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ON THE T-CONDITIONALITY OF T-POWER BASED IMPLICATIONS

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It is well known that, in forward inference in fuzzy logic, the generalized modus ponens is guaranteed by a functional inequality called the law of T-conditionality. In this paper, the T-conditionality for T-power based implications is deeply studied and the concise necessary and sufficient conditions for a power based implication I^T being T-conditional are obtained. Moreover, the sufficient conditions under which a power based implication I^T is T^*-conditional are discussed, this discussions give an ideas to construct a t-norm T^* such that the power based implication I^T is T^*-conditional.

Keywords: T-power based implications, T-conditionality, t-norms, generalized modus ponens

Classification: 03E72, 03B52

1. INTRODUCTION

Fuzzy implications are the generalization of the classical (Boolean) implications on the unit interval $[0, 1]$. They are mainly used in fuzzy logic systems [7, 29, 32]. In the fields of decision theory [8, 11, 12], image processing [13], data mining [25, 31], fuzzy DI-subsethood measure [5], etc., they are also used with an essential role. In view of its wide application, various fuzzy implications have been proposed up to now by many authors (see for instance, [1, 2, 3, 9, 15, 23, 29, 30]).

In [18], motivated by the famous example related to tomatoes in [21], a novel additional property called invariance property for fuzzy implications was introduced. Although this property is not usually required on fuzzy implications, it is closely related to approximate reasoning. As many of the most usual fuzzy implications do not have this property, thus, in the same paper, a new class of fuzzy implications named T-power based implications was introduced. Subsequent research shows that most of the members of T-power based implications have invariance property [19]. Additionally, an in-depth analysis of all binary operators satisfying invariance property with respect to powers of a continuous t-norm was performed in [20, 22].

Fuzzy implications are mainly used in fuzzy logic systems to model fuzzy conditionals, and to perform fuzzy inference processes through *generalized modus ponens* and *generalized modus tollens* rules [4, 6, 7, 10, 32]. It is well known that the *generalized*

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modus ponens can be implemented by a scheme enabled by a functional inequality called the law of T-conditionality (T-conditionality for short) [3], also known as *modus ponens inequality* [26]. Hence, the T-conditionality for many kinds of implication functions has been extensively studied (see for instance, [1, 16, 17, 23, 24, 26, 27, 28]). Recently, Li et al. [16] have studied the T-conditionality for T-power based implications. The main contributions of [16] are as follows:

Theorem 1.1. (Qin and Xie [16], Theorem 3.10) Let $T^\sharp = (\langle a_j^\sharp, b_j^\sharp, T_j^\sharp\rangle)_{j\in J}$ be a continuous t-norm and $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}$ a continuous t-norm.

(i) If $b_j^\sharp < 1$ for all $j \in J$, then I^T is a $T^\sharp$-conditional if and only if $T = T_M$.

(ii) If there exists a $j_0 \in J$ satisfying $b_{j_0}^\sharp = 1$, then I^T is a $T^\sharp$-conditional if and only if the following two statements are true.

(a) Every open generating interval (a_i, b_i) of T satisfies the condition that $(a_i, b_i) \subseteq (a_{j_0}^\sharp, 1)$.

(b) For every open generating interval (a_i, b_i) of T and its corresponding summand T_i, it holds that

$$\frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})} \le a_{j_0}^\sharp + (1-a_{j_0}^\sharp)\cdot I_{T_{j_0}^\sharp}(\frac{x-a_{j_0}^\sharp}{1-a_{j_0}^\sharp}, \frac{y-a_{j_0}^\sharp}{1-a_{j_0}^\sharp}) \tag{1}$$

for all $x, y \in [a_i, b_i]$ such that $x > y$, where t_i is an additive generator of T_i.

Corollary 1.2. (Qin and Xie [16], Corollary 3.11) Let $T^\sharp = (\langle a_j^\sharp, b_j^\sharp, T_j^\sharp\rangle)_{j\in J}$ be a continuous t-norm.

(i) If $b_j^\sharp < 1$ for all $j \in J$, then $I^{T^\sharp}$ is a $T^\sharp$-conditional if and only if $T^\sharp = T_M$.

(ii) If there exists a $j_0 \in J$ satisfying $b_{j_0}^\sharp = 1$, then $I^{T^\sharp}$ is a $T^\sharp$-conditional if and only if $T_{j_0}^\sharp$ is a nilpotent t-norm such that $T^\sharp = (\langle a_{j_0}^\sharp, 1, T_{j_0}^\sharp\rangle)$ and

$$\frac{\alpha}{\beta} \le a_{j_0}^\sharp + (1-a_{j_0}^\sharp)\cdot t_{j_0}^{-1}(\beta-\alpha) \tag{2}$$

for all $\alpha, \beta \in [0,1]$ with $\beta > \alpha$, where t_{j_0} is the normalized additive generator of $T_{j_0}^\sharp$.

Although Theorem 1.1 (ii) gives a necessary and sufficient condition on the T-conditionality for a power based implication, inequation (1) is merely an elaboration of inequation $I^T(x,y) \le I_{T^\sharp}(x,y)$ for all $x, y \in [0,1]$ with $x > y$. Note that $I^T(x,y) \le I_{T^\sharp}(x,y) \Leftrightarrow T^\sharp(x, I^T(x,y)) \le y$ for all $x, y \in [0,1]$ with $x > y$. This indicates that inequality (1) is another display of the definition on T-conditionlity for the power based implications. Hence, there still exists a problem to solve: Given a continuous t-norm T, how do we construct a t-norm T^* such that the power based implication I^T is T^*-conditional.

In addition, inequation (2) is too complicated to use.

In view of the above considerations, it is necessary to continue to investigate the T-conditionality for T-power based implications.

The paper is organized as follows. In Section 2, some concepts and results are recalled. In Section 3, new results on the T-conditionality for T-power based implications are given. Finally, the paper ends with a section devoted to the conclusions.

2. PRELIMINARIES

For convenience, in this section, the definitions and results to be used in the rest of the paper are outlined.

Definition 2.1. (Baczyński and Jayaram [3], Definition 1.1.1) A function $I : [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies, for all x, x_1, x_2, y, y_1, $y_2 \in [0,1]$, the following conditions:

if $x_1 < x_2$, then $I(x_1, y) \geq I(x_2, y)$, i. e., $I(\cdot, y)$ is decreasing, (I1)

if $y_1 < y_2$, then $I(x, y_1) \leq I(x, y_2)$, i. e., $I(x, \cdot)$ is increasing, (I2)

$I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$. (I3)

The set of all fuzzy implications will be denoted by FI.

Definition 2.2. (Klement et al. [14], Definition 1.1) An associative, commutative and increasing function $T : [0,1]^2 \to [0,1]$ is called a t-norm if it satisfies $T(x,1) = x$ for all $x \in [0,1]$.

The following are the four basic t-norms T_M, T_P, T_L, T_D given by, respectively:

$$T_M(x,y) = \min(x,y), \quad T_P(x,y) = xy, \quad T_L(x,y) = \max(x+y-1, 0),$$
$$T_D(x,y) = \begin{cases} \min(x,y), & \text{if } x = 1 \text{ or } y = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.3. (Baczyński and Jayaram [3], Definition 2.1.2.) A t-norm T is called

- continuous if it is continuous in both the arguments;
- strict, if it is continuous and strictly monotone;
- Archimedean, if for all $x, y \in (0,1)$ there exists an $n \in N$ such that $x_T^{(n)} < y$, where
$$x_T^{(1)} = x, \quad x_T^{(n)} = T(x, x_T^{(n-1)}) \text{ for all } n \geq 2,$$
with the convention $x_T^{(0)} = 1$;
- nilpotent, if it is continuous and for each $x \in (0,1)$, there exists an $n \in N$ such that $x_T^{(n)} = 0$.

Remark 2.4. (Baczyński and Jayaram [3], Remark 2.1.4) If a t-norm T is strict or nilpotent, then it is Archimedean. Conversely, every continuous Archimedean t-norm is strict or nilpotent.

Theorem 2.5. (Baczyński and Jayaram [3], Theorem 2.1.5) For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) T is a continuous Archimedean t-norm.

(ii) T has a continuous additive generator, i. e., there exists a continuous, strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))), \ \ x,y \in [0,1].$$

Remark 2.6. (Baczyński and Jayaram [3], Remark 2.1.7)

(i) T is a strict t-norm if and only if each continuous additive generator t of T satisfies $t(0) = \infty$.

(ii) T is a nilpotent t-norm if and only if each continuous additive generator t of T satisfies $t(0) < \infty$.

Theorem 2.7. (Klement et al. [14], Theorem 3.23) Let $t : [0,1] \to [0,\infty]$ be a strictly decreasing function with $t(1) = 0$ such that $t(x) + t(y) \in Ran(t) \cup [t(0^+), \infty]$ for all $(x,y) \in [0,1]^2$. The following function $T : [0,1]^2 \to [0,1]$ is a t-norm:

$$T(x,y) = t^{-1}(\min(t(x) + t(y), t(0))), \ \ x,y \in [0,1].$$

Theorem 2.8. (Klement et al. [14], Theorem 3.43) Let A be an index set and $(T_i)_{i\in A}$ a family of t-norms, let $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T : [0,1]^2 \to [0,1]$ is a t-norm:

$$T(x,y) = \begin{cases} a_i + (b_i - a_i) \cdot T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}), & \text{if } \ x,y \in [a_i,b_i], \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{3}$$

Definition 2.9. (Klement et al. [14], Definition 3.44) Let A be an index set and $(T_i)_{i\in A}$ a family of t-norms, and let $\{(a_i,b_i)\}_{i\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$.

(i) The t-norm T defined by (3) is called the ordinal sum of t-norms, also known as the ordinal sum of the summands $\langle a_i, b_i, T_i\rangle$, $i \in A$. In this case we write $T = (\langle a_i, b_i, T_i\rangle)_{i\in A}$.

(ii) $T = (\langle a_i, b_i, T_i\rangle)_{i\in A}$ is said to be trivial if $A = \{1\}$, $a_1 = 0$ and $b_1 = 1$.

(iii) $T = (\langle a_i, b_i, T_i\rangle)_{i\in A}$ is called an ordinal sum of continuous Archimedean t-norms if T_i is a continuous Archimedean t-norm for all $i \in A$.

Theorem 2.10. (Klement et al. [14], Theorem 5.11) For a function $T : [0,1]^2 \to [0,1]$ the following statements are equivalent:

(i) T is a continuous t-norm.

(ii) T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e, there exist a uniquely determined (finite or countably infinite) index set A, a family of uniquely determined pairwise disjoint open subintervals $\{(a_i, b_i)\}_{i\in A}$ of $[0,1]$ and a family of uniquely determined continuous Archimedean t-norms $(T_i)_{i\in A}$ such that

$$T = (\langle a_i, b_i, T_i\rangle)_{i\in A}.$$

Remark 2.11. For a continuous t-norm T, if $T \neq T_M$, then it is either a continuous Archimedean t-norm or a non-trivial ordinal sum of continuous Archimedean t-norms.

Definition 2.12. (Klement et al. [14], Massanet et al. [18]) Let T be a continuous t-norm. For each $x \in [0,1]$, nth roots and rational powers of x with respect to T are defined by

$$x_T^{(\frac{1}{n})} = \sup\{z \in [0,1] | z^{(n)} \leq x\}, \quad x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)},$$

where m, n are positive integers.

Definition 2.13. (Massanet et al. [18], Definition 4) A binary operator $I : [0,1]^2 \to [0,1]$ is said to be a T-power based implication if there exists a continuous t-norm T such that

$$I(x,y) = \sup\{r \in [0,1] | y_T^{(r)} \geq x\}, \text{ for all } x, y \in [0,1]. \tag{4}$$

If I is a T-power based implication, then it will be denoted by I^T.

Proposition 2.14. (Massanet et al. [18], Proposition 5) Let T be a continuous t-norm and I^T its power based implication defined by (4).

(i) If $T = T_M$, then $I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{if } x > y, \end{cases}$ is the Rescher implication I_{RS}.

(ii) If T is an Archimedean t-norm with additive generator t, then

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t(x)}{t(y)}, & \text{if } x > y, \end{cases}$$

with the convention that $\frac{a}{\infty} = 0$ for all $a \in [0,1]$.

(iii) If T is an ordinal sum t-norm of the form $T = (\langle a_i, b_i, T_i\rangle)_{i\in A}$, where T_i is an Archimedean t-norm with additive generator t_i for all $i \in A$, then

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, & \text{if } x > y \text{ and } x, y \in [a_i, b_i], \\ 0, & \text{otherwise.} \end{cases}$$

3. NEW RESULTS ON T-CONDITIONALITY OF T-POWER BASED IMPLICATIONS

A fuzzy implication I is T-conditional [3] if there exists a t-norm T such that

$$T(x, I(x, y)) \leq y \text{ for all } x, y \in [0, 1]. \tag{TC}$$

From Theorem 1.1 (i) it is easy to see that the power based implication I^{T_M} satisfies (TC) with respect to any t-norm T. Hence, in the paper, only the T-conditionality for I^T in which T is a continuous Archimedean t-norm, or a non-trivial ordinal sum of continuous Archimedean t-norms, is discussed.

Considering that the two t-norms of the power-based implication I_T and the T-conditionality may not be same, the study on T-conditionality for the power-based implication I^T is carried out from the following two aspects: one is that the pair (I^T, T) satisfies (TC), the other is that the pair (I^T, T^*) satisfies (TC), where T^* is a t-norm different from T in general.

3.1. New results of I^T satisfying (TC) with t-norm T

In this sub-section, in view of the complexity of inequation (2), we give two new results on the pair (I^T, T) satisfying (TC).

Proposition 3.1. (Qin and Xie [16], Corollary 3.11 (ii)) Let T be a continuous Archimedean t-norm and I^T its power based implication. If the pair (I^T, T) satisfies (TC), then T is a nilpotent t-norm.

Theorem 3.2. Let T be a nilpotent t-norm with continuous additive generator t. Then the pair (I^T, T) satisfies (TC) if and only if

$$\frac{t(p)}{1-p} \geq t(0) \text{ for all } p \in [0, 1).$$

Proof. (Sufficiency) Let $x, y \in [0, 1]$ with $x > y$, and let $\frac{t(x)}{t(y)} = p_1$. Obviously, $t(x) < t(y) \leq t(0)$, and $p_1 \in [0, 1)$.

Assume that $\frac{t(p)}{1-p} \geq t(0)$ for all $p \in [0, 1)$, then $\frac{t(p_1)}{1-p_1} \geq t(0)$, i.e.,

$$\frac{t\left(\frac{t(x)}{t(y)}\right)}{1 - \frac{t(x)}{t(y)}} \geq t(0) \geq t(y).$$

Hence

$$t\left(\frac{t(x)}{t(y)}\right) \geq t(y) - t(x).$$

Thus

$$t(x) + t\left(\frac{t(x)}{t(y)}\right) \geq t(y).$$

Case 1: If $t(x)+t\left(\frac{t(x)}{t(y)}\right) \geq t(0)$, then

$$T(x, I^T(x,y)) = t^{-1}\left(\min\left(t(x)+t\left(\frac{t(x)}{t(y)}\right), t(0)\right)\right) = 0 \leq y.$$

Case 2: If $t(x)+t\left(\frac{t(x)}{t(y)}\right) < t(0)$, note that $t(x)+t\left(\frac{t(x)}{t(y)}\right) \geq t(y)$, then

$$T(x, I^T(x,y)) = t^{-1}\left(t(x)+t\left(\frac{t(x)}{t(y)}\right)\right) \leq y.$$

Hence, for all $x, y \in [0,1]$ with $x > y$, we get

$$T(x, I^T(x,y)) = t^{-1}\left(\min\left(t(x)+t\left(\frac{t(x)}{t(y)}\right), t(0)\right)\right) \leq y.$$

Therefore, the pair (I^T, T) satisfies (TC).

(Necessity) Let the pair (I^T, T) satisfy (TC). Suppose that there exists a $p_0 \in (0,1)$ such that

$$\frac{t(p_0)}{1-p_0} < t(0).$$

Since t is a continuous, strictly monotonous function, then there exists a $y_0 \in (0,1)$ such that

$$\frac{t(p_0)}{1-p_0} < t(y_0) < t(0). \tag{5}$$

On the other hand, by the continuity of t, there exists an $x_0 \in (0,1)$ such that $\frac{t(x_0)}{t(y_0)} = p_0$.

From $p_0 < 1$ we get $x_0 > y_0$. From (5) we have

$$\frac{t\left(\frac{t(x_0)}{t(y_0)}\right)}{1-\frac{t(x_0)}{t(y_0)}} < t(y_0).$$

Hence

$$t(x_0)+t\left(\frac{t(x_0)}{t(y_0)}\right) < t(y_0) < t(0),$$

i. e.,

$$t^{-1}\left(t(x_0)+t\left(\frac{t(x_0)}{t(y_0)}\right)\right) > y_0,$$

then

$$\begin{aligned} T(x_0, I^T(x_0,y_0)) &= t^{-1}\left(\min\left(t(x_0)+t\left(\frac{t(x_0)}{t(y_0)}\right), t(0)\right)\right) \\ &= t^{-1}\left(t(x_0)+t\left(\frac{t(x_0)}{t(y_0)}\right)\right) \\ &> y_0, \end{aligned}$$

a contradiction to the fact that the pair (I^T, T) satisfies (TC). □

To show the application of Theorem 3.2, two examples are given.

Example 3.3. Consider the nilpotent t-norm T_L, an additive generator of it is $t(x) = 1 - x$, $x \in [0,1]$. From Proposition 2.14 we get

$$I^{T_L}(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le y \le 1, \\ \frac{1-x}{1-y}, & \text{if } 0 \le y < x \le 1. \end{cases}$$

Since $\frac{t(x)}{1-x} = 1 \ge 1 = t(0)$ for all $x \in [0,1)$, then the pair (I^{T_L}, T_L) satisfies (TC) by Theorem 3.2.

Example 3.4. Consider a nilpotent t-norm T with the following continuous additive generator

$$t(x) = \begin{cases} 1 - 1.5x, & \text{if } 0 \le x \le 0.5, \\ 0.5 - 0.5x, & \text{if } 0.5 < x \le 1. \end{cases}$$

From Proposition 2.14 we get

$$I^T(x,y) = \begin{cases} 1, & \text{if } 0 \le x \le y \le 1, \\ \frac{2-3x}{2-3y}, & \text{if } 0 \le y < x \le 0.5, \\ \frac{1-x}{2-3y}, & \text{if } 0 \le y \le 0.5 < x \le 1, \\ \frac{1-x}{1-y}, & \text{if } 0.5 < y < x \le 1. \end{cases}$$

Taking $x = 0.25$, since $\frac{t(x)}{1-x} = \frac{5}{6} < 1 = t(0)$, then the pair (I^T, T) does not satisfy (TC) by Theorem 3.2.

Proposition 3.5. Let t-norm $T = (\langle a_i, b_i, T_i \rangle)_{i \in A}$ be non-trivial and I^T its power based implication, where A is an index set, $(T_i)_{i \in A}$ is a family of continuous Archimedean t-norms, and $\{(a_i, b_i)\}_{i \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the pair (I^T, T) satisfies (TC) if and only if the following three items hold:

(i) $A = \{1\}$, $a_1 \in (0,1)$ and $b_1 = 1$.

(ii) T_1 is a nilpotent t-norm.

(iii) $\frac{t_1(p)}{1-p} \ge t_1(0)$ for all $p \in [0,1)$, where t_1 is an additive generator of t-norm T_1.

Proof. (Necessity) Although (i) and (ii) are obtained by Corollary 1.2, the reasoning is difficult to understand. Hence, we give a proof of (i) and (ii) in the following.

Let the pair (I^T, T) satisfy (TC).

(i) Suppose that $b_1 < 1$. Taking $y_1 \in (a_1, b_1)$ and assume that t_1 is an additive generator of T_1.

Since $b_1 > a_1 \ge 0$, then $0 < b_1 < 1$. Thus

$$t_1\left(\frac{y_1 - a_1}{b_1 - a_1}\right) > b_1 \cdot t_1\left(\frac{y_1 - a_1}{b_1 - a_1}\right).$$

By the continuity of t_1, there exists an $x_1 \in (a_1, b_1)$ such that

$$t_1(\frac{y_1 - a_1}{b_1 - a_1}) > t_1(\frac{x_1 - a_1}{b_1 - a_1}) > b_1 \cdot t_1(\frac{y_1 - a_1}{b_1 - a_1}). \tag{6}$$

From (6) we get

$$x_1 > y_1, \text{ and } I^T(x_1, y_1) = \frac{t_1(\frac{x_1 - a_1}{b_1 - a_1})}{t_1(\frac{y_1 - a_1}{b_1 - a_1})} > b_1.$$

Therefore,

$$T(x_1, I^T(x_1, y_1)) = \min(x_1, I^T(x_1, y_1)) = x_1 > y_1.$$

A contradiction to the fact that the pair (I^T, T) satisfies (TC).

From $b_1 = 1$ it is easy to obtain that $A = \{1\}$ and $a_1 > 0$.

(ii) From (i) we get $T = (\langle a_1, 1, T_1 \rangle)$, $a_1 \in (0, 1)$.

Suppose that T_1 is a strict t-norm with an additive generator t_1. Then

$$\frac{2t_1(\frac{1}{2})}{1 - a_1} < \infty = t_1(0).$$

By the continuity of t_1, there exists a $y_0 \in (a_1, 1)$ such that

$$\frac{2t_1(\frac{1}{2})}{1 - a_1} < t_1(\frac{y_0 - a_1}{1 - a_1}),$$

i. e.,

$$t_1(\frac{1}{2}) < \frac{1 - a_1}{2} \cdot t_1(\frac{y_0 - a_1}{1 - a_1}). \tag{7}$$

Let $x_0 = a_1 + (1 - a_1) \cdot t_1^{-1}(\frac{1+a_1}{2} \cdot t_1(\frac{y_0 - a_1}{1 - a_1}))$. Since $\frac{1+a_1}{2} < 1$, then

$$x_0 > a_1 + (1 - a_1) \cdot t_1^{-1}\left(t_1(\frac{y_0 - a_1}{1 - a_1})\right) = y_0.$$

Thus

$$I^T(x_0, y_0) = \frac{t_1(\frac{x_0 - a_1}{1 - a_1})}{t_1(\frac{y_0 - a_1}{1 - a_1})} = \frac{1 + a_1}{2} > a_1.$$

Therefore,

$$\begin{aligned}
T(x_0, I^T(x_0, y_0)) &= T(x_0, \frac{1 + a_1}{2}) \\
&= a_1 + (1 - a_1) \cdot T_1(\frac{x_0 - a_1}{1 - a_1}, \frac{\frac{1+a_1}{2} - a_1}{1 - a_1}) \\
&= a_1 + (1 - a_1) \cdot t_1^{-1}\left(t_1(\frac{x_0 - a_1}{1 - a_1}) + t_1(\frac{1}{2})\right) \\
&= a_1 + (1 - a_1) \cdot t_1^{-1}\left(\frac{1 + a_1}{2} \cdot t_1(\frac{y_0 - a_1}{1 - a_1}) + t_1(\frac{1}{2})\right)
\end{aligned}$$

$$> a_1 + (1-a_1)\cdot t_1^{-1}\left(\tfrac{1+a_1}{2}\cdot t_1(\tfrac{y_0-a_1}{1-a_1}) + \tfrac{1-a_1}{2}\cdot t_1(\tfrac{y_0-a_1}{1-a_1})\right) \quad \lceil \text{by } (7)$$
$$= a_1 + (1-a_1)\cdot t_1^{-1}\left(t_1(\tfrac{y_0-a_1}{1-a_1})\right)$$
$$= y_0.$$

A contradiction to the fact that the pair (I^T, T) satisfies (TC).

(iii) The proof is analogous to the proof of necessity for Theorem 3.2.

(Sufficiency) Analogous to the proof of sufficiency for Theorem 3.2. □

Table 1 summarizes the T-conditionality of I^T with respect to t-norm T.

t-norm T	Implication I^T	(TC)
$T = T_M$	I_{RS}	√
T is strict	I^*	×
T is a nilpotent t-norm with continuous additive generator t such that $\frac{t(p)}{1-p} \geq t(0)$ for all $p \in [0,1)$	I^*	√
T is a nilpotent t-norm with continuous additive generator t, there exists a $p \in [0,1)$ such that $\frac{t(p)}{1-p} < t(0)$	I^*	×
$T = (\langle a_1, 1, T_1\rangle), a_1 \in (0,1)$ T_1 is a nilpotent t-norm with continuous additive generator t_1 such that $\frac{t_1(p)}{1-p} \geq t_1(0)$ for all $p \in [0,1)$	$I^\star$	√
$T = (\langle a_1, 1, T_1\rangle), a_1 \in (0,1)$ T_1 is a t-norm with continuous additive generator t_1, there exists a $p \in [0,1)$ such that $\frac{t_1(p)}{1-p} < t_1(0)$	$I^\star$	×
$T = (\langle a_i, b_i, T_i\rangle)_{i\in A}$, $\exists\, b_i < 1$	$I^\bullet$	×

Tab. 1. the T-conditionality of I^T with respect to T.

Note.

(i) $I^*(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t(x)}{t(y)}, & \text{if } x > y. \end{cases}$

(ii) $I^\star(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t_1(\frac{x-a_1}{1-a_1})}{t_1(\frac{y-a_1}{1-a_1})}, & \text{if } x > y \text{ and } x, y \in [a_1, 1], \\ 0, & \text{otherwise.} \end{cases}$

(iii) $I^\bullet(x,y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{t_i(\frac{x-a_i}{b_i-a_i})}{t_i(\frac{y-a_i}{b_i-a_i})}, & \text{if } x > y \text{ and } x, y \in [a_i, b_i],\ i \in A, \\ 0, & \text{otherwise.} \end{cases}$

3.2. Characterizations of I^T satisfying (TC) with t-norm T^*

Proposition 3.1 tells us that the pair (I^T, T) does not satisfy (TC) when T is a strict t-norm, as it poses two problems: Does there exist a t-norm T^* such that the pair (I^T, T^*) satisfies (TC) when T is a strict t-norm? Can we construct a t-norm T^* such that the pair (I^T, T^*) satisfies (TC)?

Example 3.6. Let T be a strict t-norm with additive generator $t(x) = \frac{1}{x} - 1$, $x \in [0, 1]$, and I^T its power based implication, i. e.,

$$I^T(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{(1-x)y}{(1-y)x}, & \text{if } x > y. \end{cases}$$

Consider the product t-norm T_P. Let $x, y \in [0, 1]$ with $x > y$. Since

$$T_P\left(x, I^T(x, y)\right) = x \cdot \frac{(1-x)y}{(1-y)x} = \frac{(1-x)}{(1-y)} \cdot y < y,$$

then the pair (I^T, T_P) satisfies (TC).

Remark 3.7. Although T is strict, Example 3.6 shows that there may exist a t-norm T^* such that the pair (I^T, T^*) satisfies (TC).

In the following, we present the sufficient conditions under which there exist at least one t-norm T^* different from T such that the pair (I^T, T^*) satisfies (TC). That is, we give an idea to construct a t-norm T^* different from T such that (I^T, T^*) satisfies (TC). The construction is based on t-norms generated by Theorem 3.23 in [14]. The t-norm T^* is not necessarily continuous.

First, we study the case where T is a continuous Archimedean t-norm.

Theorem 3.8. Let T be a continuous Archimedean t-norm with additive generator t and I^T its power based implication. Let $m \in [t(0), \infty]$. If there exists a strictly increasing function $\psi : [0, m] \to [0, \infty]$ with $\psi(0) = 0$ such that

$$\psi(t(x)) + \psi(t(y)) \in Ran(\psi \circ t) \cup [\psi(t(0^+)), \infty] \text{ for all } x, y \in [0, 1],$$

$$\sup_{a \in [0, t(0)]} (\psi(a) - \psi(ka)) \leq \psi(t(k)) \text{ for all } k \in (0, 1),$$

then there exists a t-norm T^* given as

$$T^*(x, y) = t^{*-1}(\min(t^*(x) + t^*(y), t^*(0))), \ x, y \in [0, 1],$$

such that the pair (I^T, T^*) satisfies (TC), where $t^*(x) = \psi(t(x))$, $x \in [0, 1]$.

Proof. First, let us prove that T^* is a t-norm.

Let $x, y \in [0, 1]$. Since t is an additive generator of continuous Archimedean t-norm T, then t is a continuous, strictly decreasing function with $t(1) = 0$.

Assume that $\psi : [0,m] \to [0,\infty]$ is a strictly increasing function with $\psi(0) = 0$ such that

$$\psi(t(x)) + \psi(t(y)) \in Ran(\psi \circ t) \cup [\psi(t(0^+)), \infty] \text{ for all } x, y \in [0,1].$$

Let $t^* : [0,1] \to [0,\infty]$ be a function given by

$$t^*(x) = \psi(t(x)), \ \ x \in [0,1].$$

Obviously, t^* is a strictly decreasing function with

$$t^*(x) + t^*(y) \in \mathrm{Ran}(t^*) \cup [t^*(0^+), \infty]$$

for all $x, y \in [0,1]$. Note that $t^*(1) = 0$. Thus the following function

$$T^*(x,y) = t^{*-1}(\min(t^*(x) + t^*(y), t^*(0))), \ \ x, y \in [0,1],$$

is a t-norm by Theorem 2.7.

Assume that $\sup\limits_{a\in[0,t(0)]} (\psi(a) - \psi(ka)) \leq \psi(t(k))$ for all $k \in [0,1]$, then

$$\psi(a) - \psi(ka) \leq \psi(t(k)) \text{ for all } k \in (0,1), a \in [0,t(0)].$$

Hence

$$\psi(a) \leq \psi(ka) + \psi(t(k)) \text{ for all } k \in (0,1), a \in [0,t(0)]. \tag{8}$$

Let $x, y \in [0,1]$ with $x > y$. From Proposition 2.14 (ii) we get

$$I^T(x,y) = \frac{t(x)}{t(y)},$$

then

$$T^*(x, I^T(x,y)) = t^{*-1}\left(\min\left(t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right), t^*(0)\right)\right).$$

If $t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right) \geq t^*(0)$, then $T^*(x, I^T(x,y)) = t^{*-1}(t^*(0)) = 0 \leq y$.

If $t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right) < t^*(0)$. Let $\frac{t(x)}{t(y)} = k$, $t(y) = a$. Then

$$x = t^{-1}(ka), \ \ k \in (0,1) \text{ and } a \in (0, t(0)].$$

Therefore

$$\begin{aligned} t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right) &= t^*\left(t^{-1}(ka)\right) + t^*(k) \\ &= t^*\left(t^{-1}(ka)\right) + t^*\left(t^{-1}(t(k))\right). \end{aligned}$$

From $t^*(x) = \psi(t(x))$, $x \in [0,1]$ we get $\psi(x) = t^*(t^{-1}(x))$, $x \in Ran(t)$. Then

$$\begin{aligned} t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right) &= \psi(ka) + \psi(t(k)) \\ &\geq \psi(a) \ \ \lceil\text{by } (8) \\ &= \psi(t(y)) \\ &= t^*(y). \end{aligned}$$

Hence

$$T^*(x, I^T(x,y)) = t^{*-1}\left(t^*(x) + t^*\left(\frac{t(x)}{t(y)}\right)\right) \le y,$$

i. e., the pair (I^T, T^*) satisfies (TC). □

Corollary 3.9. Let T be a continuous Archimedean t-norm with additive generator t and I^T its power based implication. Let $m \in [t(0), \infty]$. If there exists a continuous, strictly increasing function $\psi : [0, m] \to [0, \infty]$ with $\psi(0) = 0$ such that

$$\sup_{a \in [0, t(0)]} \psi^{-1}(\psi(a) - \psi(ka)) \le t(k) \text{ for all } k \in (0,1),$$

then there exists a continuous Archimedean t-norm T^* with additive generator $t^*(x) = \psi(t(x))$, $x \in [0,1]$ such that the pair (I^T, T^*) satisfies (TC).

To show the application of Corollary 3.9, two examples are given.

Example 3.10. Let T be a continuous Archimedean t-norm with additive generator $t(x) = \frac{1}{x} - 1$, $x \in [0,1]$ and I^T its power based implication, i. e.,

$$I^T(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{(1-x)y}{(1-y)x}, & \text{if } x > y. \end{cases}$$

Consider the following function

$$\psi(x) = \frac{x}{1+x}, \ x \in [0, \infty],$$

it is continuous, strictly increasing, and $\psi(0) = 0$. By calculations, we get

$$\psi^{-1}(x) = \frac{x}{1-x}, \ x \in [0,1],$$

$$t^*(x) = \psi(t(x)) = 1 - x, \ x \in [0,1].$$

Therefore, $T^* = T_L$, and T^* corresponds to the Lukasiewicz t-norm.

Let $k \in (0,1)$. Since

$$\begin{aligned} \sup_{a \in [0, t(0)]} \psi^{-1}(\psi(a) - \psi(ka)) &= \sup_{a \in [0,\infty]} \frac{\frac{a}{1+a} - \frac{ak}{1+ak}}{1 - (\frac{a}{1+a} - \frac{ak}{1+ak})} \\ &= \sup_{a \in [0,\infty]} \frac{1-k}{\frac{1}{a} + ka + 2k} \\ &= \frac{1-k}{2\sqrt{k} + 2k} \\ &\le \frac{1-k}{k} \\ &= t(k). \end{aligned}$$

Then the pair (I^T, T_L) satisfies (TC) by Corollary 3.9.

Example 3.11. Let T be a nilpotent t-norm with continuous additive generator $t(x) = (1-x)^2$, $x \in [0,1]$ and I^T its power based implication. Since

$$\frac{t(x)}{1-x} = 1 - x < t(0) \text{ for all } x \in [0,1),$$

then the pair (I^T, T) does not satisfy (TC) by Theorem 3.2.

Consider the following function

$$\psi(x) = \frac{\sqrt{x}}{4}, \quad x \in [0,\infty].$$

Obviously, it is a continuous, strictly increasing function with $\psi(0) = 0$.

By calculations, we get $\psi^{-1}(x) = 16x^2$, $x \in [0,\infty]$, $t^*(x) = \frac{1-x}{4}$, $x \in [0,1]$. Therefore, $T^* = T_L$. Let $k \in (0,1)$. Since

$$\begin{aligned}
\sup_{a\in[0,t(0)]} \psi^{-1}(\psi(a) - \psi(ka)) &= \sup_{a\in[0,1]} \psi^{-1}\left(\frac{\sqrt{a}}{4}(1-\sqrt{k})\right) \\
&= (1-\sqrt{k})^2 \\
&\leq (1-k)^2 \\
&= t(k).
\end{aligned}$$

Then the pair (I^T, T_L) satisfies (TC) by Corollary 3.9.

In fact, let $x, y \in [0,1]$ with $x > y$, we have $x + y^2 \leq 1 + y$. Since

$$\begin{aligned}
\frac{(1-x)^2}{(1-y)^2} \leq 1 + y - x &\Leftrightarrow (1-x)^2 \leq (1+y-x)(1-y)^2 \\
&\Leftrightarrow 1 - 2x + x^2 \leq 1 + y - x - 2y - 2y^2 + 2xy + y^2 + y^3 - xy^2 \\
&\Leftrightarrow -x + x^2 \leq -y - y^2 + 2xy + y^3 - xy^2 \\
&\Leftrightarrow x^2 + y^2 - 2xy - x + y + xy^2 - y^3 \leq 0 \\
&\Leftrightarrow (x-y)^2 - (x-y) + y^2(x-y) \leq 0 \\
&\Leftrightarrow (x-y) - 1 + y^2 \leq 0 \\
&\Leftrightarrow x + y^2 \leq 1 + y,
\end{aligned}$$

then

$$T_L(x, I^T(x,y)) = \max\left(x + \frac{(1-x)^2}{(1-y)^2} - 1, 0\right) \leq y,$$

that is, the pair (I^T, T_L) satisfies TC.

Corollary 3.12. Let $\psi : [0,\infty] \to [0,\infty]$ be a continuous, strictly increasing function with $\psi(0) = 0$. If $\sup_{a\in[0,\infty]} \psi^{-1}(\psi(a) - \psi(ka)) < \infty$, $k \in (0,1)$, then there exists a strict t-norm T with additive generator

$$t(x) = \sup_{a\in[0,\infty]} \psi^{-1}(\psi(a) - \psi(ax)), \quad x \in [0,1],$$

and a t-norm T^* with additive generator $t^*(x) = \psi(t(x))$, $x \in [0,1]$ such that the pair (I^T, T^*) satisfies (TC), where I^T is a power based implication of t-norm T.

P r o o f. Obviously by Corollary 3.9, □

Example 3.13. Consider a function defined by

$$\psi(x) = \arctan(x), \ \ x \in [0, \infty].$$

It is continuous, strictly increasing with $\psi(0) = 0$. By calculations, we get

$$\psi^{-1}(x) = \tan(x), \ \ x \in [0, \frac{\pi}{2}].$$

Let $k \in (0,1)$. Then

$$\sup_{a\in[0,\infty]} \psi^{-1}(\psi(a) - \psi(ka)) = \sup_{a\in[0,\infty]} \frac{a - ak}{1 + a^2k} = \sup_{a\in[0,\infty]} \frac{1-k}{\frac{1}{a} + ka} = \frac{1}{2}\left(\frac{1}{\sqrt{k}} - \sqrt{k}\right) < \infty.$$

Taking

$$t(x) = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right), \ \ x \in [0,1].$$

Then

$$t^*(x) = \psi(t(x)) = \arctan\left(\frac{1}{2}\left(\frac{1}{\sqrt{x}} - \sqrt{x}\right)\right), \ \ x \in [0,1].$$

Let T be a t-norm with additive generator t and I^T its power based implication. Let T^* be a t-norm with additive generator t^*. Then the pair (I^T, T^*) satisfies (TC) by Corollary 3.12.

In the case that T is a non-trivial ordinal sum of continuous Archimedean t-norms, it is not an easy task to find t-norm T^* such that (I^T, T^*) satisfies (TC). Here, only the case $T = (\langle a, 1, T_1\rangle), a \in (0,1)$ is studied.

Proposition 3.14. Let $T = (\langle a, 1, T_1\rangle)$, where $a \in (0,1)$, and T_1 is a continuous Archimedean t-norm with additive generator t_1. Let $m \in [t_1(0), \infty]$. If there exists a strictly increasing function $\psi : [0, m] \to [0, \infty]$ with $\psi(0) = 0$ such that

$$\psi(t_1(x)) + \psi(t_1(y)) \in \mathrm{Ran}(\psi \circ t_1) \cup [\psi(t_1(0^+)), \infty] \ \text{ for all } \ x, y \in [0,1],$$

$$\sup_{z\in[0,t_1(0)]} (\psi(z) - \psi(kz)) \leq \psi\left(t_1(\frac{k-a}{1-a})\right) \ \text{ for all } \ k \in (a,1),$$

then there exists a t-norm T_1^* given as

$$T_1^*(x,y) = t_1^{*-1}(\min(t_1^*(x) + t_1^*(y), t_1^*(0))), \ \ x, y \in [0,1],$$

where $t_1^*(x) = \psi(t_1(x))$, $x \in [0,1]$, such that

(i) the pair (I^T, T^*) satisfies (TC), where $T^* = (\langle a, 1, T_1^*\rangle)$.

(ii) the pair $(I^T, T^{\circ *})$ satisfies (TC), where $T^{\circ *} = (\langle 0, a, T^{\circ}\rangle, \langle a, 1, T_1^*\rangle)$ and T° is a t-norm.

Proof. (i) Let $t_1^* : [0,1] \to [0,\infty]$ be a function defined by $t_1^*(x) = \psi(t_1(x))$, $x \in [0,1]$. Obviously, $t_1^*(1) = \psi(t_1(1)) = 0$, and t_1^* is a strictly decreasing function.

Define $T_1^* : [0,1]^2 \to [0,1]$ as $T_1^*(x,y) = t_1^{*-1}(\min(t_1^*(x) + t_1^*(y), t_1^*(0)))$, $x, y \in [0,1]$. Obviously, T_1^* is a t-norm.

Define $T^* : [0,1]^2 \to [0,1]$ as

$$T^*(x,y) = \begin{cases} a + (1-a)\cdot T_1^*(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & \text{if } x, y \in [a,1], \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Obviously, $T^* = (\langle a, 1, T_1^*\rangle)$, it is an ordinal sum of t-norms.

Let $x, y \in [0,1]$ with $x > y$. Consider the following cases.

Case 1: $y < a$. Then $I^T(x,y) = 0$. Hence $T^*(x, I^T(x,y)) = 0 \le y$.

Case 2: $y \ge a$. Then $I^T(x,y) = \frac{t_1(\frac{x-a}{1-a})}{t_1(\frac{y-a}{1-a})}$. Taking $z = t_1(\frac{y-a}{1-a})$, $k = \frac{t_1(\frac{x-a}{1-a})}{t_1(\frac{y-a}{1-a})}$, we get

$$k \in [0,1),\ \ y = a + (1-a)\cdot t_1^{-1}(z), z \in [0, t_1(0)],\ \text{ and }\ x = a + (1-a)\cdot t_1^{-1}(zk).$$

Case 2.1: $k \le a$. Then $T^*(x, I^T(x,y)) = \min(x,k) = k \le a \le y$.

Case 2.2: $k > a$. Then

$$\begin{aligned} T^*(x, I^T(x,y)) &= a + (1-a)\cdot T_1^*(\frac{x-a}{1-a}, \frac{k-a}{1-a}) \\ &= a + (1-a)\cdot t_1^{*-1}\left(\min\left(t_1^*(t_1^{-1}(zk)) + t_1^*(\frac{k-a}{1-a}), t_1^*(0)\right)\right). \end{aligned}$$

If $t_1^*(t_1^{-1}(zk)) + t_1^*(\frac{k-a}{1-a}) \ge t_1^*(0)$, then $T^*(x, I^T(x,y)) = a + (1-a)\cdot 0 = a \le y$.

If $t_1^*(t_1^{-1}(zk)) + t_1^*(\frac{k-a}{1-a}) < t_1^*(0)$, then

$$\begin{aligned} T^*(x, I^T(x,y)) &= a + (1-a)\cdot t_1^{*-1}\left(t_1^*(t_1^{-1}(zk)) + t_1^*(\frac{k-a}{1-a})\right) \\ &= a + (1-a)\cdot t_1^{*-1}\left(t_1^*(t_1^{-1}(zk)) + t_1^*(t_1^{-1}(t_1(\frac{k-a}{1-a})))\right) \\ &= a + (1-a)\cdot t_1^{*-1}\left(\psi(zk) + \psi(t_1(\frac{k-a}{1-a}))\right). \end{aligned}$$

Since $\sup_{z\in[0,t_1(0)]}(\psi(z) - \psi(kz)) \le \psi(t_1(\frac{k-a}{1-a}))$ for all $k \in (a,1)$, then $\psi(zk) + \psi(t_1(\frac{k-a}{1-a})) \ge \psi(z)$ for $z \in [0, t_1(0)]$. Therefore,

$$\begin{aligned} T^*(x, I^T(x,y)) &\le a + (1-a)\cdot t_1^{*-1}(\psi(z)) \\ &= a + (1-a)\cdot t_1^{-1}(z) \\ &= y. \end{aligned}$$

From the above discussion, the pair (I^T, T^*) satisfies (TC).

(ii) Note that $T^{\circ *}(x,y) \le T^*(x,y)$ for all $x, y \in [0,1]$. Hence the pair $(I^T, T^{\circ *})$ satisfies (TC). □

Corollary 3.15. Let $T = (\langle a, 1, T_1\rangle)$, where $a \in (0,1)$, T_1 is a continuous Archimedean t-norm with an additive generator t_1. Let $m \in [t_1(0), \infty]$. If there exists a continuous, strictly increasing function $\psi : [0,m] \to [0,\infty]$ with $\psi(0) = 0$ such that

$$\sup_{z\in[0,t_1(0)]} \psi^{-1}(\psi(z) - \psi(kz)) \leq t_1(\frac{k-a}{1-a}) \text{ for all } k \in (a,1),$$

then there exists a t-norm T_1^* with an additive generator $t_1^*(x) = \psi(t_1(x))$, $x \in [0,1]$, such that

(i) the pair (I^T, T^*) satisfies (TC), where $T^* = (\langle a, 1, T_1^*\rangle)$.

(ii) the pair $(I^T, T^{\circ *})$ satisfies (TC), where $T^{\circ *} = (\langle 0, a, T^\circ\rangle, \langle a, 1, T_1^*\rangle)$ and T° is a t-norm.

Remark 3.16. There exists ψ such that the assumptions in Theorem 3.8 and Proposition 3.14 hold. For instance, consider the following function

$$\psi(x) = \begin{cases} n - t^{-1}(x), & \text{if } x \in (0,m], \\ 0, & \text{if } x = 0, \end{cases}$$

where t is a continuous, strictly decreasing function with $t(1) = 0$, $m = t(0)$ and $n \geq 2$.

It is easy to verify that

$$\psi(t(x)) + \psi(t(y)) \in \mathrm{Ran}(\psi \circ t) \cup [\psi(t(0^+)), \infty] \text{ for all } x, y \in [0,1], \tag{9}$$

$$\sup_{z\in[0,t(0)]} (\psi(z) - \psi(kz)) \leq \psi(t(k)) \text{ for all } k \in (0,1), \tag{10}$$

$$\sup_{z\in[0,t_1(0)]} (\psi(z) - \psi(kz)) \leq \psi\left(t(\frac{k-a}{1-a})\right) \text{ for all } k \in (a,1). \tag{11}$$

Actually, let $k \in (0,1)$. Since $n \geq 2$, then

$$\sup_{z\in(0,t(0)]} (\psi(z) - \psi(kz)) = \sup_{z\in[0,t(0)]} \left(t^{-1}(kz) - t^{-1}(z)\right) \leq 1.$$

On the other hand, since $\psi(t(k)) = n - k > 1$ and

$$\psi\left(t(\frac{k-a}{1-a})\right) = n - \frac{k-a}{1-a} > n - \frac{1-a}{1-a} = n - k > 1.$$

Then (10) and (11) hold.

Also, since

$$\psi(t(x)) = \begin{cases} n - x, & \text{if } x \in [0,1), \\ 0, & \text{if } x = 1, \end{cases} \tag{12}$$

then $\mathrm{Ran}(\psi \circ t) \cup [\psi(t(0^+)), \infty] = \{0\} \cup (n-1, \infty]$. Thus (9) holds.

Note that (12) is an additive generator of the drastic product t-norm T_D. Thus we have the following result.

Theorem 3.17. I^T is T_D-conditional for any continuous t-norm T.

Proof. It suffices to prove that $T_D(x, I^T(x,y)) \leq y$ for all $x, y \in [0,1]$ with $x > y$.
If $1 = x > y$, then $I^T(x,y) = 0$. Thus $T_D(x, I^T(x,y)) = 0 \leq y$.
If $1 > x > y$, then $I^T(x,y) < 1$, Thus $T_D(x, I^T(x,y)) = 0 \leq y$. □

Theorem 3.17 shows that there exist at least one t-norm T_D such that the pair (I^T, T_D) satisfies (TC) for all T-power based implications.

Theorem 3.18. Let A be an index set and $\{(a_i, b_i)\}_{i\in A}$ a family of non-empty, pairwise disjoint open subintervals of $[0, a]$, where $a \in (0, 1)$. Let $\{(T_i)_{i\in A}\}$ be a family of t-norms. Let $T = (\langle a, 1, T^\star\rangle)$ and $T^\circ = (\langle a_i, b_i, T_i\rangle, \langle a, 1, T^{\circ\star}\rangle)_{i\in A}$ be ordinal sum t-norms, where $T^\star$ is a continuous Archimedean t-norm and $T^{\circ\star}$ is a t-norm. If the pair $(I^{T^\star}, T^{\circ\star})$ satisfies (TC), then the pair (I^T, T°) satisfies (TC).

Proof. Let $x, y \in [0,1]$ with $x > y$ and $t^\star$ be an additive generator of t-norm $T^\star$.
Assume that $I^{T^\star}$ is $T^{\circ\star}$-conditional. Then

$$T^{\circ\star}(x, I^{T^\star}(x,y)) = T^{\circ\star}(x, \frac{t^\star(x)}{t^\star(y)}) < y.$$

Thus

$$T^{\circ\star}(\frac{x-a}{1-a}, \frac{t^\star(\frac{x-a}{1-a})}{t^\star(\frac{y-a}{1-a})}) < \frac{y-a}{1-a} \quad \text{for all } x, y \in [a,1] \text{ with } x > y.$$

Let $T^\bullet$ be an ordinal sum t-norms of the form $(\langle a, 1, T^{\circ\star}\rangle)$, i.e., $T^\bullet = (\langle a, 1, T^{\circ\star}\rangle)$.
Since $T^\circ \leq T^\bullet$, then it suffices to prove that I^T is $T^\bullet$-conditional.
If $y < a$, then $I^T(x,y) = 0$. Thus $T^\bullet(x, I^T(x,y)) = 0 \leq y$.
If $y \geq a$ and $I^T(x,y) = \frac{t^\star(\frac{x-a}{1-a})}{t^\star(\frac{y-a}{1-a})} < a$, then $T^\bullet(x, I^T(x,y)) \leq a \leq y$.
If $y \geq a$ and $I^T(x,y) = \frac{t^\star(\frac{x-a}{1-a})}{t^\star(\frac{y-a}{1-a})} \geq a$, then

$$T^\bullet(x, I^T(x,y)) = a + (1-a)\cdot T^{\circ\star}(\frac{x-a}{1-a}, \frac{I^T(x,y)-a}{1-a}).$$

Since $\frac{I^T(x,y)-a}{1-a} \leq I^T(x,y)$, then

$$\begin{aligned} T^\bullet(x, I^T(x,y)) &\leq a + (1-a)\cdot T^{\circ\star}(\frac{x-a}{1-a}, I^T(x,y)) \\ &= a + (1-a)\cdot T^{\circ\star}(\frac{x-a}{1-a}, \frac{t^\star(\frac{x-a}{1-a})}{t^\star(\frac{y-a}{1-a})}) \\ &\leq a + (1-a)\cdot\frac{y-a}{1-a} \\ &= y. \end{aligned}$$

From the above discussions, we get I^T is $T^\bullet$-conditional. □

Corollary 3.19. Let A be an index set and $\{(a_i, b_i)\}_{i\in A}$ a family of non-empty, pairwise disjoint open subintervals of $[0, a]$, where $a \in (0, 1)$. Let $\{(T_i)_{i\in A}\}$ be a family of t-norms. Let $T = (\langle a, 1, T^\star\rangle)$ and $T^\circ = (\langle a_i, b_i, T_i\rangle, \langle a, 1, T^\star\rangle)_{i\in A}$ be ordinal sum t-norms, where $T^\star$ is a nilpotent t-norm with continuous additive generator t. Then the pair (I^T, T°) satisfies (TC) if and only if

$$\frac{t(p)}{1-p} \geq t(0) \text{ for all } p \in [0, 1).$$

Proof. Obviously by Theorem 3.18 and Theorem 3.2. □

4. CONCLUSIONS

In this paper, the T-conditionality of the T-power based implications is deeply studied. The concise necessary and sufficient conditions under which the pair (I^T, T) satisfies (TC) are obtained. Moreover, we present the sufficiency conditions under which the pair (I^T, T^*) satisfies (TC) when t-norm T^* is different from T. These results show a clue to construct a t-norm T^* such that the pair (I^T, T^*) satisfies (TC). Also, these results are beneficial to the application of the T-power based implications.

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