The directed 2-linkage problem with length constraints*

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Abstract

The WEAK 2-LINKAGE problem for digraphs asks for a given digraph and vertices s_1, s_2, t_1, t_2 whether D contains a pair of arc-disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path. This problem is NP-complete for general digraphs but polynomially solvable for acyclic digraphs [8]. Recently it was shown [3] that if D is equipped with a weight function w on the arcs which satisfies that all edges have positive weight, then there is a polynomial algorithm for the variant of the weak-2-linkage problem when both paths have to be shortest paths in D. In this paper we consider the unit weight case and prove that for every pair constants k_1, k_2, there is a polynomial algorithm which decides whether the input digraph D has a pair of arc-disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path and the length of P_i is no more than $d(s_i, t_i) + k_i$, for $i = 1, 2$, where $d(s_i, t_i)$ denotes the length of the shortest (s_i, t_i)-path. We prove that, unless the exponential time hypothesis (ETH) fails, there is no polynomial algorithm for deciding the existence of a solution P_1, P_2 to the WEAK 2-LINKAGE problem where each path P_i has length at most $d(s_i, t_i) + c \log^{1+\epsilon} n$ for some constant c.

Keywords: (arc)-disjoint paths, shortest disjoint paths, acyclic digraph, linkage

1 Introduction

Notation throughout this paper follows [2, 1]. We use $[0, i]$ to denote the set $\{0, 1, 2, \ldots, i\}$.

Problems concerning disjoint paths with prescribed end vertices in graphs and digraphs play an important role in many combinatorial problems. Among the most important such

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problems are the k-LINKAGE problem and the WEAK k-LINKAGE problem which we formulate below for digraphs.

k-LINKAGE **Input:** A digraph $D = (V, A)$ and distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ **Question:** Does D contain k vertex-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for $i \in [k]$?

WEAK k-LINKAGE **Input:** A digraph $D = (V, A)$ and not necessarily distinct vertices $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ **Question:** Does D contain k arc-disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for $i \in [k]$?

It is an easy and well-known fact that the k-LINKAGE problem and the WEAK k-LINKAGE problems are polynomially equivalent in the sense that one can easily reduce one to the other by a polynomial reduction see e.g. [1, Chapter 10].

A famous and very important result by Robertson and Seymour [13] shows that the corresponding linkage problems for undirected graphs are polynomially solvable for fixed k and that the problems are in fact FPT, meaning that there is an algorithm for each problem whose running time is of the form $O(f(k)n^c)$ for some computable function f and a constant c. It was shown in [13] that $c = 3$ will do and this has been improved to $c = 2$ in [10].

For directed graphs the situation is quite different: Fortune, Hopcroft and Wyllie [8] proved that already the 2-LINKAGE and the WEAK 2-LINKAGE problems are NP-complete. They also showed that if the input is an acyclic digraph, then both linkage problems are polynomially solvable when the number of terminals is fixed (not part of the input).

Theorem 1.1. *[8] The weak k-LINKAGE problem in acyclic digrpahs is solvable in time $O(k!n^{k+2})$.*

Eilam-Tzoreff [7] proved that for undirected graphs the 2-LINKAGE problem is also polynomially solvable if each edge of the input graph is equipped with a positive length and the goal is to check whether there is a solution P_1, P_2 such that P_i is a shortest (s_i, t_i)-path for $i = 1, 2$. This result has recently been extended by Gottschau et al. in [9] and by Kobayashi and Sako in [11] to the case where edges of length zero are allowed.

The problem has also been studied on digraphs by Bérczi and Kobayashi [3] who recently proved the following:

Theorem 1.2. *There exists a polynomial algorithm that runs in time* $|V|^{O(a_1+a_2)}$ *for the following problem as well as its arc-version: Given a digraph* $D = (V, A)$*, vertices* $s_1, s_2, t_1, t_2 \in V$*, a weight function* w *on* A *such that the weight of every directed cycle is positive and numbers* a_1 *and* a_2*; decide whether* D *has disjoint paths* $P_1^1, \ldots, P_{a_1}^1, P_1^2, \ldots, P_{a_2}^2$ *such that* P_j^i *is a shortest* (s_i, t_i)*-path for* $i = 1, 2$ *and* $1 \leqslant j \leqslant a_i$*.*

There are several other papers dealing with shortest path version of the k-LINKAGE problem, see e.g. [4, 6, 12].

In this paper we consider the following variant of the weak-2-linkage problem where the paths do not have to be shortest paths (in terms of number of arcs) but there is a bound on how far from being shortest they can be. Throughout the paper, we denote by $d(u, v)$ the length of the shortest (u, v)-path.

SHORT WEAK 2-LINKAGE SW2L$(D, s_1, s_2, t_1, t_2, k_1, k_2)$
Input: A digraph $D = (V, A)$, vertices $s_1, s_2, t_1, t_2 \in V$ and natural numbers k_1, k_2
Question: Is there a pair of arc-disjoint paths P_1, P_2 such that P_i is an (s_i, t_i)-path and $|A(P_i)| \leq d(s_i, t_i) + k_i$?

Clearly this problem is NP-complete when $k_1, k_2 = n-1$ since that puts no restriction on P_1, P_2 in a solution. The main result of our paper is that when k_1, k_2 are both constants the SHORT WEAK 2-LINKAGE problem can be solved in polynomial time. We also prove that the problem is NP-complete when there is no restriction on the length of one of the paths. Finally, we show that under the exponential time hypothesis, there is no polynomial algorithm for the SHORT WEAK 2-LINKAGE when $k_1, k_2 \in O(\log^{1+\epsilon} n)$ no matter how small the value of ϵ is as long as it is positive.

2 2-linkage with almost shortest paths

Let D be a digraph and s a vertex. The **reach** of s is the set of vertices x such that there exists a path from s to x in D. Using breadth-first-search we can partition the reach of a vertex s into **levels**, such that L_s^i denotes the set of vertices x such that the shortest path from s to x is of length i. We say that an arc uv is **between** two levels if, $d(s, u) < \infty$ and $d(s, v) = d(s, u) + 1$.

Suppose s_1 and s_2 are fixed, let A_1 denote the set of arcs between two consecutive levels from s_1 and A_2 the set of arcs between two consecutive levels from s_2. Note that both A_1 and A_2 induce acyclic digraphs. Furthermore, an arc uv is in A_i if and only if some shortest (s_i, v)-path uses the arc uv. We will use the following lemma and its analogous for A_2:

Lemma 2.1. *If* P *is a path from* s_1 *to* t_1 *of length at most* $d(s_1, t_1) + k$*, then* P *uses at most* k *arcs not belonging to* A_1*.*

Proof. Every path from s_1 to t_1 must visit every level with index smaller than $d(s_1, t_1)$ at least once. Moreover, it must use an arc of A_1 to go from one level to the next, which ends the proof. □

Theorem 2.2. *For every fixed choice of positive integers k_1, k_2 the problem* SHORT WEAK LINKAGE PROBLEM *with input $[D, s_1, s_2, t_1, t_2, k_1, k_2]$ is polynomially solvable.*

Proof: Let $k = \max\{k_1, k_2\}$, $n = |V|$ and $m = |A|$. We shall describe an algorithm that runs in time $n^{O(k)}$ for the problem. Let $E_1 = (v_1, u_1), \ldots, (v_i, u_i)$ and $E_2 = (z_1, w_1), \ldots, (z_j, w_j)$ be two ordered subsets of A of at most k arcs each and that avoid A_1 and A_2 respectively. Recall that A_1 denotes the set of arcs between two consecutive levels from s_1 and A_2 the set of arcs between two consecutive levels from s_2. Let $d_{A_\ell}(x, y)$ denote the distance from x to y in the digraph induced by the arcs in A_ℓ. We call E_1 and E_2 *feasible* if the following holds.

- $d_{A_1}(s_1, v_1) + 1 + d_{A_1}(u_1, v_2) + 1 + d_{A_1}(u_2, v_3) + \cdots + d_{A_1}(u_{i-1}, v_i) + 1 + d_{A_1}(u_i, t_1) \leqslant d_D(s_1, t_1) + k_1$
- $d_{A_2}(s_2, z_1) + 1 + d_{A_2}(w_1, z_2) + 1 + d_{A_2}(w_2, z_3) + \cdots + d_{A_2}(w_{j-1}, z_i) + 1 + d_{A_2}(w_j, t_2) \leqslant d_D(s_2, t_2) + k_2$

We will describe an $O(n^C)$ algorithm for some constant C, which decides if there exists a solution P_1, P_2 to the problem such that for $\ell = 1, 2$, P_ℓ only uses arcs of A_ℓ and E_ℓ. To solve the general question, we only need to run this algorithm for all feasible choices of E_1 and E_2. We note that there are less than $(m^k)^2 \leq n^{4k}$ (as $m \leq n^2$) ways of choosing E_1 and E_2. So the algorithm only needs to be run at most $O(n^{4k})$ times.

Let now E_1 and E_2 be fixed. We create the digraph D' by adding the vertices s'_1, t'_1, s'_2, t'_2 to D and the following paths:

- A path from s'_1 to every vertex $x \in \{s_1, u_1, u_2, \ldots, u_i\}$ of length $d_D(s_1, x) + 1$.
- A path from every vertex $x \in \{t_1, v_1, v_2, \ldots, v_i\}$ to t'_1 of length $d_D(x, t_1) + 1$.
- A path from s'_2 to every vertex $x \in \{s_2, w_1, w_2, \ldots, w_j\}$ of length $d_D(s_2, x) + 1$.
- A path from every vertex $x \in \{t_2, z_1, z_2, \ldots, z_j\}$ to t'_2 of length $d_D(x, t_2) + 1$.

All the internal vertices on all the above added paths are distinct and new vertices. However, the length of all these paths cannot be grater than n and the number of these paths is $2i + 2j + 4 = O(n)$. Hence, the number of vertices we add is still polynomial in the size of the initial graph. Let P_1^* be a (s'_1, t'_1)-path in D' and let x be the first vertex on P_1^* from $\{s_1, u_1, u_2, \ldots, u_i\}$ and let y be the last vertex on P_1^* from $\{t_1, v_1, v_2, \ldots, v_i\}$.

Then the subpath from s_1' to x has length $d_D(s_1, x) + 1$ and the subpath from y to t_1' has length $d_D(y, t_1) + 1$. This implies that the length of P_1^* is the following.

$$(d_D(s_1,x)+1)+(d_D(y,t_1)+1)+d_D(x,y) = d_D(s_1,x)+d_D(x,y)+d_D(y,t_1)+2 \geq d_D(s_1,t_1)+2$$

As there exists an (s_1', t_1')-path of length $d_D(s_1, t_1) + 2$ in D' (using the arcs $s_1's_1$ and t_1t_1' and a shortest (s_1, t_1)-path in D), we note that the shortest (s_1', t_1')-path in D' has length exactly $d_D(s_1, t_1) + 2$. Furthermore if the subpath of P_1^* from x to y only uses arcs from A_1 then it has length $d_D(x, y)$ and we have equality everywhere in the above equation, which implies that the length of P_1' is $d_D(s_1, t_1) + 2 = d_{D'}(s_1', t_1')$. Analogously if the length of P_1^* is $d_{D'}(s_1', t_1')$ then the subpath from x to y only uses arcs from A_1.

Clearly the analogous result also holds for a shortest path from s_2' to t_2'. By Theorem 1.2, we know that we can determine in polynomial time if there exist $i+1$ shortest paths from s_1' to t_1' and $j + 1$ shortest paths from s_2' to t_2' such that all $i + j + 2$ paths are arc-disjoint.

We claim that if such paths exist, then the answer to our instance of the SHORT WEAK LINKAGE PROBLEM is *true* and if there is no such $i + j + 2$ arc-disjoint paths for any feasible choice of E_1 and E_2, then the answer to our instance is *false*.

First assume that we found $i + j + 2$ arc-disjoint paths for some feasible choice of E_1 and E_2. Now remove all vertices in $V(D') \setminus V(D)$ from the (s_1', t_1')-paths and add the arcs E_1. Note that the outdegree of s_1 will be one (as it belongs to one of the paths) and the indegree of s_1 will be zero. Analogously the indegree of t_1 will be one and the outdegree will be zero. All other vertices will have indegree and out degree equal to each other (if they belong to k paths then the indegree and outdegree will both be k). Therefore the arcs in the resulting subdigraph form a path from s_1 to t_1 plus possibly a number of cycles. As the total number of arcs in the subdigraph is less than $d(s_1, t_1) + k_1$ the path from s_1 to t_1 (after discarding any cycles) also has length less than $d(s_1, t_1) + k_1$. Indeed, the graph contains all the arcs of E_1 (i arcs) and all the arcs of the $i+1$ paths of length $d(s_1, t_1) + 2$ except those of $D' \setminus D$. The number of arcs of $D' \setminus D$ in those paths is $N_1 = \sum_{x\in\{s_1,u_1,\ldots,u_i\}}(d(s_1, x) + 1) + \sum_{x\in\{v_1,\ldots,v_i,t_1\}}(d(x, t_1) + 1)$. Let us set $u_0 = s_1$ and $v_{i+1} = t_1$. The total number of arcs in our graph is:

$$\begin{aligned} i + (d(s_1,t_1) + 2) \times (i+1) - N_1 &= i + \sum_{\ell=0}^{i} [d(s_1,t_1) - d(s_1,u_\ell) - d(v_{\ell+1},t_1)] \\ &\leqslant i + \sum_{\ell=0}^{i} d(u_\ell, v_{\ell+1}) \\ &\leqslant d(s_1,t_1) + k_1 \quad \text{by definition of the feasibility of } E_1 \end{aligned}$$

Analogously we find a path from s_2 to t_2 of length less than $d(s_2, t_2) + k_2$. By our construction these paths are arc-disjoint, completing the proof of one direction.

Now assume that there exist arc-disjoint paths P_1 and P_2 in D, such that P_ℓ is a (s_ℓ, t_ℓ)-path of length less than $d(s_\ell, t_\ell) + k_\ell$. Let E_ℓ be the arcs on P_ℓ that do not belong to A_ℓ ($\ell \in [2]$). Hence, E_1 and E_2 are feasible. Using these E_1 and E_2 and the subpaths of P_1 and P_2 after removing the arcs in E_1 and E_2 we note that we can obtain the desired $i + j + 2$ arc-disjoint paths in D'. This completes the proof.

D' has size $O(n^2)$, so the existence of this path Q can be checked in polynomial time, and the overall problem can be solved in time $n^{O(k)}$.

3 Non-polynomial cases

This section is devoted to the proof of the NP-completeness of the problem of semi-short weak 2-linkage:

SEMI-SHORT WEAK 2-LINKAGE SSW2L$(D, s_1, s_2, t_1, t_2, k)$
Input: A digraph $D = (V, A)$, vertices $s_1, s_2, t_1, t_2 \in V$ and a natural number k
Question: Is there a pair of arc-disjoint paths P_1, P_2 such that P_1 is an (s_1, t_1)-path of length $\lvert A(P_1)\rvert \leq d(s_1, t_1) + k$ and P_2 is an (s_2, t_2)-path?

Eilam-Tzoreff proved in [7] that the problem of SSW2L is NP-complete if P_1 has to be a shortest (s_1, t_1)-path, which is the case $k = 0$. We observe that this can be generalized to all values of k.

Theorem 3.1. *The semi-short weak 2-linkage problem is NP-complete for all values of k.*

Proof. The proof is by reducing SSW2L for $k = 0$ to SSW2L with any value of k. Let k be fixed and let $D, s_1, s_2, t_1, t_2, 0$ be an instance of SSW2L. Let us now create D' by replacing every arc of D by a directed path of length $k+1$. Let P'_1 and P'_2 be (s_1, t_1) and (s_2, t_2)-paths respectively in D'.

Note that every path P' from s_1 to t_1 in D' is thus the image of a path P from s_1 to t_1 in D by subdivision of the edges of P, and P' is $(k+1)$ times longer than P. Let P_1 and P_2 be the preimage of P'_1 and P'_2. Hence, if $\lvert A(P'_1)\rvert \leqslant d_{D'}(s_1, t_1) + k < d_{D'}(s_1, t_1) + k + 1$, then $\lvert A(P_1)\rvert < d_D(s_1, t_1) + 1$, which means that P_1 is a shortest (s_1, t_1)-path in D. It is also easy to see that P'_1 and P'_2 arc disjoint if and only if P_1 and P_2 are. Thus, (P'_1, P'_2) is a solution of SSW2L$(D', s_1, s_2, t_1, t_2, k)$ if and only if P_1 and P_2 are solutions of SSW2L$(D, s_1, s_2, t_1, t_2, 0)$. This proves the NP-completeness of SSW2L for all k. □

Note that our algorithm in Theorem 2.2 is only polynomial for constant k, so it is natural to ask whether we could replace constant k by some function of n.

Recall the so-called Exponential Time Hypothesis (ETH) which in one of many formulations says that there exist a real number $\delta > 0$ so that no algorithm can solve 3-SAT instances with m clauses in time $O(2^{\delta m})$. This modification of the commonly known version of ETH is in fact equivalent to that, see e.g. [5, Theorem 14.4].

Theorem 3.2. *Assuming that ETH is true, then for every $\epsilon > 0$ there is no polynomial algorithm for* WEAK SHORT 2-LINKAGE *problem when the input D is a digraph on n vertices and $k_1, k_2 = \Theta(\log^{1+\epsilon} n)$.*

Proof: We give the proof when $k_1, k_2 = \log^{1+\epsilon} N$, where N is the number of vertices in the input digraph. Let $[D, s_1, s_2, t_1, t_2]$ be an instance of the weak 2-linkage problem and let n be the number of vertices of D. Let ϵ' be defined such that $2^{n^{\frac{1}{1+\epsilon'}}} = \left\lceil 2^{n^{\frac{1}{1+\epsilon}}} \right\rceil$ and note that $\epsilon' \leq \epsilon$ and that $\epsilon' > 0$ when n is large enough. Construct a new digraph D' by adding an independent set of size $2^{n^{\frac{1}{1+\epsilon'}}} - n$ so that the resulting digraph has $N = 2^{n^{\frac{1}{1+\epsilon'}}}$ vertices, implying that we have $(\log N)^{1+\epsilon'} = n$. Clearly every pair of arc-disjoint (s_1, t_1)-, (s_2, t_2)-paths, P_1, P_2 in D' use only vertices from D and hence each of their lengths is at most $n = \log^{1+\epsilon'} N \leq \log^{1+\epsilon} N$ so P_i is at most k_i longer that the shortest (s_i, t_i)-path for $i = 1, 2$.

Suppose there is an algorithm for the WEAK SHORT 2-LINKAGE problem that runs in time $O(N^c)$ for inputs on N vertices when $k_1, k_2 = \log^{1+\epsilon} N$ for some fixed constanct $c > 0$. Then we have

$$\begin{aligned} N^c &= (2^{n^{\frac{1}{1+\epsilon'}}})^c \\ &= 2^{c \cdot n^{\frac{1}{1+\epsilon'}}} \\ &< 2^{\delta \cdot n} \end{aligned}$$

for every fixed constant $\delta > 0$ provided that n is large enough. This means that we can solve the general weak 2-linkage problem in time $O(2^{\delta \cdot n})$ for every $\delta > 0$.

To see that this contradicts the ETH, we just have to observe that the reduction from 3-SAT to the 2-linkage problem in [8] (see also [1, Section 10.2]) converts a 3-SAT formula with n variables and m clauses into an instance of 2-linkage with at most dm vertices where d is a constant (it is at most 61). Furthermore, the 2-linkage problem for a digraph on n vertices reduces to the weak 2-linkage problem on a digraph with twice as many vertices. □

4 Remarks

Slivkins [14] proved that the WEAK k-LINKAGE problem is $W[1]$-hard for acyclic digraphs. We can prove that the same holds for SHORT WEAK k-LINKAGE in acyclic digraphs. Indeed, consider an instance of WEAK k-LINKAGE on an acyclic digraph D and consider a topological ordering $v_1, ..., v_n$ of the vertices of D, *i.e.* an ordering such that for every arc v_iv_j, we have $j > i$. Let us build D' from D by replacing every arc v_iv_j in D by a directed path of length $(j - i)$ in D'. Hence, D' is still acyclic and every walk between a vertex v_i and a vertex v_j in D is now replaced by a walk of length $j - i$ in D' and is thus a shortest walk. Therefore, a solution of short weak k-linkage in D' immediately provides a solution of weak k-linkage in D.

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