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INTRINSIC CHARACTERIZATION OF DISTAL SPACES

M. Kosina, P. Pták

Introduction. This part brings axioms required on $\mathcal{D} \subset \exp \exp X$ in order $\mathcal{D}$ be a collection of all uniformly discrete families of subsets of X for a uniformity. A pair $(X, \mathcal{D})$ where $\mathcal{D} \subset \exp \exp X$ satisfies those axioms is called a distal space. Given a distal space $(X, \mathcal{D})$, we construct the coarsest uniformity $\mathcal{U}$ on X among those which induce $\mathcal{D}$. The space obtained is called distally coarse. It is shown the equivalence between the category of distal spaces (with distally continuous mappings) and the category of distally coarse spaces. Some equivalent conditions on a uniform space to be distally coarse are given (e.g. $(X, \mathcal{U})$ is distally coarse iff $\mathcal{U}$ has a base consisting of finite-dimensional coverings).

Investigations implicitly using the notion of distality has been done by J.R. Isbell [5] (in connection with the dimension theory of uniform spaces) and by Z. Frolík [4] (in connection with refinements of the category of uniform spaces). The last author also kindly brought to our attention the questions examined here.

Notions and results. A quasiuniformity $\mathcal{U}$ on a set X is a family of coverings of X forming a filter in the refinement ordering (see [5]). X with $\mathcal{U}$ is a quasiuniform space. Given a covering $\mathcal{X} \in \mathcal{U}$ and a set $A \subset X$, $St(A, \mathcal{X})$ denotes the union of all $X_\alpha \in \mathcal{X}$ meeting A. A family

$\{A_\alpha \mid \alpha \in I\}$ of subsets of X is called uniformly discrete of order $\mathcal{X}$ if $St(A_\alpha, \mathcal{X}) \cap A_\beta = \emptyset$ provided that $\alpha \neq \beta$. Such a family is called uniformly discrete if it is uniformly discrete of an order $\mathcal{X} \in \mathcal{U}$.

Theorem 1. Let $(X, \mathcal{U})$ be a quasiuniform space. Denote by $d(\mathcal{U})$ the collection of all uniformly discrete families of subsets of X. The following assertions hold:

$\mathcal{D}_1$ If $\{A, B\} \in d(\mathcal{U})$ then $A \cap B = \emptyset$

$\mathcal{D}_2$ The family $d(\mathcal{U})$ is closed under formation of the distal combinations. It means, if $K = \{\{A^1_\alpha \mid \alpha \in I_1\}, \dots$ $\dots, \{A^n_\alpha \mid \alpha \in I_n\}\}$ is a finite family of elements belonging to $d(\mathcal{U})$ then $\{C_\gamma \mid \gamma \in J\}$ belongs to $d(\mathcal{U})$ provided that, given $x \in C_{\gamma_1}$, $y \in C_{\gamma_2}$, $\gamma_1 \neq \gamma_2$, there exist a k, $1 \leq k \leq n$ and some $\alpha_1 \alpha_2 \in I_k$, $\alpha_1 \neq \alpha_2$ such that $x \in A^k_{\alpha_1}$, $y \in A^k_{\alpha_2}$.

Proof: Let $(X, \mathcal{U})$ be a quasiuniform space. If $\{C_\gamma \mid \gamma \in J\}$ is a distal combination of $\{A^k_\alpha \mid \alpha \in I_k\}$, $1 \leq k \leq n$ and if $\{A^k_\alpha \mid \alpha \in I_k\}$ is discrete of order $\mathcal{X}_k \in \mathcal{U}$ then $\{C_\gamma \mid \gamma \in J\}$ is discrete of order $\bigcap_{k=1}^{n} \mathcal{X}_k$. So, $\mathcal{D}_2$ is fulfilled. $\mathcal{D}_1$ is trivial.

Before getting into the further theorems let us take use of the following convention. Given a family $\{A_\alpha \mid \alpha \in I\}$ of pairwise disjoint sets, we will write $A = \bigcup_{\alpha \in I} A_\alpha$ and call a covering $\mathcal{A}$ associated with $\{A_\alpha \mid \alpha \in I\}$ if

$\mathcal{A} = \{\bar{A}_\alpha \mid \alpha \in I\}$ where $\bar{A}_\alpha = (X - A) \cup A_\alpha$.

Theorem 2. Suppose that $D \subset \exp \exp X$ fulfils the conditions $\mathcal{D}_1$, $\mathcal{D}_2$. Then there exists a quasiuniformity $\mathcal{U}(D)$ such that D is exactly the collection of all $\mathcal{U}(D)$ -discrete families of subsets of X and, for every $\mathcal{U}'$ with the preceding property, $\mathcal{U}'$ is finer than $\mathcal{U}(D)$.

Proof: Denote by $\mathcal{U}(D)$ the quasiuniformity on X such that a subbase for $\mathcal{U}(D)$ consists of all coverings associated with the families $\{A_\alpha \mid \alpha \in I\} \in D$. Of course, $D \subset d(\mathcal{U}(D))$. We have to show that $d(\mathcal{U}(D)) \subset D$. For, let $\{C_\gamma \mid \gamma \in J\} \in d(\mathcal{U}(D))$ be a family discrete of order $\mathcal{X} = \mathcal{A}_1 \cap \mathcal{A}_2 \cap \dots \cap \mathcal{A}_m$ where $\mathcal{A}_k$ are the coverings associated with $\{A^k_\alpha \mid \alpha \in I\}$. We will prove that $\{C_\gamma \mid \gamma \in J\}$ is a distal combination of $K = \{\{A^1_\alpha \mid \alpha \in I_1\}, \dots, \{A^m_\alpha \mid \alpha \in I_m\}\}$. Take $x \in C_{\gamma_1}$, $y \in C_{\gamma_2}$ and denote by L the set of all k , $1 \leq k \leq m$ such that x belongs to some A^k_α . Obviously, L is nonvoid and for every $k \in L$ there exists exactly one $\alpha(k)$ such that $x \in A^k_{\alpha(k)}$. Therefore, $St(x, \mathcal{X}) = \bigcap_{k \in L} ((X - A^k) \cup A^k_{\alpha(k)})$ (remember the convention: $A^k = \bigcup_{\alpha \in I_k} A^k_\alpha$). For every $k \in L$ put $B_k = A^k - A^k_{\alpha(k)}$. Then $X - St(x, \mathcal{X}) = X - \bigcap_{k \in L}(X - B_k) = \bigcup_{k \in L} B_k$. Since $y \in X - St(x, \mathcal{X})$ we have a $k \in L$ such that $y \in B_k$. Thus $y \in A^k_\beta$ for some $\beta \neq \alpha(k)$ and therefore $d(\mathcal{U}(\mathcal{D})) = \mathcal{D}$. $\mathcal{U}(\mathcal{D})$ is clearly the coarsest such quasiuniformity. The proof is complete.

The pair $(X, \mathcal{D})$ where X is a set and $\mathcal{D}$ is the subset of $\exp \exp X$ with the properties $\mathcal{D}_1$, $\mathcal{D}_2$ is called a quasidistal space. A mapping $\varphi : (X, \mathcal{D}_1) \longrightarrow (Y, \mathcal{D}_2)$ between quasidistal spaces is said to be distally continuous if $\{\varphi^{-1}(A_\alpha) \mid \alpha \in I\} \in \mathcal{D}_1$ whenever $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}_2$. Let us denote by $\mathcal{Q}\mathcal{D}ist$ the category of quasidistal spaces with the distally continuous mappings.

Theorem 3. A mapping $\varphi : (X, \mathcal{D}_1) \longrightarrow (Y, \mathcal{D}_2)$ is distally continuous if and only if $\varphi : (X, \mathcal{U}(\mathcal{D})) \longrightarrow (Y, \mathcal{U}(\mathcal{D}))$ is uniformly continuous. So, the category $\mathcal{Q}\mathcal{D}ist$ is isomorphic to the full subcategory of $\mathcal{Q}\mathcal{U}nif$ of distally coarse quasiuniform spaces-spaces $(X, \mathcal{U})$ with $\mathcal{U}(d(\mathcal{U})) = \mathcal{U}$.

Proof: Suppose $\varphi : (X, \mathcal{D}_1) \longrightarrow (Y, \mathcal{D}_2)$ is distally continuous. If $\mathcal{A}$ is the covering associated with $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}_2$ then $\varphi^{-1}(\mathcal{A})$ is associated with $\{\varphi^{-1}(A_\alpha) \mid \alpha \in I\} \in \mathcal{D}_1$. The remaining part of Theorem 3 is easy.

A quasidistal space $(X, \mathcal{D})$ is a distal space if it has this additional property $\mathcal{D}_3$: If $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}$ then there exists a $\{B_\alpha \mid \alpha \in I\} \in \mathcal{D}$ such that $B_\alpha \supset A_\alpha$ for every $\alpha \in I$ and $\{A_\alpha \mid \alpha \in I\} \cup \{X - B\} \in \mathcal{D}$. The family $\{B_\alpha \mid \alpha \in I\}$ is called a distal neighbourhood of $\{A_\alpha \mid \alpha \in I\}$.

<u>Theorem 4.</u> If $(X, \mathcal{U})$ is a uniform space then $(X, d(\mathcal{U}))$ is a distal space. If $(X, \mathcal{D})$ is a distal space then $(X, \mathcal{U}(\mathcal{D}))$ is a uniform space. A base for $\mathcal{U}(\mathcal{D})$ consists of all coverings $\{X_\alpha \mid \alpha \in I\}$ satisfying the following: There exist a covering $\{A_\alpha \mid \alpha \in I\}$ and sets $I_1, I_2, \ldots, I_n$ such that $I = \bigcup_{k=1}^{n} I_k$ and such that for every k , $1 \leq k \leq n$, the family $\{A_\alpha \mid \alpha \in I_k\}$ belongs to $\mathcal{D}$ and $\{X_\alpha \mid \alpha \in I_k\}$ is a distal neighbourhood of $\{A_\alpha \mid \alpha \in I_k\}$.

Proof: The first part of the theorem may be easily proved by means of the star-refinement property of uniform spaces. To prove the second, take a covering $\mathcal{A}$ associated with a family $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}$. We have a distal neighbourhood of $\{A_\alpha \mid \alpha \in I\}$, say $\{B_\alpha \mid \alpha \in I\}$. It is easily seen that the covering $\mathcal{B} \cap \mathcal{Y}$ where $\mathcal{B}$ is associated with $\{B_\alpha \mid \alpha \in I\}$ and $\mathcal{Y}$ is associated with $\{A, X - B\}$ is a star-refinement of $\mathcal{A}$.

Let $\mathcal{B}(\mathcal{D})$ be the set of all coverings in question. The system $\mathcal{B}(\mathcal{D})$ is a base for a quasiuniformity $\mathcal{U}'$. First, $\mathcal{U}'$ is finer than $\mathcal{U}(\mathcal{D})$. Let $\mathcal{A}$ be the covering associated with $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}$. If $\{B_\alpha \mid \alpha \in I\}$ is a distal neighbourhood of $\{A_\alpha \mid \alpha \in I\}$ and if $\{C_\alpha \mid \alpha \in I\}$ is a distal neighbourhood of $\{B_\alpha \mid \alpha \in I\}$ then $\{C_\alpha \mid \alpha \in I\} \cup \{X - A\}$ is a covering belonging to $\mathcal{U}'$ which refines $\mathcal{A}$.

The uniformity $\mathcal{U}(\mathcal{D})$ is finer than $\mathcal{U}'$. Take a covering $\{X_\alpha \mid \alpha \in I\} \in \mathcal{B}(\mathcal{D})$ and, according to the theorem, a covering $\{A_\alpha \mid \alpha \in I\}$. Put $\mathfrak{X}_k = \{X_\alpha \mid \alpha \in I_k\} \cup \{X - A^k\}$.

Since $\{A_\alpha \mid \alpha \in I\}$ is a covering of X then $\bigcap_{k=1}^{n} \mathfrak{X}_k$ refines the covering $\{X_\alpha \mid \alpha \in I\}$. By easy computation, the coverings $\mathfrak{X}_k$ belong to $\mathcal{U}(\mathcal{D})$ and the proof is complete.

Let us denote by $\mathbf{Dist}$ the category of distal spaces and distally continuous mappings and by $d : \mathbf{Unif} \longrightarrow \mathbf{Dist}$ the natural functor which assigns to each uniformity the induced distality. Theorem 4 implies that the category $\mathbf{Dist}$ is isomorphic to the full subcategory of $\mathbf{Unif}$ of distally coarse uniform spaces.

For the further use we need to introduce some subspaces of the ell-infinity spaces. The ell-infinity space $l_\infty(I)$, for any set I, is the metric space whose points are the real-valued bounded functions on I with the distance $\rho(f, g) = \sup|f - g|$. Denote by $H(I)$ the set of all $f \in l_\infty(I)$ with $\rho(0, f) \leq 1$ and with at most one $\alpha \in I$ such that $f(\alpha) > 0$. The symbol $\vec{\alpha}$, $\alpha \in I$ denotes the element of $H(I)$ such that $f(\alpha) = 1$ and $f(\beta) = 0$ whenever $\alpha \neq \beta$.

<u>Statement 1:</u> If $(X, \mathcal{D})$ is a distal space and if $\{B_\alpha \mid \alpha \in I\}$ is a distal neighbourhood of $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}$ then there exists a distally continuous mapping

$\varphi : (X, \mathcal{D}) \longrightarrow d(H(I))$ such that $B_\alpha \supset \varphi^{-1}(\vec{\alpha}) \supset A_\alpha$ for every $\alpha \in I$ and moreover, $\varphi(X - B) \subset \{0\}$.

The proof of this statement is easy. It may be obtained from the metrization lemma for uniform spaces and from Theorem 4 (see also [4]). The direct proof may be got by a modification of the Urysohn's procedure.

<u>Statement 2:</u> For a distal space $(X, \mathcal{D})$ the uniformity $\mathcal{U}(\mathcal{D})$ is the coarsest uniformity among those which makes every distally continuous mapping from $(X, \mathcal{D})$ into $H(X)$ uniformly continuous.

Proof: Denote by $\mathcal{U}'$ the described uniformity. First, $\mathcal{U}'$ is finer than $\mathcal{U}(\mathcal{D})$. If $\mathcal{A}$ is the covering associated with $\{A_\alpha \mid \alpha \in I\} \in \mathcal{D}$ and if $\{B_\alpha \mid \alpha \in I\}$ is a distal neighbourhood of $\{A_\alpha \mid \alpha \in I\}$ then the covering $\{B_\alpha \mid \alpha \in I\} \cup \{X - A\}$ belongs to $\mathcal{U}'$ and refines $\mathcal{A}$.

The uniformity $\mathcal{U}(\mathcal{D})$ is finer than $\mathcal{U}'$. Observe that in $H(X)$, uniform coverings of the type $\bigcup_{k=1}^{m} \mathcal{P}_k$ where every $\mathcal{P}_k$ is uniformly discrete family form a base for uniform coverings of $H(X)$.

Now, to introduce the uniform complexes is necessary. Given an abstract simplicial complex K_a , the uniform complex K is a subspace of $l_\infty(K_a)$ whose points are those non-negative functions f on the vertices of K_a such that, for some simplex s of K_a , $f(v) = 0$ for all vertices v not in s and $\sum_{v \in s} f(v) = 1$. The dimension of a simplex is one less than the number of vertices, the dimension of a complex (abstract or uniform) is the least upper bound in $\{0, 1, \dots, \infty\}$ of the dimensions of its simplexes. For uniform complexes, the dimension in the sense above is the same as the big uniform dimension Δ (for details concerning uniform complexes consult [5]).

Statement 3: For each set I, $H(I)$ is a one-dimensional uniform complex.

The proof is evident.

Statement 4: Any closed subspace of a product of finite-dimensional complexes is an inverse limit of finite-dimensional subspaces.

The proof is evident.

Recall that a covering $\mathfrak{X}$ of X is called finite-dimensional if, for some natural number n, every $x \in X$ belongs to at most n elements of $\mathfrak{X}$.

Theorem 5. Given a uniformity $\mathfrak{U}$ on X, the following conditions are equivalent:

(i) $\mathfrak{U}$ is distally coarse

(ii) $\mathfrak{U}$ has a base consisting of finite-dimensional coverings

(iii) $(X, \mathfrak{U})$ is a subspace of a product of finite-dimensional complexes

(iv) The completion of $(X, \mathfrak{U})$ is an inverse limit of finite-dimensional uniform complexes.

Proof: (i) implies (ii): It follows immediately from

Theorem 3.

(ii) implies (i): Let $\{X_\alpha \mid \alpha \in I\}$ be a finite-dimensional covering. Take a strict strinkage of $\{X_\alpha \mid \alpha \in I\}$ i.e., a uniform covering $\{Z_\alpha \mid \alpha \in I\}$ such that $St(Z_\alpha, \mathcal{Y}) \subset X_\alpha$ for some uniform covering $\mathcal{Y}$. Of course, $\{Z_\alpha \mid \alpha \in I\}$ is also finite-dimensional. Therefore there exists a uniform covering $\mathcal{P}$ which refines $\{Z_\alpha \mid \alpha \in I\}$ and it is a finite union of uniformly discrete subcollections (see [5], p. 67). It is clear that a suitable uniform neighbourhood of $\mathcal{P}$ refines $\{X_\alpha \mid \alpha \in I\}$. Now, the proof of this implication follows from Theorem 4.

(i) implies (iii): It follows from Statement 2 and 3.

(iii) implies (iv): It follows from Statement 4 because if $(X, \mathcal{U})$ is a subspace of a complete space $\mathcal{U}$ then the completion of $(X, \mathcal{U})$ is a closed subspace of $\mathcal{U}$.

(iv) implies (ii): This is trivial.

Remarks.

A. For uniform spaces and for proximity spaces we can introduce the notions of projective generation and inductive generation (see [2], p. 679). The same notions can be introduced for distal spaces as well. One can prove that the natural functor preserves inductive generation and need not preserve projective generation. In this connection the following question seems to be of interest:

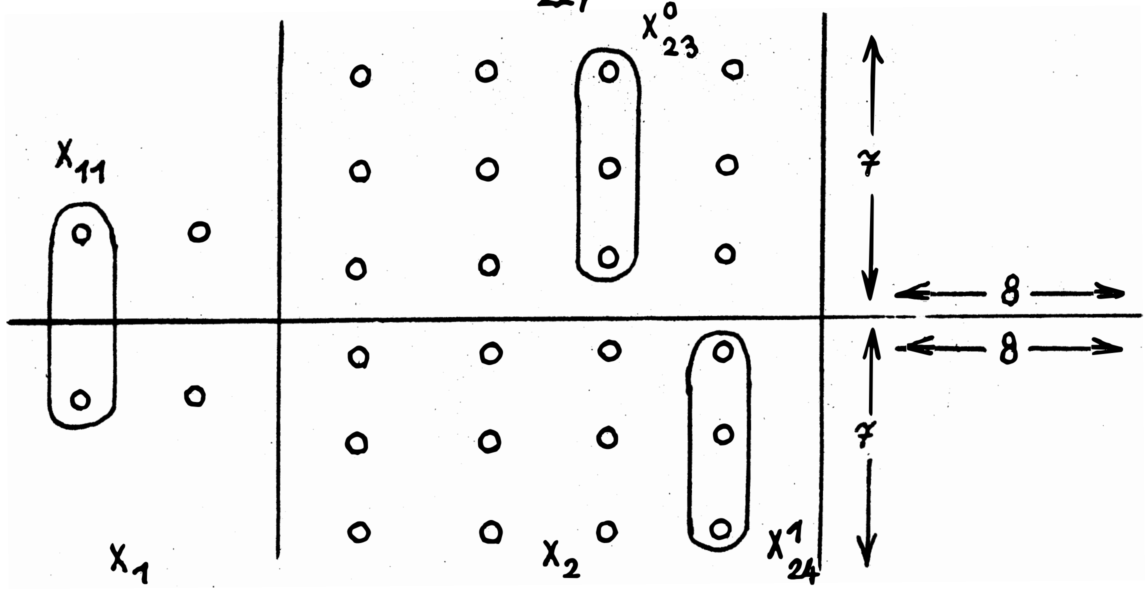
Problem: Find two uniformities on a set X, say $\mathcal{U}_1$, $\mathcal{U}_2$, so that $d(\mathcal{U}_1) = d(\mathcal{U}_2)$ but $d(\mathcal{U}_1 \wedge \mathcal{U}_2) \neq d(\mathcal{U}_1)$. Here $\wedge$ is the greatest lower bound in the lattice of uniformities. For the same for proximity spaces, see [1], [3], [6].

Is the distality $d\mathcal{U}$ of a space $\mathcal{U}$ already determined by the proximity $\mathring{p}\mathcal{U}$ and by the collection of all uniformly discrete families $\{X_\alpha \mid \alpha \in I\}$ each X_α is a one-point set? No. Let X be an uncountable set. Let $\mathcal{U}_1 = (X, \mathfrak{U})$ have a base consisting of all partitions with at most countable many elements and let $\mathcal{U}_2 = (X, \mathfrak{U})$ have a base consisting of all partitions as above with at most cardinality greater than $\aleph_0$. These uniformities $\mathcal{U}_1$, $\mathcal{U}_2$ have the same properties in question but $d\mathcal{U}_1 \neq$ $\neq d\mathcal{U}_2$.

The following question was important in the axiomatic development of distal spaces. Let $\{X_\alpha \mid \alpha \in I\}$ be a system of subsets of X, each X_α can be written as a disjoint union of two nonvoid subsets $X_\alpha^0 \cup X_\alpha^1$. Suppose that for each system of upper indexes $\{i_\alpha \mid \alpha \in I\}$, $i_\alpha \in \{0,1\}$ the system $\{X_\alpha^{i_\alpha} \mid \alpha \in I\}$ is uniformly discrete. Is then $\{X_\alpha \mid \alpha \in I\}$ uniformly discrete?

A sketch of the counterexample (by P. Simon). Let X be a disjoint union of X_k, $k \in N$ and let every X_k consist of 2^k disjoint subsets, X_{ki}, $\operatorname{card} X_{ki} = 2(2^k - 1)$. Divide all X_{ki} into two disjoint subsets, say X_{ki}^0, and X_{ki}^1, and denote $X_k^0 = \bigcup_{i=1,\dots,2^k} X_{ki}^0$, $X_k^1 = \bigcup_{i=1,\dots,2^k} X_{ki}^1$.

For the illustration a picture:



First, there exists a covering $\mathcal{Y}$ of X having the properties:

a. If $Y_\alpha, Y_\beta \in \mathcal{Y}$, $\alpha \neq \beta$ then $Y_\alpha \cap Y_\beta = \emptyset$

b. For all $Y_\alpha \in \mathcal{Y}$ it holds

i) $Y_\alpha \subset X_k$ for some $k \in N$

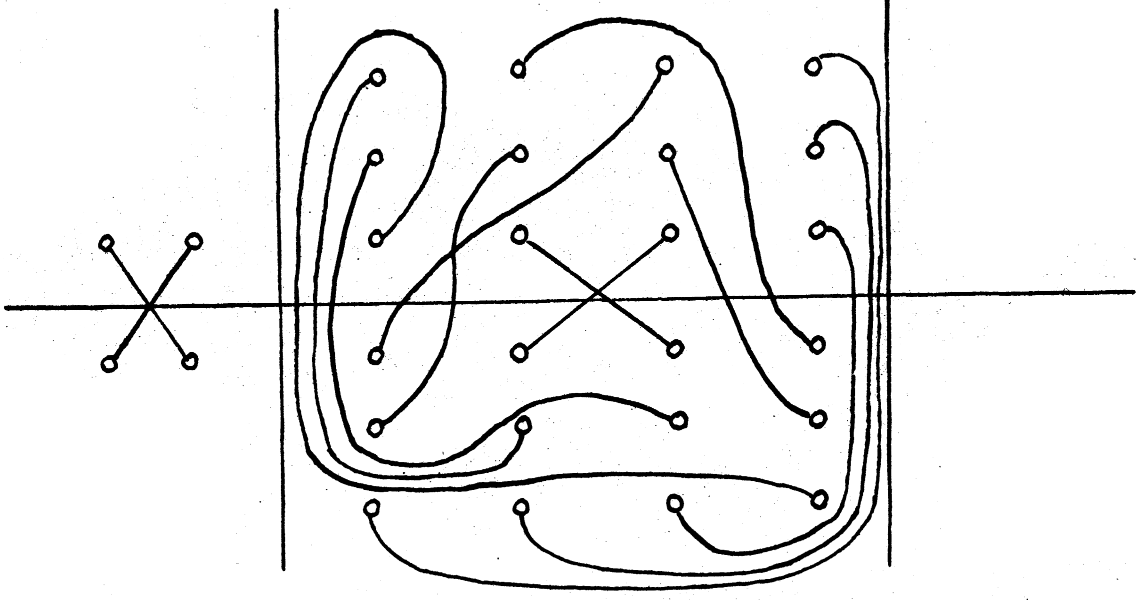
ii) $\operatorname{card} Y_\alpha = 2$

iii) $Y_\alpha - X_{ki} \neq \emptyset$ for every $k, i \in N$

iv) $Y_\alpha - X^0_k \neq \emptyset$, $Y_\alpha - X^1_k \neq \emptyset$, $k \in N$.

c. If $k \in N$, $a \in \{0, 1\}$, $1 \leq \ell$, $j \geq 2^k$ and $i \neq j$ then there is $Y_\alpha \in \mathcal{Y}$ such that $Y_\alpha \cap X^a_{ki} \neq \emptyset$ and $Y_\alpha \cap X^{1-a}_{kj} \neq \emptyset$.

In the following picture two points are connected with the line iff the corresponding two-point set belongs to $\mathcal{Y}$.



Let Λ denote the system of all collections $\mathfrak{M}$ such that for every k, i exactly one half of X_{ki} belongs to $\mathfrak{M}$. Given $\mathfrak{M} \in \Lambda$, define a covering $\mathcal{X}_{\mathfrak{M}}$ as follows:

i) $\mathcal{X}_{\mathfrak{M}} \prec \mathcal{Y}$

ii) For every $x \in X$, $St(x, \mathcal{X}_{\mathfrak{M}})$ meets at most one set belonging to $\mathfrak{M}$.

iii) If $\mathcal{Z}$ is a covering, $\mathcal{X}_{\mathfrak{M}} \prec \mathcal{Z} \prec \mathcal{Y}$ then there is a set $Z_\gamma \in \mathcal{Z}$ which meets at least two sets belonging to $\mathfrak{M}$.

Now, the system $\{\mathcal{X}_{\mathfrak{M}} \mid \mathfrak{M} \in \Lambda\}$ forms a subbase for a uniformity $\mathcal{U}$ on the set X and, in this uniformity, every $\mathfrak{M} \in \Lambda$ is a uniformly discrete collection. It remains to show that $\{X_{ki} \mid k \in N,\ 1 \leq i \leq 2^k\}$ is not uniformly discrete but it is not because every covering $\mathcal{X} \in \mathcal{U}$ contains a two-point set (proof by induction).

B. The distally coarse spaces form a reflective subcategory of uniform spaces. Denote again by $d\mathcal{U}$ the reflection of a space $\mathcal{U}$. It has for a base the finite-dimensional coverings of $\mathcal{U}$.

The functor d preserves subspaces, i.e., if $\mathcal{U}_1$ is a subspace of $\mathcal{U}$ then $d\mathcal{U}_1$ is a subspace of $d\mathcal{U}$. It follows immediately from this lemma: If $\mathcal{U}_1$ is a subspace of $\mathcal{U}$ and if $\mathfrak{X}_1$ is a n-dimensional uniform covering of $\mathcal{U}_1$ then there exists an $n+1$-dimensional uniform covering $\mathfrak{X}$ of $\mathcal{U}$ such that the trace of $\mathfrak{X}$ on $\mathcal{U}_1$ refines $\mathfrak{X}_1$.

The proof of the lemma is not difficult.

For the totally bounded reflection p^0 we have the formula: $p^0\mathcal{U} = \mathcal{U} \vee p^0 t_f \mathcal{U}$. (Here t_f is the fine co-reflection and the symbol $\vee$ is that of the least upper bound in the lattice of uniformities on a set.) The similar formula for d, namely $d\mathcal{U} = \mathcal{U} \vee d t_f \mathcal{U}$, does not hold. To exhibit it, note the following

<u>Proposition 1:</u> For a uniform space $\mathcal{U}$, $\Delta(d\mathcal{U}) = \delta(d\mathcal{U})$ where Δ means the big and δ the small uniform dimension.

The proof can be obtained from Theorem 5, p. 79 in [5].

<u>Proposition 2:</u> There exists a uniform space $\mathcal{U} = (X, \mathfrak{U})$ such that the following holds:

i) X is countable

ii) $d\mathcal{U} \neq \mathcal{U}$

iii) $t_f\mathcal{U}$ is the discrete uniformity.

Proof: There exists a separable uniform space $\mathcal{V}$ such that $t_p \mathcal{V}$ is discrete and $\Delta \mathcal{V} = \infty$, $\delta \mathcal{V} = 0$ (see [5], p. 79). $\mathcal{U}$ is a countable dense subspace of $\mathcal{V}$.

If $\mathcal{U}$ is from Proposition 2 then $\mathcal{U} \vee d\, t_p \mathcal{U} = \mathcal{U} \neq d\mathcal{U}$.

This space $\mathcal{U}$ can be also used to demonstrate that the formula $p^0(\mathcal{V} \times p^0 \mathcal{V}) = p^0(\mathcal{V}) \times p^0(\mathcal{V})$ holding for each space $\mathcal{V}$ does not hold for distal reflection. (The proof is easy because of the countability of $\mathcal{U}$.)

Finally, perhaps the most interesting difference between p^0 and d. If $\mathcal{U} = (X, \mathrm{U})$ is a space such that for $\mathcal{V} = (X, V)$, $p^0(\mathcal{U}) = p^0(\mathcal{V})$ implies that V is coarser than U, then $\mathcal{U}$ is proximally fine, i.e., for every space $\mathcal{W}$, $\varphi : p^0 \mathcal{U} \longrightarrow p^0 \mathcal{W}$ is uniformly continuous iff $\varphi : \mathcal{U} \longrightarrow \mathcal{W}$ is. It was established by M. Hušek that such a statement does not hold for the distal reflection. In fact, it was shown that for every space $\mathcal{U}$ there is a space $\mathcal{U}$ with the properties:

i) $\widetilde{\mathcal{U}}$ is distally unique. It means, if $\mathcal{V} \neq \widetilde{\mathcal{U}}$ then $d\mathcal{V} \neq d\widetilde{\mathcal{U}}$.

ii) $\mathcal{U}$ is a quotient space of $\widetilde{\mathcal{U}}$.

This construction is in this publication on the page 113.

Now, to exhibit a space in question it suffices to take the space $\widetilde{\mathcal{U}}$ for a space $\mathcal{U}$, $\mathcal{U}$ is not finest among those uniform spaces having the distality $d\mathcal{U}$ (use that the distally fine spaces form a coreflective subcategory of $\mathbf{Unif}$).

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