation assumptions (2.3.3) and (2.3.4), as well as the bounds (2.5.6) and (2.5.7) for each

$\delta \in (0,1]$, it follows that Θ also satisfies (2.3.2), (2.5.6) and (2.5.7). In other words, $\Theta \in \mathfrak{X}$. Now, by interpolating between Sobolev spaces, we get

$$\begin{aligned}\left\|\theta-\theta^{\delta}\right\|_{H^r} &\lesssim \left\|\theta-\theta^{\delta}\right\|_{L^2}^{1-\vartheta_r}\left\|\theta-\theta^{\delta}\right\|_{H^s}^{\vartheta_r},\\ \left\|\gamma-\gamma^{\delta}\right\|_{H^t} &\lesssim \left\|\gamma-\gamma^{\delta}\right\|_{L^2}^{1-\vartheta_t}\left\|\gamma-\gamma^{\delta}\right\|_{H^{s-1/2}}^{\vartheta_t},\end{aligned}$$

where $\vartheta_r := \frac{r}{s}$ and $\vartheta_t := \frac{t}{s-\frac{1}{2}}$. Observing that the quantities on the RHS all go to zero, as $\delta \to 0^+$, uniformly on $[0,T]$ implies that $(\theta,\gamma) \in C([0,T];H^r \times H^t)$ for all $1 \leqslant r < s$, $\frac{1}{2} \leqslant t < s-\frac{1}{2}$. We did not address ω or β above as we have already obtained the top-level regularity result for these terms (i.e., $\omega,\beta \in C([0,T];H^1)$) and so these lesser regularity results follow trivially.

We now turn our attention to demonstrating that $\Theta \in C_{\mathrm{W}}([0,T];X)$. Let $h > 0$ be given and take $u \in H^{-s}$ to be arbitrary. In addition, for arbitrary $1 \leqslant r < s$, let $v \in H^{-r}$ be such that

$$\|u-v\|_{H^{-s}} \leqslant \frac{h}{3}. \tag{2.7.1}$$

Such a v exists by the density of H^r in H^t whenever $r > t$. We now want to show that the pairing $\langle u, \theta-\theta^{\delta}\rangle$ can be made arbitrarily small uniformly in time. Indeed,

$$\langle u,\theta-\theta^{\delta}\rangle = \langle u,\theta\rangle - \langle u,\theta^{\delta}\rangle = \langle u-v,\theta\rangle + \langle v,\theta-\theta^{\delta}\rangle + \langle v-u,\theta^{\delta}\rangle. \tag{2.7.2}$$

Recall that the dual pairing is given by the L^2 inner product $(\cdot,\cdot)_{L^2}$. The first and third terms can be bounded above by $\frac{h}{3}$ using (2.7.1) in addition to the uniform bounds on θ and θ^{δ} in H^s. For the second term, we choose $\delta \in (0,1]$ sufficiently small so that $\left\|\theta-\theta^{\delta}\right\|_{H^r} < \frac{h}{3}$. It follows that, for $\delta \in (0,1]$ small enough,

$$\left|\langle u,\theta-\theta^{\delta}\rangle\right| \leqslant h. \tag{2.7.3}$$

Given that $h > 0$ was arbitrary, the above bounds were uniform in time and that $\theta^{\delta} \in C([0,T];H^s)$, we necessarily have $\theta \in C_{\mathrm{W}}([0,T];H^s)$. A virtually identical argument gives $\gamma \in C_{\mathrm{W}}([0,T];H^{s-1/2})$. Regarding ω and β, the weaker regularity result, $\omega,\beta \in C_{\mathrm{W}}([0,T];H^1)$, again follows from the stronger regularity result we have already obtained: $\omega,\beta \in C([0,T];H^1)$.

We are now at a point to show that $(\theta,\gamma,\omega,\beta)$ solves the initial value problem for the non-regularized

system. Notice that, as we take s to be large enough, the preliminary regularity $\Theta \in C([0,T]; X_r)$ for any $r < s$ suffices for Θ to be a solution to the system. To begin, observe that we have

$$\Theta^\delta(\alpha, t) = \Theta_0(\alpha) + \int_0^t \mathfrak{F}^\delta(\Theta^\delta(\alpha, s))\, ds.$$

We now pass to the limit in the above equation:

$$\Theta(\alpha, t) = \Theta_0(\alpha) + \int_0^t \mathfrak{F}(\Theta(\alpha, s))\, ds,$$

where $\mathfrak{F}$ is given by (2.4.10). We can differentiate with respect to time, which yields $\partial_t \Theta = \mathfrak{F}(\Theta)$, and so Θ indeed solves (2.1.4). □

Before proceeding to the top-level regularity result for solutions to the non-regularized system, we want to prove that the initial value problem for the non-regularized system is stable under small perturbations of the Cauchy data. This stability result will immediately imply the uniqueness of solutions to the non-regularized initial value problem. We have the following theorem on the dependence of the solutions on the initial data:

Theorem 2.7.2. *If the regularity index s of X is sufficiently large and $\Theta, \Theta' \in X$ are solutions of the initial value problem for the non-regularized system (again, this is the system* (2.1.4) *with right-hand side given by* (2.4.10)*) on the time interval $[0,T]$, with corresponding initial data $\Theta_0, \Theta_0' \in \mathfrak{X}$, then it holds that*

$$\sup_{t\in[0,T]} \left\|\Theta(t) - \Theta'(t)\right\|_{X_1} \lesssim_T \left\|\Theta_0 - \Theta_0'\right\|_{X_1}. \tag{2.7.4}$$

Proof. As in the proof of Theorem 2.6.4, we begin by defining an appropriate energy. In this case, it is the energy $\mathcal{E}_{\text{flow}}$ of the difference of two solutions with different Cauchy data:

$$\mathcal{E}_{\text{flow}} := \mathcal{E}^1_{\text{flow}} + \mathcal{E}^0_{\text{flow}} + \frac{1}{2}\left\|\omega - \omega'\right\|^2_{H^1} + \frac{1}{2}\left\|\beta - \beta'\right\|^2_{H^1}, \tag{2.7.5}$$

where

$$\mathcal{E}^1_{\text{flow}} := \frac{1}{2}\int_0^{2\pi} (\partial_\alpha(\theta-\theta'))^2 + \frac{1}{4\tau s_\alpha}(\gamma-\gamma')\Lambda(\gamma-\gamma') + \frac{\gamma^2}{16\tau^2 s_\alpha^2}(\gamma-\gamma')^2\, d\alpha, \tag{2.7.6}$$

$$\mathcal{E}^0_{\text{flow}} := \frac{1}{2}\int_0^{2\pi} (\theta-\theta')\Lambda(\theta-\theta') + \frac{1}{4\tau s_\alpha}(\gamma-\gamma')^2 + (\theta-\theta')^2\, d\alpha. \tag{2.7.7}$$

We denote this energy $\mathcal{E}_{\text{flow}}$ as it controls the continuity of the flow map. We note that, since Θ and Θ' satisfy different initial conditions, $\mathcal{E}_{\text{flow}}(0)$ will not vanish as was the case in Theorem 2.6.4, however $\mathcal{E}_{\text{flow}}(0) \sim \left\|\Theta_0 - \Theta_0'\right\|_{X_1}$.

We want to estimate $\frac{d\mathcal{E}_{\text{flow}}}{dt}$. The calculations are very similar to those in the proofs of Theorem 2.6.4 and Theorem 2.5.10, so we omit them. In summary, we obtain

$$\frac{d\mathcal{E}_{\text{flow}}}{dt} \lesssim \mathcal{E}_{\text{flow}}. \tag{2.7.8}$$

We then have

$$\mathcal{E}_{\text{flow}}(t) \leqslant \mathcal{E}_{\text{flow}}(0)e^{ct}, \tag{2.7.9}$$

where c is the implied constant in (2.7.8). Therefore, it follows that

$$\sup_{t\in[0,T]} \left\|\Theta(t) - \Theta'(t)\right\|_{X_1}^2 \lesssim \sup_{t\in[0,T]} \mathcal{E}_{\text{flow}}(t) \leqslant \mathcal{E}_{\text{flow}}(0)e^{cT} \lesssim e^{cT}\left\|\Theta_0 - \Theta_0'\right\|_{X_1}^2. \tag{2.7.10}$$

This is what we wanted to show.

□

Theorem 2.7.3. *Solutions of the non-regularized initial value problem* (2.1.4) *(where the right-hand side is given by* (2.4.10)*) are in* $C([0,T];X)$.

Proof. Recall that we have already obtained this desired regularity result for ω and β. Indeed, we have $\omega,\beta \in C([0,T];H^1)$. So, all that remains to do is show that $(\theta,\gamma) \in C([0,T];H^s \times H^{s-1/2})$. The proof of this result is virtually the same as the proof of Theorem 5.6 in [Amb1]. For the sake of completeness, we shall sketch the argument, which follows [Amb1] closely.

Observe that Fatou's lemma gives the following estimates:

$$\|\theta_0\|_{H^s}^2 \leqslant \liminf_{t\to 0^+}\|\theta(t)\|_{H^s}^2 \text{ and } \|\gamma_0\|_{H^{s-1/2}}^2 \leqslant \liminf_{t\to 0^+}\|\gamma(t)\|_{H^{s-1/2}}^2. \tag{2.7.11}$$

Using (2.7.11) and properties of limits inferior, we are able to conclude that $\liminf_{t\to 0^+}\mathcal{E}(t) \geqslant \mathcal{E}(0)$. On the other hand, the uniform energy estimate of Theorem 2.5.10 can be used to obtain $\limsup_{t\to 0^+}\mathcal{E}(t) \leqslant \mathcal{E}(0)$ (indeed, apply Grönwall as in Section 2.8). Ergo, the energy is right-continuous at $t = 0$. Additionally, we can determine that a number of components of the energy are continuous more or less by inspection. For example, considering $\mathcal{E}^0$, we know that $\theta \in C([0,T];L^2)$ by our preliminary regularity result. We can then consider the difference of the energy and the parts we know to be continuous. This difference will then be right-continuous at $t = 0$. Then, a bit of work will give us that $\|\theta\|_{H^s}$ and $\|\gamma\|_{H^{s-1/2}}$ are right-continuous at $t = 0$.

Now, we want to consider an arbitrary time $t_0 \in [0,T]$. The idea is that we can interpret $t = t_0$ as a new initial time. We could then run the same existence argument as in Theorem 2.6.1 to obtain the existence of a regularized solution on some time interval about t_0. Then, upon invoking Theorem 2.7.2, we see that this solution starting at time $t = t_0$ must coincide with the solution we already found starting from time $t = 0$. We can then run the above argument which showed that solutions are right-continuous at $t = 0$ to obtain the right-continuity of solutions at $t = t_0$. We also want our solutions to be left-continuous at $t = t_0 \neq 0$. However, this is actually simple now. In fact, all of the arguments given to obtain right-continuity work with time reversed, so this immediately gives us the left-continuity of solutions at $t = t_0$. Finally, as $t_0 \in [0,T]$ was an arbitrary time, we are able to conclude the desired regularity result: $\Theta \in C([0,T];X)$. □

2.8 Proof of Theorem 2.2.1

In this section, we will prove the first main theorem, Theorem 2.2.1. In the previous sections, we have shown that the model problem (2.1.4) is well-posed locally in time and that solutions are continuous from $[0,T]$ into X. What remains is to show that these results can be extended to the full water waves system (2.3.42) and then to prove the lifespan results. We shall begin by discussing how to extend the previous local well-posedness and regularity results on the model problem to the full water waves system. Then, we will prove the desired lifespan results as a corollary of the main energy estimate Theorem 2.5.10.

2.8.1 Extending the Results on the Model Problem to the Full Water Waves System

In order to extend the well-posedness and regularity results for the model problem to the full water waves system (2.3.42), we will, following the plan outlined in Remark 2.3.2, utilize mapping properties of the operator $(\mathrm{id} + \mathscr{K}[\Theta])^{-1}$. The operator $\mathrm{id} + \mathscr{K}[\Theta]$ is a Fredholm operator on X (indeed, it is a compact perturbation of the identity). In Section 5 of [AmbEtAl], it is moreover shown that the operator $\mathrm{id} + \mathscr{K}$ is invertible (see Section 2.10 below for an alternative presentation on the solvability of the integral equations). We then have the following result:

Lemma 2.8.1. *The inverse operator* $(\mathrm{id} + \mathscr{K}[\Theta])^{-1} : X \to X$ *is continuous.*

Proof. Given what we already know about $\mathrm{id} + \mathscr{K}[\Theta]$ (i.e., that it is an invertible Fredholm operator), the desired result follows from standard Fredholm theory. In particular, Theorem 1.4.15 of [Mur] should suffice. Alternatively, we can utilize the Fredholm alternative. More specifically, it is shown in [AmbEtAl] that $\mathrm{id} + \mathscr{K}[\Theta]$ is a Fredholm operator with trivial kernel and so, by the Fredholm alternative, $\mathrm{id} + \mathscr{K}[\Theta]$ is also a surjection. Hence, we know $\mathrm{id} + \mathscr{K}[\Theta]$ to be a bounded (by virtue of being Fredholm), bijective linear operator on a Hilbert space and so has a bounded inverse by the bounded inverse theorem. □

Lemma 2.8.1 is the desired mapping property and it gives us the following local-in-time well-posedness theorem, recalling that B is defined in (2.1.2):

Theorem 2.8.2. *Let s be sufficiently large. The system* (2.1.1) *is locally well-posed. Namely, there exists a unique solution* $\Theta \in C([0, T(B, |V_0|)]; \mathfrak{X})$ *to the system* (2.1.1) *and the flow map* $\Theta_0 \mapsto \Theta$ *is Lipschitz regular* $X_1 \to C([0, T]; X_1)$.

Proof. The solvability result of [AmbEtAl] (or, alternatively, Section 2.10) and Lemma 2.8.1 imply that Theorem 2.5.10, Theorem 2.6.1, Corollary 2.6.3, Theorem 2.6.4, Theorem 2.7.1, Theorem 2.7.2, Theorem 2.7.3 apply to the system (2.3.42). This then gives the desired result. □

2.8.2 Lifespan of Solutions

We have now established that the full water waves system (2.3.42) is locally well-posed. We now turn to address the question of how long these solutions persist. The theory of quasilinear hyperbolic equations suggests an $\mathcal{O}(\frac{1}{\varepsilon})$ lifespan in the small-data setting, given that our system is quadratically nonlinear [Kato2, Kato1, Maj2]. However, obtaining this existence time requires careful, delicate analysis. Our goal here is to prove that when $V_0 = 0$, we get an existence time of order $\mathcal{O}(\log \frac{1}{\varepsilon})$ as this can be done using the

energy estimates we have already obtained. On the other hand, when $V_0 \neq 0$, we simply show that the existence time is $O(\frac{1}{(1+|V_0|)^2})$. In a forthcoming paper, we will prove the quadratic $O(\frac{1}{\varepsilon})$ lifespan for small-data solutions when $V_0 = 0$. We first consider the case $V_0 = 0$ and then proceed to consider $V_0 \neq 0$.

Zero Background Current In considering the existence time of solutions, the background current V_0 plays a significant role. For example, even in the case of a flat initial free surface, the interaction of the background current with the obstacle may lead to large deviations in the free surface and the formation of splash singularities (see [AmbEtAl] for numerical simulations). Here we shall consider the lifespan of solutions in the special case where $V_0 = 0$. In that case, by Theorem 2.5.10, we have the following energy estimate:

$$\frac{d\mathcal{E}}{dt} \lesssim \mathfrak{P}(\mathcal{E}) = \mathcal{E} + \mathcal{E}^N, \tag{2.8.1}$$

where $N > 1$; recall that $\chi = 0$ when $V_0 = 0$. Further, the energy estimate (2.8.1) applies to the full water waves system as we discussed in the previous subsection.

As noted earlier, our goal here is to prove a "short" existence time using just the uniform energy estimate of Theorem 2.5.10 and some basic analysis tools. Specifically, we have the following result on the lifespan of solutions:

Lemma 2.8.3. *For s sufficiently large, the energy $\mathcal{E} = \mathcal{E}(t)$ of a solution to the full water waves system* (2.3.42) *with $V_0 = 0$ satisfies equation* (2.8.1)*. Further assume that the Cauchy data augmenting the system is small: $B = \varepsilon \ll 1$. Then, $\mathcal{E}$ remains bounded on $[0, T(\varepsilon)]$, where*

$$T(\varepsilon) \gtrsim \log \frac{1}{\varepsilon}, \tag{2.8.2}$$

which implies that solutions to the water waves system (2.3.42) *persist on a timescale of at least $O(\log \frac{1}{\varepsilon})$.*

Proof. We begin by writing $T(\varepsilon) = \frac{1}{2C} \log \varepsilon^{-1}$, where $C > 0$ is such that $\dot{\mathcal{E}} \leqslant C(\mathcal{E} + \mathcal{E}^N)$. We shall proceed by utilizing the bootstrapping principle. Namely, we assume that, for some $0 < r < 1$, we have $\mathcal{E}(t) \in [0, r]$ for all $0 \leqslant t \leqslant T(\varepsilon)$. We will then show that, for ε sufficiently small, $\mathcal{E}(t)$ is bounded above by $\frac{r}{2}$ for all $0 \leqslant t \leqslant T(\varepsilon)$. Via Grönwall's inequality, coupled with the fact that $\varepsilon \ll 1$ and $r \in (0, 1)$, one can obtain

$$\mathcal{E}(t) \leqslant K\varepsilon^{2-\frac{1}{2}(1+r^{N-1})} < K\varepsilon \ \ \forall t \in [0, T(\varepsilon)], \tag{2.8.3}$$

where $K > 0$ is such that $\mathcal{E}(0) \leqslant K\varepsilon^2$. Then, we can take ε sufficiently small so that

$$\mathcal{E}(t) < K\varepsilon < \frac{r}{2} \ \ \forall t \in [0, T(\varepsilon)]. \tag{2.8.4}$$

The bootstrapping principle then gives the desired result. □

Remark 2.8.4. *There is nothing special about $\frac{1}{2}$ and $\frac{r}{2}$ in the proof of Lemma 2.8.3. In fact, we can write $T(\varepsilon) = \frac{h}{C}\log\varepsilon^{-1}$ and, as long as $h \in (0,1)$, we can take ε sufficiently small so that*

$$\mathcal{E}(t) < K\varepsilon^{2-2h} < \varrho < r.$$

However, given that the lifespan we obtain in Lemma 2.8.3 is already far from sharp, we are not overly concerned with optimizing these constants.

Remark 2.8.5. *The proof of Lemma 2.8.3 clarifies the obstruction to obtaining the sharper $O(\frac{1}{\varepsilon})$ lifespan, videlicet the linear term $\mathcal{E}$ on the right-hand side of the energy estimate. If we could prove a slightly sharper energy estimate that eliminated this linear term, and so had an energy estimate of the form $\dot{\mathcal{E}} \lesssim \mathcal{E}^{\frac{3}{2}}$ + higher-order terms, then a couple simple modifications to the above Grönwall argument would give us the desired quadratic lifespan.*

In addition to the small-data result of Theorem 2.8.3, we also want to deduce a simple $O(1)$ lifespan in the case of large data when $V_0 = 0$. We do so presently.

Lemma 2.8.6. *Consider the energy of a solution to* (2.3.42)*, where we still take $V_0 = 0$. The energy of such a solution satisfies* (2.8.1) *as we have noted several times already. Then, $\mathcal{E}$ remains bounded on $[0, T(B)]$, where*

$$T(B) \gtrsim \frac{1}{B^{N-1}}. \tag{2.8.5}$$

In other words, solutions to (2.3.42) *with large Cauchy data have at least an $O(\frac{1}{B^{N-1}})$ lifespan. Recall again that B is defined in* (2.1.2).

Proof. Observe that, if $\mathcal{E} \sim 1$, we can rewrite (2.8.1) to obtain

$$\frac{d\mathcal{E}}{dt} \lesssim \mathcal{E}^N. \tag{2.8.6}$$

Now, write $T(B) = \frac{h}{C}\frac{1}{B^{N-1}}$, where $C > 0$ is such that $\dot{\mathcal{E}} \leqslant C\mathcal{E}$ and $h > 0$ shall be chosen shortly. Recall that, for some $K > 0$, we have $\mathcal{E}(0) \leqslant KB^2$. Assume that we have $\mathcal{E}(t) \in [0, 3KB^2]$ for all $0 \leqslant t \leqslant T$. Using Grönwall's inequality, we are able to obtain

$$\mathcal{E}(t) \leqslant KB^2 e^{(3K)^{N-1}h}. \tag{2.8.7}$$

Then, as it is straightforward to see, we can take h sufficiently small so that $0 \leqslant \mathcal{E}(t) < 2KB^2$ for all $t \in [0, T(B)]$. □

Non-Zero Background Current Here we shall suppose that $V_0 \neq 0$. In that case, our energy estimate is of the form

$$\frac{d\mathcal{E}}{dt} \lesssim \mathfrak{P}(\mathcal{E}) = \mathcal{E} + \mathcal{E}^N + (1 + |V_0|)(\sqrt{\mathcal{E}} + \mathcal{E}^M) \lesssim (1 + |V_0|)\sqrt{\mathcal{E}} + \mathcal{E}^N. \tag{2.8.8}$$

Again, we know from numerical experiments (see [AmbEtAl]) that, in this setting, splash singularities can occur in $O(1)$ time and so an $O(1)$ lifespan is the best we can hope to do. As such, we will just consider large data.

Lemma 2.8.7. *When $V_0 \neq 0$, the energy $\mathcal{E} = \mathcal{E}(t)$ of a solution to* (2.3.42) *satisfies equation* (2.8.8)*. Then, $\mathcal{E}$ remains bounded on $[0, T(B, |V_0|)]$ with*

$$T(B, |V_0|) \gtrsim \frac{1}{(1 + |V_0|)^2} \wedge \frac{1}{B^{N-1}}. \tag{2.8.9}$$

So, solutions in this setting have at least an $O(1)$ lifespan.

Proof. We begin by observing that we can use Young's inequality to rewrite the energy estimate (2.8.8) as follows:

$$\frac{d\mathcal{E}}{dt} \lesssim (1 + |V_0|)^2 + \mathcal{E} + \mathcal{E}^N \lesssim (1 + |V_0|)^2 + \mathcal{E}^N.$$

We shall again proceed by utilizing the bootstrapping principle, supposing that $\mathcal{E}(t) \in [0, 4KB^2]$ for all $t \in [0, T(B, |V_0|)]$, where $K > 0$ is such that $\mathcal{E}(0) \leqslant KB^2$. Write

$$T = \frac{h}{C}\left(\frac{1}{(1 + |V_0|)^2} \wedge \frac{1}{B^{N-1}}\right)$$

with $C > 0$ such that $\dot{\mathcal{E}} \leqslant C((1 + |V_0|)^2 + \mathcal{E}^N)$ and $h > 0$ to be chosen soon. Then, Grönwall's inequality

gives

$$\mathcal{E}(t) \leqslant \left(KB^2 + hC^{-1}\right)e^{(3K)^{N-1}h}. \tag{2.8.10}$$

Upon taking h to be sufficiently small (i.e., such that $\frac{h}{C} < KB^2$ and $e^{(3K)^{N-1}h} < \frac{3}{2}$), we have $0 \leqslant \mathcal{E}(t) < 3KB^2$ for all $0 \leqslant t \leqslant T(B, |V_0|)$. The bootstrapping principle then gives the desired result. □

2.9 The Damped System

2.9.1 Introduction

We have now proved all of the desired results for the undamped system and we are now ready to introduce the damper. Recall that the damping shall be effected via the application of an external pressure to a small portion of the free surface. We shall let $\omega \subset [0, 2\pi)$ be a connected interval on which we will damp the fluid and let χ_ω be a smooth, non-negative cut-off function, which is positive on ω. Here, we consider a single type of damping, where the external pressure is given by:

$$p_{\text{ext}} := \partial_x^{-1}(\chi_\omega \partial_x \varphi), \tag{2.9.1}$$

which we refer to as Clamond damping or linear $H^{1/2}$ damping. Recall that φ is the velocity potential. We also remind the reader that there is a Bernoulli constant $b(t)$ in (2.9.1) which we have chosen to ignore. That we have chosen to ignore it should not be taken to imply that it is unimportant in general, just that it is unimportant for our analysis.

Clamond damping, a type of modified sponge layer, was introduced by Clamond, et. al. in [CFGK2] in the context of $3d$ water waves, where the damping term was given by

$$p_{\text{ext},3d} = \nabla^{-1} \cdot (\chi_\omega \nabla \tilde{\phi}) - b(t). \tag{2.9.2}$$

In this formulation, ∇ is the horizontal gradient, χ_ω is essentially the same as above (a smooth function which is non-zero in the damping region and zero in the wave propagation region), $\tilde{\phi}$ is the velocity potential evaluated at the free surface, b is a Bernoulli constant and $\nabla^{-1} := \Delta^{-1}\nabla$; in other words,

$$\nabla^{-1} \cdot (\chi_\omega \nabla \tilde{\phi}) := -\frac{iD}{|D|^2} \cdot (\chi_\omega \nabla \tilde{\phi}). \tag{2.9.3}$$

Note that $D = (D_1, D_2)$ in (2.9.3). Equation (2.9.1) is just the analogue of (2.9.3) in $1d$.

Recall that $\xi(\alpha) = \alpha + \partial_\alpha^{-1}(s_\alpha \cos\theta(\alpha) - 1)$. Given that $x = \xi$ on $\mathcal{S}_t$, it follows that we have the following relation at the interface:

$$\partial_x = \frac{1}{s_\alpha \cos\theta(\alpha)} \partial_\alpha. \tag{2.9.4}$$

We can then invert ∂_x as follows:

$$\partial_x^{-1} u(\alpha) = \partial_\alpha^{-1}\big[s_\alpha \cos\theta(\alpha) u(\alpha)\big] \tag{2.9.5}$$

This allows us to rewrite p_{ext}:

$$p_{\text{ext}} = \partial_\alpha^{-1}\big[(s_\alpha \cos\theta)\chi_\omega (s_\alpha \cos\theta)^{-1} \partial_\alpha \varphi\big] = \partial_\alpha^{-1}(\chi_\omega \varphi_\alpha). \tag{2.9.6}$$

Note that the cut-off function χ_ω acts on $\xi(\alpha)$, not α, as we want to localize the effects of damping to a region of space. Further, we define $\varphi(\alpha) := \varphi(\xi(\alpha), \eta(\alpha))$.

2.9.2 New Evolution Equations

Given that we will effect the damping via the application of an external pressure, the damping will enter into the evolution equations via a modified pressure at the free boundary. Recall from the earlier discussion of the derivation of the evolution equations that the pressure only entered into the γ_t equation. We have, from [AmbEtAl], that

$$\gamma_t = -2p_\alpha + \frac{\partial_\alpha((V - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma)}{s_\alpha} - 2s_\alpha \mathbf{W}_t \cdot \hat{\mathbf{t}} - \frac{\gamma\gamma_\alpha}{2s_\alpha^2} - 2g\eta_\alpha + 2(V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{W}_\alpha \cdot \hat{\mathbf{t}}).$$

The damped dynamic boundary condition is given by

$$p\big|_{\mathcal{S}} = -\tau\kappa(\zeta) + p_{\text{ext}} = -\frac{\tau}{s_\alpha}\theta_\alpha + \partial_\alpha^{-1}(\chi_\omega \varphi_\alpha), \tag{2.9.7}$$

from which it follows that the damped γ_t equation is

$$\gamma_t = \frac{2\tau}{s_\alpha}\theta_{\alpha\alpha} - 2\chi_\omega\varphi_\alpha + \frac{\partial_\alpha((V - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma)}{s_\alpha} - 2s_\alpha \mathbf{W}_t \cdot \hat{\mathbf{t}} - \frac{\gamma\gamma_\alpha}{2s_\alpha^2} - 2g\eta_\alpha + 2(V - \mathbf{W}\cdot\hat{\mathbf{t}})(\mathbf{W}_\alpha \cdot \hat{\mathbf{t}}). \tag{2.9.8}$$

So, the only difference is that we have picked up a term proportional to $\chi_\omega \varphi_\alpha$.

The damped water waves system is then likewise of the form

$$\begin{cases} (\mathrm{id} + \mathscr{K}[\Theta])\partial_t \Theta = \mathfrak{F}_D(\Theta) \\ \Theta(t=0) = \Theta_0 \end{cases}, \tag{2.9.9}$$

where $\mathfrak{F}_D$ denote the right-hand side $\mathfrak{F}$ with the γ_t equation modified to effect Clamond damping; that is $\mathfrak{F}_{D,1} = \mathfrak{F}_1$, $\mathfrak{F}_{D,2} = \mathfrak{F}_2 - 2\chi_\omega \varphi_\alpha$, $\mathfrak{F}_{D,3} = \mathfrak{F}_3$ and $\mathfrak{F}_{D,4} = \mathfrak{F}_4$. Notice that the compact operator $\mathscr{K}[\Theta]$ is exactly the same as in the undamped water waves system (2.1.1). As before, we will simply prove energy estimates for a model problem and deduce the desired estimates for the full system from the mapping properties of $(\mathrm{id} + \mathscr{K}[\Theta])^{-1}$ (i.e., Lemma 2.8.1). Specifically, we consider the following damped model problem:

$$\begin{cases} \partial_t \Theta = \mathfrak{F}_D(\Theta) \\ \Theta(t=0) = \Theta_0 \end{cases}. \tag{2.9.10}$$

2.9.3 Energy Estimates and Analysis

In this section, we will show that the results obtained for the undamped model problem (2.1.4) also hold for the damped model problem (2.9.10). Given that, as noted above, the only difference in the evolution equations is a term proportional to $\chi_\omega \varphi_\alpha$ in the γ_t equation, we only need to ensure that this term does not derail the estimates. We begin by showing that Theorem 2.5.10 still holds in the presence of Clamond damping. Namely, we have the following theorem:

Theorem 2.9.1. *We define the energy $\mathcal{E}_{\text{damped}}$ of a solution to* (2.9.10) *in the same way (i.e., via Definition 2.5.1). Then, for s sufficiently large, we claim that $\mathcal{E}_{\text{damped}}$ satisfies*

$$\frac{d\mathcal{E}_{\text{damped}}}{dt} \lesssim \mathfrak{P}(\mathcal{E}_{\text{damped}}). \tag{2.9.11}$$

Proof. Notice that $\dot{\mathcal{E}}_{\text{damped}}$ contains the following terms that were not present in the undamped system:

$$-2\int_0^{2\pi} \gamma(\chi_\omega \varphi_\alpha)\, d\alpha, \tag{2.9.12}$$

$$-\frac{1}{2\tau s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha, \tag{2.9.13}$$

$$-\frac{1}{8\tau^2 s_\alpha^2}\int_0^{2\pi} \gamma(\chi_\omega\varphi_\alpha)(\partial_\alpha^{j-2}\gamma)^2\, d\alpha, \tag{2.9.14}$$

$$-\frac{1}{8\tau^2 s_\alpha^2}\int_0^{2\pi} \gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\, d\alpha, \tag{2.9.15}$$

where $2 \leqslant j \leqslant s+1$. As we noted above, the only term contributed by the damper is proportional to φ_α. The term φ_α may appear unfamiliar, but, in fact, it is a rather routine term. Indeed, we have

$$\varphi_\alpha = s_\alpha \nabla\varphi \cdot \hat{\mathbf{t}} = s_\alpha \mathbf{W}\cdot\hat{\mathbf{t}} + \frac{\gamma}{2}. \tag{2.9.16}$$

So, considering (2.9.12), we see that Lemma 2.5.4 in conjunction with the identity (2.9.16) immediately gives

$$-2\int_0^{2\pi} \gamma(\chi_\omega\varphi_\alpha)\, d\alpha \lesssim \|\gamma\|_{L^2}\|\varphi_\alpha\|_{L^2} \lesssim \|\gamma\|_{L^2}\left\|\mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{L^2} + \|\gamma\|_{L^2}^2 \lesssim \mathfrak{P}(\mathcal{E}_{\text{damped}}). \tag{2.9.17}$$

For (2.9.13), we can apply the estimate (2.5.63) and Lemma 1.3.9 to obtain

$$\begin{aligned}\frac{1}{2\tau s_\alpha}\int_0^{2\pi} (\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\Lambda(\partial_\alpha^{j-2}\gamma)\, d\alpha &\lesssim \|\chi_\omega\varphi_\alpha\|_{H^{s-1/2}}\|\gamma\|_{H^{s-1/2}} \lesssim \|\varphi_\alpha\|_{H^{s-1/2}}\|\gamma\|_{H^{s-1/2}} \\ &\lesssim \|\gamma\|_{H^{s-1/2}}\left(\left\|\mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} + \|\gamma\|_{H^{s-1/2}}\right)\end{aligned}$$

Notice that Lemma 2.5.4 allows us to control all of the terms in $\left\|\mathbf{W}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}}$ except for $\left\|\mathbf{BR}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}}$. In order to control this term, we represent the Birkhoff-Rott integral using (2.3.25) and then apply Lemmas 1.3.10, 1.3.7 and 2.4.5. Doing so gives

$$\begin{aligned}\left\|\mathbf{BR}\cdot\hat{\mathbf{t}}\right\|_{H^{s-1/2}} &\lesssim \left\|\zeta_\alpha \mathcal{H}\left(\frac{\gamma}{\zeta_\alpha}\right)\right\|_{H^{s-1/2}} + \|\zeta_\alpha K[\zeta]\gamma\|_{H^{s-1/2}} \\ &\lesssim \|\zeta_a\|_{H^{s-1/2}}\|\gamma\|_{H^{s-1/2}}(1+\|\zeta_d\|_{H^{s-1/2}}) + \|\zeta_\alpha\|_{H^{s-1/2}}\|K[z]\gamma\|_{H^s} \\ &\lesssim \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^2 + \|\gamma\|_{H^{s-1/2}}(1+\|\theta\|_{H^s})^4.\end{aligned} \tag{2.9.18}$$

We can then apply this estimate to finish estimating (2.9.13):

$$\frac{1}{2\tau s_\alpha}\int_0^{2\pi}(\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\Lambda(\partial_\alpha^{j-2}\gamma)\,d\alpha \lesssim \|\gamma\|_{H^{s-1/2}}(\|\mathbf{W}\cdot\hat{\mathbf{t}}\|_{H^{s-1/2}}+\|\gamma\|_{H^{s-1/2}}) \lesssim \mathfrak{P}(\mathcal{E}_{\text{damped}}). \tag{2.9.19}$$

We now apply the estimate (2.9.18) to (2.9.14):

$$\begin{aligned}\frac{1}{8\tau^2 s_\alpha^2}\int_0^{2\pi}\gamma(\chi_\omega\varphi_\alpha)(\partial_\alpha^{j-2}\gamma)^2\,d\alpha &\lesssim \left\|\gamma(\chi_\omega\varphi_\alpha)(\partial_\alpha^{j-2}\gamma)\right\|_{L^2}\left\|\partial_\alpha^{j-2}\gamma\right\|_{L^2}\\ &\lesssim \|\gamma\|_{H^{s-1/2}}^3\|\varphi_\alpha\|_{H^{s-1/2}}\\ &\lesssim \|\gamma\|_{H^{s-1/2}}^3(\|\mathbf{W}\cdot\hat{\mathbf{t}}\|_{H^{s-1/2}}+\|\gamma\|_{H^{s-1/2}})\\ &\lesssim \mathfrak{P}(\mathcal{E}_{\text{damped}}).\end{aligned} \tag{2.9.20}$$

Finally, we consider (2.9.15) and here we can just use (2.9.20). We have

$$\begin{aligned}\frac{1}{8\tau^2 s_\alpha^2}\int_0^{2\pi}\gamma^2(\partial_\alpha^{j-2}\gamma)(\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\,d\alpha &\lesssim \left\|\gamma(\partial_\alpha^{j-2}\gamma)\right\|_{L^2}\left\|\gamma(\partial_\alpha^{j-2}\chi_\omega\varphi_\alpha)\right\|_{L^2}\\ &\lesssim \|\gamma\|_{H^{s-1/2}}^3\|\varphi_\alpha\|_{H^{s-1/2}}\\ &\lesssim \mathfrak{P}(\mathcal{E}_{\text{damped}}).\end{aligned} \tag{2.9.21}$$

□

Remark 2.9.2. *The proofs of Corollary 2.6.3, Theorem 2.7.1 and Theorem 2.7.3 will either go through in the damped setting exactly as written or require at most minor modifications. Proving damped versions of Theorem 2.6.1, Theorem 2.6.4 and Theorem 2.7.2 require considering energy estimates for the differences. However, as in the above case, the added damping term will cause no problems in these estimates, particularly given that the term contributed by the damper can be expressed in a way that only contains terms we have already estimated. As such, we omit these calculations. Finally, given that Theorem 2.5.10 applies to the damped system, all of our results on the lifespan of solutions (Lemma 2.8.3, Lemma 2.8.6 and Lemma 2.8.7) also apply to the damped system.*

Following Remark 2.9.2, we have the following theorem:

Theorem 2.9.3. *Let s be sufficiently large. The damped model problem* (2.9.9) *is locally-in-time well-posed (in the sense of Hadamard) and the unique solution Θ is in $C([0,T(B,|V_0|)];\mathfrak{X})$, where B is defined in* (2.1.2).

The flow map is Lipschitz continuous from X_1 into $C([0,T];X_1)$. In the context of small Cauchy data $B = \varepsilon \ll 1$, we have

$$T(\varepsilon) \gtrsim \log \frac{1}{\varepsilon} \text{ for } V_0 = 0. \tag{2.9.22}$$

For large Cauchy data, we have

$$T(B,|V_0|) \gtrsim \begin{cases} B^{1-N} & V_0 = 0 \\ (1+|V_0|)^{-2} \wedge B^{1-N} & V_0 \neq 0 \end{cases}, \tag{2.9.23}$$

where N is a parameter given in equation (2.5.58).

Remark 2.9.4. *From Theorem 2.9.3, we see that the stated claims hold for the damped model problem* (2.9.10). *By the solvability result of [AmbEtAl] (or of Section 2.10) and Lemma 2.8.1, we can, exactly as in the undamped case, extend the desired results to the full damped water waves system* (2.9.9). *This proves Theorem 2.2.4.*

2.10 Invertibility of $\mathrm{id} + \mathscr{K}$

Our objective in this section is to provide a proof of the solvability of the $(\gamma_t, \omega_t, \beta_t)$ system of integral equations in a multiconnected, horizontally-periodic domain with a bottom. Solvability was proved in [AmbEtAl], but we include this result as it is achieved via alternative means and our approach can be more readily extended to higher dimensions. In proving that this system is solvable, we follow the work of Schiffer in [Sch]. However, in order to apply these results, we will need to ensure that the periodic Green function defined via the cotangent kernel shares some basic properties with the non-periodic free space Green function. We now turn our attention to this issue.

2.10.1 Properties of the Periodic Green Function

For $x, y \in \mathbb{R}^2$, we denote by $N = N(x,y)$ the fundamental solution to Laplace's equation; that is, $N(x,y) := -\frac{1}{2\pi}\log|x-y|$. For $z, w \in \mathbb{C}$, we extend the definition of N in the natural way. Then, we have

$$\partial_{n_y} N(x,y) = \frac{1}{2\pi}\frac{(x-y)\cdot n_y}{|x-y|^2} \tag{2.10.1}$$

and subsequently set

$$k(x,y) := \partial_{n_y} N(x,y),\ k(z,w) := \frac{1}{2\pi}\frac{(z-w)^* n_w^{\mathbb{C}}}{|z-w|^2}, \tag{2.10.2}$$

where $z = \mathfrak{C}(x)$, $w = \mathfrak{C}(y)$ and $n^{\mathbb{C}}$ satisfies

$$(a,b)\cdot n_y = \mathfrak{Re}\{(a+ib)^* n_w^{\mathbb{C}}\}.$$

In this case we have $k(x,y) = \mathfrak{Re}\, k(z,w)$. Using an identity of Mittag-Leffler, we can transform the integral kernel:

$$\mathrm{pv}\sum_j k(z+2\pi j, w) = \mathrm{pv}\sum_j \frac{n_w^{\mathbb{C}}}{2\pi}\frac{1}{z+2\pi j - w} = \frac{1}{4\pi} n_w^{\mathbb{C}} \cot\frac{1}{2}(z-w); \tag{2.10.3}$$

see [AbFo] or [AmbEtAl] for details.

For the sake of compactness, we introduce some new notation. Let Σ denote the boundary of Ω; that is, $\Sigma := \partial\Omega = \mathcal{S}\cup\mathcal{B}\cup\mathcal{C}$. As before, Ω denotes the fluid domain. Lastly, we make a note regarding the convention we follow with regard to the unit normal since it differs slightly from the convention used until now. In this section, we let n_P denote the inward-pointing normal at $P\in\Sigma$. Hence, for $P\in\mathcal{S}$, we have $n_P = -\hat{\mathbf{n}}(\tilde{\alpha})$, where $\zeta(\tilde{\alpha}) = P$.

Lemma 2.10.1. *It holds that*

$$\int_\Sigma k(z,P)\, d\sigma(P) = \begin{cases} 1 & z\in\Omega \\ \frac{1}{2} & z\in\Sigma \\ 0 & z\in\complement\Omega \end{cases}, \tag{2.10.4}$$

with σ denoting surface measure on Σ.

Proof. We follow the proof in [Fol] for the non-periodic free space Green function, extending it to the periodic case.

($z\in\complement\Omega$) Fix $z\in\complement\Omega$ and observe that the map $P\mapsto N(z,P)$ is C^∞ in $\overline{\Omega}$, and harmonic on Ω. We can therefore apply Green's formula to get

$$0 = \int_\Sigma \partial_{n_P} N(z,P)\, d\sigma(P) = \int_\Sigma k(z,P)\, d\sigma(P),$$

as desired.

($z\in\Omega$) Fix $z\in\Omega$, pick $\varepsilon>0$ such that $B_\varepsilon = B_\varepsilon(z) \Subset \Omega$, set $\Omega^\varepsilon = \Omega - \overline{B}_\varepsilon$ and $S_\varepsilon = S_\varepsilon(z) = \partial B_\varepsilon(z)$. Observe that the map $P\mapsto N(z,P)$ satisfies the same conditions as above on Ω^ε as opposed to Ω. Therefore,

following an application of Green's formula, we have

$$0 = \int_\Sigma k(z,P)\, d\sigma(P) + \int_{S_\varepsilon} k(z,P)\, d\sigma_\varepsilon(P),$$

with σ_ε being the surface measure on S_ε. So, we will need to evaluate $\int_{S_\varepsilon} k(z,\cdot)\, d\sigma_\varepsilon$. First, let us rewrite $k(z,\cdot)$ on S_ε. Notice that $n_P^{\mathbb{C}} = \varepsilon^{-1}(P-z)$. Write $z - P = \varepsilon e^{i\vartheta}$ for $\vartheta \in [0, 2\pi)$ and observe that

$$k(z, z-\varepsilon e^{i\vartheta}) = -\frac{e^{i\vartheta}}{4\pi}\cot\frac{\varepsilon}{2}e^{i\vartheta} = -\frac{1}{2\pi\varepsilon} + O(\varepsilon),$$

since $\cot z = \frac{1}{z} + O(|z|)$. It then follows that

$$0 = \int_\Sigma k(z,P)\, d\sigma(P) - \frac{\sigma(S_\varepsilon)}{2\pi\varepsilon} + O\left(\int_{S_\varepsilon} \varepsilon\, d\sigma\right) = \int_\Sigma k(z,P)\, d\sigma(P) - 1 + O(\varepsilon^2).$$

Sending $\varepsilon \to 0^+$ yields the desired equality.

($z \in \Sigma$) Lastly, fix $z \in \Sigma$ and let $\varepsilon > 0$. Set $B_\varepsilon = B_\varepsilon(z)$ and, recalling that $S_\varepsilon = \partial B_\varepsilon$, denote $\Sigma^\varepsilon = \Sigma - (\Sigma \cap B_\varepsilon)$, $S'_\varepsilon = S_\varepsilon \cap \Omega$ and $S''_\varepsilon = \{y \in S_\varepsilon : n_z \cdot y < 0\}$. Again, we observe that the mapping $P \mapsto N(z,P)$ is harmonic in $\Omega - \overline{B}_\varepsilon$ and C^∞ up to the boundary $\Sigma^\varepsilon \cup S'_\varepsilon$. So,

$$0 = \int_{\Sigma^\varepsilon} k(z,P)\, d\sigma(P) + \int_{S'_\varepsilon} k(z,P)\, d\sigma_\varepsilon(P).$$

We infer that

$$\begin{aligned}\int_\Sigma k(z,P)\, d\sigma(P) &= \lim_{\varepsilon\to 0^+}\int_{\Sigma^\varepsilon} k(z,P)\, d\sigma(P) = -\lim_{\varepsilon\to 0^+}\int_{S'_\varepsilon} k(z,P)\, d\sigma_\varepsilon(P) = \lim_{\varepsilon\to 0^+}\left\{\frac{\sigma_\varepsilon(S'_\varepsilon)}{2\pi\varepsilon} + O\left(\int_{S'_\varepsilon}\varepsilon\, d\sigma_\varepsilon\right)\right\}\\ &= \lim_{\varepsilon\to 0^+}\frac{\sigma_\varepsilon(S'_\varepsilon)}{2\pi\varepsilon}.\end{aligned}$$

So, we need only compute $\sigma_\varepsilon(S'_\varepsilon)$. To this end, we observe that, due to the regularity of the boundary, the symmetric difference of S'_ε and S''_ε is contained in an "equatorial strip" with measure $O(\varepsilon^2)$. Whence it follows that $\sigma_\varepsilon(S'_\varepsilon) = \sigma_\varepsilon(S''_\varepsilon) + O(\varepsilon^2) = \pi\varepsilon + O(\varepsilon^2)$. Putting this all together, we get

$$\int_\Sigma k(z,P)\, d\sigma(P) = \lim_{\varepsilon\to 0^+}\left\{\frac{\pi\varepsilon + O(\varepsilon^2)}{2\pi\varepsilon}\right\} = \frac{1}{2}.$$

This completes the proof. □

For $\phi \in C(\Sigma)$ we may define

$$u(x) := \int_\Sigma k(x,P)\phi(P)\, d\sigma(P).$$

Then, for $h \in \mathbb{R}$ small and nonzero, we define $u_h(P) := u(P + hn_P)$ for $P \in \Sigma$ and note that we have $P + hn_P \in \Omega$ for $h > 0$ and $P + hn_P \in \complement\Omega$ for $h < 0$.

Lemma 2.10.2. *For $P \in \Sigma$, set*

$$u_+(P) := \lim_{h\to 0^+} u_h(P),\ u_-(P) := \lim_{h\to 0^-} u_h(P).$$

Then, we claim that

$$u_+(P) = -\frac{1}{2}\phi(P) + \int_\Sigma k(P,Q)\phi(Q)\, d\sigma(Q),\ u_-(P) = \frac{1}{2}\phi(P) + \int_\Sigma k(P,Q)\phi(Q)\, d\sigma(Q).$$

Proof. We again follow the proof given in [Fol] to extend to the periodic case. Fix $P \in \Sigma$ and $h > 0$ sufficiently small. Then, as noted above, $P + hn_P \in \Omega$ and thus

$$\begin{aligned} u_h(P) &= \phi(P)\int_\Sigma k(P+hn_P,Q)\, d\sigma(Q) + \int_\Sigma k(P+hn_P,Q)(\phi(Q)-\phi(P))\, d\sigma(Q) \\ &= \int_\Sigma k(P+hn_P,Q)(\phi(Q)-\phi(P))\, d\sigma(Q). \end{aligned}$$

Continuity then implies that

$$\lim_{h\to 0^+} u_h(P) = -\phi(P)\int_\Sigma k(P,Q)\, d\sigma(Q) + \int_\Sigma k(P,Q)\phi(Q)\, d\sigma(Q) = -\frac{1}{2}\phi(P) + \int_\Sigma k(P,Q)\phi(Q)\, d\sigma(Q).$$

On the other hand, for $h < 0$, we have

$$\begin{aligned} u_h(P) &= \phi(P)\int_\Sigma k(P+hn_P,Q)\, d\sigma(Q) + \int_\Sigma k(P+hn_P,Q)(\phi(Q)-\phi(P))\, d\sigma(Q) \\ &= \phi(P) + \int_\Sigma k(P+hn_P,Q)(\phi(Q)-\phi(P))\, d\sigma(Q). \end{aligned}$$

It then follows, again from continuity, that

$$\lim_{h\to 0^-} u_h(P) = \phi(P) - \phi(P)\int_\Sigma k(P,Q)\,d\sigma(Q) + \int_\Sigma k(P,Q)\phi(Q)\,d\sigma(Q) = \frac{1}{2}\phi(P) + \int_\Sigma k(P,Q)\phi(Q)\,d\sigma(Q).$$

□

2.10.2 Solvability of the System

With this machinery in place, we now want to consider the Fredholm eigenvalues of the operator specialized to the water waves problem. We begin by observing that Lemma 2.10.2 holds in the case of the complexified kernel. That is, if we define $u_+(\wp)$ and $u_-(\wp)$ for complex $\wp \in \Sigma$ in the natural way, then the same jump relations at the boundary given in Lemma 2.10.2 will hold. We now define the relevant operator $T_k[\cdot]$ by $T_k[\cdot] : \phi \mapsto 2\int_\Sigma k(\cdot,\wp)\phi(\wp)\,d\sigma(\wp)$. We shall let ϕ_ν denote the eigenfunctions of $T_k[\cdot]$ on S. In other words, we take the ϕ_ν to solve

$$\phi_\nu(\cdot) = 2\lambda_\nu \int_\Sigma k(\cdot,\wp)\phi_\nu(\wp)\,d\sigma(\wp) \qquad (\text{on } \Sigma). \tag{2.10.5}$$

Observe that the λ_ν's aren't exactly the eigenvalues corresponding to the ϕ_ν's, rather the eigenvalues are of the form $\mu_\nu = \lambda_\nu^{-1}$. Additionally, we define

$$2\lambda_\nu \int_\Sigma k(z,\wp)\phi(\wp)\,d\sigma(\wp) = \begin{cases} h_\nu(z) & z \in \Omega \\ \tilde{h}_\nu(z) & z \in \complement\Omega \end{cases}. \tag{2.10.6}$$

It shall also be worthwhile to consider the complex derivatives of h_ν and $\tilde{h}_\nu$, which give rise to the dual formulation of the Fredholm eigenvalue problem. Thus, we introduce the holomorphic functions

$$v_\nu(z) = \partial_z h_\nu(z),\ \tilde{v}_\nu(z) = \partial_z \tilde{h}_\nu(z). \tag{2.10.7}$$

Then, we can apply Lemma 2.10.2 to evaluate the limit of the various h's as z tends to the boundary. In

particular,

$$\begin{aligned}
\lim_{z\to\varkappa\in\Sigma} h_\nu(z) &= -\lambda_\nu\phi_\nu(\varkappa) + \lambda_\nu \int_\Sigma k(\varkappa,\wp)\phi_\nu(\wp)\, d\sigma(\wp) \\
&= (1-\lambda_\nu)\phi_\nu(\varkappa). \\
\lim_{z\to\varkappa\in\Sigma} \tilde{h}_\nu(z) &= \lambda_\nu\phi_\nu(\varkappa) + \lambda_\nu \int_\Sigma k(\varkappa,\wp)\phi_\nu(\wp)\, d\sigma(\wp) \\
&= (1+\lambda_\nu)\phi_\nu(\varkappa)
\end{aligned} \tag{2.10.8}$$

Further, it clearly holds that

$$\partial_n h_\nu^j\big|_\Sigma = \partial_n \tilde{h}_\nu\big|_\Sigma. \tag{2.10.9}$$

If we let $z = z(s)$ parameterize Σ by arclength, then we can combine (2.10.8) and (2.10.9) into a single equation relating v_ν and $\tilde{v}_\nu$:

$$\tilde{v}_\nu(z)\frac{dz}{ds} = \frac{1}{1-\lambda_\nu} v_\nu(z)\frac{dz}{ds} + \frac{\lambda_\nu}{1-\lambda_\nu}\overline{v_\nu(z)\frac{dz}{ds}} \qquad (z = z(s)). \tag{2.10.10}$$

Utilizing (2.10.10), we can formulate a set of integral equations solved by the v's:

$$-2\lambda_\nu \int_\Omega \overline{\frac{v_\nu(w)}{(w-z)^2}}\, dm^2(w) = \begin{cases} v_\nu(z) & z\in\Omega \\ (1-\lambda_\nu)\tilde{v}_\nu(z) & z\in\complement\Omega \end{cases}, \tag{2.10.11}$$

$$2\lambda_\nu \int_{\complement\Omega} \overline{\frac{\tilde{v}_\nu(w)}{(w-z)^2}}\, dm^2(w) = \begin{cases} (1+\lambda_\nu)v_\nu(z) & z\in\Omega \\ \tilde{v}_\nu(z) & z\in\complement\Omega \end{cases} \tag{2.10.12}$$

where m^2 denotes two-dimensional Lebesgue measure. See [Sch] for further details.

We now see that the periodic h and v functions defined via the cotangent kernel satisfy the same boundary jump relations as those defined via the non-periodic free space Green function. We can utilize the boundary jump relations of (2.10.8) to prove that

$$\int_\Omega |v_\nu|^2\, dm^2 = \frac{\lambda_\nu+1}{\lambda_\nu-1}\int_{\complement\Omega} |\tilde{v}_\nu|^2\, dm^2; \tag{2.10.13}$$

see [Sch] for details. As in [Sch], we deduce from (2.10.13) that $|\lambda_\nu| \geqslant 1$. What remains then is to show that

$\lambda_v \neq 1$. Following [Sch] or [Fol], we see that, in the non-simply-connected setting, there is a nontrivial kernel corresponding to the integral equations for the h functions. In fact, the kernel is spanned by χ_C, the characteristic function of the boundary of the obstacle. However, a key point is that in the vortex sheet (or layer potential) formulation of the water waves problem, we are generally more interested in the gradient of the potentials as opposed to the potentials themselves. That is to say, the vortex sheet formulation of the water waves problem is a "v-problem" – what we are really interested in is the kernel corresponding to the v's. Given that the kernel of the h functions is spanned by a constant function, it is clear that the corresponding kernel for the v functions will be trivial. This is exactly as desired for we may now apply the Fredholm alternative to deduce that the inhomogeneous system of integral equations under consideration is solvable (via Neumann series). That is, we have now proved the following theorem.

Theorem 2.10.3. *The system of Fredholm integral equations of the second kind for $(\gamma_t, \omega_t, \beta_t)$ is solvable.*

Remark 2.10.4. *The above analysis also shows that the system arising from the Cauchy integral formulation in [AmbEtAl] is solvable, subject to a minor modification. The Cauchy integral formulation is dual to the vortex sheet formulation and corresponds to an "h-problem", which is dual to the "v-problem". This implies, as noted above, that the integral equations have a non-trivial, but finite-dimensional, kernel, which is spanned by χ_C. Thus, the system has a Fredholm pseudoinverse. In particular, the system is invertible upon applying a rank-one correction, which projects away from the kernel. This is exactly the process used to invert the system in [AmbEtAl].*

CHAPTER 3

A Toy Model for Damped Water Waves

3.1 An Introduction to the Toy Model

Up until now, we have been focused on the vortex sheet formulation of the water waves system. We are going to switch gears a bit and consider the water waves problem from the point of view of the Zakharov-Craig-Sulem formulation [Zak, CrSu], however, as we shall discuss shortly, the vortex sheet formulation is still very much part of our considerations. We shall also change a few assumptions made in Chapter 2, viz. assuming that we no longer have obstacles in the flow (i.e., the domain is simply connected), that the domain is of infinite depth and that the location of the free boundary is given by the graph of a function η. To summarize these changes, we have

$$\Omega_t \coloneqq \{(x,y) \in \mathbb{T} \times \mathbb{R} : -\infty < y < \eta(x,t)\}, \tag{3.1.1}$$

$$\mathcal{S}_t \coloneqq \partial\Omega_t = \{(x,y) \in \mathbb{T} \times \mathbb{R} : y = \eta(x,t)\}. \tag{3.1.2}$$

Of course, we shall still have a scalar potential φ such that $\mathbf{v} = \nabla\varphi$, where $\mathbf{v}$ is the fluid velocity field in (1.1.1). Let ψ denote the trace of the velocity potential φ along the free boundary $\mathcal{S}_t$. Then, (η, ψ) solves

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta - \tau H(\eta) + \dfrac{1}{2}(\partial_x \psi)^2 - \dfrac{1}{2}\dfrac{(G(\eta)\psi + \partial_x \eta \partial_x \psi)^2}{1 + (\partial_x \eta)^2} = -p_{\text{ext}} \end{cases}, \tag{3.1.3}$$

where p_{ext} is the external pressure which effects the damping and is given by (1.1.3), $G(\eta)$ is the (normalized) Dirichlet-Neumann map given by

$$\begin{aligned} G(\eta)\psi(x,t) &\coloneqq \sqrt{1 + (\partial_x \eta(x,t))^2}\,\partial_{\hat{\mathbf{n}}}\varphi(x, \eta(x,t), t) \\ &= \partial_y \varphi(x, \eta(x,t), t) - \partial_x \eta(x,t)\partial_x \varphi(x, \eta(x,t), t) \end{aligned} \tag{3.1.4}$$

and we let $\hat{\mathbf{n}}$ denote the outward unit normal vector field on $\mathcal{S}_t$. We take $H(\eta)$ to denote the mean curvature

of the free surface:

$$H(\eta) := \partial_x \left(\frac{\partial_x \eta}{\sqrt{1 + (\partial_x \eta)^2}} \right). \tag{3.1.5}$$

The system (3.1.3) is known as the Zakharov-Craig-Sulem formulation of the water waves system.

Our toy model will be built from the paradifferential equation for the water waves system. This paradifferential equation originates in the beautiful work of Alazard-Burq-Zuily [ABZ1], which considers (3.1.3) with $p_{\text{ext}} \equiv 0$. The paradifferential approach to the study of water waves began with the work of Alazard-Métivier on the regularity of three-dimensional diamond waves [AlMe]. We briefly recall that construction in the context of the $2d$ gravity-capillary water waves system, referring to [ABZ1] for the details. Let (V, B) be the trace of $\nabla\varphi$ along the free surface. The paralinearized system is

$$\begin{cases} \partial_t \eta + T_V \partial_x \eta - T_\lambda u = f_1 \\ \partial_t u + T_V \partial_x u + T_\mu \eta = f_2 \end{cases}. \tag{3.1.6}$$

In (3.1.6), $u := \psi - T_B \eta$ is the good unknown of Alinhac. For more on the good unknown, see [Ali1, Ali2]. For an exposition of paracomposition, the setting in which the good unknown arises, see [Tay5]. For an interesting application of the good unknown to the relativistic or nonrelativistic compressible Euler equations, see [Tra]. Further, $\mu = \mu(x, \xi)$ is a symbol of order 2 such that

$$H(\eta) = -T_\mu \eta + f_H. \tag{3.1.7}$$

We use f_H and f_j to denote the smooth(er) remainder terms, which obey nice estimates. Finally, $\lambda = \lambda(x, \xi)$ is (related to) the symbol of the Dirichlet-Neumann map, which we discuss further presently.

It has been known since the work of Calderón that, at least when $\eta \in C^\infty$, $G(\eta)$ is a classical, elliptic ΨDO of order one [Cal]. As such, its symbol admits an asymptotic expansion:

$$\lambda(x, \xi) \sim \sum_{j \geqslant 0} \lambda_{1-j}(x, \xi), \tag{3.1.8}$$

where $\lambda_{1-j}(x, \xi)$ is homogeneous of degree $1 - j$ in ξ. In d dimensions, the principal symbol is given by

$$\lambda_1(x, \xi) = \sqrt{(1 + \left|\nabla\eta(x)\right|^2)|\xi|^2 - (\nabla\eta(x) \cdot \xi)^2}. \tag{3.1.9}$$

The remaining symbols can be computed by induction and, as j increases, the symbols involve higher derivatives of η with λ_{1-j} depending on derivatives of η up to order $|1-j|+2$ [ABB]. Direct computation verifies that in dimension two we have $\lambda_1(x,\xi) = |\xi|$ and $\lambda_0(x,\xi) = 0$, so that $\lambda(x,\xi) = |\xi| + \text{remainder}$. So, in dimensions three and higher, $G(\eta)$ is a ΨDO, however, in $2d$, $G(\eta)$ is a Fourier multiplier which is independent of the domain (at least at the principal level)! This dramatic simplification really points up the phenomenological distinction between $2d$ and $3d$ water waves.

Given the above, we can see that the situation becomes quite a bit more complex when $\eta \notin C^\infty$. In particular, $G(\eta)$ is then a ΨDO with symbol of limited smoothness. To work with symbols of limited regularity, one must use some form of symbol smoothing (e.g., paradifferential analysis). See [Tay4, Tay5] for more on symbol smoothing. Further, the full asymptotic expansion in (3.1.8) fails to be meaningful when η is not C^∞. Nevertheless, we can extend the definition of λ to the case where $\eta \notin C^\infty$ by truncating the asymptotic expansion and only keeping the terms which are meaningful. For example, if η is C^k, but not C^{k+1}, we could set $\lambda = \lambda_1 + \ldots + \lambda_{-k+2}$. In the low-regularity setting of Alazard-Burq-Zuily, η was only at least C^2 and this led to

$$\lambda = \lambda_1 + \lambda_0. \tag{3.1.10}$$

Of course, in this setting, it is no longer the case that $G(\eta) = \mathrm{Op}(\lambda)$. Indeed, for λ as in (3.1.10), we have

$$G(\eta)\psi = T_\lambda u - T_V \partial_x \eta + f_G, \tag{3.1.11}$$

where f_G, once again, contains much smoother remainder terms.

With (3.1.6) established, one can then find a symmetrizer of the form

$$S = \begin{pmatrix} T_p & 0 \\ 0 & T_q \end{pmatrix}, \tag{3.1.12}$$

where p is of order $\frac{1}{2}$ and q is of order 0, such that

$$S \begin{pmatrix} 0 & -T_\lambda \\ T_\mu & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & -T_\gamma \\ T_\gamma^* & 0 \end{pmatrix} S \tag{3.1.13}$$

up to acceptable remainder terms. One can then show that

$$\partial_t S \begin{pmatrix} \eta \\ u \end{pmatrix} + T_V \partial_x S \begin{pmatrix} \eta \\ u \end{pmatrix} + J \begin{pmatrix} T_\gamma^* & 0 \\ 0 & T_\gamma \end{pmatrix} S \begin{pmatrix} \eta \\ u \end{pmatrix} = F, \tag{3.1.14}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ and J with $i = \sqrt{-1}$, we can restate (3.1.14) as a single equation for a complex-valued unknown. Defining $U := T_p\eta + iT_q u$, we have the following paradifferential equation for the water waves system:

$$\partial_t U + T_V \partial_x U + iT_\gamma U = f, \tag{3.1.15}$$

where $\gamma = \gamma(x, \xi, t)$ is an elliptic symbol of order $\frac{3}{2}$. Of course, the right-hand side f consists of smooth(er) remainder terms. A similar result can be obtained for the gravity water waves system, however in that case γ is of order $\frac{1}{2}$ [ABZ3].

We want to consider a damped form of (3.1.15). This leads us to consider the following toy model for the (two-dimensional) water waves system subject to Clamond damping:

$$\begin{cases} \partial_t U + W(U)\partial_x U + iLU + \chi_\omega U = 0 \\ U(t=0) = U_0 \in H^\sigma \end{cases}. \tag{3.1.16}$$

In the above, the unknown $U : \mathbb{T} \to \mathbb{C}$ and L is defined by

$$L := |D|^\alpha \text{ for } \alpha \in (0, 2], \tag{3.1.17}$$

Notice that $W(U)$ is the toy model counterpart to the paraproduct operator T_V in (3.1.15). For some integer $N \gg 0$, W is continuous from $H^{s-N} \to H^s$:

$$\|W(U)\|_{H^s} \lesssim \left\|\langle D\rangle^{-N} U\right\|_{H^s}. \tag{3.1.18}$$

Further, $W(\cdot)$ is real-valued, satisfies $W(U) = W(U^*)$ and scales linearly in U (i.e., $W(\cdot)$ is homogeneous of

degree one: $W(\lambda U) = \lambda W(U)$ for $\lambda \in \mathbb{R}$). Finally, χ_ω corresponds to the cut-off function in (1.1.3). Notice that $\alpha = \frac{3}{2}$ in (3.1.17) corresponds to capillary waves and $\alpha = \frac{1}{2}$ corresponds to gravity waves. For discussion of the Cauchy problem for a similar toy model see [Ala4].

We shall be working in the small-data setting and so assume that

$$\|U_0\|_{H^\sigma} = \varepsilon \ll 1. \tag{3.1.19}$$

Our aim will be to show that solutions to (3.1.16) exist on timescale $O(\frac{1}{\varepsilon})$. We will achieve this by rescaling U: $U(x,t) = \varepsilon v(x, \varepsilon t)$. Then, our objective will be to simply obtain uniform in ε estimates on the solution v to the equation

$$\begin{cases} \partial_t v + W_\varepsilon(v)\partial_x v + \frac{i}{\varepsilon} L v + \frac{1}{\varepsilon}\chi_\omega v = 0 \\ v(t=0) = v_0 \in H^\sigma \end{cases}, \tag{3.1.20}$$

where $W_\varepsilon(v) \coloneqq \varepsilon^{-1} W(\varepsilon v)$.

Notice that, due to the scaling linearity and (3.1.18), $W_\varepsilon(v)$ satisfies

$$\left\|\partial_x^k W_\varepsilon(v)\right\|_{L^\infty} \lesssim 1 \text{ for } k = 0, 1, \tag{3.1.21}$$

where the estimate is uniform in ε. We shall further assume that W commutes with differentiation with respect to time:

$$\partial_t W(U) = W(\partial_t U). \tag{3.1.22}$$

We justify this assumption by recourse to the properties of the paraproduct operator T_V. Observe that $\partial_t(T_V U) = T_V \partial_t U + T_{\partial_t V} U$, which can be seen by differentiating equation (1.3.41). Then, Lemma 1.3.4 gives the estimate

$$\|\partial_t(T_V U)\|_{H^s} \lesssim \|V\|_{L^\infty}\|\partial_t U\|_{H^s} + \|\partial_t V\|_{L^\infty}\|U\|_{H^s},$$

provided that V and $\partial_t V$ are L^∞. Finally, since $\partial_t U \approx |D|^\alpha U$ (at least when $\alpha > 1$), we see that $T_{\partial_t V} U$ can be considered a lower-order remainder term. If $\alpha \leqslant 1$, we have $\partial_t U \approx W(U)\partial_x U + i|D|^\alpha U$ and this will still represent the highest-order term. This justifies our assumption (3.1.22) as long as we take V and $\partial_t V$ to be L^∞.

Regarding whether V and $\partial_t V$ are L^∞, we first note that we are not considering rough solutions/data and

so we can simply assume that V has enough regularity that the desired inclusion holds via Sobolev embedding. Nevertheless, in the low-regularity context of [ABZ1], this assumption is verified. In particular, V is H^{s-1} for $s > 2 + \frac{d}{2}$. Additionally,

$$\partial_t V + (V \cdot \nabla)V + a\zeta = 0,$$

where $\zeta = \nabla\eta$ and a is the Taylor coefficient ($a = -\partial_y p\big|_{y=\eta}$ with y being the vertical coordinate). We have $(V \cdot \nabla)V \in H^{s-2}$ and $\zeta \in H^{s-1/2}$. Finally, a is $H^{s-3/2}$. Hence, $\partial_t V$ is H^{s-2}, which is just enough to have $\partial_t V \in L^\infty$. All of the above details and more can be found in [ABZ1, ABZ3].

3.1.1 Connections with the Vortex Sheet Formulation

Though our toy model (3.1.16) is most clearly related to the Zakharov-Craig-Sulem formulation, the toy model, and indeed this entire chapter, still has the vortex sheet formulation in mind. Ultimately, the objective is the $O(\frac{1}{\varepsilon})$ lifespan for the vortex sheet water waves system which was the object of discussion in the previous chapter. We would like to take a few moments to discuss how the contents of this chapter relate to achieving that objective.

To do so in the simplest setting possible, we consider the infinite-depth vortex sheet equations (with no obstacles):

$$\begin{cases} \theta_t &= \dfrac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \dfrac{1}{s_\alpha}(V - \mathbf{BR}\cdot\hat{\mathbf{t}})\theta_\alpha + \dfrac{1}{s_\alpha}\mathbf{m}\cdot\hat{\mathbf{n}} \\ \gamma_t &= \dfrac{2\tau}{s_\alpha}\theta_{\alpha\alpha} + \dfrac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) + \dfrac{\gamma_\alpha}{s_\alpha}(V - \mathbf{BR}\cdot\hat{\mathbf{t}}) + \dfrac{\gamma}{s_\alpha}(s_{\alpha t} - \mathbf{m}\cdot\hat{\mathbf{t}}). \\ & \quad -2s_\alpha\mathbf{BR}_t\cdot\hat{\mathbf{t}} - \dfrac{\gamma\gamma_\alpha}{2s_\alpha^2} + 2(V - \mathbf{BR}\cdot\hat{\mathbf{t}})\mathbf{BR}_\alpha\cdot\hat{\mathbf{t}} - 2g\eta_\alpha \end{cases} \tag{3.1.23}$$

If we similarly rescale the vortex sheet system by taking $\theta = \varepsilon\overline{\theta}$, $\gamma = \varepsilon\overline{\gamma}$ and $t = \varepsilon^{-1}\overline{t}$, and then immediately dropping the "bars", we obtain

$$\begin{cases} \theta_t &= \dfrac{1}{\varepsilon}\left(\dfrac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \dfrac{1}{s_\alpha}\mathbf{m}\cdot\hat{\mathbf{n}}\right) + \dfrac{1}{s_\alpha}(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\theta_\alpha \\ \gamma_t &= \dfrac{2}{\varepsilon}\left(\dfrac{\tau}{s_\alpha}\theta_{\alpha\alpha} - g\eta_\alpha\right) + \dfrac{1}{s_\alpha}(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\gamma_\alpha + \dfrac{\gamma}{s_\alpha}(s_{\alpha t} - \mathbf{m}\cdot\hat{\mathbf{t}}) \quad . \\ & \quad -2s_\alpha\mathbf{BR}_t\cdot\hat{\mathbf{t}} - \dfrac{\gamma\gamma_\alpha}{2s_\alpha^2} + 2(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\mathbf{BR}_\alpha\cdot\hat{\mathbf{t}} + \varepsilon\dfrac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) \end{cases} \tag{3.1.24}$$

Of course, the above systems do not include the effects of Clamond damping. Recall that the effect of adding

the damper is to contribute the term $-2\chi_\omega\varphi_\alpha$ to the evolution equation for γ and further that

$$\varphi_\alpha = s_\alpha \nabla\varphi \cdot \hat{\mathbf{t}} = s_\alpha \mathbf{BR} \cdot \hat{\mathbf{t}} + \frac{\gamma}{2}.$$

Ergo, the rescaled damped vortex sheet equations are given by

$$\begin{cases} \theta_t &= \dfrac{1}{\varepsilon}\left(\dfrac{1}{2s_\alpha^2}\mathcal{H}(\gamma_\alpha) + \dfrac{1}{s_\alpha}\mathbf{m}\cdot\hat{\mathbf{n}}\right) + \dfrac{1}{s_\alpha}(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\theta_\alpha \\ \gamma_t &= \dfrac{2}{\varepsilon}\left(\dfrac{\tau}{s_\alpha}\theta_{\alpha\alpha} - g\eta_\alpha - s_\alpha\chi_\omega \mathbf{BR}\cdot\hat{\mathbf{t}} - \chi_\omega\dfrac{\gamma}{2}\right) + \dfrac{1}{s_\alpha}(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\gamma_\alpha + \dfrac{\gamma}{s_\alpha}(s_{\alpha t} - \mathbf{m}\cdot\hat{\mathbf{t}}). \\ & \quad -2s_\alpha \mathbf{BR}_t\cdot\hat{\mathbf{t}} - \dfrac{\gamma\gamma_\alpha}{2s_\alpha^2} + 2(\varepsilon V - \mathbf{BR}\cdot\hat{\mathbf{t}})\mathbf{BR}_\alpha\cdot\hat{\mathbf{t}} + \varepsilon\dfrac{\gamma}{2s_\alpha^2}\mathcal{H}(\gamma\theta_\alpha) \end{cases} \tag{3.1.25}$$

The rescaled damped vortex sheet system (3.1.25) is undoubtedly more complicated than the rescaled toy model (3.1.20). Nevertheless, there are some similarities. To clarify these similarities, we will rewrite the system by taking $\mathbf{u} = (u_1, u_2)^t = (\theta, \gamma)^t$:

$$\partial_t \mathbf{u} = \mathbf{N}_g(\mathbf{u}, \partial_t\mathbf{u}) + \frac{1}{\varepsilon}\boldsymbol{\mathcal{L}}(\mathbf{u})\mathbf{u} + \frac{1}{\varepsilon}\boldsymbol{\mathcal{X}}_\omega\mathbf{u} + \frac{1}{\varepsilon}\mathbf{N}_b(\mathbf{u}). \tag{3.1.26}$$

In (3.1.26), $\boldsymbol{\mathcal{L}}(\mathbf{u})$ is a linear operator with coefficients depending nonlinearly and nonlocally on $\mathbf{u}$, $\mathbf{N}_g$ and $\mathbf{N}_b$ are nonlinear, nonlocal operators and $\boldsymbol{\mathcal{X}}_\omega$ is a multiplication operator. We can, with varying degrees of effort, write down all of the operators in (3.1.26) explicitly. For example, the linear operator $\boldsymbol{\mathcal{L}}(\mathbf{u})$ is given by

$$\boldsymbol{\mathcal{L}}(\mathbf{u}) := \begin{pmatrix} 0 & \frac{1}{2s_\alpha^2}\mathcal{H}\,\partial_\alpha \\ \frac{2\tau}{s_\alpha}\partial_\alpha^2 & 0 \end{pmatrix}. \tag{3.1.27}$$

Thus, it is seen that $\boldsymbol{\mathcal{L}}(\mathbf{u})$ depends on $\mathbf{u}$ via the arclength parameter s_α. On the other hand, the multiplication operator $\boldsymbol{\mathcal{X}}_\omega$, which arises due to the damping, is given by

$$\boldsymbol{\mathcal{X}}_\omega = (0, -\chi_\omega); \text{ that is } \boldsymbol{\mathcal{X}}_\omega\mathbf{u} = \begin{pmatrix} 0 \\ -\chi_\omega \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\chi_\omega u_2. \tag{3.1.28}$$

Writing down the nonlinearities is rather more involved and, given that we will not carry out any analysis of this system, we omit this step.

The connection between the toy model and the vortex sheet system is hopefully becoming clearer.

Indeed, equations (3.1.20) and (3.1.26) look a lot alike. Of course, there are differences (e.g., there is the "bad" nonlinearity $\mathbf{N}_b(\mathbf{u})$ which has no counterpart in the toy model), but it is sensible to conjecture that tools that can successfully be applied to (3.1.20) might also yield results for the vortex sheet formulation of the water waves system. To preview a bit, we build an energy for solutions to (3.1.20) using the commuting vector field $\varepsilon\partial_t$ and prove the necessary energy estimate to obtain the desired $O(1)$ lifespan for solutions to (3.1.20) (and therefore the desired $O(\frac{1}{\varepsilon})$ lifespan for solutions to (3.1.16)). We view this chapter as a sort of proof of concept, showing that commuting vector fields can be used to obtain a quadratic lifespan for solutions to a water-waves-like equation. In order to extend this approach to the gravity-capillary water waves system, we will need to overcome some difficulties. For example, due to the structure of the toy model, we clearly have $\varepsilon\partial_t \approx |D|^\alpha$ when $\alpha > 1$. The structure of the vortex sheet equations are not so simple and such arguments will be a bit more delicate. Nevertheless, we should still have $\varepsilon\partial_t \approx \mathcal{L}(\mathbf{u})$ and it is our hope that this approach will extend to the vortex sheet formulation of the $2d$ gravity-capillary water waves problem. As is the case for the toy model, the presence of surface tension is crucial for the applicability of this vector field. For an example of the application of rescaling in conjunction with the commuting vector field $\varepsilon\partial_t$ to obtain a large-time existence result for the water waves system, see [Més]. This paper also has some further discussion on the necessity of surface tension for the applicability of the $\varepsilon\partial_t$ vector field in the context of the Zakharov-Craig-Sulem formulation. This strategy, somewhat broadly speaking, has also been applied to other problems in fluid dynamics, primarily those involving singular limits (e.g., anelastic limits for Euler-type systems [BrMé], the incompressible limit of the Euler equations [MéSc, Ala1] and the low-mach-number limit of the full Navier-Stokes equations [Ala2]).

Before moving on to our analysis, we want to discuss one (potential) more connection between the toy model and the vortex sheet formulation of the water waves problem. This connection arises via the paralinearization of the water waves system. To be a bit more clear, we believe that the paralinearization of the vortex sheet formulation of the $2d$ water waves system will be of the same form (3.1.15) as that of the Zakharov-Craig-Sulem formulation. This connection is only a potential connection as, to our knowledge, the vortex sheet system has never been paralinearized. In future work, we plan to perform this paralinearization and hopefully confirm our belief that it is of the form (3.1.15). If this is indeed the case, it will provide another, very clear, connection between the toy model and the vortex sheet system as, indeed, the toy model was built from (3.1.15).

3.2 Main Results and Plan of the Chapter

As noted in the introduction, our objective is to show that solutions to (3.1.16) have an $O(\frac{1}{\varepsilon})$ lifespan. Our approach will be to consider the rescaled equation (3.1.20) and to show that solutions have an $O(1)$ lifespan. Our first task will be to define an energy $\mathcal{E}$ for solutions of (3.1.20). This will be done using carefully chosen vector fields. However, the value of α in (3.1.17) plays a key role in defining a suitable energy. In particular, the order of L, compared to the order 1 nonlinearity $W_\varepsilon(v)\partial_x v$, will determine whether L is principal or sub-principal, and this fact plays an important role in determining the analysis necessary to attack the problem. So, the marked difference in analysis is between the cases $\alpha > 1$ and $\alpha \leqslant 1$. However, rather than focusing on these more general cases, we will largely focus in on the cases $\alpha = \frac{3}{2}$ and $\alpha = \frac{1}{2}$. In either case, we have the following result:

Theorem 3.2.1. *Let v be a solution of* (3.1.20)*, where $\alpha = \frac{3}{2}$ or $\alpha = \frac{1}{2}$, and suppose that σ is sufficiently large. If $\mathcal{E}$ is the appropriate energy of a solution to* (3.1.20)*, then we have*

$$\frac{d\mathcal{E}}{dt} \lesssim \mathcal{E}. \tag{3.2.1}$$

Detailed statements of this result can be found in Theorem 3.4.5 for the case $\alpha = \frac{3}{2}$ and in Theorem 3.4.7 for the case $\alpha = \frac{1}{2}$.

Remark 3.2.2. *It is crucial that the energy estimate* (3.2.1) *is uniform in ε. With such an estimate in hand, a routine Grönwall argument will yield the desired $O(1)$ lifespan. Of course, we can then deduce that solutions to* (3.1.16) *persist on an $O(\frac{1}{\varepsilon})$ timescale.*

Remark 3.2.3. *Our definition of L clearly omits the gravity-capillary case, corresponding to $L := \sqrt{|D| + |D|^3}$. Nevertheless, we are quite confident that the same results would obtain. The only place where our arguments would not apply directly to the gravity-capillary case is in the proof of Lemma 3.3.4. However, it should not be too difficult to extend Lemma 3.3.4 to handle $L = \sqrt{|D| + |D|^3}$ or something more general like $L = \sqrt{|D|^\alpha + |D|^\beta}$.*

In Section 3, we prove some preliminary commutator estimates which will be needed to prove the main energy estimates. The main energy estimates are proved in Section 4. Finally, Section 5 contains an alternative proof of one of the main results.

3.3 ΨDO Commutator Estimates

In proving the energy estimates which are the primary focus of this chapter, we shall encounter a number of commutators involving ΨDO. Specifically, we will need to handle commutators involving positive integer powers of $P_L = \mathrm{Op}(p_L)$, which is given by

$$P_L \coloneqq iL + \chi_\omega. \tag{3.3.1}$$

Recall that L is defined in (3.1.17). In the sequel, we will primarily be concerned with two types of commutators involving P_L. They will be of the form $\left[P_L^k, f\right]$ and $\left[P_L^k, \partial_x\right]$ for $k \in \mathbb{N}$. Note that we lightly abuse notation by not distinguishing the notation for a function f and the operator $M_f : u \mapsto fu$. To avoid confusion, note that, when P is an operator and f a function, we define the commutator $[P, f] \coloneqq \left[P, M_f\right]$; that is,

$$[P, f](u) = P(fu) - fPu.$$

The challenge posed by working with P_L is that its symbol p_L is not smooth at $\xi = 0$, otherwise we could use standard ΨDO commutator estimates. However, as a Fourier multiplier, iL commutes with ∂_x and so the commutators of the form $\left[P_L^k, \partial_x\right]$ will be rather straightforward to understand. In this case, we have the following result:

Lemma 3.3.1. *Let $k \in \mathbb{N}$. Then, for all $s \geqslant 0$, we have*

$$\left\|\left[P_L^k, \partial_x\right](u)\right\|_{H^s} \lesssim \|u\|_{H^{s+k\alpha-\alpha}}. \tag{3.3.2}$$

Proof. We shall proceed by induction on k. For $k = 1$, it is straightforward to verify (3.3.2). Namely, by Lemma 1.3.9, we have

$$\|[P_L, \partial_x](u)\|_{H^s} = \|(\partial_x \chi_\omega)u\|_{H^s} \lesssim \|u\|_{H^s} \quad \forall s \geqslant 0. \tag{3.3.3}$$

Now, assume that, for some fixed $k \in \mathbb{N}$, we have

$$\left\|\left[P_L^k, \partial_x\right](u)\right\|_{H^s} \lesssim \|u\|_{H^{s+k\alpha-\alpha}} \quad \forall s \geqslant 0. \tag{3.3.4}$$

Observe that we can write

$$\big[P_L^{k+1}, Q\big](u) = P_L\big[P_L^k, Q\big](u) + [P_L, Q](P_L^k u), \tag{3.3.5}$$

where Q is an arbitrary operator (e.g., we could have $Q = \partial_x$ or $Q = M_f$). Fixing $s \geqslant 0$, we can now estimate each term on the right-hand side of (3.3.5) in the H^s norm. We apply the triangle inequality, Lemma 1.3.9 and the induction hypothesis (3.3.4) to the first term:

$$\begin{aligned}
\big\|P_L\big[P_L^k, \partial_x\big](u)\big\|_{H^s} &\leqslant \big\|L\big[P_L^k, \partial_x\big](u)\big\|_{H^s} + \big\|\chi_\omega\big[P_L^k, \partial_x\big](u)\big\|_{H^s} \\
&\lesssim \big\|\big[P_L^k, \partial_x\big](u)\big\|_{H^{s+\alpha}} + \big\|\big[P_L^k, \partial_x\big](u)\big\|_{H^s} \\
&\lesssim \|u\|_{H^{s+k\alpha}}.
\end{aligned} \tag{3.3.6}$$

The second term on the right-hand side of (3.3.5) is straightforward to estimate in H^s via Lemma 1.3.9:

$$\big\|[P_L, \partial_x](P_L^k u)\big\|_{H^s} = \big\|(\partial_x \chi_\omega) P_L^k u\big\|_{H^s} \lesssim \big\|P_L^k u\big\|_{H^s} \lesssim \|u\|_{H^{s+k\alpha}}. \tag{3.3.7}$$

Using equation (3.3.5) and the triangle inequality, we can deduce from (3.3.6) and (3.3.7) that

$$\big\|\big[P_L^{k+1}, \partial_x\big](u)\big\|_{H^s} \lesssim \|u\|_{H^{s+k\alpha}}. \tag{3.3.8}$$

This completes the proof. □

The commutators of the form $\big[P_L^k, f\big]$ will require a bit more work. Our objective is to prove the following:

Lemma 3.3.2. *Let $k \in \mathbb{N}$. Then, we have*

$$\big\|\big[P_L^k, f\big](u)\big\|_{H^s} \lesssim \|f\|_{H^r} \|u\|_{H^{s+k\alpha-1}}, \tag{3.3.9}$$

provided $r > \frac{3}{2}$, $s \geqslant 0$ and $s + k\alpha \leqslant r$.

Notice that if we could prove (3.3.9) for $k = 1$, then proceeding by induction on k and exploiting (3.3.5) would give the result much like in the proof of Lemma 3.3.1. However, L will not commute with

multiplication by a function as it did with differentiation. In fact, we have

$$[P_L, f](u) = i[L, f](u).$$

Therefore, in order to get such an argument off the ground, we will need to understand how to estimate such a commutator.

As noted above, a key detail here is that ℓ (the symbol of L) is not smooth at $\xi = 0$ and so classical ΨDO (or Fourier multiplier) commutator estimates will not apply directly. However, notice that if $\phi = \phi(\xi) \in C_c^\infty(T^*\mathbb{T})$ with $\phi \equiv 0$ in some neighborhood of $\xi = 0$, then $\phi\ell \in S^\alpha_{1,0}$ (recall the symbol class $S^m_{\rho,\delta}$ is defined in (1.3.27)). Thus, classical ΨDO commutator estimates "almost" apply to L and in fact do apply to L as long as we filter out the low frequencies. Given that we should be able to handle the low frequencies with Bernstein-type inequalities for band-limited functions, this indicates that we should be able to adapt classical ΨDO commutator estimates to handle L and this is, in fact, our next objective.

There are a great many results on estimating commutators of the form $[\mathrm{Op}(p), f](u)$ in Sobolev spaces, where $p \in S^m_{\rho,\delta}$ or some other appropriate symbol class. The book [Tay4] is an excellent resource for such estimates. The result which we will use as a basis for building the needed commutator estimate is the following:

Lemma 3.3.3. *Let $m \geqslant 0$. Further, take $r, s \in \mathbb{R}$ such that $r > \frac{3}{2}$, $s \geqslant 0$ and $s + m \leqslant r$. Then, for $p = p(x,\xi) \in \mathcal{B}S^m_{1,1}$, we have*

$$\|[\mathrm{Op}(p), f](u)\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^{s+m-1}}. \tag{3.3.10}$$

In other words, $[\mathrm{Op}(p), f]$ *is an operator of order* $m - 1$.

Proof. See Proposition 4.2 in [Tay6]. □

The symbol class $\mathcal{B}S^m_{1,1}$ contains those $p(x,\xi) \in S^m_{1,1}$ such that

$$\exists \gamma \in (0,1) : \ \operatorname{supp} \hat{p}(\theta,\xi) \subset \{(\theta,\xi) : |\theta| \leqslant \gamma|\xi|\}. \tag{3.3.11}$$

This class is related to the symbol class B^m_r introduced by Meyer in [Mey]. What is important for us about this symbol class is that

$$S^m_{1,0} \subset \mathcal{B}S^m_{1,1} + S^{-\infty}_{1,0}; \tag{3.3.12}$$

in fact, we have, for any $\delta > 0$, $S^m_{1,\delta} \subset \mathcal{B}S^m_{1,1} + S^{-\infty}_{1,0}$. In addition, $\mathrm{Op}\mathcal{B}S^m_{1,1}$ contains the paradifferential operators of [Bony]. A similar estimate that would have suited our purposes is Lemma 3.4 in [Ala2]. We could have also used the estimate (3.6.1) from [Tay4], which for $p \in S^m_{1,0}$ would give the same estimate as (3.3.10) after applying Sobolev embedding.

We are now going to use Lemma 3.3.3 to prove the needed estimate for commutators involving L.

Lemma 3.3.4. *Let L be as in* (3.1.17)*, $r > \frac{3}{2}$, $s \geqslant 0$ and $s + \alpha \leqslant r$. Then, the result of Lemma 3.3.3 holds with $p = \ell$. Namely, it holds that*

$$\|[L, f](u)\|_{H^s} \lesssim \|f\|_{H^r} \|u\|_{H^{s+\alpha-1}}. \tag{3.3.13}$$

Proof. As we noted previously, the challenge we must overcome is that ℓ is not smooth at $\xi = 0$. To deal with this fact, we decompose L into a component with smooth symbol and a low-frequency component. We will be able to control the (non-smooth) low-frequency factor via Bernstein-type inequalities, while Lemma 3.3.3 will give us control of the high-frequency, but smooth, factor.

We begin by defining a low-frequency filtering operator S_0 as in (1.3.33). The aforementioned decomposition is then $L = L_1 + L_{>1}$, where $L_1 := S_0 L$ and $L_{>1} := (\mathrm{id} - S_0)L$. Observe that in defining L_1 we have filtered out the high frequencies, only retaining frequencies ξ with $|\xi| \leqslant 1$. Likewise, for $L_{>1}$, we have filtered out the low frequencies and only retain frequencies $|\xi| > 1$.

From here, simply applying the triangle inequality gives

$$\|[L, f](u)\|_{H^s} \leqslant \|[L_1, f](u)\|_{H^s} + \|[L_{>1}, f](u)\|_{H^s}. \tag{3.3.14}$$

We now just have to estimate each term on the right-hand side of (3.3.14). We will begin with the low-frequency component. As Fourier multipliers, S_0 and L commute, so we can use Lemma 1.3.9 and a Bernstein-type inequality (e.g., Lemma 1.3.3) to obtain

$$\|L_1(fu)\|_{H^s} = \|LS_0(fu)\|_{H^s} \lesssim 2^\alpha \|f\|_{H^r} \|u\|_{H^s}, \tag{3.3.15}$$

$$\|fL_1 u\|_{H^s} \lesssim \|f\|_{H^r} \|LS_0 u\|_{H^s} \lesssim 2^\alpha \|f\|_{H^r} \|u\|_{H^s}. \tag{3.3.16}$$

Of course, (3.3.15) and (3.3.16) imply that

$$\|[L_1, f](u)\|_{H^s} \lesssim_\alpha \|f\|_{H^r}\|u\|_{H^s}. \tag{3.3.17}$$

Now we proceed to the high-frequency component. We know from (3.3.12) that we can write $L_{>1} = \mathcal{B}L_{>1} + R$, where $\mathcal{B}L_{>1} \in \operatorname{Op}\mathcal{B}S^m_{1,1}$ and $R \in \operatorname{Op}S^{-\infty}_{1,0}$. We can apply Lemma 3.3.3 to deduce that

$$\|[\mathcal{B}L_{>1}, f](u)\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^{s+\alpha-1}}. \tag{3.3.18}$$

On the other hand, since R is a smoothing operator, we have

$$\|[R, f](u)\|_{H^s} \leqslant \|R(fu)\|_{H^s} + \|fRu\|_{H^s} \lesssim \|fu\|_{H^s} + \|f\|_{H^r}\|Ru\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^s}. \tag{3.3.19}$$

Of course, we could, for any $m \in \mathbb{R}$, put $\|u\|_{H^{s+m}}$ on the right-hand side of (3.3.18). However, doing so would not gain us anything, so we do not bother. Combining the estimates (3.3.14), (3.3.17), (3.3.18) and (3.3.19) yields (3.3.13).

□

We now have all of the tools needed to prove the second main commutator estimate:

Proof of Lemma 3.3.2. To begin, observe that, by Lemma 3.3.4, we have

$$\|[P_L, f](u)\|_{H^s} = \|[L, f](u)\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^{s+\alpha-1}} \ \forall s \geqslant 0. \tag{3.3.20}$$

Now, assume that, for some fixed $k \in \mathbb{N}$, we have

$$\left\|\left[P_L^k, f\right](u)\right\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^{s+k\alpha-1}} \ \forall s \geqslant 0. \tag{3.3.21}$$

Fixing $s \geqslant 0$, we can again make use of equation (3.3.5), which gives

$$\left\|\left[P_L^{k+1}, f\right](u)\right\|_{H^s} \leqslant \left\|P_L\left[P_L^k, f\right](u)\right\|_{H^s} + \left\|[P_L, f](P_L^k u)\right\|_{H^s}. \tag{3.3.22}$$

Then, Lemma 1.3.9, (3.3.20) and (3.3.21) yield

$$\left\|P_L\left[P_L^k, f\right](u)\right\|_{H^s} \lesssim \left\|\left[P_L^k, f\right](u)\right\|_{H^{s+\alpha}} + \left\|\left[P_L^k, f\right](u)\right\|_{H^s} \lesssim \|f\|_{H^r}\|u\|_{H^{s+k\alpha+\alpha-1}}, \tag{3.3.23}$$

$$\left\|[P_L, f](P_L^k u)\right\|_{H^s} \lesssim \|f\|_{H^r}\left\|P_L^k u\right\|_{H^{s+\alpha-1}} \lesssim \|f\|_{H^r}\|u\|_{H^{s+k\alpha+\alpha-1}}.. \tag{3.3.24}$$

The desired claim then follows by induction. □

Having the above commutator estimates in hand, we are now prepared to prove the desired energy estimates pursuant to the approach outlined in Section 2.

3.4 The Main Energy Estimates

Here our objective is to prove the energy estimates of Theorem 3.2.1. This will, of course, require first defining an appropriate energy for solutions to (3.1.20). However, as we noted previously, the value of α plays an important role in defining a suitable energy. Nevertheless, there are some relevant results which we can prove for L with any value of $\alpha \in (0, 2]$. In the linear case ($W \equiv 0$), one can show that the solution v actually decays (in norm). On the other hand, in the nonlinear case ($W \not\equiv 0$), we show that v satisfies

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \lesssim \|v(t)\|_{L^2}^2.$$

After proving the above results, we break our analysis into two cases: $\alpha = \frac{3}{2}$ and $\alpha = \frac{1}{2}$. We first consider $\alpha = \frac{3}{2}$, defining an appropriate energy $\mathcal{E}_{\text{cap}}$ and then proving the desired estimate. Finally, we do the same for $\alpha = \frac{1}{2}$.

3.4.1 Linear Damping

Here we show that, when $W \equiv 0$, the external pressure p_{ext} damps the energy. In particular, we show that, for any $k \in \mathbb{N}_0$ and $\sigma \geqslant k\alpha$, the $H^{k\alpha}$ norm of v is decreasing in time and thus is bounded above by $\|v_0\|_{H^{k\alpha}}$. Recall that $v_0 \in H^\sigma$ by hypothesis. We begin with the following result:

Lemma 3.4.1. *If v solves* (3.1.20) *with $W \equiv 0$, then $\|v(t)\|_{L^2}$ is decreasing in time and, in particular,*

$$\|v\|_{L_t^\infty L_x^2} \leqslant \|v_0\|_{L^2}. \tag{3.4.1}$$

This claim is valid for any $\alpha \in (0, 2]$.

Proof. Differentiating $\|v(t)\|_{L^2}^2$ with respect to t and passing the derivative through the integral gives

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = \int_0^{2\pi} v^*\partial_t v + v\partial_t v^*\, dx. \tag{3.4.2}$$

Noting that v^* also solves (3.1.20), substituting from (3.1.20) into (3.4.2) and doing some simple computations yields

$$\begin{aligned}\frac{d}{dt}\|v(t)\|_{L^2}^2 &= -\int_0^{2\pi} v^*\left(\frac{i}{\varepsilon}Lv + \frac{1}{\varepsilon}\chi_\omega v\right) dx - \int_0^{2\pi} v\left(\frac{i}{\varepsilon}Lv^* + \frac{1}{\varepsilon}\chi_\omega v^*\right) dx \\ &= -\frac{1}{\varepsilon}\int_0^{2\pi} v^*\chi_\omega v + v\chi_\omega v^*\, dx - \frac{i}{\varepsilon}\int_0^{2\pi} v^*Lv + vLv^*\, dx\end{aligned}$$

We can then slightly rewrite the second term on the right-hand side above to see that it is purely imaginary:

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\frac{2}{\varepsilon}\left(\int_0^{2\pi} \chi_\omega |v|^2\, dx + i\int_0^{2\pi}\left|\sqrt{L}v\right|^2 dx\right). \tag{3.4.3}$$

But, the left-hand side of (3.4.3) is real-valued, so the imaginary part of the right-hand side must vanish. We therefore have

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\frac{2}{\varepsilon}\int_0^{2\pi} \chi_\omega |v|^2\, dx \leqslant 0. \tag{3.4.4}$$

Hence, $\|v(t)\|_{L^2}$ is decreasing in t, from which (3.4.1) immediately follows. □

Lemma 3.4.2. *Let $k \in \mathbb{N}_0$ be arbitrary. If v solves* (3.1.20)*, where $W \equiv 0$ and $\sigma \geqslant k\alpha$, then $\|v\|_{H^{k\alpha}}(t)$ is decreasing and is thus controlled by $\|v_0\|_{H^{k\alpha}}$:*

$$\|v\|_{L_t^\infty H_x^{k\alpha}} \lesssim \|v_0\|_{H^{k\alpha}}. \tag{3.4.5}$$

Again, we note that this result holds for any $\alpha \in (0, 2]$.

Proof. Let Z denote the vector field $\varepsilon\partial_t$ and consider Zv. Observing that $Zv = -iLv - \chi_\omega v$, we will have $Zv \in L^2$ as long as $Lv \in L^2$ (i.e., $\sigma \geqslant \alpha$ in equation (3.1.20)). In addition, since Z commutes with ∂_t, L and χ_ω, we have

$$\partial_t Zv + \frac{i}{\varepsilon}LZv + \frac{1}{\varepsilon}\chi_\omega Zv = 0.$$

Hence, by Proposition 3.4.1, $\|Zv(t)\|_{L^2}$ is decreasing; that is, $\|(iL+\chi_\omega)v(t)\|_{L^2}$ is decreasing and in particular

$$\|(iL+\chi_\omega)v\|_{L^2} \lesssim \|v_0\|_{H^\alpha}. \tag{3.4.6}$$

It then follows that $v \in H^\alpha$ with

$$\|v\|_{L_t^\infty H_x^\alpha} \lesssim \|v_0\|_{H^\alpha}. \tag{3.4.7}$$

Consider now $Z^2 v$. We will have $Z^2 v \in L^2$ whenever $\sigma \geqslant 2\alpha$. As before, $Z^2 v$ solves (3.1.20) with $W \equiv 0$ and, again, $\left\|Z^2 v(t)\right\|_{L^2}$ is decreasing in t. But,

$$Z^2 v = (iL+\chi_\omega)^2 v = (-L^2 + iL\chi_\omega + i\chi_\omega L + \chi_\omega^2)v.$$

In other words, $\left\|(L^2 - i(L\chi_\omega + \chi_\omega L) - \chi_\omega^2)v(t)\right\|_{L^2}$ is decreasing and we thus have

$$\left\|(L^2 - i(L\chi_\omega + \chi_\omega L) - \chi_\omega^2)v\right\|_{L^2} \lesssim \|v_0\|_{H^{2\alpha}}. \tag{3.4.8}$$

Therefore, we deduce from (3.4.8) that $v \in H^{2\alpha}$ with the estimate

$$\|v\|_{L_t^\infty H_x^{2\alpha}} \lesssim \|v_0\|_{H^{2\alpha}}. \tag{3.4.9}$$

We continue to iterate this argument and see that, for any $k \in \mathbb{N}_0$, $\left\|Z^k v(t)\right\|_{L^2}$ is decreasing, which implies that $\left\|(iL+\chi_\omega)^k v(t)\right\|_{L^2}$ is decreasing. We therefore conclude that, as long as the initial data is sufficiently regular ($\sigma \geqslant k\alpha$), $v \in H^{k\alpha}$ and

$$\|v\|_{L_t^\infty H_x^{k\alpha}} \lesssim \|v_0\|_{H^{k\alpha}}. \tag{3.4.10}$$

□

3.4.2 A Nonlinear L^2 Estimate

At this point, we are ready to turn on W and so we shall henceforth remove the assumption that $W \equiv 0$. We shall first obtain an a priori L^2 bound and then focus on the higher Sobolev estimates.

Lemma 3.4.3. *If v is a solution of the Clamond toy model* (3.1.20)*, then*

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \lesssim \|v(t)\|_{L^2}^2.$$

Once more, the above is valid for any $\alpha \in (0,2]$.

Proof. We begin by differentiating $\|v(t)\|_{L^2}$, passing the derivative through the integral, substituting from (3.1.20) and using the fact that $W_\varepsilon(v) = W_\varepsilon(v^*)$ to obtain

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\int_0^{2\pi} W_\varepsilon(v)\partial_x|v|^2\,dx - \frac{2i}{\varepsilon}\int_0^{2\pi}\left|\sqrt{L}v\right|^2 dx - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega|v|^2\,dx. \tag{3.4.11}$$

Since the left-hand side of (3.4.11) is real-valued, the imaginary part of the right-hand side must vanish and so we will have

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = -\Re\mathrm{e}\left\{\int_0^{2\pi} W_\varepsilon(v)\partial_x|v|^2\,dx\right\} - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega|v|^2\,dx. \tag{3.4.12}$$

We can then integrate by parts in the first term in (3.4.12) to obtain

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 = \int_0^{2\pi}\partial_x W_\varepsilon(v)|v|^2\,dx - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega|v|^2\,dx \leqslant \int_0^{2\pi}\partial_x W_\varepsilon(v)|v|^2\,dx,$$

where the final inequality follows from the fact that $-\frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega|v|^2\,dx \leqslant 0$. Via Sobolev embedding (Lemma 1.3.2), we know that $\partial_x W_\varepsilon(v) \in L^\infty$ and, due to (3.1.21), we have a uniform-in-ε estimate. It then follows that

$$\frac{d}{dt}\|v(t)\|_{L^2}^2 \lesssim \|v(t)\|_{L^2}^2. \tag{3.4.13}$$

□

Having obtained the needed L^2 estimate, we are now going to define an energy for solutions of (3.1.20) in order to obtain the desired Sobolev estimates. At this point, the analysis becomes more sensitive to the value of α, hence we shall stop considering general $\alpha \in (0,2]$ and break our consideration up into two cases, $\alpha > 1$ and $\alpha \leqslant 1$. In particular, as previously noted, we will focus in on $\alpha = \frac{3}{2}$, corresponding to capillary waves, and $\alpha = \frac{1}{2}$, corresponding to gravity waves.

3.4.3 The Energy Estimate for Capillary Waves ($\alpha = \frac{3}{2}$)

Definition 3.4.4. *Let $\alpha = \frac{3}{2}$ and define the energy for a solution of the Clamond toy model* (3.1.20) *by*

$$\mathcal{E}_{\mathrm{cap}}(t) := \sum_{k=0}^{2}\left\|Z^k v(t)\right\|_{L^2}^2, \tag{3.4.14}$$

where Z is a given vector field.

The vector field we will primarily consider in Definition 3.4.4 will be $Z = \varepsilon\partial_t$. Since the simpler choice of $Z = \varepsilon\partial_t$ works in the capillary case, we present this argument first. However, this vector field will not work in the gravity case and there we utilize $Z = P_L$, where P_L is defined in (3.3.1). In Section 4, we include an argument showing that this choice of Z also works in the capillary case, which gives a unified approach to both problems. Either choice of Z will yield

$$\mathcal{E}_{\text{cap}} \sim \|v\|_{H^3}^2. \tag{3.4.15}$$

Now that we have a suitable energy in hand, we can proceed to prove the main energy estimate in the case $\alpha = \frac{3}{2}$.

Theorem 3.4.5. *If $\alpha = \frac{3}{2}$, v is a solution of* (3.1.20) *with $\sigma \geqslant 3$ and $\mathcal{E}_{\text{cap}} = \mathcal{E}_{\text{cap}}(t)$ is given by* (3.4.14) *with $Z = \varepsilon\partial_t$, then one has*

$$\frac{d\mathcal{E}_{\text{cap}}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.4.16}$$

Proof. Write $\mathcal{E}_{\text{cap}}(t) = \mathcal{E}_{\text{cap},0}(t) + \mathcal{E}_{\text{cap},1}(t) + \mathcal{E}_{\text{cap},2}(t)$ and notice that, by Lemma 3.4.3, we have

$$\frac{d\mathcal{E}_{\text{cap},0}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.4.17}$$

Upon taking the derivative of $\mathcal{E}_{\text{cap},1}$ and substituting from (3.1.20) for v_t and v_t^*, one sees that

$$\frac{d\mathcal{E}_{\text{cap},1}}{dt} = -\int_0^{2\pi} Z(W_\varepsilon(v)\partial_x v)Zv^* + ZvZ(W_\varepsilon(v^*)\partial_x v^*)\,dx - \frac{2i}{\varepsilon}\int_0^{2\pi}\left|\sqrt{L}Zv\right|^2 dx - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega|Zv|^2\,dx. \tag{3.4.18}$$

Since the left-hand side of (3.4.18) is real-valued, the imaginary part will vanish and, after noting that the third term in (3.4.18) has a good sign, we will have

$$\frac{d\mathcal{E}_{\text{cap},1}}{dt} \leqslant -\Re\mathrm{e}\left\{\int_0^{2\pi} Z(W_\varepsilon(v)\partial_x v)Zv^* + ZvZ(W_\varepsilon(v^*)\partial_x v^*)\,dx\right\}.$$

Expanding in the remaining integral using the Leibniz rule, recalling that $W_\varepsilon(v) = W_\varepsilon(v^*)$ and integrating by parts yields

$$\frac{d\mathcal{E}_{\text{cap},1}}{dt} \leqslant \int_0^{2\pi}\partial_x W_\varepsilon(v)|Zv|^2\,dx - \Re\mathrm{e}\left\{\int_0^{2\pi} W_\varepsilon(Zv)\partial_x v\cdot Zv^* + W_\varepsilon(Zv^*)\partial_x v^*\cdot Zv\,dx\right\}. \tag{3.4.19}$$

The first integral in (3.4.19) is easily estimated:

$$\int_0^{2\pi} \partial_x W_\varepsilon(v)|Zv|^2\,dx \leqslant \|\partial_x W_\varepsilon(v)\|_{L^\infty}\|Zv\|_{L^2}^2 \lesssim \|Zv\|_{L^2}^2. \tag{3.4.20}$$

One can then bound the second integral in (3.4.19) as follows:

$$-\Re\mathfrak{e}\left\{\int_0^{2\pi} W_\varepsilon(Zv)\partial_x v\cdot Zv^* + W_\varepsilon(Zv^*)\partial_x v^*\cdot Zv\,dx\right\} \lesssim \|W_\varepsilon(Zv)\|_{L^\infty}\|v\|_{\dot H^1}\|Zv\|_{L^2}. \tag{3.4.21}$$

It then follows from (3.4.20) and (3.4.21) that

$$\frac{d\mathcal{E}_{\text{cap},1}}{dt} \lesssim \|Zv\|_{L^2}^2 + \|v\|_{\dot H^1}\|Zv\|_{L^2} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.4.22}$$

Finally, consider the derivative of $\mathcal{E}_{\text{cap},2}$ with respect to t. We compute the derivative, substitute from (3.1.20) and, much as before, we will obtain

$$\frac{d\mathcal{E}_{\text{cap},2}}{dt} \leqslant -\Re\mathfrak{e}\left\{\int_0^{2\pi} Z^2(W_\varepsilon(v)\partial_x v)Z^2v^* + Z^2vZ^2(W_\varepsilon(v^*)\partial_x v^*)\,dx\right\}. \tag{3.4.23}$$

Expanding via the Leibniz rule, one obtains

$$\begin{aligned}\frac{d\mathcal{E}_{\text{cap},2}}{dt} &\leqslant -\Re\mathfrak{e}\left\{\int_0^{2\pi} W_\varepsilon(Z^2v)\partial_x v\cdot Z^2v^* + W_\varepsilon(Z^2v^*)\partial_x v^*\cdot Z^2v\,dx\right\}\\ &\quad -2\,\Re\mathfrak{e}\left\{\int_0^{2\pi} W_\varepsilon(Zv)\partial_x Zv\cdot Z^2v^* + W_\varepsilon(Zv^*)\partial_x Zv^*\cdot Z^2v\,dx\right\}\\ &\quad -\Re\mathfrak{e}\left\{\int_0^{2\pi} W_\varepsilon(v)\partial_x Z^2v\cdot Z^2v^* + W_\varepsilon(v^*)\partial_x Z^2v^*\cdot Z^2v\,dx\right\}\\ &= I + II + III.\end{aligned} \tag{3.4.24}$$

We first consider III in (3.4.24) and observe that, since $W_\varepsilon(v) = W_\varepsilon(v^*)$, we may integrate by parts to see that

$$III = \int_0^{2\pi} \partial_x W_\varepsilon(v)\left|Z^2v\right|^2\,dx \leqslant \|\partial_x W_\varepsilon(v)\|_{L^\infty}\left\|Z^2v\right\|_{L^2}^2. \tag{3.4.25}$$

On the other hand, an appropriate bound on I is immediate:

$$I \lesssim \left\|W_\varepsilon(Z^2v)\right\|_{L^\infty}\|\partial_x v\|_{L^2}\left\|Z^2v\right\|_{L^2}. \tag{3.4.26}$$

Finally, we have

$$II \lesssim \|W_\varepsilon(Zv)\|_{L^\infty} \|\partial_x Zv\|_{L^2} \|Z^2 v\|_{L^2}. \tag{3.4.27}$$

Recall that we can control the H^r norm of v with $\mathcal{E}_{\text{cap}}$ for $r \leqslant 3$. Then, since $\|Zv\|_{\dot{H}^1} \lesssim \|v\|_{H^{5/2}}$, it follows from (3.4.24), (3.4.25), (3.4.26) and (3.4.27) that

$$\frac{d\mathcal{E}_{\text{cap},2}}{dt} \lesssim \|Z^2 v\|_{L^2}^2 + \|v\|_{\dot{H}^1} \|Z^2 v\|_{L^2} + \|Zv\|_{\dot{H}^1} \|Z^2 v\|_{L^2} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.4.28}$$

Upon combining (3.4.17), (3.4.22) and (3.4.28), we conclude that

$$\frac{d\mathcal{E}_{\text{cap}}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.4.29}$$

□

3.4.4 The Energy Estimate for Gravity Waves ($\alpha = \frac{1}{2}$)

Here we seek to obtain a result analogous to Theorem 3.4.5 when $\alpha = \frac{1}{2}$. Again, we shall first need to define a suitable energy, but the situation is complicated by the fact that L is now sub-principal. In particular, the simpler vector field $Z = \varepsilon\partial_t$ will no longer suffice and we will need a more carefully chosen Z.

Definition 3.4.6. *Let $\alpha = \frac{1}{2}$ and define the energy for a solution v of the Clamond toy model* (3.1.20) *by*

$$\mathcal{E}_{\text{grav}}(t) := \sum_{k=0}^{4} \|Z^k v(t)\|_{L^2}^2(t), \tag{3.4.30}$$

where $Z = P_L$. Recall that P_L is defined in (3.3.1).

Notice that the definition of the energy in this case requires more copies of the vector field Z. This arises from the fact that L is now only of order $\frac{1}{2}$, instead of order $\frac{3}{2}$ in the previous case and so we will need more copies in order to close the estimates. Notice that

$$\mathcal{E}_{\text{grav}} \sim \|v\|_{H^2}^2. \tag{3.4.31}$$

The vector field $\varepsilon\partial_t$, which we used to define $\mathcal{E}_{\text{cap}}$, had the benefit of commuting with ∂_x, W_ε and functions of x. However, the vector field $Z = P_L$ in Definition 3.4.6 does not have these nice commutation properties and so obtaining the desired energy estimates will require understanding the effects of commuting

powers of Z with ∂_x and functions of x (and t), such as $W_\varepsilon(v)$. This is precisely the motivation for the results obtained earlier in Section 3.

Defining the energy as in Definition 3.4.6, we can prove the following energy estimate.

Theorem 3.4.7. *If $\alpha = \frac{1}{2}$, v is a solution of* (3.1.20) *with $\sigma \geqslant 2$ and $\mathcal{E}_{\text{grav}} = \mathcal{E}_{\text{grav}}(t)$ is given by* (3.4.30)*, then one has*

$$\frac{d\mathcal{E}_{\text{grav}}}{dt} \lesssim \mathcal{E}_{\text{grav}}.$$

Proof. Again, we begin by writing $\mathcal{E}_{\text{grav}} = \mathcal{E}_{\text{grav},0} + \mathcal{E}_{\text{grav},1} + \mathcal{E}_{\text{grav},2} + \mathcal{E}_{\text{grav},3} + \mathcal{E}_{\text{grav},4}$ and noting that the desired result for $\mathcal{E}_{\text{grav},0}$ holds by Lemma 3.4.3:

$$\frac{d\mathcal{E}_{\text{grav},0}}{dt} \lesssim \mathcal{E}_{\text{grav}}. \tag{3.4.32}$$

Now, fix $1 \leqslant k \leqslant 4$ and consider $\mathcal{E}_{\text{grav},k}$. Upon differentiating with respect to t, substituting from (3.1.20) and rewriting a bit, we obtain

$$\frac{d\mathcal{E}_{\text{grav},k}}{dt} = -\int_0^{2\pi} Z^k(W_\varepsilon(v)\partial_x v)Z^k v^* + Z^k v Z^k(W_\varepsilon(v^*)\partial_x v^*)\,dx - \frac{2i}{\varepsilon}\int_0^{2\pi}\left|\sqrt{L}Z^k v\right|^2 dx - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega\left|Z^k v\right|^2 dx. \tag{3.4.33}$$

We know that the left-hand side of (3.4.33) is real-valued, so the second term, which is purely imaginary, must vanish. Moreover, the third term has a good sign. We therefore deduce that

$$\frac{d\mathcal{E}_{\text{grav},k}}{dt} \leqslant -\Re\mathrm{e}\left\{\int_0^{2\pi} Z^k(W_\varepsilon(v)\partial_x v)Z^k v^* + Z^k v Z^k(W_\varepsilon(v^*)\partial_x v^*)\,dx\right\}. \tag{3.4.34}$$

By adding and subtracting, we can rewrite (3.4.34) as

$$\begin{aligned}\frac{d\mathcal{E}_{\text{grav},k}}{dt} &\leqslant -\Re\mathrm{e}\left\{\int_0^{2\pi}[Z^k(W_\varepsilon(v)\partial_x v) - W_\varepsilon(v)Z^k\partial_x v]Z^k v^* + [Z^k(W_\varepsilon(v^*)\partial_x v^*) - W_\varepsilon(v^*)Z^k\partial_x v^*]Z^k v\,dx\right\}\\ &\quad - \Re\mathrm{e}\left\{\int_0^{2\pi} W_\varepsilon(v)Z^k\partial_x v Z^k v^* + W_\varepsilon(v^*)Z^k\partial_x v^* Z^k v\,dx\right\}\\ &= C_k + D_k.\end{aligned} \tag{3.4.35}$$

We see that C_k is a commutator term and applying the Cauchy-Schwartz inequality yields

$$C_k \lesssim \left\|\left[Z^k, W_\varepsilon(v)\right](\partial_x v)\right\|_{L^2}\left\|Z^k v\right\|_{L^2}.$$

We can now invoke Lemma 3.3.2 to finish off the estimate. We then get

$$C_k \lesssim \|W_\varepsilon(v)\|_{H^r} \|\partial_x v\|_{H^{k/2-1}} \left\|Z^k v\right\|_{L^2} \lesssim \|\partial_x v\|_{H^{k/2-1}} \left\|Z^k v\right\|_{L^2}, \tag{3.4.36}$$

where $r > \frac{3}{2}$ and $r \geqslant \frac{k}{2}$. More specifically, for $1 \leqslant k \leqslant 3$, we can use $r = \frac{3}{2}+$ and, for $k = 4$, we can use $r = 2$. At worst, since $k \leqslant 4$, the right-hand side of (3.4.36) involves $\|\partial_x v\|_{H^1} \leqslant \|v\|_{H^2}$. We thus obtain

$$C_k \lesssim \|\partial_x v\|_{H^{k/2-1}} \left\|Z^k v\right\|_{L^2} \lesssim \mathcal{E}_{\text{grav}}. \tag{3.4.37}$$

We now move on to consider D_k. Here, we make use of the fact that $W_\varepsilon(v) = W_\varepsilon(v^*)$ and commute Z^k with ∂_x, which will cause us to pick up a derivative commutator:

$$D_k = -\int_0^{2\pi} W_\varepsilon(v)\partial_x \left|Z^k v\right|^2 - \Re\mathfrak{e}\left\{\int_0^{2\pi} W_\varepsilon(v)\left[Z^k, \partial_x\right](v) Z^k v^* + W_\varepsilon(v^*)\left[Z^k, \partial_x\right](v^*) Z^k v \, dx\right\}. \tag{3.4.38}$$

We now integrate by parts in the first term and apply the Cauchy-Schwartz inequality to both terms:

$$D_k \lesssim \|\partial_x W_\varepsilon(v)\|_{L^\infty} \left\|Z^k v\right\|_{L^2}^2 + \|W_\varepsilon(v)\|_{L^\infty} \left\|\left[Z^k, \partial_x\right](v)\right\|_{L^2} \left\|Z^k v\right\|_{L^2}. \tag{3.4.39}$$

We now see that we can apply Lemma 3.3.1, which gives us

$$D_k \lesssim \left\|Z^k v\right\|_{L^2}^2 + \|v\|_{H^{k/2-1/2}} \left\|Z^k v\right\|_{L^2}. \tag{3.4.40}$$

Notice that the Sobolev norm is of order at most $\frac{3}{2}$ since $k \leqslant 4$. As such, we do not have any trouble closing the estimate:

$$D_k \lesssim \left\|Z^k v\right\|_{L^2}^2 + \|v\|_{H^{k/2-1/2}} \left\|Z^k v\right\|_{L^2} \lesssim \mathcal{E}_{\text{grav}}. \tag{3.4.41}$$

Upon combining (3.4.35), (3.4.37) and (3.4.41), we have

$$\frac{d\mathcal{E}_{\text{grav},k}}{dt} \lesssim \mathcal{E}_{\text{grav}}. \tag{3.4.42}$$

Finally, the estimates (3.4.32) and (3.4.42) give us the desired control of the time derivative of the

energy:

$$\frac{d\mathcal{E}_{\text{grav}}}{dt} \lesssim \mathcal{E}_{\text{grav}}. \tag{3.4.43}$$

□

3.5 An Alternative Proof of Theorem 3.4.5

Theorem 3.4.5 is sufficient for the purpose of obtaining $O(1)$ existence time for solutions of (3.1.20). However, in the $\alpha = \frac{1}{2}$ case, we were no longer able to use the vector field $\varepsilon\partial_t$, instead we utilized $iL + \chi_\omega$. Here, our goal is to show that one can obtain the result of Theorem 3.4.5 using the vector field $iL + \chi_\omega$ and so show that both results can be obtained using a unified approach.

Theorem 3.5.1. *If $\alpha = \frac{3}{2}$, v is a solution of* (3.1.20) *with $\sigma \geqslant 3$ and $\mathcal{E}_{\text{cap}} = \mathcal{E}_{\text{cap}}(t)$ is given by Definition 3.4.4 with $Z = iL + \chi_\omega$, then one has*

$$\frac{d\mathcal{E}_{\text{cap}}}{dt} \lesssim \mathcal{E}_{\text{cap}}.$$

Proof. As in the proof of Theorem 3.4.5, write $\mathcal{E}_{\text{cap}} = \mathcal{E}_{\text{cap},0} + \mathcal{E}_{\text{cap},1} + \mathcal{E}_{\text{cap},2}$ and notice that, by Lemma 3.4.3, we have

$$\frac{d\mathcal{E}_{\text{cap},0}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.5.1}$$

We thus begin in earnest by considering the time derivative of $\mathcal{E}_{\text{cap},k}$ for $k = 1, 2$:

$$\frac{d\mathcal{E}_{\text{cap},k}}{dt} = -\int_0^{2\pi} Z^k(W_\varepsilon(v)\partial_x v)Z^k v^* + Z^k v Z^k(W_\varepsilon(v^*)\partial_x v^*)\,dx - \frac{2i}{\varepsilon}\int_0^{2\pi}\left|\sqrt{L}Z^k v\right|^2 dx - \frac{2}{\varepsilon}\int_0^{2\pi}\chi_\omega\left|Z^k v\right|^2 dx.$$

As we've seen many times already, we can reduce this to

$$\frac{d\mathcal{E}_{\text{cap},k}}{dt} \leqslant -\Re\mathfrak{e}\left\{\int_0^{2\pi} Z^k(W_\varepsilon(v)\partial_x v)Z^k v^* + Z^k v Z^k(W_\varepsilon(v^*)\partial_x v^*)\,dx\right\}. \tag{3.5.2}$$

We now rewrite the integral on the right-hand side of (3.5.2) by adding/subtracting and making note of

the fact that $W_\varepsilon(v) = W_\varepsilon(v^*)$:

$$\begin{aligned}\frac{d\mathcal{E}_{\text{cap},k}}{dt} &\leqslant -\mathfrak{Re}\left\{\int_0^{2\pi}[Z^k(W_\varepsilon(v)\partial_x v) - W_\varepsilon(v)Z^k\partial_x v]Z^k v^* + [Z^k(W_\varepsilon(v^*)\partial_x v^*) - W_\varepsilon(v^*)Z^k\partial_x v^*]Z^k v\,dx\right\} \\ &\quad - \mathfrak{Re}\left\{\int_0^{2\pi} W_\varepsilon(v)(Z^k\partial_x v Z^k v^* + Z^k\partial_x v^* Z^k v)\,dx\right\} \\ &= C_k + D_k. \end{aligned} \tag{3.5.3}$$

We begin by considering the commutator term C_k for which we plainly have

$$C_k \lesssim \left\|\left[Z^k, W_\varepsilon(v)\right](\partial_x v)\right\|_{L^2}\left\|Z^k v\right\|_{L^2}. \tag{3.5.4}$$

We can apply Lemma 3.3.2 to estimate the right-hand side of (3.5.4):

$$C_k \lesssim \|W_\varepsilon(v)\|_{H^r}\|\partial_x v\|_{H^{3k/2-1}}\left\|Z^k v\right\|_{L^2}. \tag{3.5.5}$$

We will either take $r = \frac{3}{2}+$ for $k = 1$ or $r = 3$ for $k = 2$. In addition, $\frac{3k}{2} - 1 \leqslant 2$ and so the highest Sobolev norm of v appearing is $\|\partial_x v\|_{H^2} \leqslant \|v\|_{H^3}$. Thus, the energy estimate closes and we have

$$C_k \lesssim \|\partial_x v\|_{H^{3k/2-1}}\left\|Z^k v\right\|_{L^2} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.5.6}$$

We rewrite D_k by commuting Z with ∂_x and integrating by parts, which yields

$$D_k = \int_0^{2\pi}\partial_x W_\varepsilon(v)\cdot\left|Z^k v\right|^2 dx - \mathfrak{Re}\left\{\int_0^{2\pi} W_\varepsilon(v)\left[Z^k,\partial_x\right](v)Z^k v^* + W_\varepsilon(v^*)\left[Z^k,\partial_x\right](v^*)Z^k v\,dx\right\}. \tag{3.5.7}$$

Hence, we have the estimate

$$D_k \lesssim \|\partial_x W_\varepsilon(v)\|_{L^\infty}\left\|Z^k v\right\|_{L^2}^2 + \|W_\varepsilon(v)\|_{L^\infty}\left\|\left[Z^k,\partial_x\right](v)\right\|_{L^2}\left\|Z^k v\right\|_{L^2}. \tag{3.5.8}$$

We apply the derivative commutator estimate of Lemma 3.3.1 with $\alpha = \frac{3}{2}$ to bound the commutator term. This gives control via the energy:

$$D_k \lesssim \left\|Z^k v\right\|_{L^2}^2 + \|v\|_{H^{3k/2-3/2}}\left\|Z^k v\right\|_{L^2} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.5.9}$$

We are able to close the above estimate as $\frac{3k}{2} - \frac{3}{2} \leqslant \frac{3}{2} < 2$ given that $k = 2$ is the worst-case scenario. Ergo, upon combining (3.5.5) and (3.5.9), we obtain control of $\frac{d\mathcal{E}_{\text{cap},k}}{dt}$:

$$\frac{d\mathcal{E}_{\text{cap},k}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.5.10}$$

This gives us the desired estimate:

$$\frac{d\mathcal{E}_{\text{cap}}}{dt} \lesssim \mathcal{E}_{\text{cap}}. \tag{3.5.11}$$

□

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