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## THE RIBES-ZALESSKII PROPERTY OF SOME ONE RELATOR GROUPS

GILBERT MANTIKA, NARCISSE TEMATE-TANGANG, AND DANIEL TIEUDJO

ABSTRACT. The profinite topology on any abstract group  $G$ , is one such that the fundamental system of neighborhoods of the identity is given by all its subgroups of finite index. We say that a group  $G$  has the Ribes-Zaleskii property of rank  $k$ , or is  $RZ_k$  with  $k$  a natural number, if any product  $H_1H_2\cdots H_k$  of finitely generated subgroups  $H_1, H_2, \dots, H_k$  is closed in the profinite topology on  $G$ . And a group is said to have the Ribes-Zaleskii property or is  $RZ$  if it is  $RZ_k$  for any natural number  $k$ . In this paper we characterize groups which are  $RZ_2$ . Consequently, we obtain condition under which a free product with amalgamation of two  $RZ_2$  groups is  $RZ_2$ . After observing that the Baumslag-Solitar groups  $BS(m, n)$  are  $RZ_2$  and clearly  $RZ$  if  $m = n$ , we establish some suitable properties on the  $RZ_2$  property for the case when  $m = -n$ . Finally, since any group  $BS(m, n)$  can be viewed as a HNN-extension, then we point out the Ribes-Zaleskii property of rank two on some HNN-extensions.

### 1. INTRODUCTION AND RESULTS

Properties of the profinite topology were studied by M. Hall in [10]. A finitely generated subgroup  $H$  of a free group  $F$  is closed in the profinite topology of  $F$  if  $H$  is the intersection of subgroups of finite index that contain  $H$ . This is equivalent to the statement that for any finitely generated subgroup  $H$  of a free group  $F$ , and any element  $g \in F \setminus H$ , there exist a normal subgroup  $N$  of finite index in  $F$  such that  $g \notin HN$ . In connection with the result of Hall, some authors introduced the Ribes-Zaleskii property of rank  $k$  on an abstract group. An abstract group  $G$  satisfies the *Ribes-Zaleskii property of rank  $k$* , or is  $RZ_k$  with  $k$  a natural number, if for any finitely generated subgroups  $H_1, H_2, \dots, H_k$  and any element  $g \in G \setminus H_1H_2\cdots H_k$ , there exist a normal subgroup  $N$  of finite index in  $G$  such that  $g \notin H_1H_2\cdots H_kN$ . A group is said to have the *Ribes-Zaleskii property* or is  $RZ$  if it is  $RZ_k$  for any natural number  $k$ . It is clear that finite groups and finitely generated abelian groups are  $RZ$ . See [6]. Also, a direct product of groups which are  $RZ$  is  $RZ$ . See [7]. Using the link between the profinite topology and finitely

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approximable groups, C. Rosendal characterized countable discrete groups which are RZ. See [25].

$RZ_0$  means residually finite. Conditions under which a group  $G$  is  $RZ_0$  or  $RZ_1$  were established and some examples of groups  $RZ_0$  and  $RZ_1$  were given. See [9, 12, 13, 15]. It is easy to see that for any natural number  $k$ ,  $RZ_{k+1}$  implies  $RZ_k$ . But the inverse is not true. For example  $F_2 \times F_2$  cited by C. Rosendal in [26] is  $RZ_0$  but not  $RZ_1$ , where  $F_2$  is the free group of rank 2.

The original motivation for the study of the property RZ goes back to a problem posed by J. Rhodes on the existence of an algorithm to compute the closure of subset of finite semigroup. See [20]. Recently, M. Doucha and M. Malicki in [8] showed that the  $RZ_2$  and  $RZ_3$  properties form the lower and upper group theoretic bounds for finite approximability of actions on triangle-free graphs and  $K_n$ -free graphs,  $n \geq 3$ .

Other authors have investigated on finding conditions under which the free constructions of groups inherit the  $RZ_k$  property of all the group factors. N.S. Romanovskii [24] has proved that the free product of groups which are  $RZ_1$  is also  $RZ_1$ . Further, T. Coulbois [7] has proved that the free product of RZ groups is also RZ. Also, Ribes and Zalesskii have proved that, when  $\mathcal{C}$  is a variety of finite groups closed under extensions, the free product of groups which are  $RZ_2$  is also  $RZ_2$  relatively to  $\mathcal{C}$ . See [22].

But for a free product with amalgamation  $G = (G_1 * G_2; A = B, \varphi)$  (denoted also  $G = G_1 \underset{A=B}{*} G_2$ ) of groups  $G_1$  and  $G_2$  with amalgamated isomorphic subgroups  $A \leq G_1$  and  $B \leq G_2$ , a similar statement is not always true. Examples of free product with amalgamation of two  $RZ_1$  groups which is not  $RZ_1$  were given in the works of E. Rips [23] and R. Allenby and D. Doniz [1].

Moldavanskii and Uskova [18] proved that under some conditions, free products with amalgamation of two  $RZ_1$  groups is  $RZ_1$ . Specifically, they proved

**Proposition 1.1** ([18, Theorem 3]). *The group  $G = (G_1 * G_2; A = B, \varphi)$  where  $A$  is a normal subgroup of  $G_1$ ,  $B$  is a normal subgroup of  $G_2$  and groups  $A$  and  $B$  satisfy the maximum conditions for subgroups, is  $RZ_1$  if the groups  $G_1$  and  $G_2$  are  $RZ_1$ .*

In this paper we characterize groups which are  $RZ_2$ . We prove

**Theorem 1.1.** *Let  $G$  be a group and let  $U$  be a finitely generated subgroup contained in the center  $Z(G)$  of  $G$ .  $G$  is  $RZ_2$  if and only if the factor group  $G/U^n$  is  $RZ_2$  for any nonzero natural number  $n$ .*

From this result, we obtain a result similar to that of Moldavanskii and Uskova for the property  $RZ_2$  of groups with amalgamation. The case where the free factors in a free product amalgamated by a finite subgroup are RZ was studied by T. Coulbois in his thesis. See [6]. In this paper, we investigate the case where the amalgamated subgroup can be infinite. That is

**Corollary 1.1.** *Let  $G = G_1 \underset{A=B}{*} G_2$  be a free product of groups  $G_1$  and  $G_2$  with amalgamated subgroups  $A \leq G_1$  and  $B \leq G_2$ . If  $A$  and  $B$  are finitely generated*

subgroups contained in the centers  $Z(G_1)$  and  $Z(G_2)$  of  $G_1$  and  $G_2$  respectively, and groups  $G_1$  and  $G_2$  are  $RZ_2$ , then  $G$  is  $RZ_2$ .

It is then easy to see that if  $G_1$  and  $G_2$  are two  $RZ_2$  groups, and  $a$  and  $b$  are elements in  $G_1$  and  $G_2$  respectively with  $a \in Z(A)$  and  $b \in Z(B)$ , then the group  $G = G_1 \underset{a=b}{*} G_2$  is also  $RZ_2$ .

Also, we recall the class of two-generator one-relator groups, called the Baumslag-Solitar groups, given by the presentation  $BS(m, n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$  where  $m$  and  $n$  are nonzero integers. This class of groups deeply studied by G. Baumslag and Solitar [4], were introduced to point out a class of finitely generated non-hopfian groups. Some residual properties of  $BS(m, n)$  were studied [2, 3].

It is easily seen using the results of [21, 27]

**Proposition 1.2.** *For any nonzero integer  $n$ , group  $BS(n, n)$  is  $RZ$ .*

Since for  $|m| = n$  the group  $BS(m, n)$  is  $RZ_0$  and  $RZ_1$  (see [16]), then the case where  $m = -n$  is also for interest. Thus we investigate this case. We obtain

**Theorem 1.2.** *Let  $n$  be a nonzero natural number. If  $H_1$  and  $H_2$  are two finitely generated subgroups of  $BS(n, -n)$  contained in the free factors of  $BS(n, -n)$ , then the product  $H_1 H_2$  is closed in the profinite topology on  $BS(n, -n)$ .*

Also, any Baumslag-Solitar group  $BS(m, n) = \langle a, b \mid a^{-1}b^m a = b^n \rangle$  can be seen as an HNN-extension with associated subgroups  $\langle b^m \rangle$  and  $\langle b^n \rangle$ . So, we also focus on Ribes-Zaleskii's property of rank  $k$  of some HNN-extensions. Let  $K$  be a finitely generated abelian group and let  $A, B$  be finitely generated isomorphic subgroups of  $K$ . Since finitely generated abelian groups are  $RZ$ , it follows immediately that if  $A = B = K$ , then the HNN-extension  $G = \langle K, t \mid t^{-1}At = B \rangle$  is  $RZ$  as a finitely generated abelian group.

But if  $A \neq B$  in the HNN-extension  $G = \langle K, t \mid t^{-1}At = B \rangle$ , then  $G$  is not  $RZ_1$ . See [17, Lemma 1]. Thus,  $G$  is not  $RZ_k$  for any natural number  $k \geq 1$ .

Using the result of G. Baumslag and M. Tretkoff that can be reformulated as

**Proposition 1.3** ([2, Theorem 3.1]). *Let  $A$  be  $RZ_0$  and let  $H, K$  be isomorphic finite subgroups of  $A$ . Then the HNN-extension  $G = \langle A, t \mid t^{-1}Ht = K \rangle$  is  $RZ_0$ .*

It comes that if a group  $K$  is  $RZ$  and particularly  $RZ_0$ ,  $A$  and  $B$  isomorphic finite normal subgroups of  $K$ , then the HNN-extension  $G = \langle K, t \mid t^{-1}At = B \rangle$  is  $RZ_0$ . As in the proof of ([17, Lemma 2]), it can be pointed out a free product of  $RZ$  groups as a finite subgroup of finite index of  $G$ . Now, since any virtually  $RZ$  group is also  $RZ$  (see [7]), we obtain easily

**Proposition 1.4.** *Let  $K$  be  $RZ$ , and let  $A$  and  $B$  be isomorphic finite normal subgroups of  $K$ . Then, the HNN-extension  $G = \langle K, t \mid t^{-1}At = B \rangle$  is  $RZ$ .*

From which we get by adding Theorem 1.1

**Corollary 1.2.** *Let  $K$  be a group and let  $A$  and  $B$  be isomorphic finitely generated subgroups of  $Z(K)$ , the center of  $K$ . Let  $G = \langle K, t \mid t^{-1}At = B, \varphi \rangle$  be an HNN-extension with  $\varphi(a) = t^{-1}at$  for any  $a \in A$ . If  $K$  is  $RZ_2$  and contains a*

finitely generated subgroup of finite index  $U$  in both  $A$  and  $B$  such that  $\varphi(u) = u$  for any  $u \in U$ , then  $G$  is  $RZ_2$ .

## 2. PRELIMINARIES

In this section we collect some notions, basic properties and facts about free products of groups with amalgamation, HNN-extensions and finitely generated groups. For more details see [14].

Let us recall some notions concerned with the construction of a free product  $G = (G_1 * G_2, A = B, \varphi)$  of groups  $G_1$  and  $G_2$  with amalgamated subgroups  $A \leq G_1$  and  $B \leq G_2$  where  $\varphi: A \rightarrow B$  is an isomorphism. The group  $G = (G_1 * G_2, A = B, \varphi)$  can also be written as  $G = G_1 \underset{A=B}{*}^{\varphi} G_2$  or simply as  $G = G_1 \underset{A=B}{*} G_2$  when there is

no confusion. An element  $g$  in  $G$  can be written in a form  $g = g_1 g_2 \cdots g_r$  ( $r \geq 1$ ) where for any  $i = 1, 2, \dots, r$  element  $g_i$  belongs to one of the free factor  $G_1$  or  $G_2$ , and if  $r > 1$  any successive  $g_i$  and  $g_{i+1}$  do not belong to the same factor  $G_1$  or  $G_2$  (nor to the amalgamated subgroups  $A$  and  $B$ ). We say that  $g$  is written in a *reduced form*. In general, an element of the group  $G = G_1 \underset{A=B}{*} G_2$  can have more than one reduced form. But any two reduced forms of an element  $g$  have the same number of components, which we will call the length of the element  $g$  and denote by  $l(g)$ .

About HNN-extensions, let  $G$  be a group and let  $A$  and  $B$  be its subgroups with  $\varphi: A \rightarrow B$  an isomorphism. Let  $\langle t \rangle$  be the infinite cyclic group generated by a new element  $t$ . The HNN-extension  $G^*$  of  $G$  relative to  $A$ ,  $B$  and  $\varphi$  is the factor group  $G * \langle t \rangle / N$ , where  $N$  is the normal closure of the set  $\{t^{-1}at(\varphi(a))^{-1}, a \in A\}$ . The group  $G$  is called the basis of  $G^*$ ,  $t$  is its stable letter, and  $A$  and  $B$  are the associated subgroups. The notation  $G^* = \langle G, t; t^{-1}at = \varphi(a), a \in A \rangle$  is used.

Concerning finitely generated groups, it is not hard to obtain the following results.

**Proposition 2.1.** *Let  $G$  be a group and let  $N$  be a normal subgroup of  $G$ .*

(1) *If  $H$  is a finitely generated subgroup, then the subgroup  $\overline{H} = HN/N$  of  $G/N$  is. Particularly, if  $G$  is a finitely generated, then  $G/N$  is.*

(2) *If  $N$  and  $G/N$  are finitely generated, then the group  $G$  is.*

**Proof.** Consider the canonical epimorphism  $\pi: G \rightarrow G/N$ .

(1) Let  $H$  be subgroup and let  $X$  be its finitely generated subset. Then  $\overline{H} = HN/N = \pi(H) = \pi(\langle X \rangle) = \langle \pi(X) \rangle$ . Thus the subgroup  $HN/N$  is finitely generated.

(2) Since  $G/N$  is finitely generated, there exist elements  $g_1, g_2, \dots, g_r$  in  $G$  such that  $G/N = \langle \overline{g}_1, \overline{g}_2, \dots, \overline{g}_r \rangle$ , where each  $\overline{g}_i$  ( $1 \leq i \leq r$ ) represents the image by  $\pi$  of element  $g_i$  in  $G/N$ . Consider  $g \in G$  such that  $\overline{g} = \overline{g}_1^{s_1} \overline{g}_2^{s_2} \cdots \overline{g}_r^{s_r}$  where the  $s_k$  are integers. Then  $\overline{g} = \overline{g_1^{s_1} g_2^{s_2} \cdots g_r^{s_r}}$ , and there exists  $n \in N$  with  $g = g_1^{s_1} g_2^{s_2} \cdots g_r^{s_r} n$ ; that is  $g \in \langle g_1, g_2, \dots, g_r \rangle N$ . Finally  $G = \langle g_1, g_2, \dots, g_r \rangle N$  is finitely generated since  $N$  is.  $\square$

**Proposition 2.2.** *Any quotient of a  $RZ_2$  group by a finitely generated normal subgroup is also  $RZ_2$ .*

**Proof.** Let  $G$  be a  $\text{RZ}_2$  group and let  $N$  be a finitely generated normal subgroup of the group  $G$ . We shall prove that the factor group  $G/N$  is  $\text{RZ}_2$ .

Consider two finitely generated subgroups  $\overline{H}_1 = H_1/N$  and  $\overline{H}_2 = H_2/N$  of  $G/N$ , where  $H_1$  and  $H_2$  are subgroups containing  $N$ . Let  $g$  be an element of  $G$  such that  $\overline{g} \in G/N$  and  $\overline{g} \notin \overline{H}_1 \overline{H}_2$ . It is clear that  $g \notin H_1 H_2$ . Using Proposition 2.1, it is also clear that subgroups  $H_1$  and  $H_2$  are finitely generated. Therefore, since  $G$  is  $\text{RZ}_2$ , there exists a normal subgroup  $M$  of finite index in  $G$  such that  $g \notin H_1 H_2 M$ . Consequently we have  $\overline{g} \notin \overline{H}_1 \overline{H}_2 \overline{M}$  where  $\overline{M} = MN/N$ . If, on contrary  $\overline{g} \in \overline{H}_1 \overline{H}_2 \overline{M}$ , then  $\overline{g} = \overline{h}_1 \overline{h}_2 \overline{t}$  with  $h_1 \in H_1$ ,  $h_2 \in H_2$  and  $t \in MN$ . And then there exist  $m \in M$  and  $n \in N$  such that  $g = h_1 h_2 m n = h_1 h_2 (m n m^{-1}) m$ . Now, since  $N \triangleleft G$  and  $N \leq H_2$ , it is obvious that  $h = h_2 (m n m^{-1}) \in H_2$ . But this implies that  $g = h_1 h m \in H_1 H_2 M$  which contradicts the fact that  $g \notin H_1 H_2 M$ . So  $\overline{g} \notin \overline{H}_1 \overline{H}_2 \overline{M}$ , with  $\overline{M}$  a normal subgroup of finite index in  $G/N$ . Thus, the factor group  $G/N$  is  $\text{RZ}_2$  as required.  $\square$

**Proposition 2.3.** *Let  $G$  be a group and let  $A$  be a finitely generated subgroup in  $G$ . If  $A$  is contained in  $Z(G)$  the center of  $G$ , then for any nonzero natural number  $t$ , the subset  $A^t = \{a^t, a \in A\}$  of  $G$  is a normal subgroup of finite index in  $A$ .*

**Proof.** Assume that the subgroup  $A$  is contained in  $Z(G)$ . Then  $A$  is a finitely generated abelian group. Therefore  $A$  is equal to a direct sum  $\bigoplus_{i \leq l} A_i$ , where each  $A_i$  is cyclic. For  $i \leq l$ , let  $a_i$  be a generator of  $A_i$ . So,

$$A = \langle a_1, a_2, \dots, a_l \rangle$$

is generated by the elements  $a_1, a_2, \dots, a_l$ . Let  $t$  be a nonzero natural number.

On one hand, since  $Z(G)$  is commutative, it is obvious that  $A^t = \{a^t, a \in A\}$  is a normal subgroup of  $A$ .

On the other hand the factor group

$$A/A^t = \langle \overline{a}_1, \overline{a}_2, \dots, \overline{a}_l \mid \overline{a}_1^t = 1, \overline{a}_2^t = 1, \dots, \overline{a}_l^t = 1 \rangle$$

is finitely generated where  $\overline{a}_i = a_i A^t$  for any  $i \in \{1, 2, \dots, l\}$ . Also, the group  $A/A^t$  is commutative, so it can be written as  $\overline{A}_t = \langle \overline{a}_1 \mid \overline{a}_1^t = 1 \rangle \times \langle \overline{a}_2 \mid \overline{a}_2^t = 1 \rangle \times \dots \times \langle \overline{a}_l \mid \overline{a}_l^t = 1 \rangle$ . Finally, since the order of each group  $\langle \overline{a}_i \mid \overline{a}_i^t = 1 \rangle$ ,  $i \in \{1, 2, \dots, l\}$  is at most  $t$ , it follows that the order of  $A/A^t$  is finite.  $\square$

### 3. PROOF OF THEOREM 1.1 AND COROLLARY 1.1

**Proof of Theorem 1.1.** Since the subgroup  $U \leq Z(G)$  is finitely generated, it comes that for any nonzero natural number  $t$ , the subgroup  $U^t \leq G$  is normal and finitely generated. Thus, if  $G$  is  $\text{RZ}_2$ , then using Proposition 2.2 the factor group  $G/U^t (t \geq 1)$  is.

Conversely, suppose that any factor group  $G/U^t (t \geq 1)$  is  $\text{RZ}_2$ . Let prove that  $G$  is  $\text{RZ}_2$ . To do it, let  $H_1$  and  $H_2$  be two finitely generated subgroups of  $G$ , and let  $g$  be an element in  $G$  such that  $g \notin H_1 H_2$ .

We need to determine a normal subgroup  $N$  of finite index in  $G$  ( $N \triangleleft_f G$ ) such

that  $g \notin H_1H_2N$ . Consider for any nonzero natural number  $t$ , the factor group  $G/U^t$  and the canonical epimorphism

$$\vartheta_t: G \longrightarrow G/U^t.$$

**Case 1.** Assume that there exist a nonzero natural number  $t_0$  such that  $\vartheta_{t_0}(g) \notin \vartheta_{t_0}(H_1)\vartheta_{t_0}(H_2)$  in  $G/U^{t_0}$ . Since  $H_1$  and  $H_2$  are finitely generated, it follows using Proposition 2.1 that  $\vartheta_{t_0}(H_1)$  and  $\vartheta_{t_0}(H_2)$  are finitely generated. Now the group  $G/U^{t_0}$  is  $\text{RZ}_2$ . Therefore there exists  $\overline{N} \triangleleft_f G/U^{t_0}$  such that  $\vartheta_{t_0}(g) \notin \vartheta_{t_0}(H_1)\vartheta_{t_0}(H_2)\overline{N}$ . Let  $N$  be the preimage of  $\overline{N}$  by  $\vartheta_{t_0}$ . Clearly,  $g \notin H_1H_2N$ . Thus  $G$  is  $\text{RZ}_2$ .

**Case 2.** Assume now that for any nonzero natural number  $t$  we have  $\vartheta_t(g) \in \vartheta_t(H_1)\vartheta_t(H_2)$  in  $G/U^t$ . We need to prove that this case is not possible.

For  $t = 1$ ,  $\vartheta_1(g) = \vartheta_1(a)\vartheta_1(b)$  with  $a \in H_1$  and  $b \in H_2$ . That is  $gU = abU$  and then  $g = abu$  with  $u \in U$ . Let  $y = ab$ . Then, we have  $g = yu$ .

For any  $t \geq 2$ ,  $\vartheta_t(g) = \vartheta_t(a_t)\vartheta_t(b_t)$ , where  $a_t \in H_1$  and  $b_t \in H_2$ ; that is  $g = a_t b_t u_t$  with the elements  $a_t, b_t$  and  $u_t$  fixed respectively in  $H_1, H_2$  and  $U^t$ . Therefore for any  $t \geq 2$  we have  $g = a_t a^{-1} a b b^{-1} b_t u_t = h_t y k_t u_t$ , where  $h_t = a_t a^{-1} \in H_1$  and  $k_t = b^{-1} b_t \in H_2$ . Thus,

$$(3.1) \quad u = y^{-1} h_t y k_t u_t.$$

Set  $S = \langle \{y^{-1} h_t y k_t \mid h_t \in H_1, k_t \in H_2, t \geq 2\} \rangle$  be the subgroup generated by the elements of the form  $y^{-1} h_t y k_t$ , with  $h_t \in H_1$  and  $k_t \in H_2$ , ( $t \geq 2$ ). Since  $y^{-1} h_t y k_t = uu_t^{-1} \in U$ , then  $S$  is a subgroup of  $U$ . Also, for  $s = y^{-1} h_t y k_t$  ( $t \geq 2$ ), we have  $s^{-1} = k_t^{-1} y^{-1} h_t^{-1} y \in S$ ; and it follows that  $k_t s^{-1} = y^{-1} h_t^{-1} y$ . From  $U \leq Z(G)$  and  $s^{-1} \in U$ , we obtain  $k_t s^{-1} = s^{-1} k_t = y^{-1} h_t^{-1} y$ . The equality  $s^{-1} = y^{-1} h_t^{-1} y k_t^{-1}$  then arises. Finally,  $y^{-1} h_t^{\epsilon_t} y k_t^{\epsilon_t} \in S$  with  $\epsilon_t = \pm 1$ . Thus:

$$\begin{aligned} (y^{-1} h_{t_1}^{\epsilon_{t_1}} y k_{t_1}^{\epsilon_{t_1}})(y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}}) &= (y^{-1} h_{t_1}^{\epsilon_{t_1}} y)(k_{t_1}^{\epsilon_{t_1}} \times y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}}) \\ &= (y^{-1} h_{t_1}^{\epsilon_{t_1}} y)(y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} \times k_{t_1}^{\epsilon_{t_1}}), \quad \text{since} \\ y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} \in Z(G) &= y^{-1} h_{t_1}^{\epsilon_{t_1}} y y^{-1} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} k_{t_1}^{\epsilon_{t_1}} \\ &= y^{-1} h_{t_1}^{\epsilon_{t_1}} h_{t_2}^{\epsilon_{t_2}} y k_{t_2}^{\epsilon_{t_2}} k_{t_1}^{\epsilon_{t_1}}, \quad \epsilon_{t_i} = \pm 1. \end{aligned}$$

It comes then that the elements of  $S$  have the form:

$$(3.2) \quad y^{-1} h_{t_1}^{\epsilon_{t_1}} \dots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \dots k_{t_1}^{\epsilon_{t_1}}, \quad \epsilon_{t_i} = \pm 1, \quad i = 1, \dots, n.$$

*Subcase (a)* Suppose that  $u$  belongs to subgroup  $S$ . So, from (3.2), we have  $u = y^{-1} h_{t_1}^{\epsilon_{t_1}} \dots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \dots k_{t_1}^{\epsilon_{t_1}}$ ; that is  $yu = h_{t_1}^{\epsilon_{t_1}} \dots h_{t_n}^{\epsilon_{t_n}} y k_{t_n}^{\epsilon_{t_n}} \dots k_{t_1}^{\epsilon_{t_1}} = h_{t_1}^{\epsilon_{t_1}} \dots h_{t_n}^{\epsilon_{t_n}} a b k_{t_n}^{\epsilon_{t_n}} \dots k_{t_1}^{\epsilon_{t_1}}$ .

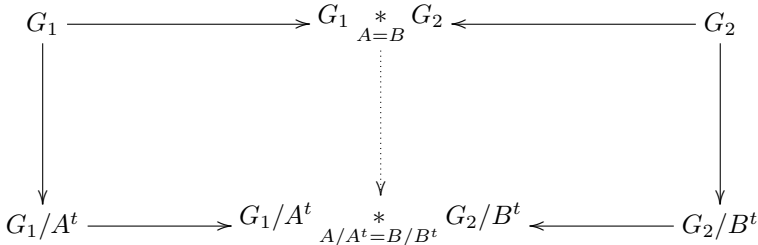
Then, since  $h_{t_1}^{\epsilon_{t_1}} \dots h_{t_n}^{\epsilon_{t_n}} a \in H_1$  and  $b k_{t_n}^{\epsilon_{t_n}} \dots k_{t_1}^{\epsilon_{t_1}} \in H_2$ , it follows that  $g = yu \in H_1H_2$ , and this result contradicts the assertion  $g \notin H_1H_2$ .

*Subcase (b)* Now  $u \notin S$ . On one hand, since the group  $U$  is commutative and finitely generated, it possesses the maximal property for groups, that is, each of its subgroups is finitely generated. Thus,  $S$  is finitely generated. On the other hand,  $U$  as a commutative and finitely generated group is  $\text{RZ}_1$ . Therefore,  $U$  possesses a normal subgroup  $M$  of finite index such that  $u \notin SM$ .

Also, since  $M \triangleleft_f U$ , all the elements of the factor group  $U/M$  have finite order. Let  $U_0$  be the finite set of representative classes modulo  $M$  in  $U$ . For any  $g \in U_0$ , there exists a natural number  $r_g$  such that  $g^{r_g} \in M$ . Also, for any  $g \in U$ , there exist  $g_0 \in U_0$  such that  $gg_0^{-1} \in M$ . Thus,  $(gg_0^{-1})^{r_{g_0}} = g^{r_{g_0}}(g_0^{r_{g_0}})^{-1}$  belongs to  $M$ , and it follows that  $g^{r_{g_0}}$  also belongs to  $M$ . Let  $t'$  be the least common multiple of the  $r_g$ , with  $g \in U_0$ . We have  $g^{t'} \in M$  for any  $g \in U$ , and then  $U^{t'} \subseteq M$ . If  $t' = 1$ , then any  $g \in U$  belongs to  $M$ . Particularly,  $u \in M$ , and it contradicts the fact that  $u \notin M$  since  $u \notin SM$ . So  $t' \geq 2$ , and  $u = y^{-1}h_{t'}yk_{t'}u_{t'}$ . Now,  $y^{-1}h_{t'}yk_{t'} \in S$  and  $u_{t'} \in U^{t'} \subseteq M$ , thus  $u \in SM$ , which is again not possible. Finally, **Case 2** is not possible as required, and we get only **Case 1**. Thus, the group  $G$  is  $RZ_2$ , and the theorem is completely demonstrated.  $\square$

We are now ready to prove Corollary 1.1.

**Proof of Corollary 1.1.** Suppose that all the assumptions of the corollary are satisfied. Since  $A = B$  coincides with the center of the amalgamated group  $G$  (see [14, Corollary 4.5]), to prove that  $G$  is  $RZ_2$ , we prove that  $G/A^t$  is  $RZ_2$  for any nonzero natural number  $t$  and conclude using Theorem 1.1. To do it, let  $t$  be a nonzero natural number and let  $\overline{H}_1$  and  $\overline{H}_2$  be two finitely generated subgroups of  $G/A^t$ . Let  $\overline{g}$  be an element of  $G/A^t$  such that  $\overline{g} \notin \overline{H}_1 \overline{H}_2$ . We need to determine  $\overline{N} \triangleleft_f G/A^t$  such that  $\overline{g} \notin \overline{H}_1 \overline{H}_2 \overline{N}$ . We recall by Proposition 2.3 that the subgroups  $A^t$  and  $B^t$  are normal with finite index in  $A$  and  $B$  in respectively. Since  $A/A^t$  and  $B/B^t$  are finite and isomorphic, the canonical homomorphisms  $G_1 \rightarrow G_1/A^t$  and  $G_2 \rightarrow G_2/B^t$  can be extended to the epimorphism  $G \rightarrow G_1/A^t \underset{A/A^t=B/B^t}{*} G_2/B^t$  with kernel  $A^t = B^t$ . See ([19, Theorem 1.1]). This situation can be illustrated by the following diagram



Let  $G(t) = G_1/A^t \underset{A/A^t=B/B^t}{*} G_2/B^t$ . It is clear that the groups  $G/A^t$  and  $G(t)$  are isomorphic.

Now, using the fact that the subgroups  $A^t$  and  $B^t$  have finite index respectively in  $A$  and  $B$  which are finitely generated, it follows by ([6, Proposition 1.1]) that  $A^t$  and  $B^t$  are finitely generated. Thus, by Proposition 2.2 the groups  $G_1/A^t$  and  $G_2/B^t$  are  $RZ_2$ . Also, the groups  $A/A^t$  and  $B/B^t$  are finite; thus, the group  $G(t)$  is  $RZ_2$  (see [6, Theorem 5.2]), and  $G/A^t$  is. Since  $G/A^t$  is  $RZ_2$  for any arbitrary nonzero natural number  $t$ , we conclude by Theorem 1.1 that  $G$  is  $RZ_2$ . Hence Corollary 1.1 is demonstrated.  $\square$



## 4. PROOF OF THEOREM 1.2

We recall a result of P. Stebe which will be used in some statement of the proof of the Theorem 1.2. It states that for any element  $h$  of a free group  $F$  and for any nonzero integer  $n$ , there exists a normal subgroup  $N$  of finite index in  $F$  such that  $N \cap \langle h \rangle = \langle h^n \rangle$  (see [28]). We establish

**Lemma 4.1.** *Let  $n$  be a nonzero natural number. For any finitely generated subgroups  $H_1$  and  $H_2$  of  $BS(n, -n) = \langle a, b \mid a^{-1}b^na = b^{-n} \rangle$ , and any normal subgroup  $U$  of finite index in  $\langle b^n \rangle$  such that  $(\langle b^n \rangle \cap H_1)U \neq \langle b^n \rangle$  and  $(\langle b^n \rangle \cap H_2)U \neq \langle b^n \rangle$ , there exists a normal subgroup  $N$  of finite index in  $BS(n, -n)$  satisfying  $N \cap \langle b^n \rangle = U$ ,  $(N \langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$  and  $(N \langle b^n \rangle) \cap NH_2 = N(\langle b^n \rangle \cap H_2)$ .*

**Proof.** Let  $H_1$  and  $H_2$  be two finitely generated subgroups of  $BS(n, -n)$ , and let  $U$  be a normal subgroup of finite index  $t$  in  $\langle b^n \rangle$  satisfying all the assumptions in the lemma. Consider  $c_1, \dots, c_t$  a system of left cosets representatives of  $U$  in  $\langle b^n \rangle$  where  $c_1 = 1$ .

Since  $BS(n, -n)$  is  $RZ_1$  and  $U$  is finitely generated as a finite index subgroup of the finitely generated group  $\langle b^n \rangle$ , there exists  $N_1 \triangleleft_f BS(n, -n)$  such that  $c_i \notin N_1U$  for any  $i = 2, \dots, t$ . Also, there exists  $i \in \{2, 3, \dots, t\}$  such that  $c_i \notin H_1U$ . Indeed: assume in contrary that for any  $i \in \{2, 3, \dots, t\}$   $c_i \in H_1U$ ; that is  $c_i = h_i u_i$  with  $h_i \in H_1$  and  $u_i \in U$ . Therefore  $h_i = c_i u_i^{-1} \in H_1 \cap \langle b^n \rangle$  for any  $i \in \{2, 3, \dots, t\}$ . Thus,  $c_i$  belongs to the subgroup  $(H_1 \cap \langle b^n \rangle)U$  of  $\langle b^n \rangle$  for any  $i \in \{1, 2, \dots, t\}$ . Consequently, it follows that  $(H_1 \cap \langle b^n \rangle)U = \langle b^n \rangle$  and this contradicts the hypothesis  $\langle b^n \rangle \neq (H_1 \cap \langle b^n \rangle)U$ . So, there exists  $i \in \{2, 3, \dots, t\}$  such that  $c_i \notin H_1U$ . Similarly, there exists  $j \in \{2, 3, \dots, t\}$  such that  $c_j \notin H_2U$ .

It is easy to see that the groups  $H_1U$  and  $H_2U$  are finitely generated in  $BS(n, -n)$  and so, again using the fact that  $BS(n, -n)$  is  $RZ_1$ , there exist normal subgroups  $N_{2i}$  and  $N_{3j}$  of finite index in  $BS(n, -n)$  such that  $c_i \notin N_{2i}H_1U$  and  $c_j \notin N_{3j}H_2U$ . Set  $I = \{i \in \{2, \dots, t\}, c_i \notin H_1U\}$  and  $J = \{i \in \{2, \dots, t\}, c_i \notin H_2U\}$ . Thus,  $N_2 = \bigcap_{i \in I} N_{2i}$  and  $N_3 = \bigcap_{i \in J} N_{3i}$  are normal subgroups of finite index in  $BS(n, -n)$  as finite intersections of normal subgroups of finite index in  $BS(n, -n)$ . Therefore,  $c_i \notin N_2H_1U$  for any  $i \in I$  and  $c_j \notin N_3H_2U$  for any  $j \in J$ . Let:

$$N = N_1U \cap N_2U \cap N_3U.$$

For any  $l \in \{1, 2, 3\}$ ,  $N_l$  is a normal subgroup of finite index in  $BS(n, -n)$ , and  $N_lU$  is. Consequently,  $N$  is also a normal subgroup of finite index in  $BS(n, -n)$ .

It is obvious that  $U \subseteq N \cap \langle b^n \rangle$ . Conversely, let  $g \in N \cap \langle b^n \rangle$ . There exist  $n_1 \in N_1$  and  $u \in U$  such that  $g = n_1u$ . If  $g \notin U$ , then there exist  $i \in \{2, 3, \dots, t\}$  and  $c_i$  in  $\langle b^n \rangle$  such that  $gU = c_iU$ . Thus  $c_i \in gU = n_1uU = n_1U$ , and this implies that  $c_i \in N_1U$ , but it contradicts the assumption that  $c_i \notin N_1U$  for any  $i \in \{2, 3, \dots, t\}$ . So  $g \in U$  and  $U = N \cap \langle b^n \rangle$ .

Let us now prove that  $(N \langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$ . On one hand, it is easy to see that  $(N \langle b^n \rangle) \cap NH_1 \supseteq N(\langle b^n \rangle \cap H_1)$ . On the other hand, let  $g \in (N \langle b^n \rangle) \cap NH_1$ .

Then  $g = kb_1 = k'h_1$ , where  $k, k' \in N$ ,  $b_1 \in \langle b^n \rangle$  and  $h_1 \in H_1$ . Since  $\langle b^n \rangle = \bigcup_{i=1}^t c_iU$

( $c_i \in \langle b^n \rangle$ ), there exist  $j \in \{1, 2, \dots, t\}$  and  $u \in U$  such that  $b_1 = c_j u$ . Thus  $c_j = k^{-1} k' h_1 u^{-1} \in NH_1 U$ . Since  $U \subseteq H_1 U$  implies  $UH_1 U = H_1 U$ , we have  $NH_1 U \subseteq N_2 UH_1 U \subseteq N_2 H_1 U$ . Recalling that  $c_i \notin N_2 H_1 U$  for any  $c_i \notin H_1 U$ , we obtain  $c_j \in H_1 U$  since  $c_j \in N_2 H_1 U$ . Therefore, there exist  $h'_1 \in H_1$  and  $u' \in U$  satisfying  $c_j = h'_1 u'$ . From  $U \leq \langle b^n \rangle$ , we have  $h'_1 = c_j u'^{-1} \in \langle b^n \rangle$ . Consequently,  $h'_1 \in \langle b^n \rangle \cap H_1$  and then

$$g = kb_1 = kc_j u = kh'_1 u' u = k(h'_1 u' u h_1'^{-1}) h_1'.$$

Furthermore  $U \leq N$  and  $N \triangleleft BS(n, -n)$ , so that  $h'_1 u' u h_1'^{-1} \in N$ . Therefore  $kh'_1 u' u h_1'^{-1} \in N$  and then  $g \in N(\langle b^n \rangle \cap H_1)$ . Thus,  $(N \langle b^n \rangle) \cap NH_1 \subseteq N(\langle b^n \rangle \cap H_1)$  and we get the equality  $(N \langle b^n \rangle) \cap NH_1 = N(\langle b^n \rangle \cap H_1)$ .

We prove similarly that  $(N \langle b^n \rangle) \cap NH_2 = N(\langle b^n \rangle \cap H_2)$ . Hence, the lemma is proven.  $\square$

**Proof of Theorem 1.2.** Let us recall that in the group  $BS(n, -n) = \langle b \rangle_{b^n=c}^*$

$BS(1, -1)$ , the subgroups  $\langle b \rangle$  and  $BS(1, -1) = \langle a, c \mid a^{-1}ca = c^{-1} \rangle$  are the free factors. Let  $H_1$  and  $H_2$  be two finitely generated subgroups of  $BS(n, -n)$  contained in the free factors, and let  $g \in BS(n, -n) \setminus H_1 H_2$ . In order to prove that the product  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ , we need to determine a normal subgroup  $N$  of finite index in  $BS(n, -n)$  such that  $g \notin H_1 H_2 N$ .

**Case 1.** Assume that  $H_1$  and  $H_2$  are subgroups of  $\langle b \rangle$ .

Since the group  $\langle b \rangle$  is commutative, it comes that  $H_1 H_2$  is an infinite cyclic group. Also,  $BS(n, -n)$  is  $RZ_1$  and  $g \in BS(n, -n) \setminus H_1 H_2$ . Thus, there exists  $M \triangleleft_f BS(n, -n)$  such that  $g \notin H_1 H_2 M$ . That is, the set  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ .

**Case 2.** Next, consider that  $H_1$  and  $H_2$  are subgroups of  $BS(1, -1)$ .

*Subcase (a)* Suppose that  $g \in BS(1, -1)$ . Since the group  $BS(1, -1)$  is polycyclic, it is  $RZ_2$ . Thus, there exists a subgroup  $M \triangleleft_f BS(1, -1)$  such that  $g \notin H_1 H_2 M$ . Let the factor groups  $\overline{H_1} = H_1/H_1 \cap M$ ,  $\overline{H_2} = H_2/H_2 \cap M$  and  $\overline{BS(1, -1)} = BS(1, -1)/M$  be considered modulo  $M$ . By Proposition 2.1 (1),  $\overline{H_1}$  and  $\overline{H_2}$  are finitely generated subgroups of  $\overline{BS(1, -1)}$ . Let  $\overline{g}$  be the class of  $g$  modulo  $M$  in  $\overline{BS(1, -1)}$ ; then  $\overline{g} \notin \overline{H_1} \overline{H_2}$  in  $\overline{BS(1, -1)}$ . Also, since  $M \cap \langle c \rangle$  is generated by one element as a subgroup of a one generated group, there exists a natural number  $t$  such that  $M \cap \langle c \rangle = \langle c^t \rangle = \langle b^{nt} \rangle$ . Therefore, by the result of P. Stebe cited previously, there exists  $L \triangleleft_f \langle b \rangle$  satisfying  $L \cap \langle b^n \rangle = \langle b^{nt} \rangle = M \cap \langle c \rangle$ .

Set  $\overline{\langle b^n \rangle} = \langle b^n \rangle / (L \cap \langle b^n \rangle)$  and  $\overline{\langle c \rangle} = \langle c \rangle / (M \cap \langle c \rangle)$  respectively subgroups of  $\overline{\langle b \rangle} = \langle b \rangle / L$  and  $\overline{BS(1, -1)}$ . Clearly, the canonical epimorphisms  $\langle b \rangle \longrightarrow \overline{\langle b \rangle}$  and  $BS(1, -1) \longrightarrow \overline{BS(1, -1)}$  induce an epimorphism  $\pi : BS(n, -n) \longrightarrow \overline{BS(n, -n)} = \overline{\langle b \rangle}_{\overline{b^n=c}}^* \overline{BS(1, -1)}$ . Since the groups  $\overline{\langle b \rangle}$  and  $\overline{BS(1, -1)}$  are finite, it comes that

the group  $\overline{BS(n, -n)}$  is a free product of finite groups amalgamated by finite subgroups. Now, using the fact that Since  $\overline{\langle b \rangle}$  and  $\overline{BS(1, -1)}$  are finite, they are  $RZ_2$ . Thus  $\overline{BS(n, -n)}$  is  $RZ_2$  as a free product of  $RZ_2$  groups amalgamated by finite subgroups. See [6, Theorem 5.3]. Also in  $\overline{BS(n, -n)}$ , we have  $\overline{g} \notin \overline{H_1} \overline{H_2}$ . Consequently, there exists a normal subgroup  $\overline{N}$  of finite index in  $\overline{BS(n, -n)}$  such

that  $\bar{g} \notin \overline{H_1 H_2 N}$ . Taking  $N$  to be the preimage of  $\bar{N}$  via  $\pi$ , we have  $g \notin H_1 H_2 N$  as desired. Again the set  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ .

*Subcase (b)* Suppose that  $g \notin BS(1, -1)$ . Let  $g = g_1 g_2 \cdots g_r$  ( $r \geq 1$ ) be a reduced form of  $g$  in the amalgamated free product of groups  $BS(n, -n) = \langle b \rangle \underset{b^n=c}{*} BS(1, -1)$ .

Suppose that  $r = 1$ . That is  $g \in \langle b \rangle \setminus \langle b^n \rangle$ , since  $g \notin BS(1, -1)$ . Recall once again that  $\overline{BS(n, -n)}$  is  $RZ_1$ . Then there exists  $M \triangleleft_f BS(n, -n)$  such that  $g \notin \langle b^n \rangle M$ , and the factor group  $\overline{BS(n, -n)}/M$  is finite. Set  $\overline{BS(n, -n)} = BS(n, -n)/M$ ,  $\overline{\langle b \rangle} = \langle b \rangle / (\langle b \rangle \cap M)$ ,  $\overline{BS(1, -1)} = BS(1, -1) / (BS(1, -1) \cap M)$ ,  $\overline{\langle b^n \rangle} = \langle b^n \rangle / (\langle b^n \rangle \cap M)$  and  $\overline{\langle c \rangle} = \langle c \rangle / (\langle c \rangle \cap M)$ . Let  $\bar{g}$  be the class of  $g$  modulo  $M$ . It is clear that in  $\overline{BS(n, -n)}$  we have  $\bar{g} \notin \overline{H_1 H_2}$ , where  $\overline{H_1} = H_1 / H_1 \cap M$ ,  $\overline{H_2} = H_2 / H_2 \cap M$ . Since  $\overline{BS(n, -n)}$  is finite, it is trivially  $RZ_2$ . Thus, there exists a normal subgroup  $\bar{N}$  which is also trivial of finite index in  $\overline{BS(n, -n)}$  and such that  $\bar{g} \notin \overline{H_1 H_2 \bar{N}}$ . Taking  $N = M$  to be the preimage of  $\bar{N}$  via  $\pi$ , we have  $g \notin H_1 H_2 N$  as desired. Therefore,  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ .

Suppose that  $r > 1$ . Let  $I$  and  $J$  be the subsets of  $\{1, 2, \dots, r\}$  consisting of indices of components of  $g$  which belong to  $\langle b \rangle \setminus \langle b^n \rangle$  and  $BS(1, -1) \setminus \langle c \rangle$  respectively. Since  $BS(n, -n)$  is  $RZ_1$ , there exists a subgroup  $M \triangleleft_f BS(n, -n)$  such that  $g_i \notin \langle b^n \rangle M$  and  $g_j \notin \langle c \rangle M$  for any  $i \in I$  and any  $j \in J$ . Considering  $\overline{BS(n, -n)} = BS(n, -n)/M$ ,  $\overline{\langle b \rangle} = \langle b \rangle / (\langle b \rangle \cap M)$ ,  $\overline{BS(1, -1)} = BS(1, -1) / (BS(1, -1) \cap M)$ ,  $\overline{\langle b^n \rangle} = \langle b^n \rangle / (\langle b^n \rangle \cap M)$  and  $\overline{\langle c \rangle} = \langle c \rangle / (\langle c \rangle \cap M)$ , we have  $\bar{g} \notin \overline{BS(1, -1)}$  and  $\bar{g} \notin \overline{H_1 H_2}$ .

Using again the fact that  $\overline{BS(n, -n)}$  is finite, and then trivially  $RZ_2$ , we obtain that there exists a normal subgroup  $\bar{N}$  the trivial subgroup of finite index in  $\overline{BS(n, -n)}$  such that  $\bar{g} \notin \overline{H_1 H_2 \bar{N}}$ . Thus, as in the previous case the desired result is obtained.

**Case 3.** Finally, suppose that  $H_1 \leq \langle b \rangle$  and  $H_2 \leq BS(1, -1)$ . Let us recall that  $g = g_1 g_2 \cdots g_r$  ( $r \geq 1$ ) is a reduced form of  $g$  in  $BS(n, -n) = \langle b \rangle \underset{b^n=c}{*} BS(1, -1)$ .

*Subcase (a)* Suppose that  $l(g) = 0$ . That is  $g \in \langle b^n \rangle = \langle c \rangle$ . It is obvious that  $g \notin (\langle b^n \rangle \cap H_1)(\langle c \rangle \cap H_2)$  since  $g \notin H_1 H_2$ . Also,  $(\langle b^n \rangle \cap H_1)(\langle c \rangle \cap H_2)$  can be viewed as a finitely generated subgroup of  $\langle c \rangle$ , and  $\langle c \rangle$  is  $RZ_1$ . Therefore, there exists  $U \triangleleft_f \langle c \rangle$  such that  $g \notin (\langle b^n \rangle \cap H_1)(\langle c \rangle \cap H_2)U$ ; and it comes that  $(\langle b^n \rangle \cap H_1)U \neq \langle b^n \rangle$  and  $(\langle b^n \rangle \cap H_2)U \neq \langle b^n \rangle$ . Thus, by Lemma 4.1, there exists a subgroup  $M \triangleleft_f BS(n, -n)$  verifying  $M \cap \langle c \rangle = U$ ,  $(M \langle b^n \rangle) \cap (MH_1) = M(\langle b^n \rangle \cap H_1)$  and  $(M \langle c \rangle) \cap (MH_2) = M(\langle c \rangle \cap H_2)$ . Define the factor group  $\overline{BS(n, -n)} = BS(n, -n)/M$ , where  $\overline{\langle b \rangle} = \langle b \rangle / (M \cap \langle b \rangle)$ ,  $\overline{BS(1, -1)} = BS(1, -1) / (M \cap BS(1, -1))$ ,  $\overline{\langle b^n \rangle} = \langle b^n \rangle / (M \cap \langle b^n \rangle)$  and  $\overline{\langle c \rangle} = \langle c \rangle / (M \cap \langle c \rangle)$ .

Since,

$$\begin{aligned} (MH_1/M) \cap (M \langle b^n \rangle / M) &= \{gM \mid g \in MH_1 \text{ and } g \in M \langle b^n \rangle\} \\ &= \{gM \mid g \in MH_1 \cap M \langle b^n \rangle\} \\ &= (MH_1 \cap M \langle b^n \rangle) / M \\ &= M(H_1 \cap \langle b^n \rangle) / M, \end{aligned}$$

we have  $\overline{H_1} \cap \overline{\langle b^n \rangle} = \overline{H_1 \cap \langle b^n \rangle}$  with  $\overline{H_1} = H_1 / (M \cap H_1 = MH_1 / M)$ . Similarly, we obtain also  $\overline{H_2} \cap \overline{\langle c \rangle} = \overline{H_2 \cap \langle c \rangle}$ , with  $\overline{H_2} = H_2 / (M \cap H_2 = MH_2 / M)$ .

We claim that  $\overline{g} \notin \overline{H_1} \overline{H_2}$ . Indeed: if  $\overline{g} \in \overline{H_1} \overline{H_2}$ , then  $\overline{g} = \overline{h_1} \overline{h_2}$  with  $\overline{h_1} \in \overline{H_1}$  and  $\overline{h_2} \in \overline{H_2}$ . Since  $g \in \langle c \rangle$ ,  $H_1 \leq \langle b \rangle$  and  $H_2 \leq BS(1, -1)$ , then  $\overline{h_1} = \overline{gh_2^{-1}} \in \overline{BS(1, -1)}$ . Consequently,  $\overline{h_1} \in \overline{H_1} \cap \overline{BS(1, -1)} \subseteq \overline{\langle b \rangle} \cap \overline{BS(1, -1)} = \overline{\langle b^n \rangle}$ . Thus  $\overline{h_1} \in \overline{H_1} \cap \overline{\langle b^n \rangle}$ . Similarly,  $\overline{h_2} \in \overline{H_2} \cap \overline{\langle c \rangle}$ , so that  $\overline{g} \in (\overline{H_1} \cap \overline{\langle b^n \rangle})(\overline{H_2} \cap \overline{\langle c \rangle}) = \overline{(H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)}$ . Thus  $g = h_1 h_2 m \in (H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)M$ , where  $m \in M$ , and it follows that  $m = h_2^{-1} h_1^{-1} g \in \langle c \rangle$ . Therefore,  $m \in M \cap \langle c \rangle = U$  so that  $g \in (H_1 \cap \langle b^n \rangle)(H_2 \cap \langle c \rangle)U$ . But this contradicts the assumption that  $g \notin (\langle b^n \rangle \cap H_1)(\langle c \rangle \cap H_2)U$ . Thus  $\overline{g} \notin \overline{H_1} \overline{H_2}$  in  $\overline{BS(n, -n)}$ . Since  $\overline{BS(n, -n)}$  is  $RZ_2$  as a finite group, there exists a subgroup  $\overline{N} \triangleleft_f \overline{BS(n, -n)}$  such that  $\overline{g} \notin \overline{H_1} \overline{H_2} \overline{N}$ . And like in the previous cases, it comes that there exists a subgroup  $N \triangleleft_f BS(n, -n)$  satisfying  $g \notin H_1 H_2 N$ . And the set  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$  as desired.

*Subcase (b)* Suppose that  $l(g) = 1$ . That is  $g \in BS(1, -1) \setminus \langle c \rangle$  (or  $g \in \langle b \rangle \setminus \langle b^n \rangle$ ).

► Suppose in addition that  $g \notin \langle c \rangle H_2$ . Since  $\langle c \rangle H_2$  is a finitely generated subgroup of  $BS(1, -1)$  which is  $RZ_2$  as a polycyclic group, there exists a subgroup  $M \triangleleft_f BS(1, -1)$  such that  $g \notin \langle c \rangle H_2 M$ . Thus  $\overline{g} \notin \overline{\langle c \rangle H_2}$  in  $\overline{BS(1, -1)} = \overline{BS(1, -1) / M}$ , where  $\overline{\langle c \rangle} = \langle c \rangle / (\langle c \rangle \cap M)$  and  $\overline{H_2} = H_2 / H_2 \cap M$ . Since  $M \cap \langle c \rangle$  can be viewed as a subgroup of  $\langle b \rangle$ , then using the P. Stebe's result cited previously, there exists  $L \triangleleft_f \langle b \rangle$  satisfying  $L \cap \langle b^n \rangle = M \cap \langle c \rangle$ . Now, consider  $\overline{\langle b \rangle} = \langle b \rangle / L$ ,  $\overline{\langle b^n \rangle} = \langle b^n \rangle / L \cap \langle b^n \rangle = \langle c \rangle / M \cap \langle c \rangle = \overline{\langle c \rangle}$ , and then  $\overline{BS(n, -n)} = \overline{\langle b \rangle} *_{\overline{b^n} = \overline{c}} \overline{BS(1, -1)}$ . In

$\overline{BS(n, -n)}$ , we have  $\overline{H_1} = H_1 / L \cap H_1$ ,  $\overline{H_2} = H_2 / M \cap H_2$  and  $\overline{g} = gM \notin \overline{\langle c \rangle H_2}$ . Also,  $\overline{g} \notin \overline{H_1} \overline{H_2}$ . Indeed: if  $\overline{g} \in \overline{H_1} \overline{H_2}$ , then  $\overline{g} = \overline{h_1} \overline{h_2}$  with  $\overline{h_1} \in \overline{H_1}$  and  $\overline{h_2} \in \overline{H_2}$ . Thus  $\overline{h_1} = \overline{gh_2^{-1}} \in \overline{BS(1, -1)}$ , and then  $\overline{h_1} \in \overline{H_1} \cap \overline{BS(1, -1)} \subseteq \overline{\langle c \rangle}$ . It comes that  $\overline{g} = \overline{h_1} \overline{h_2} \in \overline{\langle c \rangle H_2}$ , which contradicts the assumption that  $\overline{g} \notin \overline{\langle c \rangle H_2}$ . Then  $\overline{g} \notin \overline{H_1} \overline{H_2}$  in  $\overline{BS(n, -n)}$ . Using the fact that groups  $\overline{\langle b \rangle}$  and  $\overline{BS(1, -1)}$  are  $RZ_2$  as finite groups, we obtain that  $\overline{BS(n, -n)}$  is  $RZ_2$  as a free product of  $RZ_2$  groups amalgamated by finite subgroups. And the desired result is obtained like in **Case 2 (b)**.

► Suppose now that  $g \in \langle c \rangle H_2$ . Hence  $g = c^t h_2$ , with  $t \in \mathbb{Z}$  and  $h_2 \in H_2$ . From  $g \notin H_1 H_2$  we have  $c^t \notin H_1 H_2$ . Since  $l(c^t) = 0$ , so using **Case 3 Subcase (a)** there exists  $N \triangleleft_f BS(n, -n)$  such that  $c^t \notin H_1 H_2 N$ . Thus  $g \notin H_1 H_2 N$  and the set  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ .

The subcase  $g \in \langle b \rangle \setminus \langle b^n \rangle$  is treated similarly, since  $\langle b \rangle$  as a finitely generated abelian group is  $RZ$  and particularly  $RZ_2$ .

*Subcase (c)* Let finally examine the case  $l(g) \geq 2$ , with  $g = g_1 g_2 \dots g_r$  ( $r \geq 2$ ). Denote again by  $I$  and  $J$  the set of indices in  $\{1, 2, \dots, r\}$  of components of  $g$  belonging in  $\langle b \rangle \setminus \langle b^n \rangle$  and  $BS(1, -1) \setminus \langle c \rangle$  respectively. Since  $BS(n, -n)$  is  $RZ_1$ , the desired result is obtained like in *Case 2 (b)*  $r > 1$ . That is, the set  $H_1 H_2$  is closed in the profinite topology of  $BS(n, -n)$ . And the theorem is demonstrated.  $\square$

## 5. PROOF OF COROLLARY 1.2

Assume that  $K$  is  $\text{RZ}_2$  and contains a finitely generated subgroup  $U$  of finite index in both  $A$  and  $B$  such that  $\varphi(u) = u$  for any  $u \in U$ . Since  $U \leq Z(K)$  and  $t^{-1}ut = u$  for any  $u \in U$ , it comes that  $U \leq Z(G)$ . By Proposition 2.3, we have  $U^n \leq_f U$  and consequently  $U^n \leq_f A$  and  $U^n \leq_f B$ , for any nonzero natural number  $n$ . It is then obvious that  $U^n$ , for any nonzero natural number  $n$ , is finitely generated. Thus,  $K/U^n$  is  $\text{RZ}_2$ , by Proposition 2.2. Also, since  $A$  and  $B$  are isomorphic, so are the finite groups  $A/U^n$  and  $B/U^n$  for any nonzero natural number  $n$ . Thus, for any nonzero natural number  $n$  the HNN-extension  $G_n = G/U^n = \langle K/U^n, \tau \mid \tau^{-1}A/U^n\tau = B/U^n \rangle$  is  $\text{RZ}$  by Proposition 1.4 and particularly  $\text{RZ}_2$ . Consequently  $G$  is  $\text{RZ}_2$  by Theorem 1.1. So, the corollary is demonstrated.

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DEPARTMENT OF MATHEMATICS,  
HIGHER TEACHER'S TRAINING COLLEGE, THE UNIVERSITY OF MAROUA,  
P.O. BOX 55 MAROUA – CAMEROON  
*E-mail: gilbertmantika@yahoo.fr*

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,  
FACULTY OF SCIENCES, THE UNIVERSITY OF MAROUA,  
P.O. BOX 814 MAROUA – CAMEROON  
*E-mail: tematen@yahoo.fr*

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE,  
THE UNIVERSITY OF NGAOUNDERE AND AIMS CAMEROON,  
P.O. BOX 454 NGAOUNDERE – CAMEROON  
*E-mail: tieudjo@yahoo.com*