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SEMINAR UNIFORM SPACES 1975-76

On hedgehog-topologically fine uniform spaces

by Zdeněk Frolík, Jan Pelant, and Jiří Vilímovský

This paper is motivated by the following general problem and program: Given a class $\mathcal{K}$ of spaces, is there the largest coreflective subclass of $\mathcal{K}$, and if there is one, find "useful" descriptions of it. In this paper we study EXT, the class of all X such that uniformly continuous real valued functions extend from subspaces to X. The most interesting descriptions are in terms of hedgehogs, the property of the space of all uniformly continuous functions to be a ring and the distal structure.

If X is a uniform space we denote by $t_f X$ the set X endowed with the finest uniformity topologically equivalent to X. The set of all uniformly continuous functions on X is denoted by U(X). H(A) will denote the hedgehog over a set A, that is the set of all $\langle a,x \rangle, a \in A, 0 \leq x \leq 1$, where we consider $\langle a,0 \rangle = \langle b,0 \rangle$ for all $a, b \in A$ (the null of H(A)), with the metric $d(\langle a,x \rangle, \langle a,y \rangle) = |x - y|$ and $d(\langle a,x \rangle, \langle b,y \rangle) = x + y$ if $a \neq b$. Recall that H(A) is an injective uniform space (cf.[7]). $H(\omega) - t_f$ will denote the class of all X for which any uniformly continuous mapping $f: X \longrightarrow H(\omega)$ remains uniformly continuous into $t_f H(\omega)$. In an analogous way we may define $H(\alpha) - t_f$ for other cardinals, $R - t_f$, $(I \times \omega) - t_f$, where $I \times \omega$ is the uniform product of a compact interval I and ω with the discrete uniformity. It is easy to verify that all these classes are coreflective. For general coreflections of this type see [8].
Further important class of uniform spaces we want to work with is the class of all X having the property that U(X) is a ring (i.e. U(X) is closed under multiplication). One can again prove the coreflectivity of this class.

Proposition 1: The following properties of a uniform space X are equivalent:
(a) U(X) is a ring

(b) $f^2 \in U(X)$ whenever $f \in U(X)$.

The proof is easy using the identity:

$$f \cdot g = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

Theorem 1: Each of the following properties implies the next one:

(1) X is $H(\omega) - t_f$

(2) X is $R - t_f$

(3) U(X) is a ring

(4) X is $(I \times \omega) - t_f$

Proof: (1) $\Rightarrow$ (2). Because of coreflectivity of $H(\omega) - t_f$ it suffices to prove that $(H(\omega) - t_f)R = t_f R$. Let us define two mappings $f, g: R \to H(Z)$, where Z is the set of all integers. (Of course H(Z) is uniformly equivalent to $H(\omega)$.)

$$f(x) = \begin{cases} \langle n, |x-n| \rangle & \text{for } x \in [\![n, n+\frac{1}{2}]\!],\ n \in Z \\ \langle n, |n+1-x| \rangle & \text{for } x \in [\![n+\frac{1}{2}, n+1]\!],\ n \in Z \end{cases}$$

$$g(x) = \begin{cases} \langle n, |x-n+\frac{1}{2}| \rangle & \text{for } x \in [\![n-\frac{1}{2}, n]\!],\ n \in Z \\ \langle n, |x-n-\frac{1}{2}| \rangle & \text{for } x \in [\![n, n+\frac{1}{2}]\!],\ n \in Z \end{cases}$$

Both f, g are uniformly continuous, hence they are uniformly continuous from $(H(\omega) - t_f)R$ into $t_f H(Z)$.

$t_f R$ has for basis of uniformity the covers of the form $\mathcal{W} = \mathcal{W}(\{\varepsilon_n; n \in Z\})$, where ε_n are positive reals and $\mathcal{W}$ consists of metric linear covers of intervals $[\![n, n+1]\!]$ with radius ε_n. Similarly, $t_f H(A)$ has basis formed by covers $\mathcal{W} = \mathcal{W}(\varepsilon, \{\varepsilon_a; a \in A\})$, ε, ε_a positive, $\sup_{a \in A} \varepsilon_a < \varepsilon$, having an ε-ball centred in the null of H(A) and the rest consists of metric linear ε_a-covers with centres in $\{\langle a, x \rangle; \varepsilon \leq x \leq 1\}$. Let us take any $\mathcal{W}(\{\varepsilon_n\})$ from the basis of $t_f R$. We may assume that all ε_n are less than $\frac{1}{4}$. If we denote $\mathcal{U} = \{]\!] n, n+1 [\![; n \in Z\} \cup \{]\!] n-\frac{1}{2}, n+\frac{1}{2} [\![; n \in Z\}$, then the cover

$$f^{-1}[\mathcal{W}(\tfrac{1}{4}, \{\varepsilon_n\})] \wedge g^{-1}[\mathcal{W}(\tfrac{1}{4}, \{\varepsilon_n\})] \wedge$$

is uniform in $(H(\omega) - t_f)R$ and refines $\mathcal{W}(\{\varepsilon_n\})$, hence $(H(\omega) - t_f)R = t_f R$.

(2) $\Longrightarrow$ (3): Let $X \in R - t_f$, $f \in U(X)$. The mapping f can be decomposed as if', where f' is uniformly continuous into $t_f R$, $i: t_f R \to R$ is the identity. $f^2 = (if')^2 = i^2 f'$, i^2 is uniformly continuous, hence f^2 is. The rest follows from the Proposition 1.

(3) $\Longrightarrow$ (4): Take $\mathcal{W}(\{\varepsilon_n\})$ from the basis of $t_f(I \times \omega)$. The function $F(x) = \frac{x}{\varepsilon_m}$ for $x \in I \times \{n\}$ is the product of two uniformly continuous functions on $I \times \omega$, $f^{-1}[\mathcal{S}(1)] = \mathcal{W}(\{\varepsilon_n\})$, where $\mathcal{S}(1)$ is usual 1-cover of the real line. Using the fact that the property (3) is coreflective we obtain the statement.

Remark: All properties in Theorem 1 differ. As counterexamples can serve the following: $H(\omega)$ for (2) $\not\Longrightarrow$ (1), R for (4) $\not\Longrightarrow$ (3), finally the coreflection in (3) on the real line is not topologically fine. The class $H(\omega) - t_f$ is hereditary. This follows easily from the injectivity of $H(\omega)$. The other classes in Theorem 1 are not hereditary, moreover every uniform space can be embedded into some $R - t_f$ space, because every injective space has all uniformly continuous functions bounded, hence it is $R - t_f$. One can find it interesting that the following holds:

Proposition 2: $R^n - t_f = R - t_f$ for any positive integer n.

The proof will work in three steps.

1. At first we prove that $R - t_f(I^n \times \omega) = t_f(I^n \times \omega)$. We define real valued functions $p_1, \ldots, p_n$ on $I^n \times \omega$ as follows:

$$p_i(x_1, \ldots, x_n, k) = 2k + x_i \ , \ i = 1, \ldots, n$$

It is obvious that for any uniform cover $\mathcal{U}$ of $t_f(I^n \times \omega)$ there is an open cover $\mathcal{V}$ of R such that $\bigwedge_{i=1}^{n} p_i^{-1}(\mathcal{V})$ refines $\mathcal{U}$.

2. Now we shall prove that $R - t_f(I^n \times R) = t_f(I^n \times R)$. We take $R_1 = \bigcup_{n \text{ odd}} [\![n - \frac{1}{2}, n + 1]\!]$, $R_2 = \bigcup_{n \text{ even}} [\![n - \frac{1}{2}, n + 1]\!]$.

$I^n \times R_1$ is of the form $I^{n+1} \times \omega$ and the functions $p_1, \ldots, p_{n+1}$ from the step 1 corresponding to $I^n \times R_1$ have obviously uniformly continuous extensions $\bar{p}_1, \ldots, p_{n+1}$ on $I^n \times R$. Similarly we obtain functions $\bar{q}_1, \ldots, q_{n+1}$ if we start with $I^n \times R_2$. Finally $\{I^n \times R_1, I^n \times R_2\}$ is a uniform cover of $I^n \times R$, hence for any uniform cover $\mathcal{U}$ of $t_f(I^n \times R)$ we can find an open cover $\mathcal{V}$ of R such that the cover

$$\bigwedge_{i=1}^{n+1} p_i^{-1}(\mathcal{V}) \wedge \bigwedge_{i=1}^{n+1} q_i^{-1}(\mathcal{V}) \wedge \{I^n \times R_1, I^n \times R_2\}$$

refines $\mathcal{U}$.

3. In the third step e prove $R - t_f R^n = t_f R^n$ for all $n \geq 2$. For any open ball B centred in 0 in R^n there is the mapping $f: R^n \; B \longrightarrow I^{n-1} \times R_+$ which is uniformly continuous and topological homeomorphism, hence f is a uniform isomorphism between $R - t_f(R^n \setminus B)$ and $t_f(I^{n-1} \times R_+)$. (Here applies step 2.) Hence we obtain that $R - t_f(R^n \; B)$ is topologically fine for every open ball B. Now the result follows immedi tely using th lemma (see [2]) which asserts that if for two topologically homeomorphic uniform spaces their uniformities coincide outside all neighbourhoods of some point, then these spaces are uniformly isomorphic. Applying the lemma to $R - t_f R^n$ and $t_f R^n$ we finish the proof.

Remarks: In the same way we can obtain that $H(\omega) - t_f = (H(\omega) \times I^m) - t_f$ for every positive integer n. It should be noted that $R - t_f(I^\omega \times R)$ is not topologically fine, because one can easily check that $R - t_f(I^\omega \times R)$ is projectively generated by real valued functions (i.e. it has subbase of linear covers) and that $t_f(I^\omega \times R)$ has not this property. Consequently $R^\omega - t_f$ is distinct from $R - t_f$.

Theorem 2: The following properties of a uniform space X are equivalent:

(1) X is $H(\omega) - t_f$

(2) X is $H(\alpha)$ - t_f for α any infinite cardinal number
(3) X is hereditarily R - t_f
(4) For every subspace Y of X, U(Y) is a ring
(5) X is hereditarily $(I \times \omega)$ - t_f

Proof: (1) $\Rightarrow$ (2): Let α be any cardinal number, take $\mathcal{W}(\varepsilon, \{\varepsilon_a; a \in \alpha\})$ any cover from the basis of $t_f H(\alpha)$. For every $n \in \omega$ we denote A_n the set of all $\epsilon \in \alpha$ such that $\frac{1}{n+2} < \varepsilon_n \leq \frac{1}{n+1}$. The mapping $f: H(\alpha) \to H(\omega)$ defined $f(\langle a,x \rangle) = \langle n,x \rangle$ for $a \in A_n$, is uniformly continuous, hence it is uniformly continuous from $(H(\omega) - t_f)\, H(\alpha)$ into $t_f H(\omega)$. $\mathcal{W}(\varepsilon, \{\frac{1}{n+2}; n \in \omega\})$ is a uniform cover of $t_f H(\omega)$, hence its preimage under f is a uniform cover of $(H(\omega)$ $t_f)\, H(\alpha)$ and r fines $\mathcal{W}(\varepsilon, \{\varepsilon_a\})$.

(2) $\Longrightarrow$ (1) is obvious, because $H(\omega)$ is a retract of $H(\alpha)$.
The implications (1) $\Rightarrow$ (3), (3) $\longrightarrow$ ( ) and ( ) $\Longrightarrow$ (5) follow from Theorem 1. It remains to prove (5) $\Longrightarrow$ (1 (obtained also by P. Pták). Take X hereditarily $(I \times \omega)$ - t_f, $f: X \longrightarrow H(\omega)$ uniformly continuous. Further take $\mathcal{W}(\varepsilon$ $\{\varepsilon_n\})$ any uniform cover from the basis of $t_f H(\omega)$. $B = \{\langle n,x \rangle; \frac{\varepsilon}{2} \leq$ $<$ $\}$ i s s b pace of $H(\omega)$ uniformly somorphic to $I \times \omega$. We denote $Y = f^{-1}[B]$ a a subspace of X. $f_1 = f|_Y$ is uniformly continuous into $t_f B$, $\mathcal{W}(\{\varepsilon_n\})$ is a uniform cover of t and $f_1^{-1}[\mathcal{W}(\{\varepsilon_n\})] \cup f^{-}$ $[\,'\langle a,x \rangle,$ $< \varepsilon]$ s a uniform cover of X refining f^{-1} $[\mathcal{W}(\varepsilon, \{\varepsilon$ $\})]$.

Now e turn our attention to two important cl se connected w th extension of un formly continuou mappings We hall denote by EXT the clas o hose u fo m paces where uniformly continuous (real valued) f nctions extend to the hole space from arbitrary subspaces and b T_ω the class of all uniform spaces here niformly contin o s functions extend from uniformly discrete count ble subs p ce .

Let $\{k_n; n \in \omega\}$ be a sequence of natural numbers. We shall denote by $D(\{k_n\})$ the following uniform space: The underlying set is $\{\langle n,i\rangle; n \in \omega, i = 1,2,\dots,k_n\}$, the system $\{\mathcal{U}_n; n \in \omega\}$ forms a basis of a uniformity of $D(\{k_n\})$, where $\mathcal{U}_n$ is defined as
$\{\{\langle m,i\rangle\}; m<n\} \cup \{\{\langle m,i\rangle; i = 1,2,\dots,k_m\}; m \geq n\}$.
One can immediately see that this is a metrizable zerodimensional topologically discrete uniformity.

Proposition 3: Let $\mathcal{C}$ be a coreflective subcategory of uniform spaces contained in EXT_ω, F the corresponding coreflector. Then $F(D(\{k_n\}))$ is uniformly discrete.

Remark: For any coreflective subcategory $\mathcal{C}$ in uniform spaces, the class Sub $(\mathcal{C})$ consisting of all subspaces of spaces in $\mathcal{C}$ is again coreflective. (For the proof of this fact see [8].) If $\mathcal{C}$ is a subclass of EXT_ω, which is hereditary, then Sub $(\mathcal{C})$ is again contained in EXT_ω. Hence in the proof of the proposition we may assume that $\mathcal{C}$ is hereditary, or, equivalently, that F preserves embeddings.

Lemma: Let f be a continuous function on compact interval $[0,1]$ with $f(0) = 0$, $f(1) = n(k - 1)$, where n, k are given natural numbers. Then we can find points $x_1, x_2, \dots, x_k$ of $[0,1]$, fulfilling the follo ing:

$$x_1 < x_2 < \dots < x_k;\ x_k - x_1 = \frac{1}{n};\ f(x_{i+1}) - f(x_i) \geq 1$$

for all $i = 1,2,\dots,k - 1$.

Proof: The proof is elementary. Dividing $[0,1]$ into n equal intervals we find points x_k, x_1 with $x_k - x_1 = \frac{1}{n}$ and $f(x_k) - f(x_1) \geq k - 1$. The other x_i's we obtain from the Darboux property of continuous f in $[x_1,x_k]$.

Proof of Proposition 2: We define the function f_0 on uniformly discrete countable subset $\{\{0\}\times\{n\}; n \in \omega\} \cup \{\{1\}\times\{n\}; n \in \omega\}$ of $I \times \omega$ in this way: For all $n \in \omega$ $f_0(\{0\}\times\{n\}) = 0$, $f_0(\{1\}\times\{n\}) = (n + 1)(k_n - 1)$.
f_0 has a uniformly continuous extension f on $F(I \times \omega)$.

Using the Lemma we can find in every $I \times \{n\}$ some points $x_1, x_2, \dots, x_{k_n}$ such that $x_{k_n} - x_1 = \frac{1}{n+1}$ and for all $i = 1,2,\dots,k_n - 1$ there is $x_i < x_{i+1}$ and $f(x_{i+1}) - f(x_i) \geq 1$. This procedure defines the uniformly continuous embedding of the space $D(\{k_n\})$ into $I \times \omega$. It is obvious that $D(\{k_n\})$ embedded in this way in $I \times \omega$ is uniformly discrete in the uniformity inherited fro $F(I \times \omega)$.

Theorem 3: Let $\mathcal{C}$ be a coreflective class in uniform spaces contained in EXT_ω. Then all $X \in \mathcal{C}$ have the following property:

Whenever $\{B_n; n \in \omega\}$ is a countable uniformly discrete family in X, $B_n = \bigcup_{i=1}^{k_n} A_{n_i}$ and for all n the finite systems $\{A_{n_i}; i = 1,2,\dots,k_n\}$ are again uniformly discrete, then $\{A_{n_i}; n \in \omega, i = 1,2,\dots,k_n\}$ is a uniformly discrete family in X. (∗)

Proof: Again we may and shall assume that $\mathcal{C}$ is closed under arbitrary subspaces. $B = \bigcup_{n \in \omega} B_n$ with the related uniformity from X is an element of $\mathcal{C}$. The canonical mapping $q: B \longrightarrow D(\{k_n\})$ defined $q(x) = \langle n,i \rangle$ for x in A_{n_i} is uniformly continuous, hence $F(q)$ is uniformly continuous from $FB = B$ into $F(D(\{k_n\}))$, where F is the coreflector corresponding to $\mathcal{C}$. Using the foregoing proposition we obtain that $\{A_{n_i}\}_{i,n}$ is uniformly discrete family in X.

As a consequence of the foregoing theorem we obtain the following

Theorem 4: $H(\omega) - t_f$ is the lar est coreflective subcategory of uniform spaces contained in EXT_ω.

Pro f: (i) At first we prove that $H(\omega) - t_f$ is contained in EXT_ω. Assume X is $H(\omega) - t_f$, $f \in U(D)$, where D is countable uniformly discrete subspace of X. We define $j: D \longrightarrow H(D)$ by t e formula $j(x) = \langle x,1 \rangle$. j is uniformly continuous and from the injectivity of $H(D)$ we can extend it to the uniformly continuous $\bar{j}: X \longrightarrow H(D)$. From the assumption we obtain that $\bar{j}$ remains uniformly continuous into $t_f H(D)$, which is in EXT_ω, hence f extends to some $g \in U(t_f H(D))$. The composition $g\bar{j} \in U(X)$ is an extension of f on the space X.

(ii) To prove the converse we suppose a coreflective class $\mathcal{C}$ contained in EXT_ω. We denote F the corresponding coreflector. From the preceding remarks it is clear that we may assume F to preserve topology and embeddings. Uniformly discrete families in $t_f(I \times \omega)$ are exactly those of the form (*) in Theorem 3. Using this theorem we obtain that $F(I \times \omega)$ has the same uniformly discrete families as $t_f(I \times \omega)$. But obviously $t_f(I \times \omega)$ is distal (the coarsest one having the same uniformly discrete families), hence $F(I \times \omega) = t_f(I \times \omega)$. F preserves embeddings, hence the spaces in $\mathcal{C}$ are hereditarily $(I \times \omega)$-t_f. Applying Theorem 2, we complete the proof.

Theorem 5: Each of the following conditions is equivalent to the conditions (1) - (5) in Theorem 2:

(6) For $\{f_n; n \in \omega\}$ a countable family of uniformly continuous bounded functions with uniformly discrete supports in X the function $\sum_{n \in \omega} f_n$ is uniformly continuous (i.e. the family $\{f_n\}_n$ is uniformly equicontinuous).

(7) Whenever $\{B_n; n \in \omega\}$ is a countable uniformly discrete family in X, $B_n = \bigcup_{i=1}^{k_n} A_{n_i}$ and for all n the finite families $\{A_{n_i};\ i = 1,2,\dots,k_n\}$ are again uniformly discrete, then $\{A_{n_i};\ i = 1,2,\dots,k_n, n \in \omega\}$ is uniformly discrete in X. (I.e. the property (*) from the Theorem 3.)

Proof: $(1) \Longrightarrow (6)$: We may assume that all f_n are not identically 0. Take $g_n = \frac{f_n}{\|f_n\| 2^n}$, where $\| \ \|$ is a usual sup norm. g_n are uniformly continuous and converge uniformly to 0, hence $g = \sum_{n \in \omega} g_n$ is a bounded uniformly continuous function, hence g is uniformly continuous into some compact interval J. Take $b: X \longrightarrow H(\omega)$ uniformly continuous such that for $x \in \operatorname{supp} f_n$ there is $b(x) = \langle n,1 \rangle$ (it is again possible from the injectivity of $H(\omega)$). The ca e ian product ϑ of g and b defined by the formula $\vartheta(x) = \langle g(x), b(x) \rangle$ ps X into $J \times H(\omega)$ uniformly, hence it is uniformly continuous into $(H(\omega) - t_f)\ (J \times H(\omega))$. From Theorem 1, condition (3) and Theorem 4 it follows that in this space we can multiply functions and extend functions from countable uniformly discrete subspaces, hence the function $h(\langle x,y \rangle) = x \cdot h'(y)$ is uniformly continuous on it, where h' is uniformly continuous on $(H(\omega) - t_f)\ H(\omega)$ wit $h'(\langle n,1 \rangle) = 2^n \cdot \| f_n \|$. The composition ϑ is a uniformly continuous function on X and one can easily see that it is equal to $\sum_{n \in \omega} f_n$.

(1) $\longrightarrow$ (7) is immediate from Theorems 3 and 4.

(7) $\Rightarrow$ (1): Take $f: X \longrightarrow H(\omega)$ uniformly continuous. From the property (7) we obtain that f is distally continuous from X into $t_f H(\omega)$, $t_f H(\omega)$ is distally coarse, hence f is uniformly continuous into $t_f \cdot H(\omega)$. (Distal continuity means that preimages of uniformly discrete families are again uniformly discrete. Distally continuous mappings and distally coarse spaces are studied in [3].)

(6) $\Longrightarrow$ (1): One can easily verify that spaces fulfilling (6) form a coreflective clas (they are closed under uniform sums and quotients). To prove (1) it suffices to know, whether this class is contained in EXT_ω . This will be an immediate corollary of the follo ing Proposition 4 The proof i then completed by applying Theorem .

Proposition 4 . Let X be a unifor space fulfilling Condition (6) in Theorem 5. Then X is in EXT.

Proof: Let Y be any subspace of X, $f \in U(X)$. For any integer n we denote $Y_n = f^{-1}[\,]n, n+1[\,]$, t e set of all odd integers will be denoted by Z_1, t e set of all even integers by Z_2. The family $\{Y_n; n \in Z_1\}$ is uniformly discrete in X. W choose bounded $f_n' \in U(X)$ for all $n \in Z_1$ such t at $f_b' | Y_n = f | Y_n$ and $\{\operatorname{supp} f_n';\ n \in Z_1\}$ is a unifor dis- crete family in X. The function $f' = \sum_{n \in Z_1} f_n' \in U(X)$, hence $g = f - f' | Y \in U(Y)$ and $g | \bigcup \{Y_n; n \in Z_1\} = 0$.

We have $g = \sum_{n \in Z_2} g_n$, where

$$g_n(x) = \begin{cases} g(x) & \text{for } x \in Y_n \\ 0 & \text{otherwise} \end{cases}$$

Now choose $\{h_n; n \in Z_2\}$ system of bounded uniformly continuous functions on X with uniformly discrete supports such that $h_n | Y = g_n$ for all $n \in Z_2$. Then $h = \sum_{n \in Z_2} h_n \in U(X)$, $f - f' | Y - h | Y = 0$, hence $(f' + h) | Y = f$ and $f' + h \in U(X)$

Theorem 6: $H(\omega) - t_f$ is the largest coreflective subcategory of uniform spaces contained in EXT.

Proof: Using Proposition 4 and Theorem 5 we obtain that $H(\omega) - t_f$ is contained in EXT. The rest follows from Theorem 4.

Now we are able to add the following characterizations of the spaces in Theorem 2.

Theorem 7: Each of the following conditions is equivalent to the conditions (1) - (7) in Theorems 2 and 5:

(8) X is in $EXT \cap R - t_f$

(9) X is in EXT and U(X) is a ring.

Proof: The implications $(1) \longrightarrow (8)$, $(8) \Longrightarrow (9)$ follow immediately from Proposition 1 and Theorem 6. We shall show $(9) \longrightarrow (4)$.

Taking any subspace Y of X, $f \in U(Y)$, we have an extension $\bar{f} \in U(X)$ of f. $\bar{f}^2 \in U(X)$, too, hence $f^2 = \bar{f}^2 | Y \in U(Y)$. Proposition 1 completes the proof.

Concluding remarks:

1) Several coreflective conditions for X to be in EXT have been known for a while. E.g. J. Isbell showed [6] that locally fine spaces are in EXT, and the first author noted in [1] that each sub-inversion-closed (subIC) space is in EXT. The latter condition is quite fine, but it is not the finest, as shows the following

Example. Put $X = \bigvee_{n=1}^{\infty} \omega_n$, $\omega_n = \omega$ for any $n \in N$.

Consider a uniformity on X whose base is formed by the covers $\mathcal{X}_n = \{\alpha \mid \alpha \in \omega_{n'}, n' \leq n\} \cup \{\omega_{n'} \mid n' > n\}$. Then subIC(X) $\notin$ H(ω) - t_f(X).

Proof: The space X is a countable complete metric topologically discrete space. Since for complete metric spaces the subIC coreflector is the same as the IC one (inversion closed)(see [1]) we have immediately that subIC(X) is the uniformly discrete space (as in IC(X), all countable partitions consisting of cozero sets, are uniform covers, again see [1]).

On the other hand, there is a nondiscrete space Y finer than X such that Y $\in$ H(ω) - t_f. It suffices to endow the set X with the uniformity which has for basis the covers with the discrete trace on at most finite number of ω_n, say ω_{n_1}, $\omega_{n_2}, \ldots, \omega_{n_k}$, and with the finite partition trace on each $\omega_n, n \notin \{n_1, \ldots, n_k\}$.

2) There are several classes of uniform spaces which are of interest in measure theory. In these cases coreflection sufficient conditions have been studied for long time. We refer to two preprints by J. Pachl:

Free uniform measures on subinversionclosed (uniform) spaces), to appear in Comment. Math. Univ. Carolinae, 17(1976)

Katětov-Shirota Theorem in uniform spaces (to appear).

References:

[1] Frolík Z. Four functors into paved spaces, Seminar Uniform Spaces 1973-74 directed by Z. Frolík, MÚ ČSAV, Prague 1975

[2] Cozero refinement of uniform spaces, Seminar Uniform Spaces 1975 (informal seminar notes)

[3] Basic refinements of uniform spaces, Lecture notes in Mathematics 378. Springer-Verlag, 140-158

[4] Three uniformities associated with uniformly continuous functions, Symposia Mathematica, vol.17, Academic Press London and New York 1976, 69-80.

[5] Hager A.W. Some nearly fine uniform spaces, Proc. of the London Math. Soc. 28(1974), 517-546.

[6] Isbell J. Uniform spaces, Amer. Math. Soc., Providence, 1964.

[7] Pelant J., Pták P. Injectivity of polyhedra, this volume.

[8] Vilímovský J. Generation of coreflections in categories, Comment. Math. Univ. Carolinae 14,2(1973), 305-323.