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# FUNCTIONAL GAUSSIAN APPROXIMATIONS IN HILBERT SPACES: THE NON-DIFFUSIVE CASE

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ABSTRACT. We develop a functional Stein-Malliavin method in a non-diffusive Poissonian setting, thus obtaining a) quantitative central limit theorems for approximation of arbitrary non-degenerate Gaussian random elements taking values in a separable Hilbert space and b) fourth moment bounds for approximating sequences with finite chaos expansion. Our results rely on an infinite-dimensional version of Stein's method of exchangeable pairs combined with the so-called Gamma calculus. Two applications are included: Brownian approximation of Poisson processes in Besov-Liouville spaces and a functional limit theorem for an edge-counting statistic of a random geometric graph.

## 1. INTRODUCTION

The now classical Stein-Malliavin method, a combination of Stein's method with Malliavin calculus, has been very sucessful in deriving quantitative central limit theorems for non-linear approximation. Since its inception by Nourdin and Peccati in 2013 (see [NP09]), it has formed a vivid community which developed the theory further and applied it to numerous situations. An excellent exposition of the basic method is available in the monograph [NP12] , while I. Nourdin keeps a rather exhaustive and continuously updated list of references on the webpage `https://sites.google.com/site/malliavinstein`. From a theoretical point of view, one of the main remaining challenges is an adaptation of the method to the infinite-dimensional setting, with quantitative approximation of Gaussian processes as main application. For random elements taking values in a Hilbert space, and in a diffusive context, this has recently been achieved by [BC20]. In this work, we provide the natural analogue in the non-diffusive context of Poisson spaces. More specifically, let $X$ be a square-integrable measurable transformation of a Poisson process and $Z$ be a Gaussian process, both taking values in some separable Hilbert space $K$. Informally, our main results (Theorems 3 and 4 on page 9) provide bounds on a probabilistic distance between $X$ and $Z$ (metrizing convergence in law) in terms of the first four strong moments of $X$ or alternatively in terms of so called contractions. From these bounds, one can directly deduce quantitative and functional central limit theorems for convergence towards a Gaussian process, as well as an infinite-dimensional version of the Fourth Moment Theorem, which says that for a sequence of $K$-valued multiple Poisson-integrals, convergence of the second and fourth moments implies convergence towards a Gaussian process.

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It is noteworthy to observe that while the analogous diffusive statements in [BC20] look similar to our non-diffusive ones, their proofs are rather different, for the same reason as in the finite-dimensional case: No chain rule is available in the non-diffusive case, which renders the usual integration by parts argument unfeasible. Instead, one can construct an appropriate exchangeable pair and then apply a Taylor argument in order to control the term resulting from an application of Stein's method. Compared to the finite-dimensional setting, several technical issues arise which require the use of Hilbert-space techniques. A commonality with the diffusive statements is, however, that our main results subsume all known finite-dimensional Malliavin-Stein bounds in a Poissonian context as special cases (see Remark 2 on page 9 for details).

In order to illustrate our results, we provide two applications: The first one concerns the classical approximation of a Brownian motion by a normalized Poisson process with growing intensity $\lambda$. A natural class of Hilbert spaces accommodating the sample paths of both processes are the so-called Besov-Liouville spaces. In [CD13], the authors showed that convergence takes place at rate $\lambda^{-1/2}$ (as in the classical one-dimensional case). To prove this, they first transferred both processes isometrically $\ell^2(\mathbb{N})$ and then had to go through rather tedious calculations. In contrast to this, our bounds yield the same result in just a few lines, and no isometry is necessary. As a second application we illustrate, using an edge counting statistic of a random graph, how known one-dimensional central limit theorem can be made functional with very little additional effort.

Besides the already mentioned reference [BC20], the work [CD13] is concerned with quantitative functional approximation in a Malliavin-Stein context as well. As already mentioned, the authors use a different approach which crucially depends on isometrically mapping all random elements to $\ell^2(\mathbb{N})$. In applications, the need to explicitly evaluate such an isometry can be seen as a drawback. Also, our setting seems to be more general and does not rely on ad-hoc arguments depending on the Gaussian process at hand. Other related references proving functional central limit theorems using Malliavin-Stein techniques are [Kas17, Kas20, DK21, DKP19].

The rest of this paper is organized as follows. In Section 2 we introduce the necessary preliminaries, followed by the main results in Section 3. The proofs are given in Section 4 which is followed by the two aforementioned applications in Section 5. An appendix contains several technical lemmas required for the proofs.

## 2. Preliminaries

### 2.1. **Probability on Hilbert spaces.**

Let $K$ be a real separable Hilbert space, $\mathcal{B}(K)$ the Borel $\sigma$-algebra of $K$ and $(\Omega, \mathcal{F}, P)$ a complete probability space. A $K$-valued random variable $X$ is a measurable map from $(\Omega, \mathcal{F})$ to $(K, \mathcal{B}(K))$. Such random variables are characterized by the property that for any continuous linear functional $\phi \in K^*$, the function $\phi(X) : \Omega \to \mathbb{R}$ is a real-valued random variable. As usual, the distribution or law of $X$ is the push-forward probability measure $P \circ X^{-1}$ on $(K, \mathcal{B}(K))$. The set of all $K$-valued random variables is a a vector space over the field of real numbers. If the Lebesgue integral $\mathbb{E}[\|X\|_K] = \int_\Omega \|X\|_K \, dP$ exists and is finite, then the Bochner integral $\int_\Omega X dP$ exists in $K$ and is called the expectation of $X$. Slightly abusing notation, we denote this integral by $\mathbb{E}[X]$ as well, and it can always inferred from the context whether $\mathbb{E}[\cdot]$ refers to Lebesgue or Bochner integration with respect to $P$. For $p \geq 1$, $L^p(\Omega, P)$ denotes the Banach space of all equivalence classes (under almost sure equality) of $K$-valued random variables $X$ with finite $p$-th moment, i.e., such that

$$\|X\|_{L^p(\Omega,P)} = \mathbb{E}\big[\|X\|_K^p\big]^{1/p} < \infty.$$

Note that for all $X \in L^p(\Omega, P)$, the Bochner integral $\mathbb{E}[X]$ exists. In the case $X \in L^2(\Omega, P)$, the covariance operator $S : K \to K$ of $X$ is defined by

$$Su = \mathbb{E}[\langle X, u \rangle_K X].$$

$S$ is a positive, self-adjoint trace-class operator that verifies the identity

$$\operatorname{Tr} S = \mathbb{E}\Big[\|X\|_K^2\Big].$$

We denote by $\mathcal{S}_1(K)$ the Banach space of all trace-class operators on $K$, equipped with norm $\|T\|_{\mathcal{S}_1(K)} = \operatorname{Tr}|T|$, where $|T| = \sqrt{TT^*}$ and $T^*$ denotes the adjoint of $T$. The subspace of Hilbert-Schmidt operators on $K$ is denoted by $\operatorname{HS}(K)$, its inner product and norm by $\langle \cdot, \cdot \rangle_{\operatorname{HS}(K)}, \|\cdot\|_{\operatorname{HS}(K)}$ respectively. Recall that

$$\|\cdot\|_{\operatorname{op}} \leq \|\cdot\|_{\operatorname{HS}(K)} \leq \|\cdot\|_{\mathcal{S}_1(K)},$$

where $\|\cdot\|_{\operatorname{op}}$ denotes the operator norm.

## 2.2. **Gaussian measures and Stein's method.**

In this section, we introduce Gaussian measures, the associated abstract Wiener spaces and Stein characterization of Gaussian measures. The theory will be presented within a general Banach space setting. Standard references for Gaussian measures and abstract Wiener spaces are the monographs [Bog98, Kuo75], while Stein's method for Gaussian measures has been developed by Shih in [Shi11] (see also Barbour's earlier work [Bar90] for the special case of Brownian motion).

2.2.1. *Abstract Wiener spaces.* Let $H$ be a real separable Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle_H$ and $\|\cdot\|$ be a norm on $H$ weaker than $\|\cdot\|_H$. Denote $B$ the Banach space obtained via completion of $H$ with respect to $\|\cdot\|$ and $i$ the canonical embedding of $H$ into $B$. The triple $(i, H, B)$ defines an abstract Wiener space and has first been introduced by Gross in [Gro67a]. We identify $B^*$ as a dense subspace of $H^*$ under the adjoint $i^*$ of $i$, so that we have the continuous embeddings $B^* \subseteq H \subseteq B$, where, as usual, $H$ is identified with its dual $H^*$. All of this can be summarized via the diagram

$$B^* \xrightarrow{i^*} H^* = H \xrightarrow{i} B.$$

The abstract Wiener measure $p$ on $B$ is characterized as the Borel measure on $B$ satisfying

$$\int_B \exp\Big(i \langle x, \eta \rangle_{B,B^*}\Big) p(dx) = \exp\left(-\frac{\|\eta\|_H^2}{2}\right),$$

for any $\eta \in B^*$.

2.2.2. *Gaussian measures.* Let $B$ be a separable Banach space, with $\mathcal{B}(B)$ its Borel $\sigma$-algebra. A Gaussian measure $\mu$ is a probability measure on $(B, \mathcal{B}(B))$ such that every linear functional $x \in B^*$, considered as a (real-valued) random variable on $(B, \mathcal{B}(B), \mu)$, has a Gaussian distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Such a Gaussian measure is called centered and/or non-degenerate, if these properties hold for the distributions of every $x \in B^*$.

We can see that every abstract Wiener measure is a Gaussian measure, and conversely, for every Gaussian measure $\mu$ on $B$, there exists a Hilbert space $H$ such that $(i, H, B)$ forms an abstract Wiener space. The space $H$ is known as the Cameron Martin space.

2.2.3. *Stein characterization of Gaussian measures.* Let $B$ be a real separable Banach space with norm $\|\cdot\|$. Let $Z$ be a $B$-valued random variable which induces a centered Gaussian measure $\mu_Z$ on $B$ and let $(i, H, B)$ be the associated abstract Wiener space. By $\{P_t : t \geq 0\}$ we denote the Ornstein-Uhlenbeck semi-group of $Z$. It has the Mehler representation

$$P_t f(x) = \int_B f\left(e^{-t}x + \sqrt{1 - e^{-2t}}y\right)\mu_Z(dy),$$

provided such an integral exists. In [Shi11, Theorem 3.1], Shih proved the following Stein lemma for abstract Wiener measures.

**Theorem 1.** *Let $X$ be a $B$-valued random variable with distribution $\mu_X$.*

*i) If $B$ is finite-dimensional, then $\mu_X = \mu_Z$ if and only if*

$$\mathbb{E}\left[\langle X, \nabla f(X)\rangle_{B,B^*} - \Delta_G f(X)\right] = 0 \tag{1}$$

*for any twice-differentiable function $f$ on $B$ such that $\mathbb{E}\left[\left\|\nabla^2 f(Z)\right\|_{\mathcal{S}_1(H)}\right] < \infty$.*

*ii) If $B$ is infinite-dimensional, then $\mu_X = \mu_Z$ if and only if (1) holds for every twice $H$-differentiable function $f$ on $B$ such that $\nabla f(x) \in B^*$ for every $x \in B$,*
$$\mathbb{E}\left[\left\|\nabla^2 f(Z)\right\|_{\mathcal{S}_1(H)}\right] < \infty \text{ and } \mathbb{E}\left[\|\nabla f(Z)\|_{B^*}^2\right] < \infty.$$

The notion of $H$-derivative which is also known as Fréchet derivative along $H$ and appears in Theorem 1 was introduced by Gross in [Gro67b], and we briefly recall it here for the sake of self-containedness. A function $f : U \to W$ from an open set $U$ of $B$ into a Banach space $W$ is said to be $H$-differentiable at $x \in U$ if the map $\phi(h) = f(x + h), h \in H$, regarded as a function defined in a neighborhood of the origin of $H$ is Fréchet-differentiable at 0. The $H$-derivative of $f$ at $x$ in the direction $h \in H$ is denoted by $\langle \nabla f(x), h\rangle_H$. The $k$-th order $H$-derivatives of $f$ at $x$ can then be constructed inductively and are denoted by $\nabla^k f(x)$, provided they exist. If $f$ is scalar-valued, $\nabla f(x) \in H^* \simeq H$ and $\nabla^2 f(x)$ is a bounded linear operator from $H$ to $H^*$ for every $x \in U$. The notation $\langle \nabla^2 f(x)h, k\rangle_H$ or $\nabla^2 f(x)(h, k)$ will stand for the action of the linear form $\nabla^2 f(x)(h, \cdot)$ on $k$.
If $\nabla^2 f(x)$ is a trace-class operator on $H$, the Gross Laplacian $\Delta_G f(x)$ of $f$ at $x$ is defined as $\Delta_G f(x) = \text{Tr}_H(\nabla^2 f(x))$.

2.2.4. *Stein's equation.* In view of Theorem 1, the associated Stein equation is given by

$$\langle x, \nabla g(x)\rangle_{B,B^*} - \Delta_G g(x) = h(x) - \mathbb{E}[h(Z)]$$

for $x \in B$, where $h$ belongs to a suitable class of test functions. In this paper, we will assume our test functions belong to $C_b^3(K)$, the class of real-valued functions on $K$ that have bounded Fréchet derivatives up to order three. This space is equipped with the norm

$$\|h\|_{C_b^3(K)} = \sup_{j=1,2,3} \sup_{x \in K} \left\|D^j h(x)\right\|_{K^{\otimes j}}.$$

Using standard semigroup techniques, the first two authors of this work showed in [BC20] that there is a solution $g_h(x)$ for every test function $h(x)$ and that $g_h \in C_b^3(K)$ when $h \in C_b^3(K)$. Specifically, [BC20, Lemma 2.4] provides the estimates

$$\sup_{x \in K} \left\|D^j g_h(x)\right\|_{K^{\otimes j}} \leq \frac{1}{j} \|h\|_{C_b^j(K)}$$

and

$$\|g_h\|_{C_b^3(K)} \leq \|h\|_{C_b^3(K)}.$$

Thus, using the probability distance

$$d_3(X_1, X_2) = \sup_{\substack{h \in C_b^3(K) \\ \|h\|_{C_b^3(K)} \leq 1}} |\mathbb{E}[h(X_1) - h(X_2)]|,$$

Stein's equation implies that

$$d_3(X, Z) = \sup_{\substack{h \in C_b^3(K) \\ \|h\|_{C_b^3(K)} \leq 1}} |\mathbb{E}[\Delta_G g_h(X) - \langle X, Dg_h(X)\rangle_K]|.$$

## 2.3. Dirichlet structure.

This section contains an overview of Dirichlet structures, which is the framework we will be working within alongside Stein's method. We start by recalling the definition and properties of a Dirichlet structure on $L^2(\Omega; \mathbb{R})$ (full details can be found in the monographs [BGL14, BH91]) before focusing on an extension to $L^2(\Omega; K)$. Given a probability space $(\Omega, \mathcal{F}, P)$, a Dirichlet structure $(\mathbb{D}, \mathcal{E})$ on $L^2(\Omega; \mathbb{R})$ with the associated carré du champ operator $\Gamma$ consists of a Dirichlet domain $\mathbb{D}$, which is a dense subset of $L^2(\Omega; \mathbb{R})$ and a carré du champ operator $\Gamma : \mathbb{D} \times \mathbb{D} \to L^1(\Omega, \mathbb{R})$ characterized by the following properties.

- $\Gamma$ is bilinear, symmetric ($\Gamma(F, G) = \Gamma(G, F)$) and positive ($\Gamma(F, F) \geq 0$).
- the induced positive linear form $F \to \mathcal{E}(F, F)$, where $\mathcal{E}(F, G) = \frac{1}{2}\mathbb{E}[\Gamma(F, G)]$, is closed in $L^2(\Omega; \mathbb{R})$, i.e., $\mathbb{D}$ is complete when equipped with the norm

$$\|\cdot\|_{\mathbb{D}}^2 = \|\cdot\|_{L^2(\Omega; \mathbb{R})}^2 + \mathcal{E}(\cdot).$$

**Remark 1.** We do not assume that $\Gamma$ satisfies the so-called diffusion property – see [BGL14, Definition 3.1.3] – as opposed to what is being done in [BC20].

Here and in the following, $\mathbb{E}[\cdot]$ denotes the expectation on $(\Omega, \mathcal{F})$ with respect to $P$. The linear form $\mathcal{E}$ is known as a Dirichlet form and for brevity we write $\mathcal{E}(F)$ for $\mathcal{E}(F, F)$. Every Dirichlet form gives rise to a strongly continuous semigroup $\{P_t\}_{t \geq 0}$ on $L^2(\Omega; \mathbb{R})$ and an associated symmetric Markov generator $-L$, defined on a dense subset $\text{dom}(-L) \subseteq \mathbb{D}$. There are two important relations between $\Gamma$ and $L$, the first being the integration by part formula

$$\mathbb{E}[\Gamma(F, G)] = -\mathbb{E}[FLG] = -\mathbb{E}[GLF],$$

which is valid for $F, G \in \mathbb{D}$. The second relation is

$$\Gamma(F, G) = \frac{1}{2}(L(FG) - GLF - FLG),$$

which holds for all $F, G \in \text{dom}(L)$ such that $FG \in \text{dom}(L)$. If $-L$ is diagonalizable with spectrum $\mathbb{N}_0$ (the set of natural numbers plus 0) and $F_q$ is an eigenfunction corresponding to the eigenvalue $q$, then $-LF_q = qF_q$. We can define a pseudo-inverse $-L^{-1}$ by $-L^{-1}F_q = \frac{1}{q}F_q$ when $q \neq 0$ and 0 otherwise. The definition of $-L$ and $-L^{-1}$ for a general $F = \sum_{q \in \mathbb{N}_0} F_q$ follows naturally via linearity. Alternatively, $L$ can be defined as the generator of the heat semigroup $\{P_t\}_{t \geq 0}$ (on $\text{dom}(L)$) which satisfies

$$\partial_t P_t = LP_t = P_t L.$$

Next we present what is meant by a Dirichlet structure on $L^2(\Omega; K)$. Let us adopt the notations $\widetilde{\mathbb{D}}, \widetilde{\Gamma}, \widetilde{L}, \widetilde{P_t}$ for the Dirichlet domain, Dirichlet form, carré du champ operator, generator and semigroup associated with elements in $L^2(\Omega; \mathbb{R})$. Meanwhile, $\mathbb{D}, \Gamma, L, P_t$ are reserved for the counterpart objects associated with elements in $L^2(\Omega; K)$. Given a separable Hilbert space $K$, one has that $L^2(\Omega; K)$ is isomorphic to $L^2(\Omega; \mathbb{R}) \otimes K$. The Dirichlet structure on $L^2(\Omega; \mathbb{R})$ can

therefore be extended to $L^2(\Omega; K)$ via a tensorization procedure. Let $\mathbb{N}_0$ be the spectrum of $-\widetilde{L}$ and $\{k_i\}_{i \in \mathbb{N}}$ an orthonormal basis of $K$. $\mathcal{A}$ will be the set of all functions $X$ taking the form

$$X = \sum_{q,i \in I} F_{q,i} \otimes k_i$$

such that $I \subseteq \mathbb{N}^2$ is a finite set and $F_{q,i} \in \ker\left(-\widetilde{L} + qI\right)$. Assuming another element $Y = \sum_{p,j \in J} G_{p,j} \otimes k_j$ in $\mathcal{A}$, we can define $L, \Gamma, P_t, \mathcal{E}$ for $t \geq 0$ via

$$\begin{cases} LX = L \sum_{q,i \in I} F_{q,i} \otimes k_i = \sum_{q,i \in I} \left(\widetilde{L} F_{q,i}\right) \otimes k_i \\ P_t X = P_t \sum_{q,i \in I} F_{q,i} \otimes k_i = \sum_{q,i \in I} \left(\widetilde{P}_t F_{q,i}\right) \otimes k_i \\ \Gamma(X,Y) = \frac{1}{2} \sum_{q,i \in I} \sum_{p,j \in J} \widetilde{\Gamma}(F_{q,i}, F_{p,j}) \otimes (k_i \otimes k_j + k_j \otimes k_i) \end{cases}$$

and

$$\mathcal{E}(X,Y) = \mathbb{E}[\operatorname{Tr} \Gamma(X,Y)].$$

In the last line, we identify $\Gamma(X,Y)$ as an element of $L^2(\Omega; \mathbb{R}) \otimes K \otimes K \simeq L^2(\Omega, \mathcal{L}(K,K))$ via the action

$$\Gamma(X,Y)u = \frac{1}{2} \sum_{q,i \in I} \sum_{p,j \in J} \widetilde{\Gamma}(F_{q,i}, F_{p,j}) \otimes \left(\langle k_i, u \rangle_K \otimes k_j + \langle k_j, u \rangle_K \otimes k_i\right).$$

Since $\mathcal{A}$ is clearly dense in $L^2(\Omega; K)$, these operators can be extended to appropriate domains in $L^2(\Omega; K)$. This has been verified in [BC20, Proposition 2.5 and Theorem 2.6] (excluding the diffusion identity), which we restate below for the reader's convenience.

**Proposition 1** (Proposition 2.5 in [BC20]). *The operators $L$ $L^{-1}$, $\mathcal{E}$ and $\Gamma$ can be extended to* $\operatorname{dom}(L)$, $\operatorname{dom}(L^{-1})$ *and* $\operatorname{dom}(\Gamma) = \operatorname{dom}(\mathcal{E}) = \mathbb{D} \times \mathbb{D}$, *respectively, given by*

$$\operatorname{dom}(L) = \left\{ X \in L^2(\Omega; K) : \sum_{q \in \mathbb{N}_0} q^2 \widetilde{J}_q\left(\|X\|_K^2\right) < \infty \right\},$$

$\operatorname{dom}(L^{-1}) = L^2(\Omega; K)$ *and*

$$\mathbb{D} = \left\{ X \in L^2(\Omega; K) : \sum_{q \in \mathbb{N}_0} q \widetilde{J}_q\left(\|X\|_K^2\right) < \infty \right\},$$

*where $\widetilde{J}_q(\cdot)$ denotes the projection onto $\ker\left(\widetilde{L} + qI\right) \subseteq L^2(\Omega; \mathbb{R})$. In particular, one has*

$$\mathcal{A} \subseteq \operatorname{dom}(L) \subseteq \mathbb{D} \subseteq \operatorname{dom}(L^{-1}) = L^2(\Omega; K),$$

*and all inclusions are dense.*

**Theorem 2** (Theorem 2.6 in [BC20]). *For a Dirichlet structure $(\mathbb{D}, \Gamma)$ on $L^2(\Omega; K)$, the following is true.*

   (i) *$\Gamma$ is bilinear, almost surely positive, symmetric and self-adjoint with respect to $\langle \cdot, \cdot \rangle_K$.*
   (ii) *The Dirichlet domain $\mathbb{D}$ equipped with the norm*

$$\|X\|_{\mathbb{D}}^2 = \|X\|_{L^2(\Omega;K)} + \|\Gamma(X,X)\|_{L^1(\Omega;\mathcal{S}_1)}$$

   *is complete, so that $\Gamma$ is closed.*
   (iii) *The generator $-L$ acting on $L^2(\Omega; K)$ is positive, symmetric, densely defined and has the same spectrum as $-\widetilde{L}$.*

(iv) *There is a compact pseudo-inverse $L^{-1}$ of $L$ such that*

$$LL^{-1}X = X - \mathbb{E}[X]$$

*for all $X \in L^2(\Omega; K)$, where the expression on the right is a Bochner integral.*

(v) *The integration by parts formula*

$$\mathbb{E}[\operatorname{Tr} \Gamma(X, Y)] = -\mathbb{E}[\langle LX, Y \rangle_K] = -\mathbb{E}[\langle X, LY \rangle_K]$$

*is satisfied for all $X, Y \in \operatorname{dom}(-L)$.*

(vi) *The generators $\Gamma, L, \widetilde{L}$ are related via*

$$\operatorname{Tr} \Gamma(X, Y) = \frac{1}{2}\Big(\widetilde{L}\langle X, Y \rangle_K - \langle LX, Y \rangle_K - \langle X, LY \rangle_K\Big) \tag{2}$$

*for all $X, Y \in \operatorname{dom}(-L)$.*

(vii) *The identity*

$$\langle \Gamma(X, Y)u, v \rangle_K = \frac{1}{2}\Big(\widetilde{\Gamma}(\langle X, u \rangle_K, \langle Y, v \rangle_K) + \widetilde{\Gamma}(\langle Y, u \rangle_K, \langle X, v \rangle_K)\Big),$$

*is valid for all $X, Y \in \mathbb{D}$ and $u, v \in K$.*

## 2.4. **Analysis on Poisson space.**

So far we have been working with a general probability space. In this section we will get more specific and describe the Poisson space on which most of our objects of interest are defined. We direct the reader to the references [LP18, NN18] for an extensive treatment of this topic. Let $(\mathcal{Z}, \mathscr{L}, \mu)$ be a measure space such that $\mu$ is $\sigma$-finite. A Poisson random measure $\eta$ on $(\mathcal{Z}, \mathscr{L})$ with control measure $\mu$ is a family of distributions defined on some probability space $(\Omega, \mathcal{F}, P)$ that satisfies

- $\eta(B)$ is a Poisson distribution on $\Omega$ with mean $\mu(B)$,
- $\eta(B_1), \eta(B_2)$ are independent when $B_1 \cap B_2 = \emptyset$.

If such a Poisson random measure exists, the associated probability space $(\Omega, \mathcal{F}, P)$ is called a Poisson space. Next, let $\widehat{\eta}$ be the compensated Poisson random measure, that is $\widehat{\eta}(B) = \eta(B) - \mu(B)$, whenever $\mu(B)$ is finite. Denote $L_s^2(\mu^q)$ the set of all symmetric functions in $L^2(\mu^q)$. For $f \in L_s^2(\mu^q)$, $I_q^\eta(f)$ denotes a multiple (Wiener-Itô) integral of order $q$. Unless we are simultaneously dealing with two different Poisson random measures, $I_q(\cdot)$ will be understood as an integral with respect to $\widehat{\eta}$. Multiple integrals have the following isometry property: for any integers $q, p \geq 1$,

$$\mathbb{E}[I_q(f)I_p(g)] = \mathbb{1}_{\{q=p\}} q! \langle \tilde{f}, \tilde{g} \rangle_{L^2(\mu^q)},$$

where $\tilde{f}$ denotes the symmetrization of $f$, and we recall that $I_q(f) = I_q(\tilde{f})$. The contraction of two kernels $f \in L_s^2(\mu^q)$ and $g \in L_s^2(\mu^p)$, denoted by $f \star_r^l g$ for $0 \leq l \leq r \leq q \wedge p$, is obtained by identifying $r$ variables and then integrating $l$ of those:

$$f \star_r^l g(y_1, \ldots, y_{r-l}, y_{r-l+1}, \ldots, y_{q-l}, z_1, \ldots, z_{p-r})$$
$$= \int_{\mathcal{Z}^l} f(x_1, \ldots, x_l, y_1, \ldots, y_{r-l}, y_{r-l+1}, \ldots, y_{q-l}) g(x_1, \ldots, x_l, y_1, \ldots, y_{r-l}, z_1, \ldots, z_{p-r})$$
$$d\mu(x_1, \ldots, x_l)$$

provided the integral exists in $L^2(\mu^{q+p-r-l})$. Contractions are central objects for analysis on Poisson space as they appear in the product formula for multiple integrals. There are two ways of stating this product formula on Poisson space: [Las16, Proposition 6.1] and [DP18, Lemma 2.4], each having different assumptions. We will state both below.

**Lemma 1** (Proposition 6.1 in [Las16]). *Let $f \in L_s^2(\mu^q), g \in L_s^2(\mu^p)$ and assume that $f \star_r^l g \in L^2(\mu^{q+p-r-l})$. Then,*

$$I_q(f)I_p(g) = \sum_{r=0}^{q \wedge p} r! \binom{q}{r} \binom{p}{r} \sum_{l=0}^{r} \binom{r}{l} I_{q+p-r-l}(f \star_r^l g). \tag{3}$$

**Lemma 2** (Lemma 2.4 in [DP18]). *Let $f \in L_s^2(\mu^q), g \in L_s^2(\mu^p)$ and assume that $F = I_q(f), G = I_p(g) \in L^4(P)$. Then*

$$FG = \sum_{k=1}^{q \wedge p - 1} \widetilde{J}_k(FG) + I_{q+p}(f \widetilde{\otimes} g).$$

The collection of all multiple integrals of order $q$ form the so-called Poisson chaos of order $q$ in $L^2(\Omega; \mathbb{R})$, which is denoted by $\mathcal{H}_q$. Since $\mathbb{E}[I_q(f)I_p(g)] = 0$ for $q \neq p$, we have the orthogonal decomposition

$$L^2(\Omega, \mathcal{F}, P) = \bigoplus_{q=1}^{\infty} \mathcal{H}_q.$$

Similarly as what we did for Dirichlet structures, we define $\mathcal{H}_q(K)$ ($K$-valued Poisson chaos of order $q$) as the closure of $\mathcal{H}_q \otimes K$ in $L^2(\Omega, K)$. Then,

$$L^2(\Omega; K) = \bigoplus_{q=1}^{\infty} \mathcal{H}_q(K).$$

Consequently, every $X \in L^2(\Omega, K)$ can be decomposed as

$$X = \sum_{q \in \mathbb{N}_0} F_q = \sum_{\substack{i \in \mathbb{N}, \\ q \in \mathbb{N}_0}} \langle F_q, k_i \rangle_K k_i = \sum_{\substack{i \in \mathbb{N}, \\ q \in \mathbb{N}_0}} F_{q,i} k_i,$$

where $F_q \in \mathcal{H}_q(K)$, $F_{q,i} \in \mathcal{H}_q$ with $F_{q,i} = I_q(f_{q,i})$ for some $f_{q,i} \in L_s^2(\mu^q)$.

2.5. **An exchangeable pair on Poisson space.** Another tool that we will make use of alongside Stein's method is the method of exchangeable pairs, which we will describe here. Per [LP18, Corollary 3.7], since $\eta$ is a Poisson random measure on $(\mathcal{Z}, \mathcal{L}, \mu)$, we can consider $\eta$ as a proper Poisson point process written as

$$\eta = \sum_{n=1}^{\kappa} \delta_{X_n},$$

such that $X_n, \kappa$ are random elements in $\mathcal{Z}, \mathbb{N} \cup \{0, \infty\}$, respectively. It is well known that any $F \in L^2(\Omega; \mathbb{R})$ has the representation $F = f(\eta)$ for some measurable function $f : \mathbb{N} \to \mathbb{R}$, which is uniquely defined up to null sets (see [LPS16]). In [DVZ18, Section 3.1], via continuous thinning of $\eta$, the authors are able to construct a family of new Poisson point processes $(\eta^t)_{t \geq 0}$ and from there derive a path-wise representation for the semigroup $\widetilde{P}_t$ associated with $\eta$. Specifically, the action of $\widetilde{P}_t$ can be described via the Mehler formula

$$\widetilde{P}_t f(\eta) = \mathbb{E}[f(\eta^t)|\eta].$$

Following up on this result, they made the important observation that for every $t \geq 0$, $(\eta, \eta^t)$ is an exchangeable pair (i.e., $(\eta, \eta^t)$ and $(\eta^t, \eta)$ have the same distribution) and that as a result, for any kernel $g \in L_s^2(\mu^p)$, the pair $\left( I_p^\eta(g), I_p^{\eta^t}(g) \right)$ is also exchangeable.

## 3. Statement of main results

In what follows, let $K$ be a separable Hilbert space with orthonormal basis $\{k_i\}_{i \in \mathbb{N}}$, and let $X$ denote a $K$-valued centered random variable in $L^2(\Omega; K)$ with finite chaos decomposition

$$X = \sum_{q=1}^{N} F_q, \tag{4}$$

where each $F_q$ belongs to the $q$-th $K$-valued Poisson chaos. Furthermore, assume that $X$ has covariance operator $S$, which in turn decomposes as

$$S = \sum_{q=1}^{N} S_q,$$

where, for each $1 \leq q \leq N$, $S_q$ is the covariance operator of $F_q$. Finally, we will denote by $f_{q,i} \in \mathfrak{H}^{\otimes q}$ the kernel of $F_{q,i} = \langle F_q, k_i \rangle_K = I_q(f_{q,i})$.

Our first main result provides a quantitative bound on the distance between the law of $X$ and a centered $K$-valued Gaussian random variable $Z$ in terms of the first four moments of $X$.

**Theorem 3.** *Assume $X$ is a $K$-valued random variable as described above with finite fourth moment, i.e., $\mathbb{E}\left[\|X\|_K^4\right] < \infty$. Then, letting $Z$ be a centered Gaussian random variable on $K$ with covariance operator $S'$, the following estimate holds*

$$
\begin{aligned}
d_3(X, Z) \leq{}& \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}} \\
&+ \sum_{1 \leq q \leq N} \frac{2q-1}{4q}\sqrt{\mathbb{E}\left[\|F_q\|_K^4\right] - \mathbb{E}\left[\|F_q\|_K^2\right]^2 - 2\|S_q\|_{\mathrm{HS}}^2} \\
&+ \sum_{1 \leq p \neq q \leq N} \frac{p+q-1}{4p}\sqrt{\mathbb{E}\left[\|F_p\|_K^2\|F_q\|_K^2\right] - \mathbb{E}\left[\|F_p\|_K^2\right]\mathbb{E}\left[\|F_q\|_K^2\right]} \\
&+ \sqrt{N\mathbb{E}\left[\|X\|_K^2\right]}\sqrt{\sum_{1 \leq q \leq N} 2^{3q-1}(4q-3)\left(\|F_q\|_K^4 - \mathbb{E}\left[\|F_q\|_K^2\right]^2 - 2\|S_q\|_{\mathrm{HS}}^2\right)} \\
\leq{}& \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}} \\
&+ \left(\frac{N(2N-1)}{4} + \sqrt{2^{3N-1}N(4N-3)\mathbb{E}\left[\|X\|_K^2\right]}\right) \\
&\quad \sqrt{\mathbb{E}\left[\|X\|_K^4\right] - \mathbb{E}\left[\|X\|_K^2\right]^2 - 2\|S\|_{\mathrm{HS}}^2}.
\end{aligned}
$$

**Remark 2.** Note that Theorem 3 is an infinite-dimensional version of the fourth moment theorems on the Poisson space obtained in [DVZ18, Theorem 1.2, Theorem 1.7] and [DP18, Theorem 1.3]. In particular, the aforementioned results are special cases of Theorem 3 obtained by setting $K = \mathbb{R}^d$ for a positive integer $d$.

**Remark 3.** Observe that Theorem 3 can be viewed as a Poissonian counterpart of [BC20, Theorem 3.10] in the context of a non-diffusive chaos structure. The fact that we are working with a non-diffusive structure (where no chain rule is available for the Gamma calculus introduced in Section 2) forces us to use different techniques in order to obtain the above quantitative bounds than the ones used in [BC20], making these results comparable in nature, but very different in their methodologies of proof.

Whenever $X$ belongs to a single chaos, we can reformulate Theorem 3 in a more compact form:

**Corollary 1** (Quantitative Fourth Moment Theorem)**.** *Let the notation of Theorem 3 prevail. When $X$ belongs to a single chaos, i.e., $X \in \mathcal{H}_q(K)$ for some $q \geq 1$, one has*

$$d_3(X, Z) \leq \frac{1}{2} \left\| S - S' \right\|_{\mathrm{HS}}$$
$$+ \left( \frac{2q-1}{4q} + \sqrt{2^{3q-1}(4q-3)q\mathbb{E}\left[\|X\|_K^2\right]} \right) \sqrt{\mathbb{E}\left[\|X\|_K^4\right] - \mathbb{E}\left[\|X\|_K^2\right]^2 - 2\|S\|_{\mathrm{HS}}^2}.$$

As $d_3$ metrizes convergence in law, the above corollary in particular shows that within a single non-diffusive chaos, convergence of the second and fourth strong moments implies convergence towards a (Hilbert-valued) Gaussian.

A particularly useful formulation of the above moment bounds for applications uses contraction operators acting on the kernels of the multiple integrals appearing in the chaos decomposition representation of $X$ given in (4). Contractions, which are the analytic quantities defined in Section 2, allow for much simpler computation compared to dealing directly with the first four moments. Some examples of previous works that use contraction norms to obtain quantitative limit theorem for Poisson random variables include [LRP13a, LRP13b, RS13].

Our second main result is the following contraction bound.

**Theorem 4.** *Let the notation and setup of Theorem 3 prevail. Moreover, let $\mathfrak{H} = L^2(\mathcal{Z}, \mu)$ where $\mathcal{Z}$ is the $\sigma$-fnite measure space described in Subsection 2.4. Then it holds that*

$$d_3(X, Z) \leq \left( \frac{N(2N-1)}{4} + \sqrt{2^{3N-2}N(4N-3)\mathbb{E}\left[\|X\|_K^2\right]} \right) \sqrt{\beta} + \frac{1}{2}\left\|S - S'\right\|_{\mathrm{HS}},$$

*where the quantity $\beta$ is given (in terms of contraction norms) by*

$$\beta = \sum_{\substack{1 \leq p,q \leq N \\ q \neq p}} a_{p,q}(p \wedge q) \left\| f_q \star_{q \wedge p}^{q \wedge p} f_p \right\|_{\mathfrak{H}^{\otimes |q-p|} \otimes K^{\otimes 2}}^2$$

$$+ \sum_{1 \leq p,q \leq N} \sum_{r=1}^{q \wedge p - 1} b_{p,q}(r) \left\| f_q \star_r^r f_p \right\|_{\mathfrak{H}^{\otimes(q+p-2r)} \otimes K^{\otimes 2}}^2$$

$$+ \sum_{1 \leq p,q \leq N} \sum_{(r,s,l,m) \in I} c_{p,q,l,m}(r,s) \left\| f_q \star_r^l f_p \right\|_{\mathfrak{H}^{\otimes(q+p-r-l)} \otimes K^{\otimes 2}} \left\| f_q \star_s^m f_p \right\|_{\mathfrak{H}^{\otimes(q+p-r-l)} \otimes K^{\otimes 2}}.$$

*Here, the combinatorial coefficients are given by*

$$\begin{cases} a_{p,q}(r) = p!q! \binom{q}{r}\binom{p}{r} + r!^2 \binom{q}{r}^2 \binom{p}{r}^2 |p-q|! \\ b_{p,q}(r) = p!q! \binom{q}{r}\binom{p}{r} \\ c_{p,q,l,m}(r,s) = r!s! \binom{q}{r}\binom{q}{s}\binom{p}{r}\binom{p}{s}\binom{r}{l}\binom{s}{m}(p+q-r-l)! \end{cases},$$

*and the index set $I$ is defined by*

$$I = \{(r,s,l,m) \in \mathbb{N}^4 \colon 0 \leq r, s \leq q \wedge p, \ 0 \leq l \leq r, \ 0 \leq m \leq s,$$
$$r+l = s+m, \ (r,s,l,m) \notin \{(0,0,0,0), (q \wedge p, q \wedge p, q \wedge p, q \wedge p)\}\}.$$

**Example 1.** If $X$ is a sum of elements of the first two chaoses, i.e., $X = I_1(f_1) + I_2(f_2)$, Theorem 4 requires the contraction norms $\left\| f_1 \star_1^1 f_2 \right\|_{\mathfrak{H} \otimes K^{\otimes 2}}$, $\left\| f_2 \star_1^1 f_2 \right\|_{\mathfrak{H}^{\otimes 2} \otimes K^{\otimes 2}}$, $\left\| f_1 \star_1^0 f_2 \right\|_{\mathfrak{H}^{\otimes 2} \otimes K^{\otimes 2}}$,

$\left\|f_1 \star_1^0 f_2\right\|_{\mathfrak{H}^{\otimes 2} \otimes K^{\otimes 2}}$, $\left\|f_2 \star_2^0 f_2\right\|_{\mathfrak{H}^{\otimes 2} \otimes K^{\otimes 2}}$, $\left\|f_2 \star_2^1 f_2\right\|_{\mathfrak{H} \otimes K^{\otimes 2}}$ and $\left\|f_1 \star_1^0 f_1\right\|_{\mathfrak{H} \otimes K^{\otimes 2}}$ to converge to 0 to get convergence towards a Gaussian law.

**Example 2.** Let $\mu$ be a $\sigma$-finite measure on some measure space. By setting $K = \mathbb{R}, \mathfrak{H} = L^2(\mu)$ and $X = I_p(f)$ for some $p \geq 2$ in Theorem 4, we get a result comparable to [PSTU10, Theorem 5.1] and [PT08, Theorem 2]. For instance, whenever $X = I_2(f)$, Theorem 4 and [PSTU10, Example 5.2] both state that normal convergence happens if $\left\|f \star_1^1 f\right\|_{L^2(\mu^2)}$, $\|f\|_{L^4(\mu^2)}$ and $\left\|f \star_2^1 f\right\|_{L^2(\mu)}$ converge to 0, keeping in mind that $\|f\|_{L^4(\mu^2)}^2 = \left\|f \star_2^0 f\right\|_{L^2(\mu^2)}$, and $\left\|f \star_1^0 f\right\|_{L^2(\mu^3)} = \left\|f \star_2^1 f\right\|_{L^2(\mu)}$.

Another example is [PSTU10, Example 5.3], which states that $X = I_3(g)$ converges to a Gaussian distribution if $\|g\|_{L^4(\mu^3)}^2$, $\left\|g \star_1^1 g\right\|_{L^2(\mu^4)}$, $\left\|g \star_2^1 g\right\|_{L^2(\mu^3)}$, $\left\|g \star_3^1 g\right\|_{L^2(\mu^2)}$ and $\left\|g \star_3^2 g\right\|_{L^2(\mu)}$ all converge to 0, which is the same condition suggested in Theorem 4.

Further, we would like to mention [ET14, LRP13a, LRP13b] which also offer contraction bounds for normal approximation on the Poisson space.

## 4. Proof of main results

We begin with the proof of Theorem 3 which uses the method of exchangeable pairs developed in Section 2.

### 4.1. Proof of Theorem 3.

Let $G$ be a Gaussian random variable on $K$ with the same covariance operator as $X$, i.e., $G$ has covariance operator $S$. Similarly to [BC20, Corrolary 3.3], it holds that

$$d_3(G, Z) \leq \frac{1}{2} \left\|\mathcal{S} - \mathcal{S}'\right\|_{\mathrm{HS}}.$$

Therefore, it suffices to derive the desired moment bound for $d_3(X, G)$ which yields the first item in Theorem 3 as

$$d_3(X, Z) \leq d_3(X, G) + d_3(G, Z).$$

In Subsection 2.5, we constructed an exchangeable pair of the form $(F_q, F_q^t)$ based on an element of a fixed $K$-valued chaos $F_q$, where $q$ denotes the order of the Poisson chaos. Recall that $X$ has the chaos decomposition (4). It follows that, for any $t \geq 0$, if we define $X^t$ as

$$X^t = \sum_{q=1}^{N} F_q^t,$$

then the pair $(X, X^t)$ is also exchangeable, and we can apply Taylor's theorem to get

$$0 = \lim_{t \to 0} \frac{1}{2t} \mathbb{E}\left[\left\langle -L^{-1}(X^t - X), Dg(X^t) + Dg(X)\right\rangle_K\right]$$
$$= \lim_{t \to 0} \mathbb{E}\left[\frac{1}{2t}\left\langle -L^{-1}(X^t - X), Dg(X^t) - Dg(X)\right\rangle_K + \frac{1}{t}\left\langle -L^{-1}(X^t - X), Dg(X)\right\rangle_K\right]$$
$$= \lim_{t \to 0} \mathbb{E}\left[\frac{1}{2t}\left\langle -L^{-1}(X^t - X), D^2g(X)(X^t - X) + r\right\rangle_K + \frac{1}{t}\left\langle -L^{-1}(X^t - X), Dg(X)\right\rangle_K\right],$$

where $r$ denotes the remainder term. Let $R(t) = \mathbb{E}\left[\frac{1}{2t}\left\langle -L^{-1}(X^t - X), r\right\rangle_K\right]$. Note that $\mathbb{E}[\Delta_G g(X)] = \sum_{1 \leq q \leq N} \mathbb{E}\left[\mathrm{Tr}_K\left(D^2g(X)S_q\right)\right]$. Combined with part (a) and (b) of Lemma 6

and keeping in mind $F_q = \sum_{i\in\mathbb{N}} F_{q,i} k_i$, this leads to

$$
\begin{aligned}
0 = &\sum_{1\le q\le N} \mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\Gamma(F_q, -L^{-1}F_q)\big)\big] \\
&+ \sum_{1\le p\ne q\le N}\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big] - \mathbb{E}[\langle X, Dg(X)\rangle_K] + \lim_{t\to 0} R(t) \\
= &\mathbb{E}[\Delta_G g(X)] - \mathbb{E}[\langle X, Dg(X)\rangle_K] + \sum_{1\le q\le N}\mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma(F_q, -L^{-1}F_q) - S_q\big)\big)\big] \\
&+ \sum_{1\le p\ne q\le N}\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big] + \lim_{t\to 0} R(t).
\end{aligned}
$$

The above equation and the Stein equation introduced in Section 2 imply

$$
\begin{aligned}
d_3(X, G) = &\sup_{h\in C_b^3(K)} |\Delta_G g(X) - \langle X, Dg(X)\rangle_K| \\
\le &\sup_{h\in C_b^3(K)} \left\{ \sum_{1\le q\le N} \big|\mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma(F_q, -L^{-1}F_q) - \mathcal{S}_q\big)\big)\big]\big| \right. \\
&\left. + \sum_{1\le p\ne q\le N}\left|\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[\Big\langle k_i, D^2 g(X)\widetilde{\Gamma}\Big(-\widetilde{L}^{-1}F_{p,i}, F_{q,j}\Big)k_j\Big\rangle_K\Big]\right| + \left|\lim_{t\to 0} R(t)\right| \right\}. \quad (5)
\end{aligned}
$$

For the first term on the right side of (5), it holds that

$$
\begin{aligned}
\sum_{1\le q\le N}& \big|\mathbb{E}\big[\mathrm{Tr}_K\big(D^2 g(X)\big(\Gamma(F_q, -L^{-1}F_q) - S_q\big)\big)\big]\big| \\
&\le \sum_{1\le q\le N} \big\|D^2 g(X)\big\|_{L^2(\Omega;\mathrm{HS}(K))} \left\|\frac{1}{q}\Gamma(F_q, F_q) - S_q\right\|_{L^2(\Omega;\mathrm{HS}(K))} \\
&\le \sum_{1\le q\le N} \frac{1}{2q}\sqrt{\sum_{i,j\in\mathbb{N}} \mathrm{Var}(\Gamma(F_{q,i}, F_{q,j}))} \\
&\le \sum_{1\le q\le N} \frac{2q-1}{4q}\sqrt{\sum_{i,j\in\mathbb{N}} \mathbb{E}\Big[F_{q,i}^2 F_{q,j}^2\Big] - \mathbb{E}\Big[F_{q,i}^2\Big]\mathbb{E}\Big[F_{q,j}^2\Big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2} \\
&= \sum_{1\le q\le N} \frac{2q-1}{4q}\sqrt{\mathbb{E}\Big[\|F_q\|_K^4\Big] - (\mathbb{E}\Big[\|F_q\|_K^2\Big])^2 - 2\|S_q\|_{\mathrm{HS}}^2}.
\end{aligned}
$$

In particular, we have used the fact that $\|D^2 g(x)\|_{K^{\otimes 2}} = \|D^2 g(x)\|_{\mathrm{HS}(K)}$ and [BC20, Lemma 2.4] to get the third line above. The fourth line is a consequence of [DVZ18, Lemma 2.2]. Finally, the identity $\langle Sf, g\rangle_K = \mathbb{E}[\langle X, f\rangle_K \langle X, g\rangle_K]$ allows us to get the term $\|S_q\|_{\mathrm{HS}}$ in the last line.

Now we study the second term on the right side of (5). Application of [BC20, Lemma 2.4] and [DVZ18, Lemma 2.2] gives

$$
\sum_{1 \leq p \neq q \leq N} \left| \sum_{i,j \in \mathbb{N}} \mathbb{E}\left[ \left\langle k_i, D^2 g(X) \widetilde{\Gamma}\left( -\widetilde{L}^{-1} F_{p,i}, F_{q,j} \right) k_j \right\rangle_K \right] \right|
$$

$$
\leq \sum_{1 \leq p \neq q \leq N} \mathbb{E}\left[ \sqrt{ \sum_{i,j \in \mathbb{N}} \left\langle k_i, D^2 g(X) k_j \right\rangle_K } \sqrt{ \sum_{i,j \in \mathbb{N}} \widetilde{\Gamma}\left( -\widetilde{L}^{-1} F_{p,i}, F_{q,j} \right)^2 } \right]
$$

$$
\leq \sum_{1 \leq p \neq q \leq N} \sqrt{ \sum_{i,j \in \mathbb{N}} \mathbb{E}\left[ \left\langle k_i, D^2 g(X) k_j \right\rangle_K^2 \right] } \sqrt{ \sum_{i,j \in \mathbb{N}} \mathbb{E}\left[ \widetilde{\Gamma}\left( -\widetilde{L}^{-1} F_{p,i}, F_{q,j} \right)^2 \right] }
$$

$$
\leq \sum_{1 \leq p \neq q \leq N} \frac{p+q-1}{2p} \left\| D^2 g(X) \right\|_{L^2(\Omega; \mathrm{HS}(K))} \sqrt{ \sum_{i,j \in \mathbb{N}} \mathbb{E}\left[ F_{p,i}^2 F_{q,j}^2 \right] - \mathbb{E}\left[ F_{p,i}^2 \right] \mathbb{E}\left[ F_{q,j}^2 \right] }
$$

$$
\leq \sum_{1 \leq p \neq q \leq N} \frac{p+q-1}{4p} \sqrt{ \mathbb{E}\left[ \left\| F_p \right\|_K^2 \left\| F_q \right\|_K^2 \right] - \mathbb{E}\left[ \left\| F_p \right\|_K^2 \right] \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right] }.
$$

As the last step, we evaluate the remainder term in (5).

$$
\lim_{t \to 0} R(t) \leq \left\| D^3 g \right\|_\infty \lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ \left\| X^t - X \right\|_K^3 \right]
$$

$$
\leq \left( \sqrt{ \lim_{t \to 0} \mathbb{E}\left[ \frac{1}{t} \left\| X^t - X \right\|_K^2 \right] } \sqrt{ \lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ \left\| X^t - X \right\|_K^4 \right] } \right)
$$

$$
\leq \sqrt{ 2N \mathbb{E}\left[ \left\| X \right\|_K^2 \right] } \sqrt{ \sum_{1 \leq q \leq N} 2^{3q-2}(4q-3) \left( \left\| F_q \right\|_K^4 - \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right]^2 - 2 \left\| S_q \right\|_{\mathrm{HS}}^2 \right) },
$$

The second line is a consequence of Hölder's inequality and [BC20, Lemma 2.4]. The third line uses Lemma 7 (which is stated in the appendix). We can hence deduce from (5) that

$$
d_3(X, G) \leq \sum_{1 \leq q \leq N} \frac{2q-1}{4q} \sqrt{ \mathbb{E}\left[ \left\| F_q \right\|_K^4 \right] - \left( \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right] \right)^2 - 2 \left\| S_q \right\|_{\mathrm{HS}}^2 }
$$

$$
+ \sum_{1 \leq p \neq q \leq N} \frac{p+q-1}{4p} \sqrt{ \mathbb{E}\left[ \left\| F_p \right\|_K^2 \left\| F_q \right\|_K^2 \right] - \mathbb{E}\left[ \left\| F_p \right\|_K^2 \right] \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right] }
$$

$$
+ \sqrt{ N \mathbb{E}\left[ \left\| X \right\|_K^2 \right] } \sqrt{ \sum_{1 \leq q \leq N} 2^{3q-1}(4q-3) \left( \left\| F_q \right\|_K^4 - \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right]^2 - 2 \left\| S_q \right\|_{\mathrm{HS}}^2 \right) }. \quad (6)
$$

In order to obtain the second estimate in Theorem 3, observe that

$$
\mathbb{E}\left[ \left\| X \right\|_K^4 \right] - \left( \mathbb{E}\left[ \left\| X \right\|_K^2 \right] \right)^2 - 2 \left\| S \right\|_{\mathrm{HS}}^2 = \sum_{1 \leq q \leq N} \mathbb{E}\left[ \left\| F_q \right\|_K^4 \right] - \left( \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right] \right)^2 - 2 \left\| S_q \right\|_{\mathrm{HS}}^2
$$

$$
+ \sum_{1 \leq p \neq q \leq N} \mathbb{E}\left[ \left\| F_p \right\|_K^2 \left\| F_q \right\|_K^2 \right] - \mathbb{E}\left[ \left\| F_p \right\|_K^2 \right] \mathbb{E}\left[ \left\| F_q \right\|_K^2 \right],
$$

This combined with Lemma 5, the bound at (6) and the fact that

$$\sum_{1 \leq q, p \leq N} \sqrt{y_{q,p}} \leq \sqrt{N^2 \sum_{1 \leq p, q \leq N} y_{q,p}} \qquad \text{for } y_{q,p} \geq 0,$$

$$2^{3q-1} q(4q - 3) \leq 2^{3N-1} N(4N - 3) \qquad \text{for } 1 \leq p \leq N,$$

$$\frac{2q - 1}{4q} \vee \frac{p + q - 1}{4p} \leq \frac{N(2N - 1)}{4} \qquad \text{for } 1 \leq p, q \leq N,$$

, yields

$$d_3(X, G) \leq \left( \frac{N(2N - 1)}{4} + \sqrt{2^{3N-1} N(4N - 3) \mathbb{E}\Big[ \|X\|_K^2 \Big]} \right) \sqrt{\mathbb{E}\Big[ \|X\|_K^4 \Big] - \mathbb{E}\Big[ \|X\|_K^2 \Big]^2 - 2 \|S\|_{\mathrm{HS}}^2}.$$

$$\square$$

We now turn to the proof of Theorem 4, which makes use of the second estimate in Theorem 3.

4.2. **Proof of Theorem 4.** The strategy here consists of making use of the product formula (3) for Poisson multiple integrals in order to represent the quantity $\mathbb{E}\Big[ \|X\|_K^4 \Big] - \mathbb{E}\Big[ \|X\|_K^2 \Big]^2 - 2 \|S\|_{\mathrm{HS}}^2$ which appears in the second estimate of Theorem 3 in term of contraction norms. We begin by noting that this quantity can be written as

$$\mathbb{E}\Big[ \|X\|_K^4 \Big] - \mathbb{E}\Big[ \|X\|_K^2 \Big]^2 - 2 \|S\|_{\mathrm{HS}}^2 = \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p, q \leq N}} \left( \mathbb{E}\big[ F_{q,i}^2 F_{p,j}^2 \big] - \mathbb{E}\big[ F_{q,i}^2 \big] \mathbb{E}\big[ F_{p,j}^2 \big] - 2\mathbb{E}[F_{q,i} F_{p,j}]^2 \right)$$

$$= \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p, q \leq N}} \left( \mathbb{E}\big[ F_{q,i}^2 F_{p,j}^2 \big] - \mathbb{E}\big[ F_{q,i}^2 \big] \mathbb{E}\big[ F_{p,j}^2 \big] \right)$$

$$- 2 \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq q \leq N}} \mathbb{E}[F_{q,i} F_{p,j}]^2.$$

An application of the product formula (3) for Poisson multiple integrals yields

$$F_{q,i} F_{p,j} = \sum_{r=0}^{q \wedge p} r! \binom{q}{r} \binom{p}{r} \sum_{l=0}^{r} \binom{r}{l} I_{q+p-r-l}\left( f_{q,i} \widetilde{\star}_r^l f_{p,j} \right).$$

Now by the orthogonality of Poisson chaos of different orders, one has

$$\mathbb{E}\big[ F_{q,i}^2 F_{p,j}^2 \big] = \sum_{r,s=0}^{q \wedge p} \sum_{\substack{0 \leq l \leq r \\ 0 \leq m \leq s \\ r+l=s+m}} c_{p,q,l,m}(r,s) \left\langle f_{q,i} \widetilde{\star}_r^l f_{p,j}, f_{q,i} \widetilde{\star}_s^m f_{p,j} \right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}}, \qquad (7)$$

where the coefficient $c_{p,q,l,m}(r,s)$ is given by

$$c_{p,q,l,m}(r,s) = r! s! \binom{q}{r} \binom{q}{s} \binom{p}{r} \binom{p}{s} \binom{r}{l} \binom{s}{m} (p + q - r - l)!.$$

Let us define the index set $I$ as

$$I = \Big\{ (r, s, l, m) \in \mathbb{N}^4 : 0 \leq r, s \leq q \wedge p, \ 0 \leq l \leq r, \ 0 \leq m \leq s,$$
$$r + l = s + m, \ (r, s, l, m) \notin \{(0, 0, 0, 0), (q \wedge p, q \wedge p, q \wedge p, q \wedge p)\} \Big\}.$$

Then, using Lemma 8, Equation (7) can be rewritten as

$$
\mathbb{E}\left[F_{q,i}^2 F_{p,j}^2\right] = q!p! \left\|f_{q,i}\right\|_{\mathfrak{H}^{\otimes q}}^2 \left\|f_{p,j}\right\|_{\mathfrak{H}^{\otimes q}}^2 + 2q!^2 \left\langle f_{q,i}, f_{q,j}\right\rangle_{\mathfrak{H}^{\otimes q}}^2
$$
$$
+ a_{p,q}(p \wedge q) \left\|f_{q,i} \star_{q \wedge p}^{q \wedge p} f_{p,j}\right\|_{\mathfrak{H}^{\otimes |q-p|}}^2 \mathbb{1}_{\{q \neq p\}} + \sum_{r=1}^{q \wedge p - 1} b_{p,q}(r) \left\|f_{q,i} \star_r^r f_{p,j}\right\|_{\mathfrak{H}^{\otimes(q+p-2r)}}^2
$$
$$
+ \sum_{(r,s,l,m) \in I} c_{p,q,l,m}(r,s) \left\langle f_{q,i} \widetilde{\star}_r^l f_{p,j}, f_{q,i} \widetilde{\star}_s^m f_{p,j}\right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}},
$$

where the combinatorial coefficients $a_{p,q}(r)$ and $b_{p,q}(r)$ are given by

$$
\begin{cases}
a_{p,q}(r) = p!q! \binom{q}{r}\binom{p}{r} + r!^2 \binom{q}{r}^2 \binom{p}{r}^2 |p-q|! \\[2mm]
b_{p,q}(r) = p!q! \binom{q}{r}\binom{p}{r}
\end{cases}.
$$

Consequently, we hence obtain

$$
\mathbb{E}\left[\|X\|_K^4\right] - \mathbb{E}\left[\|X\|_K^2\right]^2 - 2\|S\|_{\mathrm{HS}}^2 = \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p,q \leq N}} \left( \mathbb{E}\left[F_{q,i}^2 F_{p,j}^2\right] - \mathbb{E}\left[F_{q,i}^2\right]\mathbb{E}\left[F_{p,j}^2\right] - 2\mathbb{E}\left[F_{q,i}F_{p,j}\right]^2 \right)
$$
$$
= \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p \neq q \leq N}} a_{p,q}(p \wedge q) \left\|f_{q,i} \star_{q \wedge p}^{q \wedge p} f_{p,j}\right\|_{\mathfrak{H}^{\otimes |q-p|}}^2
$$
$$
+ \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p,q \leq N}} \sum_{r=1}^{q \wedge p - 1} b_{p,q}(r) \left\|f_{q,i} \star_r^r f_{p,j}\right\|_{\mathfrak{H}^{\otimes(q+p-2r)}}^2
$$
$$
+ \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p,q \leq N \\ (r,s,l,m) \in I}} c_{p,q,l,m}(r,s) \left\langle f_{q,i} \widetilde{\star}_r^l f_{p,j}, f_{q,i} \widetilde{\star}_s^m f_{p,j}\right\rangle_{\mathfrak{H}^{\otimes(q+p-r-l)}}.
$$

Since we have

$$
\left\|f_q \star_r^l f_p\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)} \otimes K^{\otimes 2}}^2 = \sum_{i,j \in \mathbb{N}} \left\|\langle f_q, k_i\rangle_K \star_r^l \langle f_p, k_j\rangle_K\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}}^2 = \sum_{i,j \in \mathbb{N}} \left\|f_{q,i} \star_r^l f_{p,j}\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}}^2,
$$

we can sum over $i,j \in \mathbb{N}$ and apply Holder's inequality to get

$$
\mathbb{E}\left[\|X\|_K^4\right] - \mathbb{E}\left[\|X\|_K^2\right]^2 - 2\|S\|_{\mathrm{HS}}^2 \leq \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p \neq q \leq N}} a_{p,q}(p \wedge q) \left\|f_{q,i} \star_{q \wedge p}^{q \wedge p} f_{p,j}\right\|_{\mathfrak{H}^{\otimes |q-p|}}^2
$$
$$
+ \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p,q \leq N}} \sum_{r=1}^{q \wedge p - 1} b_{p,q}(r) \left\|f_{q,i} \star_r^r f_{p,j}\right\|_{\mathfrak{H}^{\otimes(q+p-2r)}}^2
$$
$$
+ \sum_{\substack{i,j \in \mathbb{N} \\ 1 \leq p,q \leq N \\ (r,s,l,m) \in I}} c_{p,q,l,m}(r,s) \left\|f_q \star_r^l f_p\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}} \left\|f_q \star_s^m f_p\right\|_{\mathfrak{H}^{\otimes(q+p-r-l)}},
$$

which concludes the proof. $\qquad\qquad\square$

## 5. Applications

### 5.1. Brownian approximation of a Poisson process in Besov-Liouville spaces.

5.1.1. *A brief overview of Besov-Liouville spaces.* For an extensive account on the current topic, we invite readers to view [SKM93]. For $f \in L^p([0,1], ds)$ and $\beta > 0$, we define the left and right fractional integrals respectively as

$$\left(I_{0+}^\beta f\right)(s) = \frac{1}{\Gamma(\beta)} \int_0^s (s-r)^{\beta-1} f(r) dr$$

and

$$\left(I_{1-}^\beta f\right)(s) = \frac{1}{\Gamma(\beta)} \int_s^1 (r-s)^{\beta-1} f(r) dr.$$

This allows us to define the Besov-Liouville spaces

$$\mathcal{I}_{\beta,p}^+ = \left\{ I_{0+}^\beta \widehat{f}, \ \widehat{f} \in L^p([0,1]) \right\},$$

which are Banach spaces when equipped with the norm $\|f\|_{\mathcal{I}_{\beta,p}^+} = \left\|\widehat{f}\right\|_{L^p([0,1])}$. The Besov-Liouville spaces $\mathcal{I}_{\beta,p}^-$ are defined accordingly with the right fractional integrals. When $\beta p < 1$, the spaces $\mathcal{I}_{\beta,p}^+$ and $\mathcal{I}_{\beta,p}^-$ are canonically isomorphic and therefore will both be denoted by $\mathcal{I}_{\beta,p}$.

**Remark 4.** As pointed out in [CD13], $\mathcal{I}_{\beta,2}$ for $\beta < 1/2$ is an appropriate class of Besov-Liouville spaces for the functional approximation of a Poisson process by a Brownian motion since they are Hilbert spaces containing both the sample paths of the Poisson process and the Brownian motion.

Similarly to the left and right fractional integrals, one can define left and right fractional derivatives as

$$\left(D_{0+}^\beta f\right)(s) = \frac{1}{\Gamma(1-\beta)} \frac{d}{ds} \int_0^s (s-r)^{-\beta} f(r) dr$$

$$\left(D_{1-}^\beta f\right)(s) = \frac{1}{\Gamma(1-\beta)} \frac{d}{ds} \int_s^1 (r-s)^{-\beta} f(r) dr$$

As the name suggests, $D_{0+}^\beta$ is the inverse of $I_{0+}^\beta$ (see [SKM93, Theorem 2.4]). Two examples for the action of this operator that will be useful later are

$$\left(D_{0+}^\beta \operatorname{Id}\right)(r) = \frac{r^{-\beta+1}}{(-\beta+1)\Gamma(-\beta+1)} \quad \text{and} \quad \left(D_{0+}^\beta 1_{[a,\infty)}\right)(r) = \frac{(r-a)_+^{-\beta}}{\Gamma(-\beta+1)}, \tag{8}$$

where Id denotes the identity function. Let us also mention a few important facts about fractional integrals and derivatives. Given $0 < \beta < 1$ and $1 < p < 1/\beta$, $I_{0+}^\beta$ is a bounded operator from $L^p([0,1])$ to $L^q([0,1])$ with $q = p(1-\beta p)^{-1}$. Moreover, for $\beta > 0$ and $p \geq 1$, $I_{0+}^\beta$ is bounded from $L^p([0,1])$ into itself (see for instance [SKM93, Equation (2.72)]). Next, fractional derivatives are the inverses of fractional integrals, in the sense that

$$\left(D_{0+}^\beta I_{0+}^\beta f\right)(s) = f(s)$$

for $f \in L^1([0,1])$. Furthermore, fractional integrals enjoy the semigroup property (see [SKM93, Theorem 2.5]), that is

$$\left(I_{0+}^\alpha I_{0+}^\beta f\right)(s) = \left(I_{0+}^{\alpha+\beta} f\right)(s)$$

as long as $\beta > 0$, $\alpha + \beta > 0$ and $f \in L^1([0,1])$.

5.1.2. *A functional central limit theorem.* We consider a Poisson process $N_\lambda(t)$ with intensity $\lambda$. It is well known (see for instance [NN18, Example 9.1.3]) that it can be represented as

$$N_\lambda(t) = \sum_{n \in \mathbb{N}} 1_{[T_n, \infty)}(t), \tag{9}$$

where $T_n = \sum_{i=1}^n \alpha_i$ and $\{\alpha_i \colon i \in \mathbb{N}\}$ are independent exponentially distributed random variables with parameter $\lambda$, i.e., $\alpha_i \sim \mathrm{Exp}(\lambda)$ for all $i \in \mathbb{N}$. This implies that $T_n$ is Gamma distributed with shape $n$ and rate $\lambda$, i.e., $T_n \sim \mathrm{Gamma}(n, \lambda)$. As pointed out in [CD13], $N_\lambda(t)$ maps into $\mathcal{I}_{\beta,2}$ for $\beta < 1/2$.

For any $t \in [0, 1]$, define

$$X_\lambda(t) = \frac{N_\lambda(t) - \lambda t}{\sqrt{\lambda}}$$

and let $Z$ be a Brownian motion on $\mathcal{I}_{\beta,2}$, that is a $\mathcal{I}_{\beta,2}$-valued Gaussian random variable with covariance operator

$$S' = I_{0^+}^\beta I_{0^+}^{1-\beta} I_{1^-}^{1-\beta} D_{0^+}^\beta, \tag{10}$$

where the expression of the covariance operator was derived in [CD13]. We are now ready to state the main result of this application, namely the Brownian approximation of a Poisson process in $\mathcal{I}_{\beta,2}$.

**Theorem 5.** *On a Besov-Liouville space $\mathcal{I}_{\beta,2}$ with $\beta < 1/2$, the distributions of $X_\lambda$ and $Z$ are asymptotically close as $\lambda \to \infty$. Their closeness can be quantified by*

$$d_3(X_\lambda, Z) \lesssim \frac{1}{\sqrt{\lambda}}.$$

*Proof.* $X_\lambda(t)$ can be represented as a Poisson multiple integral of order one. Let $\mathfrak{H} = L^2(\mathbb{R}^+, \lambda dx)$ be the underlying Hilbert space to the compensated Poisson process $N_\lambda(t) - \lambda t$. Furthermore, let $f(t) = \frac{1}{\sqrt{\lambda}} 1_{[0,t]} \in \mathfrak{H}$. We can hence write

$$X_\lambda(t) = I_1(f(t)).$$

Theorem 4 then provides us with the estimate

$$d_3(X_\lambda, Z) \lesssim \left\| f \star_1^0 f \right\|_{\mathfrak{H} \otimes K^{\otimes 2}}^2 + \left\| S_\lambda - S' \right\|_{\mathrm{HS}(K)}, \tag{11}$$

where $S_\lambda$ denotes the covariance operator of $X_\lambda$ and where $K = \mathcal{I}_{\beta,2}$. We begin by computing the contraction norm appearing above. We have

$$(f \star_1^0 f)(x) = \frac{1}{\lambda} 1_{[0,t]}(x) 1_{[0,s]}(x) = 1_{[x,\infty)}(t) 1_{[x,\infty)}(s),$$

so that

$$\left\| f \star_1^0 f \right\|_{\mathfrak{H} \otimes K^{\otimes 2}}^2 = \frac{1}{\lambda^2} \int_0^1 \int_0^1 \int_0^1 \left( \left( D_{0^+}^\beta 1_{[x,\infty)} \right)(t) \left( D_{0^+}^\beta 1_{[x,\infty)} \right)(s) \right)^2 \lambda dx ds dt$$

$$= \frac{1}{\lambda \Gamma(-\beta+1)^4} \int_0^1 \int_0^1 (t-x)_+^{-2\beta} (s-x)_+^{-2\beta} ds dt \lesssim \frac{1}{\lambda},$$

where the last inequality simply comes from the fact that $\int_0^1 \int_0^1 (t-x)_+^{-2\beta} (s-x)_+^{-2\beta} ds dt$ is finite.

In order to estimate the remaining term, namely $\|S_\lambda - S'\|_{\mathrm{HS}(K)}$, we apply Lemma 9 and Lemma 10. This yields

$$\|S_\lambda - S'\|_{\mathrm{HS}(K)}^2 = \left\|\mathbb{E}\left[\left(D_{0^+}^\beta X_\lambda\right)(r)\left(D_{0^+}^\beta X_\lambda\right)(s)\right] - \mathbb{E}\left[\left(D_{0^+}^\beta Z\right)(r)\left(D_{0^+}^\beta Z\right)(s)\right]\right\|_{L^2([0,1]^{\otimes 2})}^2$$

$$= \left\|-\frac{\lambda}{\Gamma(-\beta+1)^2(-\beta+1)^2}(r - r \wedge s)^{-\beta+1}(s - r \wedge s)^{-\beta+1}\right\|_{L^2([0,1]^{\otimes 2})}^2 = 0,$$

which concludes the proof.    $\square$

5.2. **Edge counting in random graphs.** In [LRP13a], the authors studied Gaussian fluctuations of real-valued $U$-statistics related to graphs generated by Poisson point processes. We will apply Theorem 4 to obtain a functional version of their results in all three regimes mentioned in [LRP13a, Example 4.13]. Recall from Subsection 2.5 the definition of a proper Poisson point process

$$\eta_\lambda = \sum_{i=1}^{\mathrm{Po}(\lambda)} \delta_{Y_i},$$

where $\mathrm{Po}(\lambda)$ is a Poisson distribution on $\mathbb{R}$, while $\{Y_i\}_{i\in\mathbb{N}}$ is an i.i.d. sequence of $\mathbb{R}^d$-valued random variables distributed as $\ell$ and independent from $\mathrm{Po}(\lambda)$. For simplicity and illustration purposes, let us assume $\ell$ is the Lebesgue measure on $\mathbb{R}^d$. The control measure of $\eta_\lambda$ is therefore

$$\mu_\lambda(\cdot) = \lambda\ell(\cdot).$$

Let $G$ be a graph generated by $\eta_\lambda$, so that $G$ has the vertex set $\{Y_1, \ldots, Y_{\mathrm{Po}(\lambda)}\}$. In addition, let $W \subseteq \mathbb{R}^d$ be a symmetric set which will serve as our original window in which we monitor the edges of $G$, and let $H_\lambda \subseteq \mathbb{R}^{2d}$ be a symmetric set which will serve as our original edge set. For $0 \le t \le 1$, define

$$\begin{cases} W_t = t^{\frac{1}{2d}} W \\ H_{\lambda,t} = t^{\frac{1}{2d}} H_\lambda \\ \widehat{W_t} = \{x - y : x, y \in W_t\} \\ \overline{H}_{\lambda,t} = \{x - y : x, y \in H_{\lambda,t}\} \end{cases}.$$

We will assume that any edge, written in pairs $(x, y)$, belongs to $H_{\lambda,t}$ if and only if $x - y \in \overline{H}_{\lambda,t}$. For example, this property holds for a disk graph with base edge set $\overline{H}_\lambda = B(0, r_\lambda)$, an open ball of radius $r_\lambda$ at the origin. We note that compared to the setup in [LRP13a], our window and edge set are not static but evolve with time.

We are interested in a Poissonized $U$-statistics of the form

$$F_\lambda(t) = \sum_{\substack{(x,y)\in\eta_\lambda^2 \\ x\neq y}} 1_{H_{\lambda,t}\cap W_t^2}(x, y) = \sum_{1=i_1<i_2}^{\mathrm{Po}(\lambda)} 1_{H_{\lambda,t}\cap W_t^2}(Y_{i_1}, Y_{i_2})$$

which counts edges of $G$ that belong to the set $\overline{H}_{\lambda,t}$ and lie inside the window $W_t$ at time $t$. It is clear from the hypothesis that $\{F_\lambda(t)\}_{t\in[0,1]}$ as a process belongs to $K = L^2([0,1])$. As proved in [RS13], our $U$-statistic has a finite chaos expansion given by

$$F_\lambda(t) = \mathbb{E}[F_\lambda(t)] + I_1(f_1(t)) + I_2(f_2(t)),$$

where the (functional) kernels $f_1(t)$ and $f_2(t)$ are given by

$$\begin{cases} f_1(t) = 2 \int_{\mathbb{R}^d} 1_{H_{\lambda,t} \cap W_t^2}(x,y) \lambda dy \\ f_2(t) = 1_{H_{\lambda,t} \cap W_t^2}(x,y) \end{cases}.$$

Let $\bar{F}_\lambda(t)$ denote the centered and normalized version of $F_\lambda(t)$ given by

$$\bar{F}_\lambda(t) = \frac{F_\lambda(t) - \mathbb{E}[F_\lambda(t)]}{\sigma} = I_1(g_1(t)) + I_2(g_2(t)),$$

where $\sigma^2 = \mathrm{Var}(F_\lambda(1))$, $g_1(t) = \frac{f_1(t)}{\sigma}$ and $g_2(t) = \frac{f_2(t)}{\sigma}$. For convenience, we will also write $\ell_t$ for $\ell(W_t)$ and $\psi_{\lambda,t}$ for $\ell\left(\overline{H}_{\lambda,t} \cap \widehat{W}_t\right)$. Using the scaling properties of the Lebesgue measure, we can write

$$\ell_t = \sqrt{t}\ell_1 \quad \text{and} \quad \psi_{\lambda,t} = \sqrt{t}\psi_{\lambda,1}.$$

We can actually compute $\sigma^2$ explicitly, using the orthogonality of Wiener chaos of different orders and the isometry property of Poisson multiple integrals. This yields

$$\sigma^2 = \|f_1(1)\|_{L^2(\mu_\lambda)}^2 + \|f_2(1)\|_{L^2(\mu_\lambda^2)}^2$$

$$= 4\lambda^3 \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} 1_{W_1}(x) 1_{\overline{H}_{\lambda,1} \cap \widehat{W}_1}(y-x) d(y-x) \right)^2 dx + \int_{\mathbb{R}^{2d}} 1_{H_{\lambda,1} \cap W_1^2}(x,y) \lambda^2 dx dy$$

$$= 4\ell_1 \lambda^3 \psi_{\lambda,1}^2 + \ell_1 \lambda^2 \psi_{\lambda,1}.$$

Based on the above expression for $\sigma^2$, we can consider three different regimes (similarly to what was done in [LRP13a]), namely

- Regime 1: $\lambda \psi_{\lambda,1} \to \infty$ as $\lambda \to \infty$;
- Regime 2: $\lambda \psi_{\lambda,1} \to 1$ for $c > 0$ as $\lambda \to \infty$;
- Regime 3: $\lambda \psi_{\lambda,1} \to 0$ and $\lambda \sqrt{\psi_{\lambda,1}} \to \infty$ as $\lambda \to \infty$.

Within Regime 1, $\sigma^2$ is dominated by $\|f_1(1)\|_{L^2(\mu_\lambda)}^2$ for large values of $\lambda$, which implies

$$\sigma^2 \asymp 4\ell_1 \lambda^3 \psi_{\lambda,1}^2,$$

whereas in Regime 2, we get

$$\sigma^2 \asymp 4\ell_1 \lambda^3 \psi_{\lambda,1}^2 \asymp \ell_1 \lambda^2 \psi_{\lambda,1},$$

and finally in Regime 3, it holds that

$$\sigma^2 \asymp \ell_1 \lambda^2 \psi_{\lambda,1}.$$

We are now ready to present the application of our results to edge counting in random graphs.

**Theorem 6.** *As $\lambda \to \infty$, $\bar{F}_\lambda(t)$ converges in $K = L^2([0,1])$ to a $K$-valued Gaussian random variable $Z$ with covariance function $\phi(s,t) = \mathbb{E}[Z(s)Z(t)]$. More specifically,*

- *In Regime 1, $\phi(t,s) = \sqrt{ts(t \wedge s)}$ and*

$$d_3(\bar{F}_\lambda, Z) \lesssim \lambda^{-\frac{1}{2}} + \frac{1}{\lambda \psi_{\lambda,1}};$$

- *In Regime 2, $\phi(t,s) = \frac{4\sqrt{ts(t \wedge s)} + t \wedge s}{5}$ and*

$$d_3(\bar{F}_\lambda, Z) \lesssim \lambda^{-\frac{1}{2}} + |\lambda \psi_{\lambda,1} - 1|;$$

- *In Regime 3, $\phi(t,s) = t \wedge s$ which implies that $Z$ is a Brownian motion, and*

$$d_3(\bar{F}_\lambda, Z) \lesssim \lambda^{-1} \psi_{\lambda,1}^{-1/2} + \lambda \psi_{\lambda,1}.$$

*Proof.* In order to make use of Theorem 4, we will need to evaluate contraction norms, but also the Hilbert-Schmidt norm of the difference between the covariance operators, i.e., $\|S - S'\|_{\mathrm{HS}}$. Let us start with this term before we turn to the contraction norms themselves. As before, $S_\lambda$ and $S'$ denotes the covariance operator of $\bar{F}_\lambda$ and $Z$ respectively. Based on [HE15, Theorem 7.4.3] and how Hilbert-Schmidt norms are defined for integral operators, we can use

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} = \left\|\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big] - \mathbb{E}[Z(t)Z(s)]\right\|_{L^2([0,1]^{\otimes 2})}$$
$$\leq \left\|\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big] - \mathbb{E}[Z(t)Z(s)]\right\|_\infty.$$

Our task is hence to compute $\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big]$. We have

$$\langle f_1(t), f_1(s)\rangle_{L^2(\mu_\lambda)} = 4\lambda^3\psi_{\lambda,t}\psi_{\lambda,s}\ell_{t\wedge s} = \sqrt{ts(t\wedge s)}4\ell_1\lambda^3\psi_{\lambda,1}^2$$

and

$$\langle f_2(t), f_2(s)\rangle_{L^2(\mu_\lambda^2)} = \lambda^2\psi_{\lambda,t\wedge s}\ell_{t\wedge s} = (t\wedge s)\ell_1\lambda^2\psi_{\lambda,1},$$

so that

$$\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big] = \frac{\langle f_1(t), f_1(s)\rangle_{L^2(\mu_\lambda)} + \langle f_2(t), f_2(s)\rangle_{L^2(\mu_\lambda^2)}}{\sigma^2}$$
$$= \frac{\sqrt{ts(t\wedge s)}4\lambda\psi_{\lambda,1} + t\wedge s}{4\lambda\psi_{\lambda,1} + 1}.$$

At this step, we need to differentiate our analysis depending on what regime we are in.

**Regime 1:** We assume here that $\lambda\psi_{\lambda,1} \to \infty$. The limiting covariance operator $S'$ then has covariance function $\phi(t,s) = \sqrt{ts(t\wedge s)}$. We can use the fact that for $a \ll A, b \ll B$,

$$\left|\frac{A+a}{B+b} - \frac{A}{B}\right| \lesssim \left|\frac{a}{B}\right| + \left|\frac{b}{B}\right|$$

in order to deduce that

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} \leq \sup_{1\leq s,t\leq M}\left|\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big] - \phi(t,s)\right| \lesssim \frac{1}{\lambda\psi_{\lambda,1}}. \tag{12}$$

**Regime 2:** Here, $\lambda\psi_{\lambda,1} \to 1$, so that the limiting covariance function is given by $\phi(t,s) = \frac{4\sqrt{ts(t\wedge s)}+t\wedge s}{5}$. Moreover,

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} \leq \sup_{1\leq s,t\leq M}\left|\mathbb{E}\big[\bar{F}_\lambda(t)\bar{F}_\lambda(s)\big] - \phi(t,s)\right|$$
$$= \sup_{1\leq s,t\leq M}\left|\frac{4\sqrt{ts(t\wedge s)}\lambda\psi_{\lambda,1} + t\wedge s}{4\lambda\psi_{\lambda,1} + 1} - \frac{4\sqrt{ts(t\wedge s)} + t\wedge s}{5}\right| \lesssim |\lambda\psi_{\lambda,1} - 1|. \tag{13}$$

**Regime 3:** The fact that $\lambda\psi_{\lambda,1} \to 0$ implies in this case that the limiting covariance function is given by $\phi(t,s) = t \wedge s$, and we hence have

$$\left\|S_\lambda - S'\right\|_{\mathrm{HS(K)}} \lesssim \frac{\lambda^3\psi_{\lambda,1}^2}{\lambda^2\psi_{\lambda,1}} \asymp \lambda\psi_{\lambda,1}. \tag{14}$$

We now turn to the second part of the bound appearing in Theorem 4, namely the contraction norms. We need to evaluate the norms of $g_1(t) \star_1^0 g_1(t)$, $g_1(t) \star_1^0 g_2(t)$, $g_1(t) \star_1^1 g_2(t)$, $g_2(t) \star_1^0 g_2(t)$, $g_2(t) \star_2^0 g_2(t)$ and $g_2(t) \star_1^1 g_2(t)$. The calculations we need to perform are very similar to the ones appearing in the proof of [LRP13a, Theorem 4.7], hence we will not provide full details

and proceed straight to the result. Let us still include two examples of these calculations (the cases of the contractions $g_1(t) \star_1^0 g_1(t)$ and $g_2(t) \star_1^1 g_2(t)$) for the reader's convenience and for the sake of staying self-contained. Recall that $W_t$, $H_{\lambda,t}$ are symmetric sets, $W_t$ (respectively $H_{\lambda,t}$) is contained in $W_{t'}$ (respectively $H_{\lambda,t'}$) for $t \leq t'$, and that $\psi_{\lambda,t} = \sqrt{t}\psi_{\lambda,1}$, while $\ell_t = \sqrt{t}\ell_1 < \infty$. We can then write

$$\left\| f_1(t) \star_1^0 f_1(s) \right\|_{L^2(\mu_\lambda)\otimes K^{\otimes 2}}^2$$

$$= \left\| \int_{\mathbb{R}^d} \left( 4\int_{Z^2} 1_{H_{\lambda,t}\cap W_t^2}(x,y)1_{H_{\lambda,s}\cap W_s^2}(x,u)\lambda dy\lambda du \right)^2 \lambda dx \right\|_{K^{\otimes 2}}$$

$$\leq 16\lambda^5 \left\| \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{2d}} 1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,y)1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,u)dydu \right)^2 dx \right\|_{K^{\otimes 2}}$$

$$\asymp \lambda^5 \left\| \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{2d}} 1_{W_{s\vee t}}(x)1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-x)1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(u-x)d(y-x)d(u-x) \right)^2 dx \right\|_{K^{\otimes 2}}$$

$$\asymp \lambda^5 \left\| \ell_{s\vee t}\psi_{\lambda,s\vee t}^4 \right\|_{K^{\otimes 2}} \asymp \lambda^5 \psi_{\lambda,1}^4$$

and

$$\left\| f_2(t) \star_1^1 f_2(s) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2$$

$$\leq \left\| \int_{\mathbb{R}^{2d}} \left( \int_{\mathbb{R}^d} 1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,y)1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,u)\lambda dx \right)^2 \lambda^2 dydu \right\|_{K^{\otimes 2}}$$

$$= \lambda^4 \left\| \int_{\mathbb{R}^{4d}} 1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,y)1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(x,u)1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(v,y)1_{H_{\lambda,s\vee t}\cap W_{s\vee t}^2}(v,u)dxdydudv \right\|_{K^{\otimes 2}}$$

$$\leq \lambda^4 \left\| \int_{\mathbb{R}^{4d}} 1_{W_{s\vee t}}(x)1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-x)1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(u-x)1_{W_{s\vee t}}(x)1_{\overline{H}_{\lambda,s\vee t}\cap \widehat{W}_{s\vee t}}(y-v) \right.$$
$$\left. dxd(y-x)d(u-v)d(v-y) \right\|_{K^{\otimes 2}}$$

$$\asymp \lambda^4 \left\| \ell_{s\vee t}\psi_{\lambda,s\vee t}^3 \right\|_{K^{\otimes 2}} \asymp \lambda^4 \psi_{\lambda,1}^3.$$

For the remaining contractions, performing similar calculations yields $\left\| f_1(t) \star_1^0 f_2(t) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2 \lesssim \lambda^4 \psi_{\lambda,1}^3$, $\left\| f_2(t) \star_1^0 f_2(t) \right\|_{L^2(\mu_\lambda^3)\otimes K^{\otimes 2}}^2 \lesssim \lambda^3 \psi_{\lambda,1}^2$, $\left\| f_2(t) \star_2^0 f_2(t) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2 \lesssim \lambda^2 \psi_{\lambda,1}$, and finally $\left\| f_1(t) \star_1^1 f_2(t) \right\|_{L^2(\mu_\lambda)\otimes K^{\otimes 2}}^2 \lesssim \lambda^5 \psi_{\lambda,1}^4$. We split the remainder of the proof into three cases corresponding to the three possible regimes.

**Regime 1:** Here, $\lambda\psi_{\lambda,1} \to \infty$ as $\lambda \to \infty$, and since $\sigma^2 \asymp \lambda^3\psi_{\lambda,1}^2$, we have $\left\| g_1(t) \star_1^0 g_1(t) \right\|_{L^2(\mu_\lambda)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-1}$, $\left\| g_1(t) \star_1^0 g_2(t) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-2}\psi_{\lambda,1}^{-1}$, $\left\| g_2(t) \star_1^0 g_2(t) \right\|_{L^2(\mu_\lambda^3)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-3}\psi_{\lambda,1}^{-2}$, $\left\| g_2(t) \star_2^0 g_2(t) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-4}\psi_{\lambda,1}^{-3}$, $\left\| g_2(t) \star_1^1 g_2(t) \right\|_{L^2(\mu_\lambda^2)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-2}\psi_{\lambda,1}^{-1}$ and lastly $\left\| g_1(t) \star_1^1 g_2(t) \right\|_{L^2(\mu_\lambda)\otimes K^{\otimes 2}}^2 \lesssim \lambda^{-1}$. Note that all the above estimates are asymptotically bounded from above by $\lambda^{-1}$, and using (12), the estimate in Theorem 4 yields

$$d_3\left(\bar{F}_\lambda, Z\right) \lesssim \lambda^{-\frac{1}{2}} + \frac{1}{\lambda\psi_{\lambda,1}}.$$

**Regime 2:** As in this case, we have $\lambda\psi_{\lambda,1} \to 1$ as $\lambda \to \infty$, we get $\sigma^2 \asymp \lambda^3\psi_{\lambda,1}^2 \asymp \lambda^2\psi_{\lambda,1}$.

Therefore, we can reuse the computations from Regime 1 combined with (13) to get

$$d_3(\bar{F}_\lambda, Z) \lesssim \lambda^{-\frac{1}{2}} + |\lambda \psi_{\lambda,1} - 1|.$$

**Regime 3:** In this regime, $\lambda \psi_{\lambda,1} \to 0$ and $\lambda \sqrt{\psi_{\lambda,1}} \to \infty$ as $\lambda \to \infty$, so that $\sigma^2 \asymp \lambda^2 \psi_{\lambda,1}$. This allows us to deduce that $\left\| g_1(t) \star_1^0 g_1(t) \right\|_{L^2(\mu_\lambda) \otimes K^{\otimes 2}}^2 \lesssim \lambda \psi_{\lambda,1}^2$, $\left\| g_1(t) \star_1^0 g_2(t) \right\|_{L^2(\mu_\lambda^2) \otimes K^{\otimes 2}}^2 \lesssim \psi_{\lambda,1}$, $\left\| g_2(t) \star_1^0 g_2(t) \right\|_{L^2(\mu_\lambda^3) \otimes K^{\otimes 2}}^2 \lesssim \lambda^{-1}$, $\left\| g_2(t) \star_2^0 g_2(t) \right\|_{L^2(\mu_\lambda^2) \otimes K^{\otimes 2}}^2 \lesssim \lambda^{-2} \psi_{\lambda,1}^{-1}$, $\left\| g_2(t) \star_1^1 g_2(t) \right\|_{L^2(\mu_\lambda^2) \otimes K^{\otimes 2}}^2 \lesssim \psi_{\lambda,1}$ and $\left\| g_1(t) \star_1^1 g_2(t) \right\|_{L^2(\mu_\lambda) \otimes K^{\otimes 2}}^2 \lesssim \lambda \psi_{\lambda,1}^2$. Since $\lambda^{-2} \ll \psi_{\lambda,1} \ll \lambda^{-1}$, all terms listed are asymptotically bounded by $\lambda^{-2} \psi_{\lambda,1}^{-1}$. Combining this fact with (14) yields

$$d_3(\bar{F}_\lambda, Z) \lesssim \lambda^{-1} \psi_{\lambda,1}^{-1/2} + \lambda \psi_{\lambda,1},$$

which concludes the proof. $\qquad \square$

## Appendix

This section gathers ancillary lemmas used in the proofs of our main results as well as in the different applications presented in this paper.

### 5.3. **Lemmas related to the proofs of Theorems 3 and 4.** Our first lemma is a crucial result from [DVZ18] which we restate here for convenience.

**Lemma 3.** *Let $p, q \geq 1$ be integers, and let $F_q = I_q^\eta(f_q), G_p = I_p^\eta(g_p)$ and $F_q^t = I_q^{\eta^t}(f_q), G_p^t = I_p^{\eta^t}(g_p)$ be real-valued Poisson multiple integrals as constructed in Section 2. Then, the following limits hold almost surely.*

*(a)* $\lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ F_q^t - F_q | \eta \right] = -q F$
*(b)* $\lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ (F_q^t - F_q)(G_p^t - G_p) | \eta \right] = 2\widetilde{\Gamma}(F_q, G_p)$
*(c)* $\lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ F_q^t (G_p^t - G_p) | \eta \right] = 2\widetilde{\Gamma}(F_q, G_p) - p F_q G_p$
*(d)* $\lim_{t \to 0} \frac{1}{t} \mathbb{E}\left[ (F_q^t - F_q)^4 \right] = -4q \mathbb{E}\left[ F_q^4 \right] + 12 \mathbb{E}\left[ F_q^2 \widetilde{\Gamma}(F_q, F_q) \right].$

*Proof.* The proof of part $(a)$, $(b)$ and $(d)$ are in [DVZ18, Proposition 3.2]. Part $(c)$ is a consequence of $(a)$ and $(b)$. $\qquad \square$

Our next lemma states a more general version of Lemma 3, part $(d)$.

**Lemma 4.** *Let $(X, X^t)$ be an exchangeable pair such that $X = \sum_{q \in \mathbb{N}} I_q^\eta(x_q)$ and $X^t = \sum_{q \in \mathbb{N}} I_q^{\eta^t}(x_q)$. Let the pairs $(Y, Y^t), (U, U^t)$ and $(V, V^t)$ be defined in the same way. Then, one has*

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E}\left[ (X^t - X)(Y_t - Y)(U^t - U)(V^t - V) \right] = 4\mathbb{E}\left[ \widetilde{\Gamma}(X, Y) UV + \widetilde{\Gamma}(X, V) YU \right.$$
$$\left. + \widetilde{\Gamma}(X, U) YV + \widetilde{L} XYUV \right].$$

*Proof.* This limit is a consequence of exchangeability and Lemma 3. Indeed, denoting

$$M_t = \frac{1}{t} \mathbb{E}\left[ (X^t - X)(Y_t - Y)(U^t - U)(V^t - V) \right],$$

we can write

$$
\begin{aligned}
\lim_{t\to\infty} M_t =& 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[XYUV - X^tYUV - XY^tUV - XYU^tV - XYUV^t\big] \\
&+ 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[X^tY^tUV + X^tYU^tV + X^tYUV^t\big] \\
=& 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[-(X^t - X)YUV - X(Y^t - Y)UV - XY(U^t - U)V - XYU(V^t - V)\big] \\
&+ 2\lim_{t\to\infty}\frac{1}{t}\mathbb{E}\big[(X^tY^t - XY)UV + (X^tU^t - XU)YV + (X^tV^t - XV)YU\big] \\
=& 2\mathbb{E}\big[-\widetilde{L}XYUV - X\widetilde{L}YUV - XY\widetilde{L}UV - XYU\widetilde{L}V + \widetilde{L}(XY)UV + \widetilde{L}(XU)YV \\
&+ \widetilde{L}(XV)YU\big] \\
=& 4\mathbb{E}\Big[\widetilde{\Gamma}(X,Y)UV + \widetilde{\Gamma}(X,V)YU + \widetilde{\Gamma}(X,U)YV + \widetilde{L}XYUV\Big].
\end{aligned}
$$

$\square$

**Lemma 5.** *Let $X = \sum_{q=1}^N F_q$, where $F_q \in \mathcal{H}^q(K)$ with covariance operator $S_q$. Furthermore, letting $\{k_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $K$, $F_q$ can be written as $\sum_{i\in\mathbb{N}} F_{q,i}k_i$, where $F_{q,i} = \langle F_q, k_i\rangle_K$. Then, it holds that*

$$
\mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{p,j}]^2 \geq 0,
$$

*which leads to*

$$
\mathbb{E}\Big[\|F_q\|_K^4\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\|S_q\|_{\mathrm{HS}}^2 \geq 0
$$

*and*

$$
\mathbb{E}\Big[\|F_q\|_K^2\|F_p\|_K^2\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]\mathbb{E}\Big[\|F_p\|_K^2\Big] \geq 0 \text{ when } q \neq p.
$$

*Proof.* By [DVZ18, Section 5], we have that

$$
\mathbb{E}\big[J_{p+q}(F_{q,i}F_{p,j})^2\big] \geq \mathbb{E}[F_{q,i}F_{p,j}]^2 + \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big],
$$

which implies that

$$
\begin{aligned}
\mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{p,j}]^2 \geq& \mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}[F_{q,i}F_{p,j}]^2 - \mathbb{E}\big[J_{p+q}(F_{q,i}F_{p,j})^2\big] \\
\geq& \mathbb{E}\Bigg[\sum_{m=1}^{p+q-1} J_m(F_{q,i}F_{p,j})^2\Bigg] \\
\geq& 0.
\end{aligned}
$$

The second and third inequalities in the statement of our lemma immediately follow, since

$$
\begin{aligned}
\mathbb{E}\Big[\|F_q\|_K^4\Big] - (\mathbb{E}\Big[\|F_q\|_K^2\Big])^2 - 2\|S_q\|_{\mathrm{HS}}^2 =& \sum_{i,j\in\mathbb{N}} \mathbb{E}\big[F_{q,i}^2 F_{q,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2 \\
\geq& 0
\end{aligned}
$$

and when $q \neq p$,

$$
\mathbb{E}\Big[\|F_q\|_K^2\|F_p\|_K^2\Big] - \mathbb{E}\Big[\|F_q\|_K^2\Big]\mathbb{E}\Big[\|F_p\|_K^2\Big] = \sum_{i,j\in\mathbb{N}} \mathbb{E}\big[F_{q,i}^2 F_{p,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{p,j}^2\big] \geq 0.
$$

$\square$

The upcoming lemma is a version of Lemma 3 in the setting of Hilbert-valued random variables.

**Lemma 6.** *Let $X = \sum_{q=1}^N F_q$, where $F_q \in \mathcal{H}^q(K)$ with covariance operator $S_q$. It holds that*

(a) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[\left\langle F_q^t - F_q, Dg(X)\right\rangle_K\right] = -q\mathbb{E}\left[\langle F_q, Dg(X)\rangle_K\right]$.

(b) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[\left\|F_q^t - F_q\right\|_K^2\right] = 2q\mathbb{E}\left[\|F_q\|_K^2\right]$.

(c) $\lim_{t\to 0} \frac{1}{2t}\mathbb{E}\left[\left\langle -L^{-1}\left(F_q^t - F_q\right), D^2g(X)(F_p^t - F_p)\right\rangle_K\right]$

$$= \frac{1}{q}\sum_{i,j\in\mathbb{N}} \mathbb{E}\left[\widetilde{\Gamma}(F_{q,i}, F_{p,j})\left\langle k_i, D^2g(X)k_j\right\rangle_K\right].$$

(d) $\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[\left\|F_q^t - F_q\right\|_K^4\right] = 4\sum_{i,j\in\mathbb{N}} \mathbb{E}\left[F_{q,i}^2\left(\widetilde{\Gamma}(F_{q,j}, F_{q,j}) - q\mathbb{E}[F_{q,j}]\right)\right]$

$$+ 8\sum_{i,j\in\mathbb{N}} \mathbb{E}\left[F_{q,i}F_{q,j}\left(\widetilde{\Gamma}(F_{q,i}, F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}]\right)\right]$$

$$- 4q\sum_{i,j\in\mathbb{N}} \left(\mathbb{E}\left[F_{q,i}^2 F_{q,j}^2\right] - \mathbb{E}\left[F_{q,i}^2\right]\mathbb{E}\left[F_{q,j}^2\right] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\right).$$

*In particular, when $q = p$ then part (c) becomes*

$$\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle -L^{-1}\left(F_q^t - F_q\right), D^2g(X)(F_q^t - F_q)\right\rangle_K\right] = \mathrm{Tr}_K\left(D^2g(X)\Gamma\left(F_q, -L^{-1}F_q\right)\right).$$

*Proof.* Part $(a)$ follows from

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\langle F_q^t - F_q, Dg(X)\right\rangle_K\right] = \lim_{t\to 0}\frac{1}{t}\sum_{i\in\mathbb{N}}\mathbb{E}\left[\left\langle (F_{q,i}^t - F_{q,i})k_i, Dg(X)\right\rangle_K\right]$$

$$= \sum_{i\in\mathbb{N}}\mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[F_{q,i}^t - F_{q,i}|\eta\right]\langle k_i, Dg(X)\rangle_K\right]$$

$$= -q\sum_{i\in\mathbb{N}}\mathbb{E}[F_{q,i}\langle k_i, Dg(X)\rangle_K]$$

$$= -q\mathbb{E}\left[\langle F_q, Dg(X)\rangle_K\right].$$

Part $(b)$ is a result of

$$\mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t - F_q\right\|_K^2 |\eta\right]\right] = \mathbb{E}\left[\sum_{i\in\mathbb{N}}\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[(F_{q,i}^t - F_{q,i})^2 |\eta\right]\right]$$

$$= 2\sum_{i\in\mathbb{N}}\mathbb{E}\left[\widetilde{\Gamma}(F_{q,i}, F_{q,i})\right]$$

$$= 2q\mathbb{E}\left[\|F_q\|_K^2\right].$$

For part $(c)$, we can write

$$\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle -L^{-1}\left(F_q^t - F_q\right), D^2g(X)(F_p^t - F_p)\right\rangle_K\right]$$

$$= \lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[\left\langle \sum_{i\in\mathbb{N}}\frac{1}{q}(F_{q,i}^t - F_{q,i})k_i, D^2g(X)\sum_{j\in\mathbb{N}}(F_{p,j}^t - F_{p,j})k_j\right\rangle_K\right]$$

$$= \frac{1}{q}\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[\lim_{t\to 0}\frac{1}{2t}\mathbb{E}\left[(F_{q,i}^t - F_{q,i})(F_{p,j}^t - F_{p,j})|\eta\right]\langle k_i, D^2g(X)k_j\rangle_K\right]$$

$$= \frac{1}{q}\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[\widetilde{\Gamma}(F_{q,i}, F_{p,j})\langle k_i, D^2g(X)k_j\rangle_K\right].$$

Using the above expression in the case $q = p$, along with the fact that

$$\Gamma(F_q, F_q)k_j = \Gamma\left(\sum_{i\in\mathbb{N}} F_{q,i}k_i, \sum_{m\in\mathbb{N}} F_{q,m}k_m\right)k_j = \sum_{i,m\in\mathbb{N}} \Gamma(F_{q,i}k_i, F_{q,m}k_m)k_j$$

$$= \sum_{i,m\in\mathbb{N}} \frac{1}{2}\widetilde{\Gamma}(F_{q,i}, F_{q,m})(k_i \otimes k_m + k_m \otimes k_i)k_j$$

$$= \sum_{i\in\mathbb{N}} \widetilde{\Gamma}(F_{q,i}, F_{q,j})k_i$$

yields

$$\lim_{t\to 0} \frac{1}{2t}\mathbb{E}\left[\left\langle -L^{-1}\left(F_q^t - F_q\right), D^2 g(X)(F_q^t - F_q)\right\rangle_K\right] = \mathrm{Tr}_K\left(D^2 g(X)\Gamma\left(F_q, -L^{-1}F_q\right)\right).$$

For part $(d)$, the exchangeability of $\left(F_q, F_q^t\right)$ and Lemma 4 imply

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|F_q^t - F_q\right\|_K^4\right] = \lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|\sum_{i\in\mathbb{N}}(F_{q,i}^t - F_{q,i})k_i\right\|_K^4\right]$$

$$= \lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\sum_{i,j\in\mathbb{N}}\left(F_{q,i}^t - F_{q,i}\right)^2\left(F_{q,j}^t - F_{q,j}\right)^2\right]$$

$$= 4\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}^2\left(\widetilde{\Gamma}(F_{q,j}, F_{q,j}) - q\mathbb{E}\left[F_{q,j}^2\right]\right)\right]$$

$$+ 8\sum_{i,j\in\mathbb{N}}\mathbb{E}\left[F_{q,i}F_{q,j}\left(\widetilde{\Gamma}(F_{q,i}, F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}]\right)\right]$$

$$- 4q\sum_{i,j\in\mathbb{N}}\left(\mathbb{E}\left[F_{q,i}^2 F_{q,j}^2\right] - \mathbb{E}\left[F_{q,i}^2\right]\mathbb{E}\left[F_{q,j}^2\right] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\right).$$

$\square$

The next result provides upper bounds on the limits appearing in Lemma 6, part $(b)$ and $(d)$.

**Lemma 7.** *Let $X = \sum_{q=1}^N F_q$, where $F_q \in \mathcal{H}^q(K)$ with covariance operator $S_q$. It holds that*

$$\lim_{t\to 0}\mathbb{E}\left[\frac{1}{t}\left\|X^t - X\right\|_K^2\right] \le 2N\mathbb{E}\left[\|X\|_K^2\right]$$

*and*

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|X^t - X\right\|_K^4\right] \le \sum_{1\le q\le N} 2^{3q-2}(4q-3)\left(\|F_q\|_K^4 - \mathbb{E}\left[\|F_q\|_K^2\right]^2 - 2\|S_q\|_{\mathrm{HS}}^2\right).$$

*Proof.* The first bound follows from

$$\mathbb{E}\left[\lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left\|X^t - X\right\|_K^2 \big|\eta\right]\right] = 2\sum_{\substack{i\in\mathbb{N}\\p,q\le N}}\mathbb{E}\left[\widetilde{\Gamma}(F_{p,i}, F_{q,i})\right]$$

$$= \sum_{\substack{i\in\mathbb{N}\\p\le N}} 2p\mathbb{E}\left[F_{p,i}^2\right]$$

$$\le 2N\mathbb{E}\left[\|X\|_K^2\right].$$

For the second estimate, we start by using the triangle inequality and the $c_r$-inequality (see for example [Gut13, Thm. 2.2, p.127]) to write

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\|X^t - X\|_K^4\Big] &\leq \lim_{t\to 0}\frac{1}{t}\mathbb{E}\left[\left(\sum_{1\leq q\leq N}\|F_q^t - F_q\|_K\right)^4\right]\\
&\leq \sum_{1\leq q\leq N}8^{q-1}\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\|F_q^t - F_q\|_K^4\Big].
\end{aligned}
\tag{15}
$$

Regarding the previous expression, Lemma 6 says

$$
\begin{aligned}
\lim_{t\to 0}\frac{1}{t}\mathbb{E}\Big[\|F_q^t - F_q\|_K^4\Big] =&\, 4\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}^2\Big(\widetilde{\Gamma}(F_{q,j}, F_{q,j}) - q\mathbb{E}[F_{q,j}^2]\Big)\Big]\\
&+ 8\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}F_{q,j}\Big(\widetilde{\Gamma}(F_{q,i}, F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}]\Big)\Big]\\
&- 4q\sum_{i,j\in\mathbb{N}}\Big(\mathbb{E}\big[F_{q,i}^2 F_{q,j}^2\big] - \mathbb{E}\big[F_{q,i}^2\big]\mathbb{E}\big[F_{q,j}^2\big] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2\Big).
\end{aligned}
\tag{16}
$$

We will treat each term of (16) separately. For the first term of (16), our proof will use an argument similar to the proof of [DVZ18, Lemma 2.2] or [DP18, Lemma 3.1]. First, observe that if $k$ is a fixed positive integer and $J_k$ denotes the projection into the $k$-th Poisson chaos, then

$$
\mathbb{E}\left[J_k\Big(\|F_p\|_K^2\Big)^2\right] = \sum_{i,j\in\mathbb{N}}\mathbb{E}\big[J_k\big(F_{q,i}^2\big)J_k\big(F_{q,j}^2\big)\big].
$$

In particular, the expansion in [DVZ18, Lemma 5.1] yields

$$
\begin{aligned}
\mathbb{E}\left[J_{2q}\Big(\|F_q\|_K^{\otimes 2}\Big)^2\right] &= \sum_{i,j\in\mathbb{N}}(2q)!\,\big\langle f_{q,i}\widetilde{\otimes}f_{q,i}, f_{q,j}\widetilde{\otimes}f_{q,j}\big\rangle_{\mathfrak{H}^{2q}}\\
&= \sum_{i,j\in\mathbb{N}}\left(2\mathbb{E}[F_{q,i}F_{q,j}]^2 + \sum_{r=1}^{q-1}q!^2\binom{q}{r}^2\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{2q-2r}}\right).
\end{aligned}
$$

Thus, the first term of (16) can be bounded via

$$
\begin{aligned}
\sum_{i,j\in\mathbb{N}}\mathbb{E}\Big[F_{q,i}^2\Big(\widetilde{\Gamma}(F_{q,j}, F_{q,j}) - q\mathbb{E}\big[F_{q,j}^2\big]\Big)\Big] \leq&\, \frac{1}{2}\sum_{i,j\in\mathbb{N}}\sum_{k=1}^{2q-1}(2p-k)\mathbb{E}\big[J_k\big(F_{q,i}^2\big)J_k\big(F_{q,j}^2\big)\big]\\
=&\, \frac{1}{2}\sum_{k=1}^{2q-1}(2p-k)\mathbb{E}\left[J_k\Big(\|F_q\|_K^2\Big)^2\right]\\
\leq&\, \frac{2q-1}{2}\sum_{k=1}^{2q-1}\mathbb{E}\left[J_k\Big(\|F_q\|_K^2\Big)^2\right]\\
=&\, \frac{2q-1}{2}\left(\|F_q\|_K^4 - \mathbb{E}\Big[\|F_q\|_K^2\Big]^2 - 2\|S_q\|_{\mathrm{HS}}^2\right)\\
&- \frac{2q-1}{2}\sum_{i,j\in\mathbb{N}}\sum_{r=1}^{q-1}q!^2\binom{q}{r}^2\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{2q-2r}}.
\end{aligned}
$$

The second term of (16) will receive a similar treatment. Based on [DVZ18, Lemma 5.1], we have

$$\mathbb{E}[J_{2q}(F_{q,i}F_{q,j})] = \sum_{i,j\in\mathbb{N}} (2q)! \left\| f_{q,i}\widetilde{\otimes} f_{q,j} \right\|_{\mathfrak{H}^{2q}}^2$$

$$= \sum_{i,j\in\mathbb{N}} \left( \mathbb{E}[F_{q,i}F_{q,j}]^2 + \mathbb{E}[F_{q,i}^2]\mathbb{E}[F_{q,j}^2] + \sum_{r=1}^{q} q!^2 \binom{q}{r}^2 \|f_{q,i}\star_r^r f_{q,j}\|_{\mathfrak{H}^{2q-2r}}^2 \right).$$

Hence,

$$\sum_{i,j\in\mathbb{N}} \mathbb{E}\left[ F_{q,i}F_{q,j}\left( \widetilde{\Gamma}(F_{q,i},F_{q,j}) - q\mathbb{E}[F_{q,i}F_{q,j}] \right) \right] = \frac{1}{2} \sum_{i,j\in\mathbb{N}} \sum_{k=1}^{2q-1} (2q-k)\mathbb{E}\left[ J_k(F_{q,i}F_{q,j})^2 \right]$$

$$\leq \frac{2q-1}{2} \sum_{i,j\in\mathbb{N}} \sum_{k=1}^{2q-1} \mathbb{E}\left[ J_k(F_{q,i}F_{q,j})^2 \right]$$

$$= \frac{2q-1}{2} \left( \mathbb{E}[F_{q,i}^2 F_{q,j}^2] - \mathbb{E}[F_{q,i}^2]\mathbb{E}[F_{q,j}^2] - 2\mathbb{E}[F_{q,i}F_{q,j}]^2 \right)$$

$$- \frac{2q-1}{2} \sum_{i,j\in\mathbb{N}} \sum_{r=1}^{q-1} q!^2 \binom{q}{r}^2 \|f_{q,i}\star_r^r f_{q,j}\|_{\mathfrak{H}^{2q-2r}}^2$$

$$\leq \frac{2q-1}{2} \left( \|F_q\|_K^4 - \mathbb{E}\left[ \|F_q\|_K^2 \right]^2 - 2\|S_q\|_{\mathrm{HS}}^2 \right)$$

$$- \frac{2q-1}{4} \sum_{i,j\in\mathbb{N}} \sum_{r=1}^{q-1} q!^2 \binom{q}{r}^2 \|f_{q,i}\star_r^r f_{q,j}\|_{\mathfrak{H}^{2q-2r}}^2 .$$

In addition, based on that fact that

$$\sum_{i,j\in\mathbb{N}} \left( 2\|f_{q,i}\star_r^r f_{q,j}\|_{\mathfrak{H}^{2q-2r}}^2 + 2\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{2q-2r}} \right)$$

$$= \sum_{i,j\in\mathbb{N}} \left( \|f_{q,i}\star_r^r f_{q,j}\|_{\mathfrak{H}^{2q-2r}}^2 + 2\langle f_{q,i}\star_r^r f_{q,j}, f_{q,j}\star_r^r f_{q,i}\rangle_{\mathfrak{H}^{2q-2r}} + \|f_{q,j}\star_r^r f_{q,i}\|_{\mathfrak{H}^{2q-2r}}^2 \right)$$

$$= \sum_{i,j\in\mathbb{N}} \left( \|f_{q,i}\star_r^r f_{q,j} + f_{q,j}\star_r^r f_{q,i}\|_{\mathfrak{H}^{2q-2r}}^2 \right)$$

$$\geq 0,$$

we get from (16) that

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[ \|F_q^t - F_q\|_K^4 \right] \leq (8q-6)\left( \|F_q\|_K^4 - \mathbb{E}\left[ \|F_q\|_K^2 \right]^2 - 2\|S_q\|_{\mathrm{HS}}^2 \right)$$

and from (15) that

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}\left[ \|X^t - X\|_K^4 \right] \leq \sum_{q=1}^{N} 2^{3q-2}(4q-3)\left( \|F_q\|_K^4 - \mathbb{E}\left[ \|F_q\|_K^2 \right]^2 - 2\|S_q\|_{\mathrm{HS}}^2 \right).$$

$\square$

The result below is an adaptation to our setting of a classical combinatorial identity appearing in [PT11, Proof of Proposition 11.2.2].

**Lemma 8.** *The quantity* $\left\|f_{q,i}\widetilde{\star}_0^0 f_{p,j}\right\|_{\mathfrak{H}^{q+p}}^2$ *appearing in Equation* (7) *can be written in terms of norms of non-symmetrized contractions as*

$$(q+p)!\left\|f_{q,i}\widetilde{\star}_0^0 f_{p,j}\right\|_{\mathfrak{H}^{q+p}}^2 = \Bigg( q!p!\,\|f_{q,i}\|_{\mathfrak{H}^q}^2\,\|f_{p,j}\|_{\mathfrak{H}^p}^2 + q!^2\,\langle f_{q,i},f_{q,j}\rangle_{\mathfrak{H}^q}^2\,\mathbb{1}_{\{q=p\}}$$

$$+\,q!p!\binom{q}{q\wedge p}\binom{p}{q\wedge p}\left\|f_{q,i}\star_{q\wedge p}^{q\wedge p}f_{p,j}\right\|_{\mathfrak{H}^{\otimes|q-p|}}^2\,\mathbb{1}_{\{q\neq p\}}$$

$$+\sum_{r=1}^{q\wedge p-1}q!p!\binom{q}{r}\binom{p}{r}\left\|f_{q,i}\star_r^r f_{p,j}\right\|_{\mathfrak{H}^{q+p-2r}}^2 \Bigg).$$

*Proof.* The procedure in [PT11, Proof of Proposition 11.2.2] will be slightly modified to fit our situation. Let $\mathfrak{S}_{q+p}$ be the sets of all permutations of $(q+p)$ elements and assume $\pi,\rho\in\mathfrak{S}_{q+p}$. When the intersection set $\{\pi(1),\ldots,\pi(q)\}\cap\{\rho(q+1),\ldots,\rho(q+p)\}$ contains $r$ element, this will be denoted by $\pi\overset{r}{\sim}\rho$. Since $\mathfrak{H}=L^2(\mathcal{Z},\mu)$, we have that

$$\left\|f_{q,i}\widetilde{\star}_0^0 f_{p,j}\right\|_{\mathfrak{H}^{q+p}}^2 = \left\|f_{q,i}\widetilde{\otimes} f_{p,j}\right\|_{\mathfrak{H}^{q+p}}^2$$

$$=\frac{1}{(q+p)!^2}\sum_{\pi,\rho\in\mathfrak{S}_{q+p}}\int_{\mathcal{Z}^{q+p}}f_{q,i}\big(z_{\pi(1)},\ldots,z_{\pi(q)}\big)f_{p,j}\big(z_{\pi(q+1)},\ldots,z_{\pi(q+p)}\big)$$

$$f_{q,i}\big(z_{\rho(1)},\ldots,z_{\rho(q)}\big)f_{p,j}\big(z_{\rho(q+1)},\ldots,z_{\rho(q+p)}\big)\mu(dz_1\ldots dz_{q+p})$$

$$=\frac{1}{(q+p)!^2}\sum_{\pi\in\mathfrak{S}_{q+p}}\left(\sum_{r=1}^{q\wedge p-1}\sum_{\pi\overset{r}{\sim}\rho}A_{1,r}+\sum_{\pi\overset{0}{\sim}\rho}A_2+\sum_{\pi\overset{q\wedge p}{\sim}\rho}A_3\right). \qquad (17)$$

For the second sum in (17), $\pi\overset{0}{\sim}\rho$ is equivalent to

$$\begin{cases}\{\pi(1),\ldots,\pi(q)\}\cap\{\rho(1),\ldots,\rho(q)\}=\{\pi(1),\ldots,\pi(q)\}\\\{\pi(q+1),\ldots,\pi(q+p)\}\cap\{\rho(q+1),\ldots,\rho(q+p)\}=\{\pi(q+1),\ldots,\pi(q+p)\}\end{cases},$$

which implies that

$$A_2=\int_{\mathcal{Z}^{q\wedge p}}\left(\int_{\mathcal{Z}^{q\wedge p}}f_{q,i}\big(z_{\pi(1)},\ldots,z_{\pi(q)}\big)f_{q,i}\big(z_{\pi(1)},\ldots,z_{\pi(q)}\big)\right)$$

$$\big(f_{p,j}\big(z_{\pi(q+1)},\ldots,z_{\pi(q+p)}\big)f_{p,j}\big(z_{\pi(q+1)},\ldots,z_{\pi(q+p)}\big)\big)\mu(dz_1\ldots dz_{q+p})=\|f_{q,i}\|_{\mathfrak{H}^q}^2\,\|f_{p,j}\|_{\mathfrak{H}^p}^2\,.$$

Furthermore, observe that for a fixed element $\pi\in\mathfrak{S}_{q+p}$, there are $q!$ ways to permute $\{1,\ldots,q\}$ and $p!$ ways to permute $\{q+1,\ldots,q+p\}$. Since $f_{q,i}$ and $f_{p,j}$ are symmetric functions, we have

$$\sum_{\pi\overset{0}{\sim}\rho}A_2=q!p!\,\|f_{q,i}\|_{\mathfrak{H}^q}^2\,\|f_{p,j}\|_{\mathfrak{H}^p}^2\,.$$

For the third sum in (17), there are two cases to consider. If $q=p$ then $\pi\overset{q}{\sim}\rho$ means

$$\begin{cases}\{\pi(1),\ldots,\pi(q)\}\cap\{\rho(q+1),\ldots,\rho(2q)\}=\{\pi(1),\ldots,\pi(q)\}\\\{\pi(q+1),\ldots,\pi(2q)\}\cap\{\rho(1),\ldots,\rho(q)\}=\{\pi(q+1),\ldots,\pi(2q)\}\end{cases},$$

which implies that

$$A_3 = \int_{\mathcal{Z}^q} \left( \int_{Z^q} f_{q,i}(z_{\pi(1)}, \ldots, z_{\pi(q)}) f_{q,j}(z_{\pi(1)}, \ldots, z_{\pi(q)}) \right)$$
$$f_{q,i}(z_{\pi(q+1)}, \ldots, z_{\pi(2q)}) f_{q,j}(z_{\pi(q+1)}, \ldots, z_{\pi(2q)}) \mu(dz_1 \ldots dz_{2q})$$
$$= \langle f_{q,i}, f_{q,j} \rangle_{\mathfrak{H}^q}^2 \, \mathbb{1}_{\{q=p\}},$$

and there are $q!^2$ copies like the one above. On the other hand for $q \neq p$,

$$A_3 = \int_{\mathcal{Z}^{|q-p|}} \left( \int_{\mathcal{Z}^{q\wedge p}} f_{q,i}(z_{\pi(1)}, \ldots, z_{\pi(q)}) f_{p,j}(z_{\rho(q+1)}, \ldots, z_{\rho(q+p)}) \right)$$
$$\left( \int_{\mathcal{Z}^{q\wedge p}} f_{q,i}(z_{\rho(1)}, \ldots, z_{\rho(q)}) f_{p,j}(z_{\pi(q+1)}, \ldots, z_{\pi(q+p)}) \right) \mu(dz_1 \ldots dz_{q+p})$$
$$= \int_{\mathcal{Z}^{|q-p|}} (f_{q,i} \star_{q\wedge p}^{q\wedge p} f_{p,j})^2 \mu(dz_1 \ldots dz_{|q-p|})$$
$$= \left\| f_{q,i} \star_{q\wedge p}^{q\wedge p} f_{p,j} \right\|_{\mathfrak{H}^{\otimes|q-p|}}^2 .$$

Given a fixed $\pi$ such that $\pi \overset{q\wedge p}{\sim} \rho$ and $q \neq p$, there is a total of $\binom{q}{q\wedge p}\binom{p}{q\wedge p}$ ways of choosing $q \wedge p$ elements in $\{\pi(1), \ldots, \pi(q)\} \cap \{\rho(q+1), \ldots, \rho(q+p)\}$ and $q \wedge p$ elements in $\{\pi(q+1), \ldots, \pi(q+p)\} \cap \{\rho(1), \ldots, \rho(q)\}$. In addition, there are $q!p!$ ways to organize $\{\rho(1), \ldots, \rho(q)\}$ and $\{\rho(q+1), \ldots, \rho(q+p)\}$. Therefore, combining the case $q = p$ and $q \neq p$ gives us

$$\sum_{\pi \overset{q\wedge p}{\sim} \rho} A_3 = q!^2 \langle f_{q,i}, f_{q,j} \rangle_{\mathfrak{H}^q}^2 \, \mathbb{1}_{\{q=p\}} + q!p! \binom{q}{q \wedge p}\binom{p}{q \wedge p} \left\| f_{q,i} \star_{q\wedge p}^{q\wedge p} f_{p,j} \right\|_{\mathfrak{H}^{\otimes|q-p|}}^2 \mathbb{1}_{\{q\neq p\}}.$$

We now turn to the first sum on the right side of (17), that is when $\pi \overset{r}{\sim} \rho$ for $1 \leq r \leq q \wedge p - 1$. We can write

$$A_{1,r} = \int_{\mathcal{Z}^{q+p-2r}} \left( \int_{Z^r} f_{q,i}(z_{\pi(1)}, \ldots, z_{\pi(q)}) f_{p,j}(z_{\rho(q+1)}, \ldots, z_{\rho(q+p)}) \right)$$
$$\left( \int_{\mathcal{Z}^r} f_{q,i}(z_{\rho(1)}, \ldots, z_{\rho(q)}) f_{p,j}(z_{\pi(q+1)}, \ldots, z_{\pi(q+p)}) \right) \mu(dz_1 \ldots dz_{q+p})$$
$$= \int_{\mathcal{Z}^{q+p-2r}} (f_{q,i} \star_r^r f_{p,j}(z_1, \ldots, z_{q+p-2r}))^2 \mu(dz_1 \ldots dz_{q+p-2r})$$
$$= \left\| f_{q,i} \star_r^r f_{p,j} \right\|_{\mathfrak{H}^{q+p-2r}}^2 .$$

There are $\binom{q}{r}\binom{p}{r}$ ways to choose $r$ elements in $\{\pi(1), \ldots, \pi(q)\} \cap \{\rho(q+1), \ldots, \rho(q+p)\}$ and $r$ elements in $\{\pi(q+1), \ldots, \pi(q+p)\} \cap \{\rho(1), \ldots, \rho(q)\}$. Furthermore, there are $q!p!$ ways to organize $\{\rho(1), \ldots, \rho(q)\}$ and $\{\rho(q+1), \ldots, \rho(q+p)\}$. This yields

$$\sum_{r=1}^{q\wedge p-1} \sum_{\pi \overset{r}{\sim} \rho} A_{1,r} = \sum_{r=1}^{q\wedge p-1} q!p! \binom{q}{r}\binom{p}{r} \left\| f_{q,i} \star_r^r f_{p,j} \right\|_{\mathfrak{H}^{q+p-2r}}^2 .$$

Thus, we can expand (17) as

$$
\left\| f_{q,i} \widetilde{\star}_0^0 f_{p,j} \right\|_{\mathfrak{H}^{q+p}}^2 = \frac{(q+p)!}{(q+p)!^2} \Bigg( q!p! \left\| f_{q,i} \right\|_{\mathfrak{H}^q}^2 \left\| f_{p,j} \right\|_{\mathfrak{H}^p}^2 + q!^2 \left\langle f_{q,i}, f_{q,j} \right\rangle_{\mathfrak{H}^q}^2 \mathbb{1}_{\{q=p\}}
$$
$$
+ q!p! \binom{q}{q \wedge p} \binom{p}{q \wedge p} \left\| f_{q,i} \star_{q \wedge p}^{q \wedge p} f_{p,j} \right\|_{\mathfrak{H}^{\otimes|q-p|}}^2 \mathbb{1}_{\{q \neq p\}}
$$
$$
+ \sum_{r=1}^{q \wedge p - 1} q!p! \binom{q}{r} \binom{p}{r} \left\| f_{q,i} \star_r^r f_{p,j} \right\|_{\mathfrak{H}^{q+p-2r}}^2 \Bigg),
$$

which is the desired statement. $\qquad \square$

5.4. **Lemmas related to the proof of Theorem 5.** Our first lemma expresses the Hilbert-Schmidt norm in a Besov-Liouville space as an norm in $L^2 \left([0,1]^{\otimes 2}\right)$.

**Lemma 9.** *Let $K = \mathcal{I}_{\beta,2}$ and $S$ be the covariance operator of a random variable $X \in L^2(\Omega) \otimes K$. Let $f \in K$, then*

$$
\left( D_{0+}^{\beta} S f \right)(s) = \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] \left( D_{0+}^{\beta} f \right)(r) dr \qquad (18)
$$

*is in $L^2([0,1])$. This leads to*

$$
\| S \|_{\mathrm{HS}(K)} = \left\| \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] \right\|_{L^2([0,1]^2)}. \qquad (19)
$$

*Proof.* Let $f, g \in K$. Applying Fubini's theorem to $\langle Sf, g \rangle_K = \mathbb{E}[\langle f, X \rangle_K \langle g, X \rangle_K]$ and rearranging terms yields

$$
\int_0^1 \left( D_{0+}^{\beta} S f \right)(s) \left( D_{0+}^{\beta} g \right)(s) ds
$$
$$
= \int_0^1 \left( \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] \left( D_{0+}^{\beta} f \right)(r) dr \right) \left( D_{0+}^{\beta} g \right)(s) ds,
$$

which is equivalent to

$$
\int_0^1 \left( \left( D_{0+}^{\beta} S f \right)(s) - \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] \left( D_{0+}^{\beta} f \right)(r) dr \right) \left( D_{0+}^{\beta} g \right)(s) ds = 0. \qquad (20)
$$

Let $\{g_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{I}_{\beta,2}$. Due to the isometry between $\mathcal{I}_{\beta,2}$ and $L^2([0,1])$, the set $\left\{ D_{0+}^{\beta} g_n \right\}_{n \in \mathbb{N}}$ is an orthonormal basis of $L^2([0,1])$. Then, Equation (20) implies (18). To prove (19), let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis of $L^2([0,1])$. Then, $\{e_m \otimes e_n\}_{m,n \in \mathbb{N}}$ is an orthonormal basis of $L^2([0,1]^{\otimes 2})$. Also, $\left\{ I_{0+}^{\beta} e_n \right\}_{n \in \mathbb{N}}$ is a basis of $K$. Now observe that, using (18), we can write

$$
\left\langle I_{0+}^{\beta} e_m, S I_{0+}^{\beta} e_n \right\rangle_K = \int_0^1 e_m(s) \left( D_{0+}^{\beta} S I_{0+}^{\beta} e_n \right)(s) ds
$$
$$
= \int_0^1 e_m(s) \left( \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] e_n(r) dr \right) ds
$$
$$
= \int_0^1 \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] e_m(s) e_n(r) dr ds,
$$

which leads to

$$\|S\|_{\mathrm{HS}(K)}^2 = \sum_{m,n\in\mathbb{N}} \left\langle I_{0+}^{\beta} e_m, S I_{0+}^{\beta} e_n \right\rangle_K^2 = \left\| \mathbb{E}\left[ \left( D_{0+}^{\beta} X \right)(r) \left( D_{0+}^{\beta} X \right)(s) \right] \right\|_{L^2([0,1]^{\otimes 2})}^2,$$

where the first equality comes from the identity $\|T\|_{\mathrm{HS}(K)}^2 = \sum_{m,n\in\mathbb{N}} \langle k_m, T k_n \rangle_K^2$ for an operator $T \in \mathrm{HS}(K)$ and an orthonormal basis $\{k_n\}_{n\in\mathbb{N}}$ of $K$. $\qquad\square$

**Remark 5.** Let $\zeta$ be a $L^2([0,1])$-valued random variable with covariance operator $T$. Note that the second statement in Lemma 9 is comparable to the identity

$$\|T\|_{\mathrm{HS}(L^2([0,1]))} = \|\mathbb{E}[\zeta(r)\zeta(s)]\|_{L^2([0,1]^{\otimes 2})}$$

whenever $T \in \mathrm{HS}\left( L^2\left([0,1]\right) \right)$.

The following lemma is helpful to compute the Hilbert-Schmidt norms of the Poisson process and Brownian motion appearing in Subsection 5.1.

**Lemma 10.** *Let the setting of Subsection 5.1 prevail, where $X_\lambda$ and $Z$ denoting a Poisson process and a Brownian motion in $\mathcal{I}_{\beta,2}$, respectively. Then, one has*

$$\mathbb{E}\left[ \left( D_{0+}^{\beta} Z \right)(r) \left( D_{0+}^{\beta} Z \right)(s) \right] = \frac{1}{\Gamma(-\beta+1)^2} \int_0^{r\wedge s} (r-x)^{-\beta}(s-x)^{-\beta} dx$$

*and*

$$\mathbb{E}\left[ \left( D_{0+}^{\beta} X_\lambda \right)(r) \left( D_{0+}^{\beta} X_\lambda \right)(s) \right] = \frac{1}{\Gamma(-\beta+1)^2} \int_0^{r\wedge s} (r-x)^{-\beta}(s-x)^{-\beta} dx$$
$$- \frac{\lambda}{\Gamma(-\beta+1)^2(-\beta+1)^2}(r - r\wedge s)^{-\beta+1}(s - r\wedge s)^{-\beta+1}.$$

*Proof.* According to [CD13, Section 3.1], the covariance operator of our Brownian motion is $S' = I_{0+}^{\beta} I_{0+}^{1-\beta} I_{1-}^{1-\beta} D_{0+}^{\beta}$. Substituting this into Equation (18), we get

$$\left( D_{0+}^{\beta} I_{0+}^{\beta} I_{0+}^{1-\beta} I_{1-}^{1-\beta} D_{0+}^{\beta} f \right)(s) = \int_0^1 \mathbb{E}\left[ \left( D_{0+}^{\beta} Z \right)(r) \left( D_{0+}^{\beta} Z \right)(s) \right] \left( D_{0+}^{\beta} f \right)(r) dr. \qquad (21)$$

For the left-hand side, note that $f \in \mathcal{I}_{\beta,2}$ implies that $D_{0+}^{\beta} f \in L^2 \subseteq L^1$, so that $I_{0+}^{1-\beta} I_{1-}^{1-\beta} D_{0+}^{\beta} f \in L^1$. Thus, $D_{0+}^{\beta} I_{0+}^{\beta} = I$ by [SKM93, Theorem 2.4]. Continuing with the left-hand side, we first write out $I_{0+}^{1-\beta}$ using its definition and then perform an integration by part, which yields

$$\left( I_{0+}^{1-\beta} I_{1-}^{1-\beta} D_{0+}^{\beta} f \right)(s) = \frac{1}{\Gamma(1-\beta)} \int_0^1 1_{[0,s]}(r)(s-r)^{-\beta} \left( I_{1-}^{1-\beta} D_{0+}^{\beta} f \right)(r) dr$$
$$= \frac{1}{\Gamma(1-\beta)} \int_0^1 I_{0+}^{1-\beta} \left( 1_{[0,s]}(\cdot)(s-\cdot)^{-\beta} \right)(r) \left( D_{0+}^{\beta} f \right)(r) dr$$

In particular, the integration by part is valid since [SKM93, Equation (2.20)] is satisfied for $p = q = 2$ and $0 < \beta < 1/2$. Equation (21) then becomes

$$\int_0^1 \left( \frac{1}{\Gamma(1-\beta)} I_{0+}^{1-\beta} \left( 1_{[0,s]}(\cdot)(s-\cdot)^{-\beta} \right)(r) - \mathbb{E}\left[ \left( D_{0+}^{\beta} Z \right)(r) \left( D_{0+}^{\beta} Z \right)(s) \right] \right) \left( D_{0+}^{\beta} f \right)(r) dr = 0.$$

Now, using a basis argument like the one in the proof of Lemma 9 yields

$$
\begin{aligned}
\mathbb{E}\Big[\Big(D_{0^+}^{\beta}Z\Big)(r)\Big(D_{0^+}^{\beta}Z\Big)(s)\Big] &= \frac{1}{\Gamma(1-\beta)} I_{0^+}^{1-\beta}\Big(1_{[0,s]}(\cdot)(s-\cdot)^{-\beta}\Big)(r) \\
&= \frac{1}{\Gamma(1-\beta)^2}\int_0^r (r-x)^{-\beta}(s-x)^{-\beta}1_{[0,s]}(x)dx \\
&= \frac{1}{\Gamma(1-\beta)^2}\int_0^{r\wedge s} (r-x)^{-\beta}(s-x)^{-\beta}dx,
\end{aligned}
$$

which is the first statement of our lemma.

We now turn to the second statement. Recall the representation of $X_\lambda$ given at (9). In order to use this representation in computing $\mathbb{E}\Big[\Big(D_{0^+}^{\beta}X_\lambda\Big)(r)\Big(D_{0^+}^{\beta}X_\lambda\Big)(s)\Big]$, we need the joint density of $(T_n, T_m)$. By definition, $T_{m\wedge n}$ and $T_{m\vee n} - T_{m\wedge n}$ are independent and distributed as $\Gamma(m\wedge n, \lambda)$ and $\Gamma(|m-n|, \lambda)$, respectively. Their joint density is hence given by

$$
f_{T_{m\wedge n}, T_{m\vee n}-T_{m\wedge n}}(x,y) = \frac{\lambda^{m\vee n}}{\Gamma(n\vee m)\Gamma(|m-n|)}x^{n\wedge m-1}y^{|m-n|-1}e^{-\lambda(x+y)}.
$$

Since $T_{m\vee n} = T_{m\wedge n} + (T_{m\vee n} - T_{m\wedge n})$, we can write, using a simple change of variable,

$$
f_{T_{m\wedge n}, T_{m\vee n}}(x,y) = \frac{\lambda^{m\vee n}}{\Gamma(n\wedge m)\Gamma(|m-n|)}x^{n\wedge m-1}(y-x)^{|m-n|-1}e^{-\lambda y}1_{\{x<y\}}. \tag{22}
$$

We are now ready to compute $\mathbb{E}\Big[\Big(D_{0^+}^{\beta}X_\lambda\Big)(r)\Big(D_{0^+}^{\beta}X_\lambda\Big)(s)\Big]$. We have

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(D_{0^+}^{\beta}X_\lambda\Big)(r)\Big(D_{0^+}^{\beta}X_\lambda\Big)(s)\Big] \\
&= \frac{1}{\lambda\Gamma(-\beta+1)^2}\Bigg(\sum_{n,m\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big] - \frac{\lambda s^{-\beta+1}}{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}\Big] \\
&\quad - \frac{\lambda r^{-\beta+1}}{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big] + \frac{\lambda^2}{(-\beta+1)^2}s^{-\beta+1}r^{-\beta+1}\Bigg) \\
&= \frac{1}{\lambda\Gamma(-\beta+1)^2}\Bigg(\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_n)_+^{-\beta}\Big] + \sum_{n\in\mathbb{N}}\sum_{m\neq n}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big] \\
&\quad - \frac{\lambda}{-\beta+1}s^{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(r-T_n)_+^{-\beta}\Big] - \frac{\lambda}{-\beta+1}r^{-\beta+1}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big] \\
&\quad + \frac{\lambda^2}{(-\beta+1)^2}s^{-\beta+1}r^{-\beta+1}\Bigg). \tag{23}
\end{aligned}
$$

The first sum on the right side (consisting of all diagonal terms when $m = n$) simplifies as

$$\frac{1}{\lambda\Gamma(-\beta+1)^2}\sum_{n\in\mathbb{N}}\mathbb{E}\Big[(t-T_n)_+^{-\beta}(s-T_n)_+^{-\beta}\Big]$$

$$= \frac{1}{\lambda\Gamma(-\beta+1)^2}\sum_{n\in\mathbb{N}}\int_0^\infty (r-x)_+^{-\beta}(s-x)_+^{-\beta}\frac{\lambda^n}{\Gamma(n)}x^{n-1}e^{-\lambda x}dx$$

$$= \frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}e^{-\lambda x}\left(\sum_{n\in\mathbb{N}}\frac{(\lambda x)^{n-1}}{(n-1)!}\right)dx$$

$$= \frac{1}{\Gamma(-\beta+1)^2}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}dx.$$

Next, we consider the second sum on the right side of (23). The joint density of $(T_n, T_m)$ given in (22) enables us to write

$$\sum_{n\in\mathbb{N}}\sum_{m\neq n}\mathbb{E}\Big[(r-T_n)_+^{-\beta}(s-T_m)_+^{-\beta}\Big] = \frac{\lambda^2}{-\beta+1}\int_0^{r\wedge s}(r-x)^{-\beta}(s-x)^{-\beta}(s+t-2x)dx$$

$$= \frac{\lambda^2}{(-\beta+1)^2}(s-x)^{-\beta+1}(r-x)^{-\beta+1}\Big|_0^{r\wedge s}$$

$$= \frac{\lambda^2}{(-\beta+1)^2}\Big(s^{-\beta+1}r^{-\beta+1} - (s-r\wedge s)^{-\beta+1}(r-r\wedge s)^{-\beta+1}\Big).$$

For the remaining sums in (23), observe that

$$\mathbb{E}\Big[(s-T_m)_+^{-\beta}\Big] = \frac{\lambda}{-\beta+1}s^{-\beta+1}$$

and substitute the last three calculations into (23) to obtain the second statement in Lemma 10.

$\square$

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