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CANONICAL PARTITION RELATIONS AND POINT CHARACTER OF ℓ_1- SPACES

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<u>troduction and basic notions :</u> The covering $\mathcal{U}$ of a metric space (X,ρ) is called uniform if there exists $\varepsilon > 0$ such that for every $x \in X$ there is $U \in \mathcal{U}$ so that the ε-ball $B_\varepsilon(x) = \{y ; \rho(x,y) < \varepsilon\}$ is contained in U . We say that the covering $\mathcal{U}$ is c-bounded if $\operatorname{diam} U < c$ for every $U \in \mathcal{U}$. We say that the covering is bounded if it is c-bounded for some $c > 0$. We say that the point character of a metric space (X,ρ) is bigger than α $(pc(X,\rho) > \alpha)$ if there exists $c > 0$ such that for every c-bounded uniform covering $\mathcal{U}$ of X there exists a point $x \in X$ which i contained at least in α members of $\mathcal{U}$. The question of the existence of spaces with arbitrary large point character was answered affirmatively in [P] and [S₁], where point character of ℓ_∞- spaces was investigated. Here we prove an analogous result for ℓ_1-spaces. We use combinatorial lemma proved in [B].

<u>ts :</u>

Def : The mapping $f : [U]^m \longrightarrow V$ $(|U| \leq |V|)$ is called canonical if there exists $\ell \in \{0,1,2,\dots m\}$ and $1 \leq j_1 \leq j_2 \leq \dots \leq j_\ell \leq m$ such that for every $A = \{a_1, a_2, \dots a_m\}$, $B = \{b_1, b_2, \dots b_m\}$ $(a_1 < \dots < a_m,\ b_1 < b_2 \dots < b_m)$ $A, B \in [U]^m$ $f(A) \neq f(B) \Longleftrightarrow \langle a_{j_1} \dots a_{j_\ell}\rangle \neq \langle b_{j_1}, \dots b_{j_\ell}\rangle$

From results proved in [B] it follows the consequent

<u>Lemma :</u> For every cardinal number α and a positive integer m there exists a cardinal number β_m such that the following holds : for every mapping $f : [\beta_m]^m \longrightarrow \beta_m$ there exists $X \subset \beta_m$ $|X| = \alpha^+$ such that the mapping $f/[X]^m$ is canonical. $\quad (*)$

Theorem : For every cardibal number α there exists cardinal number β such that $pc\, \ell_1(\beta) > \alpha$. (We make no attempt here to find a smallest β with above mentioned property.)

Proof : Put $\beta = \sup \beta_m$ (β_m satisfy $(*)$) and denote by T the following subset of $\ell_1(\beta)$, $T = \{ \frac{1}{|K|} \chi_K ; K \in [\beta]^{<\omega} \}$ (where χ_K denotes the characteristic function of a set K) As $\ell_1(\beta)$ is a linear space it suffices to prove that for every 1-bounded uniform cover $\mathcal{U}$ of $\ell_1(\beta)$ there exists $\varphi_0 \in \ell_1(\beta)$ so that φ_0 is contained at least in α sets of $\mathcal{U}$. As $\mathcal{U}$ is uniform, there exists $\varepsilon > 0$ such that for every $x \in \ell_1(\beta)$ there is $U \in \mathcal{U}$ so that $B_\varepsilon(x) \subset U$. Let us take now m so large that $\frac{1}{m} < \frac{\varepsilon}{2}$ and put

$$T_m = \{ \tfrac{1}{m} \chi_K ; K \in [\beta]^m \} \subset T$$

Choose $f : T_m \longrightarrow \mathcal{U}$ so that for every $x \in T_m$, $B_\varepsilon(x) \subset f(x)$ Now identify the elements of T_m and m-element subsets of β and apply the Lemma to the mapping f . We get the existenxe of a set $X \subset \beta_m$ so that the mapping f restricted to the set $[X]^m$ is canonical. The corresponding number ℓ must be positive as from $\ell = 0$ it follows $[X]^m \subset U$ for some $U \in \mathcal{U}$ and it is a contradiction as $\operatorname{diam} [X]^m = 2$ while $\operatorname{diam} U < 1$ Put $Y = \{ \{ z_1, z_2, \dots z_{j_1-1}, z_{j_1} + \gamma, z_{j_1+1}, \dots z_m \}, \gamma < \alpha \}$ where

$$z < z_2 < \dots < z_{j_1} < z_{j_1} + \alpha < z_{j_1+1} < \dots < z_m$$

Such an Y exists because $|X| = \alpha^+ > \alpha$

So we have $Y \subset [X]^m \equiv T_m \subset T \subset \ell_1$

Moreover for $\varphi, \psi \in Y$, $\varphi \neq \psi$ we have $f(\varphi) \neq f(\psi)$

and $\rho(\varphi, \psi) = \frac{2}{m} < \varepsilon$.

Let us fix a $\varphi_0 \in Y$; we have $\varphi_0 \in f(\psi)$ for every $\psi \in Y$
As $|Y| = \alpha^+$ the theorem is proved.

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