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CATEGORIAL REFINEMENTS AND THEIR RELATION TO REFLECTIVE SUBCATEGORIES

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In the general part (§ 1, § 2) there are discussed mainly the following problems: When the refinement $\mathfrak{X}$ of a given category gives (by the construction $\mathfrak{X}$-fine) a coreflective subcategory, and when the given category is reflective (under the idempotent reflector) in its refinement $\mathfrak{X}$. There appears that there exists a closed connection between these two questions. (The main results about this are 2.3, 2.8, 2.12, 2.15) From the theory also follows the theorem about decomposition of any idempotent reflection into one "full reflection" and one "refinement reflection" (2.14).

In the third paragraph the general theory is applied to the case of concrete categories and concrete refinements, namely to the category of uniform spaces. It is shown in 3.3 that there exists a bijection between reflective refinements and coreflective subcategories and between coreflective refinements and hereditary epireflective subcategories. At the end of the paper the ideas of the theory are used to make some conclusions which, I hope, put some little more light into the structure of reflexive and coreflexive subcategories of uniform spaces.

§ 1. Reflections and coreflections.

Let $\mathcal{L}$ be a category. We shall denote $|\mathcal{L}|$ the class of objects and $\mathcal{L}^{m}$ the class of morphisms of $\mathcal{L}$. Let $X, Y \in |\mathcal{L}|$, we shall denote $\mathcal{L}(X, Y)$ the set of all morphisms of $\mathcal{L}$ with domain X and range Y.

At first we recall some basic concepts and propositions which will be frequently used. For all of them we refer to [9].

1.1. Proposition: Assume $\mathcal{L}$ is a category, $X, Y \in |\mathcal{L}|$. Let further $Hom_{\mathcal{L}}(X, Z) \simeq Hom_{\mathcal{L}}(Y, Z)$ (natural equivalence) for any $Z \in |\mathcal{L}|$. Then X is $\mathcal{L}$-isomorphic to Y.

1.2. Definition. Let $\mathcal{L}$ be a subcategory of the category $\mathcal{K}$ and let $J: \mathcal{L} \hookrightarrow \mathcal{K}$ be the corresponding embedding. We shall say that $\mathcal{L}$ is reflective in $\mathcal{K}$ if there exists a functor $F: \mathcal{K} \longrightarrow \mathcal{L}$ (called reflector) which is a left adjoint to J.

1.3. Proposition: Let $\mathcal{L}$ be a subcategory of the category $\mathcal{K}$, $J: \mathcal{L} \hookrightarrow \mathcal{K}$ the embedding. $\mathcal{L}$ is a reflective subcategory of $\mathcal{K}$ if and only if for every $X \in |\mathcal{K}|$ there exist $\bar{X} \in |\mathcal{L}|$ and $\mu^X \in \mathcal{K}(X, \bar{X})$ such that for any $Y \in |\mathcal{L}|$, $f \in \mathcal{K}(X, Y)$ there exists exactly one $g \in \mathcal{L}(\bar{X}, Y)$ fulfilling $g\mu^X = f$.

1.4. Definition. Assume $J: \mathcal{L} \hookrightarrow \mathcal{K}$ is an embedding of categories, $F: \mathcal{K} \longrightarrow \mathcal{L}$ the reflector. We shall say that $\mathcal{L}$ is idempotently reflective in $\mathcal{K}$, if FX is $\mathcal{L}$-isomorphic to $FJFX$ for any $X \in |\mathcal{K}|$.

(The functor F is idempotent.)

Remark. If the embedding $J : \mathcal{L} \hookrightarrow \mathcal{K}$ is full and $\mathcal{L}$ is reflective in $\mathcal{K}$, then $\mathcal{L}$ is idempotently reflective in $\mathcal{K}$. Of course, in the general case of embedding, the statement does not hold.

1.5. Proposition: Let $\mathcal{L} \overset{U_1}{\hookrightarrow} \mathcal{K} \overset{U_2}{\hookrightarrow} \mathcal{M}$ be the embeddings of categories. Suppose $\mathcal{L}$ is reflective in $\mathcal{K}$, $\mathcal{K}$ is reflective in $\mathcal{M}$. Then $\mathcal{L}$ is reflective in $\mathcal{M}$. Moreover, if $\mathcal{L}$ is idempotently reflective in $\mathcal{K}$ and U_2 is a full embedding, then $\mathcal{L}$ is idempotently reflective in $\mathcal{M}$.

Proof: Let us define $F = F_1 F_2$, where $F: \mathcal{K} \longrightarrow \mathcal{L}$, $F_2 : \mathcal{M} \longrightarrow \mathcal{K}$ are the corresponding reflectors. For any $X \in \mathcal{M}$, $Y \in \mathcal{L}$ there is:

$$\mathcal{L}(FX, Y) = \mathcal{L}(F_1 F_2 X, Y) \cong \mathcal{K}(F_2 X, Y) \cong \mathcal{M}(X, Y) .$$

If F_1 is idempotent and U_2 is a full embedding, there is $F_1 U_1 F_1 = F_1$ and $F_2 U_2 = 1$, which immediately implies $F U_2 U_1 F = F$.

1.6. Definition. Let $\mathcal{L}$ be a subcategory of $\mathcal{K}$. $\mathcal{K}$ will be called the refinement of $\mathcal{L}$, if $|\mathcal{K}| = |\mathcal{L}|$.

Let $\mathcal{L}$ be a subcategory of the category $\mathcal{M}$. There exists exactly one category $\mathcal{K}$ such that $\mathcal{K}$ is a refinement of $\mathcal{L}$ and simultaneously a full subcategory of $\mathcal{M}$. In this manner the general embedding can be decomposed in a canonical way into one refinement and one full embedding.

1.7. Proposition: Let $\mathfrak{L}, \mathfrak{M}, \mathfrak{X}$ be as in the foregoing remark. Let $\mathfrak{L}$ be reflective in $\mathfrak{M}$, $F : \mathfrak{M} \longrightarrow \mathfrak{L}$ the corresponding reflector. Then $\mathfrak{L}$ is reflective in $\mathfrak{X}$ under the reflector $F|\mathfrak{X}$.

Proof: Let $U_2 : \mathfrak{X} \hookrightarrow \mathfrak{M}$, $U_1 : \mathfrak{L} \hookrightarrow \mathfrak{X}$ be the embeddings. $F|\mathfrak{X} = U_2 F$. For any $X \in |\mathfrak{X}|$ (hence $X \in |\mathfrak{M}|$), there exists $\mu^X \in \mathfrak{X}(X, U_2 F X)$. Whenever $Y \in |\mathfrak{L}|$, $f \in \mathfrak{X}(X, Y)$ there exists exactly one $g \in \mathfrak{L}(FX, Y)$ such that $g\mu^X = f$, hence $\mathfrak{L}$ is reflective in $\mathfrak{X}$ under the functor $F|\mathfrak{X}$ in virtue of 1.3.

1.8. Remark. In an analogous way we can define the dual concepts, the right adjoint and the coreflective subcategory. We can prove then the dual results to 1.3, 1.5, 1.7.

§ 2. Refinements.

2.1. Definition. Let $\mathfrak{X}$ be a refinement of the category $\mathfrak{L}$. We say that an object X is $\mathfrak{X}$-fine, if for any $Y \in |\mathfrak{L}|$ there is $\mathfrak{X}(X, Y) = \mathfrak{L}(X, Y)$. We say that X is $\mathfrak{X}$-coarse, if for any $Y \in |\mathfrak{L}|$ there is $\mathfrak{X}(Y, X) = \mathfrak{L}(Y, X)$. An object X is called $\mathfrak{X}$-bi-extremal if X is simultaneously $\mathfrak{X}$-fine and $\mathfrak{X}$-coarse. (The definition appears firstly in [2].)

The refinement $\mathfrak{X}$ of $\mathfrak{L}$ is called reflective (coreflective) if $\mathfrak{L}$ is a reflective (coreflective) subcategory of $\mathfrak{X}$.

2.2. Proposition: Let $\mathcal{K}$ be a reflective refinement of $\mathcal{L}$, $F: \mathcal{K} \longrightarrow \mathcal{L}$ the corresponding reflector. An object X is $\mathcal{K}$-fine if and only if $FX \overset{\mathcal{L}}{=} X$. (More precisely FX is $\mathcal{L}$-isomorphic to X.)

Proof: Let $X \overset{\mathcal{L}}{=} FX$. Whenever $Y \in |\mathcal{L}|$, there is $\mathcal{L}(X,Y) = \mathcal{L}(FX,Y) = \mathcal{K}(X,Y)$, hence X is $\mathcal{K}$-fine. Conversely, assume X is $\mathcal{K}$-fine. We have $\mathcal{L}(X,Y) = \mathcal{K}(X,Y)$ for any $Y \in |\mathcal{L}|$. But $\mathcal{K}(X,Y) = \mathcal{L}(FX,Y)$, hence by 1.1 X is $\mathcal{L}$-isomorphic to FX.

2.3. Theorem. Let $\mathcal{K}$ be an idempotently reflective refinement of $\mathcal{L}$. Then $\mathcal{K}$-fine (= the class of all objects with the property $\mathcal{K}$-fine taken like a full subcategory of $\mathcal{L}$) forms a coreflective subcategory of $\mathcal{L}$.

Proof: Let $F: \mathcal{K} \longrightarrow \mathcal{L}$ be the reflector. We shall show that $F|\mathcal{L}$ is the coreflector from $\mathcal{L}$ onto $\mathcal{K}$-fine.

Let $Y \in |\mathcal{L}|$. F is a left adjoint to the corresponding embedding, so there exist $\varphi^Y \in \mathcal{L}(FY, Y)$, $\mu^Y \in \mathcal{K}(Y, FY)$ and there is $\varphi^Y \circ \mu^Y = 1_Y$. Now we take any $f \in \mathcal{L}(X, Y)$, where X is $\mathcal{K}$-fine. In virtue of 2.2 there is $\mu^Y f \in \mathcal{L}(X, FY)$;

(i) there is $\varphi^Y \mu^Y f = 1_Y f = f$,

(ii) for any $Z \in |\mathcal{L}|$ there is $\mathcal{L}(FY, Z) \cong \mathcal{K}(Y, Z)$,

F is idempotent, so by 2.2 FY is $\mathcal{K}$-fine, hence $\mathcal{L}(FY,Z) \cong \mathcal{K}(FY,Z)$. Now we obtain from 1.1 that Y is $\mathcal{K}$-isomorphic to FY. Then $\mathcal{L}(X,Y) \cong \mathcal{L}(X,FY)$, which implies the unicity. This completes the proof.

The converse is not true in general (see further § 3).

We shall say that a family $\{f_a : X_a \longrightarrow Y\}$ in $\mathcal{L}$ inductively $\mathcal{K}$-generates Y if $g \in \mathcal{K}(Y,Z)$ belongs to $\mathcal{L}^m$ provided that all $g \circ f_a$ belong to $\mathcal{L}^m$. Similarly we define projective $\mathcal{K}$-generation.
The meaning of induction $\mathcal{K}$-stable and projection $\mathcal{K}$-stable seems to be obvious.

2.4. <u>Theorem</u>. Let $\mathcal{K}$ be a refinement of the category $\mathcal{L}$. The class of all $\mathcal{K}$-fine spaces is inductively $\mathcal{K}$-stable.

Proof: Assume that $\{f_a : X_a \longrightarrow Y\}_a$ is an inductively $\mathcal{K}$-generating family and let for any a and for any object Z: $\mathcal{L}(X_a, Z) = \mathcal{K}(X_a, Z)$. If $g \in \mathcal{K}(Y,Z)$, then for all a there is $g \circ f_a \mathcal{K}^m$, hence $g \circ f_a \in \mathcal{L}^m$, and hence $g \in \mathcal{L}^m$.

2.5. <u>Definition</u>. Let $\mathcal{L}$ be a category, $\mathcal{K}_1, \mathcal{K}_2$ two refinements of $\mathcal{L}$. We shall define the refinement $\mathcal{K}_1 \wedge \mathcal{K}_2$ by $(\mathcal{K}_1 \wedge \mathcal{K}_2)^m = \mathcal{K}_1^m \cap \mathcal{K}_2^m$. We define another refinement $\mathcal{K}_1 \vee \mathcal{K}_2$ of $\mathcal{L}$ now. The symbol

$f: X \longrightarrow Y$ will be the morphism in $\mathcal{X}_1 \vee \mathcal{X}_2$ if it is of the form $f = (f_0, \dots, f_n)$ (the finite word), such that $f_i \in \mathcal{X}_1^m \cup \mathcal{X}_2^m$ for any $i = 0, \dots, n$, the domain of f_0 is X, the range of f_n is Y and such that there are no neighbours in the word which can be composed either in $\mathcal{X}_1$ or in $\mathcal{X}_2$. The composition of two morphisms $f = (f_0, \dots, f_n)$, $g = (g_0, \dots, g_m)$ is the word $h = (h_0, \dots, h_k)$ which we obtain from the word $(f_0, \dots, f_n, g_0, \dots, g_m)$ by composing of all the neighbours in $\mathcal{X}_1$ and in $\mathcal{X}_2$ (if it is possible). Obviously $\mathcal{X}_1 \vee \mathcal{X}_2$ is a category. If we correspond to every f from $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) the word (f) we obtain the natural embedding of $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) into $\mathcal{X}_1 \vee \mathcal{X}_2$, hence $\mathcal{X}_1 \vee \mathcal{X}_2$ is the refinement of $\mathcal{L}$ containing both $\mathcal{X}_1$ and $\mathcal{X}_2$.

In a similar way we can define these operations for a family of refinements. One can easily see that these two operations form the structure of a complete lattice on the class of all refinements of $\mathcal{L}$.

2.6. <u>Proposition:</u> Let $\mathcal{X}_1, \mathcal{X}_2$ be refinements of the category $\mathcal{L}$. Then:

(i) $\mathcal{X}_1^m \subset \mathcal{X}_2^m$ implies $\mathcal{X}_1$-fine $\supset \mathcal{X}_2$-fine.

(ii) $(\mathcal{X}_1\text{-fine}) \cap (\mathcal{X}_2\text{-fine}) = (\mathcal{X}_1 \vee \mathcal{X}_2)\text{-fine}$.

Proof: (i) is evident.

(ii): from (i) it follows that $(\mathcal{K}_1\text{-}fine) \cap (\mathcal{K}_2\text{-}fine) \supset (\mathcal{K}_1 \vee \mathcal{K}_2)\text{-}fine$. Let $X \in (\mathcal{K}_1\text{-}fine) \cap (\mathcal{K}_2\text{-}fine)$, Y any object, $f = (f_0, \dots, f_n) \in (\mathcal{K}_1 \vee \mathcal{K}_2)(X, Y)$. We shall prove that the word f has only one element. Assume $n > 1$. $f \in (\mathcal{K}_1 \vee \mathcal{K}_2)^m$, the domain of f_0 is X, hence $f_0 \in \mathcal{L}^m$. So f_0 can be composed with f_1 either in $\mathcal{K}_1$ or in $\mathcal{K}_2$, which is the contradiction. So $n = 1$, hence $x \in (\mathcal{K}_1 \vee \mathcal{K}_2)\text{-}fine$.

2.7. Definition. Let $\mathcal{L}$ be a category. We shall say that the refinement $\mathcal{K}$ of $\mathcal{L}$ is f-maximal (resp. c-maximal) if for any other refinement $\mathcal{K}'$ such that $\mathcal{K}'\text{-}fine = \mathcal{K}\text{-}fine$ (resp. $\mathcal{K}'\text{-}coarse = \mathcal{K}\text{-}coarse$), we have $\mathcal{K}' \subset \mathcal{K}$.

2.8. Theorem. Let $\mathcal{K}$ be the idempotently reflective refinement of $\mathcal{L}$. Then $\mathcal{K}$ is f-maximal.

Proof: Let $F: \mathcal{K} \longrightarrow \mathcal{L}$ be the reflector, $\mathcal{K}'$ a refinement such that $\mathcal{K}'\text{-}fine = \mathcal{K}\text{-}fine$. For any object X we have $\rho^X \in \mathcal{L}(FX, X)$, $\mu^X \in \mathcal{K}(X, FX)$ such that $\rho^X \mu^X = 1_X$. Assume $f \in \mathcal{K}'(X, Y)$. There is $f\rho^X \in \mathcal{L}^m$. It follows from 2.3 that there exists exactly one $\vartheta \in \mathcal{L}(FX, FY)$ such that $\rho^Y \circ \vartheta = f \circ \rho^X$. Further we observe that $\mu^Y \circ f \circ \rho^X \in \mathcal{L}^m$ and moreover $\rho^Y \circ \mu^Y \cdot f \circ \rho^X = f \circ \rho^X$, hence $\vartheta = \mu^Y \circ f \rho^X$. Finally $\rho^Y \circ \vartheta \circ \mu^X \in \mathcal{K}^m$ and there is $\rho^Y \circ \vartheta \circ \mu^X =$

$= \varphi^Y \cdot \mu^Y \cdot f \cdot \varphi^X \cdot \mu^X = f$, hence $f \in \mathcal{X}^m$ and $\mathcal{X}$ is f -maximal.

Remark. To avoid set theoretical confusions we shall not study the class of all refinements of a given category. We restrict our attention to "relative" classes of refinements and we shall see that there will not be any lack of generality in practice. For the rest of this paragraph we shall fix the fundamental category $\mathcal{L}$ and some idempotently reflective refinement $\hat{\mathcal{L}}$ of $\mathcal{L}$. We shall denote $\mathbb{K}$ the class of all refinements $\mathcal{X}$ such that $\mathcal{L}^m \subset \mathcal{X}^m \subset \hat{\mathcal{L}}^m$. Under the symbol $\mathcal{X}_1 \vee \mathcal{X}_2$ for the elements from $\mathbb{K}$ we shall not understand exactly the refinement described in 2.5, because it need not lay in $\mathbb{K}$. To avoid this difficulty we put two morphisms $(f_0, \dots, f_m)$, $(g_0, \dots, g_m)$ from $\mathcal{X}_1 \vee \mathcal{X}_2$ identical, whenever they give, after the possible composition in $\hat{\mathcal{L}}^m$, the same word.

2.9. Theorem. $(\mathbb{K}, \wedge, \vee)$ forms a complete lattice with $0, 1$.

The proof follows immediately from 2.5 and 2.8. The role of 0 is played by $\mathcal{L}$ and the role of 1 is played by $\hat{\mathcal{L}}$.

2.10. Proposition: Suppose $\mathcal{X}$ is from $\mathbb{K}$. Let us denote

$$\mathcal{K}_m = \bigvee \{ \mathcal{K}' \mid \mathcal{K}' \in \mathbb{K} ,\ \mathcal{K}'\text{-fine} = \mathcal{K}\text{-fine} \} .$$

Then $\mathcal{K}_m$ - *fine* = $\mathcal{K}$ - *fine* and $\mathcal{K}_m$ is f -maximal in $\mathbb{K}$.

The proof of the equality is an easy consequence of 2.6, the maximality is obvious.

Suppose $\mathcal{K} \in \mathbb{K}$ and let $\mathcal{K}$ -fine form a coreflective subcategory of $\mathcal{L}$. Let F be the coreflector onto $\mathcal{K}$ -fine, $\mu^X : FX \longrightarrow X$ the corresponding morphisms (see the dual theorem to 1.3). We define the refinement $\overline{\mathcal{K}} \in \mathbb{K}$ in the following manner: For any $X, Y \in |\mathcal{L}|$ we put $\overline{\mathcal{K}}(X, Y) \subset \hat{\mathcal{L}}(X, Y)$ and $f \in \overline{\mathcal{K}}(X, Y)$ if there exists $f' \in \mathcal{L}(FX, FY)$ such that $\mu^Y \circ f' = f \mu^X$.

2.11. <u>Proposition:</u> For any $\mathcal{K} \in \mathbb{K}$ there is $\overline{\mathcal{K}} = \mathcal{K}_m$,provided $\mathcal{K}$ -fine forms a coreflective subcategory of $\mathcal{L}$.

Proof: (i) Let X, Y be any objects, $f \in \mathcal{K}(X, Y)$. FX is $\mathcal{K}$ -fine, hence $f \mu^X \in \mathcal{L}(FX, Y)$. Let us denote $f' = F(f \circ \mu^X) \in \mathcal{L}^m$. Then there is $\mu^Y \circ f' = \mu^Y \circ F(f \circ \mu^X) = f \circ \mu^X$, hence $f \in \overline{\mathcal{K}}(X, Y)$.

(ii) It suffices to prove that $\overline{\mathcal{K}}$ - *fine* = $\mathcal{K}$ - *fine* .It follows from (i) that $\overline{\mathcal{K}}$ - *fine* $\subset$ $\mathcal{K}$ - *fine* . Let $X \in \mathcal{K}$-*fine*, $f \in \overline{\mathcal{K}}(X, Y)$, there exists $f' \in \mathcal{L}(FX, FY) \cong \mathcal{L}(X, FY)$ such that $\mu^Y \circ f' = f$, hence $\overline{\mathcal{K}}$ - *fine* = $\mathcal{K}$ - *fine* , which completes the proof.

It follows immediately from this proposition:

2.12. <u>Corollary:</u> Let $\mathcal{X} \in \mathbb{K}$ and let $\mathcal{X}$-*fine* be coreflective in $\mathcal{L}$. Then $\mathcal{L}$ is idempotently reflective in $\mathcal{X}_m$.

2.13. <u>Definition and remark.</u> Assume $\mathcal{X} \in \mathbb{K}$. Let $\langle X \rangle_{\mathcal{X}}$ denote the class of all objects X' such that for any Y there is $\mathcal{X}(X, Y) \cong \mathcal{X}(X', Y)$.

We observe that $\langle X \rangle_{\mathcal{X}}$ is the class of all X' $\mathcal{X}$-isomorphic to X, hence for any $X' \in \langle X \rangle_{\mathcal{X}}$, $Y \in |\mathcal{X}|$ there is $\mathcal{X}(Y, X) \cong \mathcal{X}(Y, X')$. We denote $\mathcal{X}^*$ the category whose objects are $\langle X \rangle_{\mathcal{X}}$ and morphisms between $\langle X \rangle_{\mathcal{X}}$ and $\langle Y \rangle_{\mathcal{X}}$ will be exactly the morphisms from $\mathcal{X}(X, Y)$. It can be said that $\mathcal{X}^*$ is obtained from $\mathcal{X}$ like the factorcategory with respect to $\mathcal{X}$-isomorphisms.

Suppose that $\mathcal{X}$ is idempotently reflective, F the reflector. Then $\mathcal{X}$-*fine* is coreflective in $\mathcal{L}$, $\mathcal{X} = \mathcal{X}_m$. Let X be any object. Then for any Y there is $\mathcal{X}(X,Y) \cong \mathcal{L}(FX, FY) \cong \mathcal{X}(FX, Y)$, hence $FX \in \langle X \rangle_{\mathcal{X}}$. Then there is obviously $\langle X \rangle_{\mathcal{X}} \cap \mathcal{X}\text{-}fine = \{FX\}$ for any X. We can define $F^* : \mathcal{X}^* \longrightarrow \mathcal{X}\text{-}fine$ like $F^*(\langle X \rangle_{\mathcal{X}}) = FX$ (analogously for morphisms). One can easily see that F^* is an isomorphism of categories. (The inverse can be defined by $G(X) = \langle X \rangle_{\mathcal{X}}$ for $X \in \mathcal{X}\text{-}fine$.)

Let us decompose the general categorical embedding as in 1.6 into one refinement and one full embedding. Let $\mathcal{L}$, $\mathcal{K}$, $\mathcal{M}$ be as in 1.6. It follows from 1.5 that whenever $\mathcal{L}$ is idempotently reflective in $\mathcal{K}$ and $\mathcal{K}$ is reflective in $\mathcal{M}$, then $\mathcal{L}$ is idempotently reflective in $\mathcal{M}$. Now we are able to prove the converse:

2.14. Theorem. Let $\mathcal{L}, \mathcal{K}, \mathcal{M}$ be as in 1.6 and let $\mathcal{L}$ be idempotently reflective in $\mathcal{M}$ under the reflector $F : \mathcal{M} \longrightarrow \mathcal{L}$. Then F can be decomposed into two reflectors $F_1 : \mathcal{K} \longrightarrow \mathcal{L}$, $F_2 : \mathcal{M} \longrightarrow \mathcal{K}$, $F = F_1 F_2$.

Proof: Let $U_1 : \mathcal{L} \hookrightarrow \mathcal{K}$, $U_2 : \mathcal{K} \hookrightarrow \mathcal{M}$ be the embeddings. In virtue of 1.7 $\mathcal{L}$ is reflective in $\mathcal{K}$ under the functor $F_1 = U_2 F$. Hence $\mathcal{K} = \mathcal{K}_{\mathcal{M}}$. Let $X \in |\mathcal{M}|$, we define:

$$F_2(X) = \begin{cases} X & \text{for } X \in |\mathcal{K}| \\ F(X) & \text{for } X \notin |\mathcal{K}| \end{cases}$$

If $X \in |\mathcal{K}|$, then $F(X) \in \langle X \rangle_{\mathcal{K}}$, so there exist $\mu^X : FX \longrightarrow X$, $\varphi^X : X \longrightarrow FX$, both $\mathcal{K}$-isomorphisms.

For $f \in \mathcal{M}(X, Y)$ we define:

$$F_2(f) = \begin{cases} f & \text{for } X, Y \in |\mathcal{K}| \\ F(f) & \text{for } X, Y \notin |\mathcal{K}| \\ \varphi^Y f & \text{for } X \in |\mathcal{K}|, Y \notin |\mathcal{K}| \\ \mu^X f & \text{for } X \notin |\mathcal{K}|, Y \in |\mathcal{K}| \end{cases}$$

There is $\mu^Y \circ \varphi^Y = 1_Y$, $\varphi^Y \mu^Y = 1_{FY}$ for any

$Y \in |\mathcal{X}|$. So we obtain that F_2 is a functor from $\mathcal{M}$ into $\mathcal{X}$. Obviously $F_2 U_2 F_2 = F_2$. For any $X \in |\mathcal{M}|$; $Y \in |\mathcal{X}|$ there is:

$$(F_2 X, Y) = \begin{cases} \mathcal{X}(X,Y) = \mathcal{M}(X,Y) & \text{for } X \in |\mathcal{X}| \\ \mathcal{X}(FX,Y) = \mathcal{L}(FX,Y) \cong \mathcal{X}(X,Y) = \mathcal{M}(X,Y) & \end{cases}$$

for $X \notin |\mathcal{X}|$.

Obviously $F_1 F_2 = F$ and the proof is complete.

2.15. <u>Theorem.</u> Let $\mathcal{M}$ be a refinement of $\mathcal{L}$. Let $\mathcal{L}$ be complete, cocomplete, locally and colocally small. Let the canonical morphisms into coproducts remain collective epimorphisms also in $\mathcal{M}$ and the canonical projections from products be collective monomorphisms in $\mathcal{M}$. Further we suppose that for any $f, g \in \mathcal{L}^m$ the equalizer of the pair (f, g) in $\mathcal{L}$ is a monomorphism in $\mathcal{M}$ and the coequalizer of (f, g) is an epimorphism in $\mathcal{M}$. Then for any refinement $\mathcal{X}$ such that $\mathcal{L}^m \subset \mathcal{X}^m \subset \mathcal{M}^m$ there is:

(i) $\mathcal{X}$-*fine* is monocoreflective in $\mathcal{L}$,

(ii) $\mathcal{X}$-*coarse* is epireflective in $\mathcal{L}$.

Proof: To prove the theorem we shall use the criterion of monocoreflectivity stated in [5]. From there it suffices to prove that $\mathcal{X}$-*fine* is closed under coproducts and coequalizers of diagrams $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ where $Y \in \mathcal{X}$-*fine* .

Let $\{X_a\}$ be the family of objects in $\mathcal{X}$-*fine* ,

$\{X_a \xrightarrow{\nu_a} \Sigma X_a\}$ their coproduct in $\mathcal{L}$. Suppose $Y \in |\mathcal{L}|$, $f \in \mathcal{K}(\Sigma X_a, Y)$. For all indices a there is $f\nu_a \in \mathcal{L}(X_a, Y)$, hence there exists exactly one $f' \in \mathcal{L}(\Sigma X_a, Y)$ such that $f'\nu_a = f\nu_a$ for all a. By the assumption $\{\nu_a\}$ is a collective epimorphism, hence $f' = f$ and hence $f \in \mathcal{L}^m$.

Suppose the diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{L}$ such that $Y \in \mathcal{K}\text{-}fine$.

Let $Y \xrightarrow{c} C$ denote the coequalizer of (f, g) in $\mathcal{L}$. For $Z \in |\mathcal{L}|$, $h \in \mathcal{K}(C, Z)$ we have $hc \in \mathcal{L}(Y, Z)$. Of course, $hcf = hcg$, hence there exists exactly one $h' \in \mathcal{L}(C, Z)$ such that $h'c = hc$. But c is an epimorphism also in $\mathcal{K}$, hence $h' = h$ and hence $h \in \mathcal{L}^m$. In a similar way we can prove the dual result (ii).

We shall see later on that the assumptions of the foregoing theorem are often fulfilled. The following theorem is an easy consequence of 2.12 and 2.13. (We use the notation from the remark before the theorem 2.9.)

2.16. <u>Proposition:</u> Let for any $\mathcal{K} \in \mathbb{K}$ the corresponding full subcategory $\mathcal{K}\text{-}fine$ of the category $\mathcal{L}$ be coreflective. The following is equivalent:

(i) $\mathcal{K} = \mathcal{K}_m$

(ii) $\mathcal{K}$ is an idempotently reflective refinement of $\mathcal{L}$

(iii) For any $X \in |\mathcal{L}|$ the coreflection $\mathcal{K}^f X$ of X in the subcategory $\mathcal{K}$-*fine* is $\mathcal{K}$-isomorphic to X.

(iv) For any $X, Y \in |\mathcal{L}|$, $f \in \mathcal{K}(X, Y)$ if and only if $f = f'k$, where $f' \in \mathcal{L}^m$ and k is a $\mathcal{K}$-isomorphism.

2.17. <u>Definition.</u> Let $\mathcal{K}$ be a refinement of $\mathcal{L}$, M any class of objects. We define $\mathcal{K}\text{-fine}|_M$ the class of such objects X that whenever $Y \in M$, then $\mathcal{K}(X, Y) = \mathcal{L}(X, Y)$. In an analogous way we can define $\mathcal{K}\text{-coarse}|_M$. Thus in the special case of $M = |\mathcal{L}|$ we obtain the definition 2.1.

2.18. <u>Theorem.</u> Let $\mathcal{K}$ be a refinement of $\mathcal{L}$, M any class of objects. There exists a refinement $\mathcal{M}$ of $\mathcal{L}$, $\mathcal{M} \subset \mathcal{K}$ such that $\mathcal{M}\text{-fine} = \mathcal{K}\text{-fine}|_M$.

Proof: At first we define the class of morphisms $\mathcal{M}' \subset \mathcal{K}^m$. For any X, Y we put

$$\mathcal{M}'(X, Y) = \begin{cases} \mathcal{K}(X, Y) & \text{for } Y \in M \\ \mathcal{L}(X, Y) & \text{for } Y \notin M \end{cases}$$

Now we define the refinement $\mathcal{M}$: $f \in \mathcal{M}(X, Y)$ if there exist $f_0, f_1, \ldots, f_n \in \mathcal{M}'$, the domain of f_n being X, the range of f_0 being Y and $f = f_0 f_1 \ldots f_n$, the composition being made in $\mathcal{K}$. Let $X \in \mathcal{K}\text{-fine}|_M$, Z any object, $f \in \mathcal{M}(X, Z)$. There exist $f_0 \ldots f_n$ all in $\mathcal{M}'$, f_0 ranges in Z, f_n has its domain in X,

$f = f_0 \dots f_m$. Let us denote Z_i the domain of f_i .

1. There is — either $Z \in M$, then $f \in \mathcal{L}^m$.
— or $Z \notin M$, then $f_0 \in \mathcal{L}(Z_0, Z)$

2. There is — either $Z_0 \in M$, then $f_1 \dots f_m \in \mathcal{L}^m$, hence $f \in \mathcal{L}^m$.
— or $Z_0 \notin M$, then $f_1 \in \mathcal{L}(Z_1, Z_0)$, hence $f_0 f_1 \in \mathcal{L}^m$.

Obviously we obtain by induction that $f \in \mathcal{L}(X, Z)$, hence $X \in \mathcal{M}\text{-fine}$.

Conversely let $X \in \mathcal{M}\text{-fine}$, $Y \in M$, then $\mathcal{K}(X, Y) = \mathcal{M}(X, Y) = \mathcal{L}(X, Y)$, hence $X \in \mathcal{K}\text{-fine}|_M$.

<u>Remarks</u>. From the last theorem we can conclude that the theorem 2.15 can be generalized to the subcategories of type $\mathcal{K}\text{-fine}|_M$, M being any class of objects.

We can also obtain the dual results, substituting respectively $\mathcal{K}\text{-coarse}|_M$, coreflective, reflective, projective, c -maximal,... instead of $\mathcal{K}\text{-fine}|_M$, reflective, coreflective, inductive, f -maximal.

We shall show now that in the main theorems of this paragraph 2.8 and 2.14 the assumption of the idempotency of the reflector cannot be omitted.

2.19. <u>Example</u>: Assume the category LCS of the Hausdorff locally convex topological vactor spaces over the field of real numbers R and continuous linear mappings. On every such space we have the unique translation invariant uniformity and the continuous linear mappings are in these uniformities uniformly continuous. In this way we have given the embedding J of the category LCS into the category $\mathfrak{U}$ of separated uniform spaces and uniformly continuous mappings. Rajkov has proved in [10] that for any uniform space there exists a free locally convex space, in another words that the category LCS is a reflective subcategory of $\mathfrak{U}$. The construction is made by giving a suitable topology on the free-vector space over the underlying set of the given uniform space. Let us denote F this reflector and $LCS_{\mathfrak{U}}$ the refinement of LCS given by the embedding J. From 1.7 we see that $LCS_{\mathfrak{U}}$ is a reflective refinement of LCS (but not idempotently reflective). One can immediately see that $LCS_{\mathfrak{U}}$-*fine* $= \{0\}$. (the category involving only the null-dimensional space). Now it is evident that $LCS_{\mathfrak{U}}$ is not f-maximal. It suffices to take the refinement $\mathcal{S}$ generated by all mappings which is actually greater than $LCS_{\mathfrak{U}}$ and still $\mathcal{S}$-*fine* $= \{0\}$. Thus the theorem 2.8 does not hold generally for all reflective refinements.

The same example can serve us for the answer to the question about the general validity of the decomposition theorem 2.14. Suppose that the reflector $F : \mathfrak{U} \longrightarrow LCS$ can be decomposed into two reflectors $F_1 : LCS_{\mathfrak{U}} \longrightarrow LCS$, $F_2 : \mathfrak{U} \longrightarrow LCS_{\mathfrak{U}}$. From 1.7 we know that F_1 is naturally equivalent to the functor $F|_{LCS_{\mathfrak{U}}}$. Let $\mathfrak{D}$

denote the twopoint uniform space. Then $F\mathcal{D}$ is the two-dimensional space R^2. Suppose $\dim F_2\mathcal{D} = 0$. Then $F_1 F_2 \mathcal{D}$ is at most onedimensional, hence $F\mathcal{D} \neq F_1 F_2 \mathcal{D}$. On the other hand if the dimension of the space $F_2\mathcal{D}$ is at least 1, then $F_1 F_2 \mathcal{D}$ is of infinite dimension, which is the contradiction. Hence, the idempotency in 2.14 cannot be omitted.

Thus we have seen that the idempotent reflections are of great importance for the theory of the preceding paragraph. But in the practical cases of concrete categories the condition is often fulfilled, as will be seen from the following easy proposition:

2.20. Proposition: Let $J : \mathcal{L} \hookrightarrow \mathcal{K}$ be the embedding of concrete categories, $F : \mathcal{K} \longrightarrow \mathcal{L}$ the reflector such that for any $X \in |\mathcal{K}|$ the reflection mapping $\mu^X : X \longrightarrow FX$ can be represented by the identity mapping. Then F is idempotent. (Analogously the dual proposition.)

§ 3. Concrete refinements.

3.1. Definition. Let $\mathcal{L}$ be a concrete category, $\square$ the forgetful functor into the category of sets. We shall denote $Set_{\mathcal{L}}$ the refinement of $\mathcal{L}$, where $Set_{\mathcal{L}}(X, Y)$ consists of all triples $\langle f, X, Y \rangle$ written $f : X \longrightarrow Y$, f being a mapping of $\square X$ into $\square Y$. As usual we shall regard $\mathcal{L}$ to be a subcategory of $Set_{\mathcal{L}}$.

Let $\mathcal{K}$ be a refinement of $\mathcal{L}$. We shall call it concrete refinement, if $\mathcal{L}^m \subset \mathcal{K}^m \subset Set_{\mathcal{L}}^m$.

3.2. Theorem. Let $\mathcal{L}$ be concrete, complete, locally and colocally small category. Let for any family X_a of objects hold : $\square(\sum_{\mathcal{L}} X_a) = \sum_{Set} \square X_a$, $\square(\prod_{\mathcal{L}} X_a) = \prod_{Set} \square X_a$. Let further each equalizer in $\mathcal{L}$ be one to one and each coequalizer be onto. Let $\mathcal{K}$ be a concrete refinement of $\mathcal{L}$, then $\mathcal{K}$-*fine* is monocoreflective in $\mathcal{L}$, $\mathcal{K}$-*coarse* is epireflective in $\mathcal{L}$.

Proof: follows immediately from 2.15.

There is a lot of categories fulfilling the assumptions of the foregoing theorem. For example TOP (the category of topological spaces and continuous mappings), HAUS (the category of Hausdorff topological spaces), CR (separated comppetely regular spaces), Unif (the category of uniform spaces and uniformly continuous mappings), U (of separated uniform spaces) and others.

We shall treat the category $\mathbb{U}$ of separated uniform spaces and uniformly continuous mappings. $\mathbb{U}$ fulfils the assumptions of the theorem 3.2, hence for any concrete refinement $\mathcal{K}$ there exist two functors $\mathcal{K}^f$, $\mathcal{K}^c$, the former being the monocoreflector onto $\mathcal{K}$-*fine* , the latter being the epireflector onto $\mathcal{K}$-*coarse* . Kennison [7],[8] has proved that in $\mathbb{U}$ every nontrivial coreflection is a comodification (the coreflector preserves the underlying sets).

3.3. Theorem. Let $\mathcal{K}$ be a concrete refinement of

$\mathcal{U}$. The following holds:

(1) $\mathcal{X}$-*fine* is coreflective in $\mathcal{U}$

(2) $\mathcal{X}$-*coarse* is epireflective and hereditary in $\mathcal{U}$.

(3) Every nontrivial coreflection in $\mathcal{U}$ is of the form $\mathcal{X}$-*fine*, where $\mathcal{X}$ is a concrete refinement.

(4) Every hereditary epireflection in $\mathcal{U}$ is of the form $\mathcal{X}$-*coarse*, where $\mathcal{X}$ is a concrete refinement.

Proof: (1) and (2) are obvious from 3.2.

(3) Let F be the coreflector in $\mathcal{U}$. Let for any X the identical mapping $\mu^X \in \mathcal{U}(FX, Y)$ represent the coreflection. We define the concrete refinement $\mathcal{X}$ of $\mathcal{U}$: for any $X, Y \in |\mathcal{U}|$, $f \in \mathcal{X}(X, Y)$ if there exists $g \in \mathcal{U}(FX, Y)$ such that $g\mu^X = \mu^Y f$. In other words if f is uniformly continuous from FX into Y. It is easy to see that the functors $\mathcal{X}^f$, F are equivalent.

(4) Let $\mathcal{R}$ be epireflective and hereditary in $\mathcal{U}$ under the reflector F. Suppose for any X the morphism $\mu^X \in \mathcal{U}(X, FX)$ represent the reflection. Again we put $f \in \mathcal{X}(X, Y)$ if there exists $g \in \mathcal{U}(FX, FY)$ such that $\mu^Y f = g\mu^X$, or, equivalently, if $\mu^Y f$ is uniformly continuous. One can immediately see that $\mathcal{R} \subset \mathcal{X}$-*coarse*. Conversely, let $X \in \mathcal{X}$-*coarse*. μ^X is an epimorphism, hence the image of X in FX under μ^X is dense. But every uniform subspace of FX is again in $\mathcal{R}$, hence μ^X must be onto. Suppose that μ^X is not one to one. Then there exist two distinct points x, y in X such that $\mu^X(x) = \mu^X(y)$. Let Q denote the space of rational

numbers with the usual metric uniformity, d any irrational number. We define the mapping $f : Q \longrightarrow X$. For $a < d$ we put $f(a) = x$, for $a > d$ we put $f(a) = y$. Clearly f is not uniformly continuous, but $\mu^X f$ is a constant, hence the uniformly continuous mapping, which is a contradiction. Hence μ^X is bijective. The rest is obvious, similarly as in (3).

Remarks. One can easily see that $Set_{\mathcal{U}}$ is a (idempotently) reflective refinement, but not coreflective refinement of $\mathcal{U}$. Hence for any concrete refinement $\mathcal{X}$ its f-maximal hull $\mathcal{X}_m$ must be again concrete and may be constructed by the method used in the proof of (3) in the foregoing theorem.

For the dual case, if we want construct c-maximal hulls for concrete refinements, it need not exist in general. For example, $Set_{\mathcal{U}}$-coarse is a singleton and one can see that there exists no c-maximal refinement generating it. But if $\mathcal{X}$ is a concrete refinement such that the corresponding reflector $\mathcal{X}^c$ is a modification, then there exists a c-maximal hull of $\mathcal{X}$, it is concrete and may be constructed by the method used in the proof of (4) in the theorem 3.3.

We shall show that the assumption of concreteness in the theorems 3.2 and 3.3 cannot be omitted. We give an example of a refinement $\mathcal{X}$ of $\mathcal{U}$ such that $\mathcal{X}$-fine need not be coreflective.

3.4. Example: We shall represent the objects in $\mathcal{U}$

in the form $(X, \mathcal{U})$, where $\mathcal{U}$ is a system of all vicinities of the diagonal. We define the refinement Φ of U in the following manner: We put $f \in \Phi((X,\mathcal{U}), (Y,\mathcal{V})$, if f is a mapping from $X \times X$ into $Y \times Y$ such that for any $V \in \mathcal{V}$ there is $f^{-1}(V) \in \mathcal{U}$. (Hence $f : (X \times X, \mathcal{U}) \longrightarrow (Y \times Y, \mathcal{V})$ is a morphism in the category $S(P^-)$.) We define the composition in Φ in a natural way; obviously we obtain the category. The embedding of U into Φ may be defined, if we correspond to each $f \in \mathrm{U}^m$ the cartesian power f^2. So g is from U^m iff there exists $f \in \mathrm{U}^m$ such that $g = f^2$.

(i) At first we show that the onepoint space $\{a\}$ is Φ-fine. Let $(Y, \mathcal{V})$ be any uniform space. There is $\bigcap\{V \mid V \in \mathcal{V}\} = \Delta_Y$ ... the diagonal, hence for any $f \in \Phi(\{a\}, (Y, \mathcal{V}))$ there is $f(a,a) = (x,x)$ for some $x \in Y$. It suffices to define $g : g(a) = x$, $g \in \mathrm{U}(\{a\}, (Y, \mathcal{V}))$ and $g \times g = f$.

(ii) Let $\{a, b\}$ be the twopoint discrete space. We show that $\{a,b\}$ is not Φ-fine. We define $f \in \Phi(\{a, b\}, \{a, b\})$ in the following way:

$f(a,a) = (b,b)$; $f(b,b) = (a,a)$; $f(a,b) = (a,b)$; $f(b,a) = (b,a)$.

Obviously f cannot be obtained as $g \times g$ for some mapping $g : \{a,b\} \longrightarrow \{a,b\}$.

Let us suppose that Φ-fine is coreflective in U. It follows from (i) that the coreflection is not trivial. We notice that in that case the coreflective subcategory must contain all uniformly discrete spaces, which is the contradiction with (ii).

We have noticed that $\mathcal{S}et_{\mathcal{U}}$ is a reflective refinement, but not coreflective. The reason for it is the fact that there exists the finest uniformity on every set, but not the coarsest. If we take instead of $\mathcal{U}$ the category $\mathcal{U}nif$ of all (nonseparated) uniform spaces, then $\mathcal{S}et_{\mathcal{U}nif}$ is simultaneously reflective and coreflective. Despite of the nonexistence of the coarsest separated uniformity on every set, we show that there exists the coarsest modification $\mathcal{R}$ in $\mathcal{U}$ in the following sense: For any other modification $\mathcal{R}'$ there is $\mathcal{R} \subset \mathcal{R}'$.

3.5. Theorem. Let $\mathcal{R}$ be any modification in $\mathcal{U}$. Then $\mathcal{P}recomp \subset \mathcal{R}$, where $\mathcal{P}recomp$ denotes the epireflective subcategory of $\mathcal{U}$ generated by all precompact uniform spaces. ($\mathcal{P}recomp$ is a modification.)

Proof: Let I denote the compact unit interval. Suppose there exists the uniform space I' on the same underlying set and coarser than I. Hence the identity mapping $i : I \longrightarrow I'$ is uniformly continuous, hence it is a topological homeomorphism. But there exists only one uniformity on I compatible with the corresponding topology, hence $I = I'$.
Suppose $\mathcal{R}$ is any modification, we obtained that $I \in \mathcal{R}$. Hence the projective hull of $\{I\}$ in $\mathcal{U}$ must be contained in $\mathcal{R}$. But the projective hull of $\{I\}$ is $\mathcal{P}recomp$ and the proof is complete.

3.6. Examples: Let $\mathbf{C}$ (resp. $\mathbf{P}$) denote the refinement generated by all continuous (resp. proximally continuous) mappings. Then:

a) $\mathit{Set}_{\mathcal{U}}$-*fine* = *Discr* (the category of all uniformky discrete spaces)

b) $\mathit{Set}_{\mathcal{U}}$-*coarse* = $\{0\}$ (the singleton)

c) C-*fine* = *Fine* (the category of all fine spaces)

d) C-*coarse* = $\{0\}$

e) P-*fine* = PF (the category of all proximally fine spaces, i.e. the finest spaces generating the corresponding proximity)

f) P-*coarse* = *Precomp*

g) $\mathit{Set}_{\mathcal{U}}$, C are idempotently reflective refinements (f -maximal), P is an idempotently coreflective refinement (c -maximal).

Proof: a), b), c) are obvious.

d) Suppose there exists at least a twopoint space $X \in C$ -coarse. We choose any twopoint subspace $\{a, b\}$ of X. Let Q denote the space of rational numbers with its metric uniformity, z any fixed irrational number. We define $f(x) = a$ for $a < z$ and $f(x) = b$ for $a > z$ the mapping from Q into X. Obviously f is continuous but not uniformly continuous.

For e) we refer to 1 .

f), g) are obvious.

For further interesting examples we refer to [2],[3], [4].

3.7. Proposition: Let $\mathcal{K}$ be a concrete refinement of $\mathcal{U}$, M the class of spaces projectively generating $\mathcal{U}$ (i.e. for any $Y \in |\mathcal{U}|$ there exists a family $\{M_a\}$ of spaces from M such that Y can be uniformly embedded into the uniform product $\prod M_a$. Then $\mathcal{K}\text{-fine} = \mathcal{K}\text{-fine}|_M$.

Proof: Obviously $\mathcal{K}\text{-fine} \subset \mathcal{K}\text{-fine}|_M$.
Let $X \in \mathcal{K}\text{-fine}|_M$, Z any uniform space. There exist $M_a \in M$ such that Z is embeddable into $\prod M_a$. Let us denote $j : Z \hookrightarrow \prod M_a$ the embedding, $\pi_a : \prod M_a \longrightarrow M_a$ the projections. Let $f \in \mathcal{K}(X, Z)$. For all a we have $\pi_a j f \in \mathcal{K}(Z, M_a) = \mathcal{U}(Z, M_a)$, hence there exists exactly one $g \in \mathcal{U}(X, \prod M_a)$ such that for all a there is $\pi_a g = \pi_a j f$. Hence there is $g = jf$, and hence f is uniformly continuous.

Remark. Such classes M fulfilling the assumptions of the foregoing theorem are for example $\mathcal{M}$ (all metric spaces), $\gamma\mathcal{M}$ (complete metric spaces), Γ (complete spaces), $\mathcal{I}nj$ (injective uniform spaces). We shall use the current principle of the construction of the coreflector $\mathcal{K}^f$ onto $\mathcal{K}$-fine spaces in $\mathcal{U}$.

3.8. Let $\mathcal{K}$ be a concrete refinement of $\mathcal{U}$, X any uniform space. We define $X^{(1)}$ the set X endowed with the uniformity projectively generated by all $\mathcal{K}$-morphisms with the domain X and ranging in metric spaces. One can easily see that this corresponding is functorial and for $X \in \mathcal{K}$-fine there is $X^{(1)} = X$. Now we define by transfinite induction: $X^{(\alpha+1)} = (X^{(\alpha)})^{(1)}$ for α nonlimit ordinal, for a limit ordinal β we define $X^{(\beta)}$ the set X endowed with the uniformity equal to the infimum (in the order "finer than") of all foregoing uniformities. Obviously there exists an ordinal $\gamma = \min\{\alpha \mid X^{(\alpha+1)} = X^{(\alpha)}\}$.

<u>Theorem.</u> $X^{(\gamma)} = \mathcal{K}^f X$.

The proof is routine by transfinite induction.

The following proposition is a consequence of 2.16 and 3.8.

3.9. <u>Proposition:</u> Let $\mathcal{K}$ be a concrete refinement of $\mathcal{U}$. The following conditions are equivalent:

(i) $\mathcal{K}$ is f-maximal.

(ii) $\mathcal{K}$ is an idempotently reflective refinement of $\mathcal{U}$.

(iii) For any X: $\langle X \rangle_{\mathcal{K}}$ contains $X^{(1)}$.

(iv) For any X: $\langle X \rangle_{\mathcal{K}}$ has the finest element equal to $X^{(1)}$.

(v) For any X: $\langle X \rangle_{\mathcal{K}}$ contains $\mathcal{K}^f X$.

(vi) For any X: $\langle X \rangle_{\mathcal{K}}$ has the finest element equal to $\mathcal{K}^f X$.

(vii) For any X: $\mathcal{K}^f X = X^{(1)}$.

<u>Remark.</u> Let $P \in \mathcal{U}$, $\mathcal{D}$ the system of all uniform vicinities of the diagonal. For any $U \in \mathcal{D}$ we set: $Q_U : \Sigma\{D_{x_U} \mid x_U \in U \setminus \Delta_P$, D_{x_U} is a twopoint space $\{a_{x_U}, b_{x_U}\}$ with the indiscrete uniformity$\} \vee \Sigma\{D_{x_U} \mid$

$x_{\mathfrak{U}} \in P \times P \setminus \mathfrak{U}$, $D_{x_{\mathfrak{U}}} = \{a_{x_{\mathfrak{U}}}, b_{x_{\mathfrak{U}}}\}$ with the discrete uniformity $\}$, Δ_P being the diagonal in $P \times P$, the sum is taken in the category $\mathcal{Unif}$.
Let us denote $(Q, \mathcal{W}) = \inf\{Q_{\mathfrak{U}} ; \mathfrak{U} \in \mathcal{D}\}$. (The infimum is taken in the order "finer than".) It follows from [1], 37.A.8 that $(P, \mathcal{D})$ is a quotient of $(Q, \mathcal{W})$. We notice that $\bigcap \mathcal{D} = \Delta_P$, hence $(Q, \mathcal{W})$ is an element from $\mathfrak{U}$.

I thank for the idea of the following lemma to M. Hušek.

3.10. Lemma. $(Q, \mathcal{W})$ is the minimal object in $\langle (Q, \mathcal{W}) \rangle$ (with respect to the order "finer than").

Proof: Let $(Q, \mathcal{V})$ be a strictly finer uniform space that $(Q, \mathcal{W})$. For any $\mathfrak{U} \in \mathcal{D}$ we have at least one $x_{\mathfrak{U}} \in \mathfrak{U} \setminus \Delta_P$ such that the sets $\{a_{x_{\mathfrak{U}}}\}$, $\{b_{x_{\mathfrak{U}}}\}$ are not $\mathcal{V}$-proximal. Of course, they are proximal in $Q_{\mathfrak{U}}$. We set $A = \{a_{x_{\mathfrak{U}}} ; \mathfrak{U} \in \mathcal{D}\}$, $B = \{b_{x_{\mathfrak{U}}} ; \mathfrak{U} \in \mathcal{D}\}$. We can suppose that $(Q, \mathcal{W})$ is not uniformly discrete (because in this case $(Q, \mathcal{W})$ is even proximally fine). Let $\mathcal{U}$ be any uniform cover of $(Q, \mathcal{W})$. Then it must be $\{a_{x_{\mathfrak{U}}}\}$, $\{b_{x_{\mathfrak{U}}}\}$ proximal (with respect to $\mathcal{U}$) for some $\mathfrak{U} \in \mathcal{D}$. Hence the sets A, B are proximal in the uniformity $\mathcal{W}$. On the other hand, one can see from the construction of A, B that A, B are not proximal in the uniformity $\mathcal{V}$, and the proof is complete.

3.11. Theorem. Let $\mathcal{C}$ be a coreflective subcategory of $\mathcal{U}$ and let the corresponding coreflector c preserve the proximity (i.e. for any $X \in |\mathcal{U}|$ we have $pcX = pX$). Then $\mathcal{C} = \mathcal{U}$.

Proof: For any $X \in |\mathcal{U}|$ we construct Q_X , the space $(Q, \mathcal{W})$ from the foregoing lemma. $Q_X \in |\mathcal{C}|$ because of the minimality in $\langle Q_X \rangle_p$. Thus X , being the quotient of Q_X is again in $|\mathcal{C}|$, hence $\mathcal{C} = \mathcal{U}$.

3.12. Corollary. There are no concrete refinements of $\mathcal{U}$ simultaneously reflective and coreflective. (Except of the trivial refinement $\mathcal{U}$.)

Proof: Suppose $\mathcal{X}$ is simultaneously reflective and coreflective refinement of $\mathcal{U}$. From 3.5 $\mathcal{X}$-coarse must contain all precompact spaces, hence $\mathcal{X}$-fine preserves proximity and hence $\mathcal{X}$-fine $= \mathcal{U}$.

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