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ON A SYSTEM OF NONLINEAR WAVE EQUATIONS WITH THE KIRCHHOFF-CARRIER AND BALAKRISHNAN-TAYLOR TERMS

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Abstract. We study a system of nonlinear wave equations of the Kirchhoff-Carrier type containing a variant of the Balakrishnan-Taylor damping in nonlinear terms. By the linearization method together with the Faedo-Galerkin method, we prove the local existence and uniqueness of a weak solution. On the other hand, by constructing a suitable Lyapunov functional, a sufficient condition is also established to obtain the exponential decay of weak solutions.

Keywords: system of nonlinear wave equations of Kirchhoff-Carrier type; Balakrishnan-Taylor term; Faedo-Galerkin method; local existence; exponential decay

MSC 2020: 35L20, 35L70, 35Q74, 37B25

1. Introduction

We consider the initial-boundary value problem for the system of nonlinear wave equations with the Kirchhoff-Carrier and Balakrishnan-Taylor terms

$$(1.1)\qquad \begin{aligned} u_{tt} - \lambda u_{xxt} - \mu_1(t, \langle u_x(t), u_{xt}(t)\rangle) u_{xx} &= f_1(x,t,u,v,u_x,v_x,u_t,v_t), \\ v_{tt} - \mu_2(t, \|v(t)\|^2, \|v_x(t)\|^2) v_{xx} &= f_2(x,t,u,v,u_x,v_x,u_t,v_t), \end{aligned}$$

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$0 < x < 1$, $0 < t < T$, associated with the Robin-Dirichlet conditions

$$u(0,t) = u(1,t) = v(1,t) = v_x(0,t) - \zeta v(0,t) = 0 \tag{1.2}$$

and initial conditions

$$(u(x,0), v(x,0)) = (\tilde{u}_0(x), \tilde{v}_0(x)), \quad (u_t(x,0), v_t(x,0)) = (\tilde{u}_1(x), \tilde{v}_1(x)), \tag{1.3}$$

where $\lambda > 0$, $\zeta \geqslant 0$ are given constants and $\tilde{u}_i$, $\tilde{v}_i$, μ_i, f_i $(i = 1,2)$ are given functions satisfying conditions, which will be specified later. In (1.1), the nonlinear terms $\mu_1(t, \langle u_x(t), u_{xt}(t)\rangle)$, $\mu_2(t, \|v(t)\|^2, \|v_x(t)\|^2)$ depend on the integrals $\langle u_x(t), u_{xt}(t)\rangle = \int_0^1 u_x(x,t)u_{xt}(x,t)\,\mathrm{d}x$, $\|v(t)\|^2 = \int_0^1 v^2(x,t)\,\mathrm{d}x$ and $\|v_x(t)\|^2 = \int_0^1 v_x^2(x,t)\,\mathrm{d}x$.

The system (1.1) is regarded as the combination between the wave equation of the Kirchhoff-Carrier type and that with the Balakrishnan-Taylor damping, in which a related case of $(1.1)_1$ was proposed by Balakrishnan and Taylor in 1989, see [1]. They established the following new model for flight structures with viscous and nonlinear nonlocal damping in the one dimensional case

$$\begin{aligned}\varrho u_{tt} + EIu_{xxxxx} - cu_{xxt} \\ - \bigg(H + \frac{EA}{2L}\int_0^L |u_x|^2\,\mathrm{d}x + \tau\bigg|\int_0^L u_x u_{xt}\,\mathrm{d}x\bigg|^{2(N+\eta)}\int_0^L u_x u_{xt}\,\mathrm{d}x\bigg)u_{xx} = 0,\end{aligned} \tag{1.4}$$

where $u = u(x,t)$ represents the transversal deflection of an extensible beam of length $2L > 0$ in the rest position, $\varrho > 0$ is the mass density, E is Young's modulus of elasticity, I is the cross-sectional moment of inertia, H is the axial force (either traction or compression), A is the cross-sectional area, $c > 0$ is the coefficient of viscous damping, $\tau > 0$ is the Balakrishnan-Taylor damping coefficient, $0 \leqslant \tau < 1$, $0 \leqslant \eta < \frac{1}{2}$ and $N \in \mathbb{N}$. Equation (1.4) seems to be related to the panel flutter equation and spillover problem, we refer the reader to [2] for more information. Since then, the equation with the Balakrishnan-Taylor damping was studied in many papers in which the properties of the solution such as stability, decay and blow-up in time are considered, see [3]–[6], [8], [9], [12]–[17], [20]–[22], [27], [31], [32], [35] and references therein. Some authors were interested in the effects of time-varying delay, which appear in many applications to science because physical, chemical, biological, thermal, and economical phenomena naturally not only depend on the present state but also on some past occurrences, see examples in [17], [20], [21].

In 2011, Emmrich and Thalhammer considered a class of integro-differential equations with applications in nonlinear elastodynamics. They proposed a general model for the description of nonlinear extensible beams incorporating the weak, viscous, strong and Balakrishnan-Taylor damping as follows (see [8], equation (1.1))

$$
\begin{aligned}
(1.5)\qquad & u_{tt}+\alpha\Delta^2u+\xi u+\kappa u_t-\lambda\Delta u_t+\mu\Delta^2u_t\\
&\quad-\Big(\beta+\gamma\int_\Omega|\nabla u|^2\,\mathrm{d}x+\delta\Big|\int_\Omega\nabla u\nabla u_t\,\mathrm{d}x\Big|^{q-2}\int_\Omega\nabla u\nabla u_t\,\mathrm{d}x\Big)\Delta u=h
\end{aligned}
$$

in $\Omega\times(0,\infty)$, where $\Omega\subset\mathbb{R}^n$ is a bounded domain. The constants have physical meaning: $\alpha>0$ is the elasticity coefficient, $\gamma>0$ is the extensibility coefficient, $\lambda\geqslant 0$ is the viscous damping coefficient, $\mu\geqslant 0$ is the strong damping coefficient, $\delta\geqslant 0$ is the Balakrishnan-Taylor damping coefficient, $\beta\in\mathbb{R}$ is the axial force coefficient ($\beta>0$ means traction or $\beta<0$ compression), $\kappa\in\mathbb{R}$ is the weak damping coefficient (although without the sign condition), $\xi\in\mathbb{R}$ is source coefficient and the exponent q belongs to $[2,\infty)$. In [12], Tavares et al. worked with an alternative expression of the Balakrishnan-Taylor term

$$
(1.6)\qquad \Phi(u,u_t)=\int_\Omega\nabla u\nabla u_t\,\mathrm{d}x=-\int_\Omega(\Delta u)u_t\,\mathrm{d}x
$$

to study the well-posedness and long-time dynamics of the class of extensible beams with the Balakrishnan-Taylor and frictional damping

$$
(1.7)\quad u_{tt}+\Delta^2u-\Big(\beta+\gamma\int_\Omega|\nabla u|^2\,\mathrm{d}x+\delta|\Phi(u,u_t)|^{q-2}\Phi(u,u_t)\Big)\Delta u+\kappa u_t+f(u)=h,
$$

in $\Omega\times\mathbb{R}_+$, where Ω is a bounded domain in $\mathbb{R}^n$ with the smooth boundary $\Gamma=\partial\Omega$. In [27], the authors proved the existence and uniqueness of the initial-boundary value problem

(1.8)

$$
\begin{cases}
u_{tt}-\lambda u_{xxt}-\mu(t,\langle u_x(t),u_{xt}(t)\rangle,\|u(t)\|^2,\|u_x(t)\|^2)u_{xx}\\
\qquad =f(x,t,u,u_t,u_x,\langle u_x(t),u_{xt}(t)\rangle,\|u(t)\|^2,\|u_x(t)\|^2),\quad 0<x<1,0<t<T,\\
u(0,t)=u(1,t)=0,\\
u(x,0)=\tilde u_0(x),\ u_t(x,0)=\tilde u_1(x),
\end{cases}
$$

where μ, f, $\tilde u_0$, $\tilde u_1$ are given functions, $\lambda>0$ is a given constant. When $\mu=B(\|u_x(t)\|^2)+\sigma(\langle u_x(t),u_{xt}(t)\rangle)$ and $f=-\lambda_1u_t+f(u)+F(x,t)$, they put suitable hypotheses and sufficient conditions for the nonlinear Balakrishnan-Taylor damping $\sigma(\langle u_x(t),u_{xt}(t)\rangle)$ to get the exponential decay of solution. When $f=0$ and μ depends only on $\|v(t)\|^2$ and $\|v_x(t)\|^2$, equation $(1.8)_1$ reduces to the equation of the Kirchhoff-Carrier type, describing nonlinear vibrations of an elastic string, which were studied in [7], [25], [34].

In [19], Jamil and Fetecau studied a mathematical model describing helical flows of the Maxwell fluid in an annular region between two infinite coaxial circular cylinders of radii 1 and $R > 1$ below

$$
(1.9)\qquad \begin{cases} \lambda u_{tt} + u_t = \nu\Big(u_{xx} + \dfrac{1}{x}u_x - \dfrac{1}{x^2}u\Big), & 1 < x < R,\ t > 0, \\ \lambda v_{tt} + v_t = \nu\Big(v_{xx} + \dfrac{1}{x}v_x\Big), & 1 < x < R,\ t > 0, \\ u_x(1,t) - u(1,t) = \dfrac{f}{\mu}t,\ v_t(1,t) = \dfrac{g}{\mu}t, & t > 0, \\ u(R,t) = v(R,t) = 0, & t > 0, \\ u(x,0) = u_t(x,0) = 0, & 1 < x < R, \\ v(x,0) = v_t(x,0) = 0, & 1 < x < R, \end{cases}
$$

where λ, μ, ν, f, and g are given constants. The authors obtained an exact solution for this problem by means of finite Hankel transforms and presented it in the series form in terms of Bessel functions $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$, $J_2(x)$ and $Y_2(x)$, satisfying all imposed initial and boundary conditions. Other works on helical flows for ordinary and fractional derivative models can be found in [28], [18] and [33]. Some recent works focus on studying the porous elastic systems with nonlinear damping, see [11], [10], [24], [29] and references therein. It seems that the first research about the system of equations of the Kirchhoff type with the Balakrishnan-Taylor damping is given in [26], where its model is described as

$$
(1.10)\qquad \begin{cases} u_{tt} - \Big(a + b\|\nabla u\|^2 + \sigma\displaystyle\int_\Omega \nabla u \nabla u_t \,\mathrm{d}x\Big)\Delta u \\ \qquad\qquad + \displaystyle\int_0^t g_1(t-s)\Delta u(s)\,\mathrm{d}s = f_1(u,v), & t > 0,\ x \in \Omega, \\ v_{tt} - \Big(a + b\|\nabla v\|^2 + \sigma\displaystyle\int_\Omega \nabla v \nabla v_t \,\mathrm{d}x\Big)\Delta v \\ \qquad\qquad + \displaystyle\int_0^t g_2(t-s)\Delta v(s)\,\mathrm{d}s = f_2(u,v), & t > 0,\ x \in \Omega, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & (x,t) \in \Gamma \times [0,\infty), \end{cases}
$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary Γ. By the energy method, Mu and Ma in [26] obtained an arbitrary decay of solutions according to the relaxation functions. To the best of our knowledge, the system of equations of the Kirchhoff-Carrier type with the Balakrishnan-Taylor damping (1.1) has not been extensively studied. Motivated by the above articles, we survey the unique existence and exponential decay of weak solutions of the problem (1.1). Our plan in this paper is as follows. In Section 2, we present some preliminaries. In Section 3, by applying the linearization method together with the Faedo-Galerkin method and the weak compact method, we prove the local existence and the uniqueness of a weak solution (Theorem 3.7). In Section 4, by constructing a suitable Lyapunov functional, we prove a condition sufficient to obtain the exponential decay of weak solutions (Theorem 4.2). The results obtained here are a relative generalization of [24], [27] and [34] by improving and developing these previous works essentially.

2. Preliminaries

First, we put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leqslant p \leqslant \infty$, the Banach space of real functions $u\colon (0,T) \to X$, measurable and such that $\|u\|_{L^p(0,T;X)} < \infty$ with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X & \text{if } p = \infty. \end{cases}$$

On $H^1 \equiv H^1(\Omega)$, we use the norm

$$\|v\|_{H^1} = \sqrt{\|v\|^2 + \|v_x\|^2}.$$

We put

$$V = \{v \in H\colon\ v(1) = 0\}, \tag{2.1}$$

$$a(u,v) = \int_0^1 u_x(x) v_x(x) \, \mathrm{d}x + \zeta u(0) v(0) \quad \forall\, u, v \in V. \tag{2.2}$$

The set V is a closed subspace of H^1 and on V, the three norms $\|v\|_{H^1}$, $\|v_x\|$ and $\|v\|_a = \sqrt{a(v,v)}$ are equivalent.

We state the following lemmas, the proofs of which are straightforward, and hence we omit the details.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and for all $v \in H^1$,*

$$\|v\|_{C^0(\overline{\Omega})} \leqslant \sqrt{2} \|v\|_{H^1}.$$

Lemma 2.2. *Let $\zeta \geqslant 0$. Then the imbedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact and for all $v \in V$,*

$$\|v\|_{C^0(\overline{\Omega})} \leqslant \|v_x\| \leqslant \|v\|_a,$$
$$\frac{1}{\sqrt{2}}\|v\|_{H^1} \leqslant \|v_x\| \leqslant \|v\|_a \leqslant \sqrt{1+\zeta}\|v\|_{H^1}.$$

Lemma 2.3. *Let $\zeta \geqslant 0$. Then the symmetric bilinear form $a(\cdot,\cdot)$ defined by (2.2) is continuous on $V \times V$ and coercive on V. Furthermore,*
(i) $|a(u,v)| \leqslant (1+\zeta)\|u_x\|\|v_x\|$ *for all $u, v \in V$,*
(ii) $a(v,v) \geqslant \|v_x\|^2$ *for all $v \in V$.*

Lemma 2.4. *Let $\zeta \geqslant 0$. Then there exists a Hilbert orthonormal base $\{\widetilde{\varphi}_j\}$ of L^2 consisting of the eigenfunctions $\widetilde{\varphi}_j$ corresponding to the eigenvalue $\tilde{\lambda}_j$ such that*

$$0 < \tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots \leqslant \tilde{\lambda}_j \leqslant \ldots, \quad \lim_{j\to\infty} \tilde{\lambda}_j = \infty;$$
$$a(\widetilde{\varphi}_j, v) = \tilde{\lambda}_j \langle \widetilde{\varphi}_j, v\rangle \quad \forall v \in V,\ j = 1,2,\ldots$$

Furthermore, the sequence $\{\widetilde{\varphi}_j \tilde{\lambda}_j^{-1/2}\}$ is also a Hilbert orthonormal basis of V with respect to the scalar product $a(\cdot,\cdot)$.

On the other hand, we also have that $\widetilde{\varphi}_j$ satisfy the boundary value problems

$$\begin{cases} -\Delta\widetilde{\varphi}_j = \tilde{\lambda}_j\widetilde{\varphi}_j & \text{in } (0,1), \\ \widetilde{\varphi}_{jx}(0) - \zeta\widetilde{\varphi}_j(0) = \widetilde{\varphi}_j(1) = 0, & \widetilde{\varphi}_j \in C^\infty(\bar{\Omega}). \end{cases}$$

Proof. The proof of Lemma 2.4 can be found in [30], page 87, Theorem 7.7 with $H = L^2$ and $a(\cdot,\cdot)$ is defined by (2.2). □

3. The existence and uniqueness theorem

Considering $T^* > 0$ fixed, we make the following assumptions:
(H$_1$) $\tilde{u}_0, \tilde{u}_1 \in H_0^1 \cap H^2$, $\tilde{v}_0 \in V \cap H^2$, $\tilde{v}_1 \in V$, $\tilde{v}_{0x} - \zeta\tilde{v}_0 = 0$,
(H$_2$) $\mu_1 \in C^1([0,T^*] \times \mathbb{R})$, $\mu_2 \in C^1([0,T^*] \times \mathbb{R}_+^2)$, and there exist positive constants μ_{1*}, μ_{2*} such that $\mu_1(t,y) \geqslant \mu_{1*}$ for all $(t,y) \in [0,T^*] \times \mathbb{R}$ and $\mu_2(t,y,z) \geqslant \mu_{2*}$ for all $(t,y,z) \in [0,T^*] \times \mathbb{R}_+^2$,
(H$_3$) $f_i \in C^1([0,1] \times [0,T^*] \times \mathbb{R}^6)$, $i = 1,2$.

Let $\lambda > 0$, $\zeta \geqslant 0$. For every $T \in (0, T^*]$, we say that (u, v) is a weak solution of the problem (1.1)–(1.3) if

$$(u,v) \in \widetilde{W}_T = \{(u,v) \in L^\infty(0,T;(H_0^1 \cap H^2) \times (V \cap H^2))\colon \\ (u',v') \in L^\infty(0,T;(H_0^1 \cap H^2) \times V), \\ u'' \in L^2(0,T;H_0^1) \cap L^\infty(0,T;L^2),\ v'' \in L^\infty(0,T;L^2)\}$$

and (u, v) satisfies the variational problem

$$\text{(3.1)} \qquad \begin{cases} \langle u''(t), \varphi\rangle + \lambda \langle u_x'(t), \varphi_x\rangle + \mu_1[u](t)\langle u_x(t), \varphi_x\rangle = \langle f_1[u,v](t), \varphi\rangle, \\ \langle v''(t), \widetilde{\varphi}\rangle + \mu_2[v](t) a(v(t), \widetilde{\varphi}) = \langle f_2[u,v](t), \widetilde{\varphi}\rangle \end{cases}$$

for all $(\varphi, \widetilde{\varphi}) \in H_0^1 \times V$ and a.e. $t \in (0, T)$, together with the initial conditions

$$\text{(3.2)} \qquad (u(0), u'(0)) = (\tilde{u}_0, \tilde{u}_1), \quad (v(0), v'(0)) = (\tilde{v}_0, \tilde{v}_1),$$

where

(3.3)
$$\begin{cases} \mu_1[u](t) = \mu_1(t, \langle u_x(t), u_x'(t)\rangle), \\ \mu_2[v](t) = \mu_2(t, \|v(t)\|^2, \|v_x(t)\|^2), \\ f_i[u,v](x,t) = f_i(x,t,u(x,t),v(x,t),u_x(x,t),v_x(x,t),u'(x,t),v'(x,t)), \quad i = 1,2. \end{cases}$$

For each $M > 0$ given, we set the constants $K_M = K_M(f_1, f_2)$, $\widetilde{K}_M = \widetilde{K}_M(\mu_1, \mu_2)$ as

(3.4)
$$\begin{cases} K_M = K_M(f_1,f_2) = \sum_{j=1}^{2} \|f_j\|_{C^1(A_M)} = \sum_{j=1}^{2}\sum_{i=0}^{8} \|D_i f_j\|_{C^0(A_M)}, \\ \widetilde{K}_M = \widetilde{K}_M(\mu_1,\mu_2) = \sum_{j=1}^{2} \|\mu_j\|_{C^1(A_M^{(j)})} \\ \qquad = \|\mu_1\|_{C^0(A_M^{(1)})} + \sum_{i=1}^{2} \|D_i \mu_1\|_{C^0(A_M^{(1)})} + \|\mu_2\|_{C^0(A_M^{(2)})} + \sum_{i=1}^{3} \|D_i \mu_2\|_{C^0(A_M^{(2)})}, \end{cases}$$

where

$$\text{(3.5)} \qquad \begin{cases} \|f_j\|_{C^0(A_M)} = \sup_{(x,t,,y_1,\ldots,y_6) \in A_M} |f_j(x,t,y_1,\ldots,y_6)|, \quad j = 1,2, \\ \|\mu_1\|_{C^0(A_M^{(1)})} = \sup_{(t,y) \in A_M^{(1)}} |\mu_1(t,y)|, \\ \|\mu_2\|_{C^0(A_M^{(2)})} = \sup_{(t,y,z) \in A_M^{(2)}} |\mu_2(t,y,z)|, \\ A_M = [0,1] \times [0,T^*] \times [-M,M]^6, \\ A_M^{(1)} = [0,T^*] \times [-M^2, M^2], \\ A_M^{(2)} = [0,T^*] \times [0,M^2]^2. \end{cases}$$

For each $T \in (0, T^*]$, we put

$$(3.6)\quad V_T = \{(u,v) \in L^\infty(0,T;(H_0^1 \cap H^2) \times (V \cap H^2)) : \\ (u',v') \in L^\infty(0,T;(H_0^1 \cap H^2) \times V),\ (u'',v'') \in L^2(0,T;H_0^1 \times L^2)\};$$

it is a Banach space with respect to the norm

$$(3.7)\qquad \|(u,v)\|_{V_T} = \max\{\|(u,v)\|_{L^\infty(0,T;(H_0^1\cap H^2)\times(V\cap H^2))}, \\ \|(u',v')\|_{L^\infty(0,T;(H_0^1\cap H^2)\times V)},\ \|(u'',v'')\|_{L^2(0,T;H_0^1\times L^2)}\}.$$

For every $M > 0$, we put

$$(3.8)\qquad \begin{aligned} W(M,T) &= \{(u,v) \in V_T\colon\ \|(u,v)\|_{V_T} \leqslant M\}, \\ W_1(M,T) &= \{(u,v) \in W(M,T)\colon\ (u'',v'') \in L^\infty(0,T;L^2 \times L^2)\}. \end{aligned}$$

Now, we establish the following recurrent sequence $\{(u_m, v_m)\}$. The first term is chosen as $(u_0, v_0) \equiv (0,0)$; suppose that

$$(3.9)\qquad (u_{m-1}, v_{m-1}) \in W_1(M,T),$$

then we associate (1.1)–(1.3) with the following problem.

Find $(u_m, v_m) \in W_1(M,T)$ $(m \geqslant 1)$ which satisfies the linear variational problem

$$(3.10)\qquad \begin{cases} \langle u_m''(t), \varphi\rangle + \lambda\langle u_{mx}'(t), \varphi_x\rangle + \mu_{1m}(t)\langle u_{mx}(t), \varphi_x\rangle = \langle F_{1m}(t), \varphi\rangle, \\ \langle v_m''(t), \widetilde{\varphi}\rangle + \mu_{2m}(t) a(v_m(t), \widetilde{\varphi}) = \langle F_{2m}(t), \widetilde{\varphi}\rangle \end{cases}$$

for all $(\varphi, \widetilde{\varphi}) \in H_0^1 \times V$ and a.e. $t \in (0,T)$, together with the initial conditions

$$(3.11)\qquad (u_m(0), u_m'(0)) = (\tilde{u}_0, \tilde{u}_1), \quad (v_m(0), v_m'(0)) = (\tilde{v}_0, \tilde{v}_1),$$

where

$$(3.12)\qquad \begin{aligned} \mu_{1m}(t) &= \mu_1[u_{m-1}](t) = \mu_1(t, \langle \nabla u_{m-1}(t), \nabla u_{m-1}'(t)\rangle), \\ \mu_{2m}(t) &= \mu_2[v_{m-1}](t) = \mu_2(t, \|v_{m-1}(t)\|^2, \|\nabla v_{m-1}(t)\|^2), \\ F_{im}(x,t) &= f_i[u_{m-1}, v_{m-1}](t) \\ &= f_i(x,t,u_{m-1}(x,t), v_{m-1}(x,t), \nabla u_{m-1}(x,t), \nabla v_{m-1}(x,t), \\ &\qquad u_{m-1}'(x,t), v_{m-1}'(x,t)), \quad i = 1,2. \end{aligned}$$

Then we have the following theorem.

Theorem 3.1. *Let* (H$_1$)–(H$_3$) *hold. Then there exist constants* $M, T > 0$ *such that, for* $(u_0, v_0) \equiv (0,0)$, *there exists a recurrent sequence* $\{(u_m, v_m)\} \subset W_1(M,T)$ *defined by* (3.9)–(3.12).

P r o o f. The proof of Theorem 3.1 consists of three steps.

Step 1. The Faedo-Galerkin approximation. Let $\{\varphi_j\}$ be a basis of H_0^1 formed by the eigenfunction φ_j of the operator $-\Delta = -\frac{\partial^2}{\partial x^2}$ such that $-\Delta\varphi_j = \lambda_j\varphi_j$, $\varphi_j \in H_0^1 \cap C^\infty([0,1])$, $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, $\lambda_j = (j\pi)^2$, $j = 1, 2, \dots$, and let $\{\widetilde{\varphi}_j\}$ be the basis of V as in Lemma 2.4. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)\varphi_j, \quad v_m^{(k)}(t) = \sum_{j=1}^{k} d_{mj}^{(k)}(t)\widetilde{\varphi}_j, \tag{3.13}$$

where the coefficients $c_{mj}^{(k)}(t)$, $d_{mj}^{(k)}(t)$ satisfy the system of linear differential equations

$$\begin{cases} \langle \ddot{u}_m^{(k)}(t), \varphi_j\rangle + \lambda\langle \dot{u}_{mx}^{(k)}(t), \varphi_{jx}\rangle + \mu_{1m}(t)\langle u_{mx}^{(k)}(t), \varphi_{jx}\rangle = \langle F_{1m}(t), \varphi_j\rangle, \\ \langle \ddot{v}_m^{(k)}(t), \widetilde{\varphi}_j\rangle + \mu_{2m}(t)a(v_m^{(k)}(t), \widetilde{\varphi}_j) = \langle F_{2m}(t), \widetilde{\varphi}_j\rangle, \\ (u_m^{(k)}(0), \dot{u}_m^{(k)}(0)) = (\tilde{u}_{0k}, \tilde{u}_{1k}), \ (v_m^{(k)}(0), \dot{v}_m^{(k)}(0)) = (\tilde{v}_{0k}, \tilde{v}_{1k}), \end{cases} \tag{3.14}$$

$1 \leqslant j \leqslant k$, in which

(3.15)

$$\begin{cases} (\tilde{u}_{0k}, \tilde{u}_{1k}) = \displaystyle\sum_{j=1}^{k} (\alpha_j^{(k)}, \beta_j^{(k)})\varphi_j \to (\tilde{u}_0, \tilde{u}_1) \text{ strongly in } (H_0^1 \cap H^2) \times (H_0^1 \cap H^2), \\ (\tilde{v}_{0k}, \tilde{v}_{1k}) = \displaystyle\sum_{j=1}^{k} (\widetilde{\alpha}_j^{(k)}, \tilde{\beta}_j^{(k)})\widetilde{\varphi}_j \to (\tilde{v}_0, \tilde{v}_1) \text{ strongly in } (V \cap H^2) \times V. \end{cases}$$

The system of (3.14) and (3.15) can be rewritten in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \lambda\lambda_j \dot{c}_{mj}^{(k)}(t) + \lambda_j\mu_{1m}(t)c_{mj}^{(k)}(t) = \langle F_{1m}(t), \varphi_j\rangle, \\ \ddot{d}_{mj}^{(k)}(t) + \tilde{\lambda}_j\mu_{2m}(t)d_{mj}^{(k)}(t) = \langle F_{2m}(t), \tilde{\varphi}_j\rangle, \\ (c_{mj}^{(k)}(0), d_{mj}^{(k)}(0)) = (\alpha_j^{(k)}, \widetilde{\alpha}_j^{(k)}), \\ (\dot{c}_{mj}^{(k)}(0), \dot{d}_{mj}^{(k)}(0)) = (\beta_j^{(k)}, \tilde{\beta}_j^{(k)}), \quad 1 \leqslant j \leqslant k. \end{cases} \tag{3.16}$$

Using Banach's contraction principle, it is not difficult to prove that the system (3.16) has a unique solution $c_{mj}^{(k)}(t)$, $d_{mj}^{(k)}(t)$, $1 \leqslant j \leqslant k$, on the interval $[0, T]$.

Step 2. A priori estimate. We put

$$\begin{aligned} S_m^{(k)}(t) = {} & \|\dot{u}_m^{(k)}(t)\|^2 + \|\dot{v}_m^{(k)}(t)\|^2 + \|\dot{u}_{mx}^{(k)}(t)\|^2 + \|\dot{v}_m^{(k)}(t)\|_a^2 + \lambda\|\Delta\dot{u}_m^{(k)}(t)\|^2 \\ & + \mu_{1m}(t)(\|u_{mx}^{(k)}(t)\|^2 + \|\Delta u_m^{(k)}(t)\|^2) \\ & + \mu_{2m}(t)(\|v_m^{(k)}(t)\|_a^2 + \|\Delta v_m^{(k)}(t)\|^2) \\ & + 2\lambda\int_0^t (\|\dot{u}_{mx}^{(k)}(s)\|^2 + \|\Delta\dot{u}_m^{(k)}(s)\|^2)\,\mathrm{d}s \\ & + 2\int_0^t \|\ddot{u}_{mx}^{(k)}(s)\|^2\,\mathrm{d}s + \int_0^t \|\ddot{v}_m^{(k)}(s)\|^2\,\mathrm{d}s. \end{aligned} \tag{3.17}$$

Then, it follows from (3.14), (3.17) that

$$
\begin{aligned}
(3.18)\quad S_m^{(k)}(t) = {} & S_m^{(k)}(0) + 2\langle F_{1m}(0), \Delta\tilde{u}_{0k} + \Delta\tilde{u}_{1k}\rangle + 2\langle F_{2m}(0), \Delta\tilde{v}_{0k}\rangle \\
& + 2\mu_{1m}(0)\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle + 2\int_0^t \mu_{1m}(s)\|\Delta\dot{u}_m^{(k)}(s)\|^2\,\mathrm{d}s \\
& + \int_0^t \mu_{1m}'(s)(\|u_{mx}^{(k)}(s)\|^2 + \|\Delta u_m^{(k)}(s)\|^2 + 2\langle \Delta u_m^{(k)}(s), \Delta\dot{u}_m^{(k)}(s)\rangle)\,\mathrm{d}s \\
& + \int_0^t \mu_{2m}'(s)(\|v_m^{(k)}(s)\|_a^2 + \|\Delta v_m^{(k)}(s)\|^2)\,\mathrm{d}s \\
& + 2\int_0^t (\langle F_{1m}(s), \dot{u}_m^{(k)}(s)\rangle + \langle F_{2m}(s), \dot{v}_m^{(k)}(s)\rangle)\,\mathrm{d}s \\
& + 2\int_0^t (\langle F_{1m}'(s), \Delta u_m^{(k)}(s) + \Delta\dot{u}_m^{(k)}(s)\rangle + \langle F_{2m}'(s), \Delta v_m^{(k)}(s)\rangle)\,\mathrm{d}s \\
& - 2\langle F_{1m}(t), \Delta u_m^{(k)}(t) + \Delta\dot{u}_m^{(k)}(t)\rangle - 2\langle F_{2m}(t), \Delta v_m^{(k)}(t)\rangle \\
& - 2\mu_{1m}(t)\langle \Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle + \int_0^t \|\ddot{v}_m^{(k)}(s)\|^2\,\mathrm{d}s \\
= {} & S_m^{(k)}(0) + 2\langle F_{1m}(0), \Delta\tilde{u}_{0k} + \Delta\tilde{u}_{1k}\rangle + 2\langle F_{2m}(0), \Delta\tilde{v}_{0k}\rangle \\
& + 2\mu_{1m}(0)\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle + \sum_{j=1}^{9} I_j.
\end{aligned}
$$

□

First, we need the folowing lemma whose proof is easy, hence we omit the details.

Lemma 3.2. *We have*

(i) $|\mu_{im}(t)| \leqslant \widetilde{K}_M$, $i = 1, 2$,

(ii) $|\mu_{1m}'(t)| \leqslant \Psi_m(M, t)$,

(iii) $|\mu_{2m}'(t)| \leqslant \widetilde{K}_M(1 + 4M^2)$,

(iv) $|F_{im}(x, t)| \leqslant K_M$, $i = 1, 2$,

(v) $\|F_{im}'(t)\| \leqslant \Phi_m(M, t)$, $i = 1, 2$,

(vi) $\|F_{im}(t)\| \leqslant \|F_{im}(0)\| + \sqrt{T}K_M((1 + 4M)\sqrt{T} + 2M)$, $i = 1, 2$,

where

$$
\begin{aligned}
\Psi_m(M, t) &= \widetilde{K}_M(1 + M^2 + M\|\nabla u_{m-1}''(t)\|), \\
\Phi_m(M, t) &= K_M(1 + 4M + \|u_{m-1}''(t)\| + \|v_{m-1}''(t)\|)
\end{aligned}
$$

satisfy

(vii) $\Psi_m(M, \cdot)$, $\Phi_m(M, \cdot) \in L^2(0, T)$,

(viii) $\|\Psi_m(M, \cdot)\|_{L^2(0,T)} \leqslant \widetilde{K}_M((1 + M^2)\sqrt{T} + M^2)$,

(ix) $\|\Phi_m(M, \cdot)\|_{L^2(0,T)} \leqslant K_M((1 + 4M)\sqrt{T} + 2M)$.

Using Lemma 3.2 we estimate the terms I_j on the right-hand side of (3.18) as follows. Put $\lambda_* = \min\{\lambda, \mu_{1*}, \mu_{2*}\}$, by the inequality

$$
\begin{aligned}
S_m^{(k)}(t) \geqslant \|\dot u_m^{(k)}(t)\|^2 + \|\dot v_m^{(k)}(t)\|^2 \\
+ \lambda_*(\|\Delta \dot u_m^{(k)}(t)\|^2 + \|u_{mx}^{(k)}(t)\|^2 + \|\Delta u_m^{(k)}(t)\|^2 \\
+ \|v_m^{(k)}(t)\|_a^2 + \|\Delta v_m^{(k)}(t)\|^2)
\end{aligned} \tag{3.19}
$$

we estimate I_1, I_2, I_3, I_4, I_5, respectively, as

$$
\begin{aligned}
I_1 &= 2\int_0^t \mu_{1m}(s)\|\Delta \dot u_m^{(k)}(s)\|^2\,\mathrm{d}s \leqslant \frac{2\widetilde K_M}{\lambda_*}\int_0^t S_m^{(k)}(s)\,\mathrm{d}s,\\
I_2 &= \int_0^t \mu_{1m}'(s)(\|u_{mx}^{(k)}(s)\|^2 + \|\Delta u_m^{(k)}(s)\|^2 + 2\langle \Delta u_m^{(k)}(s), \Delta \dot u_m^{(k)}(s)\rangle)\,\mathrm{d}s\\
&\leqslant \frac{2}{\lambda_*}\int_0^t \Psi_m(M,s) S_m^{(k)}(s)\,\mathrm{d}s,\\
I_3 &= \int_0^t \mu_{2m}'(s)(\|v_m^{(k)}(s)\|_a^2 + \|\Delta v_m^{(k)}(s)\|^2)\,\mathrm{d}s \leqslant \frac{\widetilde K_M(1+4M^2)}{\lambda_*}\int_0^t S_m^{(k)}(s)\,\mathrm{d}s,\\
I_4 &= 2\int_0^t (\langle F_{1m}(s), \dot u_m^{(k)}(s)\rangle + \langle F_{2m}(s), \dot v_m^{(k)}(s)\rangle)\,\mathrm{d}s \leqslant 2TK_M^2 + \int_0^t S_m^{(k)}(s)\,\mathrm{d}s,\\
I_5 &= 2\int_0^t (\langle F_{1m}'(s), \Delta u_m^{(k)}(s) + \Delta \dot u_m^{(k)}(s)\rangle + \langle F_{2m}'(s), \Delta v_m^{(k)}(s)\rangle)\,\mathrm{d}s\\
&\leqslant 2\int_0^t (\|F_{1m}'(s)\|(\|\Delta u_m^{(k)}(s)\| + \|\Delta \dot u_m^{(k)}(s)\|) + \|F_{2m}'(s)\|\|\Delta v_m^{(k)}(s)\|)\,\mathrm{d}s\\
&\leqslant 2\sqrt{\frac{3}{\lambda_*}}\int_0^t \Phi_m(M,s)\sqrt{S_m^{(k)}(s)}\,\mathrm{d}s\\
&\leqslant \frac{3}{\lambda_*}\int_0^t \Phi_m(M,s)\,\mathrm{d}s + \int_0^t \Phi_m(M,s) S_m^{(k)}(s)\,\mathrm{d}s.
\end{aligned} \tag{3.20}
$$

Estimate of I_6. By Lemma 3.2 (v) and the inequality $2ab \leqslant \frac{1}{6}a^2 + 6b^2$ for all $a, b \geqslant 0$, we get

$$
\begin{aligned}
I_6 &= -2\langle F_{1m}(t), \Delta u_m^{(k)}(t) + \Delta \dot u_m^{(k)}(t)\rangle\\
&\leqslant 2\sqrt{\frac{2}{\lambda_*}}\|F_{1m}(t)\|\sqrt{S_m^{(k)}(t)}\\
&\leqslant \frac{1}{6}S_m^{(k)}(t) + \frac{12}{\lambda_*}\|F_{1m}(t)\|^2\\
&\leqslant \frac{1}{6}S_m^{(k)}(t) + \frac{24}{\lambda_*}\|F_{1m}(0)\|^2 + \frac{24}{\lambda_*}TK_M^2((1+4M)\sqrt{T} + 2M)^2.
\end{aligned} \tag{3.21}
$$

Estimate of I_7. Similarly,

$$
\begin{aligned}
(3.22)\qquad I_7 = & -2\langle F_{2m}(t), \Delta v_m^{(k)}(t)\rangle \\
& \leqslant \frac{\lambda_*}{6}\|\Delta v_m^{(k)}(t)\|^2 + \frac{6}{\lambda_*}\|F_{2m}(t)\|^2 \\
& \leqslant \frac{1}{6}S_m^{(k)}(t) + \frac{12}{\lambda_*}\|F_{2m}(0)\|^2 + \frac{12}{\lambda_*}TK_M^2((1+4M)\sqrt{T}+2M)^2.
\end{aligned}
$$

In order to estimate the term $I_8 = -2\mu_{1m}(t)\langle\Delta u_m^{(k)}(t), \Delta\dot{u}_m^{(k)}(t)\rangle$, we need the following lemma.

Lemma 3.3. *The term $\|\mu_{1m}(t)\Delta u_m^{(k)}(t)\|^2$ is estimated as*

$$
(3.23)\qquad \|\mu_{1m}(t)\Delta u_m^{(k)}(t)\|^2 \leqslant 2\|\mu_{1m}(0)\Delta\tilde{u}_{0k}\|^2 + \frac{2T}{\lambda_*}\int_0^t \Psi_m^2(M,s)S_m^{(k)}(s)\,\mathrm{d}s.
$$

Proof. First, we need to estimate $\|\frac{\partial}{\partial t}(\mu_{1m}(t)\Delta u_m^{(k)}(t))\|$. By Lemma 3.2 (ii) and the inequality (3.19), we obtain

$$
\begin{aligned}
(3.24)\qquad \Big\|\frac{\partial}{\partial t}(\mu_{1m}(t)\Delta u_m^{(k)}(t))\Big\| & \leqslant \|\mu_{1m}'(t)\Delta u_m^{(k)}(t)\| + \|\mu_{1m}(t)\Delta\dot{u}_m^{(k)}(t)\| \\
& \leqslant \Psi_m(M,t)(\|\Delta u_m^{(k)}(t)\| + \|\Delta\dot{u}_m^{(k)}(t)\|) \\
& \leqslant \sqrt{\frac{2}{\lambda_*}}\Psi_m(M,t)\sqrt{S_m^{(k)}(t)}.
\end{aligned}
$$

The formula

$$
\mu_{1m}(t)\Delta u_m^{(k)}(t) = \mu_{1m}(0)\Delta\tilde{u}_{0k} + \int_0^t \frac{\partial}{\partial s}(\mu_{1m}(s)\Delta u_m^{(k)}(s))\,\mathrm{d}s
$$

implies that

$$
\begin{aligned}
(3.25)\qquad \|\mu_{1m}(t)\Delta u_m^{(k)}(t)\|^2 & \leqslant \Big(\|\mu_{1m}(0)\Delta\tilde{u}_{0k}\| + \int_0^t \Big\|\frac{\partial}{\partial s}(\mu_{1m}(s)\Delta u_m^{(k)}(s))\Big\|\,\mathrm{d}s\Big)^2 \\
& \leqslant 2\|\mu_{1m}(0)\Delta\tilde{u}_{0k}\|^2 + \frac{2T}{\lambda_*}\int_0^t \Psi_m^2(M,s)S_m^{(k)}(s)\,\mathrm{d}s.
\end{aligned}
$$

Lemma 3.3 is proved completely. □

Using Lemma 3.3, the term I_8 is estimated as

$$
\begin{aligned}
(3.26)\qquad I_8 = &-2\mu_{1m}(t)\langle \Delta u_m^{(k)}(t), \Delta \dot{u}_m^{(k)}(t)\rangle \\
&\leqslant \frac{\lambda_*}{6}\|\Delta \dot{u}_m^{(k)}(t)\|^2 + \frac{6}{\lambda_*}\|\mu_{1m}(t)\Delta u_m^{(k)}(t)\|^2 \\
&\leqslant \frac{1}{6}S_m^{(k)}(t) + \frac{12}{\lambda_*}\|\mu_{1m}(0)\Delta \tilde{u}_{0k}\|^2 + \frac{12T}{\lambda_*^2}\int_0^t \Psi_m^2(M,s)S_m^{(k)}(s)\,\mathrm{d}s.
\end{aligned}
$$

Finally, we estimate $I_9 = \int_0^t \|\ddot{v}_m^{(k)}(s)\|^2\,\mathrm{d}s$ as follows. Equation $(3.14)_2$ can be rewritten as

$$
\langle \ddot{v}_m^{(k)}(t), \widetilde{\varphi}_j\rangle - \mu_{2m}(t)\langle \Delta v_m^{(k)}(t), \widetilde{\varphi}_j\rangle = \langle F_{2m}(t), \widetilde{\varphi}_j\rangle, \quad 1 \leqslant j \leqslant k.
$$

Then, it follows after replacing $\widetilde{\varphi}_j$ with $\ddot{v}_m^{(k)}(t)$ and integrating that

$$
\begin{aligned}
(3.27)\qquad I_9 &= \int_0^t \|\ddot{v}_m^{(k)}(s)\|^2\,\mathrm{d}s \\
&\leqslant 2\int_0^t \mu_{2m}^2(s)\|\Delta v_m^{(k)}(s)\|^2\,\mathrm{d}s + 2\int_0^t \|F_{2m}(s)\|^2\,\mathrm{d}s \\
&\leqslant 2\widetilde{K}_M\int_0^t S_m^{(k)}(s)\,\mathrm{d}s + 2TK_M^2.
\end{aligned}
$$

It follows from (3.18), (3.20), (3.21), (3.22), (3.26) and (3.27) that

$$
(3.28)\qquad S_m^{(k)}(t) \leqslant \overline{S}_{m,k} + \sqrt{T}D_1(M) + \int_0^t \eta_m(M,s)S_m^{(k)}(s)\,\mathrm{d}s,
$$

where

$$
\begin{aligned}
(3.29)\qquad \overline{S}_{m,k} &= 2S_m^{(k)}(0) + 4\langle F_{1m}(0), \Delta\tilde{u}_{0k} + \Delta\tilde{u}_{1k}\rangle \\
&\quad + 4\langle F_{2m}(0), \Delta\tilde{v}_{0k}\rangle + 4\mu_{1m}(0)\langle \Delta\tilde{u}_{0k}, \Delta\tilde{u}_{1k}\rangle \\
&\quad + \frac{24}{\lambda_*}(\|\mu_{1m}(0)\Delta\tilde{u}_{0k}\|^2 + 2\|F_{1m}(0)\|^2 + \|F_{2m}(0)\|^2), \\
D_1(M) &= 8\Big(1 + \frac{9}{\lambda_*}((1+4M)\sqrt{T^*} + 2M)^2\Big)\sqrt{T^*}K_M^2 \\
&\quad + \frac{6}{\lambda_*}((1+4M)\sqrt{T^*} + 2M)K_M, \\
\eta_m(M,t) &= D_2(M) + 2\Phi_m(M,t) + \frac{4}{\lambda_*}\Psi_m(M,t) + \frac{24T}{\lambda_*^2}\Psi_m^2(M,t), \\
D_2(M) &= 2 + 2\Big(2 + \frac{3+4M^2}{\lambda_*}\Big)\widetilde{K}_M.
\end{aligned}
$$

On the other hand, the functions $F_{im}(x,0) = f_i(x,0,\tilde u_0(x),\tilde v_0(x),\tilde u_{0x}(x),\tilde v_{0x}(x),$ $\tilde u_1(x),\tilde v_1(x))$, $\mu_{1m}(0) = \mu_1(0,\langle \tilde u_{0kx},\tilde u_{1kx}\rangle)$ and $\mu_{2m}(0) = \mu_2(0,\|\tilde v_0\|^2,\|\tilde v_{0x}\|^2)$ are independent of m and the constant $\overline{S}_{m,k}$ is also independent of m, because

$$
\begin{aligned}
(3.30)\quad \overline{S}_{m,k} &= 2\|\tilde u_{1k}\|^2 + 2\|\tilde v_{1k}\|^2 + 2\|\tilde u_{1kx}\|^2 + 2\|\tilde v_{1k}\|_a^2 + 2\lambda\|\Delta\tilde u_{1k}\|^2 \\
&\quad + 2\mu_{1m}(0)(\|\tilde u_{0kx}\|^2 + \|\Delta\tilde u_{0k}\|^2) + 2\mu_{2m}(0)(\|\tilde v_{0k}\|_a^2 + \|\Delta\tilde v_{0k}\|^2) \\
&\quad + 4\langle F_{1m}(0),\Delta\tilde u_{0k} + \Delta\tilde u_{1k}\rangle + 4\langle F_{2m}(0),\Delta\tilde v_{0k}\rangle + 4\mu_{1m}(0)\langle\Delta\tilde u_{0k},\Delta\tilde u_{1k}\rangle \\
&\quad + \frac{24}{\lambda_*}(\|\mu_{1m}(0)\Delta\tilde u_{0k}\|^2 + 2\|F_{1m}(0)\|^2 + \|F_{2m}(0)\|^2).
\end{aligned}
$$

By means of the convergences in (3.15), there exists a constant $M>0$, independent of k and m, such that

$$
(3.31)\qquad \overline{S}_{m,k} \leqslant \frac{1}{2}M^2 \quad \forall\, m,k\in\mathbb{N}.
$$

Now, we need the following lemmas.

Lemma 3.4. *We have*

(i) $\eta_m(M,\cdot)\in L^1(0,T)$,

(ii) $\int_0^T \eta_m(M,s)\,\mathrm{d}s \leqslant \eta_M(T)$,

where

(3.32)

$$
\begin{aligned}
\eta_M(T) &= TD_2(M) + 2\sqrt{T}K_M((1+4M)\sqrt{T}+2M) \\
&\quad + \frac{4}{\lambda_*}\sqrt{T}\widetilde{K}_M((1+M^2)\sqrt{T}+M^2) + \frac{24T}{\lambda_*^2}\widetilde{K}_M^2((1+M^2)\sqrt{T}+M^2)^2.
\end{aligned}
$$

Lemma 3.5. *For every $T\in(0,T^*]$ and $\delta>0$, we put*

$$
\begin{aligned}
(3.33)\quad \overline{\gamma}_M(\delta,T) &= \Big(3 + \frac{1}{\delta} + \frac{\widetilde{K}_M(1+4M^2)}{\mu_*}\Big)T + \frac{1}{\mu_*}\sqrt{T}\widetilde{K}_M((1+M^2)\sqrt{T}+M^2), \\
\theta_m(M,\delta,t) &= 3 + \frac{1}{\delta} + \frac{\widetilde{K}_M(1+4M^2) + \Psi_{m+1}(M,t)}{\mu_*}.
\end{aligned}
$$

Then

$$
(3.34)\qquad \int_0^T \theta_m(M,\delta,t)\,\mathrm{d}t \leqslant \overline{\gamma}_M(\delta,T).
$$

The proofs of these lemmas are easy, hence we omit the details. □

Lemma 3.6. *For every $T \in (0, T^*]$ and $\delta > 0$, we put*

$$k_T(\delta) = \Big(2 + \frac{1 + 2\sqrt{2}}{\sqrt{2\lambda_*}}\Big)\sqrt{D_M(\delta, T)\exp(\overline{\gamma}_M(\delta, T))}, \tag{3.35}$$

where

$$D_M(\delta, T) = 8TK_M^2 + (17T + \delta)M^4\widetilde{K}_M^2. \tag{3.36}$$

Let $\delta > 0$ be such that

$$\Big(2 + \frac{1 + 2\sqrt{2}}{\sqrt{2\lambda_*}}\Big)M^2\widetilde{K}_M\sqrt{\delta} < 1. \tag{3.37}$$

Then, we can choose $T \in (0, T^*]$ such that

(i) $(\frac{1}{2}M^2 + \sqrt{T}D_1(M))\exp(\eta_M(T)) \leqslant M^2$,

(ii) $k_T(\delta) < 1$.

Proof. The proof of Lemma 3.6 is easy, because

$$\lim_{T \to 0_+}\Big(\frac{M^2}{2} + \sqrt{T}D_1(M)\Big)\exp(\eta_M(T)) = \frac{M^2}{2} < M^2$$

and

$$\lim_{T \to 0_+} k_T(\delta) = \Big(2 + \frac{2}{\sqrt{\lambda_*}} + \frac{1}{\sqrt{2\lambda_*}}\Big)M^2\widetilde{K}_M\sqrt{\delta} < 1.$$

□

By (3.28), (3.31) and Lemma 3.6 (i)–(ii) we obtain

$$S_m^{(k)}(t) \leqslant M^2\exp(-\eta_M(T)) + \int_0^t \eta_m(M, s)S_m^{(k)}(s)\,\mathrm{d}s. \tag{3.38}$$

Using Gronwall's lemma, we deduce from Lemma 3.4 (ii) and (3.38) that

$$\begin{aligned} S_m^{(k)}(t) &\leqslant M^2\exp(-\eta_M(T))\exp\Big(\int_0^t \eta_m(M, s)\,\mathrm{d}s\Big) \\ &\leqslant M^2\exp(-\eta_M(T))\exp(\eta_M(T)) \leqslant M^2 \end{aligned} \tag{3.39}$$

for all $t \in [0, T]$ and for all m and k. Therefore, we have

$$(u_m^{(k)}, v_m^{(k)}) \in W(M, T) \quad \forall\, m \text{ and } k. \tag{3.40}$$

Step 3. Limit process. From (3.40), we deduce the existence of a subsequence of $\{(u_m^{(k)}, v_m^{(k)})\}$ denoted identicallly and such that

$$(3.41)\qquad \begin{cases} (u_m^{(k)}, v_m^{(k)}) \to (u_m, v_m) & \text{in } L^\infty(0,T;(H_0^1\cap H^2)\times(V\cap H^2)) \text{ weak*},\\ (\dot u_m^{(k)}, \dot v_m^{(k)}) \to (u_m', v_m') & \text{in } L^\infty(0,T;(H_0^1\cap H^2)\times V) \text{ weak*},\\ (\ddot u_m^{(k)}, \ddot v_m^{(k)}) \to (u_m'', v_m'') & \text{in } L^2(0,T;H_0^1\times L^2) \text{ weak},\\ (u_m, v_m)\in W(M,T). \end{cases}$$

Passing to limit in (3.14), we have (u_m, v_m) satisfying (3.10)–(3.12) in $L^2(0,T)$. On the other hand, it follows from (3.10) and $(3.41)_4$ that

$$u_m'' = \lambda\Delta u_m' + \mu_{1m}(t)\Delta u_m + F_{1m} \in L^\infty(0,T;L^2),$$
$$v_m'' = \mu_{2m}(t)\Delta v_m + F_{2m} \in L^\infty(0,T;L^2),$$

hence $(u_m, v_m)\in W_1(M,T)$ and the proof of Theorem 3.1 is complete. □

We note that

$$(3.42)\qquad W_1(T) = \{(u,v)\in L^\infty(0,T;H_0^1\times V)\colon\ u'\in L^\infty(0,T;L^2)\cap L^2(0,T;H_0^1),\ v'\in L^\infty(0,T;L^2)\}$$

is a Banach space with respect to the norm (see Lions [23])

$$(3.43)\qquad \|(u,v)\|_{W_1(T)} = \|u\|_{L^\infty(0,T;H_0^1)} + \|v\|_{L^\infty(0,T;V)} + \|u'\|_{L^\infty(0,T;L^2)} + \|u'\|_{L^2(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}.$$

We use the result obtained in Theorem 3.1 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of the problem (1.1)–(1.3). Hence, we get the main result in this section as follows.

Theorem 3.7. *Let* (H_1)–(H_3) *hold. Then*

(i) *the problem* (1.1)–(1.3) *has a unique weak solution* $(u,v)\in W_1(M,T)$, *where the constants* $M>0$ *and* $T>0$ *are chosen as in Theorem* 3.1.

Furthermore,

(ii) *the linear recurrent sequence* $\{(u_m, v_m)\}$ *defined by* (3.9)–(3.12) *converges to the weak solution* (u,v) *of the problem* (1.1)–(1.3) *strongly in the space* $W_1(T)$ *and we have the estimate*

$$(3.44)\qquad \|(u_m,v_m)-(u,v)\|_{W_1(T)} \leqslant C_T k_T^m \quad \forall\, m\in\mathbb{N},$$

where the constant $k_T = k_T(\delta)\in[0,1)$ *is defined as in Lemma* 3.6 *and* C_T *is a constant independent of* m.

P r o o f. (a) Existence of the solution. We prove that $\{(u_m, v_m)\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$ and $\overline{w}_m = v_{m+1} - v_m$. Then $(w_m, \overline{w}_m)$ satisfies the variational problem

(3.45)
$$\begin{cases} \langle w_m''(t), \varphi\rangle + \lambda\langle w_{mx}'(t), \varphi_x\rangle + \mu_{1,m+1}(t)\langle w_{mx}(t), \varphi_x\rangle \\ \quad = (\mu_{1,m+1}(t) - \mu_{1m}(t))\langle \Delta u_m(t), \varphi\rangle + \langle F_{1,m+1}(t) - F_{1m}(t), \varphi\rangle \quad \forall\, \varphi \in H_0^1, \\ \langle \overline{w}_m''(t), \widetilde{\varphi}\rangle + \mu_{2,m+1}(t) a(\overline{w}_m(t), \widetilde{\varphi}) \\ \quad = (\mu_{2,m+1}(t) - \mu_{2m}(t))\langle \Delta v_m(t), \widetilde{\varphi}\rangle + \langle F_{2,m+1}(t) - F_{2m}(t), \widetilde{\varphi}\rangle \quad \forall\, \widetilde{\varphi} \in V, \\ w_m(0) = w_m'(0) = \overline{w}_m(0) = \overline{w}_m'(0) = 0, \end{cases}$$

where

$$\begin{aligned} (3.46) \qquad & F_{i,m+1}(t) - F_{im}(t) = f_i[u_m, v_m](t) - f_i[u_{m-1}, v_{m-1}](t), \quad i = 1, 2, \\ & \mu_{1,m+1}(t) - \mu_{1m}(t) = \mu_1[u_m](t) - \mu_1[u_{m-1}](t), \\ & \mu_{2,m+1}(t) - \mu_{2m}(t) = \mu_2[v_m](t) - \mu_2[v_{m-1}](t). \end{aligned}$$

Taking $(\varphi, \widetilde{\varphi}) = (w_m'(t), \overline{w}_m'(t))$ in (3.45), after integrating in t we get

$$\begin{aligned} (3.47) \qquad S_m(t) = & \int_0^t (\mu_{1,m+1}'(s)\|w_{mx}(s)\|^2 + \mu_{2,m+1}'(s)\|\overline{w}_m(s)\|_a^2)\,\mathrm{d}s \\ & + 2\int_0^t (\langle F_{1,m+1}(s) - F_{1m}(s), w_m'(s)\rangle \\ & \quad + \langle F_{2,m+1}(s) - F_{2m}(s), \overline{w}_m'(s)\rangle)\,\mathrm{d}s \\ & + 2\int_0^t (\mu_{1,m+1}(s) - \mu_{1m}(s))\langle \Delta u_m(s), w_m'(s)\rangle\,\mathrm{d}s \\ & + 2\int_0^t (\mu_{2,m+1}(s) - \mu_{2m}(s))\langle \Delta v_m(s), \overline{w}_m'(s)\rangle\,\mathrm{d}s \\ = {} & J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where

$$\begin{aligned} (3.48) \qquad S_m(t) = {} & \|w_m'(t)\|^2 + \|\overline{w}_m'(t)\|^2 \\ & + \mu_{1,m+1}(t)\|w_{mx}(t)\|^2 + \mu_{2,m+1}(t)\|\overline{w}_m(t)\|_a^2 \\ & + 2\lambda\int_0^t \|w_{mx}'(s)\|^2\,\mathrm{d}s \end{aligned}$$

and all the integrals on the right-hand side of (3.47) are estimated as follows.

Integral J_1. By Lemma 3.2 and the inequality

$$(3.49) \qquad S_m(t) \geqslant \|w_m'(t)\|^2 + \|\overline{w}_m'(t)\|^2 + \mu_*(\|w_{mx}(t)\|^2 + \|\overline{w}_m(t)\|_a^2)$$

with $\mu_* = \min\{\mu_{1*}, \mu_{2*}\}$, we obtain

$$(3.50) \qquad J_1 = \int_0^t (\mu'_{1,m+1}(s)\|w_{mx}(s)\|^2 + \mu'_{2,m+1}(s)\|\overline{w}_m(s)\|_a^2)\,\mathrm{d}s \leqslant \frac{1}{\mu_*}\int_0^t (\Psi_{m+1}(M,s) + \widetilde{K}_M(1+4M^2))S_m(s)\,\mathrm{d}s.$$

Integral J_2. By (H_3) it is clear that

$$\begin{aligned}
&\|F_{i,m+1}(t) - F_{im}(t)\| \\
&\qquad \leqslant K_M(2\|\nabla w_{m-1}(t)\| + 2\|\nabla\overline{w}_{m-1}(t)\| + \|w'_{m-1}(t)\| + \|\overline{w}'_{m-1}(t)\|) \\
&\qquad \leqslant 2K_M(\|\nabla w_{m-1}(t)\| + \|\nabla\overline{w}_{m-1}(t)\| + \|w'_{m-1}(t)\| + \|\overline{w}'_{m-1}(t)\|) \\
&\qquad \leqslant 2K_M\|(w_{m-1}, \overline{w}_{m-1})\|_{W_1(T)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(3.51) \qquad J_2 &= 2\int_0^t (\langle F_{1,m+1}(s) - F_{1m}(s), w'_m(s)\rangle \\
&\qquad + \langle F_{2,m+1}(s) - F_{2m}(s), \overline{w}'_m(s)\rangle)\,\mathrm{d}s \\
&\leqslant 2\int_0^t (\|F_{1,m+1}(s) - F_{1m}(s)\|\|w'_m(s)\| \\
&\qquad + \|F_{2,m+1}(s) - F_{2m}(s)\|\|\overline{w}'_m(s)\|)\,\mathrm{d}s \\
&\leqslant 4\sqrt{2}K_M\|(w_{m-1}, \overline{w}_{m-1})\|_{W_1(T)}\int_0^t \sqrt{S_m(s)}\,\mathrm{d}s \\
&\leqslant 8TK_M^2\|(w_{m-1}, \overline{w}_{m-1})\|^2_{W_1(T)} + \int_0^t S_m(s)\,\mathrm{d}s.
\end{aligned}$$

Integrals J_3, J_4. By the inequalities

$$\begin{aligned}
|\mu_{1,m+1}(t) - \mu_{1m}(t)| &\leqslant \widetilde{K}_M|\langle\nabla u_m(t), \nabla u'_m(t)\rangle - \langle\nabla u_{m-1}(t), \nabla u'_{m-1}(t)\rangle| \\
&\leqslant \widetilde{K}_M(\|\nabla w_{m-1}(t)\|\|\nabla u'_m(t)\| + \|\nabla u_{m-1}(t)\|\|\nabla w'_{m-1}(t)\|) \\
&\leqslant M\widetilde{K}_M(\|\nabla w_{m-1}(t)\| + \|\nabla w'_{m-1}(t)\|) \\
&\leqslant M\widetilde{K}_M(\|(w_{m-1}, \overline{w}_{m-1})\|_{W_1(T)} + \|\nabla w'_{m-1}(t)\|)
\end{aligned}$$

and

$$\begin{aligned}
|\mu_{2,m+1}(t) - \mu_{2m}(t)| &\leqslant \widetilde{K}_M(|\|v_m(t)\|^2 - \|v_{m-1}(t)\|^2| + |\|\nabla v_m(t)\|^2 - \|\nabla v_{m-1}(t)\|^2|) \\
&\leqslant 2M\widetilde{K}_M(\|\overline{w}_{m-1}(t)\| + \|\nabla\overline{w}_{m-1}(t)\|) \\
&\leqslant 4M\widetilde{K}_M\|(w_{m-1}, \overline{w}_{m-1})\|_{W_1(T)}
\end{aligned}$$

we obtain that

$$(3.52)\quad \begin{aligned} J_3 &= 2\int_0^t (\mu_{1,m+1}(s)-\mu_{1m}(s))\langle \Delta u_m(s), w_m'(s)\rangle \,\mathrm{d}s \\ &\leqslant 2M^2\widetilde{K}_M\int_0^t (\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)} + \|\nabla w_{m-1}'(s)\|)\|w_m'(s)\|\,\mathrm{d}s \\ &= 2M^2\widetilde{K}_M\int_0^t \|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}\|w_m'(s)\|\,\mathrm{d}s \\ &\quad + 2M^2\widetilde{K}_M\int_0^t \|\nabla w_{m-1}'(s)\|\|w_m'(s)\|\,\mathrm{d}s \\ &\leqslant TM^4\widetilde{K}_M^2\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 + \int_0^t \|w_m'(s)\|^2\,\mathrm{d}s \\ &\quad + \delta M^4\widetilde{K}_M^2\int_0^t \|\nabla w_{m-1}'(s)\|^2\,\mathrm{d}s + \frac{1}{\delta}\int_0^t \|w_m'(s)\|^2\,\mathrm{d}s \\ &\leqslant (T+\delta)M^4\widetilde{K}_M^2\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 + \Big(1+\frac{1}{\delta}\Big)\int_0^t S_m(s)\,\mathrm{d}s \end{aligned}$$

and

$$(3.53)\quad \begin{aligned} J_4 &= 2\int_0^t (\mu_{2,m+1}(s)-\mu_{2m}(s))\langle \Delta v_m(s), \overline{w}_m'(s)\rangle \,\mathrm{d}s \\ &\leqslant 8M^2\widetilde{K}_M\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}\int_0^t \|\overline{w}_m'(s)\|\,\mathrm{d}s \\ &\leqslant 16TM^4\widetilde{K}_M^2\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 + \int_0^t S_m(s)\,\mathrm{d}s. \end{aligned}$$

Combining (3.47), (3.50)–(3.53), we obtain

$$(3.54)\quad S_m(t) \leqslant D_M(\delta,T)\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 + \int_0^t \theta_m(M,\delta,s)S_m(s)\,\mathrm{d}s,$$

where $\theta_m(M,\delta,t)$, $D_M(\delta,T)$ are as in (3.33) and (3.36).

Using Gronwall's lemma, we deduce from (3.33), (3.34) and (3.54) that

$$(3.55)\quad \begin{aligned} S_m(t) &\leqslant D_M(\delta,T)\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 \exp\Big(\int_0^T \theta_m(M,\delta,s)\,\mathrm{d}s\Big) \\ &\leqslant D_M(\delta,T)\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)}^2 \exp(\overline{\gamma}_M(\delta,T)). \end{aligned}$$

Hence, we deduce from (3.43) and (3.55) that

$$(3.56)\quad \|(w_m,\overline{w}_m)\|_{W_1(T)} \leqslant k_T(\delta)\|(w_{m-1},\overline{w}_{m-1})\|_{W_1(T)} \quad \forall\, m\in\mathbb{N},$$

where the constant $k_T = k_T(\delta) \in [0,1)$ is defined as in Lemma 3.6, which implies that

$$\|(u_{m+p}, v_{m+p}) - (u_m, v_m)\|_{W_1(T)} \leqslant \frac{M}{1-k_T} k_T^m \quad \forall\, m \text{ and } p \in \mathbb{N}. \tag{3.57}$$

It follows that $\{(u_m, v_m)\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $(u,v) \in W_1(T)$ such that

$$(u_m, v_m) \to (u, v) \text{ strongly in } W_1(T). \tag{3.58}$$

On the other hand, $(u_m, v_m) \in W(M,T)$, thus there exists a subsequence $\{(u_{m_j}, v_{m_j})\}$ of $\{(u_m, v_m)\}$ such that

$$\begin{cases} (u_{m_j}, v_{m_j}) \to (u, v) & \text{in } L^\infty(0,T; (H_0^1 \cap H^2) \times (V \cap H^2)) \text{ weak*}, \\ (u'_{m_j}, v'_{m_j}) \to (u', v') & \text{in } L^\infty(0,T; (H_0^1 \cap H^2) \times V) \text{ weak*}, \\ (u''_{m_j}, v''_{m_j}) \to (u'', v'') & \text{in } L^2(0,T; H_0^1 \times L^2) \text{ weak}, \\ (u,v) \in W(M,T). \end{cases} \tag{3.59}$$

We note that

$$\|F_{im} - f_i[u,v]\|_{L^\infty(0,T;L^2)} \leqslant 2K_M \|(u_{m-1}, v_{m-1}) - (u,v)\|_{W_1(T)}, \quad i = 1,2. \tag{3.60}$$

Hence, we deduce from (3.58) and (3.60) that

$$F_{im} \to f_i[u,v] \text{ strongly in } L^\infty(0,T;L^2), \quad i = 1,2. \tag{3.61}$$

We also note that

$$|\mu_{1m}(t) - \mu_1[u](t)| \leqslant M\widetilde{K}_M (\|\nabla u_{m-1}(t) - \nabla u(t)\| + \|\nabla u'_{m-1}(t) - \nabla u'(t)\|).$$

Therefore,

$$\begin{aligned} \|\mu_{1m} - \mu_1[u]\|_{L^2(0,T)} &\leqslant M\widetilde{K}_M \big(\sqrt{T}\|u_{m-1} - u\|_{L^\infty(0,T,H_0^1)} + \|u'_{m-1} - u'\|_{L^2(0,T,H_0^1)}\big) \\ &\leqslant \big(1 + \sqrt{T}\big) M\widetilde{K}_M \|(u_{m-1}, v_{m-1}) - (u,v)\|_{W_1(T)} \to 0, \end{aligned}$$

which implies that

$$\mu_{1m} \to \mu_1[u] \text{ strongly in } L^2(0,T). \tag{3.62}$$

We also have

$$\|\mu_{2m} - \mu_2[v]\|_{L^\infty(0,T)} \leqslant 4M\widetilde{K}_M \|(u_{m-1}, v_{m-1}) - (u,v)\|_{W_1(T)} \to 0,$$

hence

$$\mu_{2m} \to \mu_2[v] \text{ strongly in } L^\infty(0,T). \tag{3.63}$$

Finally, passing to limit in (3.10)–(3.12) as $m = m_j \to \infty$, it implies from (3.58), (3.59), (3.61), (3.62) and (3.63) that there exists $(u,v) \in W(M,T)$ satisfying the system

$$\begin{cases} \langle u''(t), \varphi\rangle + \lambda\langle u'_x(t), \varphi_x\rangle + \mu_1[u](t)\langle u_x(t), \varphi_x\rangle = \langle f_1[u,v](t), \varphi\rangle, \\ \langle v''(t), \widetilde{\varphi}\rangle + \mu_2[v](t)a(v(t), \widetilde{\varphi}) = \langle f_2[u,v](t), \widetilde{\varphi}\rangle \end{cases} \tag{3.64}$$

for all $(\varphi, \widetilde{\varphi}) \in H_0^1 \times V$ and the initial conditions

$$\begin{aligned} (u(0), u'(0)) &= (\tilde{u}_0, \tilde{u}_1), \\ (v(0), v'(0)) &= (\tilde{v}_0, \tilde{v}_1). \end{aligned} \tag{3.65}$$

Furthermore, from the assumptions (H_2) and (H_3) we obtain from $(3.59)_4$ and (3.64), that

$$\begin{cases} u'' = \lambda\Delta u' + \mu_1[u]\Delta u + f_1[u,v] \in L^\infty(0,T;L^2), \\ v'' = \mu_2[v]\Delta v + f_2[u,v] \in L^\infty(0,T;L^2). \end{cases} \tag{3.66}$$

Thus, we have $(u,v) \in W_1(M,T)$. The existence proof is completed.

(b) Uniqueness of the solution. Let $(u,v), (\tilde{u},\tilde{v}) \in W_1(M,T)$ be two weak solutions of the problem (1.1)–(1.3). Then $(w,\overline{w}) = (u-\tilde{u}, v-\tilde{v})$ satisfies the variational problem

$$\begin{cases} \langle w''(t), \varphi\rangle + \lambda\langle w'_x(t), \varphi_x\rangle + \mu_1(t)\langle w_x(t), \varphi_x\rangle \\ \qquad = \langle F_1(t) - \widetilde{F}_1(t), \varphi\rangle + [\mu_1(t) - \widetilde{\mu}_1(t)]\langle \Delta\tilde{u}(t), \varphi\rangle \quad \forall\, \varphi \in H_0^1, \\ \langle \overline{w}''(t), \widetilde{\varphi}\rangle + \mu_2(t)a(\overline{w}(t), \widetilde{\varphi}) \\ \qquad = \langle F_2(t) - \widetilde{F}_2(t), \widetilde{\varphi}\rangle + [\mu_2(t) - \widetilde{\mu}_2(t)]\langle \Delta\tilde{v}(t), \widetilde{\varphi}\rangle \quad \forall\, \widetilde{\varphi} \in V, \\ w(0) = w'(0) = \overline{w}(0) = \overline{w}'(0) = 0, \end{cases} \tag{3.67}$$

where

$$\begin{cases} \mu_1(t) = \mu_1[u](t), & \widetilde{\mu}_1(t) = \mu_1[\tilde{u}](t), \\ \mu_2(t) = \mu_2[v](t), & \widetilde{\mu}_2(t) = \mu_2[\tilde{v}](t), \\ F_1(t) = f_1[u,v](t), & \widetilde{F}_1(t) = f_1[\tilde{u},\tilde{v}](t), \\ F_2(t) = f_2[u,v](t), & \widetilde{F}_2(t) = f_2[\tilde{u},\tilde{v}](t). \end{cases} \tag{3.68}$$

We take $(\varphi, \widetilde{\varphi}) = (w'(t), \overline{w}'(t))$ in (3.67) and integrate with respect to t to get

$$Z(t) = 2\int_0^t (\langle F_1(s) - \widetilde{F}_1(s), w'(s)\rangle + \langle F_2(s) - \widetilde{F}_2(s), \overline{w}'(s)\rangle)\,\mathrm{d}s + \int_0^t (\mu_1'(s)\|w_x(s)\|^2 + \mu_2'(s)\|\overline{w}(s)\|_a^2)\,\mathrm{d}s + 2\int_0^t (\mu_1(s) - \widetilde{\mu}_1(s))\langle \Delta\tilde{u}(s), w'(s)\rangle\,\mathrm{d}s + 2\int_0^t (\mu_2(s) - \widetilde{\mu}_2(s))\langle \Delta\tilde{v}(s), \overline{w}'(s)\rangle\,\mathrm{d}s, \tag{3.69}$$

where

$$Z(t) = \|w'(t)\|^2 + \|\overline{w}'(t)\|^2 + \mu_1(t)\|w_x(t)\|^2 + \mu_2(t)\|\overline{w}(t)\|_a^2 + 2\lambda\int_0^t \|w_x'(s)\|^2\,\mathrm{d}s. \tag{3.70}$$

Put

$$\mathcal{H}(s) = \mathcal{H}_M + \frac{2}{\bar{\mu}_*}M\widetilde{K}_M\|u_x''(s)\|, \tag{3.71}$$

where $\bar{\mu}_* = \min\{1, \mu_{1*}, \mu_{2*}\}$ and $\mathcal{H}_M$ is a constant defined by

$$\mathcal{H}_M = \frac{16\sqrt{2}K_M}{\sqrt{\bar{\mu}_*}} + 2\Big(\frac{2+4\sqrt{2}}{\sqrt{\bar{\mu}_*}} + \frac{M^2\widetilde{K}_M}{\lambda}\Big)M^2\widetilde{K}_M + \frac{2}{\bar{\mu}_*}(2+5M^2)\widetilde{K}_M. \tag{3.72}$$

Then it follows from (3.69) that

$$Z(t) \leqslant \int_0^t \mathcal{H}(s)Z(s)\,\mathrm{d}s. \tag{3.73}$$

We remark that by $u'' \in L^2(0,T;H_0^1)$, we obtain $\mathcal{H} \in L^2(0,T)$.

It follows from (3.73) that

$$Z^2(t) \leqslant \|\mathcal{H}\|_{L^2(0,T)}^2 \int_0^t Z^2(s)\,\mathrm{d}s. \tag{3.74}$$

By Gronwall's lemma, we deduce $Z(t) \equiv 0$, i.e., $u - \tilde{u} = v - \tilde{v} = 0$. Theorem 3.7 is proved completely. □

4. Exponential decay of solutions

This section investigates the decay of the solution of the problem (1.1)–(1.3) corresponding to $\zeta = 0$,

$$\begin{aligned} f_1(x,t,u,v,u_x,v_x,u_t,v_t) &= -\lambda_1 u_t + f_1(u,v) + F_1(x,t),\\ f_2(x,t,u,v,u_x,v_x,u_t,v_t) &= -\lambda_2 v_t + f_2(u,v) + F_2(x,t),\\ \mu_1(t,\langle u_x(t),u_{xt}(t)\rangle) &= \mu_* + \sigma(\langle u_x(t),u_{xt}(t)\rangle),\\ \mu_2(t,\|v(t)\|^2,\|v_x(t)\|^2) &= \mu_2(\|v_x(t)\|^2). \end{aligned}$$

Then, the problem (1.1)–(1.3) becomes the problem

$$\text{(4.1)}\qquad \begin{cases} u_{tt} - \lambda u_{xxt} - (\mu_* + \sigma(\langle u_x(t),u_{xt}(t)\rangle))u_{xx} + \lambda_1 u_t \\ \qquad\qquad = f_1(u,v) + F_1(x,t), \quad 0<x<1,\ t>0,\\ v_{tt} - \mu_2(\|v_x(t)\|^2)v_{xx} + \lambda_2 v_t = f_2(u,v) + F_2(x,t), \quad 0<x<1,\ t>0,\\ u(0,t) = u(1,t) = v_x(0,t) = v(1,t) = 0,\\ (u(x,0),u_t(x,0)) = (\tilde u_0(x),\tilde u_1(x)),\\ (v(x,0),v_t(x,0)) = (\tilde v_0(x),\tilde v_1(x)), \end{cases}$$

where $\lambda > 0$, $\lambda_i > 0$, $\mu_* > 0$ are given constants and σ, μ_2, $f_i(u,v)$, $F_i(x,t)$, $\tilde u_i$, $\tilde v_i$, μ_i, f_i $(i=1,2)$ are given functions.

By the same method as in the proof of Theorem 3.1, equation (4.1) has a weak solution $u(x,t)$ such that

$$\text{(4.2)}\qquad \begin{aligned} (u,v) &\in C([0,T];(H_0^1\cap H^2)\times V)\cap C^1([0,T];H_0^1\times L^2)\\ &\qquad \cap L^\infty(0,T;(H_0^1\cap H^2)\times(V\cap H^2)),\\ (u',v') &\in C([0,T];H_0^1\times L^2)\cap L^\infty(0,T;(H_0^1\cap H^2)\times V),\\ (u'',v'') &\in L^\infty(0,T;H^2\cap H_0^1)\times L^\infty(0,T;L^2) \end{aligned}$$

for $T>0$ small enough.

We make the following assumptions.

($\overline{\mathrm{H}}_2$) $\sigma \in C^1(\mathbb{R})$ and there exists a positive constant $\sigma_* < \mu_*$ such that
- (i) $\sigma(y) \geqslant -\sigma_*$ for all $y \in \mathbb{R}$,
- (ii) $y\sigma(y) > 0$ for all $y \in \mathbb{R}$, $y \neq 0$.

($\overline{\mathrm{H}}_3$) $\mu_2 \in C^1(\mathbb{R}_+)$ and there exist positive constants μ_2^*, χ_* such that
- (i) $\mu_2(y) \geqslant \mu_2^* > 0$ for all $y \geqslant 0$,
- (ii) $y\mu_2(y) \geqslant \chi_* \int_0^y \mu_2(z)\,\mathrm{d}z$ for all $y \geqslant 0$.

($\overline{\mathrm{H}}_4$) There exist $\mathcal{F} \in C^2(\mathbb{R}^2;\mathbb{R})$ and the constants α, β, d_1, $\bar d_1 > 0$ with $\alpha > 2$, $\beta > 2$ such that

(i) $\frac{\partial \mathcal{F}}{\partial u}(u,v) = f_1(u,v)$, $\frac{\partial \mathcal{F}}{\partial v}(u,v) = f_2(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(ii) $uf_1(u,v) + vf_2(u,v) \leqslant d_1\mathcal{F}(u,v)$ for all $(u,v) \in \mathbb{R}^2$,

(iii) $\mathcal{F}(u,v) \leqslant \bar{d}_1(|u|^\alpha + |v|^\beta)$ for all $(u,v) \in \mathbb{R}^2$.

($\overline{\mathrm{H}}_5$) F_1, $F_2 \in L^\infty(\mathbb{R}_+; L^2) \cap L^1(\mathbb{R}_+; L^2)$ and there exist two constants C_0, $\gamma_0 > 0$ such that $\|F_1(t)\|^2 + \|F_2(t)\|^2 \leqslant C_0 \exp(-\gamma_0 t)$ for all $t \geqslant 0$.

($\overline{\mathrm{H}}_6$) $p > \max\{2, d_1, d_1/\chi_*\}$ and $\sigma_*/\mu_* \leqslant 1 - d_1/p$.

Example 4.1. Below we give an example of the functions σ, μ_2 satisfying assumptions ($\overline{\mathrm{H}}_3$), ($\overline{\mathrm{H}}_4$),

$$\sigma(y) = \begin{cases} \dfrac{-\sigma_* y}{y-1} & \text{if } \; y < 0, \\ (\sigma_* + |y|^{r_1-2})y & \text{if } \; y \geqslant 0, \end{cases} \qquad \mu_2(y) = \mu_{2*} + y^{r_2-1}, \tag{4.3}$$

where $\sigma_* > 0$, $\mu_{2*} > 0$; r_1, $r_2 > 2$ are constants.

First, we construct the Lyapunov functional

$$\mathcal{L}(t) = E(t) + \delta\psi(t), \quad t \geqslant 0, \tag{4.4}$$

where $\delta > 0$ will be chosen later and

$$\begin{aligned} E(t) &= \frac{1}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) + \frac{1}{2}(g*u')(t) + \frac{1}{2}(g*v')(t) + \frac{\mu_*}{2}\|u_x(t)\|^2 \\ &\quad + \frac{1}{2}\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z - \int_0^1 \mathcal{F}(u(x,t), v(x,t))\,\mathrm{d}x \\ &= \frac{1}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) \\ &\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big((g*u')(t) + (g*v')(t) + \mu_*\|u_x(t)\|^2 \\ &\quad + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z\Big) + \frac{1}{p}I(t), \end{aligned} \tag{4.5}$$

$$\psi(t) = \langle u'(t), u(t)\rangle + \langle v'(t), v(t)\rangle + \frac{\lambda}{2}\|u_x(t)\|^2 + \frac{\lambda_1}{2}\|u(t)\|^2 + \frac{\lambda_2}{2}\|v(t)\|^2, \tag{4.6}$$

where

$$(g*u')(t) = \int_0^t g(t-s)\|u'(s)\|^2\,\mathrm{d}s,$$

$$(g*v')(t) = \int_0^t g(t-s)\|v'(s)\|^2\,\mathrm{d}s$$

with $g(t) = 2\bar{\lambda}\mathrm{e}^{-2\bar{k}t}$, $\bar{k}$, $\bar{\lambda}$ are constants with $\bar{k} > 0$, $0 < \bar{\lambda} < \lambda_* = \min\{\lambda_1, \lambda_2\}$, and

$$\begin{aligned} I(t) &= (g*u')(t) + (g*v')(t) + \mu_*\|u_x(t)\|^2 \\ &\quad + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z - p\int_0^1 \mathcal{F}(u(x,t), v(x,t))\,\mathrm{d}x, \quad t \geqslant 0. \end{aligned} \tag{4.7}$$

Then we have the following theorem.

Theorem 4.2. *Assume that* $(\overline{\mathrm{H}}_2)$–$(\overline{\mathrm{H}}_6)$ *hold. Let* $(\tilde{u}_0,\tilde{v}_0)\in(H_0^1\cap H^2)\times(V\cap H^2)$, $(\tilde{u}_1,\tilde{v}_1)\in(H_0^1\cap H^2)\times V$ *such that* $I(0)>0$ *and the initial energy* $E(0)$ *satisfy*

$$\eta^*=\widetilde{\mu}_*-p\bar{d}_1(R_*^{\alpha-2}+R_*^{\beta-2})>0, \tag{4.8}$$

where $R_*^2=2pE_*/((p-2)\widetilde{\mu}_*)$, $E_*=(E(0)+\frac{1}{2}\varrho)\exp(\varrho)$, $\varrho=\int_0^\infty(\|F_1(t)\|+\|F_1(t)\|)\,\mathrm{d}t$, $\widetilde{\mu}_*=\min\{\mu_*,\mu_{2*}\}$. *Then, any global weak solution of the problem* (4.1) *is exponentially decaying, i.e., there exist positive constants* $\overline{C}$, $\overline{\gamma}$ *such that*

$$\|u'(t)\|^2+\|v'(t)\|^2+\|u_x(t)\|^2+\|v_x(t)\|^2\leqslant\overline{C}\exp(-\overline{\gamma}t)\quad\forall\,t\geqslant0. \tag{4.9}$$

First, to prove the theorem we need the following lemmas.

Lemma 4.3. *The energy functional* $E(t)$ *defined by* (4.5) *satisfies*

(i) $E'(t)\leqslant\frac{1}{2}(\|F_1(t)\|+\|F_2(t)\|)+\frac{1}{2}(\|F_1(t)\|+\|F_2(t)\|)(\|u'(t)\|^2+\|v'(t)\|^2)$,

(ii) $E'(t)\leqslant-\lambda\|u_x'(t)\|^2-(\lambda_*-\bar{\lambda}-\frac{1}{2}\varepsilon_1)(\|u'(t)\|^2+\|v'(t)\|^2)-\overline{k}((g*u')(t)+(g*v')(t))+\frac{1}{2}\varepsilon_1^{-1}(\|F_1(t)\|^2+\|F_2(t)\|^2)$

for all $t\geqslant0$, $\varepsilon_1>0$, *where* $\lambda_*=\min\{\lambda_1,\lambda_2\}$.

Proof. Multiplying $(4.1)_1$ by $u'(x,t)$, $(4.2)_2$ by $v'(x,t)$ and integrating over $[0,1]$, we get

$$\begin{aligned}E'(t)=&-\lambda\|u_x'(t)\|^2-(\lambda_1-\bar{\lambda})\|u'(t)\|^2-(\lambda_2-\bar{\lambda})\|v'(t)\|^2\\&-\overline{k}((g*u')(t)+(g*v')(t))-\langle u_x(t),u_x'(t)\rangle\sigma(\langle u_x(t),u_x'(t)\rangle)\\&+\langle F_1(t),u'(t)\rangle+\langle F_2(t),v'(t)\rangle.\end{aligned} \tag{4.10}$$

On the other hand,

$$\begin{aligned}\langle F_1(t),u'(t)\rangle&\leqslant\frac{1}{2}\|F_1(t)\|+\frac{1}{2}\|F_1(t)\|\|u'(t)\|^2,\\\langle F_2(t),u'(t)\rangle&\leqslant\frac{1}{2}\|F_2(t)\|+\frac{1}{2}\|F_2(t)\|\|v'(t)\|^2.\end{aligned} \tag{4.11}$$

As $\langle u_x(t),u_x'(t)\rangle\sigma(\langle u_x(t),u_x'(t)\rangle)\geqslant0$, it follows from (4.10) and (4.11) that Lemma 4.3 (i) holds. Similarly,

$$\begin{aligned}\langle F_1(t),u'(t)\rangle&\leqslant\frac{1}{2\varepsilon_1}\|F_1(t)\|^2+\frac{\varepsilon_1}{2}\|u'(t)\|^2,\\\langle F_2(t),v'(t)\rangle&\leqslant\frac{1}{2\varepsilon_1}\|F_2(t)\|^2+\frac{\varepsilon_1}{2}\|v'(t)\|^2\quad\forall\,\varepsilon_1>0,\end{aligned} \tag{4.12}$$

and it follows from (4.10) and (4.12) that Lemma 4.3 (ii) holds.

Lemma 4.3 is proved. □

Lemma 4.4. *Assume that* $(\overline{\mathrm{H}}_2)$–$(\overline{\mathrm{H}}_6)$, $I(0)>0$ *and* (4.8) *hold. Then* $I(t)>0$ *for all* $t\geqslant 0$.

P r o o f. By the continuity of $I(t)$ and $I(0)>0$, there exists $\widetilde{T}_1>0$ such that

$$I(t)=I(u(t),v(t))>0 \quad \forall\, t\in[0,\widetilde{T}_1], \tag{4.13}$$

which implies

$$\begin{aligned} E(t) &\geqslant \frac{1}{2}(\|u'(t)\|^2+\|v'(t)\|^2)+\Big(\frac{1}{2}-\frac{1}{p}\Big)\Big(\mu_*\|u_x(t)\|^2+\int_0^{\|v_x(t)\|^2}\mu_2(z)\,\mathrm{d}z\Big)\\ &\geqslant \frac{1}{2}(\|u'(t)\|^2+\|v'(t)\|^2)+\frac{(p-2)\widetilde{\mu}_*}{2p}(\|u_x(t)\|^2+\|v_x(t)\|^2) \quad \forall\, t\in[0,\widetilde{T}_1], \end{aligned} \tag{4.14}$$

where $\widetilde{\mu}_*=\min\{\mu_*,\mu_{2*}\}$.

Combining Lemma 4.3 (i) and (4.14) and using Gronwall's inequality, we get

$$\begin{aligned} \|u_x(t)\|^2+\|v_x(t)\|^2 &\leqslant \frac{2p}{(p-2)\widetilde{\mu}_*}E(t)\leqslant \frac{2pE_*}{(p-2)\widetilde{\mu}_*}\equiv R_*^2 \quad \forall\, t\in[0,\widetilde{T}_1],\\ \|u'(t)\|^2+\|v'(t)\|^2 &\leqslant 2E(t)\leqslant 2E_* \quad \forall\, t\in[0,\widetilde{T}_1], \end{aligned} \tag{4.15}$$

where E_* is as in (4.8). Hence, it follows from $(\overline{\mathrm{H}}_4)$ (iii) and (4.15) that

$$\begin{aligned} p\int_0^1\mathcal{F}(u(x,t),v(x,t))\,\mathrm{d}x &\leqslant p\bar{d}_1(\|u(t)\|_{L^\alpha}^\alpha+\|v(t)\|_{L^\beta}^\beta)\\ &\leqslant p\bar{d}_1(\|u_x(t)\|^\alpha+\|v_x(t)\|^\beta)\\ &\leqslant p\bar{d}_1(\|u_x(t)\|^{\alpha-2}+\|v_x(t)\|^{\beta-2})\\ &\qquad\times(\|u_x(t)\|^2+\|v_x(t)\|^2)\\ &\leqslant p\bar{d}_1(R_*^{\alpha-2}+R_*^{\beta-2})(\|u_x(t)\|^2+\|v_x(t)\|^2). \end{aligned} \tag{4.16}$$

Thus,

$$\begin{aligned} I(t) &= (g*u')(t)+(g*v')(t)+\mu_*\|u_x(t)\|^2\\ &\quad+\int_0^{\|v_x(t)\|^2}\mu_2(z)\,\mathrm{d}z-p\int_0^1\mathcal{F}(u(x,t),v(x,t))\,\mathrm{d}x\\ &\geqslant (g*u')(t)+(g*v')(t)+\mu_*\|u_x(t)\|^2+\mu_{2*}\|v_x(t)\|^2\\ &\quad-p\bar{d}_1(R_*^{\alpha-2}+R_*^{\beta-2})(\|u_x(t)\|^2+\|v_x(t)\|^2)\\ &\geqslant (g*u')(t)+(g*v')(t)+\eta^*(\|u_x(t)\|^2+\|v_x(t)\|^2)\geqslant 0 \quad \forall t\in[0,\widetilde{T}_1], \end{aligned} \tag{4.17}$$

where $\eta^*=\widetilde{\mu}_*-p\bar{d}_1(R_*^{\alpha-2}+R_*^{\beta-2})$ and $\widetilde{\mu}_*=\min\{\mu_*,\mu_{2*}\}$ as in (4.8).

Now, we prove that $I(t) > 0$ for all $t \geqslant 0$. We put $T_\infty = \sup\{T > 0\colon\ I(t) > 0$ for all $t \in [0, T]\}$. If $T_\infty < \infty$ then, by the continuity of $I(t)$, we have $I(T_\infty) \geqslant 0$.

In case of $I(T_\infty) > 0$, by the same arguments as above, we can deduce that there exists $\widetilde{T}_\infty > T_\infty$ such that $I(t) > 0$ for all $t \in [0, \widetilde{T}_\infty]$. We obtain a contradiction to the definition of T_∞.

In case of $I(T_\infty) = 0$, it follows from (4.17) that

$$0 = I(T_\infty) \geqslant (g * u')(T_\infty) + (g * v')(T_\infty) + \eta^*(\|u_x(T_\infty)\|^2 + \|v_x(T_\infty)\|^2) \geqslant 0.$$

Therefore

$$u(T_\infty) = v(T_\infty) = 0, \quad (g * u')(T_\infty) = (g * v')(T_\infty) = 0.$$

By the fact that the function $s \mapsto g(T_\infty - s)\|u'(s)\|^2$ is continuous on $[0, T_\infty]$ and $g(T_\infty - s) > 0$ for all $s \in [0, T_\infty]$, we have

$$(g * u')(T_\infty) = \int_0^{T_\infty} g(T_* - s)\|u'(s)\|^2 \,\mathrm{d}s = 0,$$

which implies that $\|u'(s)\| = 0$ for all $s \in [0, T_\infty]$. It means that u is a constant function on $[0, T_\infty]$. Then $u(0) = u(T_\infty) = 0$.

Similarly, $v(0) = v(T_\infty) = 0$. It leads to $I(0) = 0$. We get a contradiction with $I(0) > 0$. Consequently, $T_\infty = \infty$, i.e. $I(t) > 0$ for all $t \geqslant 0$.

Lemma 4.4 is proved completely. □

Lemma 4.5. *Assume that* $(\overline{\mathrm{H}}_2)$–$(\overline{\mathrm{H}}_6)$ *hold. Let* $I(0) > 0$ *and* (4.8) *hold. Put*

$$E_1(t) = (g * u')(t) + (g * v')(t) + \|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z + I(t). \tag{4.18}$$

Then there exist positive constants β_1, β_2 *such that*

$$\beta_1 E_1(t) \leqslant \mathcal{L}(t) \leqslant \beta_2 E_1(t) \quad \forall\, t \geqslant 0 \tag{4.19}$$

for δ *is small enough.*

P r o o f. It is easy to see that

$$\begin{aligned}\mathcal{L}(t) &= \frac{1}{2}(\|u'(t)\|^2 + \|v'(t)\|^2)\\ &\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big((g * u')(t) + (g * v')(t) + \mu_*\|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z\Big)\\ &\quad + \frac{1}{p}I(t) + \delta\langle u'(t), u(t)\rangle + \delta\langle v'(t), v(t)\rangle\\ &\quad + \frac{\delta}{2}(\lambda\|u_x(t)\|^2 + \lambda_1\|u(t)\|^2 + \lambda_2\|v(t)\|^2).\end{aligned} \tag{4.20}$$

From the inequalities

$$|\langle u'(t), u(t)\rangle| \leqslant \frac{1}{2}\|u'(t)\|^2 + \frac{1}{2}\|u_x(t)\|^2,$$
$$|\langle v'(t), v(t)\rangle| \leqslant \frac{1}{2}\|v'(t)\|^2 + \frac{1}{2}\|v_x(t)\|^2,$$
$$-\mu_{2*}\|v_x(t)\|^2 \geqslant -\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z$$

we deduce that

(4.21)

$$\begin{aligned}
\mathcal{L}(t) &\geqslant \frac{1}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) \\
&\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big((g * u')(t) + (g * v')(t) + \mu_*\|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z\Big) \\
&\quad + \frac{1}{p}I(t) - \frac{\delta}{2}(\|u'(t)\|^2 + \|u_x(t)\|^2) - \frac{\delta}{2}(\|v'(t)\|^2 + \|v_x(t)\|^2) \\
&\geqslant \frac{1-\delta}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) \\
&\quad + \Big(\frac{1}{2} - \frac{1}{p}\Big)\Big((g * u')(t) + (g * v')(t) + \mu_*\|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z\Big) \\
&\quad + \frac{1}{p}I(t) - \frac{\delta}{2}\|u_x(t)\|^2 - \frac{\delta}{2\mu_{2*}}\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z \\
&= \frac{1-\delta}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)((g * u')(t) + (g * v')(t)) \\
&\quad + \Big(\Big(\frac{1}{2} - \frac{1}{p}\Big)\mu_* - \frac{\delta}{2}\Big)\|u_x(t)\|^2 + \Big(\frac{1}{2} - \frac{1}{p} - \frac{\delta}{2\mu_{2*}}\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z + \frac{1}{p}I(t),
\end{aligned}$$

where we choose $\beta_1 = \min\{\frac{1}{2}(1-\delta), (\frac{1}{2} - p^{-1}), (\frac{1}{2} - p^{-1})\mu_* - \frac{1}{2}\delta, (\frac{1}{2} - p^{-1} - \delta/(2\mu_{2*})), p^{-1}\}$ with δ is small enough, $0 < \delta < \min\{1; (1-2p^{-1})\mu_*; (1-2p^{-1})\mu_{2*}\}$.

Similarly, we can prove that

$$\begin{aligned}
\mathcal{L}(t) &\leqslant \frac{1+\delta}{2}(\|u'(t)\|^2 + \|v'(t)\|^2) + \Big(\frac{1}{2} - \frac{1}{p}\Big)((g * u')(t) + (g * v')(t)) \\
&\quad + \Big(\Big(\frac{1}{2} - \frac{1}{p}\Big)\mu_* + \frac{\delta(1+\lambda+\lambda_1)}{2}\Big)\|u_x(t)\|^2 \\
&\quad + \Big(\frac{1}{2} - \frac{1}{p} + \frac{\delta(1+\lambda_2)}{2\mu_{2*}}\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z + \frac{1}{p}I(t) \\
&\leqslant \beta_2 E_1(t),
\end{aligned} \tag{4.22}$$

where $\beta_2 = \max\{\frac{1}{2}(1+\delta), (\frac{1}{2} - p^{-1})\mu_* + \frac{1}{2}\delta(1+\lambda+\lambda_1), \frac{1}{2} - p^{-1} + \delta(1+\lambda_2)/(2\mu_{2*})\}$.

Lemma 4.5 is proved completely. □

Lemma 4.6. *Assume that* $(\overline{\mathrm{H}}_2)$–$(\overline{\mathrm{H}}_6)$ *hold. Let* $I(0) > 0$ *and* (4.8) *hold. Then, the functional* $\psi(t)$ *defined by* (4.6) *satisfies*

$$\begin{aligned}(4.23)\qquad \psi'(t) \leqslant{}& \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{d_1}{p}((g*u')(t) + (g*v')(t))\\ &- \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\|u_x(t)\|^2\\ &- \Big(\chi_* - \frac{d_1}{p} + \frac{1}{\mu_{2*}}\Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\Big)\int_0^{\|v_x(t)\|^2}\mu_2(z)\,\mathrm{d}z\\ &- \frac{d_1\delta_1}{p}I(t) + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2)\end{aligned}$$

for all $\varepsilon_2 > 0$, $\delta_1 \in (0,1)$.

P r o o f. Multiplying $(4.1)_1$ by $u(x,t)$, $(4.1)_2$ by $v(x,t)$ and integrating over $[0,1]$, we obtain

$$\begin{aligned}(4.24)\qquad \psi'(t) ={}& \|u'(t)\|^2 + \|v'(t)\|^2 - \mu_*\|u_x(t)\|^2\\ &- \|u_x(t)\|^2\sigma(\langle u_x(t), u_x'(t)\rangle) - \|v_x(t)\|^2\mu_2(\|v_x(t)\|^2)\\ &+ \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle\\ &+ \langle f_1(u(t),v(t)), u(t)\rangle + \langle f_2(u(t),v(t)), v(t)\rangle.\end{aligned}$$

By the inequalities

$$\begin{aligned}(4.25)\qquad -\|u_x(t)\|^2\sigma(\langle u_x(t), u_x'(t)\rangle) &\leqslant \sigma_*\|u_x(t)\|^2,\\ -\|v_x(t)\|^2\mu_2(\|v_x(t)\|^2) &\leqslant -\chi_*\int_0^{\|v_x(t)\|^2}\mu_2(z)\,\mathrm{d}z,\\ \langle f_1(u(t),v(t)), u(t)\rangle + \langle f_2(u(t),v(t)), v(t)\rangle &\\ &\leqslant d_1\int_0^1 \mathcal{F}(u(x,t),v(x,t))\,\mathrm{d}x\\ &= \frac{d_1}{p}\Big((g*u')(t) + (g*v')(t)\\ &\quad + \mu_*\|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2}\mu_2(z)\,\mathrm{d}z - I(t)\Big),\\ I(t) &\geqslant \eta^*(\|u_x(t)\|^2 + \|v_x(t)\|^2),\\ \langle F_1(t), u(t)\rangle + \langle F_2(t), v(t)\rangle &\leqslant \frac{\varepsilon_2}{2}(\|u_x(t)\|^2 + \|v_x(t)\|^2)\\ &\quad + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2)\end{aligned}$$

for all $\varepsilon_2 > 0$, we deduce that

(4.26)

$$
\begin{aligned}
\psi'(t) &\leqslant \|u'(t)\|^2 + \|v'(t)\|^2 - \mu_*\|u_x(t)\|^2 + \sigma_*\|u_x(t)\|^2 - \chi_* \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z \\
&\quad + \frac{d_1}{p}\Big((g*u')(t) + (g*v')(t) + \mu_*\|u_x(t)\|^2 + \int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z - I(t)\Big) \\
&\quad + \frac{\varepsilon_2}{2}(\|u_x(t)\|^2 + \|v_x(t)\|^2) + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2) \\
&= \|u'(t)\|^2 + \|v'(t)\|^2 \\
&\quad + \frac{d_1}{p}((g*u')(t) + (g*v')(t)) - \frac{d_1\delta_1}{p}I(t) - \frac{d_1(1-\delta_1)}{p}I(t) \\
&\quad - \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p}\Big)\|u_x(t)\|^2 - \Big(\chi_* - \frac{d_1}{p}\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z \\
&\quad + \frac{\varepsilon_2}{2}(\|u_x(t)\|^2 + \|v_x(t)\|^2) + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2) \\
&\leqslant \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{d_1}{p}((g*u')(t) + (g*v')(t)) \\
&\quad - \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p}\Big)\|u_x(t)\|^2 - \Big(\chi_* - \frac{d_1}{p}\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z - \frac{d_1\delta_1}{p}I(t) \\
&\quad - \Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)(\|u_x(t)\|^2 + \|v_x(t)\|^2) + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2) \\
&\leqslant \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{d_1}{p}((g*u')(t) + (g*v')(t)) \\
&\quad - \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\|u_x(t)\|^2 \\
&\quad - \Big(\chi_* - \frac{d_1}{p}\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z - \frac{d_1\delta_1}{p}I(t) \\
&\quad - \Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\frac{1}{\mu_{2*}}\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2) \\
&= \|u'(t)\|^2 + \|v'(t)\|^2 + \frac{d_1}{p}((g*u')(t) + (g*v')(t)) \\
&\quad - \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\|u_x(t)\|^2 \\
&\quad - \Big(\chi_* - \frac{d_1}{p} + \frac{1}{\mu_{2*}}\Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z \\
&\quad - \frac{d_1\delta_1}{p}I(t) + \frac{1}{2\varepsilon_2}(\|F_1(t)\|^2 + \|F_2(t)\|^2).
\end{aligned}
$$

Hence, Lemma 4.6 is proved by using some simple estimates. □

P r o o f of Theorem 4.2. Now we prove Theorem 4.2. Applying the above lemmas, we have

$$
\begin{aligned}
\mathcal{L}'(t) \leqslant & -(\lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2} - \delta)(\|u'(t)\|^2 + \|v'(t)\|^2) \\
& - \Big(\overline{k} - \frac{\delta d_1}{p}\Big)((g * u')(t) + (g * v')(t)) \\
& - \delta\Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\|u_x(t)\|^2 \\
& - \delta\Big(\chi_* - \frac{d_1}{p} + \frac{1}{\mu_{2*}}\Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\Big)\int_0^{\|v_x(t)\|^2} \mu_2(z)\,\mathrm{d}z \\
& - \frac{\delta d_1\delta_1}{p} I(t) + \frac{1}{2}\Big(\frac{1}{\varepsilon_1} + \frac{\delta}{\varepsilon_2}\Big)(\|F_1(t)\|^2 + \|F_2(t)\|^2)
\end{aligned}
\tag{4.27}
$$

for all $\varepsilon_1,\ \varepsilon_2 > 0$, $\delta_1 \in (0,1)$, and with δ is small enough such that $0 < \delta < \min\{1, (1-2/p)\mu_*, (1-2/p)\mu_{2*}\}$.

Because of $p > \max\{d_1, d_1/\chi_*\}$ and $\sigma_*/\mu_* \leqslant 1 - d_1/p$, we have

$$
\begin{aligned}
\lim_{\delta_1\to 0_+,\varepsilon_2\to 0_+} & \Big(\mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big) \\
&= \mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1\eta^*}{p} \\
&= \frac{d_1\eta^*}{p} + \mu_*\Big(\Big(1 - \frac{d_1}{p}\Big) - \frac{\sigma_*}{\mu_*}\Big) > 0
\end{aligned}
\tag{4.28}
$$

and

$$
\lim_{\delta_1\to 0_+,\varepsilon_2\to 0_+}\Big(\chi_* - \frac{d_1}{p} + \frac{1}{\mu_{2*}}\Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big)\Big) = \chi_* - \frac{d_1}{p} + \frac{d_1\eta^*}{p\mu_{2*}} > 0.
\tag{4.29}
$$

Therefore, we can choose $\delta_1 \in (0,1)$ and $\varepsilon_2 > 0$ such that

$$
\begin{aligned}
\theta_1 &= \theta_1(\delta_1,\varepsilon_2) = \mu_* - \sigma_* - \frac{d_1\mu_*}{p} + \frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2} > 0, \\
\theta_2 &= \theta_2(\delta_1,\varepsilon_2) = \chi_* - \frac{d_1}{p} + \frac{1}{\mu_{2*}}\Big(\frac{d_1(1-\delta_1)\eta^*}{p} - \frac{\varepsilon_2}{2}\Big) > 0.
\end{aligned}
\tag{4.30}
$$

Moreover, we can choose $\varepsilon_1 > 0$, $\delta > 0$ small enough so that

$$
\begin{gathered}
\theta_3 = \lambda_* - \bar{\lambda} - \frac{\varepsilon_1}{2} - \delta > 0, \quad \theta_4 = \overline{k} - \frac{\delta d_1}{p} > 0, \\
0 < \delta < \min\Big\{1, \Big(1 - \frac{2}{p}\Big)\mu_*, \Big(1 - \frac{2}{p}\Big)\mu_{2*}\Big\}.
\end{gathered}
\tag{4.31}
$$

By (4.27), (4.30), (4.31), we get

$$\mathcal{L}'(t) \leqslant -\beta_3 E_1(t) + \overline{C}_1 \mathrm{e}^{-\gamma_0 t} \leqslant -\frac{\beta_3}{\beta_2}\mathcal{L}(t) + \overline{C}_1 \mathrm{e}^{-\gamma_0 t} \leqslant -\overline{\gamma}\mathcal{L}(t) + \overline{C}_1 \mathrm{e}^{-\gamma_0 t}, \tag{4.32}$$

where $\beta_3 = \min\{\theta_3, \theta_4, \delta\theta_1, \delta\theta_2, \delta d_1 \delta_1/p\}$, $0 < \overline{\gamma} < \min\{\beta_3/\beta_2, \gamma_0\}$, $\overline{C}_1 = \frac{1}{2}(1/\varepsilon_1 + \delta/\varepsilon_2)C_0$.

Therefore, we have

$$\mathcal{L}(t) \leqslant \Big(\mathcal{L}(0) + \frac{\overline{C}_1}{\gamma_0 - \overline{\gamma}}\Big)\mathrm{e}^{-\gamma t} = \overline{C}_2 \mathrm{e}^{-\overline{\gamma} t}. \tag{4.33}$$

On the other hand, we have

$$E_1(t) \geqslant \min\{1, \mu_{2*}\}(\|u'(t)\|^2 + \|v'(t)\|^2 + \|u_x(t)\|^2 + \|v_x(t)\|^2). \tag{4.34}$$

Combining (4.33) and (4.34) we get (4.9). Theorem 4.2 is proved completely. □

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