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GROWTH CONDITIONS FOR THE STABILITY OF A CLASS OF TIME–VARYING PERTURBED SINGULAR SYSTEMS

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In this paper, we investigate the problem of stability of linear time–varying singular systems, which are transferable into a standard canonical form. Sufficient conditions on exponential stability and practical exponential stability of solutions of linear perturbed singular systems are obtained based on generalized Gronwall inequalities and Lyapunov techniques. Moreover, we study the problem of stability and stabilization for some classes of singular systems. Finally, we present a numerical example to validate the effectiveness of the abstract results of this paper.

Keywords: linear time–varying singular systems, standard canonical form, consistent initial conditions, Gronwall inequalities, Lyapunov techniques, practical exponential stability

Classification: 37B55, 34D20

1. INTRODUCTION

Singular systems are a combination of differential equations with algebraic constraints. In that sense, singular systems represent the constraints of the solution of the differential part. This class of systems is also called differential–algebraic equations, degenerate systems, descriptor systems, generalized systems, semi-state systems, or generalized state-space systems.

In [22], Rosenbrock introduced singular systems and handled the transformation of differential–algebraic equations (DAEs). Later on, Luenberger [18, 19] investigated the problem of the existence and uniqueness of solutions of singular systems.

Singular systems representation has been used as a perfect tool to model different problems, such as aircraft dynamics, electrical engineering, robotics, optimization problems, economics, chemistry, biology, etc.

The stability theory of linear differential–algebraic equations is an active research topic. The problem of stability analysis of linear time–invariant singular systems has attracted the attention of different authors. We mention here the references [8, 9, 10, 13], among others. However, few results are concerned with the stability theory of linear time–varying singular systems because of the difficulty arising in analysis, see [6, 7, 23].

Having the motivations stated above and encouraged by these pioneering works, this paper is concerned with the stability of the perturbed linear time–varying differential–

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algebraic system, when its associated unperturbed nominal system is transferable into a standard canonical form. The technique developed in this paper relies on Gronwall type inequalities and the Lyapunov functions.

Gronwall-type lemmas play a primary role in ordinary differential equations and singular systems to prove the results on existence, uniqueness, boundedness, comparison, perturbation, and stability, etc. Naturally, Gronwall-type inequalities will play a central role in this paper. Additionally, the method of Lyapunov functions is one of the most powerful tools to study the stability of singular systems. Algebraic–differential systems are more complicated than standard state-space systems because of the existence of algebraic equations. That's why it is not easy to calculate the derivative of the Lyapunov function.

When the origin is not necessarily an equilibrium point, we can study the stability of solutions of singular systems in a small neighborhood of the origin. This property is referred to as practical stability. Several results on the stability of the nontrivial solution re proposed in [1, 2, 10, 11, 12]. To our knowledge, no work has been reported about the practical stability of linear time–varying singular systems.

The main goal of the present paper is to establish criteria for exponential stability and practical uniform exponential stability of perturbed linear time–varying differential–algebraic equations, when its associated unperturbed nominal system is transferable into the standard canonical form. The main technical tools for deriving our results are the Gronwall inequalities and the Lyapunov direct method. That is, the aim of this paper is twofold.

In the first part, we establish some criteria for exponential stability and practical uniform exponential stability of perturbed linear time–varying singular systems, when its associated unperturbed nominal system is transferable into a standard canonical form. The main tool for deriving stability results is the generalized transition matrix based on Gronwall inequalities.

The second part of this paper is mainly devoted to establishing some criteria for uniform exponential stability and practical uniform exponential stability of a class of perturbed linear time–varying singular systems via Lyapunov functions under the presumption that the initial conditions were consistent. Furthermore, we exhibit a numerical example to show the applicability of our abstract theory. Finally, some conclusions are drawn.

2. PRELIMINARY RESULTS

Throughout this paper, unless otherwise specifies, we use the following notations.

Notations:

$\ker B$: The kernel of the matrix $B \in \mathbb{R}^{m\times n}$.

$\operatorname{im} B$: The image of the matrix $B \in \mathbb{R}^{m\times n}$.

$\mathbb{GI}_n(\mathbb{R})$: The general linear group of degree n, i. e., the set of all invertible $n \times n$ matrices over $\mathbb{R}$.

$\mathbb{C}(I, S)$: The set of continuous functions $g : I \to S$ from an open set $I \subseteq \mathbb{R}$ to a vector space S.

$\mathbb{C}^k(I, S)$: The set of k-times continuously differentiable functions $g : I \to S$ from an open set $I \subseteq \mathbb{R}$ to a vector space S.

dom g : The domain of the function g.

I_n : Identity matrix in $\mathbb{R}^{n\times n}$.

$||x|| := \sqrt{x^T x}$: Euclidean norm of $x \in \mathbb{R}^n$.

$||B|| := \sup\{||Bx|| \mid ||x|| = 1\}$, induced matrix norm of $B \in \mathbb{R}^{m\times n}$.

Now, we consider the following linear time–varying continuous singular system:

$$E(t)\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $x(t_0) = x_0 \in \mathbb{R}^n$ is the initial condition, with $(E, A) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$, where E is a singular matrix.

2.1. Standard canonical form

In this paragraph, we introduce the concept of consistent initial values, and the subclass of DAEs (E, A) which are transferable into a standard canonical form. Furthermore, we recall some needed properties for the subsequent sections. For extra details, we refer the reader to see [6, 17].

Definition 2.1. A pair $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ is called a pair of consistent initial values of the linear time–varying singular system (2.1) if there exists a solution $x(\cdot)$ of (2.1), with $t_0 \in$ dom $x(\cdot)$ and $x(t_0) = x_0$.

In the sequel, we denote by $\mathcal{W}$ the set of all pairs of consistent initial values of the linear time–varying singular system (2.1). In addition, for $t_0 \in \mathbb{R}_+$,

$$\mathcal{W}(t_0) = \{x_0 \in \mathbb{R}^n : (t_0, x_0) \in \mathcal{W}\}.$$

$\mathcal{W}(t_0)$ is the linear subspace of initial values, which are consistent at time t_0.

Notice that, if $x : I \to \mathbb{R}^n$ is a solution of the system (2.1), then $x(t) \in \mathcal{W}(t)$ for all $t \in I$.

Definition 2.2. The DAEs (E_1, A_1) and $(E_2, A_2) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$ are said to be equivalent, if there exist $S \in \mathbb{C}(\mathbb{R}_+, \mathbb{GI}_n(\mathbb{R}))$ and $T \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{GI}_n(\mathbb{R}))$, such that

$$E_2 = SE_1T, \qquad A_2 = SA_1T - SE_1\dot{T},$$

and we write $(E_1, A_1) \sim (E_2, A_2)$.

Definition 2.3. The system (2.1) is said to be transferable into a standard canonical form (SCF), if there exist $S \in \mathbb{C}(\mathbb{R}_+, \mathbb{GI}_n(\mathbb{R}))$, $T \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{GI}_n(\mathbb{R}))$, $b > 0$, such that

$$(E, A) \sim \left[\begin{pmatrix} I_b & 0 \\ 0 & N \end{pmatrix}, \begin{pmatrix} J & 0 \\ 0 & I_{n-b} \end{pmatrix} \right], \tag{2.2}$$

where $J : \mathbb{R}_+ \to \mathbb{R}^{b\times b}$ and $N : \mathbb{R}_+ \to \mathbb{R}^{(n-b)\times(n-b)}$, which is pointwise strictly lower triangular matrix. A matrix N is called pointwise strictly lower triangular, if all entries of $N(t)$ on the diagonal and above are zero for all $t \in \mathbb{R}_+$.

Now, we are in a position to characterize the set of consistent initial conditions DAEs (E, A), for the class which are transferable into a standard canonical form.

Proposition 2.1. (Berger and Ilchmann [6]) Suppose that the DAE $(E, A) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$ is transferable into a standard canonical form. Then,

$$(t_0, x_0) \in \mathcal{W} \Longleftrightarrow x_0 \in \text{ im } T(t_0) \begin{pmatrix} I_b \\ 0 \end{pmatrix}.$$

We recall some properties for the generalized transition matrix, which will be needed for the next section.

Proposition 2.2. (Berger and Ilchmann [6]) Let $(E, A) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$ be transferable into SCF for (S, T) as in definition (2.3). Hence, any solution of the initial value problem $x(t_0) = x_0$ where $(t_0, x_0) \in \mathcal{W}$, extends uniquely to a global solution $x(\cdot)$. This solution fulfills:

$$x(t) = U(t, t_0)x_0,$$

with

$$U(t, t_0) = T(t) \begin{pmatrix} \phi_J(t, t_0) & 0 \\ 0 & 0 \end{pmatrix} T^{-1}(t_0),$$

and $\phi_J(\cdot, \cdot)$ denotes the transition matrix of $\dot{z} = J(t)z$. $U(\cdot, \cdot)$ is called the generalized transition matrix of the linear system (2.1).

If the linear time–varying singular system (2.1) is transferable into SCF, then for all $t, r, s \in \mathbb{R}_+$, the generalized transition matrix satisfies the following properties:

(i) $E(t)\frac{d}{dt}U(t, s) = A(t)U(t, s)$,

(ii) $\text{im}U(t, s) = \mathcal{W}(t)$,

(iii) $U(t, s) = U(t, r)U(r, s)$,

(iv) $U^2(t, t) = U(t, t)$,

(v) $\forall x \in \mathcal{W}(t) : U(t, t)x = x$,

(vi) $\frac{\mathrm{d}}{\mathrm{d}t}U(s, t) = -U(s, t)T(t)S(t)A(t)$.

3. STABILITY ANALYSIS FOR A CLASS OF TIME–VARYING PERTURBED SINGULAR SYSTEMS

In this section, we restrict ourselves to study the problem of stability of certain classes of perturbed singular systems (2.1), when its associated unperturbed nominal system is a linear time–varying singular system transferable into a standard canonical form. The principal mathematical technique employed is generalized Gronwall inequalities.

3.1. Non-Linear integral inequalities

Gronwall–type lemmas play a crucial role in the area of the integral (and differential) equations. There exist many lemmas which carry the name of Gronwall's lemma. The original lemma established by T. H. Gronwall [16] is the following:

Lemma 3.1. Let $z : [\eta, \eta + h] \to \mathbb{R}$ be a continuous function that satisfies the following inequality:

$$0 \leq z(x) \leq \int_{\eta}^{x} A + Bz(s)\,\mathrm{d}s, \quad \text{for} \quad x \in [\eta, \eta + h],$$

where A, B are nonnegative constants. Then,

$$0 \leq z(x) \leq Ahe^{Bh}, \quad \text{for} \quad x \in [\eta, \eta + h].$$

Lemma 3.2. Let $u(t)$ be a continuous function defined on the interval $[t_0, t_1]$, satisfying the following inequality:

$$u(t) \leq a + b \int_{t_0}^{t} u(s)\,\mathrm{d}s,$$

where a, b are nonnegative constants. Then, for all $t \in [t_0, t_1]$, we have

$$u(t) \leq ae^{b(t-t_0)}.$$

Bellman [3] extended the above inequality, which reads in the following form.

Lemma 3.3. Let $u(t)$ and $b(t)$ be nonnegative continuous functions for $t \in [t_0, t_1]$, that satisfy

$$u(t) \leq a + \int_{t_0}^{t} b(s)u(s)\,\mathrm{d}s, \quad t \in [t_0, t_1],$$

where $a \geq 0$ is a constant. Then,

$$u(t) \leq a \exp\left(\int_{t_0}^{t} b(s)\,\mathrm{d}s\right), \quad t \in [t_0, t_1].$$

Before proving the stability results in this section, we need the following Integral inequality of type Gronwall lemma, which is a slight modification of the one given by [24].

Lemma 3.4. Let $u(t)$, $v(t)$, $\omega(t)$ be nonnegative continuous functions for $t \geq t_0$, and suppose

$$u(t) \leq c + \int_{t_0}^{t} \left(u(s)v(s) + \omega(s)\right)\,\mathrm{d}s, \tag{3.1}$$

where c is a positive constant. Then,

$$u(t) \leq e^{\int_{t_0}^{t} v(s)\,\mathrm{d}s}\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right)\right], \quad \forall t \geq t_0. \tag{3.2}$$

Proof. The proof of this theorem is based upon the use of the inequality $e^z \geq z+1$.

For $z = \int_{t_0}^{t} \omega(s)\,\mathrm{d}s \geq 0$, we obtain $e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} \geq \int_{t_0}^{t} \omega(s)\,\mathrm{d}s + 1$.

Thus, it follows that

$$u(t) \leq c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right) + \int_{t_0}^{t} u(s)v(s)\,\mathrm{d}s. \tag{3.3}$$

The latter inequality implies that,

$$\Big[u(t)\Big]\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right) + \int_{t_0}^{t} u(s)v(s)\,\mathrm{d}s\right]^{-1} \leq 1.$$

Multiplying this with $v(\cdot) \geq 0$, it yields that,

$$\Big[u(t)v(t)\Big]\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right) + \int_{t_0}^{t} u(s)v(s)\,\mathrm{d}s\right]^{-1} \leq v(t),$$

and

$$\begin{aligned}&\Big[u(t)v(t) + \omega(t)e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s}\Big]\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right) + \int_{t_0}^{t} u(s)v(s)\,\mathrm{d}s\right]^{-1} \\ &\leq v(t) + \Big[\omega(t)e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s}\Big]\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right)\right]^{-1}.\end{aligned}$$

Integrating both sides of the above inequality between t_0 and t, we obtain

$$\begin{aligned}&\ln\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right) + \int_{t_0}^{t} u(s)v(s)\,\mathrm{d}s\right] - \ln(c) \\ &\leq \int_{t_0}^{t} v(s)\,\mathrm{d}s + \ln\left[c + \left(e^{\int_{t_0}^{t} \omega(s)\,\mathrm{d}s} - 1\right)\right] - \ln(c).\end{aligned}$$

Converting this to the exponential form and taking into account inequality (3.3), then the inequality becomes (3.2). □

3.2. Stability analysis

This paragraph aims to state sufficient conditions for the stability of a class of perturbed singular system, when its associated homogeneous system (2.1) is transferable into a standard canonical form. The principal tool for deriving stability results is the generalized transition matrix based upon Gronwall inequalities.

Assume that some parameters of the linear time–varying singular system (2.1) are excited or perturbed, and the perturbed singular system is described by the following equation:

$$E(t)\dot{x}(t) = A(t)x(t) + E(t)\Pi(t)g(t,x(t)), \quad x(t_0) = x_0, \tag{3.4}$$

where $\Pi \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, such that $\text{im}\Pi(t) = \mathcal{W}(t)$ for all $t \in \mathbb{R}_+$, and $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function in (t, x), Lipschitz in x, uniformly in t. Further, we assume that (E, A) are transferable into a standard canonical form.

Thanks to the existence of a generalized transition matrix of (E, A). We obtain the following solution's representation (3.4).

Theorem 3.5. $x : \mathbb{R}_+ \to \mathbb{R}^n$ is a solution of (3.4), if and only if, $x(\cdot)$ satisfies the following integral equation:

$$x(t) = U(t, t_0)x_0 + \int_{t_0}^{t} U(t, s)\Pi(s)g(s, x(s))\,\mathrm{d}s, \quad t \geq t_0.$$

P r o o f. "$\Rightarrow$" Fix $t_0 \in \mathbb{R}_+$, and consider the following functions:

$$\begin{aligned} &\xi : \mathbb{R}_+ \to \mathbb{R}^n, \quad t \mapsto x(t) - \int_{t_0}^{t} U(t, s)\Pi(s)g(s, x(s))\,\mathrm{d}s, \\ &\chi : \mathbb{R}_+ \to \mathbb{R}^n, \quad t \mapsto U(t, t_0)x_0. \end{aligned}$$

Then, for all $t \geq t_0$, we have

$$E(t)\dot{\xi}(t) = E(t)\dot{x}(t) - \int_{t_0}^{t} E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t, s)\Pi(s)g(s, x(s))ds - E(t)U(t, t)\Pi(t)g(t, x(t)).$$

From (i) and (v) of Proposition 2.2, it follows that

$$\begin{aligned} E(t)\dot{\xi}(t) &= A(t)x(t) + E(t)\Pi(t)g(t, x(t)) \\ &\quad - \int_{t_0}^{t} A(t)U(t, s)\Pi(s)g(s, x(s))\,\mathrm{d}s - E(t)\Pi(t)g(t, x(t)) \\ &= A(t)x(t) - A(t)\int_{t_0}^{t} U(t, s)\Pi(s)g(s, x(s))\,\mathrm{d}s \\ &= A(t)\xi(t). \end{aligned}$$

Consequently, ξ and χ are both solutions of (2.1). Therefore, $x(t_0) = \xi(t_0) \in \mathcal{W}(t_0)$ and (v) of Proposition 2.2 gives

$$\chi(t_0) = U(t_0, t_0)x(t_0) = x(t_0) = \xi(t_0).$$

Hence, ξ and χ are both solutions of the initial value problem

$$E(t)\dot{y} = A(t)y, \quad y(t_0) = x(t_0) \in \mathcal{W}(t_0).$$

Then, from the uniqueness of the solution, we deduce that $\xi = \chi$.

"$\Leftarrow$" For fixed $t_0 \in \mathbb{R}_+$, differentiation immediately gives for all $t \geq t_0$,

$$E(t)\dot{x}(t)=E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,t_0)x(t_0)+\int_{t_0}^{t}E(t)\frac{\mathrm{d}}{\mathrm{d}t}U(t,s)\Pi(s)g(s,x(s))\,\mathrm{d}s+E(t)U(t,t)\Pi(t)g(t,x(t)).$$

Using (i) and (v) of Proposition 2.2, we obtain

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)U(t,t_0)x(t_0) + A(t)\int_{t_0}^{t}U(t,s)\Pi(s)g(s,x(s))\,\mathrm{d}s \\ &\quad + E(t)\Pi(t)g(t,x(t)) \\ &= A(t)x(t) + E(t)\Pi(t)g(t,x(t)). \end{aligned}$$

Thus, proving the theorem. □

Remark 3.6. Different authors tackled the approach of structured perturbation for linear differential–algebraic equations, see [5, 10]. Indeed, $E\Pi g$ is a structured perturbation that guarantees "consistency" with the DAE, this becomes trivial by considering the equivalent integral equation Theorem 3.5, which justifies the term $E\Pi$ in front of g. That is,

$$E(t)\Pi(t)g(t,x(t)) \in \mathcal{W}(t), \text{ for all } t \geq 0.$$

We state the definition of the exponential stability of the perturbed singular system (3.4).

Definition 3.1. The perturbed singular system (3.4) is said to be uniformly exponentially stable, if there exist positive constants α and β, such that for all $(t_0, x_0) \in \mathcal{W}$,

$$||x(t)|| \leq \beta||x_0||e^{-\alpha(t-t_0)}, \quad \forall t \geq t_0.$$

We recall the following characterization theorem for the exponential stability of linear time–varying singular systems (2.1), which are transferable into SCF.

Theorem 3.7. (Berger and Ilchmann [6]) Let (2.1) be transferable into SCF, then (2.1) is exponentially stable, if and only if, there exist positive constants k and γ, such that for all $(t_0, x_0) \in \mathcal{W}$,

$$\|U(t,t_0)x_0\| \leq ke^{-\gamma(t-t_0)}||x_0||, \quad \forall t \geq t_0.$$

Let us now state some assumptions, which we will impose later on:

(H_1) The unperturbed nominal singular system (2.1) is uniformly exponentially stable.

(H_2) Π is a continuous bounded matrix.

(H_3) There exist a continuous nonnegative function $\varphi(t)$ and a nonnegative constant $\widetilde{\varphi}$, such that

$$\|g(t,x)\| \leq \varphi(t)||x||, \quad \forall (t,x) \in \mathcal{W},$$

with $\displaystyle\int_0^{+\infty}\varphi(s)\,\mathrm{d}s \leq \widetilde{\varphi}$. Moreover, $g(t,0) = 0$ for all $t \geq t_0 \geq 0$.

Theorem 3.8. Under assumptions $(H_1) - (H_3)$, the perturbed singular system (3.4) is uniformly exponentially stable.

Proof. Invoking Theorem 3.5, we can write the solution of the perturbed singular system (3.4) as

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,s)\Pi(s)g(s,x(s))\,\mathrm{d}s, \quad \forall t \geq t_0.$$

Since the unperturbed nominal system (2.1) is uniformly exponentially stable. Then, from Theorem 3.7 there exist positive constants k and γ, such that for all $(t_0, x_0) \in \mathcal{W}$,

$$\|U(t,t_0)x_0\| \leq k e^{-\gamma(t-t_0)}||x_0||, \quad \forall t \geq t_0.$$

In addition, $\Pi(s)g(s,x(s)) \in \mathcal{W}(s)$, then it yields that for all $t \geq t_0$,

$$||x(t)|| \leq k e^{-\gamma(t-t_0)}||x_0|| + \int_{t_0}^{t} k e^{-\gamma(t-s)}||\Pi(s)g(s,x(s))||\,\mathrm{d}s.$$

Multiplying this with $e^{\gamma t}$, we obtain

$$\begin{aligned} ||x(t)||e^{\gamma t} &\leq k e^{\gamma t_0}||x_0|| + \int_{t_0}^{t} k e^{\gamma s}||\Pi(s)g(s,x(s))||\,\mathrm{d}s \\ &\leq k e^{\gamma t_0}||x_0|| + \int_{t_0}^{t} k e^{\gamma s}||\Pi(s)||||g(s,x(s))||\,\mathrm{d}s. \end{aligned}$$

On the other side, $t \mapsto \Pi(t)$ is bounded, then there exists $M > 0$, such that

$$||\Pi(t)|| \leq M, \quad \forall t \geq t_0. \tag{3.5}$$

One can deduce from the standing assumptions that,

$$||x(t)||e^{\gamma t} \leq k e^{\gamma t_0}||x_0|| + \int_{t_0}^{t} kM e^{\gamma s}\varphi(s)||x(s)||\,\mathrm{d}s.$$

Thanks to Lemma 3.3, the inequality above implies that,

$$\begin{aligned} ||x(t)||e^{\gamma t} &\leq k e^{\gamma t_0}||x_0|| \exp\left(kM \int_{t_0}^{t} \varphi(s)\,\mathrm{d}s\right) \\ &\leq k e^{\gamma t_0}||x_0|| \exp\left(kM \int_{0}^{+\infty} \varphi(s)\,\mathrm{d}s\right). \end{aligned}$$

That is,

$$||x(t)|| \leq k \exp\left(kM\widetilde{\varphi}\right)||x_0||e^{-\gamma(t-t_0)}, \quad \forall (t_0,x_0) \in \mathcal{W}, \quad \forall t \geq t_0.$$

Setting, $\beta = k\exp(kM\widetilde{\varphi})$ and $\alpha = \gamma$, we close that the perturbed singular system (3.4) is uniformly exponentially stable. □

Now, we consider the following time–varying perturbed singular system associated to the system (3.4) of the form:

$$E(t)\dot{x}(t) = A(t)x(t) + E(t)\Pi(t)g(t,x(t)) + E(t)\Pi(t)h(t), \quad x(t_0) = x_0. \tag{3.6}$$

Assume that $h(t)$ is a continuous function. That is, the origin is no longer an equilibrium point of the perturbed singular system (3.6). In this case we study the stability of solutions with respect to the small neighborhood of the origin.

The study of the asymptotic behavior of solutions led back to the study of stability of a ball centered at the origin: $B_r := \{x \in \mathbb{R}^n : \|x\| \leq r\}\,,\ r > 0$.

Definition 3.2. *i)* B_r is uniformly exponentially stable, if there exist positive constants $\lambda_1,\ \lambda_2$, such that for all $(t_0, x_0) \in \mathcal{W}$,

$$\| x(t,t_0,x_0) \| \leq \lambda_1 \| x_0 \| \exp(-\lambda_2(t-t_0)) + r, \quad \forall t \geq t_0.$$

ii) The perturbed singular system (3.6) is said to be practically uniformly exponentially stable, if there exists $r > 0$, such that $B_r \subset \mathbb{R}^n$ is uniformly exponentially stable.

Sufficient conditions on practical uniform exponential stability of the perturbed singular system (3.6) will be provided next.

Let us now consider the following assumption:

(H_4) There exists a positive constant M', such that

$$\int_0^{+\infty} ||h(t)||e^{\gamma t}\,\mathrm{d}t \leq M' < +\infty.$$

Theorem 3.9. Under assumptions $(H_1) - (H_4)$, the perturbed singular system (3.6) is practically uniformly exponentially stable.

Proof. Thanks to Theorem 3.5, the solution of the perturbed singular system (3.6) expressed as the following:

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^{t} U(t,s)\Pi(s)\left(g(s,x(s)) + h(s)\right)\,\mathrm{d}s, \quad \forall t \geq t_0.$$

Since the unperturbed nominal system (2.1) is uniformly exponentially stable, then it follows from Theorem 3.7 that there exist positive constants k and γ, such that for all $(t_0, x_0) \in \mathcal{W}$,

$$\|U(t,t_0)x_0\| \leq ke^{-\gamma(t-t_0)}||x_0||, \quad \forall t \geq t_0.$$

Furthermore, $\Pi(s)\left(g(s,x(s)) + h(s)\right) \in \mathcal{W}(s)$, then for all $t \geq t_0$, it yields that,

$$||x(t)|| \leq ke^{-\gamma(t-t_0)}||x_0|| + \int_{t_0}^{t} ke^{-\gamma(t-s)}||\Pi(s)\left(g(s,x(s)) + h(s)\right)||\,\mathrm{d}s. \tag{3.7}$$

Multiplying both sides of (3.7) by $e^{\gamma t}$, we obtain

$$\begin{aligned} ||x(t)||e^{\gamma t} &\leq ke^{\gamma t_0}||x_0|| + \int_{t_0}^{t} ke^{\gamma s}||\Pi(s)\left(g(s,x(s)) + h(s)\right)||\,\mathrm{d}s \\ &\leq ke^{\gamma t_0}||x_0|| + \int_{t_0}^{t} ke^{\gamma s}||\Pi(s)||\ ||g(s,x(s)) + h(s)||\,\mathrm{d}s \\ &\leq ke^{\gamma t_0}||x_0|| + \int_{t_0}^{t} ke^{\gamma s}||\Pi(s)||\left(||g(s,x(s))|| + ||h(s)||\right)\,\mathrm{d}s. \end{aligned}$$

Assumptions (H_2) and (H_3), implies

$$||x(t)||e^{\gamma t} \leq ke^{\gamma t_0}||x_0|| + \int_{t_0}^{t} kMe^{\gamma s}\left(\varphi(s)||x(s)|| + ||h(s)||\right)\,\mathrm{d}s.$$

By applying the Gronwall Lemma 3.4, it yields,

$$\begin{aligned} ||x(t)||e^{\gamma t} &\leq \exp\left(kM\int_{t_0}^{t}\varphi(s)\,\mathrm{d}s\right)\left(ke^{\gamma t_0}||x_0|| + \exp\left(kM\int_{t_0}^{t}e^{\gamma s}||h(s)||\,\mathrm{d}s\right) - 1\right) \\ &\leq \exp\left(kM\int_{0}^{+\infty}\varphi(s)\,\mathrm{d}s\right)\left(ke^{\gamma t_0}||x_0|| + \exp\left(kM\int_{0}^{+\infty}e^{\gamma s}||h(s)||\,\mathrm{d}s\right)\right). \end{aligned}$$

Combining the last inequality with (H_3) and (H_4), one obtains

$$||x(t)||e^{\gamma t} \leq e^{kM\widetilde{\varphi}}\left(ke^{\gamma t_0}||x_0|| + e^{kMM'}\right).$$

Consequently, we obtain

$$\begin{aligned} ||x(t)|| &\leq ke^{kM\widetilde{\varphi}}||x_0||e^{-\gamma(t-t_0)} + e^{-\gamma t}e^{kM\widetilde{\varphi}}e^{kMM'} \\ &\leq ke^{kM\widetilde{\varphi}}||x_0||e^{-\gamma(t-t_0)} + e^{kM(\widetilde{\varphi}+M')}. \end{aligned}$$

That is,

$$||x(t)|| \leq ke^{kM\widetilde{\varphi}}||x_0||e^{-\gamma(t-t_0)} + e^{kM(\widetilde{\varphi}+M')}, \quad \forall (t_0,x_0) \in \mathcal{W}, \quad \forall t \geq t_0.$$

Hence, the ball of radius $r = e^{kM(\widetilde{\varphi}+M')}$ with respect to the perturbed singular system (3.6), B_r is uniformly exponentially stable. Thus, the perturbed singular system (3.6) is practically uniformly exponentially stable. □

4. STABILITY ANALYSIS FOR A CLASS OF TIME–VARYING PERTURBED SINGULAR SYSTEMS IN THE SENSE OF LYAPUNOV

Wealthy historic background recorded in the investigation of the stability of linear time–invariant singular systems, where $E \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}$ via the method of Lyapunov, see [8, 10, 13, 14, 20].

For linear time–invariant singular systems $(E, A) \in (\mathbb{R}^{n\times n})^2$ it is well known that one seeks for solutions $P, Q \in \mathbb{R}^{n\times n}$ of the Lyapunov equation,

$$A^T PE + E^T PA = -Q, \tag{4.1}$$

and the corresponding Lyapunov function candidate is the following:

$$V : \mathcal{W} \setminus \{0\} \to \mathbb{R}, \quad x \longmapsto (Ex)^T P(Ex),$$

where $\mathcal{W} \setminus \{0\} = \mathcal{W}(t)$, for all $t \in \mathbb{R}_+$.

For linear time–varying singular systems $(E, A) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$, the analogous Lyapunov function is the following:

$$V : \mathcal{W} \to \mathbb{R}, \quad (t, x) \mapsto (E(t)x)^T P(t)(E(t)x). \tag{4.2}$$

T. Berger and A. Ilchman [7] generalized the Lyapunov equation (4.1) for linear time–varying singular systems (2.1), which are transferable into a standard canonical form (SCF), such that for all $Q(\cdot) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, there exists $P(\cdot) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ which solves the following equation:

$$x^T \left[A^T(t)P(t)E(t) + E^T(t)P(t)A(t) + \frac{\mathrm{d}}{\mathrm{d}t}(E^T(t)P(t)E(t)) \right] x = -x^T Q(t)x, \forall (t, x) \in \mathcal{W}. \tag{4.3}$$

4.1. Vanishing perturbation

In this paragraph, we assume that the perturbation g vanishes at zero, that is, $g(t, 0) = 0,\ \forall t \geq 0$, which allows us to investigate the stability properties of the null solution to the perturbed singular system (3.4) within the method of Lyapunov.

Our approach structured in this section is to use the Lyapunov function (4.2) for the unperturbed nominal linear time–varying singular system (2.1) as a Lyapunov function candidate for the perturbed singular system (3.4) under some assumptions in the perturbation term.

Now, we are in a position to state our main result in this paragraph.

Theorem 4.1. Consider the perturbed singular system (3.4). Assume that there exists $P \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ being the solution of the generalized time–varying Lyapunov equation (4.3), with $Q = Q^T \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, such that

$$\exists q_1, q_2 > 0 : \quad q_1 x^T x \leq x^T Q(\cdot)x \leq q_2 x^T x, \quad \forall (t, x) \in \mathcal{W}. \tag{4.4}$$

As well as, $E^T PE \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$, such that

$$\exists p_1, p_2 > 0 : \quad p_1 x^T x \leq x^T E^T(\cdot)P(\cdot)E(\cdot)x \leq p_2 x^T x, \quad \forall (t, x) \in \mathcal{W}. \tag{4.5}$$

Additionally, we assume that Π is bounded and

$$||g(t, x)|| \leq c(t)||x||, \quad \forall (t, x) \in \mathcal{W}, \tag{4.6}$$

where $c(\cdot)$ is a continuous positive function, with

$$c(t) \to 0 \text{ as } t \to +\infty. \tag{4.7}$$

We assume that the matrix Q is chosen such that $q_1 > 2cp_2M$. Then, the perturbed singular system(3.4) is uniformly exponentially stable.

Proof. Consider the following Lyapunov-like function:

$$V : \mathcal{W} \to \mathbb{R}, \quad (t, x) \mapsto (E(t)x)^T P(t)(E(t)x).$$

The derivative of $V(\cdot)$ where $x(\cdot)$ is the trajectory (solution) of the perturbed singular system (3.4) is given by the following:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t, x(t)) &= \dot{x}^T(t)E^T(t)P(t)E(t)x(t) + x^T(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(E^T(t)P(t)E(t)\right)x(t) \\
&\quad + x^T(t)E^T(t)P(t)E(t)\dot{x}(t) \\
&= (E(t)\dot{x}(t))^T P(t)E(t)x(t) + x^T(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(E^T(t)P(t)E(t)\right)x(t) \\
&\quad + x^T(t)E^T(t)P(t)(E(t)\dot{x}(t)) \\
&= (A(t)x(t) + E(t)\Pi(t)g(t, x(t)))^T P(t)E(t)x(t) \\
&\quad + x^T(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(E^T(t)P(t)E(t)\right)x(t) \\
&+ x^T(t)E^T(t)P(t)\left(A(t)x(t) + E(t)\Pi(t)g(t, x(t))\right) \\
&= x^T(t)\left(A^T(t)P(t)E(t) + \frac{\mathrm{d}}{\mathrm{d}t}(E^T(t)P(t)E(t)) + E^T(t)P(t)A(t)\right)x(t) \\
&+ 2x^T(t)E^T(t)P(t)E(t)\Pi(t)g(t, x(t)).
\end{aligned}$$

Based upon (4.3), (4.4) and (4.5), we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t, x(t)) &= -x^T(t)Q(t)x(t) + 2x^T(t)E^T(t)P(t)E(t)\Pi(t)g(t, x(t)) \\
&\le -q_1||x(t)||^2 + 2||x(t)||\;||E^T(t)P(t)E(t)||\;||\Pi(t)||\;||g(t, x(t))|| \\
&\le -q_1||x(t)||^2 + 2p_2||x(t)||\;||\Pi(t)||\;||g(t, x(t))||.
\end{aligned}$$

One can deduce from the assumption (4.6) that:

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t, x(t)) \le -q_1||x(t)||^2 + 2p_2c(t)||\Pi(t)||\;||x(t)||^2. \tag{4.8}$$

As $c(t) \to 0$ as $t \to +\infty$, then there exists $c > 0$, such that

$$||c(t)|| \le c, \quad \forall t \ge t_0. \tag{4.9}$$

As well as, from (3.5), it yields that,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t, x(t)) \le -q_1x^T(t)x(t) + 2cp_2Mx^T(t)x(t)$$

$$= -(q_1 - 2cp_2M)x^T(t)x(t).$$

Since, $q_1 > 2cp_2M$. Consequently, it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) &\leq -\frac{q_1 - 2cp_2M}{p_2}(E(t)x(t))^T P(t)(E(t)x(t)) \\ &= -\left(\frac{q_1}{p_2} - 2cM\right)V(t,x(t)).\end{aligned}$$

That is, we obtain for all $t \geq t_0 \geq 0$,

$$V(t,x(t)) \leq V(t_0,x(t_0))\exp\left(-\left(\frac{q_1}{p_2} - 2cM\right)(t-t_0)\right).$$

Now, we are in a position to derive an estimate for the norm of $x(\cdot)$.

$$\begin{aligned}p_1||x(t)||^2 &\leq x^T(t)E^T(t)P(t)E(t)x(t) \leq V(t_0,x(t_0))\exp\left(-\left(\frac{q_1}{p_2} - 2cM\right)(t-t_0)\right) \\ &\leq x^T(t_0)E^T(t_0)P(t_0)E(t_0)x(t_0)\exp\left(-\left(\frac{q_1}{p_2} - 2cM\right)(t-t_0)\right) \\ &\leq p_2||x(t_0)||^2\exp\left(-\left(\frac{q_1}{p_2} - 2cM\right)(t-t_0)\right).\end{aligned}$$

Consequently, we deduce that for all $(t_0,x_0) \in \mathcal{W}$,

$$||x(t)|| \leq \sqrt{\frac{p_2}{p_1}}||x(t_0)||\exp\left(-\left(\frac{q_1}{2p_2} - cM\right)(t-t_0)\right), \quad t \geq t_0.$$

Thus, the singular perturbed system (3.4) is uniformly exponentially stable. □

4.2. Non-vanishing perturbation

In this paragraph, we proceed to investigate the exponential stability of a nontrivial solution of the perturbed singular system (3.4). So, we suppose that $g(t,0)$ is not necessarily zero.

Theorem 4.2. Consider the perturbed singular system (3.4). Assume that there exists $P \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ being the solution of the generalized time–varying Lyapunov equation (4.3), with $Q = Q^T \in \mathbb{C}(\mathbb{R}_+,\mathbb{R}^{n\times n})$ satisfying condition (4.4), and $E^TPE \in \mathbb{C}^1(\mathbb{R}_+,\mathbb{R}^{n\times n})$ fulfills condition (4.5). Furthermore, we suppose Π is bounded, and $g(t,x)$ satisfies the following condition:

$$||g(t,x)|| \leq c(t)||x|| + \psi(t), \quad \forall (t,x) \in \mathcal{W}, \tag{4.10}$$

where $c(\cdot)$ is a nonnegative continuous function which fulfills condition (4.7) and $\psi(\cdot)$ is a nonnegative continuous bounded function. Assume that the matrix Q is chosen such that $q_1 > p_2M(2c+1)$.

Hence, the perturbed singular system (3.4) is practically uniformly exponentially stable.

To prove this theorem, we need to recall an essential Gronwall lemma established in [21].

Lemma 4.3. Let $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous function, ε is a positive real number and λ is a strictly positive real number. Assume that for all $t \in [0, +\infty)$ and $0 \leq v \leq t$, we have

$$\phi(t) - \phi(v) \leq \int_v^t (-\lambda\phi(s) + \varepsilon)\,\mathrm{d}s.$$

Then,

$$\phi(t) \leq \frac{\varepsilon}{\lambda} + \phi(0)\exp(-\lambda t).$$

Note that, the previous result remains true if we replace $\phi(0)$ by $\phi(t_0)$ with $0 \leq t_0 \leq v \leq t$.

P r o o f. Consider the following Lyapunov function:

$$V : \mathcal{W} \to \mathbb{R}, \quad (t, x) \mapsto (E(t)x)^T P(t)(E(t)x).$$

The total derivative $\dot{V}(\cdot)$ along the trajectory $x(\cdot)$ of the singular perturbed system (3.4), is given by

$$\begin{aligned}
\frac{\mathrm{dd}}{\mathrm{d}t} V(t, x(t)) &= \dot{x}^T(t)E^T(t)P(t)E(t)x(t) + x^T(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(E^T(t)P(t)E(t)\right)x(t) \\
&\quad + x^T(t)E^T(t)P(t)E(t)\dot{x}(t) \\
&= x^T(t)\left(A^T(t)P(t)E(t) + \frac{\mathrm{d}}{\mathrm{d}t}(E^T(t)P(t)E(t)) + E^T(t)P(t)A(t)\right)x(t) \\
&\quad + 2x^T(t)E^T(t)P(t)E(t)\Pi(t)g(t, x(t)).
\end{aligned}$$

Taking into account the assumptions, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(t, x(t)) &= -x^T(t)Q(t)x(t) + 2x^T(t)E^T(t)P(t)E(t)\Pi(t)g(t, x(t)) \\
&\leq -q_1 x^T(t)x(t) + 2||x(t)||\, ||E^T(t)P(t)E(t)||\, ||\Pi(t)||\, ||g(t, x(t))|| \\
&\leq -q_1 x^T(t)x(t) + 2p_2||x(t)||\, ||\Pi(t)||\, ||g(t, x(t))|| \\
&\leq -q_1 x^T(t)x(t) + 2p_2 M||x(t)||\,(c(t)||x(t)|| + \psi(t)) \\
&= -q_1 x^T(t)x(t) + 2p_2 Mc(t)||x(t)||^2 + 2p_2 M||x(t)||\psi(t) \\
&\leq -q_1 x^T(t)x(t) + 2p_2 Mc(t)||x(t)||^2 + p_2 M||x(t)||^2 + p_2 M\psi^2(t).
\end{aligned}$$

On the other side, the function $\psi(t)$ is a continuous nonnegative bounded function, then there exists $m > 0$, such that

$$\psi(t) \leq m, \quad \forall t \geq t_0.$$

Moreover, from (4.9) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) \leq -\left(q_1 - p_2M(2c+1)\right)x^T(t)x(t) + p_2Mm^2.$$

Since, $p_2M(2c+1) \leq q_1$. As a result, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) &\leq -\frac{q_1 - p_2M(2c+1)}{p_2}(E(t)x(t))^T P(t)(E(t)x(t)) + p_2Mm^2 \\ &= -\widetilde{\mu}V(t,x(t)) + p_2Mm^2 \qquad (4.11)\end{aligned}$$

where, $\widetilde{\mu} = \dfrac{q_1 - p_2M(2c+1)}{p_2} > 0$.

Integrating (4.11) from $v \in [t_0, t]$ to $t \geq t_0$, on both sides of the inequality, we obtain

$$V(t,x(t)) - V(v,x(v)) \leq \int_v^t \left(-\widetilde{\mu}V(s,x(s)) + p_2Mm^2\right)\mathrm{d}s.$$

Using once more the Gronwall lemma (Lemma 4.3), the above inequality implies,

$$V(t,x(t)) \leq \frac{p_2Mm^2}{\widetilde{\mu}} + V(t_0,x(t_0))\exp\left(-\widetilde{\mu}(t-t_0)\right).$$

Now, we are at a point to estimate the norm of $x(\cdot)$ as follows,

$$\begin{aligned}p_1\|x(t)\|^2 &\leq x^T(t)E^T(t)P(t)E(t)x(t) \leq V(t_0,x(t_0))\exp(-\widetilde{\mu}(t-t_0)) + \frac{p_2Mm^2}{\widetilde{\mu}} \\ &= x^T(t_0)E^T(t_0)P(t_0)E(t_0)x(t_0)\exp(-\widetilde{\mu}(t-t_0)) + \frac{p_2Mm^2}{\widetilde{\mu}} \\ &\leq p_2\|x(t_0)\|^2\exp(-\widetilde{\mu}(t-t_0)) + \frac{p_2Mm^2}{\widetilde{\mu}}.\end{aligned}$$

Consequently, we obtain

$$\|x(t)\|^2 \leq \frac{p_2}{p_1}\|x(t_0)\|^2\exp\left(-\widetilde{\mu}(t-t_0)\right) + \frac{p_2Mm^2}{\widetilde{\mu}p_1}.$$

Based on the fact that $(a+b)^\varepsilon \leq a^\varepsilon + b^\varepsilon$, for all $a,b \geq 0$ and $\varepsilon \in]0,1[$. We conclude that for all $(t_0,x_0) \in \mathcal{W}$,

$$\|x(t)\| \leq \left(\frac{p_2}{p_1}\right)^{\frac{1}{2}}\|x(t_0)\|\exp\left(-\frac{\widetilde{\mu}}{2}(t-t_0)\right) + m\left(\frac{p_2M}{\widetilde{\mu}p_1}\right)^{\frac{1}{2}}, \quad \forall t \geq t_0.$$

Hence, the perturbed singular system (3.4) is practically uniformly exponentially stable. □

Both of Theorems 4.1 and 4.2 show that not the solution $P(\cdot)$ of (4.3) is the target of interest, but $E^T(\cdot)P(\cdot)E(\cdot)$ it is. Symmetry, differentiability, and the boundary conditions are not deserving for $P(\cdot)$, but for $E^T(\cdot)P(\cdot)E(\cdot)$. In the remainder of this section, we aim to derive an extra requirement upon $E(\cdot)$ under which one might get a sequel, where all conditions made on $P(\cdot)$. This means that our target in the next lemma is to state a relationship between $P(\cdot)$ and $E^T(\cdot)P(\cdot)E(\cdot)$.

To this end, we add the following notation:

$$\xi\mathcal{W} := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n : x \in E(t)\mathcal{W}(t)\}.$$

Lemma 4.4. Consider the perturbed singular system (2.2), and let $E^TE \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, such that

$$\exists e_1, e_2 > 0 : \quad e_1 x^T x \leq x^T E^T(t)E(t)x \leq e_2 x^T x, \quad \forall (t,x) \in \mathcal{W}. \tag{4.12}$$

As well as, $P \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ is a symmetric matrix.
Then, the following statements are equivalent:

$$(i)\ \exists \widetilde{p}_1, \widetilde{p}_2 > 0 : \quad \widetilde{p}_1 x^T x \leq x^T P(t)x \leq \widetilde{p}_2 x^T x, \quad \forall (t,x) \in \xi\mathcal{W}. \tag{4.13}$$

$$(ii)\ \exists p_1, p_2 > 0 : \quad p_1 x^T x \leq x^T E^T(t)P(t)E(t)x \leq p_2 x^T x, \quad \forall (t,x) \in \mathcal{W}. \tag{4.14}$$

Proof. $(i) \Rightarrow (ii)$ Let $(t,x) \in \mathcal{W}$, that is $E(t)x \in E(t)\mathcal{W}(t)$.

From (4.12) and (4.13), it yields that

$$\widetilde{p}_1 e_1 x^T x \leq \widetilde{p}_1 x^T E^T(t)E(t)x \leq x^T E^T(t)P(t)E(t)x \leq \widetilde{p}_2 x^T E^T(t)E(t)x \leq \widetilde{p}_2 e_2 x^T x.$$

Hence, for $p_1 = \widetilde{p}_1 e_1$ and $p_2 = \widetilde{p}_2 e_2$ condition (4.14) holds.

$(ii) \Rightarrow (i)$ Let $(t,x) \in \xi\mathcal{W}$, so $x \in E(t)\mathcal{W}(t)$ and therefore there exists $z \in \mathcal{W}(t)$, such that $x = E(t)z$.

Taking into account (4.12) and (4.14), it follows that

$$\begin{aligned}\frac{p_1}{e_2} x^T x &\leq \frac{p_1}{e_2}(E(t)z)^T(E(t)z) \leq p_1 z^T z \leq z^T E^T(t)P(t)E(t)z \\ &= x^T P(t)x \leq p_2 z^T z \leq \frac{p_2}{e_1}(E(t)z)^T(E(t)z) = \frac{p_2}{e_1} x^T x.\end{aligned}$$

Thus, we obtain

$$\frac{p_1}{e_2} x^T x \leq x^T P(t)x \leq \frac{p_2}{e_1} x^T x, \quad \forall (t,x) \in \xi\mathcal{W}.$$

Setting, $\widetilde{p}_1 = \dfrac{p_1}{e_2}$ and $\widetilde{p}_2 = \dfrac{p_2}{e_1}$, hence (4.13) holds. □

Now, we are ready to state the following corollary.

Corollary 4.5. Consider the perturbed singular system (3.4), let $E \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ satisfying (4.12). If there exist $Q = Q^T \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ that satisfies (4.4) and $P \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, satisfying (4.13), such that (4.3) holds. In addition, that suppose Π is bounded, $g(t,x)$ satisfies condition (4.6) and the matrix Q is chosen such that $q_1 > 2cp_2M$. Then, the perturbed singular system (3.4) is uniformly exponentially stable.

Proof. The proof is straightforward. Owing to Lemma 4.4, we have $E^TPE \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ which satisfies condition (4.5). Hence, all conditions of Theorem 4.1 are fulfilled and then the perturbed singular system (3.4) is uniformly exponentially stable. □

Remark 4.6. Under the same assumptions of Corollary 4.5 with $g(t,x)$ satisfies condition (4.10). The perturbed singular system (3.4) is practically uniformly exponentially stable.

5. PRACTICAL EXPONENTIAL STABILITY OF A CLASS OF NONLINEAR SINGULAR SYSTEMS WITH UNCERTAINTIES

In this section, we discuss the problem of stabilization for a class of nonlinear singular systems with uncertainties.

Consider the following system:

$$\begin{cases} \bar{E}(t)\dot{x}(t) = \bar{A}(t)x(t) + B(t)(\psi(t,x,u) + u) \\ \quad x(t_0) = x_0, \end{cases} \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ is the control input. $(\bar{E}, \bar{A}) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})^2$, with $\bar{E}$ is a singular matrix, and $B \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times m})$. Assume that the pair $(\bar{E}, \bar{A})$ is transferable into a standard canonical form, and the function $\psi : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ represents uncertainties in the plant.

Let us state now some assumptions which will be imposed later on.

$(\mathcal{A}_\infty)$ For all $\bar{Q}(\cdot) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$, there exists a symmetric matrix $\bar{P}(\cdot) \in \mathbb{C}(\mathbb{R}_+, \mathbb{R}^{n\times n})$ which solves the following equation:

$$x^T \left[\bar{A}^T(t)\bar{P}(t)\bar{E}(t) + \bar{E}^T(t)\bar{P}(t)\bar{A}(t) + \frac{\mathrm{d}}{\mathrm{d}t}(\bar{E}^T(t)\bar{P}(t)\bar{E}(t))\right] x = -x^T\bar{Q}(t)x, \ \ \forall (t,x) \in \mathcal{W}, \tag{5.2}$$

with $\bar{Q} = \bar{Q}^T$ satisfies the following condition:

$$\exists \bar{q}_1, \bar{q}_2 > 0 : \quad \bar{q}_1 x^T x \le x^T\bar{Q}(\cdot)x \le \bar{q}_2 x^T x, \quad \forall (t,x) \in \mathcal{W}. \tag{5.3}$$

$(\mathcal{A}_\in)$ $\bar{E}^T\bar{P}\bar{E} \in \mathbb{C}^1(\mathbb{R}_+, \mathbb{R}^{n\times n})$ fulfills the following condition:

$$\exists \bar{p}_1, \bar{p}_2 > 0 : \quad \bar{p}_1 x^T x \le x^T\bar{E}^T(\cdot)\bar{P}(\cdot)\bar{E}(\cdot)x \le \bar{p}_2 x^T x, \quad \forall (t,x) \in \mathcal{W}. \tag{5.4}$$

$(\mathcal{A}_\ni)$ There exists a nonnegative continuous function $\phi : \mathbb{R}^n \to \mathbb{R}_+$, such that for all $t \in \mathbb{R}_+,\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$:

$$||\psi(t,x,u)|| \le \phi(x).$$

Theorem 5.1. Suppose that assumptions $(\mathcal{A}_\infty) - (\mathcal{A}_\ni)$ hold, then the feedback law

$$u(t,x) = -\frac{B^T(t)\bar{P}(t)\bar{E}(t)x\phi^2(x)}{||B^T(t)\bar{P}(t)\bar{E}(t)x||\phi(x) + \rho(t)},$$

where $\rho(t)$ is a continuous nonnegative function, with $\lim_{t\to+\infty} \rho(t) = 0$, uniformly practically exponentially stabilizes the system (5.1).

P r o o f . Consider the following Lyapunov-like function:

$$V : \mathcal{W} \to \mathbb{R}, \quad (t,x) \mapsto (\bar{E}(t)x)^T \bar{P}(t)(\bar{E}(t)x).$$

The derivative of $V(\cdot)$ along the trajectory $x(\cdot)$ of the singular system (5.1) is given by,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) &= \dot{x}^T(t)\bar{E}^T(t)\bar{P}(t)\bar{E}(t)x(t) + x^T(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\bar{E}^T(t)\bar{P}(t)\bar{E}(t)\right)x(t) \\
&\quad + x^T(t)\bar{E}^T(t)\bar{P}(t)\bar{E}(t)\dot{x}(t) \\
&= x^T(t)\left(\bar{A}^T(t)\bar{P}(t)\bar{E}(t) + \frac{\mathrm{d}}{\mathrm{d}t}(\bar{E}^T(t)\bar{P}(t)\bar{E}(t)) + \bar{E}^T(t)\bar{P}(t)\bar{A}(t)\right)x(t) \\
&\quad + 2x^T(t)\bar{E}^T(t)\bar{P}(t)B(t)(\psi(t,x,u) + u) \\
&= -x^T(t)\bar{Q}(t)x(t) - \frac{2x^T(t)\bar{E}^T(t)\bar{P}(t)B(t)B^T(t)\bar{P}(t)\bar{E}(t)x(t)\phi^2(x(t))}{||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t)) + \rho(t)} \\
&\quad + 2x^T(t)\bar{E}^T(t)\bar{P}(t)B(t)\psi(t,x,u).
\end{aligned}$$

Using $(\mathcal{A}_\infty)$ and $(\mathcal{A}_\ni)$, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) &\leq -x^T\bar{Q}(t)x - \frac{2x^T(t)\bar{E}^T(t)\bar{P}(t)B(t)B^T(t)\bar{P}(t)\bar{E}(t)x(t)\phi^2(x(t))}{||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t)) + \rho(t)} \\
&\quad + 2||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t)) \\
&\leq -x^T\bar{Q}(t)x + \frac{2||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t))\rho(t)}{||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t)) + \rho(t)}.
\end{aligned}$$

Thanks to the following inequality:

$$\frac{||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t))\rho(t)}{||B^T(t)\bar{P}(t)\bar{E}(t)x(t)||\phi(x(t)) + \rho(t)} \leq \rho(t).$$

Thus, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) \leq -x^T(t)\bar{Q}(t)x(t) + 2\rho(t).$$

Since $\rho(t) \to 0$ as $t \to +\infty$ then there exists $\bar{\rho} > 0$, such that $||\rho(t)|| \leq \bar{\rho}, \quad \forall t \geq t_0 \geq 0$. Moreover, from inequality (5.3) it yields that,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) \leq -\bar{q}_1 x^T(t)x(t) + 2\bar{\rho}.$$

Next, we deduce from assumption $(\mathcal{A}_{\in})$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x(t)) \leq -\frac{\bar{q}_1}{\bar{p}_2}(\bar{E}(t)x(t))^T\bar{P}(t)(\bar{E}(t)x(t)) + 2\bar{\rho}.$$

Integrating the above inequality from $v \in [t_0,t]$ to $t \geq t_0 \geq 0$, on both sides, we obtain

$$V(t,x(t)) - V(v,x(v)) \leq \int_v^t -\frac{\bar{q}_1}{\bar{p}_2}V(s,x(s)) + 2\bar{\rho}\,\mathrm{d}s.$$

By applying the Gronwall Lemma 4.3, it yields

$$V(t,x(t)) \leq V(t_0,x(t_0))\exp\left(-\frac{\bar{q}_1}{\bar{p}_2}(t-t_0)\right) + \frac{2\bar{\rho}\,\bar{p}_2}{\bar{q}_1}.$$

Taking into consideration (5.4), we may deduce

$$\begin{aligned} ||x(t)||^2 &\leq \frac{1}{\bar{p}_1}(\bar{E}(t)x(t))^T\bar{P}(t)(\bar{E}(t)x(t)) \\ &\leq \frac{1}{\bar{p}_1}V(t_0,x(t_0))\exp\left(-\frac{\bar{q}_1}{\bar{p}_2}(t-t_0)\right) + \frac{2\bar{\rho}\,\bar{p}_2}{\bar{q}_1} \\ &\leq \frac{\bar{p}_2}{\bar{p}_1}||x_0||^2\exp\left(-\frac{\bar{q}_1}{\bar{p}_2}(t-t_0)\right) + \frac{2\bar{\rho}\,\bar{p}_2}{\bar{q}_1}. \end{aligned}$$

Indeed, we have derived that for all $(t_0,x_0) \in \mathcal{W}$,

$$||x(t)|| \leq \sqrt{\frac{\bar{p}_2}{\bar{p}_1}}||x_0||\exp\left(-\frac{\bar{q}_1}{2\bar{p}_2}(t-t_0)\right) + \left(\frac{2\bar{\rho}\bar{p}_2}{\bar{q}_1}\right)^{\frac{1}{2}}.$$

Hence, the uncertain closed-loop singular system (5.1) is uniformly practically exponentially stable. □

6. EXAMPLE

Consider the following singular system:

$$E(t)\dot{x}(t) = A(t)x(t) + E(t)\Pi(t)g(t,x(t)) + E(t)\Pi(t)h(t), \tag{6.1}$$

where $x = (x_1,x_2) \in \mathbb{R}^2$, $E(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A(t) = \begin{pmatrix} -t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}_+$,

$$g(t,x) = \begin{pmatrix} g_1(t,x) \\ g_2(t,x) \end{pmatrix},\ h(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix},$$

with

$$\begin{cases} g_1(t,x) = \dfrac{1}{\mathrm{ch}(t)}\dfrac{x_1^2}{1+\sqrt{x_1^2+x_2^2}} \\ g_2(t,x) = \dfrac{1}{\mathrm{ch}(t)}\dfrac{x_2^2}{1+\sqrt{x_1^2+x_2^2}}, \end{cases}$$

where $\mathrm{ch}(t) := \dfrac{e^t + e^{-t}}{2}$ is the hyperbolic cosine function, and

$$\begin{cases} h_1(t) = te^{-\frac{3}{2}t} \\ h_2(t) = 0. \end{cases}$$

System (6.1) might be regarded as a perturbed singular system of

$$E(t)\dot{x}(t) = A(t)x(t). \tag{6.2}$$

It is clear that (E, A) is transferable into a standard canonical form with $S(\cdot) = T(\cdot) = I_2$. Subsequently, from Proposition 2.1, we obtain $\mathcal{W} = \mathbb{R}_+ \times \mathrm{im}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
Let $t_0 \in \mathbb{R}_+$ and $x_0 \in \mathcal{W}(t_0) = \mathrm{im}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. This means that $x_0 = \begin{pmatrix} \varrho \\ 0 \end{pmatrix}$ for some $\varrho \in \mathbb{R}$.
Since $\mathrm{im}\Pi(t) = \mathcal{W}(t)$, then for instance we may choose $\Pi(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
Consequently, we obtain $E\Pi g = \begin{pmatrix} g_1 \\ 0 \end{pmatrix}$ and $E\Pi h = \begin{pmatrix} h_1 \\ 0 \end{pmatrix}$.
Using Theorem 3.7, it follows that the unperturbed nominal system (6.2) is uniformly exponentially stable, as shown in Figure 1. Indeed, the transition matrix $U(t, t_0)$ associated to (6.2) satisfies:

$$U(t, t_0) = e^{-\frac{1}{2}(t^2 - t_0^2)}.$$

On the other side, we have

$$||g(t, x)||^2 = g_1^2(t, x) + g_2^2(t, x) \leq \frac{1}{ch^2(t)}(x_1^2 + x_2^2).$$

Consequently, we obtain for all $(t, x) \in \mathcal{W}$,

$$||g(t, x)|| \leq \varphi(t)||x||,$$

where $\varphi(t) = \dfrac{1}{\mathrm{ch}(t)}$. It is easy to check that, $\displaystyle\int_0^{+\infty} \varphi(t)\,\mathrm{d}t = \widetilde{\varphi} = \frac{\pi}{2}$. Furthermore, Π is bounded. Hence, all assumptions of Theorem 3.8 are satisfied, and then the exponential stability of the following perturbed singular system:

$$E(t)\dot{x}(t) = A(t)x(t) + E(t)\Pi(t)g(t, x(t)), \tag{6.3}$$

may be deduced as shown in Figure 2.

For $\gamma = \frac{1}{2}$, we have

$$\int_0^{+\infty} ||h(t)||e^{\frac{1}{2}t} = \int_0^{+\infty} te^{-t} = 1 < +\infty.$$

Finally, all assumptions of Theorem 3.9 are fulfilled. Then, the ball of radius $r = \exp\left(\dfrac{\pi + 2}{2}\right)$ with respect to the perturbed singular system (6.1), B_r is uniformly exponentially stable. That is, the perturbed singular system (6.1) is practically uniformly exponentially stable, as we can see in Figure 3.

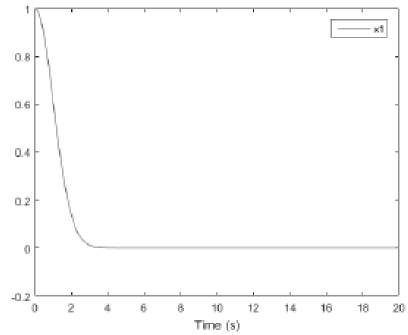

Fig. 1. The initial response of the unperturbed nominal system (6.2), with the initial condition $x_0 = [1,\ 0]^T$.

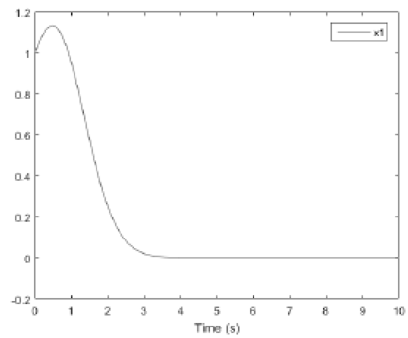

Fig. 2. The initial response of the perturbed singular system (6.3), with the initial condition $x_0 = [1,\ 0]^T$.

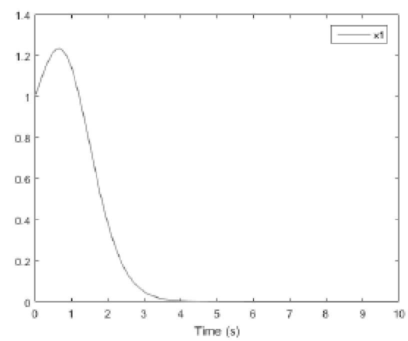

Fig. 3. The initial response of the system (6.1), with the initial condition $x_0 = [1,\ 0]^T$.

7. CONCLUSION

In this paper, we dealt with the analysis problem of stability of some classes of perturbed linear time–varying singular systems, when its associated unperturbed nominal system is transferable into a standard canonical form. Some stability criteria for exponential stability and practical uniform exponential stability have been established. The main technical tools for deriving results are generalized Gronwall inequalities and the Lyapunov direct method. A numerical example is presented to demonstrate the theoretical analysis.

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