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Globalization and inequality in an agent-based wealth exchange model

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GLOBALIZATION AND INEQUALITY IN AN AGENT-BASED WEALTH EXCHANGE MODEL

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ABSTRACT

Agent-based asset exchange models serve as an interesting and tractable means by which to study the emergence of an economy's wealth distribution. Although asset exchange models have reproduced certain features of real-world wealth distributions, previous research has largely neglected the effects of economic growth and network connectivity between agents. In this work, we study the effects of globalization on wealth inequality in the Growth, Exchange, and Distribution (GED) model [Liu *et al.*, Klein *et al.*] on a network or lattice that connects potential trading partners. We find that increasing the number of trading partners per agent results in higher levels of wealth inequality as measured by the Gini coefficient and the variance of the agent wealth distribution. However, if globalization is accompanied by a proportionate increase in the economic growth rate, the level of inequality can be held constant. We present a mean-field theory to describe the GED model based on the Fokker-Planck equation and derive the stationary wealth distributions of the network GED model. For large Ginzburg parameter for which mean-field theory is applicable, the wealth distributions for the fully connected model are found to be Gaussian; however, for sparse trade networks, a non-Gaussian "hyperequal" phase is found even for large Ginzburg parameter. It is shown that several networks (Erdős-Renyi, Barabsi-Albert, one-dimensional and two-dimensional lattices) have

the mean-field critical exponents when the Ginzburg parameter is large and held constant and the system parameters are scaled properly.

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GED . . . . Growth, Exchange, and Distribution

CHAPTER 1: Introduction

1.1 Rising Inequality

A society with a high level of economic inequality is more likely to have a physically and mentally unhealthy population, a higher rate of criminal violence, a higher rate of recreational drug use, longer working hours, a lack of community life, and larger prison populations [1]. Historically, the rise of extreme wealth inequality has led to the collapse of many advanced civilizations such as the Roman, Han, and Gupta empires [2].

Wealth inequality has increased sharply over the past century. This trend holds both within the United States [3–5] and globally [6, 7]. For example, an American family in the ninetieth wealth percentile (*i.e.*, richer than 90% of all other Americans) increased their net wealth five-fold from 1963 to 2017. In 1963, families in the lower tenth percentile had a net wealth of about zero; by 2017, the tenth-percentile American family fell into debt for an average wealth of $-\$1,000$. This fact is despite a dramatic overall increase in the average wealth in the United States over the same time period. Short time intervals over the last century during which total inequality decreased have almost always been due to a deflation of asset prices (such as a crashing stock market) which causes the net worth of the richest to fall; however, during these periods there was still very little economic mobility between classes [3].

Income inequality has also been rising. Since the 1960s, wages for the lowest-paying jobs (bottom 10%) in the U.S. have remained stagnant when adjusted for inflation. This no-growth trend even holds for the median American wage. Meanwhile, wages for the highest earning jobs have nearly doubled over the same period [4]. The stagnation of wages possibly contributes to the rising average household debt seen in the U.S. since 1980 as consumers must borrow money to maintain or improve their standard of living without an increase an income [8].

Rising inequality is not limited to the United States. There is a global trend toward wealth condensation: The top 1% wealthiest households have increased their share of the economy's total wealth by 5% since 1980 in China, Europe, as well as the United States. It is very likely that the current level of world-wide wealth inequality is drastically underestimated as the complexities of the globalized financial system provide the richest individuals with ever-expanding means of hiding their true wealth [6, 9].

The increasing inequality around the world and the negative consequences of such inequality motivates a more substantial research effort to understand the causes of wealth disparity [3, 5]. Such an understanding would be valuable in guiding public policy. A goal of the present work is to develop a simple quantitative framework in which to analyze the causes of wealth disparity.

1.2 Globalization and Inequality

Rising inequality over the past century has coincided with an explosion of international trade volume [10, 11]. This inter-connection of the world's economies and its resulting effects are known collectively as *globalization*.

The effect of globalization on global wealth disparity is an active topic of economic research [12–14]. Many researchers believe that globalization contributes to inequality within nations [15], and between nations themselves [16]. It is often argued that low-skilled workers within rich countries see their wages decreased due to competition with low-skilled workers in poorer countries. Meanwhile, high-skilled workers within rich nations see an increase in wages due to the increased demand worldwide for their products and services. Within a certain industry such as manufacturing, Krugman *et al* [16] argues that low transportation costs may increase competition between nations such that the majority of countries see a reduction in earnings while only a few countries emerge as winners.

1.3 Problem Statement

In this thesis, I study the effect of globalization on wealth inequality using an agent-based asset exchange model that includes a mechanism for economic growth. We modify an existing model, the Growth, Exchange, and Distribution (GED) model, to simulate an arbitrary trade network that connects potential trading partners. By varying the properties of this network, we can simulate the process of globalization within an economy.

We seek to shed light on a controversial question in economics using the tools of statistical physics. Although one can always question the applicability of a simplified model to the real world economy, we hope to provide a quantitative perspective that may prove useful to other researchers.

This dissertation is structured as follows. In Sec. 2, we present a brief review of influential agent-based models such as the Yard Sale model, and we introduce the GED model and its simulation results for the fully connected case. In Sec. 3, we present new simulation results for the network GED model which show that globalization exacerbates wealth inequality. We also find that the network GED model displays the same mean-field critical behavior as the fully connected model. In Sec. 4, we develop a mean-field theory for the fully connected GED model and derive its critical exponents. In Sec. 5, we use a Fokker-Planck approach to derive the wealth distribution and Ginzburg parameter of the network GED model. This theory explains the simulation results of Sec. 3. In Sec. 6, we derive the GED model's phase transition directly from the master equation. Finally, in Sec. 7, we conclude with a summary of results and suggestions for future work.

CHAPTER 2: Background

2.1 Real-World Wealth and Income Distribution

In real-world economies, both the wealth and income distributions are known to follow an exponential for the lower 97 to 99% of individuals [17–20]. That is, the number density of people $P(w)$ with wealth w is proportional to $\exp[-w/w_0]$ for some parameter w_0. This trend holds for income data across 67 different countries studied [17]. Because these wealth distributions are exponential, we can associate an inverse temperature with each economy $\beta = w_0$. This temperature has been found to gradually increase over time in the United States as the population becomes more wealthy [19].

In contrast, the income and wealth distributions of the richest members of the economy are not exponential, but follow a power law: $P(w) \propto w^{-\alpha}$, with $\alpha \approx 2$ across different data sets [18–20]. This empirical phenomenon was famously observed by Pareto in 1897 when studying the wealth distribution of rich Italian land owners [21] and is known as Pareto's law. Reference [19] shows empirically that, in the United States, the wealth of these richest people is highly correlated to the S&P500.

2.2 Agent-Based Modeling

Agent-based modeling holds much potential for advancing our understanding of economics [22–25]. Classical economic theory rests on many assumptions about the behavior of free markets. Traditionally, economies are often assumed to be in equilibrium, or perfectly efficient, or unchanging over time. Because it addresses the fundamental dynamics underlying the broader economy, agent-based modeling has the potential to reveal insights not predicted by classical economics. Instead of directly modeling an economy as a whole, agent-based modeling is concerned with the behavior of individual interacting agents. The economy, in turn, is modeled by the aggregate behavior of these many agents.

In an agent-based asset exchange model, a large number of agents, each with a certain amount of wealth w, interact with other agents to update their wealth. The resulting system-wide wealth distribution can be analyzed. Usually the exchange rule is stochastic such that agents randomly give and take wealth from each other for no underlying reason. The final stationary wealth distribution of the system, if there is one, typically depends on the specific exchange mechanism used between agents.

Often, reproducing the real-world wealth distribution is the goal of an agent-based model, but agent-based models can be used to simulate many macroeconomic phenomena such as unemployment [25] and the effect of social programs [26] and is particularly useful as a policy-guiding tool for monetary authorities. Because there are many seemingly arbitrary choices that a researcher can make to develop even a simple agent-based model, there exists many varieties of similar models that yield qualitatively different results.

Many simple exchange rules such as random additive exchange and multiplicative exchange have been studied by physicists. For simple additive exchange, in which agents randomly lose or gain a small, fixed amount of wealth from each other, some unfortunate agents will attain negative total wealth as $t \to \infty$. If we remove these agents from the economy after reaching zero wealth, the resulting wealth distribution has been shown to be a Gaussian [27]. As discussed in Sec. 2.1, this distribution is not realistic.

Many researchers believe multiplicative exchange, in which the amount of wealth traded by each agent is proportional to their current wealth, is a better approximation of economic behavior. For example, if the amount of wealth exchanged between two agents is proportional to the losing agent's wealth, the resulting wealth distribution is exponential for a wide range of parameters [27], a more realistic result [17, 19]. A similar random, multiplicative agent-based model, in which winning agents receive a fraction of the losing agent's wealth, but the agents' wealth is also modified by an additive Gaussian noise, has been solved using mean-field theory [28]. The resulting wealth distribution is a power-law for the richest agents. This feature is consistent with the real-world Pareto's law described in Sec. 2.1. This same distribution was found to hold even upon adding a simple tax mechanism to the random, multiplicative model as well [28].

Pellicer-Lostao and Lopez-Ruiz [29] introduced an interesting multiplicative agent-based model in which winners and losers of an exchange are not determined randomly (unlike most agent-based models), but *chaotically* according to a two-dimensional chaotic system: the logistic bimap. The logistic bimap is a deterministic means of selecting successive pairs of winning and losing agents to exchange wealth. This chaotic system allows the researchers to vary the level of symmetry for each pair of agents; for low symmetry, a particular agent is more likely to always win or always lose the exchanges, but a high degree of symmetry results in an effectively random win or loss for each exchange. By decreasing the level of symmetry of the logistic bimap, Pellicer-Lostao and Lopez-Ruiz found that the resulting stationary agent wealth distribution transitions from an exponential to a power law. In other words, when the winner of an exchange is determined effectively randomly (high symmetry), the resulting wealth distribution is exponential, but when agents have different likelihoods of winning or losing their exchanges, the wealth distribution is a power law. This striking result suggests that the poorest members of the economy (the bottom 97%) *might* have their wealth determined randomly, but the richest members of the economy (upper 3%) achieve their levels of wealth due to a tendency to consistently win (or lose) trades. A similar finding about the difference between rich and poor agents is discussed in Ref. [30].

There have also been efforts to model economic growth while neglecting the effect of exchange between agents [31,32]. In Ref. [31], Vallejos *et al.* employed a discrete, stochastic growth and distribution rule in which agent i receives new wealth with probability

$$\psi_i = \frac{w_i^\lambda}{\sum_j w_j^\lambda}, \tag{2.1}$$

where the parameter λ determines how much more likely the rich are to receive new wealth than the poor. This growth and distribution rule is similar to the rule originally introduced by Kang *et al.* in Ref. [33]. Vallejos *et al.* show that Eq. (2.1) alone is capable of reproducing the exponential and power-law wealth distribution of real economies.

Evers *et al.* [32] model the process of globalization within a one-dimensional lattice of economic agents. These agents do not exchange wealth with each other, but instead each receives a "wage" at each time step proportional to the average agent's wealth. [xx average over all agents?xx] To keep the system's wealth from increasing exponentially, Evers *et al.* rescale all the agents' wealth in the vicinity of the agent who receives a wage. In other words, receiving new wealth in the form of a wage results in the agent's and its neighbors' wealth being rescaled such that the total amount of wealth in each neighborhood remains constant over time. The size of this neighborhood, in units of the lattice spacing, is the model's proxy for globalization. As the size of this neighborhood of rescaling is increased, the results were found to be consistent with Kuznets's hypothesis regarding globalization and inequality [34]: At first, the process of globalization results in increased inequality between agents; however, as globalization increases to encompass all agents in the system, inequality between agents is ultimately reduced. This hypothesis is often referred to as Kuznets's "U-shaped" hypothesis. Although the work of Evers *et al.* is interesting, in our work we model globalization not as a range over which wealth is rescaled, but as a measure of the amount of long-distance trade that occurs between agents. We believe that our measure of globalization directly quantifies the defining characteristics of real-world globalization as discussed in Sec. 1.2.

In the following sections we will discuss some particular multiplicative exchange models and comment on their usefulness to address the problem of globalization and inequality and their success in reproducing real-world wealth distributions.

2.3 The Yard Sale Model

The most influential agent-based asset exchange model has been the Yard Sale model [35] in which pairs of agents exchange an amount of wealth proportional to the wealth of the poorer agent:

$$\Delta w = f \min(w_1, w_2). \tag{2.2}$$

Here Δw is the change in wealth of the winning agent. The parameter f is a number less than 1 and greater than 0, and w_1 and w_2 are the wealth of agents 1 and 2, respectively. Agents 1 and 2 are the two agents undergoing an exchange.

The winner and loser of the Yard-Sale exchange are determined randomly, typically without regard to the wealth of either agent. This randomness is a formulation of the *efficient market hypothesis*: each agent has equal knowledge of the true worth of the good or security being traded. However, the Yard Sale model assumes that, because no one has perfect knowledge of the true value of the traded item, one agent is bound to make a "good deal," while the other makes a "bad deal;" *i.e.*, either the seller has undervalued the item being traded, or the buyer has overvalued the item being traded. The extent by which the agents have missed the true value of the item is encoded through the parameter f in Eq. (2.2).

The Yard Sale model is somewhat realistic because the "transaction amount" of each trade is a small fraction of the poorer agent's wealth. Intuitively, it is rare for people or institutions to conduct a single trade in which a large amount of their net worth is at stake. Furthermore, a typical customer at a grocery store has much less wealth than the company operating the grocery store. Consequently, the value of the exchange between the two agents is proportional to the poorer agent's wealth regardless of the wealth of the seller.

The most striking result of the Yard Sale model is *wealth condensation*. After many exchanges according to Eq. (2.2) such that any agent can exchange wealth with any other agent, one agent inevitably ends up holding the vast majority of all wealth in the system. This phenomenon is called wealth condensation. The longer the simulation progresses, the closer the richest agent comes to holding all the wealth; other agents will hold only infinitesimal amounts.

Consider a system of N total agents with W total wealth. The expected value of the wealth of a randomly chosen agent remains fixed at $w = W/N$, while every agent's wealth has only an infinitesimal chance of not approaching zero in the long-time limit. This behavior is similar to how an ensemble of random walkers undergoing geometric Brownian motion will have an exponentially increasing expected value as $t \to \infty$; at the same time, an increasing majority of the wealth of the walkers will approach zero as $t \to \infty$ [36]. This apparent paradox is due to the nature of randomness and multiplicative noise. Because the noise is multiplied by the current level of wealth (or current position) of the agent (or walker), it is possible to lose a large amount of wealth if an agent is very rich, but it is impossible for a very poor agent to gain a large amount of wealth. Therefore, even if the probabilities of gaining and losing wealth are equal for each agent, poverty remains a trap that is very difficult to escape. See Refs. [37] and [33] for further discussion on this topic.

Much work has been done on the Yard-Sale model [26, 38–43]. Boghosian *et al.* [39] proved that the Gini coefficient [defined in Eqs. (2.22) and (2.23)] must increase over time. The result is constantly increasing inequality approaching total wealth condensation as $t \to \infty$. To analyze the steady-state wealth distribution of modified versions of the Yard Sale model, Boghosian *et al.* have also developed a Fokker-Planck description of the Yard Sale model that has greatly influenced the research in my thesis [38, 40]. Boghosian *et al.* showed that, when the Yard Sale model is modified to include a simple redistributive tax, the resulting wealth distribution has certain features consistent with Pareto's law[1]

The Yard Sale model has also been simulated on networks [42, 43]. In Ref. [42], Montaña *et al.* find that wealth condensation still occurs when agents are only allowed to interact with their neighbors on a network. Only when the network is disconnected such that there is no path from one group of agents to another group of agents is full wealth condensation avoided. In this disconnected case, a local wealth condensation occurs for each separate cluster of agents.

Reference [43] confirms the findings of Ref. [42] for a wide variety of network types. In this work, Bustos-Guajardo and Moukarzel employ a modification to the Yard Sale model in which the poorer agent wins the exchange with probability p.

[1]While the distribution obtained in Ref. [40] is only valid in the limit $w \to 0$, Boghosian notes that the distribution approaches a power law in the limit $w \to \infty$. This power-law behavior is consistent with the simulation results of Ref. [38].

This modification introduces a critical point into the Yard Sale model at

$$p^*(f) = \frac{\log(1-f)}{\log(1-f) - \log(1+f)}. \tag{2.3}$$

For $p < p^*$, the modified Yard Sale model progresses toward wealth condensation just as it does for $p = 1/2$. However, for $p > p^*$, wealth condensation is avoided. Note that in the original Yard Sale model,

$$p = \frac{1}{2} = \lim_{f \to 0} p^*(f), \tag{2.4}$$

which results in wealth condensation. This critical point at p^* is found to be the same for all network types studied in Ref. [43].

Bustos-Guajardo and Moukarzel find that for the network Yard Sale model, in the unstable phase ($p < p^*$), wealth does not condense onto a single agent, but instead there arise a set of "locally rich agents" that hold the vast majority of the wealth of their immediate neighbors. In other words, a few rich agents will accumulate wealth until their neighbors can no longer provide them with new wealth. Because it is very difficult (statistically impossible) for a very poor agent to regain a large portion of its wealth, the connection between a locally rich agent and its neighbors is effectively killed as the neighbors' wealth approaches zero. Therefore, this type of local wealth condensation serves as a fixed point for the entire system. The number of these locally rich agents is determined by the properties of the network.

In Ref [26], Cardaso *et al.* introduced a similar modification to the Yard Sale model in which the probability of an agent winning an exchange is proportional to the difference in wealth between the two agents:

$$p_p = \frac{1}{2} + s\frac{w_r - w_p}{w_r + w_p}, \tag{2.5}$$

where w_r is the wealth of the richer agent, w_p is the wealth of the poorer agent, and p_p is the probability that the poorer agent wins the exchange. The parameter s is referred to as the *social protection factor* and is meant to encode the effect of social programs on an agent's tendency to gain or lose wealth. When this probability is used in tandem with the Yard Sale exchange mechanism in Eq. (2.2), the resulting wealth distribution becomes more equal for higher values of s and less equal for lower values of s as expected. Interestingly, Cardaso *et al.* find hysteresis upon perturbation of the social protection factor during the course of a simulation. For example, after the wealth distribution has reached a steady state at $s = 0.5$, Cardaso *et al.* lower s by an amount Δs such that the economy approaches a less equal wealth distribution. For $\Delta s > 0.1$, reinstating the initial, higher value of $s = 0.5$ does not result in the previously high level of equality among agents. This result suggests that the effect of even a temporary suspension of social programs can lead to irreversible economic consequences. Cardaso *et al.* suggest that this hysteresis is due to a significant fraction of the population reaching a negligible level of wealth while the social protection factor is removed or reduced; after reaching infinitesimal wealth, an agent can no longer obtain a significant fraction of wealth even from winning trades under the Yard Sale exchange rule. The simulations of Ref. [26] seem to have been run for only $\sim 10^4$ Monte Carlo steps. For comparison, simulations of the GED model in Secs. 2.6.2 and 3 have been run for at least 10^6 Monte Carlo steps before taking data in order to avoid measuring transient behavior [44]. Cardaso *et al.* claim to have run their simulations "long enough to verify equilibrium;" however, it is possible that the system, after perturbation, only appeared to be in equilibrium, but rather was in a metastable state. It is possible that the economy would return to true equilibrium and to the previous level of equality after a longer simulation time.

Despite its success as an elegant and somewhat realistic agent-based model, the Yard Sale model has many limitations. Perhaps the most obvious limitation of the Yard Sale model is that it assumes a closed system in which no new wealth can enter and no wealth can exit. The exchange rule in Eq. (2.2) conserves the total wealth between the two agents, so the total wealth of the system remains constant. In the real global economy, wealth is not conserved. For example, world GDP has doubled six times since 1960 [45]. And in the case of an economic downturn, it is possible for total wealth to decrease as well. Furthermore, we may wish not to model the world economy as a whole, but instead model companies within a certain industry, or the economy of a particular region, or some other subset of the global economy. It is clear that wealth can flow in or out of these subsystems.

The Yard Sale model is also ill-suited to address the question of globalization and its effect on inequality. It is not sufficient to simply run the Yard Sale model on a lattice or network. Globalization is characterized by a large increase in the total amount of trade worldwide [10, 11]. It would be impossible to meaningfully model this increasing trade within the original Yard Sale model because the only natural time scale in the system is provided by the exchanges themselves. In other words, time is naturally defined as the number of exchanges that have taken place since the start of the simulation. If we arbitrarily define a time independent of the number of exchanges in the original Yard Sale model, then the results are trivial; we would just be observing the same dynamics sped up or slowed down at an arbitrary rate.

2.4 The Extended Yard Sale Model

Boghosian *et al.* introduced the Extended Yard Sale Model in Ref [41]. This model adds two competing mechanisms to the original Yard Sale model: *wealth-attained advantage* and a flat, *redistributive tax* on all agents' wealth. These features of the model result in critical behavior: when the wealth-attained advantage overpowers the redistribution of wealth, the economy tends toward wealth condensation. However, wealth condensation is avoided when the redistribution tax is stronger than the effect of wealth-attained advantage.

Instead of Eq. (2.2), the amount of wealth exchanged between agents 1 and 2 in the Extended Yard Sale Model is given by

$$\Delta w = \sqrt{\gamma \Delta t} \min(w_1, w_2)\eta + f_{\text{tax}} \left(\frac{W}{N} - w_1 \right) \Delta t \tag{2.6}$$

such that

$$w_1 \to w_1 + \Delta w \tag{2.7}$$

and

$$w_2 \to w_2 - \Delta w. \tag{2.8}$$

In Eq. (2.6), we have assumed that agent 1 has initiated the exchange with agent 2, and not vice-versa.

The first term on the left-hand side of Eq. (2.6) encodes the wealth-attained advantage of the richer agent via the random variable η. Unlike in the original Yard

Sale model, here the expectation value $\langle \eta \rangle$ is nonzero:

$$\langle \eta \rangle = a\sqrt{\frac{\Delta t}{\gamma}}\left(\frac{w_1 - w_2}{W/N}\right). \tag{2.9}$$

The second moment $\langle \eta^2 \rangle$ is chosen to be 1. The parameter a determines the strength of the wealth-attained advantage in deciding the winner of an exchange.

The second term of Eq. (2.6) is the redistributive tax. W is the total amount of wealth held by all agents, and N is the total number of agents. The parameter f_{tax} effectively determines the tax rate for each exchange. The initiating agent will receive or lose some extra wealth proportional to the difference between his wealth and the average wealth of all agents in the system W/N.

Note that the exchange rule of Eq. (2.6) is not symmetric upon interchange of agents 1 and 2. That is, there is a difference between agent 1 initiating an exchange with agent 2 and agent 2 initiating an exchange with agent 1. Although the wealth-attained advantage term of Eq. (2.6) is symmetric about agents 1 and 2 (due to the fact that Δw is defined as the change in wealth of the initiating agent), the redistributive term of Eq. (2.6) depends only on the wealth of the initiating agent. This asymmetry stands in contrast to the exchange rule of the original Yard Sale model.

It is interesting that this "tax" mechanism of the Extended Yard Sale Model is pairwise in nature. In real economies, the redistribution of wealth from taxes and social programs happens at a system-wide level using government as an intermediary between the rich and poor. In Refs. [41] and [46], the authors point out that the net effect of this tax term on agent 1 is equivalent to agent 1 contributing an amount of wealth $f_{\text{tax}} w_1 \Delta t$ to a large pool of collected wealth, and then receiving $f_{\text{tax}} W \Delta t/N$ wealth back. The argument is that after N exchanges, the pool of collected wealth has grown to a total of, on average, $f_{\text{tax}} W \Delta t$. Then this large pool of wealth is divided by the total number of agents N and distributed to each one. However, this narrative considers only the effect of Eq. (2.6) on agent 1, and assumes that agent 1 engages in exactly one trade per time step. In practice, however, Eq. (2.6) results in the initiating agent's effective tax rate and tax return being dependent on its number of exchanges per time step. Furthermore, because Δw is also the amount of wealth lost by agent 2, the "passive" agents (who do not initiate the exchange) have their tax rates and tax returns dependent on the wealth of their trading partners, regardless of their own wealth. Essentially, the model punishes people for accepting trades with poor agents and rewarding people for accepting trades with rich agents.

Nevertheless, the Extended Yard Sale Model displays interesting phenomenology as there exists a critical point at $f_{\text{tax}}/a = 1$. This critical behavior can be seen by solving for the model's wealth distribution using a Fokker-Planck approach. For $f_{\text{tax}} < a$, wealth condensation develops as one agent comes to hold almost all the wealth in the system. However, for $f_{\text{tax}} > a$, *partial wealth condensation* arises in which one agent comes to hold, on average of $1 - f_{\text{tax}}/a$ of the total wealth as $t \to \infty$. This interesting feature of the model is similar to the real-world wealth distribution of the United States; about 30% of all wealth in the country is held by a small fraction of the population [46].

For high redistribution and low wealth-attained advantage, the agent density as a function of wealth $P(w)$ approaches a log-normal distribution. As the wealth-attained advantage a is increased approaching the critical point, the wealth distribution becomes increasingly skewed until the vast majority of agents have near-zero wealth, but there exists a long, thin tail as $w \to \infty$. Exceeding $a/f_{\text{tax}} = 1$ results in partial wealth condensation.

We note that the Extended Yard Sale Model is not suitable for the study of globalization for the same reasons as the pure Yard Sale model: it lacks a natural way to increase the total amount of trade per unit time, and it does not take into account the effect of economic growth. However, as in the case of the Yard Sale model, it is straightforward to modify the Extended Yard Sale Model to run on a trade network that would restrict the availability of trading partners for each agent.

2.5 The Affine Wealth Model

The Affine Wealth Model [46] extends the Extended Yard Sale Model to allow for agents with negative wealth. The existence of such agents allows the model to better represent a real-world economy because the poorest members of the economy often have debt in excess of their total net worth. Including these agents allows the Affine Wealth Model to reproduce the United States' wealth distribution to remarkable accuracy.

The Affine Wealth Model supports values of agent wealth in the range $(-d, \infty)$. Here d is a positive constant chosen as the maximum amount of debt an agent can acquire. Although this arbitrary cutoff may seem unnatural at first, it is somewhat realistic. If an individual keeps falling into ever larger debt, eventually it will be impossible to acquire more debt. For example, if someone never pays their credit card bill, eventually their credit card will stop working. Eventually, creditors will no longer be willing to lend money.

The Affine Wealth Model uses the same exchange rule as the Extended Yard Sale Model in Eq. (2.6) except that the wealth of each agent w_1 and w_2 are replaced by w_1+d and w_2+d. This allows the Affine Wealth Model to use the same exchange rule as the Extended Yard Sale Model, while allowing for agents with negative wealth. The resulting wealth distribution for the Affine Wealth Model is identical to that of the Extended Yard Sale Model except shifted by $-d$.

The most interesting result of the Affine Wealth Model is its ability to reproduce the real-world wealth distribution of the United States. Using data from the U.S. Survey of Consumer Finances conducted by the Federal Reserve Board in cooperation with the U.S. Department of the Treasury [47], Li *et al.* compared the Lorenz curve of the Affine Wealth Model to the United State's Lorenz curve. Ten different Lorenz curves were available for comparison from the years 1989 to 2016. Li *et al.* tuned the three parameters of the Affine Wealth Model (d, f_{tax}, and a) to fit the empirical data. The resulting wealth distributions are remarkably accurate for each year.

Interestingly, each of the Affine Wealth Model's wealth distributions used to fit real-world data was found to be in the model's supercritical phase; *i.e.*, the best-fit economy is partially wealth-condensed. Consistently across all the years studied, about 30% of the system's wealth was held by the richest agent. According to Li *et al.*, this fit suggests that the real wealth distribution of the United States is likely also partially wealth-condensed. A few comparisons were also made between the Affine Wealth Model's wealth distribution and the real wealth distribution of several European countries using data from the European Central Bank.

Because the Affine Wealth Model is merely the Extended Yard Sale Model modified to include negative-wealth agents, the Affine Wealth Model is not suited to address the problem of globalization and inequality for the same reasons as the Extended Yard Sale Model and the pure Yard Sale model as discussed.

2.6 The GED Model

2.6.1 Fully connected GED algorithm

Here we introduce the Growth, Exchange, and Distribution (GED) model [33,37,44] in the case of a fully connected economy. In this fully connected case, every agent is capable of exchanging wealth with any other agent in the system. The GED algorithm comprises three steps: wealth exchange, distribution of new wealth due to growth, and rescaling of the wealth.

Suppose we have a set of N individual agents each with wealth w_i. For each simulation time unit, we choose N random pairs of agents to exchange wealth. For each pair, a winner is chosen at random to receive a fraction of the other's wealth. We suppose that the agents have equal knowledge of the true value of goods traded, and therefore winning a trade is purely a matter of chance, regardless of an agent's wealth. This assumption is a formulation of the efficient market hypothesis. If agent i wins the exchange and agent j loses, then

$$w_i \to w_i + f \min(w_i, w_j) \tag{2.10}$$

$$w_j \to w_j - f \min(w_i, w_j), \tag{2.11}$$

where f is a fraction less than one. This exchange rule is identical to the Yard-Sale model [35] [Eq. (2.2)].

After exchanging wealth between agents, we add new wealth to the system. We assume exponential growth governed by the parameter μ:

$$W(t) = Ne^{\mu t}, \tag{2.12}$$

where $W(t)$ is the total system wealth at time t. We will consider only the *rescaled wealth* of each agent and keep the total rescaled wealth fixed at N for all t. Because we will take $\mu \ll 1$, we add $Ne^{\mu} \approx \mu N$ *rescaled* wealth to the system after each unit of simulation time. The amount of the new wealth received by each agent is given by the distribution rule introduced by Kang et al. [44]:

$$w_i \to w_i + \mu N \frac{w_i^{\lambda}}{\sum_j w_j^{\lambda}}, \tag{2.13}$$

where λ is the *distribution parameter*. This growth rule is similar to the discrete, stochastic growth rule in Eq. (2.1) of Ref. [31].

Finally, at the end of each time step we rescale all wealth:

$$w_i \to \frac{w_i}{1+\mu}. \tag{2.14}$$

This scaling is for computational convenience because it prevents the wealth of each agent from growing without bound.

To summarize the GED algorithm:

1. Initialize N agents with uniform random wealth $w_i : w_i > 0, \sum_i w_i = N$.

2. Choose a random pair of agents and exchange wealth by Eqs. (2.10) and (2.11).

3. Repeat step 2 N times.

4. Assign additional wealth to each agent according to Eq. (2.13).

5. Rescale all agent's wealth for a total of N wealth in the system [Eq. (2.14)].

6. Increment the time by one simulation time unit.

7. Repeat steps 3–6 until arriving at a stationary wealth distribution (for $\lambda < 1$) and measure quantities of interest.

2.6.2 Fully connected GED simulation results

The results in this section are taken from Ref. [44] and were also reproduced as part of the work of this dissertation.

The fully connected GED model has been investigated numerically and shown to have a phase transition at $\lambda = 1$. For $\lambda < 1$, the economy does not experience wealth condensation, but as $\lambda \to 1^-$, the wealth distribution becomes more unequal and approaches total wealth condensation. However, for all $\lambda < 1$, there is some economic mobility: poorer agents can become richer, and richer agents can become poorer. In addition, every agent's wealth increases exponentially due to economic growth as the system evolves with time. In contrast, for $\lambda \geq 1$ there is wealth condensation as was found in the original Yard-Sale model ($\mu = 0$), and there is no economic mobility.

This critical point in λ is similar to the critical point in p for the modified Yard Sale model of Ref. [43] [Eq. (2.3)] as it marks the boundary beyond which wealth condensation forms. In Sec. 3, we find that this critical point holds for all network types studied which is consistent with the results of Ref. [43]. The GED model produces steady-state wealth distributions that appear qualitatively similar to the steady-state wealth distributions of the Extended Yard Sale Model discussed in Sec. 2.4; however, the fully connected GED wealth distribution is found to be Gaussian in the case of high wealth equality, rather than log-normal.

The energy E, heat capacity C, order parameter ϕ and susceptibility per agent

χ are defined in Refs. [37, 44]:

$$E \equiv \sum_{i}^{N} (1 - w_i)^2 , \tag{2.15}$$

$$C \equiv \langle E^2 \rangle - \langle E \rangle^2. \tag{2.16}$$

These averages are taken over time.

The order parameter ϕ is taken to be the fraction of system wealth held by all but the richest agent. Then the susceptibility per agent is simply the fluctuations of this order parameter. However, for simulations it is useful to replace the variance of the order parameter by the variance of the wealth distribution as a whole:

$$\chi \equiv N\mathrm{Var}\left(P(w)\right). \tag{2.17}$$

For a physical system, we expect the critical exponents to follow the Rushbrooke scaling relation [48]:

$$\alpha + 2\beta + \gamma = 2. \tag{2.18}$$

In the fully connected case, if we keep the number of agents N constant as we approach the critical point ($\lambda \to 1^-$), we obtain critical exponents that do not satisfy Eq. (2.18). Additionally, the energy of the system in Eq. (2.15) is found to diverge with a critical exponent equal to 1 which is unphysical. In Sec. 3.4 we will see that these results also hold for network topologies when N is held constant.

However, if we keep the Ginzburg parameter $G \equiv \mu(1-\lambda)/f^2$ constant as we approach the critical point, while scaling f and μ by N, we obtain from simulations the critical exponent $\alpha = 1$ for the heat capacity C. Under these conditions, we also find that the energy per agent no longer diverges as we approach the critical point.

These results are consistent with the predictions of mean-field theory. We will derive the GED Ginzburg parameter in Sec. 5.1, and we will derive these exponents for the fully connected model in Sec. 4. We will obtain similar critical exponents for the network GED model in Sec. 3.4.

2.6.3 Network Topologies

We will consider random graphs such as the Erdős-Renyi and scale-free Barabasi-Albert graphs to simulate the connectivity of real-world trade networks. The Erdős-Renyi algorithm is a common, simple means of constructing a random network. To construct an Erdős-Renyi network, take every pair of agents $(i, j) : i \neq j$ and create a bond between them with constant probability p. The *degree* of a node in a network is its number of neighbors. This process of randomly connecting pairs of agents results in a binomial degree distribution for Erdős-Renyi graphs. The average degree per node $\langle k \rangle$ can be set in the thermodynamic limit ($N \to \infty$) through the relation $p = \langle k \rangle/(N-1)$.

In contrast, Barabasi-Albert networks are created via *preferential attachment.* To make a Barabasi-Albert graph of size N, first initialize a fully connected network of $n = \langle k \rangle + 1 \ll N$ connected agents each with degree $\langle k \rangle$ for some even integer $\langle k \rangle$. $\langle k \rangle$ will also be the average number of neighbors per node. Connect new agents to this initial cluster one-at-a-time by creating $\langle k \rangle/2$ bonds between each new agent and $\langle k \rangle/2$ randomly chosen agents from the cluster. These random agents are chosen based on their current number of neighbors:

$$p_i = \frac{k_i}{\sum_{j \in Q} k_j}, \tag{2.19}$$

where p_i is the probability that agent i, who is not already connected to the new, incoming agent, is chosen to connect to the new, incoming agent. k_i is the degree of agent i, and Q is the set of all agents with whom the new, incoming agent has not connected. (We must be careful not to connect to the same agent more than once.) Networks created via the Barabasi-Albert algorithm result in a power-law degree distribution: $P(k) \propto k^{-3}$. The long tail of the power-law distribution allows for the existence of "hubs," very well connected agents with many trading partners.

We also consider one and two-dimensional lattices with periodic boundary conditions, which do not have a random structure, but encode the influence of geographical proximity in determining an agent's trading partners.

It is helpful to define the *edge density* D of the trade network as its total number of bonds divided by the number of bonds in a fully connected network of equal size. In other words,

$$D \equiv \frac{2|E|}{N(N-1)} = \frac{\langle k \rangle}{N-1}, \tag{2.20}$$

where E represents the set of all edges, or connections, in the exchange network, and N is the number of nodes, or agents, in the network. Therefore, $|E| = N\langle k \rangle/2$, where $\langle k \rangle$ is the average degree (or average number of neighbors) per agent in the network. For example, for a 10×10 two-dimensional (2D) nearest-neighbor lattice, with each agent having four trading partners (with periodic boundary conditions), $\langle k \rangle = 4$. Because there are 100 total agents in the lattice, $D = 4/99 \approx 1/25$.

2.6.4 Network GED algorithm

Here we generalize the GED model to allow for a trade network that links potential trading partners. In the network GED model, agents may only exchange wealth with their immediate neighbors on the trade network.

As in the fully connected case [Sec. 2.6.1], we again have a set of N individual agents each with wealth w_i. The network GED model keeps the same growth and distribution rules from Eq. (2.13). However, for each simulation time unit, we now choose DN pairs of connected agents randomly from the set of all pairs of connected agents to exchange wealth via Eqs. (2.10) and (2.11). Consequently, an assumption of the network GED model is that an agent will trade in proportion to its number of available trading partners. This assumption motivates our choice of DN exchanges per unit time. This choice guarantees that a highly connected trade network will result in a large amount of trade, while a very sparse network will allow for very few trades. This trend is borne out by history: Since the year 1800, as the feasibility of long-distance trading has rapidly expanded, the total value of global exports has surged roughly exponentially [10, 11]. This feature of the GED model lends itself to the study of globalization and is one point of differentiation from related work [43] in which the rate of trade is held constant regardless of network density.

To summarize the network GED algorithm:

1. Construct a network of size N to link potential trading partners.

2. Initialize N agents with uniform random wealth $w_i : w_i > 0, \sum_i w_i = N$.

3. Choose a random pair of agents connected via the trade network and exchange wealth by Eqs. (2.10) and (2.11). Because highly connected agents are more likely to be present in randomly selected pairs, agents trade in proportion to their total number of connections.

4. Repeat step 3 DN times.

5. Assign additional wealth to each agent according to Eq. (2.13).

6. Rescale all agent's wealth for a total of N wealth in the system [Eq. (2.14)].

7. Increment the time by one simulation time unit.

8. Repeat steps 4–7 until arriving at a stationary wealth distribution (for $\lambda < 1$) and measure quantities of interest.

The network GED model is particularly well suited to the study of globalization because it offers a unified framework for considering the effects of both trade network and growth. Economic growth and globalization are deeply intertwined phenomena as many studies show a strong positive correlation between the two [49–51]. Furthermore, from an algorithmic perspective, growth provides a separate time scale independent of the number of exchanges that occur between agents.

In the present work, simulations were run for 10^6 time units to avoid the effects of transients, and then data was collected over another 10^6 time units. Unless otherwise indicated, simulation results in this dissertation are for $\mu = f = 0.1$ and $N = 4900$.

2.7 Quantifying Inequality

For a discussion of wealth inequality, it is important to define the metrics by which inequality is measured. One common metric is the *Gini coefficient* [52]. This quantity ranges from 0 to 1 with 0 being total equality among all agents in the economy, A Gini coefficient of 1 means that the population is maximally unequal; only one agent holds the entirety of society's wealth. According to the World Bank [53], the Gini coefficient in the United States has risen substantially from 1980 to 2018.

There are two equivalent methods of defining the Gini coefficient. The first is graphical and requires introducing the *Lorenz curve.* The Lorenz curve is a common visual representation of an economy's wealth distribution. The horizontal axis of the plot marks the wealth rank of each agent, and the vertical axis gives the cumulative

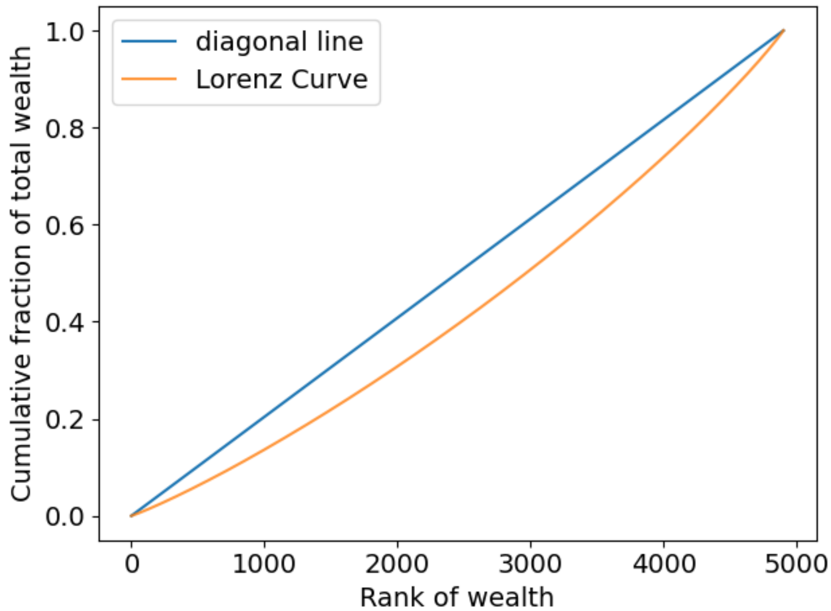


Figure 2.1: Lorenz curve for the wealth distribution of a simulation of the fully connected Growth, Exchange, and Distribution (GED) model with 4900 total agents. A perfectly equal wealth distribution would conform to a straight line (shown for reference).

fraction of wealth held by agents at or below a certain wealth rank. For an economy of N total agents, the wealth rank of the richest agent would be N, while that of the poorest would be 1. The Lorenz curve is plotted by successively summing the wealth of each agent, from poorest to richest, while moving along the horizontal axis and evaluating the fraction of the system's total wealth at each point. Therefore the slope of the Lorenz curve at each point is directly proportional to the wealth of the agent of the specified wealth rank, and both the Lorenz curve and its derivative monotonically increase along the horizontal axis. See Fig. 2.1 for an example of a Lorenz plot.

The Gini coefficient $\mathcal{G}$ is the normalized area between the diagonal line (representing a perfectly equal distribution) and the Lorenz curve:

$$\mathcal{G} \equiv \frac{1}{I}\int_0^N \frac{x}{N} - \mathcal{L}(x)dx, \tag{2.21}$$

$$I \equiv \int_0^N \frac{x}{N}dx = \frac{N}{2}, \tag{2.22}$$

where x is the wealth rank of an agent, $\mathcal{L}$ is the Lorenz curve, and we have assumed N total agents. This relationship to the Gini coefficient is why the Lorenz curve is useful for visualizing inequality; one can observe the area between the Lorenz curve and a straight line connecting the Lorenz curve's endpoints. The larger this area, the more inequality exists in the economy.

Alternatively, we can define the Gini coefficient as half the relative mean absolute difference in agent wealth [52]. Assuming W total system wealth and N total agents, $\mathcal{G}$ can be expressed as an explicit functional of the number density $P(w)$ of agents having a particular wealth w:

$$\mathcal{G}[P] \equiv \frac{1}{2NW}\int_0^\infty \int_0^\infty P(w_i)P(w_j)|w_i - w_j|dw_i dw_j. \tag{2.23}$$

This formulation of the Gini coefficient will be useful in our master equation derivation of the GED model's distribution phase transition in Sec. 6.

Although the Gini coefficient is a traditional metric by which to measure inequality, we can also make use of the variance of the rescaled wealth number density function $P(w)$. In general, we take w to be the *rescaled wealth* of an agent, defined as N times the true wealth of an agent divided by the total number of agents in the system. Therefore, in a perfectly equal economy, each agent has rescaled wealth $w = 1$. Then we can quantify the system's inequality by taking the second moment of the wealth density function:

$$\mathrm{Var}(P(w)) = \frac{1}{N}\int_0^{\infty}(1-w)^2 P(w)dw, \tag{2.24}$$

where we have used the fact that $\langle w \rangle = 1$ for the rescaled wealth w.

In addition to measuring inequality by the distribution of wealth in an economy, it is also important to characterize the underlying dynamics of the economy. For example, would a society in which each individual took turns being the richest for a day truly be unequal? In the opposite extreme, if only a few rich families dominate the economy for many generations at a time, then we might think this situation very unequal.

To characterize the *economic mobility* of the system, we define the *mixing time* τ_M for an effectively ergodic system:

$$\tau_M/t = \frac{\Omega(t)}{\Omega(0)}, \tag{2.25}$$

where

$$\Omega(t) \equiv \frac{1}{N}\sum_i^N [\overline{w}_i(t) - \overline{w}(t)]^2 \tag{2.26}$$

is the Thirumalai-Mountain metric [54], and

$$\overline{w}_i(t) = \frac{1}{t}\int_0^t w_i(t')dt' \tag{2.27}$$

and

$$\overline{w}(t) = \frac{1}{N}\sum_i^N \overline{w}_i(t) \tag{2.28}$$

are the time-averaged wealth of a single agent and the agent-averaged time average of wealth, respectively.

As each individual agent's average past wealth approaches the average past wealth of all agents, the Thirumalai-Mountain metric Ω in Eq. (2.26) approaches zero. More precisely, for an *effectively ergodic* system,

$$\Omega(t) \propto t^{-1}. \tag{2.29}$$

Therefore, the mixing time τ_M, which is only defined for an effectively ergodic system, will remain constant. The mixing time measures the time scale over which the historical wealth averages of each agent tend to equalize. Shorter mixing times indicate a higher level of economic mobility.

CHAPTER 3: Simulation Results for Network GED Model

3.1 One-Dimensional and Two-Dimensional Lattices

We find from simulations of the GED model that short-range lattices produce much more equal wealth distributions than do long-range, highly connected lattices.

Figure 3.1 shows the wealth distribution for a one-dimensional lattice with $N = 4900$ agents at different exchange ranges R. For a nearest-neighbor lattice ($R = 1$), the probability density $P(w)$ is strongly peaked – and distinctively non-Gaussian – at $w = 1$. We refer to this behavior as "hyperequality." This hyperequal distribution contrasts with the Gaussian wealth distribution of the fully connected GED model discussed in Secs. 4 and 5 and found in network topologies with low edge density. From our simulations we find that hyperequal wealth distributions arise only for sparse trade networks in which $D \lesssim \mu(1 - \lambda)$. See Sec. 5 for a derivation of this behavior.

As we increase the exchange range of the lattice to $R \geq 4$ (see Fig. 3.1), the wealth distribution for the 1D lattice behaves similarly to that of the fully connected model, and the wealth distribution is well approximated by a Gaussian for $R = 4$. As we increase R to $O(N)$, we are simulating the process of globalization because each agent trades with nearly all other agents in the system. We see that globalization results in a dramatic increase in the variance of the agent wealth distribution and thus increased inequality between agents. For large values of R, the probability density is not fit well by a Gaussian. For $R = 1024$, we see an extreme shift toward inequality and a strongly peaked agent wealth distribution at zero wealth. These characteristics match those of the fully connected model.

Figure 3.2 shows the Lorenz curve for the same wealth distributions as in Fig. 3.1.

A linear plot would represent a system in which all agents hold identical wealth, and such a line is included in Fig. 3.2 for reference. The *Gini coefficient*, a common measure of socioeconomic inequality, is proportional to the area between this diagonal line and the Lorenz curve [52]. We see that the Gini coefficient increases substantially as the system becomes more highly connected.

Similar trends hold for two-dimensional square lattices as well: increasing the exchange range dramatically increases inequality (see Fig. 3.1). In fact, given the same edge density D in each network, the wealth distributions for 1D and 2D lattices almost exactly overlap.

3.2 Random Networks

For random Erdős-Renyi and Barabasi-Albert networks, we simulate the globalization not by increasing an exchange range (because there is none), but by increasing the total number of bonds in the trade network. Again, we see that globalization leads to higher inequality. Wealth distributions across all network types studied are very similar for the same edge density D.

We have seen that an economy becomes less equal as it globalizes. Now we further consider the perspective of an individual agent: are agents with more trading partners richer than their competitors? Figure 3.3 shows the average wealth of agents of degree k in a Barabasi-Albert exchange network. In the figure "average wealth" refers to an average across all agents of degree k and an average over time. Separate simulations, each with a different instantiation of a random scale-free network, were run for each value of λ shown. We see a large negative correlation between wealth and degree as $\lambda \to 1^-$. The more trading partners an individual agent has, the less the agent's expected wealth. Similar results hold for Erdős-Renyi graphs.

3.3 Economic Mobility

In addition to the probability distribution of agent wealth, an important aspect of wealth inequality is *economic mobility*: the rate at which an individual is likely to become richer or poorer relative to other agents. We find that denser networks consistently allow for less economic mobility. This phenomenon can be quantified by the *mixing time* τ_m defined in Eq. (2.26). Shorter mixing times indicate a higher level of economic mobility.

As in the fully connected case, mixing times for the network GED model also diverge at the critical point for all network types considered. This divergence is indicative of critical slowing down. However, we find that the mixing time (at constant λ and system size N) depends on the network edge density D (see Fig. 3.4). As the number of trading partners per agent increases, mixing times increase substantially. We conclude that globalization decreases economic mobility in the GED model. This effect is a means by which globalization exacerbates wealth inequality independent of the stationary agent wealth distribution.

3.4 Mean-field Critical Behavior for Networks

The GED model's phase transition at $\lambda = 1$ is found to hold for all network types studied. At $\lambda = 1$, the GED model gives rise to many "locally rich agents" as in the network Yard Sale model of Ref. [42] (see Sec. 2.3). In fact, we will show that the GED model reduces to the original Yard Sale model at $\lambda = 1$. However, for values of $\lambda > 1$, the GED model still experiences total wealth condensation onto one agent due to the global nature of the growth and distribution rules.

With proper scaling of the rates f and μ, the GED model shows mean-field critical behavior for large, constant Ginzburg parameter G. We will derive the Ginzburg parameter in Sec. 5. Following previous work [37], we define the constants f_0 and μ_0 in order to scale f and μ by N. However, for an arbitrary network of edge density D, we must scale the rates f and μ by an additional factor of D.

$$f = f_0/(DN) \tag{3.1}$$

$$\mu = \mu_0/(DN) \tag{3.2}$$

Then from Eq. (5.30) we obtain for an arbitrary network an identical Ginzburg parameter as for the fully connected model:

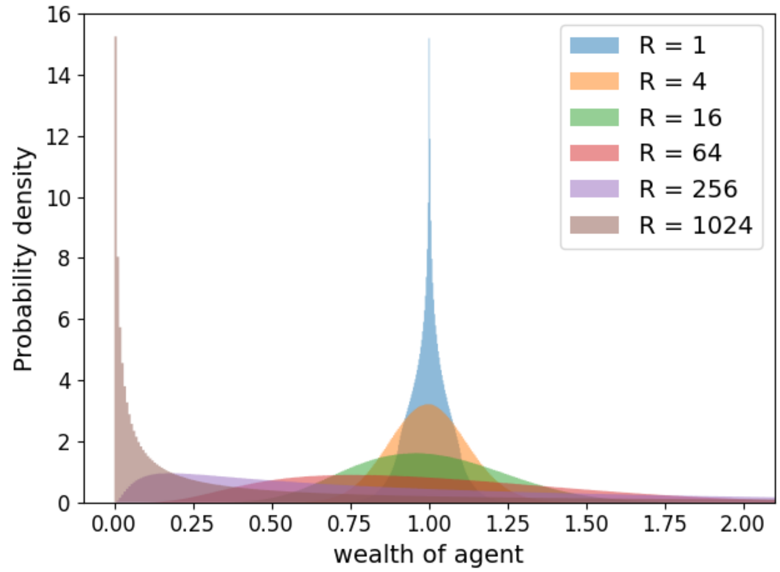
$$G = \frac{N\mu_0(1-\lambda)}{f_0^2}. \tag{3.3}$$

To see why we must scale by DN, see the mean-field theory developed in Sec 4. In this theory, time is rescaled such that, on average, each agent experiences $O(1)$ trades per infinitesimal amount of time $dt \sim 1/N$. However, for an arbitrary network, exchanges between agents occur less frequently than in the fully connected case by a factor of D [Eq. 2.20] because we have chosen DN total trades per unit time. Therefore, our infinitesimal time scale over which each agent experiences $O(1)$ trades is necessarily longer: $dt \sim 1/(DN)$. If we substitute this expression for dt into the original theory, we recover the same Ginzburg parameter for an arbitrary network as for the fully connected model. Therefore we expect the same critical behavior for the network GED model as for the fully connected model, provided G is kept constant and f and μ are scaled as in Eqs. (3.1) and (3.2).

As in the fully connected case, if we keep the number of agents N constant as we approach the critical point ($\lambda \to 1^-$), we obtain non-mean-field critical exponents that do not satisfy scaling (2.18). The energy of the system (Eq. (2.15)) is found to diverge with a critical exponent about equal to 1 which is unphysical.

However, if we keep the Ginzburg parameter constant as we approach the critical point, while scaling f and μ by DN, we obtain from simulations divergences of C and χ that are consistent with the critical exponents $\alpha = \gamma = 1$ and across all topologies studied, including 1D and 2D lattices and scale-free Barabasi-Albert networks (see Fig. 3.5). These are the same critical exponents predicted in the mean-field theory of Sec 4 and Ref. [37] for the fully connected GED model. For constant, large G, ϕ remains equal to 1 even as $\lambda \to 1$, and therefore the critical exponent $\beta = 0$. This is due to the fact that the wealth distribution remains constant for constant Ginzburg parameter in the limit of mean-field. We will derive this result from the Fokker-Planck equation in Sec 5.1. This value for β can also be seen from the fixed-point analysis of Sec 4.1 in which it is shown that, for rescaled wealth w, $w = 1$ is an attractive stationary state for all agents for all $\lambda < 1$.

Although it is unusual for a localized topology such as a nearest-neighbor 2D square lattice to follow mean-field theory, recall that the limit $G \to \infty \implies Df^2 \ll \mu(1-\lambda)$ [see Eq. (5.30)]. This limit implies that the magnitude of the growth and distribution interaction between agents is much larger than the magnitude of the exchange interaction between agents. Only the exchange interaction is affected by a change in trade network topology; the growth and distribution rules are global interactions between all agents regardless of network structure. Therefore, it is reasonable that all topologies, even if highly localized, would show mean-field behavior for large Ginzburg parameter.



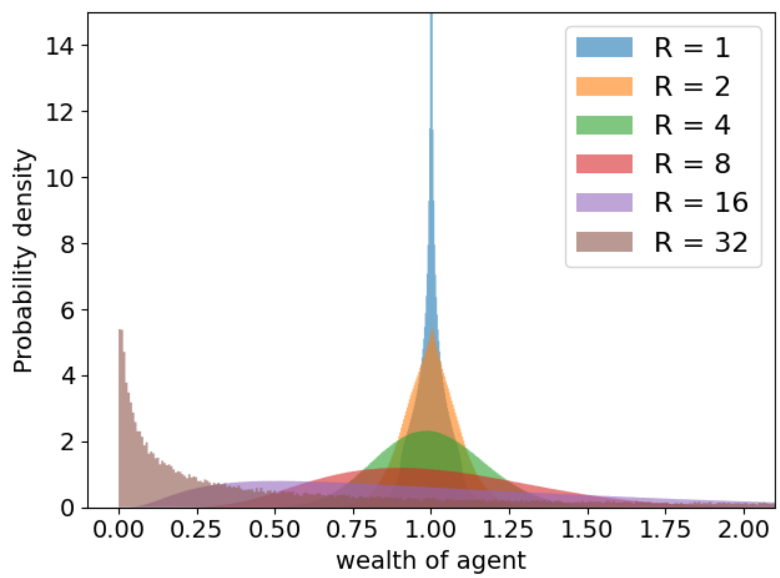


Figure 3.1: Wealth distributions for a one-dimensional lattice with exchange range R at $\lambda = 0.99$ (top) and a two-dimensional lattice at $\lambda \approx 0.97$ (bottom), both near the phase transition at $\lambda_c = 1$. For $R = 1$, the distributions are *hyperequal*, but as we increase the exchange range, we see a shift toward inequality and an increasing number of very poor agents for both lattice types.

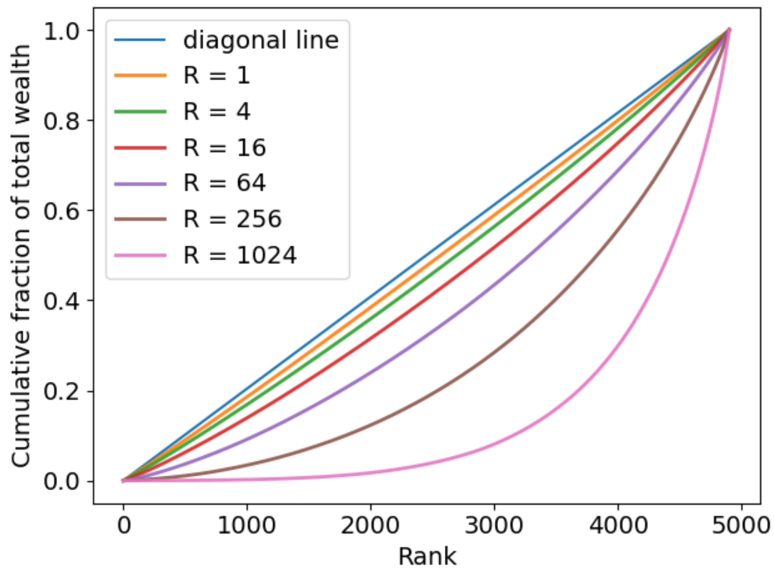


Figure 3.2: Lorenz curves for the wealth distributions of Fig. 3.1. The Gini coefficient, which is proportional to the area between the Lorenz curve and the diagonal line, increases dramatically as R increases.

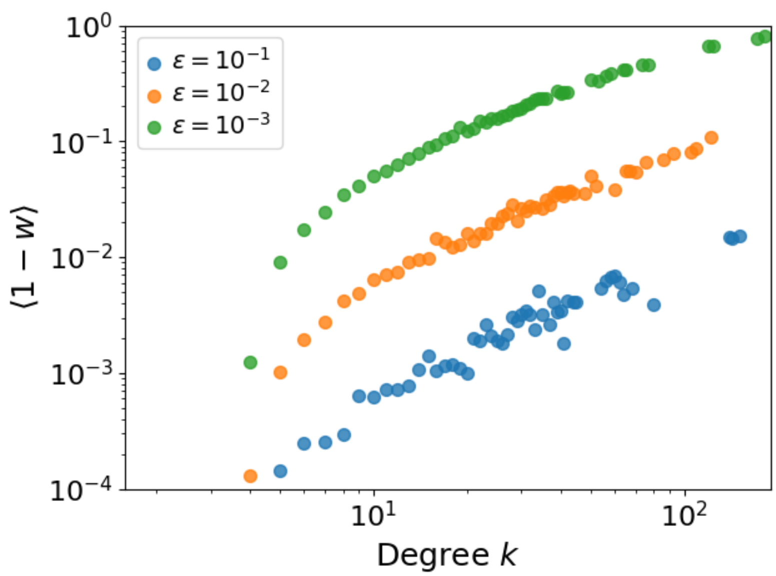


Figure 3.3: The difference between the average wealth of all agents ($\langle w \rangle = 1$) and the average wealth of all agents of degree k for Barabasi-Albert random graphs for different values of $\epsilon = 1 - \lambda$. On average, highly connected agents hold only a very small amount of wealth, especially as λ approaches $\epsilon = 0$). Values of k for which $\langle 1 - w \rangle \leq 0$ are not plotted.

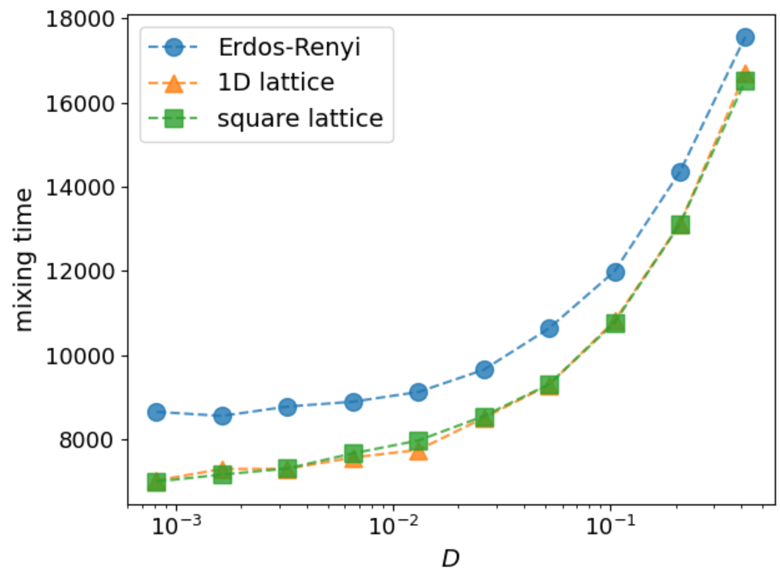


Figure 3.4: The mixing time τ_M (2.25) increases substantially as we increase the density $D = \langle k \rangle/(N-1)$ of several networks for constant $N = 4900$. Greater numbers of trading partners per agent result in longer mixing times. We see that globalized networks result in less economic mobility than do sparse networks. Results are averaged over ten simulations at each value of D for each topology at $\lambda = 1 - 10^{-2.5}$. For $\lambda = 1 - 10^{-3}$, 1D and 2D lattices are no longer effectively ergodic at $D \approx 0.4$.

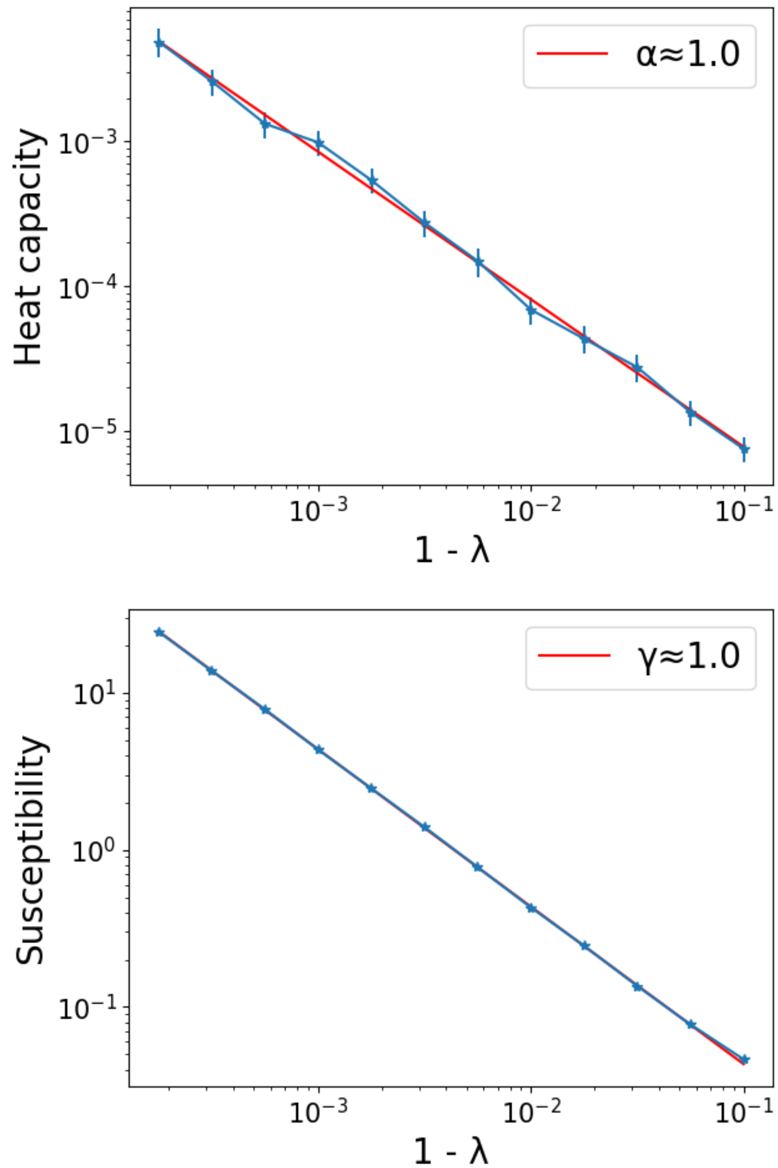


Figure 3.5: The heat capacity (top), and susceptibility per agent (bottom) for simulations of the GED model on a nearest-neighbor 2D lattice at constant Ginzburg parameter $G = 2.5 \times 10^4$ with f and μ scaled as in Eqs. (3.1) and (3.2). The measured critical exponents are consistent with mean-field theory which is unusual for a nearest-neighbor 2D lattice. Similar results are obtained for 1D lattices and Barabasi-Albert networks.

CHAPTER 4: Mean-field Theory for Fully Connected GED Model

We present a mean-field theory for the fully-connected GED model. Our discussion is based on Ref. [37]. We obtain the model's Ginzburg parameter, and show that this parameter must be held constant as $\lambda \to 1^-$ to obtain mean-field critical exponents consistent with the scaling law in Eq. (2.18). We use a method developed by Parisi and Sourlas [55] and find that the GED model's wealth distribution is a Gaussian in the limit of large Ginzburg parameter. Moreover, we can define an energy and a Hamiltonian that allows us to obtain an equilibrium (Boltzmann) description of the GED model in the limit that the number of agents $N \to \infty$. The mean-field theory results are consistent with the simulations of the fully connected GED model [44].

4.1 Exact and Mean-Field Equations

The rate of change of the wealth of agent i is given by a formally exact stochastic difference equation

$$\begin{aligned} \frac{\Delta w_i(t)}{1} &= f \sum_j \Theta\big[w_i(t) - w_j(t)\big] \eta_{ij}(t) w_j(t) \\ &\quad + f \sum_j \Big\{1 - \Theta\big[w_i(t) - w_j(t)\big]\Big\} \eta_{ij(t)} w_i(t) + \mu W(t) \frac{w_i(t)^\lambda}{S(t)}. \end{aligned} \tag{4.1}$$

The denominator on the left-hand side of Eq. (4.1) is written as 1 to emphasize that Eq. (4.1) is a difference equation rather than a differential equation. Here

$$\Theta\big(w_i - w_j\big) = \begin{cases} 1 & \big(w_i \geq w_j\big) \\ 0 & \big(w_i < w_j\big), \end{cases} \tag{4.2}$$

and

$$S(t) = \sum_i w_i^\lambda(t). \tag{4.3}$$

The parameter f is the fraction of the poorer agents's wealth that is exchanged, μ is the fraction of the total wealth that is added after N exchanges, the parameter λ determines the distribution of the added economic growth, and $\eta_{ij}(t)$ for $i \neq j$ is a time-dependent random matrix element such that

$$\eta_{ij}(t) = \begin{cases} 0 & \text{agents } i \text{ and } j \text{ do not exchange wealth} \\ 1 & \text{wealth is transferred from agent } j \text{ to agent } i \\ -1 & \text{wealth is transferred from agent } i \text{ to agent } j. \end{cases} \tag{4.4}$$

($\eta_{ij} = 0$ if $i = j$.) The matrix elements of η can be chosen from any probability distribution with the constraint that if $\eta_{ij} = \pm 1$, then $\eta_{ji} = \mp 1$. This condition imposes the constraint that the exchange conserves the total wealth.

To obtain a differential equation we multiply and divide the denominator on the left-hand side of Eq. (4.1) by N, the number of agents. Because we will take the limit $N \to \infty$ and take one time unit to correspond to N^2 exchanges, we have that $1/N \to dt$. Note that in the simulations of Ref. [44], N exchanges was chosen as the unit of time. In this case each agent will, on the average, exchange wealth with only one other agent and hence one exchange described by the difference equation would not take place in an infinitesimal amount of time. One exchange per agent does take place in an infinitesimal time if one time unit corresponds to N^2 exchanges during which each agent exchanges wealth with every other agent on the average.

The parameters f and μ in Eq. (4.1) are the rates of exchange and growth, respectively, and are defined per N exchanges (to be consistent with the simulations).

Hence to obtain a consistent differential equation, these rates need to be scaled by N. We let

$$f = f_0/N \text{ and } \mu = \mu_0/N, \tag{4.5}$$

and assume that f_0 and μ_0 are independent of N. Note that these theoretical considerations imply that the parameters f and μ in the simulations must be scaled with N if the Ginzburg parameter is held fixed.

Because wealth is added to the system after every N exchanges, the total wealth grows exponentially by Eq. (2.12).

With these considerations Eq. (4.1) becomes

$$\begin{aligned}\frac{dw_i(t)}{dt} &= f_0 \sum_j \Theta\big[w_i(t) - w_j(t)\big]\eta_{ij}(t)w_j(t) \\ &\quad + f_0 \sum_j \Big\{1 - \Theta\big[w_i(t) - w_j(t)\big]\Big\}\eta_{ij}(t)w_i(t) + \mu_0 W(t)\frac{w_i(t)^\lambda}{S(t)}. \end{aligned} \tag{4.6}$$

To obtain a mean-field theory, we choose an agent whose wealth is $w(t)$ and let $w_{\mathrm{mf}}(t)$ be the mean wealth of the remaining agents. That is,

$$w_{\mathrm{mf}}(t) = \frac{W(t) - w(t)}{N - 1}. \tag{4.7}$$

The mean-field version of Eq. (4.6) is

$$\begin{aligned}\frac{dw(t)}{dt} &= f_0\Theta\big[w(t) - w_{\mathrm{mf}}(t)\big]\eta w_{\mathrm{mf}}(t) + f_0\Big[1 - \Theta\big[w(t) - w_{\mathrm{mf}}(t)\big]\Big]\eta w(t) \\ &\quad + \mu_0 W(t)\frac{w(t)^\lambda}{S(t)}. \end{aligned} \tag{4.8}$$

The quantity $S(t)$ defined in Eq. (4.3) becomes

$$S(t) = w^{\lambda}(t) + (N-1)w_{\rm mf}^{\lambda}(t). \tag{4.9}$$

To obtain a mean-field description we have effectively coarse grained the exchanges between the chosen agent and the remaining $N-1$ agents in time, which implies a coarse graining of the noise associated with the coin flips that determine the exchange of wealth. By using the central limit theorem, we can take the noise in Eq. (4.8) to be random Gaussian. This assumption would not be valid if the chosen agent interacted with only one other agent in one unit of time. However, because the unit of time corresponds to N^2 exchanges, the chosen agent interacts with $N-1$ other agents and coarse graining in time makes sense. The coarse graining of the noise is another reason why it is necessary to choose N^2 exchanges to be one unit of time in the mean-field theory.

It will be convenient to write the growth term in Eq. (4.8) as

$$\mu_0 W(t)\frac{w^{\lambda}(t)}{S(t)} = \mu_0 W(t)\frac{[w(t)/W(t)]^{\lambda}}{[w(t)/W(t)]^{\lambda} + (N-1)^{1-\lambda}[1 - w(t)/W(t)]^{\lambda}}, \tag{4.10}$$

where we have used Eqs. (4.7) and (4.9) and divided the numerator and denominator by $W^{\lambda}(t)$.

To simplify Eq. (4.8), we first assume that $w(t) < w_{\rm mf}(t)$; that is, the wealth of the chosen agent is less than the mean wealth of the remaining $N-1$ agents. We use Eqs. (4.6) and (4.10) to obtain

$$\frac{dw(t)}{dt} = f_0\eta(t)w(t) + \mu_0 W(t)\frac{[w(t)/W(t)]^{\lambda}}{[w(t)/W(t)]^{\lambda} + (N-1)^{1-\lambda}[1 - w(t)/W(t)]^{\lambda}}. \tag{4.11}$$

We divide both sides of Eq. (4.11) by W and rewrite Eq. (4.11) as

$$\frac{d}{dt}\Big(\frac{w(t)}{W(t)}\Big) = f_0\eta(t)\frac{w(t)}{W(t)} + \mu_0\frac{[w(t)/W(t)]^\lambda}{[w(t)/W(t)]^\lambda + (N-1)^{1-\lambda}[1-w(t)/W(t)]^\lambda} - \mu_0\frac{w(t)}{W(t)}, \tag{4.12}$$

where we have used the relation [see Eq. (5.30)]

$$\frac{1}{W(t)}\frac{dw(t)}{dt} = \frac{d}{dt}\Big(\frac{w(t)}{W(t)}\Big) + \mu_0\frac{w(t)}{W(t)}. \tag{4.13}$$

We next introduce the scaled wealth fraction

$$x(t) \equiv \frac{w(t)}{W(t)}, \tag{4.14}$$

and rewrite Eq. (4.12) as

$$\frac{dx(t)}{dt} = R(x,\eta,t) \tag{4.15a}$$

with

$$R(x,\eta,t) \equiv f_0\eta(t)x(t) + \mu_0\frac{x(t)^\lambda}{x(t)^\lambda + (N-1)^{1-\lambda}[1-x(t)]^\lambda} - \mu_0 x(t). \tag{4.15b}$$

Equation (4.15) expresses the time-dependence of the wealth of the chosen agent in contact with an agent representing the mean wealth of the remaining agents. Hence, the wealth of the chosen agent is not conserved.

For $\mu_0 = 0$, the total wealth W is a constant because the noise $\eta(t)$ that determines the wealth transfer from the mean-field wealth to the chosen agent is the negative of the noise that governs the wealth transfer from the chosen agent to the mean field.

It is easy to show that for zero noise, $R(x,\eta = 0,t) = 0$ for $x = 0$, 1, and $1/N$, and that these are the only fixed points of Eq. (4.15) for $\lambda \neq 1$. To determine the stability of the fixed points, we calculate the derivative $dR(x, \eta = 0, t)/dx$ and obtain

$$\frac{dR(x,0,t)}{dx} = \mu_0 \frac{\lambda x^{\lambda-1}}{x^\lambda + (N-1)^{1-\lambda}(1-x)^\lambda} - \mu_0 \frac{x^\lambda[\lambda x^{\lambda-1} - \lambda(N-1)^{1-\lambda}(1-x)^{\lambda-1}]}{[x^\lambda + (N-1)^{1-\lambda}(1-x)^\lambda]^2} - \mu_0, \tag{4.16}$$

where $x \equiv x(t)$. For $\lambda < 1$, the derivatives at $x = 0$ and $x = 1$ are equal to ∞, which implies that these fixed points are unstable. The derivative at $x = 1/N$ is equal to $\mu_0(\lambda - 1)$, and hence the fixed point at $x = 1/N$ is stable for $\lambda < 1$. For $\lambda > 1$ the derivative at $x = 1/N$ is positive so that this fixed point is unstable. The derivative at $x = 0$ and $x = 1$ equals -1, and hence these fixed points are stable.

We next return to Eq. (4.6) and consider the case for which $w(t) > w_{\rm mf}(t)$. The growth term is the same as before. The exchange term in Eq. (4.1), $f_0 \sum_j \Theta\big(w_i - w_j\big)\eta_{ij} w_j$, becomes $f_0 \eta w_{\rm mf}$. We use Eq. (4.7) to write

$$\frac{dw(t)}{dt} = f_0 \eta(t) \frac{W(t) - w(t)}{N-1} + \mu_0 W(t) \frac{[w(t)/W(t)]^\lambda}{[w(t)/W(t)]^\lambda + (N-1)^{1-\lambda}[1 - w(t)/W(t)]^\lambda}. \tag{4.17}$$

From Eq. (4.13) and the definition of $x(t)$ in Eq. (4.14) we have

$$\frac{dx(t)}{dt} = f_0 \eta(t) \frac{1 - x(t)}{N-1} + \mu_0 \frac{x(t)^\lambda}{x(t)^\lambda + (N-1)^{1-\lambda}[1 - x(t)]^\lambda} - \mu_0 x(t). \tag{4.18}$$

Equation (4.15) for the poorer agent and Eq. (4.18) for the richer agent are the same except for the noise term, and hence the fixed points are the same. We again use Eq. (4.7) to rewrite Eq. (4.18) as

$$\frac{dx(t)}{dt} = f_0 \eta(t) x_{\rm mf}(t) + \mu_0 \frac{x(t)^\lambda}{x(t)^\lambda + (N-1)^{1-\lambda}[1 - x(t)]^\lambda} - \mu_0 x(t), \tag{4.19}$$

where $x_{\rm mf}(t) = w_{\rm mf}(t)/W(t)$ is the fraction of the mean field agent's rescaled wealth.

Note that $x(t)$ is of order $1/N$ as is $x_{\rm mf}(t)$. Equation (4.19) will be used in Sec. 4.2 to discuss the phase transition and the critical exponents.

In summary, the fixed points for all values of λ are $x = 0$, 1, and $1/N$ for the mean-field equations describing the wealth evolution of either the richer or poorer agent. For $\lambda < 1$, the fixed points at 0 and 1 are unstable, and the fixed point at $1/N$ is stable, corresponding to all agents having an equal share of the total wealth on average. For $\lambda > 1$, the fixed points at 0 and 1 are stable, and the fixed point at $x = 1/N$ is unstable, which implies that if all the agents are assigned an equal amount of wealth at $t = 0$, one agent will eventually accumulate all the wealth in a simulation of the model. Note that if we use the equation for which the chosen agent is richer than the "mean field" agent, then the stable fixed point reached when $\lambda > 1$ is $x = 1$; similarly, if we chose the equation for which the chosen agent is poorer than the mean field agent, the stable fixed point reached for $\lambda > 1$ is $x = 0$.

4.2 The Phase Transition

To analyze the phase transition at $\lambda = 1$, we investigate Eq. (4.19), the mean-field differential equation for the richer agent, for $x \sim 1/N$ and λ close to 1^-. We let

$$x(t) = \frac{1}{N} - \delta(t), \tag{4.20}$$

assume $N\delta \ll 1$, and expand the second term on the right-hand side of Eq. (4.15b) to first order in $N\delta$. After some straightforward algebra we find that

$$\frac{d\delta(t)}{dt} = f_0\eta(t)x_{\rm mf}(t) - \mu_0(1-\lambda)\delta(t), \tag{4.21}$$

We multiply both sides of Eq. (4.21) by N to obtain

$$\frac{dN\delta(t)}{dt} = f_0\eta(t)\tilde{w}_{\rm mf} - \mu_0(1-\lambda)N\delta(t), \tag{4.22}$$

where $\tilde{w}_{\rm mf} = Nx_{\rm mf}$. We write $\tilde{w}_{\rm mf} = 1 + N\delta$, let

$$\phi = N\delta, \tag{4.23}$$

and rewrite Eq. (4.22) as

$$\frac{d\phi(t)}{dt} = f_0\eta(t)\big[1-\phi(t)\big] - \mu_0(1-\lambda)\phi(t). \tag{4.24}$$

Because $\phi(t) \sim 1/\sqrt{N} \ll 1$ for $N \gg 1$, we can ignore $\phi(t)$ compared to one in Eq. (4.24) and obtain

$$\frac{d\phi(t)}{dt} = f_0\eta(t) - (1-\lambda)\mu_0\phi(t). \tag{4.25}$$

We note that the multiplicative noise term vanishes if the limit $N \to \infty$ is taken before the critical point is approached, that is, if the mean-field limit is taken before $\lambda \to 1$. However, for finite N the situation is more subtle.

The starting point for the derivation of Eq. (4.25) was Eq. (4.19), the mean-field equation for the richer chosen agent. If the chosen agent is poorer than the average of the other agents, similar arguments lead to the same equation as Eq. (4.25).

The form of Eq. (4.25) is identical to the linearized version of the Landau-Ginzburg equation [56–58] with ϕ as the fluctuating part of the order parameter. Hence, $\lambda = 1$ corresponds to a phase transition as was found in simulations of the GED model [44]. As for the usual Landau-Ginzburg equation, the factor of $(1-\lambda)$

sets the time scale for $\mu_0 \neq 0$. That is, as $\lambda \to 1^-$, there is critical slowing down, and the system decorrelates on the time scale

$$\tau \sim \frac{1}{\mu_0(1-\lambda)}. \tag{4.26}$$

Because the stable fixed point of the poorer agent is zero for $\lambda > 1$ [see Eq. (4.15)] and is one for the richer agent [see Eq. (4.18)], the order parameter is constant for all $\lambda > 1$. Hence, the exponent β, which characterizes the way the order parameter approaches its value at the transition, is equal to zero.

As mentioned, we can assume the noise $\eta(t)$ to be associated with a random Gaussian distribution of coin flips. Note that η is the average over N coin flips and hence should scale as $\sqrt{N}/N \sim 1/\sqrt{N}$. Hence $\eta(t)$ in Eq. (4.29) is order $1/\sqrt{N}$, which implies that $\phi(t) \sim 1/\sqrt{N}$ and justifies our neglect of terms higher than first order. Simulations in Ref. [44] show that the fluctuations are dominated by those near the $1/N$ fixed point.

To obtain the critical exponent γ, we adopt an approach introduced by Parisi and Sourlas [55] and note that the measure of a random Gaussian noise is given by [55]

$$P(\{\eta_j\}) = \frac{\exp\left[\int_{-\infty}^{\infty} -\beta \sum_j \eta_j^2(t)\, dt\right]}{\int \prod_j \delta\eta_j \exp\left[\int_{-\infty}^{\infty} -\beta \sum_j \eta_j^2(t) dt\right]}. \tag{4.27}$$

or

$$P(\eta) = \frac{\exp\left[\int_{-\infty}^{\infty} -\beta N \eta^2(t)\, dt\right]}{\int \delta\eta \exp\left[\int_{-\infty}^{\infty} -\beta N \eta^2(t) dt\right]}. \tag{4.28}$$

The factor of N in the argument of the exponential in Eq. (4.28) comes from the fact that $\eta_j(t) = \eta(t)$ for all j in the mean-field approach. This factor of N is consistent with the argument that $\eta(t) \sim 1/\sqrt{N}$. (In Ref. [59] the factor of N is not explicit, but is implicit in the integral over all space.)

We rewrite Eq. (4.25) as

$$\frac{1}{f_0}\frac{d\phi(t)}{dt} + \frac{(1-\lambda)}{f_0}\mu_0\phi(t) = \eta(t), \tag{4.29}$$

and replace $\eta(t)$ in Eq. (4.28) by the left-hand side of Eq. (4.29). This replacement requires a Jacobian, but in this mean-field case the Jacobian is unity [57]. Hence, the probability of ϕ is given by

$$P(\phi) = \frac{\exp\Big\{-\beta N\int_{-\infty}^{\infty}\Big[\frac{1}{f_0}\frac{d\phi(t)}{dt} + \frac{\mu_0(1-\lambda)}{f_0}\phi(t)\Big]^2 dt\Big\}}{\int\delta\phi(t)\exp\Big\{-\beta N\int_{-\infty}^{\infty}\Big[\frac{1}{f_0}\frac{d\phi(t)}{dt} + \frac{\mu_0(1-\lambda)}{f_0}\phi(t)\Big]^2 dt\Big\}}. \tag{4.30}$$

We now assume that the system is in a steady state so that $d\phi(t)/dt = 0$ over a time scale of the order of $1/\mu_0(1-\lambda)$. Hence, the average $\langle\phi^2\rangle$ is given by

$$\langle\phi^2\rangle = \frac{\int\delta\phi\,\phi^2\exp\Big\{-\beta N\int_{-\infty}^{\infty}dt\Big[\frac{\mu_0(1-\lambda)}{f_0}\phi\Big]^2\Big\}}{\int\delta\phi\,\exp\Big\{-\beta N\int_{-\infty}^{\infty}dt\Big[\frac{\mu_0(1-\lambda)}{f_0}\phi\Big]^2\Big\}} \tag{4.31}$$

$$= \frac{\int\delta\phi\,\phi^2\exp\Big[-\beta N\frac{\mu_0(1-\lambda)}{f_0^2}\phi^2\Big]}{\int\delta\phi\,\exp\Big[-\beta N\frac{\mu_0(1-\lambda)}{f_0^2}\phi^2\Big]}, \tag{4.32}$$

where the range of integration over time is limited to the interval $1/\mu_0(1-\lambda)$.

Because we have assumed a steady state, the functional integral becomes a standard integral over ϕ. We can take the limits of the integrals to be $\pm\infty$ because the factor of $N \gg 1$ in the exponential keeps ϕ of order $1/\sqrt{N}$. Hence, Eq. (4.32) now becomes

$$\langle\phi^2\rangle = \frac{\int_{-\infty}^{\infty} d\phi\,\phi^2 \exp\Big\{-\beta N\dfrac{\mu_0(1-\lambda)}{f_0^2}\phi^2\Big\}}{\int_{-\infty}^{\infty} d\phi\,\exp\Big\{-\beta N\dfrac{\mu_0(1-\lambda)}{f_0^2}\phi^2\Big\}}. \tag{4.33}$$

By using simple scaling arguments we see that the second moment of the probability distribution diverges as

$$\langle\phi^2\rangle \sim \frac{f_0^2}{N\mu_0(1-\lambda)}. \tag{4.34}$$

The fluctuating part of the order parameter $\phi = N\delta$ is analogous to the fluctuating part of the order parameter $m = M/N$ of the fully connected Ising model, where M is the total magnetization of the system and N is the number of spins. To determine the susceptibility (per spin) of the Ising model, we need to multiply $[\langle m^2\rangle - \langle m\rangle^2]$ by N. Because $\langle\phi^2\rangle = f_0^2[N\mu_0(1-\lambda)]^{-1}$ [see Eq. (4.34)], the susceptibility (per agent) of the GED model is given by

$$\chi \sim \frac{f_0^2}{\mu_0(1-\lambda)}. \tag{4.35}$$

We conclude that the susceptibility diverges near the phase transition with the exponent $\gamma = 1$.

Note that we can relate the variance of ϕ to the variance of the rescaled wealth. From the definition of $\delta(t)$ in Eq. (4.20) and the fact that $x(t) = w(t)/W(t)$ is the rescaled wealth [see Eq. (4.14)], we have

$$\phi(t) = 1 - Nx(t) = 1 - N\frac{w(t)}{W(t)} = 1 - N\tilde{w}(t). \tag{4.36}$$

We rescale the total wealth and hence the wealth of each agent so that $W(t) = N$ after the increased wealth due to economic growth has been assigned. Hence $\tilde{w}$ in

Eq. (4.36) is the rescaled wealth of a single agent. Equation (4.36) will be useful in Sec. 4.4 where we compare the predictions of the theory to the results of the simulations in Ref. [44].

4.3 The Energy and Specific Heat Exponents

From Eq. (4.30) we have that

$$P(\phi) = \frac{\exp\Big\{-\beta N\mu_0 \dfrac{(1-\lambda)}{f_0^2}\phi^2\Big\}}{\int d\phi \exp\Big\{-\beta N\mu_0 \dfrac{(1-\lambda)}{f_0^2}\phi^2\Big\}}, \tag{4.37}$$

assuming that the system is in a steady state. From the expression of the action or Hamiltonian in Eq. (4.37), where ϕ^2 is multiplied by $\beta N\mu_0(1-\lambda)/f_0^2$, we see that the Ginzburg parameter for the GED model is given by (up to numerical factors)

$$G = \frac{N\mu_0(1-\lambda)}{f_0^2}. \tag{4.38}$$

The inverse temperature β (not to be confused with the order parameter critical exponent), which arises from the amplitude of the Gaussian noise, will be absorbed in the parameter f_0. The association of β with f_0 is consistent with Eq. (4.29) in that we are relating the temperature to the amplitude of the noise and indicates that increasing the fraction of the poorer agent's wealth transferred in an exchange is equivalent to increasing the amplitude of the noise.

The total energy for the GED model can be seen from the form of the action or the Hamiltonian in Eq. (4.37)

$$E = N\phi^2, \tag{4.39}$$

in analogy with the Landau-Ginzburg-Wilson free field or Gaussian action for the fully connected Ising model [60]. Equations (4.36) and (4.39) imply that the total energy of a system of N agents is given by

$$E = \sum_{i=1}^{N} (1 - \tilde{w}_i)^2 \tag{4.40a}$$

$$= -N + \sum_{i=1}^{N} \tilde{w}_i^2, \tag{4.40b}$$

where we have used that fact that $\sum_i \tilde{w}_i = N$.

The existence of a quantity that can be interpreted as an energy implies that the probability density of the energy is given by the Boltzmann distribution for $\lambda < 1$. The latter is consistent with simulations of the GED model [44]. The existence of the Boltzmann distribution also implies that the system is in thermodynamic equilibrium and is not just in a steady state for $\lambda < 1$.

From Eq. (4.37) we find that $\langle \phi^2 \rangle \sim f_0^2/[N\mu_0(1-\lambda)]$. Hence, we conclude from Eq. (4.39) that the mean energy per agent of the GED model scales as

$$\frac{\langle E \rangle}{N} \sim \frac{f_0^2}{N\mu_0(1-\lambda)}. \tag{4.41}$$

Equation (4.41) suggests that the mean energy per agent diverges as $(1-\lambda)^{-1}$ as $\lambda \to 1$ for fixed N, which is not physical. However, if we hold the Ginzburg parameter G constant as $\lambda \to 1$, we find no divergence (the exponent is zero), which removes the apparent nonphysical behavior. That is,

$$\frac{\langle E \rangle}{N} \sim \begin{cases} (1-\lambda)^{-1} & \text{(fixed } N) \\ G^{-1} & \text{(constant } G). \end{cases} \tag{4.42}$$

Equation (4.42) implies that the energy per agent is finite as we approach the critical point only if we hold G constant.

Near the critical point the nonanalytic behavior of the mean energy per agent can be expressed as $(1-\lambda)^{1-\alpha}$, where α is the specific heat exponent. Equation (4.42) for $\langle E\rangle/N$ for constant Ginzburg parameter implies that $\alpha = 1$. This result for α is what we would find if we require that β, γ, and α to satisfy the scaling relation in Eq. (2.18) with $\beta = 0$ and $\gamma = 1$.

We can also calculate α directly using the probability distribution in Eq. (4.37). To calculate the fluctuations in the total energy, we need to calculate the average of ϕ^4. If we apply the probability in Eq. (4.37), we find that the fluctuations in the energy per agent, and hence the specific heat is proportional as $Nf^4[\mu_0(1-\lambda)]^{-2}$, where we have multiplied by N as we did for the susceptibility per spin of the fully connected Ising model. Hence, the specific heat c scales as

$$c \sim \frac{f_0^4}{N\mu_0^2(1-\lambda)^2}, \tag{4.43}$$

and

$$c \sim \begin{cases} (1-\lambda)^{-2} & \text{(fixed } N\text{)} \\ (1-\lambda)^{-1} & \text{(constant } G\text{).} \end{cases} \tag{4.44}$$

We see that if we keep the Ginzburg parameter constant, we find $c \sim f_0^2/[G\mu_0(1-\lambda)]$ and hence $\alpha = 1$. Note that if we do not keep G constant, we would find $\alpha = 2$, which does not satisfy Eq. (2.18). As a consistency check, we can use Eqs. (4.41) and (4.43) to construct the Ginzburg parameter by comparing the fluctuations of the energy, that is, the heat capacity, to the mean energy:

$$\frac{Nc}{\langle E\rangle^2} \propto \frac{f_0^2}{N\mu_0(1-\lambda)} = G^{-1}. \tag{4.45}$$

4.4 Comparison with Simulations

The mean-field theory predictions for the exponents $\alpha = 1$, $\beta = 0$, and $\gamma = 1$ are consistent with the simulation results reported in Ref. [44] for fixed G. As discussed in Sec. 4.2, mean-field theory also predicts that there is only one time scale near the phase transition and that the time scale diverges as $(1-\lambda)^{-1}$ for fixed Ginzburg parameter, an example of critical slowing down [see Eq. (4.26)]. This prediction is consistent with the simulation results for the mixing time associated with the wealth metric [44] and the energy decorrelation time, which were both found to diverge as $(1-\lambda)^{-2}$ for fixed G. The apparent discrepancy between the $(1-\lambda)^{-2}$ divergence found in the simulations and the $(1-\lambda)^{-1}$ divergence predicted by Eq. (4.26) is due to the difference in the choice of the unit of time in the simulation (N exchanges) and the theory (N^2 exchanges). To account for the difference in time units, we need to divide the simulation result for the mixing time and energy decorrelation time by N with the result that $N^{-1}(1-\lambda)^{-2} \sim (1-\lambda)(1-\lambda)^{-2} = (1-\lambda)^{-1}$, where we have used the relation $N \propto (1-\lambda)^{-1}$ for fixed G [see Eq. (4.38)].

The simulations for fixed G indicate that the energy per agent approaches a constant as $(1-\lambda) \to 0$. This behavior is associated with the nonanalytic part of the energy per agent. This result for the λ-independence of the nonanalytic part of the energy per agent is inconsistent with the relation between the energy per agent and the specific heat, $c \propto \partial\langle E(\lambda)\rangle/\partial\lambda$. The $(1-\lambda)^{-1}$ dependence of the specific heat for fixed G near $\lambda = 1$ suggests that the mean energy per agent could include a logarithmic dependence on λ. For example, the form, $\langle E\rangle/N \sim a_0 + a_L/\log(1-\lambda)$, where a_0 and a_L are independent of λ, implies that the specific heat scales as $c \sim [\log(1-\lambda)]^{-2}(1-\lambda)^{-1}$, thus yielding $\alpha = 1$ with logarithmic corrections, which standard mean-field theory cannot predict and are very difficult to detect in simulations.

There is also agreement between the exponents predicted by mean-field theory and those determined in the simulations when the measurements are done at fixed N. From Eq. (4.41) we see that if N is held constant, the mean energy per agent is predicted to diverge as $(1 - \lambda)^{-1}$, which is consistent with the simulations [44] although this divergence is unphysical because it implies that the mean energy per agent would become infinite. The exponent α is predicted to be two for fixed N, which is also in agreement with the simulations [44].

CHAPTER 5: Fokker-Planck Theory for Network GED Model

5.1 Derivation of Wealth Distributions

Unlike for the fully connected GED model, a non-Gaussian, highly equal stationary wealth distribution can arise from the network GED model (see Figs. 3.1 and 5.3). This non-Gaussian probability density is strongly peaked at ($w = 1$) and only appears for very sparse trade networks. We refer to this behavior of the GED model as *hyperequality.* In this section, we derive the existence of hyperequality for sparse networks with $D \lesssim \mu(1-\lambda)$, and show that the wealth distribution approaches a Gaussian for $D \gg \mu(1-\lambda)$ and large Ginzburg parameter.

We develop a Fokker-Planck formalism for the GED model on an arbitrary exchange network based on the work of Boghosian *et al.* on the Yard-Sale model [40, 41]. We start by considering Δ, the change in an agent's wealth w at time t. We can model Δ as a random variable whose probability distribution depends only on the agent's current wealth w.

The probability density functions for stochastic, memory-less, time-invariant quantities such as Δ are well known to follow the Fokker-Planck equation as they evolve over time [61]. For the GED model, the Fokker-Planck equation becomes

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial w}[\langle \Delta \rangle P] + \frac{\partial^2}{\partial w^2}\left[\frac{\langle \Delta^2 \rangle}{2} P\right], \tag{5.1}$$

where $P = P(w,t)$ is the number density of agents with wealth w, and Δ indicates the change of w over one unit of time (defined such that there are DN exchanges per unit time). The single derivative term of Eq. (5.1) is the drift term, and the double derivative term is the diffusive term.

To derive the wealth distribution for the GED model, we must first evaluate $\langle \Delta \rangle$

and $\langle \Delta^2 \rangle$. If an agent of wealth w exchanges wealth with an agent of wealth w', then

$$w \to w + rf \min(w, w') \tag{5.2}$$

where f is the exchange fraction, and r is a random number drawn from $\{-1, 1\}$ with equal probability. The average change in wealth due to exchange is found by averaging over r. Therefore the exchange term contributes a net-zero drift in agent wealth.

In contrast to the Yard-Sale model, the GED model contains mechanisms beyond exchange and therefore has a generally nonzero drift term. We define K and L such that

$$\langle \Delta \rangle = K - L, \tag{5.3}$$

where K is the drift in wealth due to the growth and distribution rule introduced by Kang *et al.* [33, 44], and L represents an effective loss of wealth due to rescaling. (We rescale each agent's wealth after each simulation step.)

From the GED model's growth and distribution rules we have

$$K(w) = \mu N \frac{w^\lambda}{\int_0^\infty w'^\lambda P(w') dw'}, \tag{5.4}$$

and

$$L(w) = \left| \frac{wN}{W(t)} - w \right|. \tag{5.5}$$

$W(t)$ is the total unnormalized wealth at time t. Due to the nature of the growth rule, $W(t)$ grows exponentially as in Eq. (2.12), so from Eq. (5.5), we have

$$L(w) = \left| \frac{w}{e^{\mu t}} - w \right| \approx \mu w, \tag{5.6}$$

where we have assumed μ is sufficiently small for one time step.

To solve for stationary $P(w)$, we must also express $\langle\Delta^2\rangle$ in terms of $P(w)$. Unlike for the drift term ($\langle\Delta\rangle$), for the diffusive term we cannot neglect the effects of exchange and the exchange network topology. If two agents of wealth w and w' exchange wealth, then by the GED model's exchange rule,

$$\langle\Delta^2\rangle_X = f^2 w^2 \theta(w' - w) + f^2 w'^2 \theta(w - w'), \tag{5.7}$$

where the subscript X denotes the change due to exchange only, and θ is the Heaviside step function. For a total of DN exchanges per unit time, each agent will be selected on average $2D$ times. Then, for one time unit,

$$\mathrm{Var}(\Delta) = 2\frac{D}{N}\int_0^\infty \tilde{P}(w'|w)\langle\Delta^2\rangle_X dw' \tag{5.8}$$

gives the variance of an agent's change in wealth. The variance of Δ is due only to the exchange mechanism of the GED model; the growth and distribution rules do not contribute to the variance because they are not stochastic. (However, the growth, distribution, and rescaling of wealth do contribute to $\langle\Delta^2\rangle$ via Eq. (5.13).) In Eq. (5.8), $\tilde{P}(w'|w)$ is the density (normalized to N) of the neighboring agents with wealth w' given that the chosen agent has wealth w.

Previous work [38–40] has assumed $\tilde{P}(w'|w) = P(w')$, so that the joint probability distribution $P_2(w, w') = P(w)P(w')$ and there is no correlation between the wealth of trading partners. We also use this approximation in the present work. Although this approximation is justified for fully connected systems in the thermodynamic limit, it is not usually warranted when each agent might have only a small number of trading partners. However, we will only use this approximation in the limit that $P(w)$ is sharply peaked at $w = 1$ and near zero everywhere else. As $P(w) \to N\delta(1-w)$, $\tilde{P}(w'|w)$ must also approach $N\delta(1-w')$ independently of w. Although substituting $P(w')$ for $\tilde{P}(w'|w)$ neglects the effect of inter-agent correlations, this approximation becomes better as we increase equality among agents. Under this approximation, we find that

$$\mathrm{Var}(\Delta) = 2\frac{D}{N}\int_0^\infty P(w')\langle\Delta^2\rangle_X dw' = 2D\langle\Delta^2\rangle_Y \tag{5.9}$$

where $\langle\Delta^2\rangle_Y$ denotes the averaged squared change in wealth in the fully connected Yard-Sale model. We will test this approximation by comparing with our simulation results.

In his analysis of the Yard-Sale model [40], Boghosian finds

$$\langle\Delta^2\rangle_Y = 2f^2\left(\frac{w^2}{2}A(w) + B(w)\right), \tag{5.10}$$

where A and B are given by

$$A(w) = \frac{1}{N}\int_w^\infty P(w')dw' \tag{5.11}$$

and

$$B(w) = \frac{1}{N}\int_0^w \frac{w'^2}{2}P(w')dw'. \tag{5.12}$$

Equation (5.9) together with Eqs. (5.10)–(5.12) give an expression for the variance of the change in wealth of an agent. To obtain $\langle\Delta^2\rangle$, we use

$$\mathrm{Var}(\Delta) = \langle\Delta^2\rangle - \langle\Delta\rangle^2 \tag{5.13}$$

and Eq. (5.3) to obtain

$$\langle \Delta^2 \rangle = 2D \langle \Delta^2 \rangle_Y + (K - L)^2, \tag{5.14}$$

with $\langle \Delta^2 \rangle_Y$, K, and L given by Eqs. (5.10), (5.4), and (5.5) respectively.

Equations (5.1), (5.3), and (5.14) represent the Fokker-Planck description of the GED model. Note from Eqs. (5.5) and (5.17) that for $\lambda = 1$, $K = L$, and therefore $\langle \Delta \rangle = 0$ and $\langle \Delta^2 \rangle = 2D \langle \Delta^2 \rangle_Y$. The dynamics of the GED model are thus equivalent to those of the Yard-Sale model (apart from a trivial rescaling of time by a factor of $2D$) when λ is set to 1.

Because we are interested in stationary states, we set $\partial P/\partial t = 0$, rearrange the Fokker-Planck equation [Eq. (5.1)], and cancel the derivative on both sides to arrive at

$$\langle \Delta \rangle P = \frac{\partial}{\partial w} \left[\frac{\langle \Delta^2 \rangle}{2} P \right]. \tag{5.15}$$

To solve for P, we must evaluate K in Eq. (5.4). For $\lambda = 0$ and $\lambda = 1$, the integral in Eq. (5.4) can be evaluated analytically:

$$K(w)|_{\lambda=0} = \mu \tag{5.16}$$

$$K(w)|_{\lambda=1} = \mu w. \tag{5.17}$$

To evaluate $K(w)$ for general λ, we replace $P(w')$ by $P_{\rm mf}(w')$:

$$P_{\rm mf}(w') = \delta(w' - w) + (N - 1)\delta \left(w' - \frac{N - w}{N - 1} \right). \tag{5.18}$$

In other words, we assume that all agents other than the one experiencing growth have wealth $(N - w)/(N - 1)$. This quantity is the mean value of their wealth given the total wealth is N. The growth term then becomes

$$K(w) = \mu N \frac{w^{\lambda}}{w^{\lambda} + \frac{N-1}{(N-1)^{\lambda}}(N-w)^{\lambda}}. \tag{5.19}$$

From Eqs. (5.5), (5.19), and (5.3), we find

$$\lim_{N\to\infty} \langle \Delta \rangle = \mu \left(w^{\lambda} - w \right), \tag{5.20}$$

where we have taken the large N limit to simplify the expression. Because we have taken the limit $N \to \infty$ before approaching the critical point ($\lambda \to 1$), our theory neglects the effect of multiplicative noise as discussed in Ref. [37].

Next we make an ansatz for the structure of $P(w)$ inspired by the results of numerical simulations. Suppose $P(w)$ is sharply peaked at $w = 1$ and near zero everywhere else; *i.e.*, the agents have nearly equal wealth. Then we can solve the Fokker-Planck equation in the neighborhood of $w = 1$ by approximating $A(w) \approx \frac{1}{2}$ and $B(w) \approx \frac{1}{4}$ [see Eqs. (5.11) and (5.12)]. We must verify that our resulting distribution $P(w)$ is in fact sharply peaked at $w = 1$ after solving the Fokker-Planck equation with these assumptions. This self-consistency check will reveal that our Gaussian solution for $P(w)$ is valid only for large Ginzburg parameter G.

For a very sparse trade network (small D), we will assume that

$$2D\langle \Delta^2 \rangle_Y \ll (K - L)^2, \tag{5.21}$$

which allows us to drop the second term on the right-hand-side of the diffusive term (Eq. 5.14) such that

$$\langle \Delta^2 \rangle \approx \langle \Delta \rangle^2. \tag{5.22}$$

In Sec 5.2 we show that this approximation is valid only when $D \ll \mu^2(1-\lambda)^2$. The solution of Eq. (5.15) using Eqs. (5.3) and (5.22) yields the GED model's hyperequal wealth distribution:

$$P(w) \propto \left(1 - w^{1-\lambda}\right)^{-\frac{2}{\mu(1-\lambda)}}. \tag{5.23}$$

The distribution (5.23) is sharply peaked at $w = 1$ and is near zero everywhere else, for $\lambda < 1$. Thus we expect all the wealth of the agents to quickly converge to 1 during a simulation if $D \ll \mu^2(1-\lambda)^2$. Clearly our assumptions ($A \approx \frac{1}{2}$, and $B \approx \frac{1}{4}$) used to derive Eq. (5.23) are consistent with the resulting distribution.

For a dense trade network (large D), we can make the opposite assumption of (5.21):

$$2D\langle\Delta^2\rangle_Y \gg (K-L)^2, \tag{5.24}$$

which allows us to write [via Eq. 5.14]

$$\langle\Delta^2\rangle \approx 2D\langle\Delta^2\rangle_Y. \tag{5.25}$$

In Sec 5.2 we show that this approximation is valid only for networks with $D \gtrsim \mu(1-\lambda)$. After applying these approximations for a dense network ($A \approx \frac{1}{2}$, $B \approx \frac{1}{4}$, and Eq. (5.25)), and after substituting the exact value for $K(w)$ at $\lambda = 0$ (Eq. (5.16)), we obtain

$$P(w)|_{\lambda=0} \propto (1+w^2)^{-1-\frac{\mu}{Df^2}} \exp\left[2\frac{\mu}{Df^2}\arctan(w)\right]. \tag{5.26}$$

To solve for general λ, we use our mean-field solution for the growth term K (Eq. (5.19)) and obtain as a stationary state

$$P(w) \propto (1+w^2)^{-1-\frac{\mu}{Df^2}} \exp\left[X(w)\right], \tag{5.27}$$

where the exponent is given by

$$X(w) = \frac{2\mu w^{1+\lambda} {}_2F_1(1, \frac{1+\lambda}{2}; \frac{3+\lambda}{2}; -w^2)}{Df^2(1+\lambda)}. \tag{5.28}$$

Here ${}_2F_1$ is the hypergeometric function.

This distribution for general λ in Eq. (5.27) reduces to Eq. (5.26) after letting $\lambda = 0$. Because Eq. (5.26) was derived for $\lambda = 0$ without a mean-field approximation, the agreement between Eqs. (5.26) and (5.27) helps validate our mean-field approximation (5.18) used to solve for wealth distributions Eqs. (5.23) and (5.27). This distribution for general λ is plotted in Figs. 5.1 and 5.2.

As mentioned, the stationary wealth distribution transitions from the hyperequal distribution of Eq. (5.23) to the distribution of Eq. (5.27) as we increase the networks edge density D. For more on this transition, see Fig. 5.3 and Sec. 5.2.

In Sec. 4, we showed that the fully connected GED model's wealth distribution is Gaussian in the mean-field limit. To demonstrate the consistency of Eq. (5.27) with this previous result, we adopt the same scaling rules for f and μ as in Ref. [37]. We replace f and μ in our solution for $P(w)$ by f_0 and μ_0 using Eqs. (3.1) and (3.2). Then, the limit $N \to \infty$ gives a Gaussian wealth distribution:

$$\lim_{N\to\infty} P(w) = \frac{N}{\sqrt{2\pi G}} \exp\left[-\frac{G}{2}(w-1)^2\right], \tag{5.29}$$

with G given by Eq. (3.3), or equivalently

$$G \equiv \frac{\mu(1-\lambda)}{Df^2}. \tag{5.30}$$

We recognize Eq. (5.30) as the Ginzburg parameter of the fully connected GED model divided by the exchange network's edge density D (see Eq. (2.20)). From

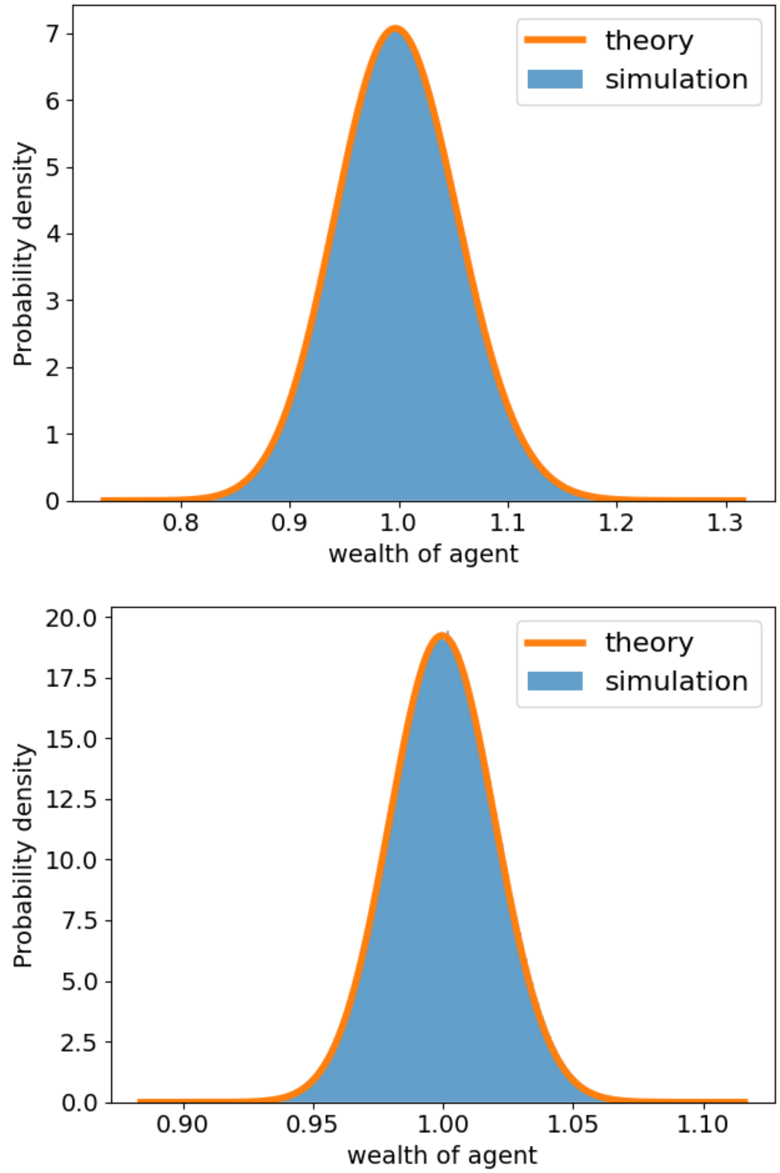


Figure 5.1: The probability density function $P(w)$ in Eq. (5.27) is compared to the probability density histogram of the GED model obtained from computer simulations for a fully connected network (top) and a nearest-neighbor 1D lattice (bottom). Results are shown for $\mu = 0.1$, $f = 0.01$, $N = 4900$. The values for λ are chosen to be approximately 0.68 (top) and 0.999 (bottom) such that $G \approx 316$ (top), and $G = 1225$ (bottom). The same function also characterizes the stationary wealth distribution of all other topologies studied in the limit $G \to \infty$. The error bars for each histogram are too small to be visible on the graphs.

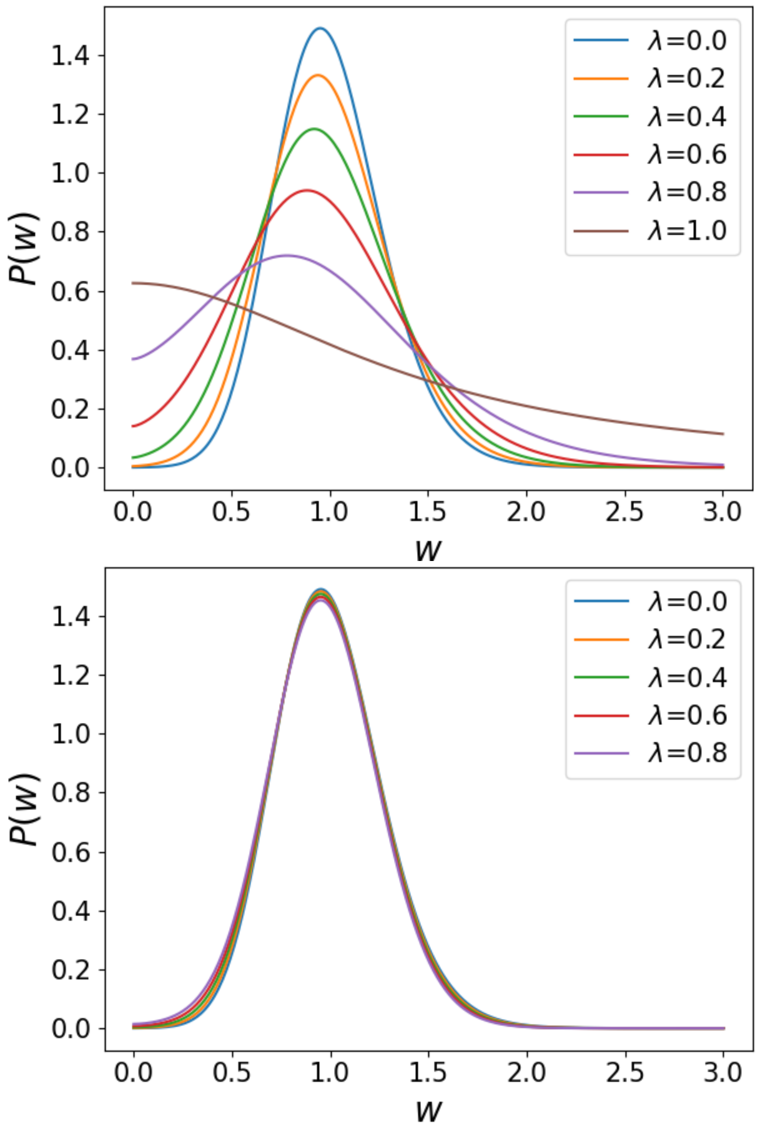


Figure 5.2: Top: The wealth distribution derived from the Fokker-Planck equation (5.27) for $\mu = f = 0.1$, $D = 1$. The distribution is not applicable for $\lambda \geq 1$ because there is no longer a peak near $w = 1$. Bottom: The wealth distribution (5.27) collapses to a single curve for different values of λ and f chosen such that $G = 10$ is held constant. The curve collapse becomes exact in the limit $G \to \infty$. Shown for $D = 1$.

Eq. (5.29), the variance σ^2 is simply G^{-1}:

$$\sigma = \sqrt{\frac{Df^2}{\mu(1-\lambda)}}. \tag{5.31}$$

From this expression for the standard deviation we see directly that an economy's wealth distribution becomes less equal as its exchange network becomes more globalized ($D \to 1^-$). In fact, G determines the width of the wealth distribution of Eq. (5.27) as well (see Fig. 5.2). Because the wealth distributions of Eqs. (5.27) and (5.29) decay sharply away from $w = 1$ only for large G, our assumptions $A \approx 1/2$ and $B \approx 1/4$ used to derive Eqs. (5.27) and (5.29) are valid only in the limit $G \to \infty$. Therefore we interpret G as the Ginzburg parameter for the GED model on an arbitrary exchange network. Because $D = 1$ for a fully connected graph, Eq. (5.30) is applicable to fully connected systems as well; substituting f_0 and μ_0 into Eq. (5.30) recovers the Ginzburg parameter of Eq. (3.3) as discussed in Sec. 4 and Refs. [37,44].

5.2 Conditions for Hyperequality

In Sec. 5.1 we stated that the hyperequal wealth distribution (Eq. (5.23)) is valid for sparse networks with density $D \ll \mu^2(1-\lambda)^2$. Meanwhile the near-Gaussian wealth distribution (Eq. (5.27)) arises for $D > \mu(1-\lambda)$. In this section, we derive these relations between D, μ, and λ, and comment on the region $\mu^2(1-\lambda^2) \lesssim D \lesssim \mu(1-\lambda)$.

First we examine the inequality (5.24) used to derive the near-Gaussian solution for $P(w)$ in Eq. (5.27). For a highly equal economy, we define

$$\delta \equiv 1 - w, \tag{5.32}$$

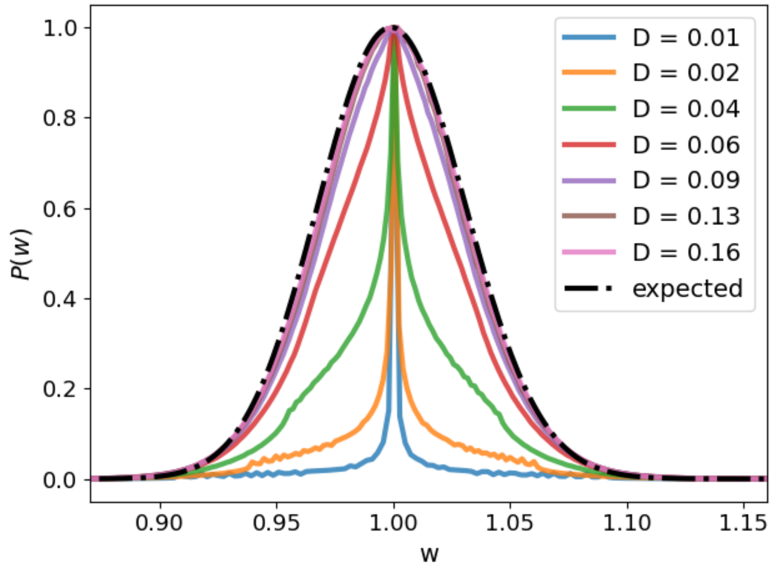


Figure 5.3: Simulation results for different exchange ranges R, and thus for different edge densities D, on a 70×70 square lattice for $\lambda = 0.5$ and $\mu = 0.2$ and constant $G = 10^3$. Each distribution $P(w)$ has been normalized such that its maximum is 1. For sparse trade networks (low D), the stationary wealth distribution is strongly peaked at $w = 1$ as in Eq. (5.23). However, when the edge density crosses the threshold $D = \mu(1 - \lambda) = 0.1$, the wealth distribution is no longer strongly peaked and is well approximated by Eq. (5.27). Similar results hold for all topologies studied.

such that $|\delta| \ll 1$ in the neighborhood of $w = 1$. From Eq. (5.20) we have,

$$\langle \Delta \rangle = \mu[(1+\delta)^\lambda - (1+\delta)] \tag{5.33}$$

$$\approx \mu(1 + \lambda\delta - (1+\delta)) \tag{5.34}$$

$$= -\mu(1-\lambda)\delta. \tag{5.35}$$

We recognize from Eqs. (5.35) and (5.3) that

$$(K - L)^2 = \langle \Delta \rangle^2 = \mu^2(1-\lambda)^2\delta^2, \tag{5.36}$$

and from Eq. (5.10) that

$$\langle \Delta^2 \rangle_Y \sim f^2 \tag{5.37}$$

for $A \approx 1/2$ and $B \approx 1/4$. Therefore, for Eq. (5.24) to hold, and to validate our Gaussian wealth distribution in Eq. (5.27), we must have

$$\delta^2 \ll \frac{Df^2}{\mu^2(1-\lambda)^2}, \tag{5.38}$$

which is a condition on the width of the stationary wealth distribution centered at $w = 1$. Only then can we drop the second term on the right-hand side of Eq. (5.14) as we did in Sec. 5 to derive Eq. (5.27).

If we assume that the δ for each agent (as defined in Eq. (5.32)) is on the order of the standard deviation σ, then we have $\delta \sim G^{-1/2}$ (by Eq. (5.31)). This consideration allows us to rewrite (5.38) as

$$\mu(1-\lambda) \ll 1, \tag{5.39}$$

which is easily satisfied for all $\lambda \geq 0$ and small μ. This successful self-consistency check affirms the integrity of the Gaussian solution in Eq. (5.27) for large G only if $\delta \sim \sigma$.

However, an inconsistency arises for $\sigma \ll f$. We must enforce (5.38) for all δ. It would normally be appropriate to assume $\delta \sim \sigma$, but, even in the limit of very small standard deviation ($\sigma \ll f$), there will always be at least DN agents with $|\delta| \geq f$. These anomalous agents arise simply because f is the discrete amount of wealth that is exchanged between two agents of wealth 1, and there are DN exchanges per time step.

Therefore, in addition to requiring the inequality (5.39), we must also replace δ

by f and require

$$Df^2 \gg \mu^2(1-\lambda)^2 f^2 \tag{5.40}$$

to safely drop the $\langle\Delta\rangle^2$ term from Eq. (5.14) and validate our Gaussian solution. This condition simplifies to

$$\frac{\mu^2(1-\lambda)^2}{D} \ll 1. \tag{5.41}$$

We see that for $D \gtrsim \mu(1-\lambda)$, the second condition (5.41) is satisfied if the first condition (5.39) is satisfied.

Alternatively, we can require that any stationary wealth distribution $P(w)$ for the GED model must satisfy

$$DN \lesssim N - \int_{1-f}^{1+f} P(w)dw, \tag{5.42}$$

where as usual we have taken the number density $P(w)$ to be normalized to N. In other words, there must be at least DN agents with $|\delta| > f$.

For a Gaussian with standard deviation σ, the requirement (5.42) simplifies to

$$D \lesssim \text{erfc}\left(\frac{f}{\sigma}\right), \tag{5.43}$$

where erfc is the complimentary error function. For $f \gg \sigma$, this condition is not necessarily satisfied. After approximating erfc(f/σ) by the first term of its Taylor expansion, (5.43) becomes

$$1 \lesssim \frac{\exp\left[-\frac{\mu(1-\lambda)}{D}\right]}{\sqrt{D\mu(1-\lambda)}}, \tag{5.44}$$

where we have substituted Eq. (5.31) for σ and have dropped a constant factor of $\sqrt{\pi}$. From Eq. (5.44) we see that, for small $\mu(1-\lambda)/D$, our approximately Gaussian solution for $P(w)$ (Eq. (5.27)) is valid. However, for $D < \mu(1-\lambda)$, it quickly becomes invalid. This is consistent with our alternative condition (5.41) that we derived *a priori*.

We see from the restrictions ((5.41) and (5.44)) that $D \sim \mu(1-\lambda)$ marks the minimum edge density above which the near-Gaussian wealth distribution (Eq. (5.27)) is a valid stationary solution to the Fokker-Planck equation.

For $D < \mu(1-\lambda)$, $P(w)$ is found from simulations to be strongly peaked at $w = 1$. To see why this happens, consider a very sparse network with $D \ll \mu^2(1-\lambda)^2$. In this case, the direction of the inequalities in Eqs. (5.41) and (5.40) would be reversed such that, even for agents with $|\delta| = f$, we could safely drop $2D\langle\Delta^2\rangle_Y$ from Eq. (5.14) to arrive at the hyperequal distribution in Eq. (5.23).

Because the Gaussian-like solution of the GED model becomes applicable at $D \gtrsim \mu(1-\lambda)$, and because the hyperequal distribution of Eq. (5.23) arises for $D \ll \mu^2(1-\lambda)^2$, we might expect the region

$$\mu^2(1-\lambda)^2 \lesssim D \lesssim \mu(1-\lambda) \tag{5.45}$$

to be a middle ground between the two solutions such that there remains a well defined peak at $w = 1$ and a variance about the mean. In fact, these are the characteristics we see from simulations in this density region. For all $D < \mu(1-\lambda)$, we do see a sharp peak at $w = 1$, regardless of network type (see Fig. 5.3).

Consequently, the hyperequal phase and its divergent wealth distribution is inherently inaccessible for fully connected or globalized systems (except in the unrealistic case of discontinuous growth: $\mu \sim 1$). The incompatibility of globalization and hyperequality is yet another mechanism by which globalization is found to contribute to wealth inequality.

CHAPTER 6: Master Equation Derivation of GED Phase Transition

In Sec. 4, we used a mean-field approximation to derive the fixed points of the fully connected GED model's growth, distribution, and rescaling mechanisms without exchange [Eq. (4.15b)]. Then, treating the effect of exchange as a noise, we showed that wealth condensation is a stable fixed point only for $\lambda \geq 1$. In this section, we derive this phase transition directly from the master equation. Although we use one for convenience, this method does not rely on a mean-field approximation. We also demonstrate how this method can easily be generalized to the case of a network when the Ginzburg parameter is large.

The master equation for the fully connected GED model is

$$\begin{aligned}\frac{\partial P(w)}{\partial t} = \frac{1}{N}\int_0^\infty dw' \int_0^\infty dw'' P(w')P(w'')\Bigg[\\ &- \delta(w'' - w) \\ &+ \frac{1}{2}\Theta(w'' - w')\delta\left(T_+(w'', fw') - w\right) \\ &+ \frac{1}{2}\Theta(w' - w'')\delta\left(T_+(w'', fw'') - w\right) \\ &+ \frac{1}{2}\Theta(w'' - w')\delta\left(T_-(w'', fw') - w\right) \\ &+ \frac{1}{2}\Theta(w' - w'')\delta\left(T_-(w'', fw'') - w\right)\Bigg],\end{aligned} \tag{6.1}$$

where we have approximated the joint probability distribution $P(w', w'')$ as the product of two single-agent distributions $P(w')P(w'')$. Equivalently, we have assumed $P(w''|w') = P(w'')$ and that an agent's wealth is independent of the wealth of its neighbors. To generalize this mean-field approximation to the case of a network of density D, we simply substitute $P(w''|w') = DP(w'')$ instead, so that the total number of neighbors per agent is no longer N, but is $\langle k \rangle = \int_0^\infty DP(w''|w')$.

This approximation is only appropriate for the network GED model in the large Ginzburg parameter limit (as discussed in Sec. 5); there is no such restriction for the fully connected model. In this section we will consider only the fully connected case, $D = 1$.

$T_{\pm}(w, \Delta)$ represents the transformation of w due to exchange, growth, and rescaling.

$$T_{\pm}(w, \Delta) = w \pm \Delta + K[w, P] - R(w). \tag{6.2}$$

Here Δ is the exchange amount which, in the Yard-Sale and GED models, is taken to be $f \min(w', w'')$ as shown in Eq. (6.1). The subscript of $T_{\pm}$ determines whether the agent in question wins or loses the exchange. Each outcome is chosen with equal probability in the master equation; note the factors of $1/2$ in each term. In this section we write $K[w, P]$ explicitly as a functional of both the agent wealth and probability density function. As in our Fokker-Planck formulation, $K[w, P]$ represents the amount of wealth gained due to growth and is given by Eq. (5.4). $L(w)$ is an effective loss of wealth due to rescaling. We again assume that μ and f are sufficiently small so that we can approximate

$$L(w) = \left(1 - e^{-\mu t}\right)(w + \Delta + K[w, P]) \approx \mu w. \tag{6.3}$$

For the special case $\lambda = 1$, we can easily evaluate the integral in Eq. (5.4) for a general distribution P. The integral

$$\int_0^\infty x P(x) dx = N \tag{6.4}$$

is the total system wealth which is conserved due to rescaling. Equation (6.4) yields

$K[w, P]|_{\lambda=1} = \mu w = L(w)$, and thus

$$T_{\pm}(w, \Delta)|_{\lambda=1} = w \pm \Delta. \tag{6.5}$$

Therefore, the dynamics of the GED model are completely equivalent to those of the Yard-Sale model for $\lambda = 1$.

A fully wealth-condensed probability distribution can be represented as

$$P(w) = (N-1)\delta(w) + \delta(w - N) \tag{6.6}$$

for a system with total wealth N. This state is trivially stationary as only a single agent posses nonzero wealth. It is impossible for a system to ever reach this wealth distribution because each agent's wealth can only be lost as a fraction of its current wealth; therefore, each agent retains an infinitesimal amount of wealth at all times. With these considerations, we take

$$P(w) = (N-1)\delta\left(w - \frac{\epsilon}{N-1}\right) + \delta\left(w - N + \epsilon\right) \tag{6.7}$$

as the wealth condensed state with some small amount of wealth $\epsilon : 0 < \epsilon \ll 1$. This distribution is a mean-field approximation because we have assumed that each agent other than the richest has equal wealth. We have taken this approximation for convenience; however, it is possible to derive the phase transition from the master equation as long as we assume that the wealth of all agents other than the richest agent are much less than one. In principle, instead of Eq. (6.7), we could substitute any

$$P(w) = \delta\left(w - N + \sum_{i=1}^{N-1} \epsilon_i\right) + \sum_{i=1}^{N-1} \delta\left(w - \epsilon_i\right) \tag{6.8}$$

with $\epsilon_i \ll 1/N$ as our wealth condensed state.

To quantify the wealth inequality among our agents, it is helpful to use the Gini coefficient $\mathcal{G}$, a metric of inequality used in economics. The Gini coefficient can be thought of as a functional of the wealth distribution $P(w)$. It is defined as half the relative mean absolute difference in agent wealth [52] [see Eq. (2.23)]. For N total wealth and N agents, $\mathcal{G}$ can be expressed as

$$\mathcal{G}[P] = \frac{1}{2N^2} \int_0^\infty \int_0^\infty P(w_i)P(w_j)|w_i - w_j| dw_i dw_j. \tag{6.9}$$

A fully wealth-condensed state (6.6) is the global maximum of $\mathcal{G}[P]$, so we can use the Gini coefficient as a metric to see how far a system is from wealth condensation. (For full wealth condensation, $\mathcal{G} = 1$).

The functional derivative of $\mathcal{G}$ is [39]

$$\frac{\delta \mathcal{G}}{\delta P(w)} = \frac{2}{N}\left[-w + \frac{1}{N}\int_0^w dx P(x)(w - x)\right]. \tag{6.10}$$

Equation (6.10) allows us to evaluate the change in $\mathcal{G}$ over time using

$$\frac{d\mathcal{G}}{dt} = \int_0^\infty dw \frac{\delta \mathcal{G}}{\delta P(w)} \frac{\partial P(w)}{\partial t}. \tag{6.11}$$

We can obtain $\partial P(w)/\partial t$ from the master equation (6.1).

As explained earlier, the substitution of (6.6) into Eqs. (6.1) and (6.11) trivially results in $\partial P(w)/\partial t = d\mathcal{G}/dt = 0$. Full wealth condensation is therefore a fixed point of the GED model. By adopting the density function (6.7) as our test case, we can perturb the system slightly away from wealth condensation to see if the fixed point is stable. We will find that the stability of (6.6) depends on the value of λ.

From Eq. (6.7) we expand the product $P(w')P(w'')$ term by term to use in the master equation (6.1):

$$
\begin{aligned}
P(w')P(w'') = & \\
& (N-1)^2\delta\left(w' - \frac{\epsilon}{N-1}\right)\delta\left(w'' - \frac{\epsilon}{N-1}\right) \\
& + (N-1)\delta\left(w' - \frac{\epsilon}{N-1}\right)\delta(w'' - (N-\epsilon)) \\
& + (N-1)\delta(w' - (N-\epsilon))\delta\left(w'' - \frac{\epsilon}{N-1}\right) \\
& + \delta(w' - (N-\epsilon))\delta(w'' - (N-\epsilon)).
\end{aligned} \tag{6.12}
$$

The first term of Eq. (6.12) represents the interaction between two agents both with infinitesimal wealth. The second and third terms represent the interaction between the super-rich agent and an agent with infinitesimal wealth. The final term represents an interaction of the rich agent with itself which is unphysical. This term results from the approximation $P(w', w'') \approx P(w')P(w'')$ and will therefore be neglected.

The first term of Eq. (6.12) when evaluated in Eq. (6.1) reduces to

$$
\begin{aligned}
\left(\frac{\partial P}{\partial t}\right)_1 = \frac{(N-1)^2}{N^2}\Bigg[& -\delta\left(\frac{\epsilon}{N-1} - w\right) \\
& + \frac{1}{2}\sum_{+,-}\delta\left(T_{\pm}\left(\frac{\epsilon}{N-1}, \frac{f\epsilon}{N-1}\right) - w\right)\Bigg].
\end{aligned} \tag{6.13}
$$

Here the subscript 1 indicates the contribution to the master equation of the first term of Eq. (6.12).

From Eq. (6.11), we obtain

$$\left(\frac{d\mathcal{G}}{dt}\right)_1 = \frac{(N-1)^3}{N^3}\Bigg[\frac{2N}{N-1}\left(\frac{\mu\epsilon}{N-1} - K[\frac{\epsilon}{N-1}, P]\right) + \sum_{+,-}\left(T_\pm - \frac{\epsilon}{N-1}\right)\Theta(T_\pm - \frac{\epsilon}{N-1})\Bigg], \tag{6.14}$$

where we have let $T_\pm = T_\pm(\frac{\epsilon}{N-1}, \frac{f\epsilon}{N-1})$. We are interested in the limit $N \to \infty$, so we can approximate

$$K[\frac{\epsilon}{N-1}, P] \approx \left(\frac{\epsilon}{N-1}\right)^\lambda \frac{N\mu}{(N-\epsilon)^\lambda}. \tag{6.15}$$

using Eq. (5.4) and the mean-field distribution (6.7). From Eq. (6.2) we also find

$$T_\pm - \frac{\epsilon}{N-1} = \frac{\epsilon}{N-1}(\pm f - \mu) + \left(\frac{\epsilon}{N-1}\right)^\lambda \frac{N\mu}{(N-\epsilon)^\lambda} \tag{6.16}$$

We see that the evolution of the Gini coefficient crucially depends on the value of λ. In Eq. (6.14) there are terms of order ϵ and terms of order $\epsilon^\lambda N^{2(1-\lambda)}$. The terms that dominate in the limits $N \to \infty$ and $\epsilon \to 0$ depend only on whether $\lambda < 1$. Therefore, in the thermodynamic limit, for $\mu \neq 0$, we can write

$$\left(\frac{d\mathcal{G}}{dt}\right)_1 = \begin{cases} -2\mu\epsilon^\lambda N^{-2\lambda} & \lambda < 1 \\ f\epsilon N^{-1} & \lambda = 1 \\ 2\epsilon N^{-1}\left(\mu + \frac{f-\mu}{2}\Theta(f-\mu)\right) & \lambda > 1 \end{cases} . \tag{6.17}$$

Note that the rate of change of the Gini coefficient when $\lambda = 1$ can be obtained from Eq. (6.14) by setting $\mu = 0$, regardless of the choice of λ. Thus again we see that the dynamics of the GED model is identical to the traditional Yard-Sale model

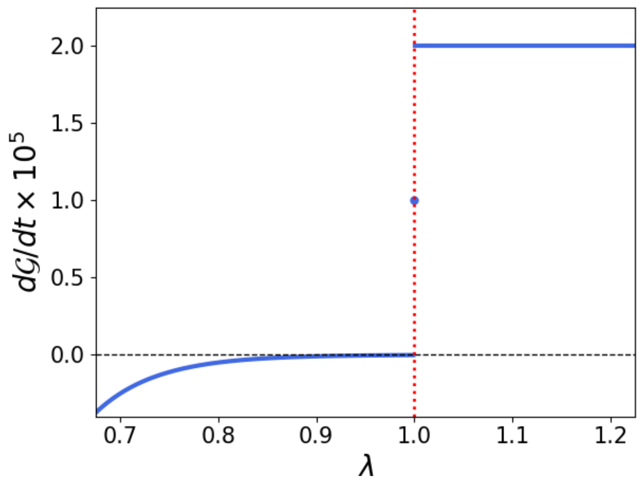


Figure 6.1: The rate of change of the Gini coefficient $\mathcal{G}$ for $\epsilon = \mu = f = 0.1$, and $N = 1000$. $d\mathcal{G}/dt$ experiences a discontinuous jump from negative to positive values at the critical point $\lambda_c = 1$.

when $\lambda = 1$.

Following similar calculations for the second and third terms of Eq. 6.12, we see that $(d\mathcal{G}/dt)_2$ and $(d\mathcal{G}/dt)_3$ are of order $\frac{1}{N}(d\mathcal{G}/dt)_1$ and thus can be neglected in the thermodynamic limit.

Finally we have

$$\frac{d\mathcal{G}}{dt} = \left(\frac{d\mathcal{G}}{dt}\right)_1, \tag{6.18}$$

and from (Eq. 6.17) we see that the Gini coefficient decreases for all $\lambda < 1$ and increases for all $\lambda \geq 1$.

Because a fully wealth-condensed state is the global maximum of $\mathcal{G}[P]$, and because the chosen distribution (6.7) is infinitesimally close to total wealth condensation (6.6) as $\epsilon \to 0$ and $N \to \infty$, an increase in $\mathcal{G}$ moves the system toward wealth condensation, and a decrease in $\mathcal{G}$ leads the system away from wealth condensation.

Therefore we have shown that wealth condensation is an attractive stationary state of the GED model if and only if $\lambda \geq 1$ (for $\mu \neq 0$). If $\mu = 0$, the GED model is simply the Yard-Sale model, and wealth condensation is attractive by Eq. (6.14) as if λ were equal to 1.

CHAPTER 7: Summary and Discussion

7.1 Summary of Results

Simulation results strongly suggest that increased globalization within the GED model results in increased wealth inequality and lower levels of economic mobility. We derived the stationary wealth distributions of the GED model on an arbitrary exchange network using a mean-field Fokker-Planck approach. This wealth distribution was found to depend on the network's edge density, and our theoretical predictions match the results of computer simulations. We conclude that the GED model implies that globalization increases inequality. However, the increase in inequality caused by globalization can be offset if accompanied by an increase in the growth rate. These results only hold for $\lambda < 1$; for $\lambda \geq 1$, no amount of growth or globalization is capable of affecting inequality because the economy is doomed to wealth condensation.

We found that all network topologies display mean-field critical exponents for large and fixed Ginzburg parameter when the exchange fraction and growth rate are scaled by the number of exchanges per time step (DN). This result holds even for highly localized topologies such as nearest-neighbor lattices, presumably due to the global nature of the growth and distribution rules.

For sparse trade networks, we found a "hyperequal" phase of the GED model for which the number density $P(w)$ diverges at $w = 1$. The existence of this behavior for networks with $D < \mu(1 - \lambda)$ was found using our Fokker-Planck formalism. Finally, we demonstrated in Sec. 6 that the GED model's phase transition at $\lambda = 1$ can be derived directly from the master equation.

7.2 Future Work

Our computational and theoretical results show the richness of agent-based economic models. There are many avenues of further research that can be explored. The GED model can be generalized to allow for a more detailed description of an economy. In its current form, the GED model does not produce wealth distributions that are qualitatively similar to known real-world wealth distributions of major economies. However, several straightforward modifications to the GED model have the potential to yield more realistic results.

One generalization of the GED model is to impose different growth rates μ for different locations on the trade network, corresponding to the various growth rates of different countries or industries, for example. Perhaps certain agents experience higher growth due to some innate characteristic or because of their geographic location on a lattice. We could also add noise to the growth of each agent which we would expect in real economies. This noise would cause growth to contribute to the diffusive term in the Fokker-Planck equation [Eq. (5.1)]. The parameters μ and f could be made to vary between agents whether as a function of their current wealth or historical wealth. In this case f_i would be a measure of the risk-takingness of each agent, and the exchange rule [Eq. (5.2)] would need to be modified to take the value of f of both agents into account (perhaps by replacing $f \to \min(f_1, f_2)$). It would be interesting to see how an agent's wealth correlated with their risk-takingness f_i.

A further modification of the GED model would be to differentiate between invested wealth and income. In this case, an agent would not receive new wealth due to growth for all of his assets. Rather, an agent would hold some wealth as "income" that is traded with other agents via the Yard Sale exchange rule, and some wealth as "investments" that would not normally be traded directly with other agents, but would yield growth via the GED growth and distribution rules. This modification would result in a more complicated model that must make some assumptions about how agents invest their wealth (*e.g.*, the percentage of income do agents invest to capture growth).

It is also possible that an economy's growth rate could be a function of its wealth distribution. For example, an economy with very high inequality could result in low productivity and low growth. In general, the growth rate could be any arbitrary functional $\mu[P(w)]$.

We could also imagine that a regulatory authority is responsible for splitting up monopolies as they arise naturally within the economy. How would simulation results change if, whenever an agent exceeds a certain level of wealth, their wealth were divided in half, either by creating a new agent, or giving half of their wealth to a pre-existing agent? What if there were some bankruptcy protection as well such that an agent who falls below a certain wealth threshold would be refunded their losing trades? There are many interesting possibilities to explore.

Finally, in reality, trade networks are not static. As supply and demand shifts globally, trade routes can open or close, consumers can start shopping at different stores, or a restaurant can lose customers. However, the GED model as described in this dissertation assumes an unchanging network such that agents have no agency to decide with whom they trade. It would be interesting to modify the GED model to allow for a dynamic network that naturally evolves as a function of the wealth dynamics on the network. For example, perhaps as an agent becomes richer, then they accrue more trading partners. This action would necessitate breaking some bonds in the trade network and connecting new pairs of agents.

This work has shown the potential of simple, stochastic agent-based models to tackle questions traditionally left to the field of economics. It is our hope that economists will adopt agent-based models developed by the physics community (*e.g.*, Refs. [27, 33, 37, 41, 44]) to further their own research. We also hope that economic policy makers might make use of agent-based modeling to guide their decision making in the near future. This area of research can allow physics to benefit society in a new way, and we are happy to be some small part of this process.

CHAPTER A: The Fokker-Planck Equation Near Zero Wealth

In Sec. 5, we derived the stationary wealth distribution of the network GED model in the neighborhood of $w \approx 1$. We then argued that this solution for the wealth distribution near $w = 1$ approximates the entire wealth distribution for all w because we restricted ourselves to large Ginzburg parameter such that $P(w)$ was sharply peaked at $w = 1$ and near zero elsewhere.

Additionally, we can use the steady-state Fokker-Planck equation to solve for the approximate wealth distribution in other regions of w as well. In this appendix, we consider the neighborhood of small $w \approx 0$. We will find that the Fokker-Planck equation strongly suggests the existence of the GED model's phase transition at $\lambda = 1$.

We make an approximation used by Boghosian *et al.* in Ref [40] and assume that in the neighborhood of small $w \approx 0$, $P(w)$ is negligibly small. This assumption implies from Eqs. (5.11) and (5.12) that $A \approx 1$ and $B \approx 0$. The resulting $P(w)$ obtained from the Fokker-Planck equation will then either validate or invalidate our assumption. By using this approximation, Eq. (5.15) can now be solved exactly for $\lambda = 0$. After substituting $\lambda = 0$, Eq. (5.15) reduces to

$$\mu(1-w)P = \frac{\partial}{\partial w}\left[\left(D\langle\Delta^2\rangle_Y + \mu^2(1-w)^2\right)P\right], \tag{A.1}$$

where we have made use of Eqs. (5.5) and (5.16). By assuming $A = 1$ and $B = 0$, the solution to Eq. (A.1) does smoothly approach zero as $w \to 0$, which in turn validates our assumptions for A and B.

If we further assume, for the sake of simplicity, that $\mu^2(1-w)^2 \ll D\langle\Delta^2\rangle_Y$ (such that the most of the change in the agents' wealth is due to exchange rather than growth), then we can drop $\mu^2(1-w)^2$ from Eq. (A.1). This high-exchange limit is not necessary to solve the equation, and the qualitative conclusions drawn from this assumption hold even if we leave both terms on the right-hand-side of Eq. (A.1). Nevertheless, under this assumption we obtain for the stationary probability density

$$P(w) \propto w^{-2+\frac{\mu}{Df^2}} \exp\left[-\frac{\mu}{Df^2 w}\right] \tag{A.2}$$

for $\lambda = 0$ and small w only.

We see that the high-exchange, small-w probability distribution in Eq. (A.2) has the same form as that of the Yard-Sale model modified by a flat, redistributive tax applied to all agents [40]. This formulation of the Yard Sale model is equivalent to the Extended Yard Sale Model with $a = 0$ as discussed in Sec. 2.4 [41]. Note that we have set $\lambda = 0$ to derive Eq. (A.2), so in general the GED model is *not* equivalent to a Yard-Sale model plus a simple tax rule.

If we repeat the above calculation Eqs. (A.1–Eqs.A.2) for $\lambda = 1$ instead of $\lambda = 0$, we find as a small w solution $P(w) \propto w^{-2}$, which does not smoothly go to zero at $w = 0$ and therefore invalidates our assumptions $A \approx 1$ and $B \approx 0$. We conclude that for $\lambda = 1$, there exists no neighborhood around $w = 0$, no matter how small, that contains a negligible number of agents. The fact that there exist agents with wealth infinitesimally close to zero suggests that the GED model at $\lambda = 1$ (and hence the unmodified Yard-Sale model) exhibits wealth condensation as a stationary state.

We have seen that $\lambda = 1$ reduces to the original Yard-Sale model and $\lambda = 0$ is equivalent to a flat, redistributive tax on agent wealth when $\mu^2 \ll Df^2$. We expect $\lambda \in (0, 1)$ to be a middle ground where wealth is redistributed, but where agents with higher wealth have an advantage in obtaining redistributed wealth.

We now solve for a stationary $P(w)$ for general λ in the limit of small w using our mean-field expression for $\langle\Delta\rangle$ in Eq. (5.3). After assuming $A \approx 1$ and $B \approx 0$, the Fokker-Planck equation yields a solution that validates this assumption. Furthermore, the solution for general $\lambda \in (0,1)$ reduces to the solution of Eq. (A.1) for $\lambda = 0$, which lends credence to the validity of our mean-field approximation Eq. (5.18).

As in Eq. (A.2), if we assume that $\mu^2 \ll Df^2$, we obtain as a stationary solution to the Fokker-Planck equation:

$$P(w) \propto w^{-2+\frac{\mu}{Df^2}} \exp\left[-\frac{\mu w^{\lambda-1}}{Df^2(1-\lambda)}\right]. \tag{A.3}$$

This solution and its derivatives approach zero as $w \to 0$ for all $\lambda \in (0,1)$ and therefore validate our previous assumption for A and B. Furthermore, we see that the distribution (A.3) is consistent with our exact, non-mean-field derivation for $\lambda = 0$ in Eq. (A.2). Again, this consistency check helps validate the mean-field approximation. For $\lambda \geq 1$, the solution in Eq. (A.3) blows up as $w \to 0$, so for these values of λ, we do have wealth condensation as a theoretical possibility. Crucially, all of these results hold even if we do not impose the condition $\mu^2 \ll Df^2$. We present this small μ limit in Eqs. (A.2) and (A.3) only because the solutions $P(w)$ for general μ, D, and f are unwieldy.

In this appendix, we have shown that, for the network GED model, wealth condensation does not develop when $\lambda < 1$ and that a non-negligible number of agents have infinitesimal wealth when $\lambda \geq 1$. To show that wealth condensation does develop when $\lambda \geq 1$, see the master equation derivation of the network GED phase transition in Sec. 6.

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