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OSCILLATION PROPERTIES OF SECOND-ORDER QUASILINEAR
DIFFERENCE EQUATIONS WITH UNBOUNDED DELAY
AND ADVANCED NEUTRAL TERMS

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Abstract. We obtain some new sufficient conditions for the oscillation of the solutions of the second-order quasilinear difference equations with delay and advanced neutral terms. The results established in this paper are applicable to equations whose neutral coefficients are unbounded. Thus, the results obtained here are new and complement some known results reported in the literature. Examples are also given to illustrate the applicability and strength of the obtained conditions over the known ones.

Keywords: oscillation; quasilinear difference equation; delay and advanced neutral terms

MSC 2020: 39A10

1. INTRODUCTION

We are dealing with the oscillatory properties of solutions of a second-order quasilinear difference equation with delay and advanced neutral terms of the form

$$(1.1) \quad \Delta(\zeta(i)(\Delta\chi(i))^\alpha) + \varrho(i)\psi^\beta(\sigma(i)) = 0, \quad i \geq i_0 > 0,$$

where $\chi(i) = \psi(i) + \varrho_1(i)\psi(i - \kappa) + \varrho_2(i)\psi(i + l)$, subject to the following conditions:

(C₁) $\{\zeta(i)\}$ and $\{\varrho(i)\}$ are real positive sequences with $\sum_{i=i_0}^{\infty} \zeta^{-1/\alpha}(i) = \infty$;

(C₂) α, β are ratios of odd positive integers and l and κ are positive integers;

(C₃) $\{\sigma(i)\}$ is a sequence of integers and $\lim_{i \rightarrow \infty} \sigma(i) = \infty$;

- (C₄) $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are real sequences with $\varrho_1(i) \geq 0$, $\varrho_2(i) \geq 1$, and $\varrho_2(i) \neq 1$ eventually;
- (C₅) $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are real sequences with $\varrho_2(i) \geq 0$, $\varrho_1(i) \geq 1$, and $\varrho_1(i) \neq 1$ eventually.

We say a real sequence $\{\psi(i)\}$ is a solution of (1.1) if it is defined and satisfies (1.1) for all $i \geq i_0$. We consider only those solutions of $\{\psi(i)\}$ of (1.1) that satisfy $\sup\{|\psi(i)| : i \geq N\} > 0$ for all $N \geq i_0$; moreover, we assume tacitly that (1.1) possesses such solutions. Such a solution $\{\psi(i)\}$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. Equation (1.1) is said to be oscillatory if all solutions of (1.1) are oscillatory.

The problem of oscillation and asymptotic behavior solutions to various classes of delay and advanced type neutral difference equations have been widely investigated in the literature, see for example [1], [2], [4], [5], [9], [10], [13], [14], [16], [17], [21]–[27] and the references cited therein. However, oscillation results for mixed type neutral difference equations are relatively scarce in the literature; some results can be found, for example, in [3], [6], [7], [8], [11], [12], [15], [19], [18], [20] and the references cited therein.

From the review of literature, we note that results obtained in [3], [6], [7], [8], [11], [12], [15], [18], [19], [20] require both of $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ to be constant or bounded sequences, and hence, the results established in these papers cannot be applied to the cases where $\lim_{i \rightarrow \infty} \varrho_1(i) = \infty$ and/or $\lim_{i \rightarrow \infty} \varrho_2(i) = \infty$. Motivated by this observation, we wish to develop new sufficient conditions which can be applied to the cases where $\lim_{i \rightarrow \infty} \varrho_1(i) = \infty$ and/or $\lim_{i \rightarrow \infty} \varrho_2(i) = \infty$. Therefore, the results obtained in the present paper are new and complement some existing results in the literature. Thus, we hope that the present paper will contribute significantly to the study of oscillation of the solutions of the second-order mixed type neutral difference equations.

2. AUXILIARY LEMMAS

In this section, we present some lemmas that will play a significant role in establishing our main results. For the sake of convenience, we define the following notation:

$$F(i) = \sum_{s=N}^{i-1} \zeta^{-1/\alpha}(s), \quad \xi(i) = \frac{1}{\varrho_2(i-l)} \left(1 - \frac{1}{\varrho_2(i-2l)} - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)} \right) > 0,$$

$$\varphi(i) = \frac{1}{\varrho_1(i+\kappa)} \left(1 - \frac{1}{\varrho_1(i+2\kappa)} \frac{F(i+2\kappa)}{F(i+\kappa)} - \frac{\varrho_2(i+\kappa)}{\varrho_1(i+2\kappa+l)} \frac{F(i+2\kappa+l)}{F(i+\kappa)} \right) > 0,$$

for any $N \geq i_0$ and for all sufficiently large i .

Lemma 2.1 ([28]). *If $E > 0$, $D \geq 0$ and $\alpha > 0$, then*

$$Du - Eu^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha + 1)^{\alpha+1}} \frac{D^{\alpha+1}}{E^\alpha},$$

where equality holds if and only if $D = E$.

Lemma 2.2. *Assume that (C_1) – (C_4) (or (C_1) – (C_3) and (C_5)) hold, and let $\{\psi(i)\}$ be an eventually positive solution of (1.1). Then there is an integer $i_1 \geq i_0$ such that, for $i \geq i_1$,*

$$(2.1) \quad \chi(i) > 0, \Delta\chi(i) > 0 \quad \text{and} \quad \Delta(\zeta(i)(\Delta\chi(i))^\alpha) < 0.$$

Proof. The proof is standard and so we omit the details. □

Lemma 2.3. *Assume that (C_1) – (C_4) (or (C_1) – (C_3) and (C_5)) hold, and $\{\psi(i)\}$ is a positive solution of (1.1) such that (2.1) holds. Then*

$$(2.2) \quad \chi(i) \geq F(i)\zeta^{1/\alpha}(i)\Delta\chi(i)$$

and

$$(2.3) \quad \left\{ \frac{\chi(i)}{F(i)} \right\} \text{ is decreasing for all } i \geq N \geq i_1.$$

Proof. Since $\{\zeta(i)(\Delta\chi(i))^\alpha\}$ is decreasing for all $i \geq i_1$, we have

$$\chi(i) = \chi(i_1) + \sum_{s=i_1}^{i-1} \frac{(\zeta(s)(\Delta\chi(s))^\alpha)^{1/\alpha}}{\zeta^{1/\alpha}(s)} \geq \zeta^{1/\alpha}(i)F(i)\Delta\chi(i).$$

Furthermore,

$$\Delta\left(\frac{\chi(i)}{F(i)}\right) = \frac{1}{\zeta^{1/\alpha}(i)} \frac{F(i)\zeta^{1/\alpha}(i)\Delta\chi(i) - \chi(i)}{F(i)F(i+1)} \leq 0, \quad \text{since } \Delta F(i) = \frac{1}{\zeta^{1/\alpha}(i)}.$$

The proof is now completed. □

Lemma 2.4. *Assume that (C_1) – (C_4) hold. If $\{\psi(i)\}$ is an eventually positive solution of (1.1) such that (2.1) holds, then $\{\chi(i)\}$ satisfies the inequality*

$$(2.4) \quad \Delta(\zeta(i)(\Delta\chi(i))^\alpha) + \varrho(i)\xi^\beta(\sigma(i))\chi^\beta(\sigma(i) - l) \leq 0$$

for sufficiently large i .

P r o o f. Let $\{\psi(i)\}$ be an eventually positive solution of (1.1) such that $\psi(i) > 0$, $\psi(i - \kappa) > 0$, $\psi(i + l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$ for an integer $i_1 \geq i_0$. From the definition of $\chi(i)$ we obtain

$$(2.5) \quad \psi(i) = \frac{1}{\varrho_2(i-l)}(\chi(i-l) - \psi(i-l) - \varrho_1(i-l)\psi(i-\kappa-l))$$

and

$$(2.6) \quad \psi(i) < \frac{1}{\varrho_2(i-l)}\chi(i-l).$$

Using (2.6) in (2.5), we have

$$(2.7) \quad \begin{aligned} \psi(i) &\geq \frac{1}{\varrho_2(i-l)}\left(\chi(i-l) - \frac{1}{\varrho_2(i-2l)}\chi(i-2l) - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)}\chi(i-\kappa-2l)\right) \\ &\geq \frac{1}{\varrho_2(i-l)}\left(1 - \frac{1}{\varrho_2(i-2l)} - \frac{\varrho_1(i-l)}{\varrho_2(i-\kappa-2l)}\right)\chi(i-l) \end{aligned}$$

for $i \geq i_1$, where we have used $\{\chi(i)\}$ is strictly increasing. Since $\lim_{i \rightarrow \infty} \sigma(i) = \infty$, we can choose an integer $i_2 \geq i_1$ such that $\sigma(i) \geq i_2$ for all $i \geq i_2$. From (2.7) we have

$$(2.8) \quad \psi(\sigma(i)) \geq \xi(\sigma(i))\chi(\sigma(i)-l), \quad i \geq i_2.$$

Combining (1.1) with (2.8), we conclude that (2.4) is satisfied. The proof of the lemma is complete. \square

Lemma 2.5. *Assume that (C₁)–(C₃) and (C₅) hold. If $\{\psi(i)\}$ is an eventually positive solution of (1.1) such that (2.1) holds, then $\{\chi(i)\}$ satisfies the inequality*

$$(2.9) \quad \Delta(\zeta(i)(\Delta\chi(i))^\alpha) + \varrho(i)\varphi^\beta(\sigma(i))\chi^\beta(\sigma(i) + \kappa) \leq 0$$

for sufficiently large i .

P r o o f. Let $\{\psi(i)\}$ be an eventually positive solution of (1.1) such that $\psi(i) > 0$, $\psi(i - \kappa) > 0$, $\psi(i + l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$ for an integer $i_1 \geq i_0$. Following a similar argument as in the proof of Lemma 2.4 and taking into account that $\{\chi(i)/F(i)\}$ is decreasing for all $i \geq i_2$ for an integer $i_2 \geq i_1$, we obtain

$$(2.10) \quad \begin{aligned} \psi(i) &\geq \frac{1}{\varrho_1(i+\kappa)}\left(\chi(i+\kappa) - \frac{\chi(i+2\kappa)}{\varrho_1(i+2\kappa)} - \frac{\varrho_2(i+\kappa)\chi(i+2\kappa+l)}{\varrho_1(i+2\kappa+l)}\right) \\ &\geq \frac{1}{\varrho_1(i+\kappa)}\left(1 - \frac{F(i+2\kappa)}{F(i+\kappa)\varrho_1(i+2\kappa)} - \frac{\varrho_2(i+\kappa)F(i+2\kappa+l)}{\varrho_1(i+2\kappa+l)F(i+\kappa)}\right)\chi(i+\kappa). \end{aligned}$$

Since $\lim_{i \rightarrow \infty} \sigma(i) = \infty$, we choose an integer $i_3 \geq i_2$ such that $\sigma(i) \geq i_3$ for all $i \geq i_3$. Thus, from (2.10) we obtain

$$(2.11) \quad \psi(\sigma(i)) = \varphi(\sigma(i))\chi(\sigma(i) + \kappa), \quad i \geq i_3.$$

Combining (1.1) and (2.11), we conclude that (2.9) is satisfied. The proof of the lemma is complete. \square

3. MAIN RESULTS

In this section, we present several sufficient conditions for the oscillation of all solutions of (1.1).

Theorem 3.1. *Assume that (C₁)–(C₄) hold and $i + l \geq \sigma(i)$. If $\beta = \alpha$ and there exists a positive nondecreasing sequence $\{\eta(i)\}$ such that for all sufficiently large integer $N \geq i_1$,*

$$(3.1) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(\eta(s)\varrho(s)\xi^\alpha(\sigma(s)) \frac{F^\alpha(\sigma(s) - l)}{F^\alpha(s)} - \frac{\Delta\eta(s)}{F^\alpha(s+1)} \right) = \infty,$$

then every solution of (1.1) is oscillatory.

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geq i_0$ such that $\psi(i) > 0$, $\psi(i - \kappa) > 0$, $\psi(i + l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$. Proceeding as in the proof of Lemmas 2.3 and 2.4, we see that (2.2), (2.3) and (2.4) hold for all $i \geq i_1$. Define

$$(3.2) \quad \omega(i) = \eta(i) \frac{\zeta(i)(\Delta\chi(i))^\alpha}{\chi^\alpha(i)}, \quad i \geq i_1.$$

Clearly $\omega(i) > 0$ for $i \geq i_1$, and from (2.4) we obtain

$$(3.3) \quad \begin{aligned} \Delta\omega(i) \leq & -\eta(i)\varrho(i) \frac{\xi^\alpha(\sigma(i))\chi^\alpha(\sigma(i) - l)}{\chi^\alpha(i)} + \frac{\Delta\eta(i)\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)} \\ & - \frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i)\chi^\alpha(i+1)} \Delta\chi^\alpha(i). \end{aligned}$$

From $i \geq \sigma(i) - l$ and by (2.3) we get

$$(3.4) \quad \frac{\chi(\sigma(i) - l)}{\chi(i)} \geq \frac{F(\sigma(i) - l)}{F(i)}, \quad i \geq i_2 \geq i_1.$$

Using (3.4) in (3.3) yields

$$(3.5) \quad \Delta\omega(i) \leq -\eta(i)\varrho(i)\xi^\alpha(\sigma(i))\frac{F^\alpha(\sigma(i)-l)}{F^\alpha(i)} + \Delta\eta(i)\frac{\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)} \\ - \frac{\eta(i)\zeta(i)(\Delta\chi(i))^\alpha}{\chi^\alpha(i)\chi^\alpha(i+1)}\Delta\chi^\alpha(i), \quad i \geq i_2.$$

From (2.2), it is easy to see that

$$(3.6) \quad \frac{1}{F^\alpha(i+1)} \geq \zeta(i+1)\frac{(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)}.$$

In view of (3.6), $\chi(i) > 0$ and $\Delta\chi(i) > 0$, (3.5) yields

$$(3.7) \quad \Delta\omega(i) \leq -\eta(i)\varrho(i)\xi^\alpha(\sigma(i))\frac{F^\alpha(\sigma(i)-l)}{F^\alpha(i)} + \frac{\Delta\eta(i)}{F^\alpha(i+1)}$$

for $i \geq i_2$. Summing up (3.7) from i_2 to i , we get

$$\sum_{s=i_2}^i \left(\eta(s)\varrho(s)\xi^\alpha(\sigma(s))\frac{F^\alpha(\sigma(s)-l)}{F^\alpha(s)} - \frac{\Delta\eta(s)}{F^\alpha(s+1)} \right) \leq \omega(i_2),$$

which contradicts (3.1). The proof of the theorem is complete. \square

Theorem 3.2. Assume that (C₁)–(C₄) hold and $i+l \geq \sigma(i)$. If there exists a positive nondecreasing sequence $\{\eta(i)\}$ such that for all sufficiently large integers $N \geq i_1$,

$$(3.8) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(E_1(s) - \left(\frac{\alpha}{\beta}\right)^\alpha \frac{(\Delta\eta(i))^{\alpha+1}f(s)}{(\alpha+1)^{\alpha+1}\eta^\alpha(s)\delta^\alpha(s)} \right) = \infty,$$

where

$$E_1(i) = \eta(i)\varrho(i)\xi^\beta(\sigma(i))\frac{F^\beta(\sigma(i)-l)}{F^\beta(i)} \quad \text{and} \quad \delta(i) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ M_1 & \text{if } \alpha < \beta, \\ M_2F^{-1+\beta/\alpha}(i+1) & \text{if } \alpha > \beta \end{cases}$$

for all $M_1 > 0$, $M_2 > 0$, then every solution of (1.1) is oscillatory.

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geq i_0$ such that $\psi(i) > 0$, $\psi(i-\kappa) > 0$, $\psi(i+l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$. Define

$$(3.9) \quad \omega(i) = \eta(i)\frac{\zeta(i)(\Delta\chi(i))^\alpha}{\chi^\beta(i)}, \quad i \geq i_1.$$

Clearly $\omega(i) > 0$ for $i \geq i_1$, and from (2.4) and (3.9) we get

$$(3.10) \quad \begin{aligned} \Delta\omega(i) \leq & -\eta(i)\varrho(i)\xi^\beta(\sigma(i))\frac{\chi^\beta(\sigma(i)-l)}{\chi^\beta(i)} + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) \\ & - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^\beta(i)}{\chi^\beta(i)}, \quad i \geq i_2 \geq i_1. \end{aligned}$$

By discrete mean value theorem (see [1]), we have

$$\Delta\chi^\beta(i) \geq \beta\frac{\chi^\beta(i)}{\chi(i+1)}\Delta\chi(i).$$

Using this and (3.4) in (3.10) gives

$$(3.11) \quad \Delta\omega(i) \leq -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi(i)}{\chi(i+1)}.$$

Since $f^{1/\alpha}(i)\Delta\chi(i)$ is decreasing, we have from (3.9) and (3.11)

$$(3.12) \quad \begin{aligned} \Delta\omega(i) & \leq -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)}{\eta^{1+1/\alpha}(i+1)}\omega^{1+1/\alpha}(i+1)\chi^{-1+\beta/\alpha}(i+1) \\ & \leq -E_1(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \beta\frac{\eta(i)\delta(i)}{\eta^{1+1/\alpha}(i+1)\zeta^{1/\alpha}(i)}\omega^{1+1/\alpha}(i+1), \end{aligned}$$

where we have used that $\chi(i)$ is increasing for $\beta > \alpha$ and $\chi(i)/F(i)$ is decreasing for $\beta < \alpha$. Applying Lemma 2.1 with

$$D = \frac{\Delta\eta(i)}{\eta(i+1)}, \quad E = \frac{\beta\eta(i)\delta(i)}{\eta^{1+1/\alpha}(i+1)\zeta^{1/\alpha}(i)},$$

we have from (3.12) that

$$\Delta\omega(i) \leq -E_1(i) + \frac{(\alpha/\beta)^\alpha(\Delta\eta(i))^{\alpha+1}\zeta(i)}{(\alpha+1)^{\alpha+1}\eta^\alpha(i)\delta^\alpha(i)}, \quad i \geq i_2.$$

Summing up the last inequality from i_2 to i , we get

$$\sum_{s=i_2}^i \left(-E_1(s) + \frac{(\alpha/\beta)^\alpha(\Delta\eta(s))^{\alpha+1}\zeta(s)}{(\alpha+1)^{\alpha+1}\eta^\alpha(s)\delta^\alpha(s)} \right) \leq \omega(i_2),$$

which contradicts (3.8). The proof of the theorem is complete. □

Theorem 3.3. Assume that (C₁)–(C₄) hold, $\alpha = \beta$ and $i + l \leq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large $N \geq i_1 \geq i_0$,

$$(3.13) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(\eta(s) \varrho(s) \xi^\alpha(\sigma(s)) - \frac{\Delta \eta(s)}{F^\alpha(s+1)} \right) = \infty,$$

then every solution of (1.1) is oscillatory.

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geq i_0$ such that $\psi(i) > 0$, $\psi(i - \kappa) > 0$, $\psi(i + l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$. Proceeding as in the proof of Theorem 3.1, we arrive at (3.3) for $i \geq i_2 \geq i_1$. From $i + \kappa \leq \sigma(i)$ we have $i \leq \sigma(i) - l$ and so

$$(3.14) \quad \frac{\chi(\sigma(i) - l)}{\chi(i)} \geq 1.$$

Using (3.14) in (3.3) yields

$$(3.15) \quad \begin{aligned} \Delta \omega(i) \leq & -\eta(i) \varrho(i) \xi^\alpha(\sigma(i)) + \Delta \eta(i) \frac{\zeta(i+1)(\Delta \chi(i+1))^\alpha}{\chi^\alpha(i+1)} \\ & - \frac{\eta(i) \zeta(i)(\Delta \chi(i))^\alpha \Delta \chi^\alpha(i)}{\chi^\alpha(i) \chi^\alpha(i+1)}, \quad i \geq i_2. \end{aligned}$$

Taking into account that (2.2) holds and using the fact that $\Delta \chi(i) > 0$, (3.15) takes the form

$$\Delta \omega(i) \leq -\eta(i) \varrho(i) \xi^\alpha(\sigma(i)) + \frac{\Delta \eta(i)}{F^\alpha(i+1)}, \quad i \geq i_2.$$

The remaining part of the proof is similar to that of Theorem 3.1 and the details are omitted. The proof of the theorem is complete. \square

Theorem 3.4. Assume that (C₁)–(C₄) hold and $i + l \leq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large $N \geq i_1 \geq i_0$,

$$(3.16) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(\eta(s) \varrho(s) \xi^\beta(\sigma(s)) - \left(\frac{\alpha}{\beta} \right)^\alpha \frac{(\Delta \eta(s))^{\alpha+1} \zeta(s)}{(\alpha+1)^{\alpha+1} \eta^\alpha(s) \delta^\alpha(s)} \right) = \infty,$$

where $\delta(i)$ is defined as in Theorem 3.3, then every solution of (1.1) is oscillatory.

Proof. The proof follows from Theorem 3.2 and (3.14) and so the details are omitted. \square

Theorem 3.5. Assume that $\alpha = \beta$, (C_1) – (C_3) and (C_5) hold, and $i - \kappa \geq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large $N \geq i_1 \geq i_0$,

$$(3.17) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(E_2(s) - \frac{\Delta\eta(s)}{F^\alpha(s+1)} \right) = \infty,$$

where

$$E_2(i) = \eta(i)\varrho(i)\varphi^\alpha(\sigma(i)) \frac{F^\alpha(\sigma(i) + \kappa)}{F^\alpha(i)},$$

then every solution of (1.1) is oscillatory.

Proof. Let $\{\psi(i)\}$ be a nonoscillatory solution of (1.1). With no loss of generality, we may assume that there is an integer $i_1 \geq i_0$ such that $\psi(i) > 0$, $\psi(i - \kappa) > 0$, $\psi(i + l) > 0$, $\psi(\sigma(i)) > 0$ and $\chi(i)$ satisfies (2.1) for all $i \geq i_1$. Proceeding as in the proof of Lemmas 2.3 and 2.5, we have (2.2), (2.3) and (2.9) hold for $i \geq i_2$ for an integer $i_2 \geq i_1$. Define $\omega(i)$ by (3.2). Then it follows from (3.2) and (2.9) that

$$(3.18) \quad \Delta\omega(i) \leq -\eta(i)\varrho(i)\varphi^\alpha(\sigma(i)) \frac{\chi^\alpha(\sigma(i) + \kappa)}{\chi^\alpha(i)} + \Delta\eta(i) \frac{\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)} \\ - \frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^\alpha \Delta\chi^\alpha(i)}{\chi^\alpha(i)\chi^\alpha(i+1)}, \quad i \geq i_2.$$

Since $i - \kappa \geq \sigma(i)$, we have $i + 1 \geq i \geq \sigma(i) + \kappa$, and from (2.3) we get

$$(3.19) \quad \frac{\chi(\sigma(i) + \kappa)}{\chi(i)} \geq \frac{F(\sigma(i) + \kappa)}{F(i)}.$$

Substituting (3.19) into (3.18) yields

$$\Delta\omega(i) \leq -E_2(i) + \Delta\eta(i) \frac{\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)} - \frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^\alpha \Delta\chi^\alpha(i)}{\chi^\alpha(i)\chi^\alpha(i+1)},$$

where $i \geq i_2$. The rest of the proof is similar to that of Theorem 3.1 and hence the details are not repeated. The proof of the theorem is complete. \square

Theorem 3.6. Assume that (C_1) – (C_3) and (C_5) hold and $i - \kappa \geq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large integers $N \geq i_1$,

$$(3.20) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(E_3(s) - \left(\frac{\alpha}{\beta}\right)^\alpha \frac{(\Delta\eta(s))^{\alpha+1} \zeta(s)}{(\alpha+1)^{\alpha+1} \eta^\alpha(s) \delta^\alpha(s)} \right) = \infty,$$

where

$$E_3(i) = \eta(i)\varrho(i)\varphi^\beta(\sigma(i)) \frac{F^\beta(\sigma(i) + \kappa)}{F^\beta(i+1)},$$

then every solution of (1.1) is oscillatory.

Proof. The proof follows from Theorem 3.2 by using (3.19) instead of (3.4), and so the details are not repeated. This completes the proof. \square

Theorem 3.7. Assume that (C₁)–(C₃) and (C₅) hold $\alpha = \beta$ and $i - \kappa \leq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large integers $N \geq i_1 \geq i_0$,

$$(3.21) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(E_4(s) - \frac{\Delta\eta(s)}{F^\alpha(s+1)} \right) = \infty,$$

where

$$E_4(i) = \eta(i)\varrho(i)\varphi^\alpha(\sigma(i)),$$

then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.5, we arrive at (3.18). For $i - \kappa \leq \sigma(i)$, we see that $i \leq \sigma(i) + \kappa$, and so since using $\{\chi(i)\}$ is increasing, we have

$$(3.22) \quad \frac{\chi(\sigma(i) + \kappa)}{\chi(i)} \geq 1.$$

Using (3.22) in (3.18), we obtain

$$\begin{aligned} \Delta\omega(i) \leq & E_4(i) + \Delta\eta(i) \frac{\zeta(i+1)(\Delta\chi(i+1))^\alpha}{\chi^\alpha(i+1)} \\ & - \frac{\eta(i)\zeta(i+1)(\Delta\chi(i+1))^\alpha \Delta\chi^\alpha(i)}{\chi^\alpha(i)\chi^\alpha(i+1)}, \quad i \geq i_2. \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 3.5 and so the details are omitted. The proof of the theorem is complete. \square

Theorem 3.8. Assume that (C₁)–(C₃) and (C₅) hold and $i - \kappa \leq \sigma(i)$. If there exists a positive nondecreasing real sequence $\{\eta(i)\}$ such that for all sufficiently large integers $N \geq i_1 \geq i_0$,

$$(3.23) \quad \limsup_{i \rightarrow \infty} \sum_{s=N}^i \left(E_5(s) - \left(\frac{\alpha}{\beta}\right)^\alpha \frac{(\Delta\eta(i))^{\alpha+1} \zeta(s)}{(\alpha+1)^{\alpha+1} \eta^\alpha(s) \delta^\alpha(s)} \right) = \infty,$$

where $\delta(i)$ is defined as in Theorem 3.2 and $E_5(i) = \eta(i)\varrho(i)\varphi^\beta(\sigma(i))$, then every solution of (1.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.2 and using (2.9) instead of (2.4) we obtain

$$(3.24) \quad \begin{aligned} \Delta\omega(i) \leq & -\eta(i)\varrho(i)\varphi^\beta(\sigma(i))\frac{\chi^\beta(\sigma(i)+\kappa)}{\chi^\beta(i)} + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) \\ & - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^\beta(i)}{\chi^\beta(i)}, \quad i \geq i_2. \end{aligned}$$

Using (3.22) in (3.24) yields

$$\Delta\omega(i) \leq -E_5(i) + \frac{\Delta\eta(i)}{\eta(i+1)}\omega(i+1) - \frac{\eta(i)}{\eta(i+1)}\omega(i+1)\frac{\Delta\chi^\beta(i)}{\chi^\beta(i)}.$$

The rest part of the proof is similar to that of Theorem 3.2 and so the details are omitted. The proof of the theorem is complete. \square

4. EXAMPLES

In this section, we present several examples to illustrate the importance of the main results.

Example 4.1. Consider the second-order neutral difference equation

$$(4.1) \quad \Delta((\Delta(\psi(i) + \psi(i-1) + i\psi(i+2)))^3) + (i^4 + 1)\psi^3(i+1) = 0, \quad i \geq 10.$$

Here, $\alpha = \beta = 3$, $\zeta(i) = \varrho_1(i) = 1$, $\varrho_2(i) = n$, $\varrho(i) = n^4 + 1$, $\kappa = 1$, $l = 2$ and $\sigma(i) = i + 1$. It is clear that (C₁)–(C₄) hold, $i + \kappa \geq \sigma(i)$, and

$$\begin{aligned} \xi(i) &= \frac{1}{i-2} \left(1 - \frac{1}{i-4} - \frac{1}{i-5} \right) = \frac{1}{i-2} \frac{i^2 - 11i + 29}{(i-4)(i-5)} > 0, \\ F(i) &= \sum_{s=N}^{i-1} \frac{1}{\zeta^{1/\alpha}(s)} = \sum_{s=10}^{i-1} \Delta s = i - 10. \end{aligned}$$

With $\eta(i) = 1$, we see that (3.1) becomes

$$\limsup_{i \rightarrow \infty} \sum_{s=10}^i \frac{(s^4 + 1)(s^2 - 11s + 29)^3 (s - 11)^3}{(s - 2)^3 (s - 4)^3 (s - 5)^3 (s - 10)^3} = \infty,$$

which, in view of Theorem 3.1, means that all solutions of (4.1) are oscillatory.

Example 4.2. Consider the second-order neutral difference equation

$$(4.2) \quad \Delta\left(\frac{1}{i^{1/3}}(\Delta(\psi(i) + 2\psi(i-1) + 8\psi(i+2)))^{1/3}\right) + (i^2 + 1)\psi^{1/3}(i+3) = 0, \quad i \geq 2.$$

Here, $\alpha = \beta = \frac{1}{3}$, $\zeta(i) = i^{-1/3}$, $\varrho_1(i) = 2$, $\varrho_2(i) = 8$, $\varrho(i) = i^2 + 1$, $\kappa = 1$, $l = 2$, and $\sigma(i) = i + 3$. It is clear that (C₁)–(C₄) hold, $\sigma(i) \geq (i + \kappa)$ and

$$\xi(i) = \frac{1}{8}\left(1 - \frac{1}{8} - \frac{2}{8}\right) = \frac{5}{64} > 0, \quad F(i) = \sum_{s=2}^{i-1} s = \frac{i^2 - i - 2}{2}.$$

With $\eta(i) = 1$, (3.13) becomes

$$\limsup_{i \rightarrow \infty} \sum_{s=2}^i (s^2 + 1) \left(\frac{5}{64}\right)^{1/3} = \infty,$$

which, in view of Theorem 3.3, means that all solutions of (4.2) are oscillatory.

Example 4.3. Consider the second-order neutral difference equation

$$(4.3) \quad \Delta((\Delta(\psi(i) + 3i\psi(i-1) + i\psi(i+2)))^3) + i^5\psi^3(i-2) = 0, \quad i \geq 2.$$

Here, $\alpha = \beta = 3$, $\zeta(i) = 1$, $\varrho_1(i) = 3i$, $\varrho_2(i) = i$, $\varrho(i) = i^5$, $\kappa = 1$, $l = 2$, and $\sigma(i) = i - 2$. It is clear that (C₁)–(C₃) and (C₅) hold, $i - \kappa \geq \sigma(i)$ and

$$\begin{aligned} \varphi(i) &= \frac{1}{3(i+1)} \left(1 - \frac{1}{3(i+2)} \frac{i}{i-1} - \frac{i+1}{3(i+4)} \frac{(i+1)}{(i-1)}\right) \\ &= \frac{1}{3(i+1)} \frac{6i^4 + 24i^3 - 39i^2 - 69i + 14}{9(i^4 + 4i^3 - 3i^2 - 10i + 8)} > 0. \end{aligned}$$

With $\eta(i) = 1$, we see that (3.17) holds for $N > 2$. Therefore, in view of Theorem 3.5, every solution of (4.3) is oscillatory.

Example 4.4. Consider the second-order neutral difference equation

$$(4.4) \quad \Delta^2(\psi(i) + 2^i\psi(i-2) + \psi(i+1)) + 9(2^i)\psi(i-1) = 0, \quad i \geq 5.$$

Here, $\alpha = \beta = 1$, $\zeta(i) = \varrho_2(i) = 1$, $\varrho_1(i) = 2^i$, $\varrho(i) = 9(2^i)$, $\kappa = 2$, $l = 1$, and $\sigma(i) = i - 1$. It is clear that (C₁)–(C₃) and (C₅) hold and $i - \kappa < \sigma(i)$. Also $F(i) = i - 5$ and so

$$\varphi(i) = \frac{1}{4^{i+1}} \left(2^i - \frac{9i-8}{32(i-3)}\right) > 0.$$

With $\eta(i) = 1$, it is easy to see that (3.21) holds. Therefore, in view of Theorem 3.7, every solution of (4.4) is oscillatory. In fact, $\{\psi(i)\} = \{(-1)^i\}$ is such a solution.

5. CONCLUSION

In this paper, we have established several new oscillation theorems for equation (1.1) by using Ricatti transformation technique and summation averaging method. Furthermore, none of the results obtained in the literature can be used for the above examples to get any conclusion since the coefficients $\{\varrho_1(i)\}$ and $\{\varrho_2(i)\}$ are unbounded. Thus, the results established in this paper are new and complement the existing results.

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