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SCATTERING DIAGRAMS FROM HOLOMORPHIC DISCS IN LOG CALABI-YAU SURFACES

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ABSTRACT. We construct special Lagrangian fibrations for log Calabi-Yau surfaces, and scattering diagrams from Lagrangian Floer theory of the fibres. Then we prove that the scattering diagrams recover the scattering diagrams of Gross-Pandharipande-Siebert [25] and the canonical scattering diagrams of Gross-Hacking-Keel [21]. With an additional assumption on the non-negativity of boundary divisors, we compute the disc potentials of the Lagrangian torus fibres via a holomorphic/tropical correspondence. As an application, we provide a version of mirror symmetry for rank two cluster varieties.

CONTENTS

1. INTRODUCTION

A *Looijenga pair* (Y, D) is a projective rational surface Y with an anticanonical cycle D. Gross-Hacking-Keel [21] following the SYZ heuristic provided a pure algebraic method to construct the mirror family for (Y, D). First, a certain affine manifold B with a singularity was constructed from (Y, D) as an alternative to the base of an SYZ fibration, capturing the expected asymptotic behavior of a putative SYZ fibration near D.[1] One can intuitively view it as the limiting affine structure obtained by squeezing all the singular SYZ fibres together. Then Gross-Hacking-Keel used the relative Gromov-Witten invariants counting $\mathbb{A}^1$-curves, rational curves in Y intersecting D exactly at a single point, to serve as substitutes for quantum correction from holomorphic discs of Maslov index zero in $X := Y \backslash D$ with boundaries on the SYZ fibres. The enumeration of the $\mathbb{A}^1$-curves decodes the so-called *canonical scattering diagram* on B. The canonical scattering diagram intuitively tells how the local charts glue together to form a subset of the mirror family, which can be viewed as the mirror family for X, a non-compact Calabi-Yau surface.

The symplectic counterpart of the canonical scattering diagram has not been fully understood previously. The major difficulty lies in analyzing the moduli spaces of holomorphic discs, which can be highly obstructed, especially when their Maslov indices are zero, due

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[1]In the case of a rational elliptic surface, this is indeed verified in [6].

to possible contributions of multiple covers. In [39], the fourth author established a general scheme for constructing a scattering diagram out of an SYZ fibration in dimension 2 using *Fukaya's trick* [13], which will be summarized in Theorem 2.16 (see [55] for a comprehensive exposition). The scattering diagram sits in the base of the SYZ fibration, and the rays (or walls) consist of torus fibres on which boundaries of Maslov zero discs lie. Throughout, it will be denoted by $\mathfrak{D}^{LF}$ where the superscript "*LF*" highlights the role of Lagrangian Floer theory (of SYZ fibres) in its construction.

Following the SYZ spirit, we construct a special Lagrangian fibration on $X := Y \setminus D$ (see Lemma 4.3), but this does not immediately reproduce the canonical scattering diagram in [21]. The main obstacle here is that the moduli space of Maslov index zero stable discs may lose its compactness after removing D. A priori, it is possible that the limiting configurations involve negative Chern number spheres lying in D. Note that X itself is neither geometrically bounded nor complete. Understanding such contributions is already a challenging task that usually requires intricate virtual techniques. To remedy the situation, we make use of the fact that (Y, D) admits a toric model $(\bar{Y}, \bar{D})$, i.e. that there exists a non-toric blowup $Y \to \bar{Y}$ up to a birational modification of the boundary divisor D. Let us call SYZ fibres in $Y \setminus D$ pulled-back from toric fibres (and sitting away from small neighborhoods of singular fibres) *admissible SYZ fibres*. Their Floer theory is somewhat easier to deal with, as Maslov index zero discs with boundaries on an admissible fibre correspond injectively to holomorphic discs with boundary on a toric fibre in $\bar{Y}$. The latter are essentially determined by their intersection profile with the toric divisor $\bar{D}$, which is prescribed by the singular fibres that the Maslov index zero discs pass through. Then we argue that the limiting configuration of such discs with a given intersection profile cannot have a sphere bubble, as otherwise we would have a disc with components of higher genus. See Section 4.2.1 for more details.

Having demonstrated compactness, we can now implement the argument in [39] to produce a scattering diagram on the portion of the SYZ base parametrizing admissible fibres. The source of *initial* rays are singular fibres associated with each point in the blowup center for $Y \to \bar{Y}$. This can be seen from the local model of a non-toric blowup in [2], which we incorporate into our construction of the SYZ fibration on $Y \setminus D$. One can easily see that these initial rays coincide with the initial data of the scattering diagram of [21], but another subtlety arises at this point. While ideally one would expect that the initial rays described above are the only rays coming out from a neighborhood of a singular fibre, as is the case in [21], this is not automatically true in our geometric situation, since we do not have control over the holomorphic discs with boundaries on non-admissible SYZ fibres. In other words, there might be a Maslov index zero disc with boundary on an admissible fibre that crosses a neighborhood of a singular fibre, in which case one would find an additional ray seemingly sourced from this region. The resolution of this subtlety comes from the ideas of tropical geometry. Having in mind the one-to-one correspondence between discs with boundary on admissible fibres in Y and those with boundary on toric fibres in $\bar{Y}$, we assign a tropical disc to each holomorphic disc in $\bar{Y}$ that has precisely the same intersection profile with $\bar{D}$. The existence of such a tropical disc, in turn, provides a strong obstruction to the existence of holomorphic discs with boundaries on admissible SYZ fibres. With the help of a tropical analogue of Gromov compactness (see Section 5.1), we have the following theorem.

Theorem 1.1 (see Theorem 5.4 and discussion in Section 2.4). *One can recover the Gross-Hacking-Keel canonical scattering diagram from the Lagrangian Floer theory of the admissible SYZ fibres in $Y \setminus D$.*

Finally, following the construction of the canonical scattering diagram from $\mathfrak{D}^{GPS}$ (see [21, Section 3.4]), we obtain an algorithm entirely based on Lagrangian Floer theory that derives the canonical scattering diagram of Gross-Hacking-Keel. An immediate corollary of the above theorem is to establish the equivalence between open Gromov-Witten invariants and log Gromov-Witten invariants via tropical geometry.

Corollary 1.2 (Corollary 5.5). *For a Looijenga pair (Y, D) and the SYZ fibration on $Y \setminus D$ in Theorem 1.1, the sum of open Gromov-Witten invariants of a fixed relative class for admissible SYZ fibres is the log Gromov-Witten invariant for a corresponding curve class.*

Notice that the former invariants are defined via symplectic geometry, whereas the latter is purely algebro-geometric. The novelty of our approach of establishing a tropical/holomorphic correspondence is that we can bypass the direct comparison of virtual fundamental cycles for moduli spaces in symplectic geometry and algebraic geometry, which would cause a huge amount of technicalities.

The mirror of Y is a Landau-Ginzburg superpotential W, which is a holomorphic function on some complex manifold[2], expected to capture the geometry of Y. For instance, in the case Y is a toric Fano manifold, then W is simply the generating functions of Maslov index two discs with boundaries on the moment fibres [8]. For the mirror of the compactification Y, [21] introduced the notion of broken lines, which are expected to be the tropicalization of Maslov index two discs with boundaries on SYZ fibres. Enumeration of broken lines results in the theta functions, which are well-defined global functions on the mirror of $Y \setminus D$, due to their compatibility with the wall-crossing behavior with respect to the canonical scattering diagram. Moreover, the spectrum of the algebra generated by the theta functions provide a partial compactification of the mirror family of X, which gives the mirror family of (Y, D). On the mirror side, Pascaleff [45] constructed a non-canonical correspondence between theta functions and certain generators of $SH^0(X)$ assuming X is an affine surface (or an exact 4-dimensional symplectic manifold). Later, Ganatra-Pomerleano [24] generalized the result to higher dimensional cases.

We will prove that W can be calculated tropically, again by establishing tropical/holomorphic correspondence for the Maslov index two discs. Namely, the tropical counterpart of the Maslov index two discs is shown to be exactly the broken lines in [21], which confirms the general expectation mentioned at the beginning. For instance, wall-crossing in the holomorphic setting is caused by the bubbling of a Maslov index two disc into a union of Maslov index two and zero discs, and bending of a broken line precisely reflects this phenomenon. Hence, our method provides an explicit (algorithmic) count of Maslov index two discs, which would be very difficult if directly looking at holomorphic discs themselves. The following is a consequence of Theorem 5.8 and Lemma 5.11:

Theorem 1.3. *Let (Y, D) be a Looijenga pair such that D contains no negative Chern number spheres and L is an admissible SYZ fibre. Then the number of Maslov index two discs with boundaries on L in class $\beta \in H_2(Y, \mathbb{Z})$ equals to the weighted count of broken lines with stop at u in class β and suitable infinity directions.*

Theorem 1.3 implies that the theta functions associated with D_i in [21], an irreducible component of D, agree with the generating functions of Maslov index two discs that intersect D_i. We remark that the computation of the mirror of toric semi-Fano surfaces will be significantly used as a stepping stone in the proof of Theorem 1.3. It has been already

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[2]In symplectic setting, a function on the rigid analytic variety over Λ

done by [7], but we will examine in Section 3.2 a tropical interpretation of their calculation which is of its own interest. It is worthwhile to mention that for the symplectic form chosen here, Y is not a monotone symplectic manifold nor we impose the monotone assumption on the SYZ fibres. Therefore, the wall-crossing results of Seidel [49] or Pascaleff-Tonkonog [47] does not directly apply here. As an application, we compute the LG mirror of the degree 5 del Pezzo surface, and verify the closed-string mirror symmetry (Theorem 7.1).

Additionally, we explore a few more examples that exactly fit into the setting of Theorem 1.3, including a non-Fano surface. More specifically, we find a tropical (and hence holomorphic, through our correspondence) argument that can cover classic examples of Auroux [1,2] and cubic surfaces [23]. Mirror symmetry can further extend to non-Fano manifolds with similar pictures. The superpotential is much harder to compute in the non-Fano cases due to the possible bubbling phenomena. Nevertheless, we expect the method here provide an algorithm to generalize our method beyond the semi-Fano case by understanding the degeneration of Looijenga pairs.

Last but not least, Theorem 1.1 has a nice application to cluster varieties of rank 2. There are two types of cluster varieties, namely the $\mathcal{A}$ and the $\mathcal{X}$ cases. Fock-Goncharov [11] conjectured that the $\mathcal{A}$ cluster varieties are mirror to the Langland dual of the $\mathcal{X}$ varieties, and vice versa. On the $\mathcal{A}$ side, there is a natural torus action of T_{K° on the $\mathcal{A}$ cluster variety while there is a fibration of the $\mathcal{X}$ to a torus T_{K^*} on the $\mathcal{X}$ side. By understanding the toric models of the rank 2 cases, we employ the scattering diagram construction in Gross-Pandharipande-Siebert [25]. Then we identify the GPS scattering diagrams with the fibers of $\mathcal{X}$ cluster scattering diagrams. Hence we obtain the following theorem, and refer readers to Theorem 6.2 for the precise statement.

Theorem 1.4. *Consider the rank 2 cluster varieties. The quotient of the $\mathcal{A}$ cluster varieties are mirror to the the Langland dual $^L\mathcal{X}_e$ cluster varieties, where $^L\mathcal{X}_e$ is fibre of the Langland dual $\mathcal{X}$ family at e.*

The organization of the paper is as follows. After reviewing preliminary materials in Section 2, we revisit Floer theory of Lagrangian fibres in toric surfaces in Section 3, where we mostly focus on *tropicalization* of holomorphic discs in a sense that will be clarified therein. Section 4 is devoted to constructing an SYZ fibration on a non-toric blowup of a toric surface and analyzing the moduli of Maslov index zero discs, which leads to the scattering diagram $\mathfrak{D}^{LF}$. We then prove our main theorem in Section 5, stating that $\mathfrak{D}^{LF}$ coincides with $\mathfrak{D}^{GPS}$. Section 6 provides a cluster duality interpretation of the mirror symmetry studied in this paper. Finally, as an application, we compute the LG mirror and show the closed string mirror symmetry for the del Pezzo surface of degree 5 in Section 7.

Notation.
- Let $N \cong \mathbb{Z}^2$ be a lattice and $M := \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. Let $N_\mathbb{R} := N \otimes \mathbb{R}$ and $M_\mathbb{R} := M \otimes \mathbb{R}$. Let $\Sigma \subseteq M_\mathbb{R}$ be the toric fan and $\bar{Y} := Y_\Sigma$ be the associated toric surface. We will denote $\bar{D}$ be the toric boundary divisor and $\bar{D}_i$ for its irreducible

component. We will write m_i for the primitive generator of the 1-cone corresponding to $\bar{D}_i$ in Σ.

- Let $p \colon \bar{Y} \to P$ be the moment map fibration, and let P denote the moment polytope. For simplicity, we will abuse the same notation p to denote its Legendre transform $p \colon \bar{Y} \backslash \bar{D} \cong (\mathbb{C}^*)^2 \to \mathbb{R}^2$, $(X, Y) \mapsto (\log |X|, \log |Y|)$. For each $u \in \text{Int} P$, we denote L_u the moment fibre over u.
- Fixing a Lagrangian L, we denote by $\mathcal{M}_{k+1,l}(L, \beta, J)$ be the moduli space of stable bordered J-holomorphic discs representing the class β in $H_2(X, L)$ with $k+1$ boundary marked points and l interior marked points with respect to the *standard toric complex structure J on X* in [28].
- Λ_0 denotes the Novikov ring over the real numbers $\mathbb{R}$,

$$\Lambda_0 := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i} \mid \lambda_i \geq 0, \lim_{i \to \infty} \lambda_i = \infty \text{ and } a_i \in \mathbb{C} \right\}.$$

There is a non-Archimedean valuation $\text{val} \colon \Lambda_0 \to \mathbb{R}$,

$$\text{val} \left(\sum_i a_i T^{\lambda_i} \right) = \inf\{\lambda_i \mid a_i \neq 0\} \text{ and } \text{val}(0) = \infty.$$

The maximal ideal of Λ_0 is denoted by $\Lambda_+ := \text{val}^{-1}((0, \infty))$.

2. Preliminaries

2.1. **Tropical Geometry.** The modified SYZ conjecture [37] suggests that a Calabi-Yau manifold collapses to an integral affine manifold (possibly with singularities) toward the large complex structure limit. It has been folklore that holomorphic curves in Calabi-Yau manifolds should converge to 1-skeletons in the integral affine manifolds satisfying a certain balancing condition. These 1-skeletons are referred to as tropical curves. The pioneering work of Mikhalkin [41] shows that the enumeration of holomorphic curves in toric surfaces equals the weighted count of tropical curves. Since tropical curves are purely combinatorial objects, establishing a correspondence between tropical and holomorphic curves provides a powerful tool for enumerative geometry.

In this section, we recall some knowledge of tropical geometry that we will use later. Let us first give the definition of tropical discs.

Definition 2.1. *A parametrized tropical disc with end at u in $M_{\mathbb{R}}$ is a triple (h, T, w) satisfying the following properties:*

(i) *T is a rooted tree that possibly contains unbounded edges. If x is the root, then T has only trivalent vertices besides x. The set of vertices is denoted by $T^{[0]}$, and the set of edges is denoted by $T^{[1]}$.*

(ii) *$h : T \to M_{\mathbb{R}}$ is a map such that $h(x) = u$, and $h(e)$ is an embedding of an affine line segment (resp. an affine ray) if $e \in T^{[1]}$ is bounded (resp. unbounded).*

(iii) *The map $w : T^{[1]} \to \mathbb{Z}_{>0}$ assigns a weight to each edge with the following balancing condition. For any vertex besides x, the three adjacent edges e_1, e_2 and e_3 satisfy*

$$w(e_1)v(e_1) + w(e_2)v(e_2) + w(e_3)v(e_3) = 0,$$

where $v(e_i)$ is the primitive vector tangent to $h(e_i)$ that is pointed away from v.

Furthermore, the triple (h, T, w) is called a parametrized tropical disc of $\bar{Y}$ if every unbounded edge is of the form $m' + \mathbb{R}_{\geq 0} m$ where $m' \in M_{\mathbb{R}}$ and $m \in M$ is the primitive generator of an 1-cone in Σ. A tropical curve/disc is the image of a parametrized tropical curve/disc.

Remark 2.2. *Notice that the balancing conditions of tropical curves/discs follow directly from that of the parametrized tropical curves/discs.*

Motivated by [1, Lemma 3.1], the Maslov index of a tropical disc is defined as follows.

Definition 2.3. *Given a tropical disc* (h, T, w), *its Maslov index* $MI(h)$ *is defined to be the twice the sum of weights of unbounded edges.*

We next recall the Mikhalkin weight [41] for the trivalent tropical discs.

Definition 2.4. *Given a tropical disc* (h, T, w) *with end at* $u \in M_{\mathbb{R}}$. *Let* $v \in T^{[0]}$ *be a trivalent vertex with adjacent edges* e_1, e_2, e_3. *Then the (Mikhalkin) weight at* v *denoted as* $Mult_v$ *is given by*

$$Mult_v := |w(e_1)v(e_1) \wedge w(e_2)v(e_2)| \in \wedge^2 M^2 \cong \mathbb{Z}.$$

Then the weight $Mult(h)$ *of the tropical disc* (h, T, w) *is defined as*

$$Mult(h) = \prod_{v \in T^{[0]}, v \neq x} Mult_v.$$

We will mainly consider tropical discs appearing naturally on the toric setting (or its slight variants). In this case, they are drawn on the Legendre transform $\mathbb{R}^2$ of the moment polytope, and their unbounded edges are required to be parallel to normal directions to the toric divisor (or 1-cones in the fan). One can intuitively think of each unbounded edge intersecting the toric divisor normal to it, as the divisors are sitting at infinity (keeping their directions) after taking Legendre transform. More generally, any affine line segment in the base of a SYZ fibration with a rational slope can be completed to a cylinder in the total space with help of a complex structure (to be canonically chosen in our geometric setting below), whose symplectic area obviously makes sense.

On the other hand, the intersection patterns of a holomorphic disc with toric divisors completely determine its topological type, and in particular its symplectic area and Maslov index. Thus it is natural to define the relative class of a given tropical disc to be that of the holomorphic disc which intersects the toric divisors in accordance with unbounded edges of the tropical disc. Throughout, we will frequently use the terms such as relative classes, symplectic areas, Maslov indices, etc., for tropical discs in this spirit.

We will sometimes need to impose further constraints to tropical discs. Namely, we fix a set of generic points in $\mathbb{R}^2$, and consider the counting invariant concerning tropical discs that pass through these points. Obviously, the constraints make discs more rigid, i.e., they behave like discs with lower Maslov indices. For this reason, we define *generalized Maslov index* of a constrained tropical disc by

$$(2.1) \qquad MI'(h, T, w) := |\text{unbounded edges of } T| - 2 |\text{point-constraints}|.$$

This is of course consistent with the generalized Maslov index of a holomorphic disc defined in [32] (see Section 2.5.1, also). Note that if there is no constraint, then MI' agrees with the Maslov index of the tropical disc above.

2.2. **Scattering diagrams.** Fix R be an Artin local $\mathbb{C}$-algebra or a complete local $\mathbb{C}$ algebra and $\mathfrak{m}_R$ be the unique maximal ideal of R. For our purpose, we will take R to be the Novikov ring, $\mathbb{C}[[t]]$ or $\mathbb{C}[[NE(Y)]]$[3], where $NE(Y)$ is the effective curve cone of a surface Y. We now define a dimension 2 scattering diagram.

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[3]This is the completion of $\mathbb{C}[NE(Y)]$ with respect to some filtration.

Definition 2.5. *A scattering diagram $\mathfrak{D}$ is a collection of walls $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$ where*

- *$\mathfrak{d} \subseteq M_{\mathbb{R}}$ is either a ray of the form $\mathfrak{d} = m'_{\mathfrak{d}} + \mathbb{R}_{\geq 0} m_{\mathfrak{d}}$ or a line of the form $\mathfrak{d} = m'_{\mathfrak{d}} + \mathbb{R} m_{\mathfrak{d}}$, for some $m'_{\mathfrak{d}} \in M_{\mathbb{R}}$ and $m_{\mathfrak{d}} \in M \setminus \{0\}$. The set $\mathfrak{d}$ is called the support of the line or ray.*
- *$f_{\mathfrak{d}} \in \mathbb{C}[[z^{m_{\mathfrak{d}}}]] \hat{\otimes}_{\mathbb{C}} R \subseteq \mathbb{C}[M] \hat{\otimes}_{\mathbb{C}} R$, called wall functions, satisfy $f_{\mathfrak{d}} \equiv 1 \mod z^{m_{\mathfrak{d}}} \mathfrak{m}_R$.*

such that for every power $k > 0$, there are only a finite number of $(\mathfrak{d}, f_{\mathfrak{d}})$ with $f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}_R^k$.

If $\mathfrak{D}$ is a scattering diagram, we write

$$\mathrm{Sing}(\mathfrak{D}) = \bigcup_{\mathfrak{d} \in \mathfrak{D}} \{m'_{\mathfrak{d}}\} \cup \bigcup_{\mathfrak{d}_1, \mathfrak{d}_2 \dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = 0} \mathfrak{d}_1 \cap \mathfrak{d}_2,$$

Now consider a smooth immersion $\phi : [0, 1] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ such that endpoints avoid the support of the scattering diagram $\mathfrak{D}$ and all intersection of ϕ with the walls are transverse. We then define the path-ordered product as follows: For each power $k > 0$, ϕ will cross only a finite number s_k of walls with $f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}_R^k$. We label them by $\mathfrak{d}_i$, where $i = 1, \ldots s_k$ with respect to the order of the path intersecting the walls. Each wall $\mathfrak{d}$ determines an automorphism

$$\theta_{\phi, \mathfrak{d}_i}(z^m) = z^m f_{\mathfrak{d}_i}^{\langle n_0, m \rangle},$$

where $n_0 \in N$ is primitive normal to $\mathfrak{d}_i$, and $\langle n_0, \phi'(t_i) \rangle > 0$, where t_i is the moment γ crosses the wall $\mathfrak{d}_i$. Then we define $\theta_{\phi, \mathfrak{D}}^k = \theta_{\mathfrak{d}_{s_k}} \circ \cdots \circ \theta_{\mathfrak{d}_1}$. Then we define $\theta_{\phi, \mathfrak{D}} = \lim_{k \to \infty} \theta_{\phi, \mathfrak{D}}^k$.

Theorem 2.6. *[27, 36] Let $\mathfrak{D}'$ be a scattering diagram. Then there exists a scattering diagram $\mathfrak{D}$ containing $\mathfrak{D}'$ such that $\mathfrak{D} \setminus \mathfrak{D}'$ consists only of rays, and such that $\theta_{\phi, \mathfrak{D}} = \mathrm{Id}$ for any closed loop γ for which $\theta_{\phi, \mathfrak{D}}$ is defined. We will call such a scattering diagram consistent. After combining $(\mathfrak{d}, f_{\mathfrak{d}}), (\mathfrak{d}', f_{\mathfrak{d}'})$ into $(\mathfrak{d}, f_{\mathfrak{d}} f_{\mathfrak{d}'})$ if $\mathfrak{d} = \mathfrak{d}'$, the resulting $\mathfrak{D}$ is unique.*

Example 2.7. *Figure 1 illustrated the first example of scattering diagrams. It corresponds to the cluster algebra of type A_2. From the geometry perspective, the diagrams is given by the del Pezzo of degree 5 with a cycle of five (-1)-curves.*
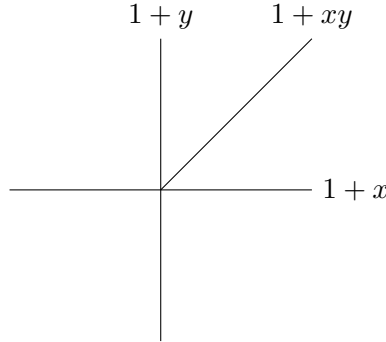

FIGURE 1. A_2 scattering diagram

Broken lines. We recall the notion of the broken line, which will be important in studying the mirror of the compactification Y of a Looijenga pair (Y, D) in Section 5.2. Recall that there is a natural short exact sequence from toric geometry

(2.2) $$0 \to K_{\Sigma} \to T_{\Sigma} := \bigoplus_{i=1}^{n} \mathbb{Z} t_i \xrightarrow{r} M \to 0,$$

where $r(t_i) = m_i$ and $K_\Sigma = \mathrm{Ker}(r)$.

Definition 2.8. *Let $\mathfrak{D}$ be a consistent scattering diagram on $M_\mathbb{R}$. A broken line with stop at $u \in M_\mathbb{R}$ is a continuous map*

$$\mathfrak{b} : (-\infty, 0] \to M_\mathbb{R}$$

such that $\mathfrak{b}(0) = u$ and with the below properties: there exist

$$-\infty = t_0 < t_1 < \cdots < t_n = 0$$

such that $\mathfrak{b}|_{[t_{i-1}, t_i]}$ is affine. For each $i = 1, \cdots, n$, there is $c_i z^{m_i^{\mathfrak{b}}(t)} \in T_\Sigma$ such that

(1) *For each i, $r(m_i^{\mathfrak{b}})$ is positively proportional to $\beta'(t)$.*
(2) *$m_1^{\mathfrak{b}} = m_j$ for some $j \in \{1, \cdots, n\}$ and $c_1 = 1$.*
(3) *$\mathfrak{b}(t_i) \in Supp(\mathfrak{D}) \backslash Sing(\mathfrak{D})$.*
(4) *If $\mathfrak{b}(t_i) \in \cap_i \mathfrak{d}_i$ (hence in a 1-dimensional intersection of walls), then $c_{i+1} z^{m_{i+1}^{\mathfrak{b}}(t_{i+1})}$ is a term in*

(2.3)
$$\big(\prod_i (\theta_{\mathfrak{b}, \mathfrak{d}_i})^{\epsilon_i} \big)(c_i z^{m_i^{\mathfrak{b}}(t_i)}),$$

   where $\epsilon_i = sgn\langle m_i^{\mathfrak{b}}(t_i), \gamma_{\mathfrak{d}_i} \rangle$.

The short exact sequence (2.2) naturally identified with

$$0 \to H_2(\bar{Y}, \mathbb{Z}) \to H_2(\bar{Y}, L_u) \to H_1(L_u, \mathbb{Z}) \to 0,$$

for any $u \in \mathrm{Int}(P) \cong M_\mathbb{R}$ since the moment fibration has no monodromy.

Definition 2.9. *Given a broken line $\mathfrak{b}$, we say that n is the length of the broken line $\mathfrak{b}$ and it has homology $[\mathfrak{b}] := m_n^{\mathfrak{b}}(u) \in H_2(\bar{Y}, L_u)$ using the above identification of short exact sequence and weight $Mono(\mathfrak{b}) := c_n$.*

The following is the simplest example of the broken line.

Example 2.10. *For any $u \in M_\mathbb{R}$ and m_i, there exists a broken line $\mathfrak{b} : (-\infty, 0] \to B_0$ with image of the form $u + \mathbb{R}_{\leq 0} m_i$. In this case, one has $n = 1$ with the z^{β_i}, where β_i is the unique disc class of Maslov index in a tubular neighborhood of $\bar{D}_i$. The weight of this broken line is 1.*

2.3. **The scattering diagrams of Gross-Pandharipande-Siebert.** The most relevant example of a scattering diagram to the purpose of this paper is the one constructed in [25] (a combination of Theorem 2.8, Theorem 3.4, Theorem 4.4 and Proposition 5.2 therein), which we will denote by $\mathfrak{D}^{GPS}$. We give a detailed review on it, as one of our goals is to retrieve $\mathfrak{D}^{GPS}$ from holomorphic disc counting in the realm of Lagrangian Floer theory.

2.3.1. *Scattering Diagrams from Simple Blowups.* Let $\pi : \tilde{Y} \to \bar{Y}$ be the blowup of $\bar{Y}$ at distinct generic[4] non-toric points $q_{ij}, i = 1, \cdots, N, j = 1, \cdots, l_i$, for some $l_i \in \mathbb{Z}_{\geq 0}$. Denote by E_{ij} the exceptional divisor corresponding to q_{ij}, and by $\tilde{D}$ the proper transform of $\bar{D}$, which is anticanonical in $\tilde{Y}$. Then there is a natural consistent scattering diagram $\mathfrak{D}^{GPS}$ associated to the above geometry constructed as follows.

Let $\mathfrak{D}_{in}^{GPS}$ be the scattering diagram consists of lines $\{(\mathfrak{d}_{ij}, f_{\mathfrak{d}_{ij}}\}$ where $\mathfrak{d}_{ij} = m'_{ij} + \mathbb{R}m_i$ and m'_{ij} is chosen such that the closure of the preimage of $\mathfrak{d}_{ij}$ under the moment map contains

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[4]This is only to avoid the situation a ray from scattering falls in the support of an initial ray.

q_{ij}. For simplicity, we will often use the expression *the wall determined by q_{ij}'* to indicate such a wall $\mathfrak{d}_{ij}$.

The associated wall function is given by $f_{\mathfrak{d}_{ij}} = 1 + t^{-[E_{ij}]}z^{m_i}$, where t is the formal variable in R is the completion of $\mathbb{C}[NE(\tilde{Y})]$ with respect to some filtration. From Theorem 2.6, $\mathfrak{D}_{in}^{GPS}$ uniquely determines a consistent scattering diagram $\mathfrak{D}^{GPS}$. Below we will explain the geometric interpretation of its wall functions. If $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}^{GPS} \setminus \mathfrak{D}_{in}^{GPS}$, then $\mathfrak{d}$ must be a ray, i.e. $\mathfrak{d} = m'_{\mathfrak{d}} + \mathbb{R}_{\geq 0}m_{\mathfrak{d}}$, and the wall function $f_{\mathfrak{d}}$ is given as the generating series of certain algebraic curves counts. Let $\bar{Y}_{\mathfrak{d}}$ be the toric blowup obtained by adding a new ray in the direction $-m_{\mathfrak{d}}$ to the fan of $\bar{Y}$. The exceptional divisor in $\bar{Y}_{\mathfrak{d}}$ appears as a toric boundary component that we denote by $\bar{D}_{\mathfrak{d}}$. (The toric blowup given in Definition 2.12 slightly generalizes this.) Let $\tilde{Y}_{\mathfrak{d}}$ be the fibre product

$$
\begin{array}{ccc}
\tilde{Y}_{\mathfrak{d}} & \xrightarrow{\pi_{\mathfrak{d}}} & \tilde{Y} \\
\downarrow & & \downarrow{\scriptstyle\pi} \\
\bar{Y}_{\mathfrak{d}} & \longrightarrow & \bar{Y}
\end{array}
$$

and $\tilde{D}_{\mathfrak{d}} = \pi_{\mathfrak{d}}^{-1}(\tilde{D})$ the corresponding boundary divisor. Then the wall function is given as

$$(2.4) \qquad f_{\mathfrak{d}} = \sum_{\beta} k_{\beta} N_{\beta} t^{(\pi_{\mathfrak{d}})_* \beta} z^{k_{\beta} m_{\mathfrak{d}}},$$

where β runs over the classes in $H_2(\tilde{Y}_{\mathfrak{d}}, \mathbb{Z})$ supporting $\mathbb{A}^1$-curves. k_{β} is the intersection of β with the component $\tilde{D}_{\mathfrak{d}}$, the proper transform of $\bar{D}_{\mathfrak{d}}$.

The coefficient N_{β} computes the associated log Gromov-Witten invariant. Intuitively, N_{β} counts the number of rational curves C in $\tilde{Y}$ of the class $\beta_{\mathbf{P}} \in H_2(\tilde{Y}, \mathbb{Z})$ ($\mathbf{P} = \{p_{ij}\}$) such that

(1) $\pi(C)$ is tangent to $\bar{D}_i$ at q_{ij} with tangency multiplicity p_{ij}.
(2) C intersects $\tilde{D}$ at exactly one point.
(3) $\sum_{i,j} p_{ij} m_i = k_{\beta} m_{\mathfrak{d}}$ for some $k_{\beta} \in \mathbb{Z}_{>0}$.
(4) $\beta = \pi^* \beta_{\mathbf{P}} - \sum_{i,j} p_{ij}[E_{ij}]$.

We will refer readers to [25, Section 4] (or [21, Definition 3.1]) for the precise definition of N_{β}.

2.3.2. *Scattering Diagrams from Orbifold Blowups.*

Gross-Pandharipande-Siebert further considered the orbifold blowup $\pi : \tilde{Y} \to Y$ at non-toric point q_{ij} with multiplicity r_{ij} and studied the enumerative meaning of the wall-functions in the associate scattering diagram. We will discuss the heuristic picture in Lagrangian Floer theory in Section 4.3 and use this to explain the mirror symmetry of cluster varieties of rank two in Section 6.

In this case, $\tilde{Y}$ has an $A_{r_{ij}-1}$-singularity for each point q_{ij} and admits a unique structure as a non-singular Deligne-Mumford stack. The exceptional divisor E_{ij} over q_{ij} contains the orbifold point due to higher multiplicity given at the blowup point q_{ij}. Denote by $\mathfrak{D}_{in}^{GPS}$ the scattering diagram consisting of $\{(\mathfrak{d}_{ij}, f_{\mathfrak{d}_{ij}})\}$, where $\mathfrak{d}_{ij}$ is the wall determined by q_{ij} and $f_{\mathfrak{d}_{ij}} = 1 + t^{[E_{ij}]} z^{r_{ij} m_i}$. We will refer the readers to Section 4.3 for the heuristic symplectic analogue. Again we write $\mathfrak{D}^{GPS}$ for the unique consistent scattering diagram constructed from $\mathfrak{D}_{in}^{GPS}$. For each ray $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}^{GPS} \setminus \mathfrak{D}_{in}^{GPS}$, one can similarly define $\bar{Y}_{\mathfrak{d}}$ and $\tilde{Y}_{\mathfrak{d}} = \bar{Y}_{\mathfrak{d}} \times_{\bar{Y}} \tilde{Y}$.

Then the wall function $f_{\mathfrak{d}}$ is given by similar formula as (2.4) with N_β the orbifold Gromov-Witten invariant counting the number of rational curves C in $\tilde{Y}$ of the class $\beta_{\mathbf{P}} \in H_2(\tilde{Y}, \mathbb{Z})$ ($\mathbf{P} = \{p_{ij}\}$) such that

(1) $\pi(C)$ passes through q_{ij} exactly once with tangency multiplicity p_{ij}.
(2) C intersects $\tilde{D}$ at exactly one point.
(3) $\sum_{i,j} p_{ij} m_i = k_\beta m_{\mathfrak{d}}$ for some $k_\beta \in \mathbb{Z}_{>0}$.
(4) $\beta = \pi^* \beta_{\mathbf{P}} - \sum_{i,j} p_{ij}[E_{ij}]$.

2.4. **Canonical Scattering Diagram of Gross-Hacking-Keel.** We begin with the same geometric setup in [21] and then summarize the construction of the canonical scattering diagram.

Definition 2.11. *A Looijenga pair* (Y, D) *is a smooth projective rational surface* Y *with* $D \in |-K_Y|$ *a reduced rational curve which has at least one singular point.*

If D is irreducible, then it is a genus one curve with a single nodal point. Otherwise, it is a cycle of smooth rational curves. The following is a useful observation from the classification of surfaces.

Proposition 2.12. [21, Proposition 1.3] *For any Looijenga pair* (Y, D), *there exist two other Looijenga pairs* $(\tilde{Y}, \tilde{D})$ *and* $(\bar{Y}, \bar{D})$ *such that*

(1) $\pi' : \tilde{Y} \to Y$ *is a blowup of nodal point(s) of* D, *called the toric blowup, and* $\tilde{D} = \pi'^* D$;
(2) $(\bar{Y}, \bar{D})$ *is a toric pair and,* $\pi : \tilde{Y} \to \bar{Y}$ *is a blowup at smooth points of* $\bar{D}$, *called a non-toric blowup, with* $\tilde{D}$ *being the proper transform of* $\bar{D}$.

We write $X := Y \setminus D$, and similarly $\tilde{X} := \tilde{Y} \setminus \tilde{D}$ and $\bar{X} := \bar{Y} \setminus \bar{D}$. There exists a meromorphic form Ω on Y restricts to a holomorphic volume form on X, unique up to $\mathbb{C}^*$-scaling. We will denote the corresponding holomorphic volume forms by $\tilde{\Omega}$ and $\bar{\Omega}$, respectively. A straightforward calculation shows that $\tilde{\Omega} = \pi'^* \Omega = \pi^* \bar{\Omega}$, where we abuse notations by putting $\pi' : \tilde{X} \to X$ and $\pi : \tilde{X} \to \bar{X}$.

Given a Looijenga pair (Y, D), Gross-Hacking-Keel [21] associated an integral affine manifold B_{GHK} with a singularity. Topologically, it is $\mathbb{R}^2$ with a unique singularity at the origin. There is a cone decomposition $B_{GHK} = \cup_{i=1}^n \sigma_i / \sim$, where σ_i is a cone bounded by two rays $\mathbb{R}_{\geq 0} v_i$ and $\mathbb{R}_{\geq 0} v_{i+1}$, and each ray $\mathbb{R}_{\geq 0} v_i$ corresponds to a component D_i of the boundary divisor D. Here the relation $\sim$ glues $\mathbb{R}_{\geq 0} v_1$ with $\mathbb{R}_{\geq 0} v_{n+1}$. The canonical scattering diagram $\mathfrak{D}^{can}$ is then a collection $\{(\mathfrak{d}, f_{\mathfrak{d}})\}$ satisfying the followings. If $\mathfrak{d}$ is a ray generated by $a v_i + b v_{i+1}$ for $a, b \in \mathbb{Z}_{\geq 0}$, then

$$\log f_{\mathfrak{d}} = \sum_{k \geq 1} k c_k X_i^{-ak} X_{i+1}^{-bk} \in R[[X_i^{-a} X_{i+1}^{-b}]],$$

where $NE(Y)$ is the monoid generated by effective curve classes in Y. The coefficient c_k is the generating function of the log Gromov-Witten invariants

$$c_k = \sum_\beta N_\beta z^\beta,$$

where the summation is over all possible classes $\beta \in H_2(Y, \mathbb{Z})$ with incidence relation $\beta.D_i = ak, \beta.D_{i+1} = bk$ and $\beta.D_j = 0$, for $j \neq i, i+1$. The coefficient N_β is the algebraic count of $\mathbb{A}^1$-curves in class β as in Section 2.3.

The integral affine manifold with singularity B_{GHK} and the canonical scattering diagram $\mathfrak{D}^{can}$ are independent under deformation of the Looijenga pair (Y, D) as stated in [21, Lemma

3.9], thanks to the deformation invariance of log Gromov-Witten invariants. If $(\tilde{Y}, \tilde{D}) \to$ (Y, D) is a toric blowup, then their corresponding integral affine manifolds with singularities are naturally isomorphic, and so are their associate canonical scattering diagrams by [21, Lemma 1.6].

If $(Y, D) \to (\bar{Y}, \bar{D})$ is a non-toric blowup, from the deformation invariance of the canonical scattering diagram, we may assume that Y is the simple blowup of $\bar{Y}$ at mutually distinct points $q_{ij} \in \bar{D}_i$. Recall that these data determine the scattering diagram $\mathfrak{D}^{GPS}$. The relation between the canonical scattering diagram $\mathfrak{D}^{can}$ of (Y, D) and the scattering diagram $\mathfrak{D}^{GPS}$ is as follows: for any $av_i + bv_{i+1}$, $a, b \in \mathbb{Z}_{\geq 0}$ and $(a, b) \neq (1, 0), (0, 1)$, we have a ray $\mathfrak{d} = \mathbb{R}_{\geq 0}(av_i + bv_{i+1})$ in $\mathfrak{D}^{can}$ with the wall function

$$f_{\mathfrak{d}} = \prod_{(\mathfrak{d}', f_{\mathfrak{d}'}) \in \mathfrak{D}^{GPS}} f_{\mathfrak{d}'},$$

here $(\mathfrak{d}', f_{\mathfrak{d}'})$ runs through all possible rays in $\mathfrak{D}^{GPS}$ with corresponding curve class $\beta \in H_2(\tilde{Y}_{\mathfrak{d}'}, \mathbb{Z})$ such that $(\pi_{\mathfrak{d}'})_* \beta \cdot D_i = a, (\pi_{\mathfrak{d}'})_* \beta \cdot D_{i+1} = b$. If $(a, b) = (1, 0)$, then wall function $f_{\mathfrak{d}}$ attached to the ray $\mathfrak{d} = \mathbb{R}_{\geq 0} v_i$ is given by

$$f_{\mathfrak{d}} = \prod_{j=1}^{j=l_i} (1 + t^{[E_{ij}]} z^{-m_i}) \prod_{(\mathfrak{d}', f_{\mathfrak{d}'})} f_{\mathfrak{d}'}.$$

Here $\mathfrak{d}_{ij}$ are the rays in $\mathfrak{D}_{in}^{GPS}$ and $(\mathfrak{d}', f_{\mathfrak{d}'})$ runs through all possible rays in $\mathfrak{D}^{GPS}$ with the corresponding curve class $\beta \in H_2(\tilde{Y}_{\mathfrak{d}'}, \mathbb{Z})$ such that $(\pi_{\mathfrak{d}'})_* \beta \cdot D_i = 1, (\pi_{\mathfrak{d}'})_* \beta \cdot D_j = 0$ for $j \neq i$ but β is not a multiple of E_{ij} for any $j = 1 \cdots, l_j$. To sum up, given any Looijenga pair (Y, D), one can construct the asscociate canonical scattering diagram $\mathfrak{D}^{can}$ of (Y, D) from the scattering diagram $\mathfrak{D}^{GPS}$ of $(\tilde{Y}, \tilde{D}) \to (\bar{Y}, \bar{D})$[5]. We refer readers to [21, Section 3.4] for further information.

Remark 2.13. *It is expected that the orbifold analogue of the work of Gross-Hacking-Keel mirror construction also hold. However, the author is not aware of that in the literature.*

2.5. **Lagrangian Floer theory.** We briefly review Lagrangian Floer theory of a compact Lagrangian and its bulk deformation following [15]. We begin by fixing a generic almost complex structure J. For a compact Lagrangian L in a symplectic manifold Y, one can assign a possibly curved A_∞-algebra structure on $H^*(L; \Lambda)$ as follows. If we write $\mathcal{M}_{k+1}(Y, L, \beta; J)$ for the moduli space of J-homomorphic discs in class $\beta \in \pi_2(Y, L)$ with $k + 1$ boundary marked points $z_0, \cdots, z_k$ then one has

(2.5) $\quad m_{k,\beta}(h_1, \cdots, h_k) := (ev_0)_!(ev_1, \cdots, ev_k)^*(h_1 \times \cdots \times h_k), \quad m_k = \sum_{\beta \in \pi_2(X,L)} m_{k,\beta} T^{\omega(\beta)}$

for $k \geq 0$ and $h_i \in H^*(L; \Lambda)$ where ev_i is the evaluation at z_i. We additionally put $m_{1,0}(h) = (-1)^{n + \deg x + 1} d_{dR} h$. The degree of $m_{k,\beta}$ is given by $2 - k - \mu(\beta)$ where $\mu(\beta)$ is the Maslov index of the class $\beta \in \pi_2(X, L)$, which directly follows from

(2.6) $$\dim \mathcal{M}_{k+1}(Y, L, \beta; J) = n + \mu(\beta) + k - 2.$$

The family $\{m_k\}_{k \geq 0}$ of multilinear operations satisfies the quadratic relations

$$\sum_{k_1 + k_2 = k+1} (-1)^\star m_{k_2}(h_1, \cdots, m_{k_1}(h_i, \cdots, h_{i+k_1-1}), \cdots, h_k) = 0$$

---

[5]Here we identify m_i with $-\phi_i(v_i)$ in [21].

called the A_∞-relations, where $\star = |h_1|' + \cdots + |h_{i-1}|'$. When $m_1^2 = 0$, then the resulting homology is referred to as the Lagrangian Floer cohomology of L.

The formula (2.5) includes the case where the term $m_0(1) := m_0 \in H^*(L; \Lambda_+)$ is nontrivial, which gives an obstruction for m_1 to become a differential. In order to derive any meaningful homological invariants, one should deform the A_∞-structure by so called a weak bounding cochain $b \in H^1(L; \Lambda_+)$, which satisfies

$$(2.7) \qquad m_0(1) + m_1(b) + m_2(b, b) + \cdots = W(b) \cdot 1_L$$

where $W(b)$ is a Novikov constant depending on b, known as the superpotential of L. If (2.7) has a solution, then L is said to be weakly unobstructed. One can easily see from (2.6) that $W(b)$ is contributed by the Maslov index 2 discs.

When L is a Lagrangian in $Y \setminus D$ for a log Calabi-Yau pair $(Y, D = \cup_i D_i)$ with D_i irreducible and $c_1(D_i)$ positive, those discs are precisely the ones that hit D exactly once. If some of D_i is not positive, then one should additionally take into account the higher Maslov discs attached with negative or zero Chern number sphere bubbles. Typical examples are toric manifolds with L being a toric fibre, see for e.g., [16]. It is conventional that one uses the exponential coordinate to write $W(b)$ in such a case. That is, we set

$$(2.8) \qquad z_i := \exp x_i \qquad i = 1, \cdots n$$

where we take $b = \sum x_i d\theta_i$ where $\{d\theta_i : i = 1, \cdots n\}$ is a basis of $H^1(L)$. Throughout, we will use the notation

$$z^{\partial\gamma} := z_1^{(\partial\gamma, d\theta_1)} \cdots z_n^{(\partial\gamma, d\theta_n)},$$

for $\partial\gamma \in H_1(L, \mathbb{Z})$, where (,) is the natural pairing between homology and cohomology.[6] It is worthwhile to mention that $z^{\partial\gamma}$ is independent of the choice of the basis $\{d\theta_i\}$ and the notation is thus intrinsic.

The exponential coordinate also helps us to include holonomy variables for flat $\mathbb{C}^*$-connections on L more naturally, which enables us to expand the set of solutions b of (2.7) over Λ_0. We remark that the coordinate change (2.8) makes sense, provided the divisor axiom [13]:

$$(2.9) \qquad \sum_{\sum n_i = n} m_{k+n, \beta}(b^{\otimes n_0}, h_1, b^{\otimes n_2}, \cdots, b^{\otimes n_{k-1}}, h_k, b^{\otimes n_k}) = \frac{1}{n!}(\partial\beta, b)^n m_k(h_1, \cdots, h_k)$$

for $\beta \in \pi_2(Y, L)$, which tells us that the variable x_i in $W(b)$ always appears as the exponential.

Let us denote by $n_\beta(L) = n_\beta^Y(L)$ the number of the holomorphic discs in class β with $\mu(\beta) = 2$ whose boundary passes through a generic point in L, or in other words, $n_\beta(L)$ is the degree of the map $ev_0 : \mathcal{M}_1(Y, L; \beta) \to L$ provided the weakly unobstructedness (2.7) (and all necessary transversality). It is easy to see that the coefficient of $z^{\partial\beta}$ in W is given by $n_\beta(L)T^{\omega(\beta)}$. Alternatively, one can define W as a function on the 'moduli space' of the pair (L, ∇) where L varies over torus fibres and ∇ is a flat $U(1)$-connection on the trivial line bundle $L \times \mathbb{C}$ by

$$(2.10) \qquad W(L, \nabla) = \sum_{\beta, \mu(\beta)=2} n_\beta(L)T^{\omega(\beta)} hol_{\partial\beta}\nabla,$$

which fits more into the SYZ setup (see [1] for more details). The superpotential presented here can be thought of as a local restriction of (2.10).

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[6]We use $\partial\gamma$ to denote a general element in $H_1(L, \mathbb{Z})$, as we will frequently look at classes in $H_1(L, \mathbb{Z})$ appearing as the boundary class of a holomorphic disc.

2.5.1. *Bulk-Deformed A_∞ Structures and Bulk-Deformed Superpotentials.* One can further deform the A_∞-structure on $H^*(L; \Lambda)$ along the direction of $H^*(X)$ by inserting interior marked points to holomorphic discs and requiring them to pass through (Poincare duals of) given ambient cocycles in $H^*(X)$. It is called the bulk-deformation of the Floer theory of L.

Suppose an ambient cocycle $\mathfrak{b} \in H^{even}(X; \Lambda_+)$ is given. We first consider for each $\beta \in \pi_2(X, L)$ $\mathcal{M}_{k+1,l}(L, \beta, J)$, the moduli space of stable J-holomorphic discs with $k+1$ boundary marked points and l interior marked points representing the class β . Its dimension is given by $n + \mu(\beta) + k - 2 + 2l$ where the additional $2l$ compared with (2.6) comes from the freedom of locations of interior markings. We then impose the incidence condition to the interior markings by taking fibre product to obtain

$$(2.11) \qquad \mathcal{M}_{k+1,l}(L, \beta, J; \mathfrak{b}^{\otimes l}) := \mathcal{M}_{k+1,l}(L, \beta, J)_{ev^{int}} \times_{X^l} \prod_{i=1}^{l} PD[\mathfrak{b}].$$

where ev^{int} is the evaluation map associated with interior marked points. Finally, the bulk-deformed A_∞-operations are defined by

$$\mathfrak{m}_{\beta,k}^{\mathfrak{b}}(h_1, \cdots, h_k) = \sum_l \frac{1}{l!} (ev_0)_! (ev_1, \cdots, ev_k)^*(h_1 \wedge \cdots \wedge h_k), \quad \mathfrak{m}_k^{\mathfrak{b}} = \sum_\beta \mathfrak{m}_{\beta,k}^{\mathfrak{b}} T^{\omega(\beta)}$$

Notice that $\mathfrak{m}_{\beta,k}^{\mathfrak{b}}$ is contributed by $\mathcal{M}_{k+1,l}(L, \beta, J; \mathfrak{b})$ with l varying over $\mathbb{Z}_{\geq 0}$.[7]

The superpotential $W(b)$ also deforms accordingly now by solving the analogue equation for $m_k^{\mathfrak{b}}$:

$$m_0^{\mathfrak{b}}(1) + m_1^{\mathfrak{b}}(b) + m_2^{\mathfrak{b}}(b, b) + \cdots = W^{\mathfrak{b}}(b) \cdot 1_L$$

and the resulting $W^{\mathfrak{b}}$ is called the bulk-deformed potential. In [32], the second and third authors considered a special type of a bulk-deformation of the Floer theory of toric fibres in a Fano surface, for which $\mathfrak{b}$ is taken to be a linear combination $\sum t_i q_i$ of generic points with $t_i^2 = 0$. One needs to impose l-many point-constraints to discs of Maslov index $\mu = 2l + 2$ in order for them to contribute to $W^{\mathfrak{b}}$. The dimension formula of the disc moduli (see Section 2.5.1) tells us that these constrained discs behave like Maslov 2 discs, and more generally, the *generalized Maslov index* μ' of a holomorphic disc was introduced in [32] as twice the number of point-constraints subtracted from the ordinary Maslov index μ (i.e., $\mu' = \mu - 2l$). Notice that (2.1) is precisely its tropical analogue. $W^{\mathfrak{b}}$ shows certain discontinuity when L varies over toric fibres due to the existence of *generalized* Maslov zero discs. Such a phenomenon is generally called a wall-crossing, which we recall below in more general context.

2.5.2. *Pseudo-isotopies and wall-crossing.* Consider two Lagrangian submanifold L and L' in (X, ω) which are related by smooth isotopy ϕ_t. Namely, ϕ_t is a diffeomorphism on X such that $\phi_0 = id$ and $\phi_1(L) = L'$. We further assume that for each t, the almost complex structure $J_t = (\phi_t)^* J$ is tame with respect to ω. We can relate the superpotentials for L and L' in the following way.

Notice that $J_0(= J)$-holomorphic discs bounding L' isotope to J_1-holomorphic discs bounding L thorough ϕ. The Floer theory of L does not depend on the choice of almost complex

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[7]Strictly speaking, (2.11) makes sense when $\mathfrak{b}$ is a single geometric cycle, not a linear combination of such. In the latter case, however, we can simply defined $\mathfrak{m}_{\beta,k}^{\mathfrak{b}}$ by expanding it linearly in accordance with the linear combination forming $\mathfrak{b}$.

structures in the following sense: the isotopy results in an A_∞-algebra isomorphism

$$\left\{ f_k = \sum_\beta f_{k,\beta} T^{\omega(\beta)} \right\}_{k \geq 0} : H^*(L; \Lambda) \to H^*(L'; \Lambda),$$

where $f_{k,\beta}$ is obtained by counting discs in

$$\cup_{t \in [0,1]} \mathcal{M}_{k+1}(L, \beta; J_t)$$

whose boundary marked points are subject to suitable incidence conditions. The degree of $f_{k,\beta}$ is given by $1 - k - \mu(\beta)$.

The A_∞-homomorphism $\{f_k\}_{k \geq 0}$ induces a map between associated sets of weak bounding cochains

$$(2.12) \qquad\qquad f_* : b \mapsto f_0(1) + f_1(b) + f_2(b, b) + \cdots .$$

This construction is now broadly referred to as *Fukaya's trick* since it has first appeared in [13]. See [55] for more recent development.

Lemma 2.14. [13] *Let L and L' be two Lagrangians that can be interpolated by an isotopy ϕ.*

(1) *the superpotential W and W' of L and L' satisfy*

$$W'(f_*(b)) = W(b)$$

after rescaling the term by suitable powers of the Novikov parameter.[8]

(2) *The map f_* only depends on the homotopy type of the isotopy ϕ_t. Namely, if there is an isotopy $\Phi_{s,t}$ over $[0,1]^2$ between two isotopies $(\phi_1)_t = \Phi_{1,t}$ and $(\phi_2)_t = \Phi_{1,2}$ from L and L', then ϕ_1 and ϕ_2 induced the same map f_* on the weak Maurer-Cartan space of L and L'.*

The contributions to (2.12) are from discs with $\mu(\beta) = 0$, and by dimension reason, a generic Lagrangian torus fibre does not bound such discs. Collecting Lagrangians bounding Maslov zero discs forms a structure called the wall in the base B of the torus fibration, which is conjecturally of real codimension 1 in B for generic almost complex structures.

Let us now focus on the situation where X admits a special Lagrangian torus fibration $\pi : X \to B$ with respect to a holomorphic volume for Ω, and L, L' are two different fibres of π. Given a reference point $u_0 \in B_0$ and a choice of the basis $\check{e}_1, \check{e}_2 \in H_1(L_{u_0}, \mathbb{Z})$, we will define the local affine coordinates around u_0. For any $u \in B_0$ in a small neighborhood of u_0, one choose a path ϕ contained in B_0 connecting u, u_0. Let C_k to be the S^1-fibration over the image of ϕ such that the fibres are in the homology class of parallel transport of $\check{e}_k$ along ϕ. Then the local complex symplectic affine coordinates are defined by

$$(2.13) \qquad\qquad x_{\check{e}_k}(u) = \int_{C_k} \text{Im}\,\Omega.$$

It is straight-forward to check that the transition functions fall in $GL(2, \mathbb{Z}) \ltimes \mathbb{R}^2$, and thus the above coordinates give an integral affine structure on B_0. We will say that the affine line $x_k = const$ is an affine line defined by $\check{e}_k$.

One can show that the loci of torus fibres bounding Maslov zero discs form an affine line in B with respect to these affine coordinates (see, for e.g., [39, Proposition 5.6]). Such a line is called the wall, and the above discussion tells us that f_* is nontrivial only if the isotopy goes

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[8]We refer readers to [51, Lemma 4.2] for details on the rescaling factors. This will not be needed for our purpose, since L and L' in the application are infinitesimally close to each other.

across some wall, in which case we call f_* a wall-crossing transformation. Strictly speaking, the wall should be defined as the loci of fibres L_u which admit nontrivial open Gromov-Witten invariants $\tilde{\Omega}(\gamma; u)$ for some relative class $\gamma \in H_2(X, L_u)$. See [38, Definition 4.12] for the precise definition of the invariants. If the underlying isotopy goes across the wall at the point u, one can roughly think of f_* as the generating series of $\tilde{\Omega}(\gamma; u)$.

Theorem 2.15. [39, Theorem 6.15] *Let $\pi : X \to B$ be a special Lagrangian fibration on a symplectic manifold M with $\dim_{\mathbb{R}} M = 4$. Suppose there exists a wall in B contributed by a disc class $\gamma \in \pi_2(X, L)$ of Maslov index zero. Then the wall is an affine line (ray) along the direction corresponding to $\partial\gamma \in H^1(L)$, and the associated wall-crossing transformation is of the form*

$$(2.14) \qquad\qquad z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_\gamma^{\langle \gamma', \gamma \rangle}.$$

where $f_\gamma \in 1 + \Lambda[[T^{\omega(\gamma)} z^{\partial\gamma}]]$.

The last author has shown that these walls attached with f_γ (as wall functions) all together form a scattering diagram defined in Section 2.2. Its construction is purely based on Lagrangian Floer theory, as was speculated in [14,36]. Notice that the wall-crossing transformations in (2.14) preserve $\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$. When a new wall produced by the collision of walls, the new walls and wall functions can thus be captured by (2) of Lemma 2.14 together with the so-called Kontesvich-Soibelman lemma (see e.g. [36, Theorem 6]). See [39] for more details. The precise statement is given as follows.

Theorem 2.16. *Let (X, ω) be a symplectic manifold of dimension 4 and let J be a tamed almost complex structure. Assume that $X' \subseteq X$ with $X' \to B$ be a Lagrangian fibration such that*

(1) *there are no singular fibres over B;*
(2) *there exists an affine structure on B such that the loci of fibres bounding holomorphic discs in a fixed class (up to parallel transport) are constrained in an affine line;*
(3) *the moduli space $\mathcal{M}(X, L, \gamma; J)$ of stable discs is compact for a fibre L of $X' \to B$ and $\gamma \in H_2(X, L)$;*
(4) *no fibres bound J-holomorphic discs of negative Maslov index.*

Then there exists a consistent scattering diagram $\mathfrak{D}^{LF}$ on B constructed from pseudo-isotopies between fibres.

The consistency of $\mathfrak{D}^{LF}$ is a direct consequence of (2) of Lemma 2.14 provided the absence of negative Maslov discs. In fact, the path-ordered product in this context is nothing but the composition of f_* (2.12) for each wall that the loop goes across. The condition (1) tells us that the isotopy along the loop can be contracted to a trivial loop, and (4) ensures that the contraction gives rise to a trivial homotopy between the composition of f_*'s and the identity map.

3. Floer theory on toric surfaces

In this section, we prove some useful facts about holomorphic discs whose boundaries lie on the moment fibre $L_u = p^{-1}(u)$, which will be the crucial ingredient for the proof of our main theorem in Section 5. We remark that holomorphic discs bounding L_u are completely classified by the work of Cho-Oh [8], and they admit very concrete coordinate-wise descriptions from which we can read off much of topological information of discs. Our task here is to assign a tropical disc to each of these holomorphic discs, whose end lies at u,

and whose intersection pattern with the boundary divisors precisely match with the original holomorphic disc.

In the second half, we compute the superpotentials for semi-Fano toric surfaces by means of tropical counting. While they have already been computed in the previous work of Chan-Lau [7], the method we use here is completely different, and is of independent interest. These toric potentials serve as a stepping stone to compute potentials for general log Calabi-Yau surfaces thorough blowup and blowdown process (under some non-negativity assumption).

3.1. **Construction of tropical discs from holomorphic discs in toric surfaces.** We will establish a weak correspondence between holomorphic and tropical discs for toric surfaces in this subsection. More precisely, to a holomorphic disc with a fixed set of point-constraints (on the toric boundary), we assign a tropical disc whose unbounded edges have directions and locations determined by the point-constraints. This will impose an obstruction for existence of certain holomorphic discs to exist, which will be crucial in proving Theorem 5.4.

Let us begin with the following useful fact about holomorphic discs in toric manifolds. Given a moment torus fibre L in a toric manifold $\bar{Y}$, up to $(\mathbb{C}^*)^n$-action, we may assume that L is the fibre over the origin in $\mathbb{R}^n$ which is identified with $\text{Int } P$ via the Legendre transformation. Then L is the fixed locus of the anti-holomorphic involution $\sigma : (z_1, \cdots, z_n) \mapsto (\frac{1}{\bar{z}_1}, \cdots, \frac{1}{\bar{z}_n})$ on $(\mathbb{C}^*)^n$. Any holomorphic disc with boundary on L restricted to $(\mathbb{C}^*)^n(\subset \bar{Y})$ can be doubled, and then extend to a rational curve in $\bar{Y}$. This gives the following lemma.

Lemma 3.1. *Let $\bar{Y}$ be a toric manifold, and L a moment fibre. Given a holomorphic disc $f : (D^2, \partial D^2) \to (\bar{Y}, L)$, there exists a unique rational curve C_f in $\bar{Y}$ such that it contains the image of f, and that $C_f \cap (\mathbb{C}^*)^n$ is invariant under σ.*

We now restrict to the case $n = 2$. Suppose that a curve C in $(\mathbb{C}^*)^2$ is defined to be the zero loci of a Laurent polynomial $F = F(z_1, z_2)$. Consider the amoeba $\mathcal{A}$ of C, the image of C under the log map

$$(3.1) \qquad \begin{array}{rcl} \text{Log}: \quad (\mathbb{C}^*)^2 & \to & \mathbb{R}^2 \\ (z_1, z_2) & \mapsto & (\log|z_1|, \log|z_2|). \end{array}$$

We recall the *Ronkin function* $N_F(u)$ [46, 48] defined on $\mathbb{R}^2$. It is given as the push-forward of $\log|F|$, i.e.

$$N_F(u) = \frac{1}{(2\pi i)^2} \int_{L_u} \log|F| \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}$$

for $u \in \mathbb{R}^2$. Notice that N_F is well-defined even for u lying in $\mathcal{A}$. It is also known that $N_F : \mathbb{R}^2 \to \mathbb{R}$ is a convex function, strictly convex on $\mathcal{A}$ and is linear (up to translation) on each connected component of the complement of $\mathcal{A}$. For E a component of $\mathbb{R}^2 \setminus \mathcal{A}$, we write $N_F^E := N_F|_E$ for this (restricted) linear function, and finally set $N_F^\infty := \max_E N_F^E$.

The spine $\mathcal{S}$ of $\mathcal{A}$ (or of C) is now defined as the corner locus of N_F^∞, i.e. the locus where N_F^∞ is not locally linear. On a component E of $\mathbb{R}^2 \setminus \mathcal{A}$, N_F^E can be written as $N_F^E = \nu_1^E u_1 + \nu_2^E u_2 + a_E$, and in this case, $\nu_E := (\nu_1^E, \nu_2^E) \in \mathbb{Z}^2$ is called the index of the component E[9]. Then $\mathcal{S}$ can be also described as the zero loci of the tropical polynomial $\sum_{\nu_E} t^{a_E} z_1^{\nu_1^E} z_2^{\nu_2^E}$ and hence, $\mathcal{S}$ is a tropical curve by definition.

Now we are ready to prove that every holomorphic disc with boundary on the moment map fibre has a (not necessarily unique) corresponding tropical disc. The idea is to use the

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[9]From the Jensen's formula [3], the index ν_E can be determined by the intersection number of C_f with certain holomorphic discs.

spine $\mathcal{S}$ of the doubling C_f of a holomorphic disc, but the tricky part here is that in general, the genus of the spine can be strictly larger than that of the original curve (see Remark 2.3 [44] for such an example). For this reason, we will make a further modification of $\mathcal{S}$ in the proof.

Lemma 3.2. *Let $\bar{Y}$ be a toric surface and consider a moment fibre L_u over $u \in \mathbb{R}^2$. For each holomorphic disc $f : (D^2, \partial D^2) \to (\bar{Y}, L_u)$, there exists a tropical disc with stop at u that satisfies the following:*

- *if the holomorphic disc f intersects a component of the toric boundary divisor at a point of multiplicity m, then the corresponding tropical disc has an unbounded edge defined by the vanishing cycle of the toric boundary and of weight m.*
- *the edge adjacent to u is defined by $\partial[Im(f)]$.*

Proof. Lemma 3.1 states that there exists a unique rational curve $C_f \subseteq \bar{Y}$ containing $\mathrm{Im}f$. Let $\mathcal{S}$ be the corresponding spine. The $\mathbb{Z}_2$-symmetry on the rational curve C_f implies that

(1) $\mathcal{A}$ is symmetric under $u \mapsto -u$,
(2) the defining equation F of C_f satisfies $F(z_1^{-1}, z_2^{-1}) = z_1^a z_2^b F(z_1, z_2)$ for some $a, b \in \mathbb{Z}$.

Then straightforward calculation using the definition of N_F shows that $N_F(u) = N_F(-u)$ if $u \notin \mathcal{A}$. Thus, the tropical polynomial of $\mathcal{S}$ is also $\mathbb{Z}_2$-invariant. As mentioned, we will modify $\mathcal{S}$ into another tropical curve $\mathcal{S}'$ in order to ensure that the resulting tropical curve is of genus 0.

Let Δ' be the Newton polytope of the tropical polynomials of $\mathcal{S}$, that is, the convex hull of the set $\{\nu_E\}$, where E varies over all the components of $\mathbb{R}^2 \setminus \mathcal{A}$. Then we take $\mathcal{S}'$ to be the tropical curve defined by the tropical polynomial $\sum_{\nu_E \in \partial\Delta'} t^{a_E} z_1^{\nu_1^E} z_2^{\nu_2^E}$. From [41, Proposition 3.11], the tropical curve $\mathcal{S}'$ is dual to a subdivision of Δ'. Since the coefficient corresponding to any interior lattice point of Δ' is zero, we see that $\mathcal{S}'$ is a tree (it has genus 0). On the other hand, the coefficients of $\mathcal{S}$ and $\mathcal{S}'$ over the lattice points on $\partial\Delta'$ still remain the same. Hence, the sets of affine lines obtained by extending the unbounded edges of $\mathcal{S}$ and $\mathcal{S}'$ agree with each other, which are determined by the intersection of C_f with the toric boundary of $\bar{Y}$. Lastly, it is obvious that $\mathcal{S}'$ is $\mathbb{Z}_2$-invariant.
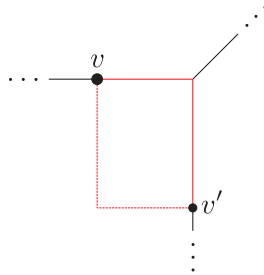

FIGURE 2. $\mathbb{Z}_2$-symmetry of a tropical curve

We claim that u is contained in $\mathcal{S}'$. Let v be a vertex in $\mathcal{S}'$, and v' its reflection image, $v = \underline{\sigma}v'$, which also lies in $\mathcal{S}'$. We may assume $v \neq v'$. There exists a path in $\mathcal{S}'$ connecting v and v'. If $u \notin \mathcal{S}$, then the image of the path under $\underline{\sigma}$ provides another path between v and v', which contradicts the fact that $\mathcal{S}'$ is a tree (see Figure 2). Since $\mathcal{S}'$ is a tropical curve of genus zero which is symmetric to u, one can halve it to gain a tropical disc with stop at u. The last part of the lemma is a consequence of balancing condition at all vertices. □

We emphasize again that the lemma provides a strict obstruction of existence of holomorphic discs.

3.2. **Tropical interpretation of the potentials for toric semi-Fano surfaces.** We end this section with finding out the relationship between the bulk-deformed superpotential of a toric Fano surface and the superpotential of a toric semi-Fano surface obtained from the Fano surface by toric blowup. Any toric semi-Fano surface can be obtained in this way, with the exception of the Hirzebruch surface $\mathbb{F}_2 = \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-2))$. This process provides a means for calculating the superpotential on any toric semi-Fano surface (except $\mathbb{F}_2$) using a tropical calculation on a toric Fano surface.

Let Y be a semi-Fano toric surface with $\tilde{Y}$ a semi-Fano toric blowup of Y (at a single point). Note that Y and $\tilde{Y}$ are necessarily smooth. Let $\tilde{w}$ be the primitive vector in the 1-cone of the fan for $\tilde{Y}$ not appearing in the fan for Y. We have that $\tilde{w} = w_1 + w_2$, where w_1 and w_2 are primitive vectors in adjacent 1-cones of the toric fan for Y. Let p be a point in $M_{\mathbb{R}}$ and consider the scattering diagram $\mathfrak{D}_t$ associated with the point $p - t\tilde{w}$. That is, $\mathfrak{D}_t$ is the scattering diagram with one ray $(\mathfrak{d}_v, f_{\mathfrak{d}}) = (p - t\tilde{w} + v\mathbb{R}_{\geq 0}, 1 + z^{-v}t)$ for each primitive vector v in a 1-cone of the fan for Y.

As t approaches infinity, one chamber of $\mathfrak{D}_t$ exhausts $M_{\mathbb{R}}$. This is the open cone spanned by w_1 and w_2 based at $p - t\tilde{w}$, and we will let U denote it. Letting $p_1, \ldots, p_n$ be generic points in U, we will study the generalized Maslov index 2 tropical discs (with respect to the points $p, p_1, \ldots, p_n$) with stop in U.

We establish some facts about the toric fan of Y. Since Y is assumed to be smooth, w_1 and w_2 form an integral basis of M.

Lemma 3.3. *Let Y, $\tilde{Y}$, w_1 and w_2 be as above. That is, let Y be a toric semi-Fano surface with $\tilde{Y}$ a semi-Fano toric blowup of Y (at a single point). Let $\tilde{w}$ be the primitive vector in the 1-cone of the fan for $\tilde{Y}$ not appearing in the fan for Y, and let w_1 and w_2 be the primitive vectors in the adjacent 1-cones with $w_1 + w_2 = \tilde{w}$. Given any primitive vector w lying in a 1-cone of the toric fan of Y, we have that $w = bw_1 + cw_2$, where $b, c < 2$.*

Given the exhaustive description of semi-Fano toric surfaces given by Chan-Lau [7], this could be checked directly. We present an elementary inductive proof here.

Proof. By the smoothness of Y, we can write $w = bw_1 + cw_2$ for some integers b, c. By assumption, the 1-cones corresponding to w_1, w_2 are adjacent in the fan of Y, in the sense that they bound a 2-cone of the fan, so b and c cannot both be positive. Without loss of generality, we will assume $c > 0$ and show $c = 1$. We will also assume that $\det(w_1, w_2) = 1$, i.e. that w_2 is counterclockwise from w_1 (this allows us to fix a choice of signs).

Let $w_3 = b_3 w_1 + c_3 w_2$ be the primitive vector in the next 1-cone after that of w_2, proceeding counterclockwise. We assume that $c_3 > 0$ and hence that $b_3 < 0$. Since $\tilde{Y}$ is semi-Fano, the self intersection number in $\tilde{Y}$ of the divisor corresponding to the 1-cone containing w_2 is at least -2. It is known that this self intersection number is equal to $-\det(w_1 + w_2, w_3) = -c_3 + b_3 \leq -2$, with equality only when $b_3 = -1$ and $c_3 = 1$. Hence, $w_3 = -w_1 + w_2$. We induct, assuming $w_n = -(n-2)w_1 + w_2$. Let $w_{n+1} = b_{n+1} w_1 + c_{n+1} w_2$ be the primitive vector in the next 1-cone counterclockwise from that of w_n, and assume that $c_{n+1} > 0$. Because w_{n+1} is counterclockwise from w_n, we get that that $b_{n+1} < -(n-2)$. From the semi-Fano condition on $\tilde{Y}$, it follows that the self intersection number of the divisor in $\tilde{Y}$ corresponding to the 1-cone containing w_n is greater than or equal to -2, so $-\det(w_{n-1}, w_{n+1}) = -\det \begin{pmatrix} -(n-3) & b_{n+1} \\ 1 & c_{n+1} \end{pmatrix} = b_{n+1} + (n-3)c_{n+1} \geq -2$.

Since w_{n+1} is counterclockwise from w_{n-1}, we have that $0 < \det \begin{pmatrix} -(n-3) & b_{n+1} \\ 1 & c_{n+1} \end{pmatrix}$, so $-1 \geq b_{n+1} + (n-3)c_{n+1} \geq -2$. Since $b_n \leq -(n-1)$ and $c_n \geq 1$, it follows that $b_{n+1} = -(n-1)$ and $c_n = 1$.

Returning to our original $w = bv_1 + cv_2$, since we assumed $c > 0$, we see that w must eventually appear in this way, and hence that $c = 1$, as desired.

Note that the divisor corresponding to w_n in $\tilde{Y}$ must have self intersection -2 for n greater than 1 and less than the maximum attained value. In the maximum case, i.e. when $w_n - w_1$ does not lie in a 1-cone of $\tilde{Y}$, we see that the self intersection of the divisor corresponding to w_n is strictly greater than -2, since our smoothness condition gives $\det \begin{pmatrix} -(n-2) & b \\ 1 & c \end{pmatrix} = -c(n-2) - b = 1$, and the self intersection will equal $-\det \begin{pmatrix} -(n-3) & b \\ 1 & c \end{pmatrix} = c(n-3) + b = -1 - c$, and $c \leq 0$. $\qquad\square$

The proof of this lemma immediately gives the following lemma.

Lemma 3.4. *Let Y, $\tilde{Y}$, w_1 and w_2 be as in Lemma 3.3, and let $w = bw_1 + cw_2$ be a primitive vector lying in a 1-cone of the toric fan of Y. If $c > 0$, then $w_1 + w$ is a primitive vector lying in a 1-cone of the toric fan of $\tilde{Y}$. Likewise, if $b > 0$, then $w_2 + w$ is a primitive vector lying in a 1-cone of the toric fan of $\tilde{Y}$.*

Proof. In the notation of the proof of Lemma 3.3, our first statement amounts to observing, for $n \geq 3$, that $w_1 + w_n = w_{n-1}$. $\qquad\square$

Let $h : T \to M_{\mathbb{R}}$ be a generalized Maslov index 2 disc (with respect to the points $p, p_1, \ldots, p_n$) with stop in the chamber U of $\mathfrak{D}_t$ described above. This is the open (affine) cone based at p determined by w_1 and w_2. We establish the following lemma regarding tropical discs with multiple unbounded edges entirely outside of a closed half plane (which will, in our uses, contain the chamber U).

Lemma 3.5. *Consider generic points $p, p_1, \ldots, p_n$, all lying outside an open half plane H, and h a generalized Maslov index 2 tropical disc with stop outside H. Then each connected component of $h^{-1}(H)$ can contain at most one unbounded edge of T.*

Intuitively, this says that, to get from one unbounded edge contained in H to any other, our path must leave H.

Proof. Since our points $p, p_1, \ldots, p_n$ are generic, our moduli space of generalized Maslov index 2 tropical discs with a given stop must be finite. It thus suffices to show that, given any generalized Maslov index 2 disc h, if a connected component of $h^{-1}(H)$ has multiple unbounded edges with image contained entirely in H, then h lies in a positive dimensional family of tropical discs with the same stop and marked point constraints.

Choose a connected component of $h^{-1}(H)$. We obtain a new object $h' : T' \to M_{\mathbb{R}}$ by removing every edge and vertex of h that does not intersect our chosen component of $h^{-1}(H)$ and modifying the map so that it maps the newly unbounded edges onto rays. By our connectedness assumption, T' a tree. This new object h' is like a tropical curve with all vertices contained in H, except its unbounded edges that are not contained in H may be in non-toric directions, since their directions may have been inherited from bounded edges of h. This can be understood as a tropical curve of degree Δ in some toric blowup of Y. We let $q_1, \ldots, q_m$ be points in the complement of H lying on unbounded edges of h'. Let $\mathcal{M}$ be the moduli space of these tropical curves of degree Δ with image covering all points $q_1, \ldots, q_m$.

Note that each such tropical curve has $m + \ell$ unbounded edges. By a standard argument, we find that the dimension of each connected component of $\mathcal{M}$ is $\ell - 1$. This then gives us an $\ell - 1$ dimensional family of generalized Maslov index 2 discs relative to the original point constraints. Since ℓ in the case of h' is precisely the number of unbounded edges with image entirely contained in H, we have the desired result. $\qquad\square$

Assume h has some edge e with p lying on $h(e)$, and let $v(e)$ be the primitive vector tangent to $h(e)$ pointing away from the vertex that comes "before" e as one moves outward from the stop. We consider two cases, depending on whether or not $v(e)$ is in the closed cone generated by w_1 and w_2.

If $v(e)$ lies outside this closed cone, we will see that e must be an unbounded edge. Writing $v(e) = bw_1 + cw_2$, we have that either $b < 0$ or $c < 0$, and we will assume without loss of generality that $c < 0$. We then assume that e is bounded, so $v(e)$ is pointing toward another vertex, which lies inside the open half plane H of points $p + a_1 w_1 + a_2 w_2$ with $a_2 < 0$. We will derive a contradiction, and conclude that e must be unbounded.

Since we have this vertex lying in H, we must have at least one unbounded edge lying entirely in H. Noting that the complement of H contains all of our marked points, by Lemma 3.5 we get that any path along the disc from one unbounded edge lying entirely in H to another must leave H. Let e' be the unbounded edge contained in H that can be connected to e by a path in the disc lying entirely in H, and let $e_1, \ldots, e_m$ be all of the edges leaving H, starting at e'. Next, let $v(e_i)$ be the primitive vector tangent to $h(e_i)$ pointing into H, and let $v(e')$ be the primitive vector in the unbounded $h(e')$ direction. Letting $w(e_i)$ be the weight of the edge e_i, the vertex balancing condition gives that $v(e') = \sum_{i=1}^{m} w(e_i) v(e_i)$. Here we are using that e' will have weight 1. Since each $v(e_i)$ has negative w_2 coordinate, and $m \geq 2$, it follows that $v(e')$ has w_2 coordinate less than or equal to -2. By Lemma 3.3, we have a contradiction. Thus, e must be unbounded.

If instead $v(e)$ lies inside the closed cone generated by w_1 and w_2, then we apply the above reasoning to $-v(e)$. This would show that e is unbounded, with $-v(e)$ pointing in the unbounded direction, which contradicts our definition of $v(e)$. Thus, we can conclude that $v(e)$ cannot lie inside this closed cone, and we have only the previous case.

To recap, we have shown that e is unbounded, and that the vector $v(e)$ pointing in the unbounded direction does not lie in the closed cone generated by w_1 and w_2. Now, if $v(e)$ is not equal to either $-w_1$ or $-w_2$, we have $v(e) = bw_1 + cw_2$ with (without loss of generality) $b < 0$ and $c > 0$. This uses the fact that the 1-cones corresponding to w_1 and w_2 are adjacent. We can then apply the above reasoning again with $-v(e)$ and the open half plane with negative w_2 coordinate to derive a contradiction. Thus, we have that $v(e)$ equals either $-w_1$ or $-w_2$.

Next, considering the unique vertex adjacent to e, we see that the image of the vertex lies on the ray in the w_i direction (for some i) starting at p. One of the two other edges adjacent to this vertex must have image lying out of the open cone based at p and spanned by w_1, w_2. Applying the above argument again gives that this edge must be unbounded. Combining all of these observations with Lemma 3.4 gives the following lemma.

Lemma 3.6. *Let Y, w_1, and w_2 be as above. Let $p, p_1, \ldots, p_n$ be point constraints positioned as above, so that $p_1, \ldots, p_n$ all lie in the exhausting chamber of the scattering diagram determined by the point $p - t(w_1 + w_2)$ for large t. Let h be a tropical disc in Y with generalized Maslov index 2 with respect to the point constraints $p, p_1, \ldots, p_n$, and assume there is some edge e of h with $h(e)$ containing p. Then e is unbounded, with its outward primitive tangent vector equal $-w_i$ for i equal to 1 or 2, and one of the two edges sharing a vertex with e is*

also unbounded, with outward pointing primitive tangent vector having negative w_j component ($j \neq i$). The third edge e' adjacent to this vertex must then have weight 1, and the primitive vector $v(e')$ tangent to $h(e')$ and pointing toward the vertex is either the vector $-w_1 - w_2$ or the negative of a primitive vector belonging to a 1-cone of the toric fan of Y.

This lemma allows us to completely describe the generalized Maslov index 2 discs in Y with respect to the point constraints $p, p_1, \ldots, p_n$ in terms of the generalized Maslov index 2 discs in $\tilde{Y}$ with respect to the point constraints $p_1, \ldots, p_n$. Every such tropical disc in Y through the point p can be "clipped" by removing the (unbounded) edge through p and its neighboring vertex, and extending the only surviving edge (now unbounded) to infinity.

We now describe a procedure for obtaining the superpotential on $\tilde{Y}$ from the superpotential on Y using a single point constraint p. Let U be our chamber of interest, as before, and let W_Y be the superpotential for Y. Since Y is toric semi-Fano, Chan-Lau [7] tells us that W_Y has a term $a_v z^v$ for each primitive vector v in a toric direction, where a_v is 1 if the toric divisor corresponding to v has self intersection not equal to -2, and is otherwise determined in the following way: Let m_v be the (total) number of consecutive -2 toric divisors in the run of consecutive -2 toric divisors containing the divisor corresponding to v, and let ℓ_v be the position of the divisor corresponding to v within this run (counting from either end). Then, in the terminology of Chan-Lau, a_v is equal to the number of admissible sequences with center ℓ_v of length at most m_v. Numerically, we have $a_v = \binom{m_v+1}{\ell_v}$.

Let h be a generalized Maslov index 2 disc in Y with respect to our single point constraint p with stop in U. We will assign this disc a weight $w(h)$. If h does not go through p, then it consists of a single ray in a toric direction v of Y. In this case, we let $w(h)$ be the coefficient of z^v in W_Y. If h does go through p, then by Lemma 3.6, h consists of three edges, one adjacent to the stop and the other two unbounded. The edge through p has outward pointing primitive tangent vector $-w_i$. Let $-v$ be the other outward pointing primitive tangent vector to the other unbounded edge. Then the primitive tangent vector on the compact edge adjacent to the stop, pointing toward the stop, is $w_i + v$. We let $w(h)$ be the product of the coefficient of z^{w_i} (which is 1) and the coefficient of z^v (which may not be 1).

Letting $v(h)$ be the primitive vector tangent to the edge adjacent to the stop, pointing toward the stop, we claim that $W_{\tilde{Y}} = \sum_h w(h) z^{v(h)}$. If h has no vertices, then $w(h) = \binom{m_{v(h)}+1}{\ell_{v(h)}}$. If h has a single vertex, then $w(h) = \binom{m_{v(h)}+1}{\ell_{v(h)}+1}$, where here we choose $\ell_{v(h)}$ so that the divisor adjacent to w_i is in position 1. Here we are using Lemma 3.4 and the last observation in the proof of Lemma 3.3 to find $w(h)$. The coefficient of $z^{v(h)}$ is then precisely $\binom{m_{v(h)}+2}{\ell_{v(h)}+1}$, as desired.

Finally, let Y be a toric Fano surface and consider a sequence $\tilde{Y}_n \to \cdots \to \tilde{Y}_1 \to Y$ of toric blowups, such that each $\tilde{Y}_j$ is semi-Fano. Using the above procedure, we can relate the superpotential on $\tilde{Y}_n$ to the bulk-deformed superpotential on Y corresponding to appropriately chosen marked points.

Let w'_i now denote the primitive vector lying in the 1-cone of the toric fan of $\tilde{Y}_i$ that does not appear in the fan of $\tilde{Y}_{i-1}$ (this corresponds to the ith blowup of Y). Let $\mathcal{D}_i$ be the scattering diagram associated with the single point p_i and $\tilde{Y}_i$. This has a ray in the v direction for each primitive vector v in a 1-cone of the fan of $\tilde{Y}_i$. Letting the point p_i tend to infinity in the $-w'_i$ direction, we get one chamber U_i exhausting $M_\mathbb{R}$. Position the points $p_1, \ldots, p_n$ such that the chamber U_i contains all points p_j with $j > i$. We claim that the bulk-deformed superpotential on Y at a point in $\bigcap_{i=1}^n U_i$ is equal to the superpotential on $\tilde{Y}_n$.

This follows inductively from our above procedure involving single blowups. Since Y is Fano, the above procedure gives us that the superpotential for $\tilde{Y}_1$ equals the bulk-deformed superpotential on Y in U_1 with the point p_1. Considering now the point p_2 in the chamber U_1, the above procedure gives us the superpotential on $\tilde{Y}_2$ using the superpotential on $\tilde{Y}_1$ and a weighted count of generalized Maslov index 2 tropical discs with respect to p_2. This weighted count corresponds exactly to the number of ways of modifying such a tropical disc by "unclipping" (replacing an unbounded edge by a generalized Maslov index 2 tropical disc with stop on that edge) to get a generalized Maslov index 2 tropical disc with respect to p_1 and p_2, so the bulk-deformed superpotential on Y with the points p_1 and p_2 in the region $U_1 \cap U_2$ agrees with the superpotential on $\tilde{Y}_2$. Inducting gives the desired general result, repeated here:

Theorem 3.7. *Let Y be a toric Fano surface, and let $\tilde{Y}_n \to \cdots \to \tilde{Y}_1 \to Y$ be a sequence of toric blowups, such that each $\tilde{Y}_i$ is semi-Fano. Let $p_1, \ldots, p_n$ be points in $M_\mathbb{R}$ arranged as described above, and let $U_1, \ldots, U_n$ be the open cones with vertex at those points described above. The bulk-deformed superpotential for Y in the chamber $\bigcap_{i=1}^n U_i$ with respect to these points is equal to the superpotential for $\tilde{Y}_n$.*

4. SYZ fibrations on Log Calabi-Yau Surfaces and wall-crossing

In this section, we construct a special Lagrangian torus fibration on $Y \setminus D$ for a log Calabi-Yau surfaces (Y, D), and prove that the fibration satisfies conditions in Theorem 2.16 away from singular fibres. Main difficulty lies in the noncompleteness of $Y \setminus D$ as it may spoil the compactness of relevant disc moduli spaces. This occurs because on $Y \setminus D$, we use the restricted symplectic form (or the associated metric) from Y. Once we have the compactness of the moduli spaces, the fibration will automatically produce a scattering diagram well-defined away from neighborhoods of singular fibres, which is one of the main subject of the paper. It will be discussed thoroughly in Section 5.

4.1. **Construction of a SYZ fibration on non-toric blowup.** Let us first look into the toric surface $\bar{Y}$ and its non-toric blowup $\tilde{Y} \to \bar{Y}$. We will equip $\bar{Y}$ with a Kähler form $\bar{\omega}$ which is invariant under the T^2-action.

In general, for given a T^2-action on $\bar{Y}$ compatible with the complex structure and a Kähler class, we can take any Kähler form, say $\bar{\omega}_0$, in the class, and average it with respect to the T^2-action. This leads to the T^2-invariant Kähler form $\bar{\omega}$ on $\bar{Y}$ that belongs to the Kähler class we began with. Since the first Betti number of $\bar{Y}$ vanish, the T^2-action is Hamiltonian. It is worth mentioning that for the fixed T^2-action on $\bar{Y}$, the base of the moment map fibration (with respect to the Kähler forms in different Kähler classes) with the complex affine coordinates are canonically isomorphic.

Let $\{q_i \in \bar{D}_{\sigma(i)}\}_i \subset \bar{Y}$ be the blowup center for $\tilde{Y} \to \bar{Y}$. Namely, $\tilde{Y}$ is obtained by taking a simple blowup at each q_i. We denote by $E_i := \pi^{-1}(q_i)$ the exceptional divisor associated with q_i. Notice that we changed the notation from 2.3 for simplicity. q_i and E_i were previously written in the form of q_{ij} and E_{ij} indicating they are attached to the toric divisor indexed by $j = \sigma(i)$. We will first construct a particular Kähler form $\tilde{\omega}_\epsilon$ on $\tilde{Y}$ from the torus-invariant Kähler form $\bar{\omega}$ downstairs.

Lemma 4.1. *Given $\epsilon > 0$, there exists a Kähler form $\tilde{\omega}_\epsilon$ on $\tilde{Y}$ and a pair of neighborhoods $\tilde{U}_i \subseteq \tilde{U}'_i$ of E_i with $\tilde{U}'_i \cap \tilde{U}'_j = \emptyset$ for $i \neq j$ that satisfy the followings.*

  (1) *$\tilde{\omega}_\epsilon = \pi^* \omega$ outside $\cup_i \tilde{U}_i$;*
  (2) *Symplectic areas of exceptional divisors with respect to $\tilde{\omega}_\epsilon$ sum up to ϵ.*

(3) *For each $q_i \in \bar{D}_{\sigma(i)}$, the sub-toric S^1-action that fixes $\bar{D}_{\sigma(i)}$ locally lifts to $\tilde{U}'_i \subseteq \tilde{Y}$.* *(In particular, $\tilde{U}'_i$ is S^1-invariant.)*

Moreover, we have $\tilde{U}'_i \searrow \{q_i\}$ as $\epsilon \to 0$.

Proof. Set $p_i := p(q_i) \in \partial P$ where $p : \bar{Y} \to P$ is the moment map. Let $V_i \subseteq P$ be a small neighborhood of p_i such that $V_i \cap V_j = \emptyset$ if $p_i \neq p_j$. Consider the subgroup $S^1 \subseteq T^2$ fixing $\bar{D}_{\sigma(i)}$. If we write the toric action locally as $(e^{i\theta_1}, e^{i\theta_2}) \cdot (x, y) = (e^{i\theta_1} x, e^{i\theta_2} y)$ for some local coordinates (x, y) near q_i with $\bar{D}_{\sigma(i)} = \{y = 0\}$, then the mentioned S^1-action is given by $(x, y) \mapsto (x, e^{i\theta} y)$. Let us choose S^1-invariant neighborhoods U_i and U'_i of q_i such that $U_i \subsetneq U'_i \subseteq p^{-1}(V_i)$. We have $U'_i \cap U'_j = \emptyset$ for $i \neq j$. We finally take $\tilde{U}_i := \pi^{-1}(U_i) \subsetneq \tilde{U}'_i := \pi^{-1}(U'_i)$ as neighborhoods of the exceptional divisor E_i.

Since U'_i's are mutually disjoint, the local S^1-action lifts to $\tilde{U}'_i$ by the universal property of blowup, which we denote by $a_\theta : \tilde{U}'_i \to \tilde{U}'_i$. Observe that the complement of $\cup_i \tilde{U}_i$ in $\tilde{Y}$ admits a T^2-action lifted from the toric action on $\bar{Y}$. The local S^1-action can be identified with the corresponding factor of this T^2-action away from E_i, in particular, on $\tilde{U}'_i \setminus \tilde{U}_i$.

It is a standard fact that there exists a Kähler metric $\tilde{\omega}$ on $\tilde{Y}$ which coincides with $\pi^* \bar{\omega}$ outside $\tilde{U}_i$ (see, for e.g., [20, p. 185]). Since $\bar{\omega}$ is invariant under the toric action, so is $\tilde{\omega}|_{\tilde{Y} \setminus \cup \tilde{U}'_i}$ under the lifted T^2-action on $\tilde{Y} \setminus \cup \tilde{U}'_i$. On the other hand, one can average $\tilde{\omega}|_{\tilde{U}'_i}$ by the local S^1-action with respect to the Harr measure $d\mu(\theta)$ on S^1. This produces an S^1-invariant local Kähler form $\tilde{\omega}_i = \frac{\int_{S^1} a_\theta^* \tilde{\omega} \, d\mu(\theta)}{\int_{S^1} d\mu(\theta)}$ on $\tilde{U}'_i$. Notice that the local form $\tilde{\omega}_i$ agrees with the global form $\tilde{\omega}$ on the overlapped region $\tilde{U}'_i \setminus \tilde{U}_i$ by T^2-invariance of $\tilde{\omega}$.

We conclude that there exists a Kähler form on $\tilde{Y}$ which equals $\tilde{\omega} = \pi^* \omega$ on the complement of $\cup_i \tilde{U}_i$ and restricts to $\tilde{\omega}_i$ on $\tilde{U}'_i$. Repeating the same procedure for all q_i's, we obtain a global Kähler form, say $\tilde{\omega}_\epsilon$, on $\tilde{Y}$ which certainly satisfies the conditions (1) and (3) in the statement. The condition (2) can be easily achieved by adjusting the size of the Kähler potential for $\tilde{\omega}|_{\tilde{U}_i}$. $\qquad \square$

Remark 4.2. *We remark that the local S^1-action on $\tilde{U}_i$ above is Hamiltonian, since it is symplectic (preserving $\tilde{\omega}_\epsilon$) and the first Betti number of $\tilde{U}_i$ vanishes. Also, the S^1-action can actually be defined on a larger region away from $\cup_{j \neq i} E_j$, as it is induced from the toric action on $\bar{Y}$ which lifts to $\tilde{Y} \setminus \cup_i \tilde{U}'_i$. In particular, we can find a moment map $\mu_{S^1} : (\pi \circ p)^{-1}(V_i) \to \mathbb{R}$ for this action.*

With a Kähler form $\tilde{\omega}_\epsilon$ on $\tilde{Y}$ constructed in the lemma and the meromorphic volume form $\tilde{\Omega}$ on $\tilde{Y}$ having simple pole along $\tilde{D}$, we adapt the constructions in [2, 19] into our setting in order to obtain an almost special Lagrangian fibration on $\tilde{X} = \tilde{Y} \setminus \tilde{D}$.

Lemma 4.3. *There exists a Lagrangian fibration $\tilde{X} \to B$ with respect to $\tilde{\omega}_\epsilon$ such that*

(1) *the fibration coincides with the pull-back of toric fibration on $\bar{X}$ outside $(\pi \circ p)^{-1}(V_i)$;*

(2) *$Im \tilde{\Omega}|_{\tilde{L}} = 0$ for any Lagrangian torus fibre $\tilde{L}$ after a suitable normalization of the phase;*

(3) *all the singular Lagrangian torus fibres are immersed S^2, which are in one-to-one correspondence with blowup points $\{q_i\}$.*

Proof. As in the proof of Lemma 4.1, let us take local coordinates (x, y) around q_i such that the toric divisor $D_{\sigma(i)}$ (that contains q_i) is locally given by $\{y = 0\}$, and the toric action trivializes as $(x, y) \mapsto (e^{i\theta_1} x, e^{i\theta_2} y)$. We normalize the moment map $\mu_{S^1} : (\pi \circ p)^{-1}(V_i) \to \mathbb{R}$

in Remark 4.2 so that it vanishes on the proper transform of the x-axis, and $\mu_{S_1} = \epsilon_i < \epsilon$ at the isolated S^1-fixed point in E_i.

Notice that $\{|x| = const.\}$ projects to a circle in the S^1-reduction which is automatically a Lagrangian by the dimension reason. Therefore [19, Theorem 1.2] applies to produce a Lagrangian fibration

$$(4.1) \qquad\qquad (\pi \circ p)^{-1}(V_i) \to \mathbb{R}^2 \qquad z \mapsto (|x(\pi(z))|, \mu_{S^1}(z))$$

with respect to $\tilde{\omega}_\epsilon$ on $\tilde{U}_i$, where $x(\pi(z))$ denotes the x-coordinate of $\pi(z)$. Since $\tilde{\omega}_\epsilon = \pi^*\omega$, the fibration coincides with the pull-back of the moment map fibration from $\bar{X}$ outside of $(\pi \circ p)^{-1}((\pi \circ p)(\tilde{U}_i))$. Therefore, the fibration above can be glued with the pull-back of the moment fibration of $\bar{X} \to \text{Int}\,P$. Instead of using explicit coordinates as in (4.1), one may take a splitting of T^2 (giving the toric action on $\bar{Y}$) into the product of the stabilizer of $D_{\sigma(i)}$ and its complementary circle, and replace $|x(\pi(z))|$ by the pull-back of the moment map for the latter factor. This way, one can identify the target of the fibration (4.1) coherently with the Lie algebra of T^2 for every i.

To see the second part of the lemma, we first show that $\tilde{\Omega}$ is invariant under the S^1-action in a neighborhood of the proper transform of $\bar{D}_{\sigma(i)}$. Since $\tilde{\Omega} = \pi^*\bar{\Omega}$ outside of $(\pi \circ p)^{-1}(V_i)$ and $\bar{\Omega}$ is torus-invariant, $\tilde{\Omega}$ is S^1-invariant outside $(\pi \circ p)^{-1}(V_i)$. Together with the fact that the S^1-action is holomorphic (i.e., it preserves the complex structure), this implies that the holomorphic $(2,0)$-form $a_\theta^*\tilde{\Omega}$ on $(\pi \circ p)^{-1}(V_i)$ can be extended as a nowhere vanishing holomorphic $(2,0)$-form on the entire $\tilde{X}$, which we still denote by $a_\theta^*\tilde{\Omega}$. Then the ratio $\tilde{\Omega}/a_\theta^*\tilde{\Omega}$ is a holomorphic function on $\tilde{X}$ which is 1 on an open set, and we conclude that $a_\theta^*\tilde{\Omega} = \tilde{\Omega}$ by maximum principle. Then (2) of the lemma again follows from [19, Theorem 1.2].

Finally, the fibre of (4.1) over $(r, \lambda) \in \mathbb{R}^2$ is singular if and only $(r, \lambda) = (|a_i|, \epsilon_i)$ where $\epsilon_i = \int_{E_i} \tilde{\omega}_\epsilon$ and $a_i = x(q_i)$. It has nodal singularities only, and hence the fibre can be at worst immersed. The singular point of the Lagrangian fibration happens at the isolated fixed point of the S^1-action, and hence is one-to-one corresponding to q_i.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Away from a neighborhood of the discriminant, the base B of the Lagrangian fibration constructed in Lemma 4.3 admits the affine structure which can be described as follows. Let us first identify the interior of the polytope $\text{Int}\,P$ with $M_\mathbb{R}\mathbb{R}^2$ via the Legendre transform so that the complex affine coordinates from the toric fibration $p : \bar{X} \to \text{Int}\,P$ agrees with the standard coordinates on $\mathbb{R}^2$. Recall that the Lagrangian fibration on $\tilde{X} = \tilde{Y} \setminus \tilde{D}$ in Lemma 4.3 is the pull-back of the moment fibration of X away from neighborhoods $(\pi \circ p)^{-1}(\bigcup_i V_i)$ of the exceptional divisors. Hence one can consider the pull-back of the complex affine structure on a subset B' of B which is diffeomorphic to $\mathbb{R}^2 \setminus \bigcup_i V_i$, and we may still call it the complex affine structure.

Lemma 4.4. *Let L be a moment fibre in $\bar{Y}$, and fix $\pi^{-1}(L)$ as a reference fibre for the complex affine structure (so that it sits over the origin). Given any $C > 0$, there exists $\epsilon > 0$ such that the affine coordinate $x_{\partial\gamma_i} > C$ of the singular fibre corresponds to q_i, if $\epsilon_i < \epsilon$.*

Proof. This is because the fibration constructed in Lemma 4.3 coincides with the pull-back of the moment fibration outside of $(\pi \circ p)^{-1}(\bigcup_i V_i)$. The holomorphic volume form $\bar{\Omega}$ near the interior of the boundary divisor component $\{x = 0\}$ (in some local coordinates) is of the form $f\frac{dx}{x} \wedge dy$, where f is a non-zero function. On the other hand, recall that the complex affine coordinates are given by $x_{\partial\gamma_i} = \int_{C_{\partial\gamma_i}} \text{Im}\tilde{\Omega}$ where $C_{\partial\gamma_i}$ is the 2-chain swept by the cycle $\partial\gamma_i \in H_1(L)$, starting from L and moving toward the singular fibre corresponding

to q_i. As $\epsilon_i \to 0$, one of the boundary components of $C_{\partial \gamma_i}$ approaches $x = 0$, and hence $\int_{C_{\partial \gamma_i}} \mathrm{Im} \bar{\Omega} \to \infty$. $\qquad \square$

In particular, as ϵ approaches 0, one can take V_i in the construction arbitrarily small so that $B' \cong (\mathbb{R}^2 \setminus \bigcup_i V_i) \nearrow \mathbb{R}^2$ as $\epsilon \to 0$. When needing to emphasize this dependence of B' on ϵ, we will write B_ϵ instead of B' in what follows.

4.2. Holomorphic discs in non-toric blowups.

We now look into holomorphic discs bounding a fibre $\tilde{L}$ of the fibration Lemma 4.3, when $\tilde{L}$ is away from $(\pi \circ p)^{-1}(\bigcup_i V_i)$. Since we are using the standard (toric) complex structure on $\bar{Y}$ and the one on $\tilde{Y}$ naturally induced through the blowup process, $\pi : \tilde{Y} \to \bar{Y}$ is holomorphic. This enables us to establish the following one-to-one correspondence between the holomorphic discs bounding by L and $\tilde{L}$. If $f : (D^2, \partial D^2) \to (\bar{Y}, L)$ is a holomorphic map, then its proper transform gives a holomorphic map $\tilde{f} : (D^2, \partial D^2) \to (\tilde{Y}, \tilde{L})$. On the other hand, if $\tilde{f} : (D^2, \partial D^2) \to (\tilde{Y}, \tilde{L})$ is holomorphic, then composition with π gives a holomorphic map $f = \pi \circ \tilde{f} : (D^2, \partial D^2) \to (\bar{Y}, L)$. See [53] for related discussions.

Suppose that the disc $\tilde{f} : (D^2, \partial D^2) \to (\tilde{Y}, \tilde{L})$ has Maslov index zero. This happens if and only if $\mathrm{Im}(\tilde{f}) \cap \tilde{D} = \emptyset$. Then its projection f can only intersect D at q_i for some i. If $|\mathrm{Im}(\tilde{f}) \cap \tilde{D}| = l$, then $\mathrm{Im}(f)$ passes l-many point-constraints in D counted with multiplicity, and its Maslov index is given by $2l$. With these point-constraints being regarded as bulk-insertions, f has generalized Maslov index 0. More generally, the same argument shows that the Maslov index of $\tilde{f}$ is the same as the generalized Maslov index of f allowing multiple insertions of q_i. To sum up, we have the following lemma.

Lemma 4.5. *There exists a bijective correspondence between the Maslov index μ holomorphic discs for $(\tilde{X}, \tilde{L})$ and the generalized Maslov index μ holomorphic discs for (X, L) viewing q_i as bulk insertions. In particular, L_u bounds a generalized Maslov index zero disc if and only if $\tilde{L}_u$ bounds for a Maslov index zero disc.*

The following corollary is a direct consequence of (2) of Lemma 4.3, since fibres in $\tilde{X}$ are special with respect to $\tilde{\Omega}$ ((2) of Lemma 4.3). Alternatively, one can use the above lemma combined with the fact that walls for the bulk-deformed potential limit to straight lines as point-constraints approach the toric divisor, whose proof is elementary.

Corollary 4.6. *If $\tilde{L}_t \subseteq \tilde{X}$ is a family of torus fibre over B' bounding holomorphic discs of Maslov index zero in a fixed relative class, then $\tilde{L}_t$ sit over an affine line in B'.*

Notice that the statement holds only for genuine holomorphic discs, but not for stable maps having nontrivial bubbled-off components that may appear in the compactification $\tilde{Y}$, especially when there exists holomorphic spheres with negative Chern numbers. We can get rid of such contributions simply by restricting ourselves to the open part $(\tilde{X}, \tilde{\omega}_\epsilon)$, but the price to pay is that the compactness of the disc moduli is not automatically guaranteed, which we discuss now.

4.2.1. *Compactness of the disc moduli for $\tilde{X}$.* In order to make sense of Floer theory for Lagrangian fibre $\tilde{L}$ in $\tilde{X}$, one needs to separately show the compactness of the corresponding disc moduli spaces since $\tilde{X}$ is non-compact and the metric is incomplete. Once we have the compactness, Theorem 2.16 automatically produces a scattering diagram $\mathfrak{D}_\epsilon^{LF}$ on B_ϵ. We will see later that it converges to a limiting scattering diagram as ϵ approaches 0.

Remark 4.7. *One may construct a scattering diagram from the geometry of $\tilde{Y}$ instead, for which Theorem 2.16 directly applies. However, there can be a lot more Maslov index zero discs due to negative Chern number spheres in $\tilde{D}$ and, none of existing techniques in Floer theory seem allowing us to explicitly compute such contributions.*

Lemma 4.8. *Let $\tilde{L} := \pi^{-1}(L)$ be a torus fibre in $\tilde{X}$ which is a pre-image of a moment map fibre in X. Then the moduli space $\mathcal{M}(\tilde{X}, \tilde{L}, \tilde{\beta}; J)$ of holomorphic discs of class $\tilde{\beta} \in \pi_2(\tilde{X}, \tilde{L})$ is compact for any $\tilde{\beta}$.*

Throughout (as well as in the statement), J denotes the *standard* complex structure unless specified otherwise, and by abuse of notation, it will also stand for that on $\bar{X}$. We remark that the proof heavily relies on the usage of the standard complex structure.

Proof. Recall from Lemma 4.5 that the set of Maslov index zero discs $\tilde{f} : (D^2, \partial D^2) \to (\tilde{X}, \tilde{L})$ is in one-to-one correspondence with that of generalized Maslov index zero discs $\bar{f} : (D^2, \partial D^2) \to (\bar{Y}, \bar{L})$. We denote by $\mathcal{M}''$, the subspace of $\mathcal{M}(\bar{Y}, L, \beta; J)$ consisting of such discs $\bar{f}$. Hence, the moduli space of stable discs $\mathcal{M}(\tilde{X}, \tilde{L}, \tilde{\beta}; J)$ is homeomorphic to $\mathcal{M}''$. Notice that $\mathcal{M}''$ does not contain stable maps with sphere-bubble components.

We show that $\mathcal{M}''$ is compact in the following. Let Σ' be a refinement of Σ by adding $\mathbb{R}_{\leq 0} v$ for each 1-cone $\mathbb{R}_{\geq 0} v$ of Σ. Let Y' be the toric (orbi-)surface associated Σ', and $\pi' : Y' \to \bar{Y}$ the birational modification on the boundary corresponding to the refinement of the fan. Then the $(\mathbb{C}^*)^2$-action on $\bar{Y}$ lifts to Y' by the universal property of blowups. In particular, the Hamiltonian T^2-action lifts to Y', and hence is again Hamiltonian since $H^1(Y') = 0$. In other words, the moment maps of Y' and $\bar{Y}$ are compatible, and one can naturally identifies the interior of the moment polytopes P' and that of P.

Take $u \in \mathbb{R}^2$ over which the fibre $\bar{L}$ sits, where we identify both $\operatorname{Int} P$ and $\operatorname{Int} P'$ with $\mathbb{R}^2$ via the Legendre transform. Up to a $(\mathbb{C}^*)^2$-action, we may assume that $u = (0,0) \in \mathbb{R}^2 \cong \operatorname{Int} P$. Consider a holomorphic disc $\bar{f} : (D^2, \partial D^2) \to (\bar{Y}, \bar{L})$ which lies in $\mathcal{M}''$. Recall from the discussion in Section 3.1 that with help of the anti-holomorphic involution σ on $\bar{X}(\cong (\mathbb{C}^*)^2)$, one can double $\bar{f}$ into a unique rational curve $C_{\bar{f}}$ in Y' containing $\operatorname{Im}\bar{f}$. We denote its proper transform in Y' by $C'_{\bar{f}}$.

Observe that the involution σ acting on $(\mathbb{C}^*)^2 \subseteq Y'$ extends to the whole Y' due to its symmetry. Then $C'_{\bar{f}}$ is a rational curve in Y' whose intersection with the toric boundary divisors can occur only at q'_i and $\sigma(q'_i)$, where $q'_i = \pi'^{-1}(q_i)$. It is shown in [25, Lemma 4.2] that the moduli space rational curves in Y' of a fixed homology class which can intersect the toric boundary only at prescribed non-toric points is compact and has no sphere bubble. This moduli space admits an induced action of σ, and the fixed loci of the action precisely consists of the proper transform $C'_{\bar{f}}$ for some $\bar{f}$. Therefore, $\mathcal{M}''$ can be identified with the fixed loci, and hence is compact. $\square$

Corollary 4.9. *There exists a consistent scattering diagram $\mathfrak{D}^{LF}_\epsilon$ on B_ϵ whose walls consist of Lagrangian fibres that bound Maslov 0 discs (or, more precisely, their corresponding open Gromov-Witten invariants are nonzero).*

4.2.2. *Computing wall-crossing transformations.* Lemma 4.8 enables us to apply Theorem 2.16 to the Lagrangian fibration $\tilde{X}$ with small neighborhoods of singular fibres being removed. It results in a scattering diagram, whose "initial rays" are originated from these singular fibres. Here, the initial rays refer to the ones induced by Maslov 0 discs emanating from these singular fibres, and we often call these discs *initial (Maslov 0) discs* for this reason. We warn readers that a priori there may exist more Maslov 0 discs coming out from neighborhoods of the

singular fibres since we did not established the compactness of the disc moduli spaces for fibres near the singular fibres and the Floer theory cannot apply directly. Later we will see that one can use Lemma 3.2 to exclude such possibility.

Consider a Lagrangian torus fibre $\tilde{L} \subseteq (\pi \circ p)^{-1}(V_i)$ in a neighborhood of the singular fibre. In terms of local coordinates (x, y) around the blowup point $q_i = (a_i, 0)$ as in (4.1), $\tilde{L}$ bounds Maslov 0 discs if and only if it sits over $|x \circ \pi| = |a_i|$. For later use, we write γ for the relative class of the contributing Maslov 0 disc, which are portions of (the proper transform of) the parallel to y-axis. We are interested in those lying above the singular fibre (i.e., $\mu_{S_1} > \epsilon_i$), since we are considering the case when the location of the singular fibre is arbitrarily close to the divisor $y = 0$ (i.e., $\epsilon_i \to 0$). Let us denote the corresponding ray (wall) by l_i, i.e., in local coordinates,

$$(4.2) \qquad l_i = \{(r, \lambda) \mid r = |x \circ \pi| = |a_i|, \ \lambda > \epsilon_i\}.$$

Notice that l_i forms a ray starting from (the image of) q_i, whose direction is normal to the toric divisor that q_i. Hence, $\{l_i\}$ and the initial rays(walls) in Section 2.3 coincide with each other, after identifying the interior of the polytope $\mathrm{Int}\, P$ with $\mathbb{R}^2$ via the Legendre transform.

We show that $\{l_i\}$ coupled with wall functions from Floer theory agrees with the initial scattering diagram $\mathfrak{D}_{in}^{GPS}$ in Section 2.3. Let us compute the wall-crossing transformation associated with the wall l_i. Our strategy is to first symplectically embed the neighborhood of the singular fibre $(\pi \circ p)^{-1}(V_i \setminus \partial P))$ into a simpler local model in [2, Example 3.1.2], the blowup $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ at a generic point in the x-axis. It is crucial to have a S^1-invariant Kähler from on this region for this purpose. Our actual geometric situation is possibly different from this local model, in that the divisor containing q_i may serve as a nontrivial sphere bubble in disc counting. However, this does not affect the Fukaya's trick across the wall l_i by the topological reason, as l_i is located on the opposite side of the divisor with respect to the singular fibre. Therefore the wall-crossing for l_i should agree with that for the corresponding wall in [2, Example 3.1.2].

The advantage of the latter is that one can detect the wall-crossing via the change of superpotentials whose computation is simpler. A similar idea was used in [55]. One may regard the local coordinates (x, y) above as coordinates on $\mathbb{C}^2$. By abuse of notation, let γ denote the relative class in $\pi_2(\widetilde{\mathbb{C}^2}, \tilde{L})$ of the proper transform of the basic disc intersecting x-axis once. Similarly, we denote by δ the relative disc class of the one intersecting y-axis once. Then $\partial \gamma$ and $\partial \beta$ form a basis of $H_1(\tilde{L})$, and we can take $z^{\partial \gamma}$ and $z^{\partial \delta}$ as variables for the superpotential. It is not difficult to see that the wall-crossing transformation going across l_i can be written in the form of

$$(4.3) \qquad (z^{\partial \gamma}, z^{\partial \delta}) \mapsto (z^{\partial \gamma}, z^{\partial \delta}(1 + f(z))).$$

The first component is the identity map since $\partial \gamma$ does not intersect the boundary of the Maslov zero disc at all, and hence contributes trivially to the pseudo-isotopy, and the second follows from Theorem 2.15. On the other hand, the explicit calculation in [2] shows that the superpotential for $\tilde{L}$ with $|x \circ \pi| < a_i$ is given by

$$T^{\omega(\gamma)} z^{\partial \gamma} + T^{\omega(\delta)} z^{\partial \delta}$$

whereas $\tilde{L}$ with $|x \circ \pi| > a_i$ has

$$T^{\omega(\gamma)} z^{\partial \gamma} + T^{\omega(\delta)} z^{\partial \delta} + T^{\omega(\gamma+\delta)} z^{\partial \gamma} z^{\partial \delta}.$$

By (1) of Lemma 2.14, the two should be compatible with the coordinate transition (4.3). Therefore we have

$$T^{\omega(\gamma)}z^{\partial\gamma} + T^{\omega(\delta)}z^{\partial\delta}(1 + f(z)) = T^{\omega(\gamma)}z^{\partial\gamma} + T^{\omega(\delta)}z^{\partial\delta} + T^{\omega(\gamma+\delta)}z^{\partial\gamma}z^{\partial\delta},$$

which leads to $f(z) = T^{\omega(\gamma)}z^{\partial\gamma}$. Thus we have shown

Lemma 4.10. *Suppose $\tilde{Y}$ is a non-toric blowup of a toric surface $\bar{Y}$, and a point q_i in the blowup center lies in the irreducible toric divisor $D_{\sigma(i)}$. Let l_i be the wall associated to q_i (see (4.2)) for the Lagrangian fibration on $\tilde{Y}$ constructed in Lemma 4.3. Then its wall-function is given by $1 + T^{\omega(\gamma)}z^{\partial\gamma}$ where γ is the relative class of the proper transformation of the holomorphic disc in $\bar{Y}$ intersecting $D_{\sigma(j)}$ exactly once.*

A direct consequence is that $\{l_i\}$ together with the wall functions (of the form $1 + f(z)$) calculated above forms the (initial) scattering diagram $\mathfrak{D}^{GPS}_{in}$ in Section 2.3.

4.3. Orbifold blowups and Floer theory.
Although we will be only concerned with the case of simple blowups in the later part of the article, we believe that majority of our result can extend to blowups with higher multiplicities (orbifold blowups) provided Fukaya's trick in the orbifold setting. Here, we briefly examine the wall-crossing in Floer theory on some standard local model for such a blowup.

Recall from 2.3.2 that a non-toric blowup with a higher multiplicity results in the nontrivial exponents in the wall functions attached to the corresponding wall in $\mathfrak{D}^{GPS}$. It creates an orbifold singularity in the total space, and the Maslov zero disc responsible for the wall-crossing passes through this singular point. Hence it is natural to consider the orbifold Floer theory of the Lagrangian torus fibres sitting over the wall. Let us examine this in a local model which will turn out to be given as the *global quotient orbifold*.

We begin with $\mathbb{C}^2$ with coordinates (x, y) (equipped with the standard toric structure). As the blowup we are interested in is non-toric, let us take a nonreduced scheme supported at the point $(1, 0)$, that is, the subscheme corresponding to the ideal generated by $(x-1)^r$ and y. (r corresponds to $r_{ij} - 1$ in 2.3.2.) The blowup will produce an orbifold projective line E_r with exactly one orbifold singular point of order r. It admits a homogeneous coordinate $[a : b]$ where $[a : b] = [\rho a : \rho^r b]$ for $\rho \in \mathbb{C}^\times$. Now, the blowup can be identified as a subspace of $\mathbb{C}^2 \times E_r$ given as

$$\tilde{Y}_{r,loc} = \{((x, y), [a : b]) \in \mathbb{C}^2 \times E_r : (x-1)^r b = ya^r\}.$$

Like in the case of ordinary (non-toric) blowup, $\tilde{Y}_{r,loc}$ admits a S^1-action. The action has exactly one isolated fixed point $((1, 0), [0 : 1])$. $((1, 0), [0 : 1])$, away from which $\tilde{Y}_{r,loc}$ is smooth. It also fixes the proper transform of x-axis, $\{((x, y), [a : b]) : b = 0\}$. We denote its complement in $\tilde{Y}_{r,loc}$ by $\tilde{W}_{r,loc}$. This is isomorphic to the affine variety

$$\{(x, y, \tilde{a}) : x^r = \tilde{a}y\} \cong \mathbb{C}^2/\mathbb{Z}_r$$

via $\tilde{a} = \frac{a^r}{b}$, under which the point $((1, 0), [0 : 1])$ in $\tilde{W}_{r,loc}$ maps to the (unique) orbifold singular point of order r. In fact, there exists a (global) quotient map

$$\tilde{W}_{1,loc} \to \tilde{W}_{r,loc} \qquad ((\underline{x}, \underline{y}), [\underline{a} : \underline{b}]) \to ((\underline{x}, \underline{y}^r), [\underline{a} : \underline{b}^r])$$

from the local model for the ordinary non-toric blowup, which is nothing but the quotient map $\mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}_r$ by the identification above.

Notice that the quotient map sends a torus fibre $L_{1,u}$ in $\tilde{W}_{1,loc}$ to the torus fibre $L_{r,u}$. Restricting to the individual torus fibre, the quotient map gives a r-fold cover, where the loop $|\underline{y}| = const.$ in $L_{1,u}$ wraps the loop $|y| = const$ in $L_{r,u}$ r-times through the covering, yet

the complementary loop $|\underline{x}| = const.$ maps isomorphically $|x| = const.$ in $L_{r,u}$. On the local model $\tilde{W}_{r,loc}$, Floer theory of its torus fibres can be completely understood by $\mathbb{Z}_r$-invariant part of Floer theory of their liftings in $\tilde{W}_{1,loc}$, which has already been discussed in Section 4.2 in detail. For our purpose, we fist need to inspect the effect of the group action on the Maurer-Cartan space of the upstair torus fibre $L_{1,u}$. Let $e_{\underline{x}}$ and $e_{\underline{y}}$ denote the standard generator of $H^1(L_{1,u}; \mathbb{Z})$, i.e., $\int_{|i|=const.} e_j = \delta_{ij}$ $(i, j \in \{\underline{x}, \underline{y}\})$.

Since the action rotates the loop $|\underline{y}| = const.$ by $2\pi/r$, the generator of the first (integral) cohomology of the torus fibre down stairs can be identified with $e_x = e_{\underline{x}}$ and $e_y = \frac{1}{r}e_{\underline{y}}$ (so that e_y integrates over $|y| = const.$ to give 1). A general element of $H^1(L_{r,u}; \mathbb{R})$ is then a linear combination $x_1 e_x + x_2 e_y = x_1 e_{\underline{x}} + (x_2/r)e_{\underline{y}}$. Recall that the potential is written in terms of the exponential coordinates $z_i = \exp(x_i)$, and the above discussion tells us that the exponential coordinates $(\underline{z_1}, \underline{z_2})$ for $L_{1,u}$ and (z_1, z_2) for $L_{r,u}$ are related by

$$z_1 = \underline{z_1}, \quad z_2 = (\underline{z_2})^r.$$

On the other hand, we have computed in Lemma 4.10 the wall-crossing formula for the unique wall $\{|\underline{x}| = 1\}$ in the standard local model $\tilde{W}_{1,loc}$ as

$$(4.4) \qquad\qquad (\underline{z_1}', \underline{z_2}') = (\underline{z_1}, \underline{z_2}(1 + T^{\omega(\gamma)}\underline{z_1})),$$

where γ is the class of the Maslov index 0 disc responsible for the wall and where $(\underline{z_1}, \underline{z_2})$ and $(\underline{z_1}', \underline{z_2}')$ are the (exponential) coordinates on the Maurer-Cartan spaces of the torus fibres in the chambers $\{|\underline{x}| < 1\}$ and $\{|\underline{x}| > 1\}$, respectively. In perspective of Floer theory of $\tilde{W}_{r,loc}$ yet without turning on the bulk insertions from twisted sectors, the formula (4.4) is still valid if we write it in terms of coordinates (z_1, z_2), which amounts to taking $\mathbb{Z}/r$-invariant part of Floer operations upstairs. It leads to the following wall-crossing formula

$$(z_1', z_2') = (z_1, z_2(1 + T^{\omega(\gamma)}z_1^r)))$$

that is valid in the local model $\tilde{W}_{r,loc}$ whose Floer theory is taken as the $\mathbb{Z}_r$-invariant part of that on $\tilde{W}_{1,loc}$ along the same spirit as [5]. The corresponding global statement should immediately follow, once the orbifold analogue of the Fukaya's trick is established (i.e., Lemma 2.14 in the orbifold setting). We remark that for our purpose, it does not require a full package of the orbifold Floer theory in the sense that the bulk-deformation by nontrivial twisted sectors is not at all involved.

4.4. **Toric blowups.**

Assume that $\pi' : \tilde{Y} \to Y$ is a toric blowup, i.e. all of $q_i \in Y$ are located at corners of D. When Y is a smooth toric surface, then so is $\tilde{Y}$ whose moment polytope is obtained by chopping off the corresponding corner of the polytope. We set $\tilde{D} := \pi'^{-1}(D)$, which represents $c_1(\tilde{X})$, and $\tilde{X} = \tilde{Y} \setminus \tilde{D}, X = Y \setminus D$.

Since $\tilde{X}$ are X are biholomorphic, the holomorphic volume forms Ω and $\tilde{\Omega}$ on X and $\tilde{X}$ are related by $\tilde{\Omega} = \pi'^*\Omega$. Any Kähler form $\tilde{\omega}$ on $\tilde{X}$ naturally induces a Kähler form $\omega' = (\pi'^{-1})^*\tilde{\omega}$ on X.[10] In particular, $L \subseteq X$ is a Lagrangian with respect to ω' if and only if $\pi'^{-1}(L) \subseteq \tilde{X}$ is a Lagrangian with respect to $\tilde{\omega}$. However, ω' does not necessarily extend to the compactification Y. Nevertheless, we can use the following lemma to link Floer theory on X and that on Y.

---

[10]Later, we will apply this to $\tilde{\omega} = \tilde{\omega}_\epsilon$ constructed in the previous subsection.

Lemma 4.11. *Assume that D supports an ample divisor.*[11] *Given any ω' on X as above, there exists a sequence of Kähler form ω_i on Y such that $\omega_i = \omega'$ on U_i for some sequence $\{U_i\}_{i \geq 1}$ of relatively compact open subsets with $U_i \nearrow X$.*

Proof. Assume that $D' = \sum_j a_j D_j$ is an ample divisor with support on D for some $a_j \geq 0$. Then by replacing D' by $lD' + \sum_{j:a_j=0} D_j$, for $l \gg 0$, we may assume that $a_j > 0$ for all j. Then for $k \gg 0$, $\sum_j ka_j D_j$ is a very ample divisor and thus X is Stein.

Choose a Hermitial metric $e^{-\psi''}$ of the line bundle $\mathcal{O}_Y(D')$ such that the associate curvature is a Kähler form ω''. In particular, ω'' is defined globally on Y. Since X is Stein and hence the $\partial\bar{\partial}$-lemma holds, we have $\omega'|_X = i\partial\bar{\partial}\psi'\omega''|_X = i\partial\bar{\partial}\psi''$ for some smooth functions ψ', ψ'' on X. Take $\omega_i = i\partial\bar{\partial}\psi$, where $\psi_i = \widetilde{max}(\psi' + C_i, \psi'')$ is the regularized max function (see, for instance, [10, Theorem 5.21]) and $C_i \nearrow \infty$. One may take a regularized max function ψ_i to be $\psi_i = \max\{\psi' + C_i, \psi''\}$ if $|(\psi' + C_i) - \psi''| > \epsilon$ for some $\epsilon > 0$. Thus, $\omega_i = \omega'$ on $U_i := \{\psi_i' + C_i - \psi'' > \epsilon\}$. Since $\psi'' \sim \log|w|$ near D, where $D = \{w = 0\}$ locally, we have $\psi'' \nearrow \infty$ near D. Thus U_i is relatively compact, and $i\partial\bar{\partial}\psi = \omega''$ is globally defined. Therefore ω_i is a globally defined Kähler form which coincides with ω' on U_i. $\qquad\square$

5. Scattering diagrams and the Landau-Ginzburg mirrors of log Calabi-Yau surfaces

Continuing the setting in the previous section, let (Y, D) be a Looijenga pair, and consider its toric blowup $(\tilde{Y}, \tilde{D})$. Recall from Corollary 4.9 that the special Lagrangian fibration on $\tilde{Y} \setminus \tilde{D} \cong Y \setminus D$ (Lemma 4.3) induces a consistent the scattering diagram $\mathfrak{D}_\epsilon^{LF}$ on B_ϵ, away from ϵ-neighborhoods of the singular fibres of the fibration $\tilde{X} = \tilde{Y} \setminus \tilde{D} \to \mathbb{R}^2$. Yet, we do not have a good control of rays emanating from these neighborhood, except that there are canonical ones l_i induced by initial Maslov 0 discs from singular fibres. In this section we show that $\mathfrak{D}_\epsilon^{LF}$ behaves nicely as $\epsilon \to 0$, and its limit agrees with $\mathfrak{D}^{GPS}$ in Section 2.3.

We also give an explicit calculations of the Landau-Ginzburg potential which counts Maslov 2 discs in $\tilde{Y}$ and Y bounding torus fibres under some nonnegativity assumption (the same assumption of Section 4.4). This will be done by establishing the correspondence between tropical and holomorphic discs. Interestingly, the same technique can apply to some non-Fano examples to recover previous computation by Auroux [1] and [2] using tropical geometry.

5.1. $\mathfrak{D}^{LF}$ and $\mathfrak{D}^{GPS}$. We now prove that $\mathfrak{D}_\epsilon^{LF}|_{B_{\epsilon'}}$ coincides with $\mathfrak{D}_{\epsilon'}^{LF}$ for $\epsilon' < \epsilon$, which enables us to obtain a well-defined limiting scattering diagram $\mathfrak{D}^{LF}$ on $\mathbb{R}^2 = \lim_{\epsilon \to 0} B_\epsilon$. Our strategy is to compare $\mathfrak{D}_\epsilon^{LF}$ with the known one $\mathfrak{D}^{GPS}$ given in Section 2.3 modulo a certain energy level depending on ϵ. Then we proceed inductively as the energies (symplectic areas) of contributing discs increase.

For later argument, we fix, once and for all, $u_i \in \mathfrak{d}_i$ for each initial ray in $\mathfrak{D}^{GPS}$ close enough to infinity, and denote by $\mathfrak{d}_i'$ the part of the ray starting from u_i with $\omega(\gamma_\mathfrak{d})$ decreasing from u_i as it moves away to infinity. We also choose an open neighborhood K_i of for each $\mathfrak{d}_i'$ such that $V_i \cap V_j = \emptyset$ if $i \neq j$, and set $K := \cup K_i$. See Figure 3.

Lemma 5.1. *In the above setting, choose two arbitrary points $x_i \in K_i$ and $x_j \in K_j$ with $i \neq j$ such that the affine line segment connecting x_i and x_j has a rational slope (Figure 3). Then there exists $\hbar > 0$, independent of the choice of x_i and x_j, such that the symplectic area of the affine line segment is bounded below by $\hbar$.*

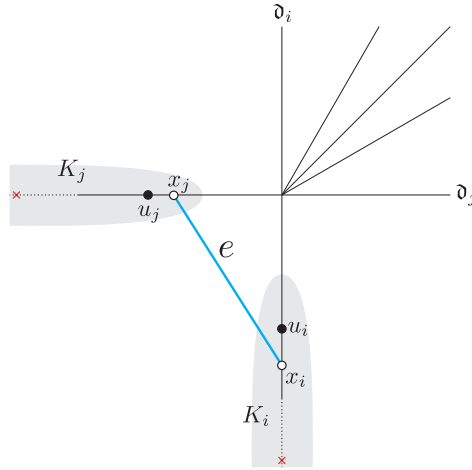[11]Under the same assumption, Gross-Hackiing-Keel proved that the mirror family is algebraic.

FIGURE 3. The areas of affine line segments between K_i and K_j are bounded below.

Here, the symplectic area of the affine line segment means that of the holomorphic cylinder lying over the segment. Notice that the usage of the terminology is consistent with the way we define the symplectic area of a tropical disc, see Section 2.1.

Proof. Notice that if the affine line segment connecting x_i, x_j has a rational slope, there exists a holomorphic cylinder with boundaries on $\tilde{L}_{x_i}$ and $\tilde{L}_{x_j}$, whose symplectic area equals that of the line segment by definition. Hence, the lemma is a direct consequence of Gromov compactness theorem. $\square$

The key step in the induction on energy is the following statement, which roughly tells us that any disc traveling through the complement of B_ϵ should have a large enough energy unless it is one of l_i's.

Lemma 5.2. *Given $\epsilon > 0$, let $\mathfrak{D}_\epsilon^{LF}$ denote the scattering diagram constructed from the admissible Lagrangian fibration over B_ϵ. There exists $\lambda = \lambda(\epsilon) > 0$ and a convex (bounded) region $B_{\lambda,\epsilon} \subseteq B_\epsilon$ such that a ray entering $B_{\lambda,\epsilon}$ other than l_i's (induced by initial Maslov 0 discs) has symplectic area at least λ. Moreover, we have $\lim_{\epsilon \to 0} \lambda = \infty$ and $\lim_{\epsilon \to 0} B_{\lambda,\epsilon} = M_\mathbb{R}$.*

For the proof, we will use the following sub-lemma, which can be viewed as a tropical analogue of Gromov compactness theorem.

Lemma 5.3. *There exists a sequence $\{\lambda_n\}$ of real numbers with $\lim_{n \to \infty} \lambda_n = \infty$ such that for each $n \in \mathbb{Z}_{>0}$, λ_n gives a lower bound for the symplectic areas of tropical discs with n non-root vertices whose unbounded edges are contained in the rays of $\mathfrak{D}_{in}^{GPS}$ and whose edge adjacent to the root is contracted.*

Proof. Since there are only finitely many initial rays and hence finitely many intersection points among the initial rays, every tropical disc with exactly one non-root vertex has a symplectic area lower bound, say $\lambda_1 > 0$. Assume that the lemma holds for $n \leq k$. Given a tropical disc (h, T, w) with $k+1$ non-root vertices with symplectic area less than λ. Deleting the edge adjacent to the root induces sub-tropical discs (h_i, T_i, w_i) with k_i non-root vertices such that $\sum_i k_i = k$. Then by induction hypothesis, one has $\lambda > \sum \lambda_{k_i}$, where we set $\lambda_{k_i} = 0$ for $k_i = 0$.

If there are at least two k_i which are positive, then one has $\lambda > \lambda_{\lfloor k/2 \rfloor} + \lambda_1$. Otherwise, there exists a sub-tropical disc (h_1, T_1, w_1) with k non-root vertices. Let λ' denote its symplectic area. From the previous argument, one has $\lambda' > \lambda_{\lfloor (k-1)/2 \rfloor} + \lambda_1$ when T_1 has at least two sub-tropical discs with positive numbers of non-root vertices. Otherwise, $T \setminus T_1$ has a bounded edge e both of whose end points are adjacent to unbounded edges. If both vertices of e lie in $K = \cup K_i$ (see Figure 3), the symplectic area of e is bounded below by $\hbar$ from Lemma 5.1. If one of the vertices is not in K, then the adjacent unbounded edge of T has a symplectic area lower bound which is independent of k. The complement of e and this unbounded edge in T has at least $k - 2$ non-root vertices, and hence its area is bigger than λ_{k-2}. To summarize, we have

$$\lambda > \min\{\lambda_{\lfloor \frac{k-2}{2} \rfloor} + \lambda_1, \lambda_{k-2} + \hbar\},$$

and one can take λ_{k+1} to be slightly bigger than the quantity on the right hand side. Furthermore, λ_n can be chosen so that $\{\lambda_n\}$ is monotone increasing, since every tropical disc with n non-root vertices contains a sub-tropical disc with $n - 1$ non-root vertices (possibly with the edge adjacent to the root contracted). Therefore we have

$$\lambda_{2k} > \min\{\lambda_{k-1} + \lambda_1, \lambda_{2k-2} + \hbar\} > \lambda_{k-1} + \min\{\lambda_1, \hbar\}.$$

In particular, the sequence $\{\lambda_n\}$ is unbounded. $\square$

proof of Lemma 5.2. Assume that there exists a ray $\mathfrak{d}$ of $\mathfrak{D}_\epsilon^{LF}$ passing through $u \in \partial B_\epsilon$. By definition of the scattering diagram $\mathfrak{D}_\epsilon^{LF}$, there exists a holomorphic disc $\tilde{f} : (D^2, \partial D^2) \to (\tilde{X}, \tilde{L}_u)$ of Maslov 0. Lemma 3.2 produces a tropical disc with stop at u whose edge adjacent to u is parallel to $\mathfrak{d}$ and whose unbounded edges are contained inside rays in $\mathfrak{D}_{in}^{GPS}$ (i.e. initial rays of $\mathfrak{D}^{GPS}$, see Section 2.3). We claim that there are only finitely many such tropical discs (up to elongation of the edge adjacent to the root) if we further require their symplectic areas to be less than a fixed number $\lambda > 0$. Once the claim is shown, one can simply take λ such $B_{\lambda,\epsilon}$ to be a convex region in B_ϵ containing all the vertices of such tropical discs. The last part of the lemma holds because $\lim_{\epsilon \to 0} B_\epsilon = M_\mathbb{R}$.

Lemma 5.3 tells us that for any tropical disc (h, T, w) with symplectic area less than a fixed constant λ, the number of non-root vertices is bounded above by some $N \in \mathbb{Z}_{>0}$. Hence, the claim will follow if we can prove that for each $n \in \mathbb{Z}_{>0}$, there are only finitely many tropical discs with symplectic areas less than λ which has precisely n non-root vertices. We use induction on n.

Let us first consider the case $n = 1$. Since all the unbounded edges are contained in finitely many initial rays, there are only finitely many intersections among these initial rays. Initial rays cut out at their common intersection points give tropical discs without non-root vertices (whose roots lie at the intersection points), and their symplectic areas are bounded below, say, by $\lambda' > 0$, since there are only finitely many such. Observe that each unbounded edge of a tropical disc with a single non-root vertex is precisely such a portion of an initial ray. These unbounded edges may additionally carry nontrivial weights (the edge adjacent to the root is automatically determined by these data, up to elongation), but the symplectic area bound λ on the whole disc forces the sum of the weights to be bounded above, since each unbounded edge contribute at least λ' to the total area. Therefore, there are only finitely many possible combinations between weights, as is desired for $n = 1$.

Suppose we know that there are only finitely many tropical discs with at most k non-root vertices whose symplectic areas are less than λ, and consider tropical discs with $k+1$ non-root vertices. Removing the edge adjacent to the root, one obtains sub-tropical discs (h_i, T_i, w_i) with k_i non-root vertices such that $\sum_i k_i = k$. If there are at least two (h_i, T_i, w_i) which do

not have any non-root vertices, then their intersection can only happen outside the region K chosen before Lemma 5.1. Hence their symplectic areas are bounded below by some constant $\hbar'$. In particular, the weight of those edges are bounded by $\lambda/\hbar'$ and thus there are finitely many of such tropical discs.

Assume now that there is only one (h_i, T_i, w_i) without non-root vertices among the sub-tropical discs. Then the other sub-tropical discs have at most k non-root vertices and symplectic areas less than λ. From induction hypothesis, there are only finitely many such tropical discs. Extending the edge adjacent to the root beyond the root if necessary, (union of) these tropical discs intersect the initial rays at finitely many points. We see that $(h_i, T_i, 1)$ should reach one of these points to form the original disc with $k + 1$ non-root vertices. Therefore, there exists a lower bound $\hbar''$ for the symplectic area of $(h_i, T_i, 1)$, independent of k. Again the weight w_i for the edge is bounded and thus there are finitely many tropical discs in this case as well, which finishes the proof. $\qquad\square$

We finally prove the main theorem of the paper, which provides a recipe to retrieve the scattering diagram of Gross-Pandharipande-Siebert from Lagrangian Floer theory. Recall that the fibration over $B' = B_\epsilon$ is obtained by removing neighborhoods of singular fibre from the fibration constructed in Theorem 4.3. Since B_ϵ is simply-connected by construction, the fibration has no monodromy over B_ϵ, and hence, the relative classes are well-defined up to parallel transport along B_ϵ.

Theorem 5.4. *Given $\epsilon > 0$, the scattering diagram $\mathfrak{D}_\epsilon^{LF}$ defined in Corollary 4.9 coincides with $\mathfrak{D}^{GPS}$ on $B_{\lambda,\epsilon}$ for $\epsilon \ll 1$ up to contributions from discs with area$> \lambda$. In particular, one can recover $\mathfrak{D}^{GPS}$ from Lagrangian Floer theory by*

$$\lim_{\epsilon \to 0} \mathfrak{D}_\epsilon^{LF}(mod\ T^\lambda)|_{B_{\lambda,\epsilon}} = \mathfrak{D}^{GPS}.$$

Proof. Let γ_i be the relative homology class of the disc associated with the blowup point q_i in Lemma 4.10, which can be represented as a holomorphic disc with boundary in a torus fibre over B_ϵ. It is easy to see that $\{[E_i]\}$ (and $\{\gamma_i\}$) forms a linearly independent subset in the cohomology. From Lemma 4.10 and the identification $t^{-[E_i]}z^{m_{\sigma(i)}} \leftrightarrow T^{\omega(\gamma_i)}z^{\partial\gamma_i}$, each blowup point q_i contributes to the same initial ray to the scattering diagram of $\mathfrak{D}_\epsilon^{LF}$ and $\mathfrak{D}^{GPS}$. From Lemma 5.2, the above are all the initial rays in the scattering diagrams $\mathfrak{D}_\epsilon^{LF}(\text{mod } T^\lambda)|_{B_{\lambda,\epsilon}}$ and $\mathfrak{D}^{GPS}|_{B_{\lambda,\epsilon}}$. Thus, we have

$$\mathfrak{D}_\epsilon^{LF}(\text{mod } T^\lambda)|_{B_{\lambda,\epsilon}} = \mathfrak{D}^{GPS}|_{B_{\lambda,\epsilon}}$$

for every $\lambda, \epsilon > 0$ from Theorem 2.6, and this completes the proof since $B_{\lambda,\epsilon} \nearrow \mathbb{R}^2$. $\qquad\square$

As a byproduct, we can verify the folklore conjecture

"the open Gromov-Witten invariants = the log Gromov-Witten invariants."

in our geometric setup, by comparing the coefficients for the wall functions of the two scattering diagrams in Theorem 5.4. Our argument passing to tropical geometry has a great advantage that we can avoid the difficulty of comparing the virtual fundamental classes in the algebraic and symplectic version of log Gromov-Witten invariants.

We can identify homology classes of the curves that the two invariants count in the following way. To a given relative class γ of a disc in $\tilde{X}$, one can associate a homology class $\bar{\gamma} \in H_2(\tilde{Y}, \mathbb{Z})$ as follows. Fix $u_0 \in B_\epsilon$ and choose a representative C_γ of $\gamma \in H_2(\tilde{X}, \tilde{L}_{u_0})$. Let l be an affine ray starting from u_0 parallel to the direction defined by $\partial\gamma$ (it moves away from the disc γ). Let $\text{Cyc}_{\partial\gamma}$ be a trivial S^1-fibration over l with fibres in the class $\partial\gamma$. Since $[\text{Cyc}_{\partial\gamma} \cap L_u] = [C_\gamma \cap L_u] \in H_1(L_u, \mathbb{Z})$, there exists C' in L_u homologous to $C_\gamma \cup \text{Cyc}_{\partial\gamma}$ where

both are thought of as non-compact 2-chains without boundary in $\tilde{X}$. The explicitly chosen such a 2-chain has a one-point compactification in $\tilde{Y}$ as a 2-cycle, whose homology class will be denoted by $\bar{\gamma}$.

Corollary 5.5. *Given a relative class γ, there exists a proper open set $U_\gamma \subseteq B_\epsilon$ (for any small enough ϵ) such that*

   (1) *if $\tilde{\Omega}(\gamma; u) = 0$ for all $u \in B_\epsilon \setminus U_\gamma$, then $N_{\bar{\gamma}} = 0$.*
   (2) *if $u \in B_\epsilon \setminus U_\gamma$ and $\tilde{\Omega}(\gamma; u) \neq 0$, then $\tilde{\Omega}(\gamma; u) = N_{\bar{\gamma}}$.*

Recall from Section 2.5.2 and Section 2.3 that $\tilde{\Omega}(\gamma; u)$ and $N_{\bar{\gamma}}$ are the open Gromov-Witten invariant and the log Gromov-Witten invariant, respectively.

Proof. The first part of the corollary is a direct consequence of Theorem 5.4 combined with the discussion in Section 2.3.

To see the second part, notice that every ray in the scattering diagram has a corresponding tropical curve. Thus, there are finitely many rays in $\mathfrak{D}^{GPS}$ with the wall functions in $\mathbb{C}[[t^{\bar{\gamma}}]][[M]]$ by Lemma 5.3. Projecting down γ by $\pi : \tilde{Y} \to \bar{Y}$, the intersection pairing of $\pi_*(\gamma)$ with each boundary component $\bar{D}_i$ is determined. All the intersections of $\pi_*(\gamma)$ with $\bar{D}$ occur at $\{q_1, \cdots, q_m\}$, since $\mu(\gamma) = 0$.

Moreover, the multiplicities and the positions of unbounded edges of the tropical discs of relative class γ are determined topologically by $\bar{\gamma}$. Indeed, the multiplicity p_i of the unbounded edge determined by q_i can be calculated via the relation

$$\bar{\gamma} = \pi^* \pi_*(\bar{\gamma}) - \sum_i p_i E_i.$$

Fix positions and multiplicities of all the edges of a tropical curve (say, in class $\bar{\gamma}$), but one unbounded edge. By balancing condition, the multiplicity and the direction of the last unbounded edge is automatically determined. Actually the position of the last unbounded edge is also determined by [42, Proposition 6.12] (see [34, Theorem 3.3.10], also). [12]

Suppose the edge adjacent to the root of a tropical disc of relative class γ lies in some affine line l. We essentially showed in the proof of Lemma 5.2 that there are finitely many tropical discs (up to elongation of the edge adjacent to the root) in the class γ. Therefore $\tilde{\Omega}(\gamma; u')$ is constant long l near infinity towards the direction along which $\omega(\gamma) \gg 0$. It suffices to take U_γ such that $U_\gamma \cap l$ is contained the part of l where $\tilde{\Omega}(\gamma)$ is constant, then the second part of the corollary holds from Theorem 5.4. □

Remark 5.6. *The unique rational curve in Lemma 3.1 corresponding to relative γ is different from those $\mathbb{A}^1$-curves contributing to $N_{\bar{\gamma}}$ even topologically. We refer the readers to Example 5.3 [32].*

Remark 5.7. *Although the toric model $(\bar{Y}, \bar{D})$ of a Looijenga pair (Y, D) is always a smooth toric surface, theorems in Section 5.1 generalize to the case when $\bar{Y}$ only have orbifold singularity at the corners. This is because X is still a smooth symplectic manifold and the local model for the non-toric blowup remains the same.*

5.2. **The Landau-Ginzburg mirror of the compactification Y.** For the rest of the section, we will assume that $\tilde{Y}$ is semi-Fano and, consider an admissible SYZ fibre $\tilde{L}_u$ in $\tilde{Y}$ (i.e., a pull-back of a moment map torus L_u in Y). We will show that the superpotential of $\tilde{L}_u$ can be calculated tropically. More specifically, we have the following theorem.

---

[12]This is a tropical analogue of [25, Lemma 4.1].

Theorem 5.8. *Assume that $\tilde{D}$ contains no negative Chern number spheres. Then $n_\beta(u) := n_\beta^{\tilde{Y}}(\tilde{L}_u)$ coincides with the weighted count of the broken lines in with respect to the scattering diagram $\mathfrak{D}^{LF}$ in Theorem 2.16.*

Here, $n_\beta^{\tilde{Y}}(\tilde{L}_u)$ is the algebraic count of Maslov index 2 discs in class β. See Section 2.5. We first need a few preliminary lemmas.

Lemma 5.9. *Assume that l is an affine ray from u in the direction of $\partial\beta$ for some $\beta \in H_2(\tilde{Y}, \tilde{L}_u)$ such that $\tilde{\omega}(\beta)$ is decreasing along the ray. There exists a constant $\hbar > 0$ such that if*

(1) *$n_\beta(u)$ is non-zero and constant along the ray, or*
(2) *$\omega(\beta) < \hbar$ and $n_\beta(u) \neq 0$,*

then $\beta = \beta_i + \alpha$ for some $\alpha \in H_2(\tilde{Y}, \mathbb{Z})$ and some i.

Proof. First we assume that n_β is non-zero along l, then there exists a stable holomorphic disc of relative class β with boundary on L_u for for each $u \in l$. Applying Lemma 3.2 to the disc component of such a stable disc, we obtain a tropical disc with stop at u whose edge adjacent to u is parallel to l. From the hypothesis, we may assume that $\tilde{\omega}(\beta) < \lambda$ for some constant $\lambda > 0$. Recall that any broken lines except those without bending (see Example 2.10), the corresponding Maslov index two tropical disc will contain a Maslov index zero tropical sub-disc, which has area less than $\omega(\beta)$. Since Lemma 5.2 tells us that there are only finitely many tropical discs on Maslov index zero in $\tilde{Y}$ with symplectic area less than λ, there are only finitely many intersections of the tropical discs of Maslov index zero with symplectic area less than $\omega(\beta)$ with l. Therefore, if $u \in l$ is close enough to infinity, there exists no tropical discs with stop on l between u and the infinity while the symplectic area is less than λ other than the (multiple) of the initial disc. In particular, l is defined by the vanishing cycle of a toric boundary divisor. Since β is of Maslov index two, the corresponding tropical disc has the same relative class as the basic disc. The second case follows from the same line of the argument of the first case. $\square$

We next show a weak version of tropical/holomorphic correspondence theorem.

Lemma 5.10. *If $n_\beta^{\tilde{Y}}(u) \neq 0$ for a generic point u, there exists a broken line $\mathfrak{b}$ with end at u such that $[\mathfrak{b}] = \beta$.*

Proof. For a generic u, $n_\beta^{\tilde{Y}}(u)$ is well-defined, i.e. $u \notin \text{Supp}(\mathfrak{D}^{LF})(\text{mod } T^\lambda)$ where $\lambda > 0$ is a constant, since there are finitely many rays in $\text{Supp}(\mathfrak{D}^{LF})(\text{mod } T^\lambda)$. We will take $\lambda < \int_{\beta_u} \tilde{\omega} + \lambda_1$, where λ_1 is the lower bound of tropical discs of Maslov index zero (see Lemma 5.3). Consider an affine ray l emanating from u in the direction of $\partial\beta$ such that $\tilde{\omega}(\beta)$ is decreasing along the ray. By genericity of the position of u, we may assume that the affine ray intersects all the rays in $\text{Supp}(\mathfrak{D}^{LF})(\text{mod } T^\lambda)$ transversally. Travelling along the ray, the following two scenarios can occur.

(i) $n_\beta^{\tilde{Y}}$ is constant along the whole affine ray, then the lemma directly follows from Lemma 5.9.

(ii) $n_\beta^{\tilde{Y}}$ jumps at some point $u_1 \in \text{Supp}(\mathfrak{D}^{LF})(\text{mod } T^\lambda)$, which means that a certain bubbling phenomenon occurs at $\tilde{L}_{u_1}$. Notice that by Gromov compactness theorem, there are only finitely many rays in the scattering diagram $\mathfrak{D}^{LF}(\text{mod } T^\lambda)$. Let u' on the affine ray close to u_1 such that $n_\beta^{\tilde{Y}}(u') \neq n_\beta^{\tilde{Y}}(u)$ and by the assumption of the

generic position of u we can assume that exactly one ray $l_{\mathfrak{d}_1}$ pass through u_1 with $\omega(\gamma_{\mathfrak{d}_i}) < \lambda$. Apply Lemma 2.14 (1) to the path connecting u', u along l implies that

$$W(u) = \mathcal{K}_{\mathfrak{d}_1} W(u') \bmod T^\lambda.$$

Therefore, there exists $\beta' \in H_2(\tilde{Y}, L_{u'})$ with $n_{\beta'}(u') \neq 0$ such that the coefficient of z^β in $\mathcal{K}_{\mathfrak{d}_1}(z^{\beta'})$ is non-zero.

We then replace (u, β) by (u', β'), and repeat the above procedure. Notice that $\tilde{\omega}(\beta_u) - \tilde{\omega}(\beta'_{u'}) \geq \lambda_1$, and the process ends in finite process, say at $u_1, \cdots, u_n$ and then it reduces to the first case.

Finally, we define the broken line $\mathfrak{b} : (-\infty, 0] \to N_\mathbb{R}$ with $\mathfrak{b}(0) = u$ such that $\mathfrak{b}(t_i) = u_{n-i}$ for $-\infty = t_0 < t_1 < \cdots < t_n = 0$ (here we set $u = u_0$). Then (1) in Definition 2.3 follows from the choice of the affine rays above, (2) follows from Lemma 5.9, (3) can be achieved by genericity of the position of u, and (4) follows from Lemma 2.14.

$\square$

Proof of Theorem 5.8. Fix $\beta \in H_2(Y, L_u)$. Notice that each bending of the broken line will increase its corresponding area by at least $\lambda_1 > 0$ (see the notation in Lemma 5.3). Thus a broken line $\mathfrak{b}$ such that $[\mathfrak{b}] = \beta$ can bend at most $\frac{\tilde{\omega}(\beta)}{\lambda_1}$ times. Together with Lemma 5.3 implies that there are finitely many places the broken lines can bend. Thus, there are finitely many broken lines $\mathfrak{b}$ such that $[\mathfrak{b}] = \beta$ by Lemma 3.2. Let $n(\beta; u)$ be the maximal number of edges of a broken line that can represent β. We will prove the theorem by induction on $n(\beta; u)$. The theorem reduces to Lemma 5.9 when $n(\beta; u) = 1$.

Assume that the theorem is true, i.e., $n_\beta(u) = n_\beta^{trop}(u)$, for all pairs of $(\beta; u)$ if $n(\beta; u) \leq n$ where $n_\beta^{trop}(u)$ is the number of broken lines in class β with end on u. Given a pair $(\beta; u)$ such $n(\beta; u) = n + 1$. There exists an affine ray emanating from u corresponds to $\partial\beta$ such that $\tilde{\omega}(\beta)$ is decreasing along the affine ray.

Suppose that n_β jumps at u_1. Using the same argument as in the proof of Lemma 5.10, one can find $\beta_1 \in H_2(Y, L_{u_1})$ with $n(\beta_1; u_1) < n(\beta; u)$. By the induction hypothesis, we have $n_{\beta_1}(u_1) = n_{\beta_1}^{trop}(u_1)$, and we can derive $n_\beta(u)$ and n_β^{trop} from their compatibility with the wall-crossing formula. The two wall-crossing formulas are identified in Theorem 5.4, and this finishes the proof. $\square$

We now consider the effect of the toric blowup $\tilde{Y} \to Y$ on the counting n_β. Note that Y is at worst semi-Fano, since contracting exceptional divisors in $\tilde{D}$ do not decrease the self-intersections of the other divisors.

Lemma 5.11. *Let $\pi' : (\tilde{Y}, \tilde{D}) \to (Y, D)$ is a simple toric blowup. Let $\tilde{L} \subseteq \tilde{X}$ be an admissible SYZ fibre in the toric model of $\tilde{Y}$ and $L = \pi'(\tilde{L})$. If the superpotential of $\tilde{L}$ is $\sum_{\beta \in H_2(\tilde{Y}, \tilde{L}):\mu(\beta)=2} n_\beta^{\tilde{Y}} T^{\tilde{\omega}(\beta)} z^{\partial\beta}$, then the superpotential of L is given by*

$$\sum_{\substack{\beta \in H_2(\tilde{Y}, \tilde{L}) \\ \beta \cdot E = 0, \, \mu(\beta) = 2}} n_\beta^{\tilde{Y}} T^{\omega(\pi'_* \beta)} z^{\partial\beta}.$$

Proof. Let $u : (\Sigma, \partial\Sigma) \to (Y, L)$ be a stable disc. From the semi-Fano condition of Y, the Lagrangian L does not bound any holomorphic discs of negative Maslov index. If it is of Maslov index 0 or 2 of relative class $\beta \in H_2(Y, L)$, then its image does not hit the corner of D. Therefore, its proper transform $\tilde{u} : (\Sigma, \partial\Sigma) \to (\tilde{Y}, \tilde{L})$ is a stable disc of relative

class $\tilde{\beta} \in H_2(\tilde{Y}, \tilde{L})$ avoiding the exceptional divisor E of $\tilde{Y} \to Y$ and the corners of $\tilde{D}$. In particular, the proper transform preserves Maslov indices, in this case.

Conversely, if $\tilde{u} : (\Sigma, \partial\Sigma) \to (\tilde{Y}, \tilde{L})$ is a stable disc of Maslov index 2. Then the projection $u = \pi' \circ \tilde{u} : (\Sigma, \partial\Sigma) \to (Y, L)$ is a stable disc. Notice that for every non-exceptional curve $\tilde{C} \subseteq \tilde{Y}$, one has

$$-K_{\tilde{Y}} \cdot [\tilde{C}] = -K_Y \cdot [\pi'(\tilde{C})] - [\tilde{C}] \cdot [E] \leq -K_Y \cdot [\pi'(\tilde{C})].$$

Therefore, the Maslov index of projection is bigger than or equal to that of the original disc $\tilde{u}$, and the equality holds if and only if the image of $\tilde{u}$ does not contains a component intersecting the exceptional divisor.

Consequently, the stable discs of (Y, L) of Maslov index 2 are in one-to-one correspondence with the stable discs of $(\tilde{Y}, \tilde{L})$ of Maslov index 2 which do not intersect the exceptional divisor. In other words, all these discs are contained in biholomorphic open subsets of $\tilde{Y}$ and Y. In particular, the corresponding counts $n_\beta^{\tilde{Y}}$ and $n_{\pi'_* \beta}^Y$ are the same.

□

Together with Theorem 5.8, we have Theorem 1.3 in the introduction.

Remark 5.12. *With the admissible special Lagrangian fibration in $Y \setminus D$, Tu [51] constructed the family Floer mirror of the total space of the admissible special Lagrnagian fibration. The superpotential for these admissible fibres satisfying the wall-crossing formula and thus is a well-defined function on the mirror as explained by Yuan [55].*

Motivated by Gross-Hacking-Keel [21, Construction 2.27], we may define the theta functions $\theta_{[D_i]}(u)$ by

$$\theta_{[D_i]}(u) = \sum_\beta n_\beta^Y(u) T^{\omega(\beta)} z^{\partial\beta}, \tag{5.1}$$

where β runs over all relative classes in $H_2(Y, L_u)$ with $\mu(\beta) = 2$ and the (unique) disc components of their holomorphic representatives intersect D_i once. Then Theorem 5.8 implies that (5.1) coincides with the original definition in [21].

The superpotential computed here are expected to be the mirror of Y. In fact, we can prove that the quantum period theorem holds under the additional assumption that Y is Fano. Let $\mathcal{M}_{0,1}(Y, A)$ be the moduli space of stable maps of genus zero with a marked point and image have class A for $A \in H_2(Y, \mathbb{Z})$. Write $d_A := A \cdot (-K_Y)$. The virtual dimension of the moduli space $\mathcal{M}_{0,1}(Y, A)$ is d_A, and it has a virtual fundamental class $[\mathcal{M}_{0,1}(Y, A)]^{vir}$. Let $ev : \mathcal{M}_{0,1}(Y, A) \to Y$ be the evaluation map, and ψ the psi-class insertion at the marked point. Then the regularized quantum period of Y is given by $\hat{G}_Y := \sum_{A \in H_2(Y, \mathbb{Z})} \langle \psi_{d_A - 2} pt \rangle_A z^A$, where $\langle \psi_{d-2} pt \rangle_A$ is the descendant Gromov-Witten invariants

$$\langle \psi_{d_A - 2} pt \rangle_A = \int_{[\mathcal{M}_{0,1}(Y, A)]^{vir}} \psi^{d_A - 2} ev^*[pt].$$

We also write $\langle \psi_{d-2} pt \rangle_d = \sum_{A : d_A = d} \langle \psi_{d_A - 2} pt \rangle_{d_A}$. The quantum period theorem is a direct consequence of [43, Theorem 1.12] (see also [52, Theorem 1.13]) combined with Theorem 5.8.

Theorem 5.13. *Assume that Y is a Fano surface and*
  (1) *Y is a non-toric blowup of a toric surface or*
  (2) *each component of the boundary divisor is nef and*
then the constant term of $W(L)^d$ is simply $(d)! \langle \psi_{d-2} pt \rangle_d$, where L is any admissible SYZ fibre in Y.

5.3. **Further examples.** Below, we use the new machinery developed in this paper to recover classical examples in earlier works of Auroux, and provide a tropical explanation of the calculations.

(i) $(\mathbb{P}^2, D)$ *with* $D = C \cup L$. We first look into the example of $Y \cong \mathbb{P}^2$ with the anticanonical divisor given by $D = C \cup L$, where C is a conic, and L is a line L in Y. The corresponding mirror potential was first computed in [1, Proposition 5.5, Proposition 5.8]. On the other hand, (Y, D) viewed as a Looijenga pair, the associated non-toric blowup $(\tilde{Y}, \tilde{D})$ and its toric model $(\bar{Y}, \bar{D})$ are computed in [30, Example 2.9], which we now explain.
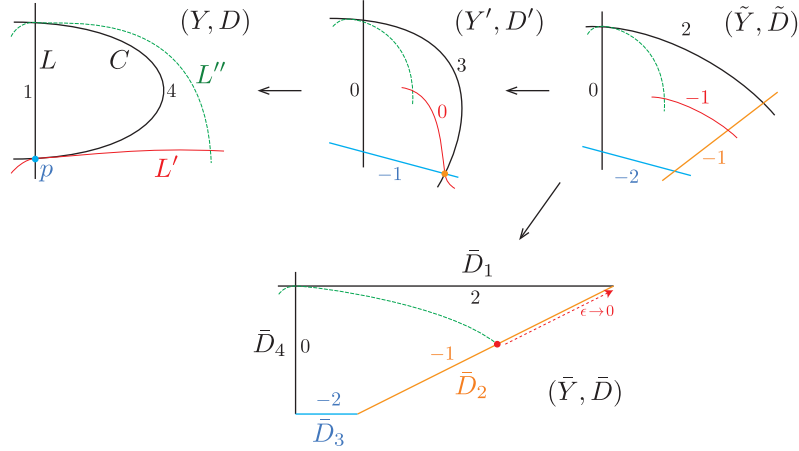
FIGURE 4. Toric model for (a toric blowup of) the pair $(\mathbb{P}^2, D)$

Let p be one of the nodes in D, and L' the tangent of C at p. We take the blowup $Y' \cong \mathbb{F}_1 \to Y$ of Y at p, and write D' for the preimage of D. Then $\tilde{Y}$ is obtained as the blowup of Y' at the intersection of the proper transform of L and the exceptional divisor for $Y' \to Y$. We denote the preimage of D' by $\tilde{D}$. The proper transform of L is now a (-1)-curve in $\tilde{Y}$, and its blow down gives rise to the Hirzebruch surface $\bar{Y} \cong \mathbb{F}_2$. The push-forward $\bar{D}$ of $\tilde{D}$ is the toric boundary divisor (see Figure 4). One may choose the fan of $\bar{Y}$ whose 1-cones are spanned by $v_1 = (0, -1)$, $v_2 = (-1, 2)$, $v_3 = (0, 1)$ and $v_4 = (1, 0)$. We will denote by $\bar{D}_i$ the component of the toric boundary divisor normal to v_i.

Given a moment fibre $\bar{L}$ of $\bar{Y}$, one has the basic disc class $\beta_i \in H_2(\bar{Y}, \bar{L})$ for each $\bar{D}_i$, which intersects $\bar{D}_i$ exactly once. The superpotential $W_{\mathbb{F}_2}$ for $\mathbb{F}_2$, however, has more contributions than those from β_i's. It is well-known by, for e.g., [1, 17, 7]

$$W_{\mathbb{F}_2} = t^{\beta_1} y^{-1} + t^{\beta_2} x^{-1} y^2 + (1 + t^{[\bar{D}_3]}) t^{\beta_3} y + t^{\beta_4} x$$

where we set $t^\beta := T^{\omega(\beta)}$ in order to keep track of the relative classes easily. Notice the sphere bubble contributions as seen from the term $t^{[\bar{D}_3] + \beta_3} y$.

Applying Theorem 5.4, the blowup introduces a single ray (wall) for the scattering diagram $\mathfrak{D}^{LF}$ for $(\tilde{Y}, \tilde{D})$. Let us denote by α the Maslov zero disc class corresponding to the ray. By Theorem 5.8, the superpotential in each chamber can be computed by counting broken lines as illustrated in Figure 5. The sphere bubble contributions are omitted in the figure, which are $t^{[\bar{D}_3]}$ multiplies of the broken lines with infinity ends parallel to the negative y-axis.

During the sequence of blowdowns $\tilde{Y} \to Y' \to Y$, geometry of the complement of the boundary divisor is unchanged, and hence, the scattering diagram remains the same. Finally,
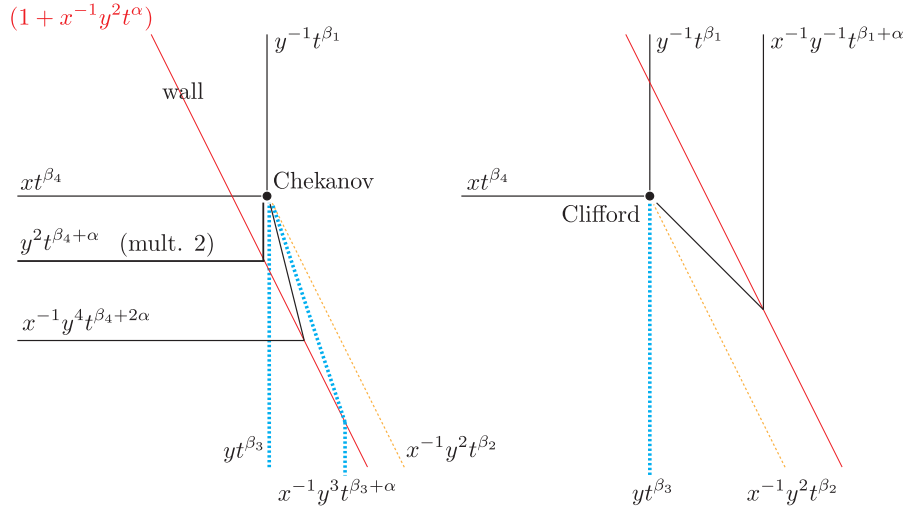
FIGURE 5. Broken line counts on the two chambers.[a] Dashed are the contributions that disappear after blowdowns to Y. The broken lines with infinity ends parallel to the negative y-axis survive until Y', and one additionally has $t^{[\bar{D}_3]}$ multiplies of these for $\tilde{Y}$, which correspond to sphere-bubble configurations.

[a]We use β_i to denote the proper transform of the relative class for simplicity.

we can derive the superpotentials for (Y', D') and (Y, D) with help of Lemma 5.11. See Figure 5 for the precise terms in the potential on each chamber.

Observe that the two superpotentials are exactly those for the Clifford torus and the Chekanov torus in $\mathbb{P}^2$ after a suitable coordinate change. The reason that the two chambers support Clifford and Chekanov tori can be explained as follows. When the blowup point $\tilde{Y} \to \bar{Y}$ is approaching $\bar{D}_1 \cap \bar{D}_2$, then the unique ray in the scattering diagram is shifting upward. At the same time, the conic C degenerates into the union of L' and L'', where L'' is the tangent of C at the node of D other than p. The resulting anticanonical divisor $L \cup L' \cup L''$ is the toric boundary of $Y \cong \mathbb{P}^2$.

Remark 5.14. *In the above procedure, we do not need to know the precise contributions of the sphere bubble configurations in $\mathbb{F}_2$. For our purpose, it suffices to start with*

$$W_{\mathbb{F}_2} = t^{\beta_1} y^{-1} + t^{\beta_2} x^{-1} y^2 + (1 + \sum_k a_k t^{k[\bar{D}_3]}) t^{\beta_3} y + t^{\beta_4} x,$$

with some unknown $a_k \in \mathbb{Q}$. Any term involving $t^{[\bar{D}_3]}$ eventually vanishes in the superpotential for (Y, D) since the corresponding discs turn to have higher Maslov indices in Y. Once we computed the superpotential for $(\mathbb{P}^2, C \cup L)$, we can derive the superpotential of $(\mathbb{F}_2, \bar{D})$ reversing the procedures in the previous example. In particular, we see $a_1 = 1$ and all other a_k vanish.

(ii) *The third Hirzebruch surface $\mathbb{F}_3$ (non-Fano).* Auroux [2, 3.2] (see Figure 4 therein) constructed special Lagrangian fibrations on $Y' \setminus D'$ where we continue to use the notation in the previous example. There are two chambers in the base of the fibration such that fibres over the points in the same chambers have the same superpotential (with suitable correction of symplectic flux).

Fibres over one chambers are hamiltonian isotopic to Clifford tori, and the other ones are hamiltonian isotopic to Chekanov tori. Auroux further showed that on the Chekanov

side, the superpotential for (Y', D') matches the superpotential for the moment fibres of $\mathbb{F}_3$. Therefore, the latter can be read off from broken lines in (left) Figure 5. The difference from the previous example of Y is that we need to take into account two more broken lines whose infinity ends are parallel to the negative y-axis. Hence, in our choice of coordinates, the potential for $\mathbb{F}_3$ is given as

$$xt^{\beta_4} + 2y^2t^{\beta_4+\alpha} + x^{-1}y^4t^{\beta_4+2\alpha} + yt^{\beta_3} + x^{-1}y^3t^{\beta_3+\alpha} + y^{-1}t^{\beta_1}$$
$$= yt^{\beta_3} + xt^{\beta_4}(1 + x^{-1}y^2t^\alpha)^2 + (yt^{-\beta_1})^{-1}(1 + x^{-1}y^4t^{\beta_3-\beta_1+\alpha}).$$

It is not very difficult to match the above with equation (3.7) of [2] up to some corrections by flux (powers of t).

(iii) *Cubic surface.* The mirror superpotentials of del Pezzo surfaces were written down by Galkin-Usnich [29]. In particular, there are 21 terms in the superpotential for a cubic surface. Therefore, Sheridan [50] conjectured that there are 21 discs with boundaries on certain Lagrangian in a cubic surface, as the open analogue of the classical result of 21 lines in a cubic surface. The conjecture was verified by Pascaleff-Tonkonog [47] and Vanugopalan-Woodward [54]. Here we provide a different method: consider the cubic surface Y as the non-toric blowup at 6 points on $\bar{Y} = \mathbb{P}^2$. We will choose the blowup loci such that there are two points on each toric boundary divisors close to a corner and exactly two points in a small neighborhood of each corner. The corresponding GPS scattering diagram has a chamber in the middle and the 21 broken lines with ends in the middle chamber are listed in Figure 6. Then we recover the superpotential of a cubic surface from Theorem 5.8.
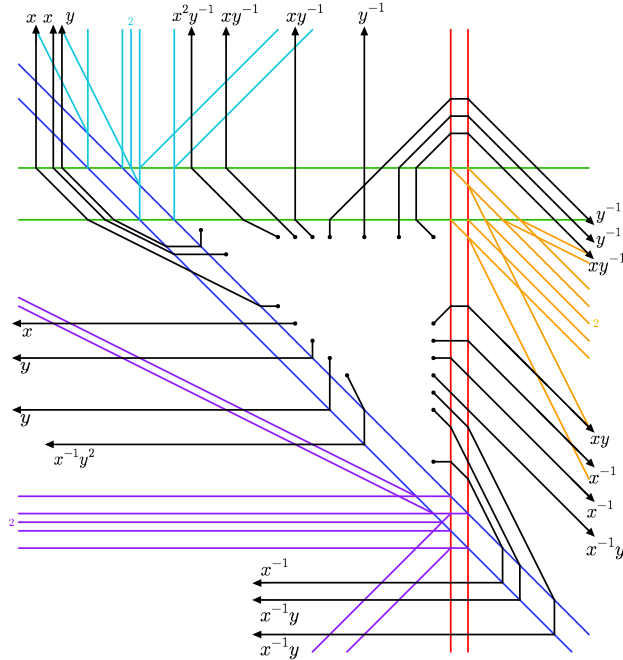

FIGURE 6. The 21 broken lines in cubic surface with ends in the middle chamber.

5.4. Towards the Local Open Gromov-Witten Invariants of Singular Fibre of Type I_0^*.
Let $\bar{Y} \cong \mathbb{P}^2$, $\bar{D}$ is the toric boundary and $\bar{D}_i, i = 1, 2, 3$, be the components of $\bar{D}$. Let $Y \to \bar{Y}$ be a non-toric blowup of six points, two points on each components of $\bar{D}$. Then Y is a

del Pezzo surface of degree three and thus a cubic surface. Denote D (or D_i) to be the proper transform of $\bar{D}$ (or $\bar{D}_i$ respectively), we have $D \in |-K_Y|$. Denote $E_{ij}, i = 1, 2, 3, j = 1, 2$ be the exceptional divisor of the blowup on $\bar{D}_i$. For each D_i, there are eight lines on the cubic surface intersection D_i and we denote them by $L_{ij}, i = 1, 2, 3, j = 1, \cdots, 8$. Notice that $\{E_{i1}, E_{i2}\} \subseteq \{L_{ij} | 1 \le i \le 8\}$. The affine monodromy of the singularity of B_{GHKS} is conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. From Theorem 2.5 [23], there is a ray $\mathfrak{d} = \mathbb{R}_{\ge 0}(av_i + bv_{i+1})$ in the canonical scattering diagram and the wall function is

$$f_{\mathfrak{d}} = \frac{\prod_{j=1}^8 (1 + z^{C_j} X_i^{-a} X_{i+1}^{-b})}{(1 - z^{C_9} X_i^{-2a} X_{i+1}^{-2b})^4},$$

for some curve class $C_i, i = 1, \cdots, 9$. From the relation between the canonical scattering diagram $\mathfrak{D}^{can}$ of (Y, D) and the scattering diagram $\mathfrak{D}^{GPS}$ of $(Y, D) \to (\bar{Y}, \bar{D})$ discussed in Section 2.4, we may get the wall function of the generic ray.

6. Application to Mirror Symmetry for Rank 2 Cluster Varieties

In this section, we will apply Theorem 5.4 to rank 2 cluster varieties. Here 'rank 2' refers to the skew-symmetric form in fixed data (defined in the next section) are of rank 2. In this section, we are going to show that the quotients of the $\mathcal{A}$ cluster varieties are mirror to the fibers of the Langland dual $\mathcal{X}$ cluster varieties.

There have been many works on the mirror symmetry for cluster varieties in the literature. From the algebro-geometric perspective, Keel-Yu [33] made use of Berkovich non-archimedean methods to construct mirror and compared their mirror algebra with the cluster algebra in [22]. Note that Keel-Yu assumed that the skew-symmetric forms are non-degenerate, while our skew-symmetric forms are of rank 2, i.e. we allow those are degenerate. We further work on the general skew-symmetrizable case instead of skew-symmetric case. Mandel [40] showed that theta bases for cluster varieties are determined by descendant log Gromov-Witten invariants of the symplectic leaves of the Langland dual cluster variety. From the symplectic perspective, Gammage-Le [18] constructed a symplectic manifold from a given transacted cluster variety such that the homological mirror symmetry holds. Kim-Lau-Zheng [35] used Lagrangian Floer theory to prove mirror symmetry for cluster varieties of type A. The following discussion would be the first to start with the $\mathcal{A}$ side and obtain mirrors in the Langland dual dual of the $\mathcal{X}$ side.

6.1. **Cluster varieties.** Let us begin by defining the *fixed data* Γ for a pair of cluster varieties without frozen variables $(\mathcal{A}, \mathcal{X})$, which consists of the following:

- a finite set I with $|I| = n$;
- a lattice N of rank $|I| = n$;
- a saturated sublattice $N_{\mathrm{uf}} \subseteq N$ of rank n;
- a skew-symmetric bilinear form $\{\cdot, \cdot\} : N \times N \to \mathbb{Q}$;
- a sublattice $N^\circ \subseteq N$ of finite index satisfying $\{N_{\mathrm{uf}}, N^\circ\} \subset \mathbb{Z}$ and $\{N, N_{\mathrm{uf}} \cap N^\circ\} \subset \mathbb{Z}$;
- a tuple of positive integers $(d_i : i \in I)$ with greatest common divisor 1;
- $M = \mathrm{Hom}(N, \mathbb{Z})$ and $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$.

Given this fixed data, a *seed data* for this fixed data is $\mathbf{s} := (e_i \in N \mid i \in I)$, where $\{e_i\}$ is a basis for N. The basis for M° would then be $f_i = \frac{1}{d_i} e_i^*$. Define the map[13]

$$p^* : N \to M^\circ$$
$$n \mapsto (N^\circ \ni n' \mapsto \{n, n'\}).$$

We will then define $v_i = p^*(e_i) \in M^\circ$. One can then associate the seed tori

$$\mathcal{A}_\mathbf{s} = T_{N^\circ} = \operatorname{Spec}\Bbbk[M^\circ], \qquad \mathcal{X}_\mathbf{s} = T_M = \operatorname{Spec}\Bbbk[N].$$

We will denote the coordinates as $X_i = z^{e_i}$ and $A_i = z^{f_i}$ and they are called the cluster variables. Similar to the definition of cluster algebras, there is a procedure, called *mutation*, to produce a new seed data $\mu(\mathbf{s})$ from a given seed $\mathbf{s}$ which is stated in [26, Equation 2.3]. The essence is that we will obtain new seed tori $\mathcal{A}_{\mu(\mathbf{s})}$, $\mathcal{X}_{\mu(\mathbf{s})}$ from the mutated seed. We will consider the oriented rooted tree $\mathfrak{T}$ with n outgoing edges from each vertex, labeled by the elements of I. Let v be the root of the tree and then attach the seed $\mathbf{s}$ to the vertex v. By associating each path in the tree with a sequence of mutation then we can then attach a seed to each vertex of $\mathfrak{T}$.

Between the tori, there are birational maps, called $\mathcal{A}$-mutation and $\mathcal{X}$-mutation, $\mu_\mathcal{X} : \mathcal{X}_\mathbf{s} \dashrightarrow \mathcal{X}_{\mu(\mathbf{s})}$, $\mu_\mathcal{A} : \mathcal{A}_\mathbf{s} \dashrightarrow \mathcal{A}_{\mu(\mathbf{s})}$ which are defined by pull-back of functions

$$(6.1) \qquad\qquad \mu_{\mathcal{X},k}^* z^n = z^n (1 + z^{e_k})^{-[n, e_k]},$$

$$(6.2) \qquad\qquad \mu_{\mathcal{A},k}^* z^m = z^m (1 + z^{v_k})^{-\langle d_k e_k, m \rangle},$$

for $n \in N$, $m \in M^\circ$. Note that those birational maps are basically the mutations of cluster variables as in Fomin and Zelevinsky [12].

Let $\mathcal{A}$ be an union of tori glued by $\mathcal{A}$-mutations. While one can define the cluster varieties as gluing of tori, however cluster varieties are defined up to certain ambiguity in codimension 2 as follows:

Definition 6.1. [9] *A smooth scheme V is a cluster variety of type $\mathcal{A}$ if there is a birational map $\mu : V \dashrightarrow \mathcal{A}$ which is an isomorphism outside codimension two subsets of the domain and range. We define a cluster variety of type $\mathcal{X}$ analogously.*

By definition, the ring of regular functions of $\mathcal{A}$ cluster varieties are upper $\mathcal{A}$ cluster algebras $\operatorname{up}(\mathcal{A})$. Similarly the ring of regular functions of the $\mathcal{X}$ cluster varieties are the (upper) $\mathcal{X}$ cluster algebras $\operatorname{up}(\mathcal{X})$. Note that by [26, Theorem 3.14], the canonical maps

$$\mathcal{A} \to \operatorname{Spec}(\operatorname{up}(\mathcal{A}))$$

are open immersions.

For the later discussion, we will note the following structure of the cluster $\mathcal{A}$ and $\mathcal{X}$ cluster varieties. Let $K = \ker p^*$[14]. Then the inclusion $K \subseteq N$ gives a map $\mathcal{X}_\mathbf{s} \to T_{K^*}$. The map is compatible with mutation maps so that there is a canonical map

$$(6.3) \qquad\qquad\qquad \lambda : \mathcal{X} \to T_{K^*}.$$

Let ℓ be the rank of the p^* map. Then λ is a flat family of ℓ-dimensional schemes. The $\ell = 2$ case is indicated in [26, Section 5].

---

[13]Note that we do not consider the cases with frozen variables in this article. So there would only be one p^* map for each cluster algebra.

[14]Again as there is no frozen variables, $\ker p_2^* = \ker p^*$ if one follows the notation in [26].

Similarly we consider the inclusion $K^\circ := K \cap N^\circ \hookrightarrow N^\circ$ induces a map of tori $T_{K^\circ} \to \mathcal{A}_{\mathbf{s}}$. This gives a torus action T_{K° on $\mathcal{A}_{\mathbf{s}}$. The action is compatible with the mutations so that there is a canonical action of T_{K° on $\mathcal{A}$.

6.1.1. *Langland dual.* Consider a fixed data Γ. The Langlands dual cluster varieties are defined with the Langlands dual fixed data $^L\Gamma$ as stated [22, appendix A]. The fixed data $^L\Gamma$ is defined by having $^Ld_i = d_i^{-1}D$, where $D = \mathrm{lcm}(d_1, \ldots d_n)$. The Langland dual lattice is $^LN = N^\circ$, sublattice $(^LN)^\circ = D{\cdot}N$, and the skew-symmetric form on ^LN as $\{\cdot, \cdot\} = D^{-1}\{\cdot, \cdot\}$. A seed data $^L\mathbf{s} = (^Le_i = d_i e_i)$ gives a basis for ^LN and so the dual basis for $^LM = M^\circ$ is $\{^Le_i^* = \frac{1}{d_i}e_i^*\}$. The basis for $^LM^\circ = \frac{1}{D}M$ can then be given by $^Lf_i = \frac{1}{^Ld_i}{}^Le_i^* = \frac{d_i}{D}{\cdot}\frac{1}{d_i}e_i^* = \frac{1}{D}e_i^*$. One can similarly define the $^Lp^*$ map. Note that now $^Lv_i = {}^Lp^*(^Le_i) = \frac{1}{D}p^*(d_i e_i) = \frac{d_i}{D}v_i$.

Repeat the same procedure and associate the tori

$$^L\mathcal{A}_{\mathbf{s}} = T_{(^LN^\circ)} = \mathrm{Spec}\Bbbk[^LM^\circ], \qquad {}^L\mathcal{X}_{\mathbf{s}} = T_{(^LM)} = \mathrm{Spec}\Bbbk[^LN].$$

Again there are birational maps between the tori, $\mu_{^L\mathcal{X}} : {}^L\mathcal{X}_{\mathbf{s}} \dashrightarrow {}^L\mathcal{X}_{\mu(\mathbf{s})}$, $\mu_{^L\mathcal{A}} : {}^L\mathcal{A}_{\mathbf{s}} \dashrightarrow {}^L\mathcal{A}_{\mu(\mathbf{s})}$ which are defined by pull-back of functions

$$\mu_{^L\mathcal{X},k}^* z^{^Ln} = z^{^Ln}(1 + z^{^Le_k})^{-^L[^Ln, {}^Le_k]},$$

$$\mu_{^L\mathcal{A},k}^* z^{^Lm} = z^{^Lm}(1 + z^{^Lv_k})^{-\langle {}^Ld_k{}^Le_k, {}^Lm\rangle},$$

for $^Ln \in {}^LN$, $^Lm \in {}^LM^\circ$. Let us unfold the mutation maps and express it in terms of the fixed data Γ:

$$(6.4) \qquad \mu_{^L\mathcal{X},k}^* z^{^Ln} = z^{^Ln}(1 + z^{d_k e_k})^{-^L[^Ln, d_k e_k]},$$

$$(6.5) \qquad \mu_{^L\mathcal{A},k}^* z^{^Lm} = z^{^Lm}(1 + z^{^Lv_k})^{-\langle De_k, {}^Lm\rangle}.$$

One can then define the Langland daul cluster varieties $^L\mathcal{A}$ and $^L\mathcal{X}$ by repeating the same gluing construction.

The mirror conjecture is that $\mathcal{A}$ and $^L\mathcal{X}$ are mirror to each other. We will give a rough digression here. Recall the $\mathcal{A}$ and $^L\mathcal{X}$ mutation maps as in (6.1) and (6.4):

$$\mu_{\mathcal{A},k}^* z^m = z^m(1 + z^{v_k})^{-\langle d_k e_k, m\rangle},$$

$$\mu_{(^L\mathcal{X}),k}^* z^{^Ln} = z^{^Ln}(1 + z^{d_k e_k})^{-^L[^Ln, d_k e_k]}.$$

While the notation may be confusing, we recall that $^LX_i = z^{d_i e_i}$. Let us illustrate the duality with a more explicit example. Consider a fixed data with a dimension 2 lattice N, skew-form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $d_1 = k$, $d_2 = l$, where k and l are coprime. Then the mutation function of $\mu_{\mathcal{A}}^*$ are

$$1 + A_1^{-k} \quad \text{and} \quad 1 + A_2^l.$$

On the other hand, the mutation functions for the $\mathcal{X}$ Langland dual $\mu_{^L\mathcal{X}}$ are

$$(1 + {}^LX_1)^l \quad \text{and} \quad (1 + {}^LX_2)^k.$$

Note there are interchanging of exponent vectors. We are going to see the geometric reason later.

6.1.2. *Cluster scattering diagrams.* Gross-Hacking-Keel-Kontsevich [22] constructed bases for cluster algebras by using ideas in the Gross-Hacking-Keel mirror construction. The gluing construction of cluster varieties can be seen as gluing tori associated to each chamber of the scattering diagrams and then the collections of theta functions give the bases for cluster algebras.

We start by first defining the $\mathcal{A}_{\mathrm{prin}}$ cluster scattering diagrams. The fixed data for cluster algebras with principal coefficients can be obtained by 'doubling' the original fixed data, i.e. considering the lattice $\widetilde{N} = N \oplus M^\circ$ as described in [26, Construction 2.11] and notation as in [9]. The initial scattering diagram is of the form

$$\mathfrak{D}_{in,\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}} := \left((e_i, 0)^\perp, 1 + z^{\tilde{p}^*((e_i,0))} \right).$$

The $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$ scattering diagrams are the consistent scattering diagrams containing $\mathfrak{D}_{in,\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ [15]. Theta functions are then defined on the cluster scattering diagrams which are similar to the one in (5.1). One can refer to [22, Section 3] for the precise definition.

Let $\mathrm{can}(V)$ be the vector spaces generated by the theta functions for $V = \mathcal{A}, \mathcal{X}$, and $\mathrm{mid}(V)$ be the algebra generated by those theta functions which are polynomials (see [22, Definition 7.2] for more precise definition). Note that $\mathrm{can}(V)$ are not algebras in general. Then it is called the full Fock–Goncharov conjecture holds when $\mathrm{mid}(V) = \mathrm{up}(V) = \mathrm{can}(V)$. The existence of positive polytope [22, Proposition 8.17] give algebra structures to $\mathrm{can}(V)$. With the enough global monomial condition satisfying, $\mathrm{can}(V)$ are finitely generated $\Bbbk$-algebras. and the Fock-Goncharov conjecture holds.

Note that even though the walls of $\mathfrak{D}^{\mathcal{A}_{\mathrm{prin}}}$ rely on a choice of the seed $\mathbf{s}$; however, Gross-Hacking-Keel-Konsevich [22] showed that the construction is mutation invariance, i.e., the algebras generated by the collections of theta functions are the same.

The '$\mathcal{A}$ and $\mathcal{X}$ scattering diagrams' are the diagrams such that we can compute the corresponding theta functions. The $\mathcal{A}$ scattering diagrams can be obtained by considering the projection map $\rho^T : {}^L\mathcal{X}_{\mathrm{prin}}(\mathbb{Z}^T) = M^\circ \times N \to {}^L\mathcal{X}(\mathbb{Z}^T) = M^\circ, (m,n) \mapsto m$. On the other hand, the $\mathcal{X}$ theta functions can be obtained by considering broken lines on the slices

$$\left\{ (m, n) \in \widetilde{M}_\mathbb{R}^\circ \,\middle|\, m = p^*(n) \right\} \subseteq \widetilde{M}_\mathbb{R}^\circ$$

in the $\mathfrak{D}_{\mathbf{s}}^{\mathcal{A}_{\mathrm{prin}}}$ scattering diagram which we will call the diagrams on the slices as the $\mathcal{X}$ scattering diagrams $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}}$.

On the other hand, the scattering diagrams for the Langland dual cluster algebras are defined in the same way. We will state the initial walls of the ${}^L\mathcal{X}$ scattering diagrams for the later discussion. The ${}^L\mathcal{X}$ scattering diagrams can be obtained by the slicing in the ${}^L\mathcal{A}_{\mathrm{prin}}$ scattering diagram. One can also see the initial wall functions by using (6.4):

$$(6.6) \qquad \left(\mathfrak{d} = \{ (p^*({}^Ln), {}^Ln) \mid {}^Lp^*({}^Ln) \in d_i e_i^\perp \}, (1 + z^{d_i e_i})^{\mathrm{ind}v_i} \right),$$

where $\mathrm{ind}v_i$ denotes the index for v_i in ${}^LM = M^\circ$. With a change of variables, we can see the scattering diagrams lie in ${}^LN_\mathbb{R} = N_\mathbb{R}^\circ$.

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[15] There is an unfortunate sign switch between the setups in GPS [21] and cluster [22] scattering diagrams. It would be resulted in the flipping scattered walls but not the inital walls of the diagram. Also the resulting algebras given by the collection of theta functions would be the same.

6.1.3. *Toric models for cluster varieties.* In this section, we are going to describe the relation between the cluster varieties and the blowups of toric varieties [26]. Let $\mathbf{s}$ be a seed and consider the fans

$$(6.7) \qquad \Sigma_{\mathbf{s},\mathcal{A}} := \{0\} \cup \{\mathbb{R}_{\geq 0} d_i e_i | i \in I\} \subseteq N^\circ,$$

$$(6.8) \qquad \Sigma_{\mathbf{s},\mathcal{X}} := \{0\} \cup \{-\mathbb{R}_{\geq 0} d_i v_i | i \in I\} \subseteq M.$$

We can then consider the toric varieties $\mathrm{TV}_{\mathbf{s},\mathcal{A}}$, $\mathrm{TV}_{\mathbf{s},\mathcal{X}}$ associated to the fans respectively. Let D_i, $i \in I$, be the toric divisor corresponding to the one-dimensional ray of the fan. Then we consider the closed subschemes

$$Z_{\mathcal{A},i} := D_i \cap \overline{V}(1 + z^{v_i}) \subseteq \mathrm{TV}_{\mathbf{s},\mathcal{A}},$$

$$Z_{\mathcal{X},i} := D_i \cap \overline{V}\left((1 + z^{e_i})^{\mathrm{ind} d_i v_i}\right) \subseteq \mathrm{TV}_{\mathbf{s},\mathcal{X}},$$

where $\mathrm{ind} d_i v_i$ denotes the greatest degree of divisibility of $d_i v_i$ in M. We will consider $(\widetilde{\mathrm{TV}}_{\mathbf{s},\mathcal{A}}, D)$ and $(\widetilde{\mathrm{TV}}_{\mathbf{s},\mathcal{X}}, D)$, the blowups of $\mathrm{TV}_{\mathbf{s},\mathcal{A}}$ and $\mathrm{TV}_{\mathbf{s},\mathcal{X}}$ along $Z_{\mathcal{A},i}$ and $Z_{\mathcal{X},i}$ respectively, where D is the proper transform of the toric boundaries. As the construction for the $\mathcal{A}$ and $\mathcal{X}$ are parallel to each other, we will denote $\mathcal{A}$ or $\mathcal{X}$ as V. Define $U_{\mathbf{s},V} = \widetilde{\mathrm{TV}}_{\mathbf{s},V} \setminus D$. We will further consider $U'_{\mathbf{s},V} \subset V$, the union of the tori $V_{\mathbf{s}}$ and $V_{\mu_i(\mathbf{s})}$, for all $i \in I$.

Note that the construction of $U_{\mathbf{s},V}$ may appear to depend on a choice of seed data. Gross-Hacking-Keel [26, Lemma 3.8 (1)] showed that for a seed $\mathbf{s}_w$ and a mutated seed $\mu_i(\mathbf{s})$, then one has $U_{\mathbf{s}_w,V}$ and $U_{\mathbf{s}_{\mu(w)},V}$ are isomorphic outside codimension two for $V = \mathcal{A}_{\mathrm{prin}}$, $\mathcal{X}$. For the $\mathcal{A}$ case, they are still isomorphic outside codimension two if the seed $\mathbf{s}_w$ is coprime ([26, Definition 3.7]).

6.2. **Mirror symmetry for rank 2 cluster varieties.** Gross-Hacking-Keel [26] indicated the Looijenga pairs constructed in [21] agree with the cluster $\mathcal{X}$ varieties in the rank 2 case. In this section, we wll investigate the T_{K° action on $\mathcal{A}$ in the rank 2 case.

Fix a seed $\mathbf{s} = (e_i)$. We will assume without lost of generality that $p^*(e_i) \neq 0$, together with, for $i \neq j$, $p^*(e_i)$ and $p^*(e_j)$ are not scalar multiple of each other. Note that this is not an entirely an extra condition.

Since we will consider the lattice quotient by the kernel of p^* map. If $p^*(e_i) = p^*(e_j)$, we can consider a lower rank lattice and then increase the degree of the blow-up's. The case when $p^*(e_i) = \alpha p^*(e_j)$ for some $\alpha \neq 1$ are similar but require more subtle combinatorics.

Consider a choice of splitting of $N \cong N/K \oplus K$ and this induces a dual splitting $M = K^\perp \oplus K^*$. This gives us $M^\circ = (K^\circ)^\perp \oplus (K^\circ)^*$. Then we define the rank two lattices: $\bar{N} = N/K$, $\bar{N}^\circ = N^\circ/K^\circ$, $\bar{M} = M/K^* = K^\perp$ and $\bar{M}^\circ = M^\circ/(K^\circ)^*$. We further consider the descended p^* map

$$\bar{p}^* : \bar{N} \to \bar{M}^\circ.$$

Let $\bar{e}_k \in \bar{N}$, $\bar{v}_k \in \bar{M}^\circ$ be the image of e_k, v_k respectively in the quotient spaces. By construction, $\bar{v}_k = \bar{p}^*(\bar{e}_k)$ for all k. Therefore, we still have $\langle \bar{e}_k, \bar{v}_k \rangle = 0$.

Recall the inclusion $K^\circ \hookrightarrow N^\circ$ give us the T_{K° action on $\mathcal{A}_{\mathbf{s}}$. Note that the torus action commutes with the mutation of seed data, i.e. for $k \in I$, the following diagram commutes

$$\begin{array}{ccc} T_{K^\circ} & \longrightarrow & \mathcal{A}_{\mathbf{s}} \\ \Big\downarrow{=} & & \Big\downarrow{\mu_k} \\ T_{K^\circ} & \longrightarrow & \mathcal{A}_{\mu_k(\mathbf{s})} \end{array}$$

Consider the tori $\mathcal{A}_{k,\mathbf{s}} = \mathrm{Spec}\Bbbk[\bar{M}^\circ] \cong \mathcal{A}_{\mathbf{s}}/T_{K^\circ}$. Since the T_{K° action on $\mathcal{A}_{\mathbf{s}}$ commutes with mutation, we can define the birational maps

$$(6.9) \qquad \bar{\mu}_k : \mathcal{A}_{k,\mathbf{s}} \dashrightarrow \mathcal{A}_{k,\mu_k(\mathbf{s})}$$

via

$$\bar{\mu}_k^*(z^{\bar{m}}) = z^{\bar{m}} \left(1 + z^{\bar{v}_k}\right)^{-\langle d_k \bar{e}_k, \bar{m} \rangle},$$

for $\bar{m} \in \bar{M}^\circ$. Hence, by [26, Proposition 2.4] we can define the scheme $\mathcal{A}_k$ as the scheme by gluing all $\mathcal{A}_{k,\mu_k(\mathbf{s})}$ via the mutation map $\bar{\mu}_k$.

Next we would describe the scheme $\mathcal{A}_k$ as blowup of toric surfaces by following [26, Construction 3.4]. Define $\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}$ as the quotient of the fan $\Sigma_{\mathbf{s},\mathcal{A}}$ in (6.7) as

$$\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k} = \{0\} \cup \{\mathbb{R}_{\geq 0} d_i \bar{e}_i\} \subseteq \bar{N}^\circ.$$

Note that $\mathbb{R}_{\geq 0} d_i \bar{e}_i \neq \mathbb{R}_{\geq 0} d_i \bar{e}_j$ for $i \neq j$ from our assumption. Let $\mathrm{TV}_{\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}}$ be the toric variety associated to the fan $\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}$. We denote $\bar{D}_i$ to be the toric boundary divisor of $\mathrm{TV}_{\overline{\Sigma}}$ corresponding to the ray generated by $d_i \bar{e}_i \in \overline{\Sigma}$. Let $\bar{v}_i$ be the image of v_i. Note that $\bar{v}_i \neq 0$ as it is in the image of the p^* map.

For $i = 1, \ldots, n$, consider the closed subscheme

$$\bar{Z}_i = \bar{D}_i \cap \bar{V}\left((1 + z^{\bar{v}_i})^{\mathrm{ind}\, d_i \bar{e}_i}\right),$$

where $\mathrm{ind}\, d_i \bar{e}_i$ denotes the greatest degree of divisibility of $d_i \bar{e}_i$ in $\bar{N}^\circ$. Note that even though $d_i e_i$ are primitive in N°, however, they may not be primitive in the quotient lattice $\bar{N}^\circ$. We will then consider the blow up $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}}$ at $\bar{Z}_i$'s. Let $\tilde{D}_i$ be the proper transform of the toric boundaries $\bar{D}_i$ and define $U_{\mathbf{s},\mathcal{A}_k} := \widetilde{\mathrm{TV}}_{\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}} \setminus \cup_i \tilde{D}_i$. We further define $U'_{\mathbf{s},\mathcal{A}_k}$ as the union of the tori $\mathcal{A}_{k,\mathbf{s}}$ and $\mathcal{A}_{k,\mu_k(\mathbf{s})}$ for $i \in I$. Note that the condition $p^*(e_i)$ and $p^*(e_j)$ not scalar multiple of each other for $i \neq j$, so we can apply both [26, Lemma 3.5(i)] and [26, Lemma 3.6]. Hence by [26, Lemma 3.5(i)], $U_{\mathbf{s},\mathcal{A}_k}$ and $U'_{\mathbf{s},\mathcal{A}_k}$ are isomorphic outside a codimension two set. We also have $U_{\mathbf{s},\mathcal{A}_k}$ and $U_{\mu(\mathbf{s}),\mathcal{A}_k}$ isomorphic outside a codimension two set by [26, Lemma 3.6].

6.2.1. *The GPS scattering diagrams associated to rank 2 cluster $\mathcal{A}$ varieties.* We can now run the machinery discussed in Section 2.3. First we would like to associate $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}}$ with a scattering diagram which the walls carry enumerative interpretation. Given the fan $\overline{\Sigma} = \overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}$. Let $\overline{\Sigma}'$ be a complete fan containing $\overline{\Sigma}$ and $\mathrm{TV}_{\overline{\Sigma}'}$ be the corresponding toric surface. Then we have $\mathrm{TV}_{\overline{\Sigma}} \hookrightarrow \mathrm{TV}_{\overline{\Sigma}'}$. We denote the Gross-Pandharipande-Siebert scattering diagram for the blow up $\widetilde{\mathrm{TV}}_{\overline{\Sigma}'}$ of $\mathrm{TV}_{\overline{\Sigma}'}$ at the proper transform of Z_i by $\mathfrak{D}^{GPS}_{\overline{\Sigma},\mathbf{s}}$. Note that the additional divisors associated to $\overline{\Sigma}' \setminus \overline{\Sigma}$ do not intersect the Z_i's. Hence the $\mathbb{A}^1$-curve countings in the wall-functions of $\mathfrak{D}^{GPS}_{\overline{\Sigma},\mathbf{s}}$ are independent of the choice of $\overline{\Sigma}' \supseteq \overline{\Sigma}$. We would then switch back to denote $\overline{\Sigma}$ as the complete fan containing $\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}$.

We are going to define the initial scattering diagram associated to $\mathrm{TV}_{\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k}}$. First note that all the inital walls will pass through the origin. Denote $d_i \bar{e}_i = (\mathrm{ind}\, d_i \bar{e}_i) q_i$, where q_i is primitive in $\bar{N}^\circ$. The reason is the blow-up points are at the zero locus of the equations $1 + z^{d_i \bar{e}_i}$. Hence we have $z^{d_i \bar{e}_i} = -1$ which then means $|z^{q_i}| = 1$. If one uses z_1, z_2 as the toric coordinates on $(\mathbb{C}^*)^2 \subseteq TV_{\overline{\Sigma}_{s,\mathcal{A}_k}}$, then there exist $a, b \in \mathbb{Z}$ such that $z^{q_i} = z_1^a z_2^b$. Therefore, under the Log-map, the wall induced by the non-toric blow up at any point on $\{|z^{q_i} = 1|\} \cap \bar{D}_i$ is a line passing through the origin.

The wall functions from one point blowup is divided into two cases: $d_i e_i$ is primitive or not in $\bar{N}^\circ$. If $d_i e_i$ is primitive, we can apply the discussion in Section 2.3 to define the corresponding wall

$$\left(\mathbb{R} \cdot d_i \bar{e}_i, 1 + z^{d_i \bar{e}_i} \right).$$

In Section 2.3, the associated wall functions are of the form $1 + t^{-[E_{ij}]} z^{m_i}$. Here we put $t^{-[E_{ij}]} = 1$ in the cluster setting.

If $d_i \bar{e}_i$ is not primitive, i.e. $\operatorname{ind} d_i \bar{e}_i > 1$, the corresponding blow-up would be a $(\operatorname{ind} d_i \bar{e}_i)$-orbifold blowup of the toric variety along the points $\bar{D}_i \cap V(1 + z^{d_i \bar{e}_i})$. We can apply [25, Theorem 5.6] to associate the wall

$$\left(\mathbb{R} \cdot q_i, 1 + z^{(\operatorname{ind} d_i \bar{e}_i) q_i} \right) = \left(\mathbb{R} \cdot d_i \bar{e}_i, 1 + z^{d_i \bar{e}_i} \right)$$

is again in the same form.

Roughly speaking, the construction in Section 2.3 is done by associating each point of blowing-up with an initial wall. Let us work over $\mathbb{C}$ or any algebraically closed characteristic 0 fields. For each i, the subscheme $\bar{D}_i \cap \bar{V}(1 + z^{\bar{v}_i})$ consists of $\operatorname{ind}\bar{v}_i$ many distinct points. From the discussion above, to each point in the subscheme $\bar{D}_i \cap \bar{V}(1 + z^{\bar{v}_i})$, we associate the wall

$$\left(\mathbb{R} \cdot d_i \bar{e}_i, 1 + z^{d_i \bar{e}_i} \right).$$

Hence the initial walls associated to the blowup loci Z_i are

$$\left(\mathbb{R} \cdot d_i \bar{e}_i, (1 + z^{d_i \bar{e}_i})^{\operatorname{ind}\bar{v}_i} \right).$$

Therefore, the initial scattering diagram associated to $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{\mathbf{s}}, \mathcal{A}}$ would be

$$(6.10) \qquad \mathfrak{D}_{\operatorname{in}, \overline{\Sigma}, \mathbf{s}}^{GPS} = \left\{ \left(\mathbb{R} \cdot d_i \bar{e}_i, (1 + z^{d_i \bar{e}_i})^{\operatorname{ind}\bar{v}_i} \right) \mid i = 1 \ldots n \right\}.$$

Let $\mathfrak{D}_{\overline{\Sigma}, \mathbf{s}}^{GPS}$ be the consistent scattering diagram containing $\mathfrak{D}_{\operatorname{in}, \mathbf{s}}$. As described in (2.4), the wall functions of $\mathfrak{D}_{\overline{\Sigma}, \mathbf{s}}^{GPS}$ would be expressed in terms of log Gromov-Witten invariants.

Matching the cluster scattering diagrams. Now we would like to obtain the scattering diagrams (6.10) from the cluster viewpoint.

Recall when the p^* map is not injective, there is a morphism $\lambda : \mathcal{X} \to T_{K^*}$ as stated in 6.3. In the discussion above, we see that there is a flat family of surface $\mathcal{Y} \to T_K^*$ obtained by blowing up toric varieties such that $\mathcal{Y} \setminus \mathcal{D}$ are isomorphic to $U'_{\mathcal{X}}$ up to codimension two, where $\mathcal{D}$ is the proper transform of the toric boundary under the blowup. Gross-Hacking-Keel [26] indicated that the family $\mathcal{Y} \to T_K^*$ is the universal family of Looijenga pair constructed in [21]. Let $\phi \in T_K^*$, we denote $\mathcal{X}_\phi$ for the fibre over $\phi \in T_{K^*}$. The $\mathcal{X}_\phi$ scattering diagram $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}_\phi}$, is defined in [9, Section 2.2.3], can be described as the quotient of the $\mathfrak{D}_{\mathbf{s}}^{\mathcal{X}}$ diagram in $\bar{N} \otimes \mathbb{R}$. This construction is well-defined as laid out in [9, Section 2.2.3]. Besides, Zhou [56] also gave a description of quotient scattering diagrams.

We will explicitly write down $\mathfrak{D}_{\mathbf{s}}^{{}^L\mathcal{X}_e}$, where e is the identity of $T_{{}^L K^*}$. Recall ${}^L N = N^\circ$. With a choice of the splitting $N^\circ = \bar{N}^\circ \oplus K^\circ$ as in the earlier section, there is the inclusion

$$K^\circ \hookrightarrow \bar{N}^\circ \oplus K^\circ$$

$$k \mapsto (0, k).$$

Hence similar to the $\mathcal{A}$ scattering diagrams defined from the $\mathcal{A}_{\operatorname{prin}}$ scattering diagrams, we can consider a projection of the $\mathfrak{D}^{{}^L\mathcal{X}}$ scattering diagram descended from $N_{\mathbb{R}}^\circ \to \overline{N}_{\mathbb{R}}^\circ$. Note that

in the two-dimensional lattice N°, the normal space to $(d_i\bar{e}_i)^\perp$ is the simply one-dimensional ray $\mathbb{R}d_i\bar{e}_i$. Combining with (6.6), the initial $^L\mathcal{X}_e$ scattering diagrams are of the form

$$\left(\mathbb{R} \cdot d_i\bar{e}_i, \, (1 + z^{d_i\bar{e}_i})^{\operatorname{ind}\bar{v}_i}\right).$$

The $^L\mathcal{X}_e$ scattering diagrams are the consistent scattering diagrams containing the initial diagram. Similar to the discussion in cluster scattering diagram, if there exist positive polytopes and the enough global monomials conditions are satisfied, the vector space $\operatorname{can}(^L\mathcal{X}_e)$ are finitely generated algebra. Then we have $\operatorname{mid}(^L\mathcal{X}_e) = \operatorname{can}(^L\mathcal{X}_e) = H^0(^L\mathcal{X}_e, \mathcal{O})$.

Note that the initial $^L\mathcal{X}_e$ scattering diagram is exactly the same as the GPS scattering diagram in (6.10). In other words, we have

(6.11) $$\mathfrak{D}^{GPS}_{\overline{\Sigma},\mathbf{s}} \cong \mathfrak{D}^{^L\mathcal{X}_e}_{^L\mathbf{s}},$$

for every seed $\mathbf{s}$. Since $\mathfrak{D}^{^L\mathcal{X}_e}_{^L\mathbf{s}}$ and $\mathfrak{D}^{^L\mathcal{X}_e}_{\mu(^L\mathbf{s})}$ are differed by mutation by [22, Theorem 1.24], we have $\mathfrak{D}^{GPS}_{\overline{\Sigma},\mathbf{s}}$ and $\mathfrak{D}^{GPS}_{\overline{\Sigma},\mu(\mathbf{s})}$ are also differed by a mutation.

Let us recall the ideas of the Gross-Hacking-Keel [21] mirror construction. Once the canonical scattering diagram in Section 2.4 is constructed, the mirrors are constructed as the spectrums of the algebras generated by the collections of theta functions. Further the algebras given by the canonical scattering diagrams and the GPS scattering diagrams are identified in [21]. Now we see that the GPS scattering diagram and the $^L\mathcal{X}_e$ scattering diagrams are the same. Hence the mirror spaces are $^L\mathcal{X}_e$.

Now we are ready to combine Floer theory into this story. Theorem 5.4 tells us that the limiting Langrangian Floer scattering diagrams agree with the GPS scattering diagrams. Hence we showed that quotients of the $\mathcal{A}$ cluster varieties are Floer mirror to the fibers at $e \in T_{K^*}$ of the $\mathcal{X}$ cluster varieties.

Theorem 6.2. *Consider a cluster fixed data Γ without frozen direction where the skew-symmetric form is of rank 2. We further fix a seed data $\mathbf{s}$. Assume either the $p^*(e_i)$ are not scalar multiple of each other. Consider the quotient map $N^\circ \to N^\circ/K^\circ$ and let $\bar{e}_i$ be the image of e_i under this quotient map. Define*

$$\overline{\Sigma}_{\mathbf{s},\mathcal{A}_k} = \{0\} \cup \{\mathbb{R}_{\geq}d_i\bar{e}_i\} \subseteq \bar{N}^\circ.$$

For $i = 1,\ldots,n$, let $v_i = p^*(e_i)$ and let $\bar{v}_i$ be the imagine of v_i under the quotient map $M^\circ \to M^\circ/(K^\circ)^*$. Define

$$\bar{Z}_i = \bar{D}_i \cap \bar{V}\left((1 + z^{\bar{v}_i})^{\operatorname{ind}d_i\bar{e}_i}\right),$$

where $\operatorname{ind}d_i\bar{e}_i$ denotes the greatest degree of divisibility of $d_i\bar{e}_i$ in $\bar{N}^\circ$.

Let $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{s,\mathcal{A}_k}}$ be the surface obtained by blowing up the subschemes $\bar{Z}_i \subseteq \mathrm{TV}(\overline{\Sigma}_{s,\mathcal{A}_k})$. Let $\tilde{D}_i$ be the proper transform of the toric boundaries $\bar{D}_i$ and define $U_{s,\mathcal{A}_k} := \widetilde{\mathrm{TV}}_{\overline{\Sigma}_{s,\mathcal{A}}} \setminus \cup_i \tilde{D}_i$. Then $U_{s,\mathcal{A}_k}$ is isomorphic to $\mathcal{A}/T_{K^\circ}$ outside a set of codimension at least two, where $\mathcal{A}/T_{K^\circ}$ is the quotient of the $\mathcal{A}$ variety under the T_{K°-action.

Then the corresponding GPS scattering diagram of $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{s,\mathcal{A}_k}}$ is the $^L\mathcal{X}_{e,s}$ scattering diagram, where $^L\mathcal{X}_e$ is the fiber of the Langland dual $\mathcal{X}$ family $^L\mathcal{X} \to T_{(^LK)^*}$ at $e \in T_{(^LK)^*}$.

Remark 6.3. *If $\operatorname{can}(^L\mathcal{X}_e)$ are finitely generated algebras, we obtain the mirror algebras of $\mathcal{A}/T_{K^\circ}$ are rings of regular functions of $^L\mathcal{X}_e$.*

Remark 6.4. *We use the algebraic geometric enumerative invariants (from Gross-Siebert-Pandharipande) of $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{s,\mathcal{A}_k}}$ to recover the $\mathcal{X}$-scattering diagram of $^L\mathcal{X}_{e,s}$ in Theorem 6.2. If*

we further assume $d_i \bar{e}_i$ are primitive in N°/K°, i.e. excluding the case about orbifold blowup, we can recover $^L \mathcal{X}_{e,s}$ from the open Gromov-Witten invariants of $\widetilde{\mathrm{TV}}_{\overline{\Sigma}_{s,\mathcal{A}_k}}$ by Theorem 5.8.

6.3. **Example.** Let us consider an example with rank $N = 3$ and the skew symmetric form is of the form

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

and $d_i = 1$ for all i. Note that since $d_i = 1$ for all i, the fixed data would be the same for the Langland dual side. Consider the initial seed $\mathbf{s} = \{e_i\}$, where e_i are the standard basis of $\mathbb{Z}^3$. Then we have

$$\Sigma_{\mathbf{s},\mathcal{A}} = \{0\} \cup \mathbb{R}_{\geq 0}(1,0,0) \cup \mathbb{R}_{\geq 0}(0,1,0) \cup \mathbb{R}_{\geq 0}(0,0,1),$$
$$\Sigma_{\mathbf{s},\mathcal{X}} = \{0\} \cup \mathbb{R}_{\geq 0}(0,-1,1) \cup \mathbb{R}_{\geq 0}(1,0,-1) \cup \mathbb{R}_{\geq 0}(-1,1,0).$$

Let $\mathrm{TV}_{\mathbf{s},\mathcal{A}}$ be the toric variety associated to the fan $\Sigma_{\mathbf{s},\mathcal{A}}$. One can see immediately that $\mathrm{TV}_{\mathbf{s},\mathcal{A}} = \mathbb{A}^3 \setminus \{\text{codimension 2 toric strata}\}$. In this example, we have $K = \ker p^* = \langle e_1 + e_2 + e_3 \rangle \subseteq N$. We then consider $K^\circ = K \cap N^\circ \subset N^\circ$. In this example $K^\circ = K$. The T_{K° action is of the form

$$\alpha \cdot (A_1, A_2, A_3) = (\alpha A_1, \alpha A_2, \alpha A_3),$$

where $\alpha \in \mathbb{C}^*$. One can see that this is the standard torus action to obtain $\mathbb{P}^2$ from $\mathbb{A}^3$. In this case, $\mathrm{TV}_{\mathbf{s},\mathcal{A}}/T_{K^\circ}$ will give us $\mathbb{P}^2 \setminus \{3 \text{ points}\}$. Alternately, consider the quotient map $N^\circ \to N^\circ/K^\circ$. Then the imagine of the fan $\Sigma_{\mathbf{s},\mathcal{A}}$ is of the form

$$\overline{\Sigma}_{\mathbf{s},\mathcal{A}} = \{0\} \cup \mathbb{R}_{\geq 0}\bar{e}_1 \cup \mathbb{R}_{\geq 0}\bar{e}_2 \cup \mathbb{R}_{\geq 0} \cdot (-\bar{e}_1 - \bar{e}_2).$$

Hence the toric variety is the projective space $\mathbb{P}^2$ up to codimension 2. The space we considered would be $\mathbb{P}^2$ blow up at the 3 points. The corresponding initial scattering diagram would be

It is easy to see this is the initial walls of the $^L \mathcal{X}_e$ scattering diagram of the dual seed of $\mathbf{s}$.

7. Mirror Symmetry

In this section, we focus on the concrete example of the del Pezzo surface of degree 5, and show that the Landau-Ginzburg model constructed by disc counting fulfills the closed-string mirror symmetry. For more general examples beyond this, the calculation of critical points becomes too complicated to carry out by hands. We speculate that there is a more effective argument that avoids a direct calculation. The main theorem in this section is the following.

Theorem 7.1. *Let $\tilde{Y}$ be a del Pezzo surface of degree 5 represented by the non-toric blowup of a toric surface $\bar{Y}$ as in Figure 8 (note that $\tilde{Y}$ with the symplectic form from Lemma 4.1 is not monotone), and let $(\check{Y}, W)$ be its LG mirror constructed from the SYZ fibration on $\tilde{Y} \setminus \tilde{D}$*

(see 4). Then the quantum cohomology ring of $\tilde{Y}$ is isomorphic to the Jacobian ideal ring of W.

We will give a concrete description of the LG mirror $(\check{Y}, W)$ in 7.1. The proof of Theorem 7.1 is essentially to show that W only has on $\check{Y}$ as many nondegenerate critical points as the rank of the cohomology of $\tilde{Y}$, since $QH(\tilde{Y})$ is known to be semisimple [4]. This will be done in 7.7.

Recall from Section 5.2 that once we realize the surface as a non-toric blowup of a toric surface, we automatically obtain a LG model whose chamber structure is recorded in the scattering diagram $\mathfrak{D}^{LF}$. However, this LG model is not enough to capture the full mirror. In fact, it is crucial to enlarge it to include nodal fibres, as the mirror potential would miss a few geometrically meaningful critical points otherwise. Recall that in the local form of the SYZ fibration on X (4.1), the nodal fibre is given as $|x| = |x(q_i)|$ and $\mu_{S^1} = \epsilon_i$ where q_i is the blowup point, and ϵ_i is the value of μ_{S^1} at the isolated S^1-fixed point $\tilde{q}_i$ in the exceptional divisor E_i. See Figure 7. We remark that even if the loop $|x| = |x(q_i)|$ in the reduction is slightly deformed, it still gives an immersed Lagrangian S^2 as long as it passes through $x(q_i)$. This way we have a small freedom in choosing the valuation of this Lagrangian (along the direction perpendicular to the divisor $\{x = 0\}$).
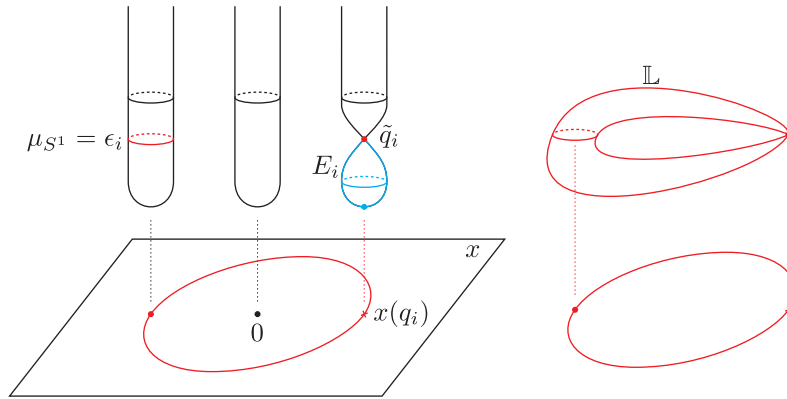

FIGURE 7. The nodal fibre in the local SYZ fibration on a non-toric blowup

7.1. Degree 5 del Pezzo surface and its LG mirror.

We realize the degree 5 del Pezzo surface $\tilde{Y}$ as the blowup of $\mathbb{P}^2$ at two of the torus-fixed points and the other two on the interior of the toric boundary. The toric blowups correspond to chopping off two corners of the moment polytope of $\mathbb{P}^2$ as shown in Figure 8, and the resulting surface $\bar{Y}$ is a (toric) del Pezzo surface of degree 7. We set $(a', 0)$ and $(0, b')$ to be the locations of the latter two blowup points in the moment polytope of $\mathbb{P}^2$.

The non-toric blowup in this case produces $\mathfrak{D}^{LF}$ which coincides with the A_2-scattering diagram. It has five chambers that support different mirror potentials, which agree with each other after suitable cluster transforms. For instance, the shaded chamber in Figure 8 (that contains the origin)[16] carries the potential given by

$$(7.1) \quad W = z_1 + z_2 + (T^a + T^{c \pm \epsilon_1})\frac{1}{z_1} + (T^b + T^{c \pm \epsilon_2})\frac{1}{z_2} + T^{a \pm \epsilon_1}\frac{z_2}{z_1} + T^{b \pm \epsilon_2}\frac{z_1}{z_2} + T^c \frac{1}{z_1 z_2}$$

---

[16]To be more precise, the genuine scattering diagram should be drawn on the log-base.

For computational simplicity, we will consider the limiting situation in which $\epsilon_1 = \epsilon_2 = 0$ from now on. On the adjacent chambers, the potential takes the form of

$$W_{up} = z_1' + (1 + T^a)z_2' + z_1'z_2' + (T^a + T^c)\frac{1}{z_1'} + T^b\frac{1}{z_2'} + T^a\frac{z_2'}{z_1'} + T^c\frac{1}{z_1'z_2'},$$

$$W_{right} = (1 + T^b)z_1'' + z_2'' + z_1''z_2'' + T^a\frac{1}{z_1''} + (T^b + T^c)\frac{1}{z_2''} + T^b\frac{z_1''}{z_2''} + T^c\frac{1}{z_1 z_2},$$

where the corresponding coordinate changes are given by

(7.2) $$z_1 = z_1' = z_1''(1 + z_2'')^{-1}, \quad z_2 = z_2'(1 + z_1')^{-1} = z_2''$$

These coordinates cover the region

(7.3)
$$0 < val(z_1) < a', \ 0 < val(z_2) < b'$$
$$0 < val(z_1') < a', \ b' < val(z_2') < b$$
$$a' < val(z_1'') < a, \ 0 < val(z_2'') < b'$$

(the same inequalities also describe corresponding chambers drawn in Figure 8). The computation for the other two chambers can be done similarly, and is omitted. By the consistency of $\mathfrak{D}^{LF}$, the five local charts are glued without ambiguity. In order to glue these local LG models, one needs to perturb slightly a complex structure to make local patches overlapped, and use the same coordinate transition as given in (7.2). Notice that (7.2) preserves the T-adic valuation.
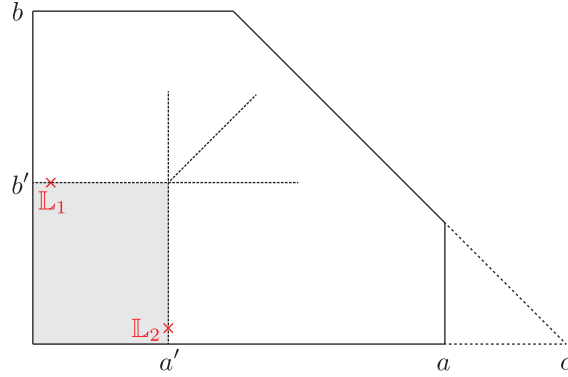

FIGURE 8. The degree 5 del Pezzo surface as a non-toric blowup

In what follows, we enlarge the above LG model to obtain the full mirror $(\check{Y}, W)$ by adding two more charts, each isomorphic to $(\Lambda_+)^2$. Observe that the fibration comes with two immersed Lagrangians $\mathbb{L}_1$ and $\mathbb{L}_2$. We denote by U_i the immersed generators of the Floer complex of $\mathbb{L}_i$ whose associated smoothing produces the torus fibre sitting in the shaded chamber in Figure 8. Let V_i denote the immersed generator complementary to U_i. For $u_i, v_i \in \Lambda_+^1$, one can take a linear combination $b_i = u_i U_i + v_i V_i$ for $\mathbb{L}_i$, and boundary-deform $\mathbb{L}_i$ into a new Lagrangian brane $(\mathbb{L}_i, b_i)$. One should think of $(\mathbb{L}_i, b_i)$ as a family of objects in the Fukaya category parametrized by the weak Maurer-Cartan space of $\mathbb{L}_i$. It is shown in [31, 3.1] that $b_i = (u_i, v_i)$ solves the weak Maurer-Cartan equation for $\mathbb{L}_i$ using reflection symmetry.

Observe that we have a freedom to choose $\mathbb{L}_i$ by adjusting the size of the base circle drawn in Figure 7 (since the circle does not have to be concentric about 0). By suitably choosing its size, one may assume that $(\mathbb{L}_i, b_i = (u_i, v_i))$ is quasi-isomorphic to the Lagrangian torus fibre

represented by $z_1 = T^{x_1}\rho_1$, $z_2 = T^{x_2}\rho_2$ (with $x_i = val(z_i)$ and $val(\rho_i) = 0$) where (u_1, v_1) and (z_1, z_2) are related by the coordinate change

$$(7.4) \qquad\qquad u_1 v_1 - 1 = \rho_1, \quad u_1 = T^{x_2 - b''}\rho_2.$$

Here, b'' is the symplectic area of the holomorphic disc bounded by $\mathbb{L}_1$ whose smoothing at the U_1-corner lies in the same class as the z_2-disc. While it can be chosen to be very close to b'', a further adjustment may be needed in order to have a genuine torus fibre satisfying $val(u_1) = x_2 - b'' > 0$ (note that we do have such a flexibility in the construction of $\mathbb{L}_1$). One can rewrite (7.4) as

$$(7.5) \qquad\qquad u_1 v_1 - 1 = T^{-val(z_1)}z_1, \quad u_1 = T^{-b''}z_2.$$

Analogously defined coordinates (u_2, v_2) on the weak Maurer-Cartan deformation space of $\mathbb{L}_2$ are related with others by

$$(7.6) \qquad\qquad u_2 v_2 - 1 = T^{-val(z_2)}z_2, \quad u_2 = T^{-a''}z_1.$$

Notice that there exists (u_1, v_1) satisfying (7.5) as long as $val(T^{val(z_1)} + 1) > val(z_2) - b''$ and $val(z_2) > b''$ regardless of whether or not (z_1, z_2) corresponds to an actual torus fibre, and the similar is true for (u_2, v_2). For instance, (z_1, z_2) with $val(z_1) = 0$ or $val(z_2) = 0$ can represent a geometric object that is supported over an immersed Lagrangian. Obviously, torus fibres themselves cannot have this feature. In other words, the chart $(\Lambda_+)^2$ from (u_i, v_i) nontrivially enlarges the original region parametrized (z_1, z_2) that supports torus fibres.

One can argue similarly to find relations between this $(\Lambda_+)^2$-chart with the chambers adjacent to the shaded one in Figure 8, which are compatible with (7.5) and (7.6) (see [31, 3.2] for more details). We conclude that $\mathbb{L}_i$ strictly extends the LG model obtained from torus fibres, i.e. the LG model constructed by gluing (7.1) with (local) ones from the other four chambers in Figure 8. In summary, the domain of the mirror LG model $\check{Y}$ are glued from the five open subsets given as in (7.3) (two of them were omitted in the equation) together with two $(\Lambda_+)^2$-charts parametrized by (u_1, v_1) and (u_2, v_2) via the transition maps given in (7.5) and (7.6), respectively.

Remark 7.2. *Suppose that two blowup points are so close, and that $\mathbb{L}_1$ and $\mathbb{L}_2$ can be made quasi-isomorphic to each other. Assuming the convergence, we may put $T = 1$ to work over $\mathbb{C}$. Then one can derive from (7.5) and (7.6) the following coordinate transition between complex coordinates (u_1, v_1) and (u_2, v_2)*

$$u_1 v_1 - 1 = u_2, \quad u_2 v_2 - 1 = u_1.$$

Setting $x = u_1, y = -v_1, z = v_2$, we have $-xy - 1 = u_2 = (x + 1)z^{-1}$,

$$xyz + x + z + 1 = 0,$$

which describes the A_2 cluster variety as an affine hypersurface.

7.2. **Critical points of the mirror potential.** Let us now compute the critical points of the potential

$$(7.7) \qquad W = z_1 + z_2 + (A + C)\frac{1}{z_1} + (B + C)\frac{1}{z_2} + A\frac{z_2}{z_1} + B\frac{z_1}{z_2} + C\frac{1}{z_1 z_2},$$

where we put $A := T^a$, $B := T^b$ and $C := T^c$ for simplicity. Elementary computation tells us that the critical points of W should satisfy

$$(7.8) \qquad \begin{aligned} z_1^2 z_2 - (A + C)z_2 - A z_2^2 + B z_1^2 - C &= 0, \\ z_1 z_2^2 - (B + C)z_1 - B z_1^2 + A z_2^2 - C &= 0. \end{aligned}$$

Subtracting the bottom multiplied by z_1 from the above multiplied by z_2, we obtain

$$(z_1 + z_2 + 1)(z_1(C + Bz_1) - z_2(C + Az_2)) = 0.$$

We then proceed by dividing into two cases depending on which of the two factors vanishes.

Case 1: If $z_1 + z_2 + 1 = 0$, then the sum of two equations in (7.8) implies $z_1z_2 + Bz_1 + Az_2 + C = 0$, and hence

$$z_1^2 + (A - B + 1)z_1 + A - C = 0.$$

Thus we obtain two solutions

$$z_1 = \frac{-A + B - 1 \pm \sqrt{1 - 2A - 2B - 2AB + A^2 + B^2 + 4C}}{2}$$

$$z_2 = \frac{A - B - 1 \mp \sqrt{1 - 2A - 2B - 2AB + A^2 + B^2 + 4C}}{2}$$

Using the expansion

$$\sqrt{1 - 2A - 2B - 2AB + A^2 + B^2 + 4C} = 1 - A - B + T^{higher},$$

one of the solutions takes the form of

$$(z_1, z_2) = (-1 + B + T^{higher}, -B + T^{higher}),$$

but $val(z_1) = 0$ is not valid if we restrict to torus fibres only. However, transferring to (u_1, v_1)-coordinate charts via

$$u_1 v_1 - 1 = z_1, \quad u_1 = T^{-b'} z_2,$$

we have a legitimate Lagrangian brane supported over the immersed Lagrangian $\mathbb{L}_1$, since $val(u_1) > 0$ and $val(v_1) > 0$. Likewise, the other solution

$$(z_1, z_2) = (-A + T^{higher}, -1 + A + T^{higher})$$

does not make sense in the original coordinates from torus fibres, but making use of

$$u_2 v_2 - 1 = z_2, \quad u_2 = T^{-a'} z_1,$$

one can replace it by the Lagrangian brane supported over $\mathbb{L}_2$.

Case 2: Let us now look into the case of $z_1(C + Bz_1) - z_2(C + Az_2) = 0$. We introduce an extra variable λ to keep the symmetry of the equation,

(7.9) $$(Bz_1 + C) = \lambda z_2, \quad (Az_2 + C) = \lambda z_1$$

which makes sense since neither of z_1, z_2 is zero. From the sum of two equations in (7.8), we have

$$(z_1 z_2 + 1 - \lambda)(z_1 + z_2) = 0.$$

If $z_1 + z_2 = 0$, then $(z_1, z_2) = (\frac{2C}{A-B}, \frac{2C}{B-A})$, whose valuation lives outside the allowed region, nor can it be replaced by appropriate boundary deformation of $\mathbb{L}_i$. Thus this critical point is not geometric in our SYZ setting. We speculate that there is *no* nontrivial object in the Fukaya category which represents this point, as otherwise its Floer cohomology with any of torus fibres would vanish.

Therefore geometrically meaningful is to examine the case $\lambda = z_1 z_2 + 1$. Making use of (7.9), one can easily find that the critical points are given as

(7.10) $$(z_1, z_2) = \left(\frac{1}{\lambda^2 - AB}(\lambda C + AC), \frac{1}{\lambda^2 - AB}(\lambda C + BC) \right)$$

where λ satisfies the degree-5 polynomial equation

$$\lambda^5 - \lambda^4 - 2AB\lambda^3 + (2AB - C^2)\lambda^2 + (A^2B^2 - C^2(A+B))\lambda - ABC^2 - A^2B^2 = 0.$$

Hence, for generic A, B and C, we obtain 5 mutually distinct critical points. Given the two critical points in the previous case, it only remains to check that the five points are represented by torus fibres. Observe that even if the valuation of a critical point escapes the chamber that supports our W (7.7), one can interpret it as the critical point from some other chamber (and vice versa) by applying the valuation-preserving coordinate change (7.2) as long as its valuation lies in the moment polytope. Note that the Jacobian ideal ring is not affected by coordinate changes. Therefore it is enough to work with W after enlarging its domain to include the other four chambers and two more $(\Lambda_+)^2$-charts.

Taking valuations of both sides of $\lambda = z_1 z_2 + 1$,

$$(7.11) \qquad\qquad\qquad val(z_1) + val(z_2) = val(\lambda)$$

which we will use quite often below. We subdivide the argument into the following four cases.

(i) If $val(z_1) \leq val(C) - val(B)$ and $val(z_2) \leq val(C) - val(A)$, then (7.9) leads to

$$val(B) + val(z_1) = val(\lambda) + val(z_2)$$
$$val(A) + val(z_2) = val(\lambda) + val(z_1)$$

so that

$$val(z_1) - val(z_2) = \frac{1}{2}(val(A) - val(B)).$$

On the other hand, combining with (7.11) we see that

$$val(z_1) + val(z_2) = val(\lambda) = \frac{1}{2}(val(A) + val(B)).$$

Hence $val(z_1) = \frac{1}{2}val(A)$ and $val(z_2) = \frac{1}{2}val(B)$, which lie inside the allowed region.

(ii) If $val(z_1) \leq val(C) - val(B)$ and $val(z_2) > val(C) - val(A)$, we have

$$val(B) + val(z_1) = val(\lambda) + val(z_2)$$
$$val(C) = val(\lambda) + val(z_1),$$

With (7.11), the above implies

$$val(z_1) = \frac{1}{2}val(C) - \frac{1}{4}val(B), \quad val(z_2) = \frac{1}{2}val(B).$$

It is not difficult to see that the corresponding point lie inside the region.

(iii) The case where $val(z_1) > val(C) - val(B)$ and $val(z_2) \leq val(C) - val(A)$ is symmetric to (ii), and we omit.

(iv) Finally, if $val(z_1) > val(C) - val(B)$ and $val(z_2) > val(C) - val(A)$, then

$$val(C) = val(\lambda) + val(z_2)$$
$$val(C) = val(\lambda) + val(z_1)$$

Using (7.11), we find that

$$val(z_1) = val(z_2) = \frac{1}{3}val(C).$$

Thus all the five critical points given by (7.10) admit Lagrangian torus fibres with suitable Λ_U-holonomies. Note, however, that the above calculation does not tell us exactly the chamber which contains a given critical torus fibre. In fact, such a chamber can vary if we change the Kähler parameter A, B and C.

Nondegeneracy of the critical points of W and the mirror symmetry for the degree 5 del Pezzo.
It is well-known that the quantum cohomology of the degree 5 del Pezzo is semisimple, and
that its rank is 7 (see [4]). Therefore, it suffices to show that all the seven critical points of
the potential W (7.7) are nondegenerate to have Theorem 7.1. The following is an elementary
calculation proving their nondegeneracy, which reflects (and is essentially equivalent to) the
fact that the choice of generic A, B and C in Section 7.2 is to achieve the maximal number
of critical points of W.

Let $f := \partial_{z_1} W$ and $g := \partial_{z_2} W$. The nondegeneracy of a critical point of W can be
reformulated as the transversality of the intersection between $\{f = 0\}$ and $\{g = 0\}$. More
concretely, we need to check whether or not df_p and dg_p is linearly independent for each
intersection point p (i.e. a critical point of W). Consider two linear combinations $\lambda_1 f + \lambda_2 g$
and $\eta_1 f + \eta_2 g$ with λ_i and η_i generic enough in the sense that $\det \begin{pmatrix} \lambda_1 & \lambda_2 \\ \eta_1 & \eta_2 \end{pmatrix}$ does not vanish
at any $p \in \{f = 0\} \cap \{g = 0\}$. Then it is easy to see that the transversality between $\{f = 0\}$
and $\{g = 0\}$ at p is equivalent to that between $\{\lambda_1 f + \lambda_2 g = 0\}$ and $\{\lambda_1 f + \lambda_2 g\}$ at p. The
strategy is to find such linear combinations which factor into lower degree polynomials so
that checking transversality becomes a bit simpler.

Going back to our situation, we may begin with

(7.12)
$$f := z_1^2 z_2 - (A + C)z_2 - Az_2^2 + Bz_1^2 - C,$$
$$g := z_1 z_2^2 - (B + C)z_1 - Bz_1^2 + Az_2^2 - C$$

as in (7.8). Recall that

$$z_1 f - z_2 g = (z_1 + z_2 + 1)(z_1(C + Bz_1) - z_2(C + Az_2)).$$

On the other hand, we have

$$f + g = (z_1 + z_2)(z_1 z_2 - C) - Bz_1 - Az_2 - 2C = 0.$$

Set $\alpha := z_1 + z_2 + 1$, $\beta := z_1(C + Bz_1) - z_2(C + Az_2)$ and $\gamma := (z_1 + z_2)(z_1 z_2 - C) - Bz_1 - Az_2 - 2C$.
By the above discussion, a sufficient condition for W being Morse is the transversality between
$\{\alpha = 0\}$ and $\{\gamma = 0\}$ and that between $\{\beta = 0\}$ and $\{\gamma = 0\}$. The former can be checked
straightforwardly, whereas the latter involves higher order terms. Similarly to before, let us
introduce new variables $\lambda = \frac{Bz_1 + C}{z_2}$ and $\lambda' = \frac{Az_2 + C}{z_1}$. In these variables, one has

$$\beta = (\lambda - \lambda')z_1 z_2$$

One can freely replace β by $\lambda - \lambda'$, and λ can be taken as a coordinate of $\{\beta = 0\}$.
$(\lambda(z_1, z_2), \lambda'(z_1, z_2))$ is singular along some affine plane, which finitely many critical points of
W can easily avoid by choosing generic parameters.

Observe that $\gamma|_{\{\beta=0\}}$ can be rewritten as

$$(z_1 + z_2)(z_1 z_2 - C) - \lambda z_1 - \lambda' z_2 = \gamma = (z_1 + z_2)(z_1 z_2 + 1 - \lambda)$$

with $z_1 = C\frac{\lambda + A}{\lambda^2 - AB}$ and $z_2 = C\frac{\lambda + B}{\lambda^2 - AB}$. Denote the last two factors by γ_1 and γ_2, respectively.
Finally, the question of W being Morse boils down to ask if 0 is a critical value of $\gamma_i|_{\{\beta=0\}}$
for $i = 0, 1$. From our earlier discussion in Section 7.2, we can actually ignore $\gamma_1|_{\{\beta=0\}}$ since
0 is not a critical value for a critical point lying in the domain of definition of W. On the
other hand, $(\lambda^2 - AB)^2 \gamma_2|_{\{\beta=0\}}$ is explicitly given as

$$-\lambda^5 + \lambda^4 + 2AB\lambda^3 + (C^2 - 2AB)\lambda^2 + (C^2(A + B) - A^2 B^2)\lambda + ABC^2 + A^2 B^2,$$

which has five zeroes for generic A, B and C, and hence 0 is not a critical for such A, B and
C.

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