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Samer Israwi and Henrik Kalisch

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Associated to the KdV Equation**

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Partial differential equations / *Équations aux dérivées partielles*

# A Mathematical Justification of the Momentum Density Function Associated to the KdV Equation

Samer Israwi<sup>a</sup> and Henrik Kalisch<sup>\*, b</sup>

<sup>a</sup> Lebanese University, Faculty of Sciences 1, Department of Mathematics,  
Hadath-Beirut, Lebanon

<sup>b</sup> Department of Mathematics, University of Bergen, Postbox 7800, 5020 Bergen,  
Norway

E-mails: [s\\_israwi83@hotmail.com](mailto:s_israwi83@hotmail.com), [henrik.kalisch@uib.no](mailto:henrik.kalisch@uib.no)

**Abstract.** Consideration is given to the KdV equation as an approximate model for long waves of small amplitude at the free surface of an inviscid fluid. It is shown that there is an approximate momentum density associated to the KdV equation, and the difference between this density and the physical momentum density derived in the context of the full Euler equations can be estimated in terms of the long-wave parameter.

**Résumé.** L'équation de KdV est considérée comme un modèle approximatif pour des ondes longues de faible amplitude à la surface libre d'un fluide non visqueux. On montre qu'il y a une densité de moment approximative associée à l'équation de KdV, et que la différence entre cette densité et la densité de moment physique dérivée dans le contexte du système d'Euler peut être estimée en fonction du paramètre d'onde longue.

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## Version française abrégée

Dans le présent travail, nous prouverons qu'au moins pour la densité de moment, l'approximation entre l'équation de KdV et système d'Euler peut être rendue mathématiquement rigoureuse. Le résultat principal montre que la densité  $I$  dans l'équation de KdV converge vers la densité de moment physique définie en fonction de la solution du système d'Euler pour un fluide parfait lorsque les paramètres physiques  $\mu$  et  $\varepsilon$  tendent vers zéro. L'énoncé précis de notre théorème est le suivant.

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\* Corresponding author.

**Théorème.** Soit  $(\zeta, \phi)$  une solution du système d'Euler, avec condition initiale  $(\zeta_0, \phi_0)$  suffisamment régulière. Soit  $\eta$  une solution de l'équation de KdV (1) avec une condition initiale  $\eta_0 = \zeta_0$ . Alors, il existe une constante  $C$  tel que :

$$\left\| \int_{-1}^{\varepsilon\zeta(\cdot, t)} \partial_x \phi(z, \cdot, t) dz - \eta(\cdot, t) - \varepsilon \frac{3}{4} \eta^2(\cdot, t) - \mu \frac{1}{6} \eta_{xx}(\cdot, t) \right\|_{L^\infty} \leq C \mu^2 (1 + t).$$

La preuve de ce théorème est basée sur l'utilisation du système de Peregrine (9) comme un modèle intermédiaire entre l'équation de KdV et le problème d'Euler. Le point clé est que la densité de moment définie dans le contexte des équations d'Euler est reliée à la vitesse moyenne  $\bar{V}$ . En effet, nous avons

$$\int_{-1}^{\varepsilon\zeta(x, t)} \partial_x \phi(z, x, t) dz = (1 + \varepsilon\zeta(x, t)) \bar{V}(x, t),$$

et cette relation peut être utilisée d'une manière favorable dans la preuve car le système Peregrine (9) est également défini en fonction de la vitesse horizontale moyenne.

On définit la vitesse variable associée à une solution  $\eta$  de l'équation de KdV

$$v_{KdV} = \eta - \frac{\varepsilon}{4} \eta^2 - \frac{\mu}{6} \eta_{xt}$$

(voir [11]) par suite on obtient l'estimation suivante

$$\|\bar{V} - v_{KdV}\|_{L^\infty} \leq C \mu^2 t,$$

qui est l'ingrédient principal utilisé dans la preuve du résultat final.

### 1. Introduction

In the present contribution, we consider the question of momentum conservation in the context of the Korteweg–de Vries equation. The KdV equation

$$\eta_t + \eta_x + \varepsilon \frac{3}{2} \eta \eta_x + \mu \frac{1}{6} \eta_{xxx} = 0 \tag{1}$$

is known to yield a valid description of surface waves for waves of small amplitude and large wavelength at the free surface of an incompressible, inviscid fluid running in a narrow open channel where transverse effects can be neglected.

Suppose  $h_0$  is the depth of the undisturbed fluid, and let  $\lambda$  denote a typical wavelength and by  $a$  a typical amplitude of a wavefield to be described. The nondimensional number  $\varepsilon = a/h_0$  then represents the relative amplitude. If we define the long-wave parameter by  $\mu = h_0^2/\lambda^2$ , then the KdV equation is known to be a good model for waves at the free surface of a fluid if the relations  $\mu \ll 1$  and  $\varepsilon = \mathcal{O}(\mu)$ . The approximation can be made rigorous using the techniques developed in [7, 10, 11, 16, 17] and others.

It is well known that the KdV equation has an infinite number of formally conserved integrals (indeed the conservation can be made rigorous by following the work of [8]). If the equation is given in the non-dimensional form (1), the first three conserved integrals are

$$\int_{-\infty}^{\infty} \eta dx, \quad \int_{-\infty}^{\infty} \eta^2 dx, \quad \text{and} \quad \int_{-\infty}^{\infty} \left( \frac{\mu}{3\varepsilon} \eta_x^2 - \eta^3 \right) dx. \tag{2}$$

The first integral is found to be invariant with respect to time  $t$  as soon as it is recognized that the KdV equation can be written in the form

$$\frac{\partial}{\partial t} (\eta) + \frac{\partial}{\partial x} \left( \eta + \varepsilon \frac{3}{4} \eta^2 + \mu \frac{1}{6} \eta_{xx} \right) = 0, \tag{3}$$

where the quantity appearing under the time derivative is interpreted as excess mass density, and the term appearing under the spatial derivative is the mass flux through a cross section of unit width due to the passage of a surface wave. The second and third integral are sometimes

called momentum and energy, but this terminology may be misleading since these integrals are not readily interpreted as approximations of the physical momentum and energy appearing in the context of the Euler equations. Indeed, the authors of [1] already state clearly that they do not believe these integrals to be approximations of the physical momentum and density, and further doubt was cast on this interpretation in more recent work [2, 12–14].

On the other hand, in physical flow problems, mass flux is often identical with momentum density, so one might think that the term  $\eta + \varepsilon \frac{3}{4} \eta^2 + \mu \frac{1}{6} \eta_{xx}$  in (3) might be interpreted as momentum flux. This is indeed correct as shown in the recent work [4], where based on ideas developed in [3], it was shown how to find integral quantities that do represent approximations to the physically relevant momentum and energy densities. In particular, following the procedure laid out in [4] gives the expression for momentum density as

$$I = \eta + \varepsilon \frac{3}{4} \eta^2 + \mu \frac{1}{6} \eta_{xx}. \tag{4}$$

Since the analysis in [4] was based on a formal asymptotic analysis, the question of whether this identity can be made mathematically rigorous has so far remained open. In the present work we will prove that a firm mathematical proof can indeed be given. The main result to be proved thus states that the density  $I$  converges to the physical momentum density defined in terms of a solutions of the governing Euler equation for a perfect fluid if  $\mu$  and  $\varepsilon$  tend to zero. The precise statement is as follows.

**Theorem 1.** *Let  $(\zeta, \phi)$  be a solution of the water-wave problem defined below, with regular initial data  $(\zeta_0, \phi_0)$ . Let  $\eta$  be a solution of the KdV equation (1) with initial data  $\eta_0 = \zeta_0$ . Then there exists a constant  $C$ , so that we have the estimate*

$$\left\| \int_{-1}^{\varepsilon \zeta(\cdot, t)} \partial_x \phi(z, \cdot, t) dz - \eta(\cdot, t) - \varepsilon \frac{3}{4} \eta^2(\cdot, t) - \mu \frac{1}{6} \eta_{xx}(\cdot, t) \right\|_{L^\infty} \leq C \mu^2 (1 + t). \tag{5}$$

## 2. Auxiliary results

Denoting the original (dimensional) variables with a tilde, we introduce a scaling to make the small amplitude and long wavelength relative to the undisturbed depth explicit. Thus we define new variables (without a tilde) by  $\tilde{x} = \lambda x$ ,  $\tilde{z} = h_0 z$ ,  $\tilde{\zeta} = a \zeta$ ,  $\tilde{t} = \frac{\lambda}{c_0} t$ ,  $\tilde{\phi} = \frac{a \lambda g}{c_0} \phi$ .

Then we obtain the system

$$\begin{cases} \mu \partial_x^2 \phi + \partial_z \phi^2 = 0 & \text{in } \Omega_t, \\ \partial_z \phi = 0, & \text{at } z = -1, \\ \partial_t \zeta - \frac{1}{\mu} (-\mu \partial_x \zeta \partial_x \phi + \partial_z \phi) = 0 & \text{at } z = \varepsilon \zeta, \\ \partial_t \phi + \zeta + \frac{\varepsilon}{2} (\partial_x \phi)^2 + \frac{\varepsilon}{2\mu} (\partial_z \phi)^2 = 0 & \text{at } z = \varepsilon \zeta, \end{cases} \tag{6}$$

where  $\Omega_t = \{(x, z), -1 < z < \varepsilon \zeta(x, t)\}$  is the fluid domain delimited by the free surface  $\{z = \zeta(x, t)\}$ , and the flat bottom  $\{z = -1\}$ , and where  $\phi(x, z, t)$ , defined on  $\Omega_t$  is the velocity potential associated to the flow (that is, the two-dimensional velocity field  $\mathbf{v}$  is given by  $\mathbf{v} = (\partial_x \phi, \partial_z \phi)^T$ ). As is well known, the existence of the velocity potential is guaranteed by the assumption of irrotational flow. The equations above are formally equivalent to the Zakharov–Craig–Sulem (ZCS) equations. They are written in terms of the trace  $\Phi(x, t) = \phi(x, \varepsilon \zeta(x, t), t)$  and the Dirichlet–Neumann operator  $G(\zeta)$  as

$$\begin{cases} \zeta_t - \frac{1}{\mu} G(\varepsilon \zeta) \Phi = 0, \\ \Phi_t + \zeta + \frac{\varepsilon}{2} \Phi_x^2 - \frac{\varepsilon}{\mu} \frac{[G(\varepsilon \zeta) \Phi + \varepsilon \mu \zeta_x \Phi_x]^2}{2(1 + \varepsilon^2 \mu \zeta_x^2)} = 0. \end{cases} \tag{7}$$

Given a solution of this system, we reconstruct the potential  $\phi$  by solving the Laplace equation in the domain  $\Omega_t$  (cf. [15, 19]), and then define the average velocity in the context of the full water-wave problem by

$$\bar{V}(x, t) = \frac{1}{1 + \varepsilon\zeta} \int_{-1}^{\varepsilon\zeta} \partial_x \phi(x, z, t) dz. \tag{8}$$

From [16, Theorem 4.16], we have the following result.

**Theorem 2.**

- (1) For large enough  $s$ , there exists a unique solution  $(\zeta, \Phi) \in C(0, T/\varepsilon, H^s \times \dot{H}^{s+1})$  of the ZCS water-wave system.
- (2) For the average velocity, we have  $\bar{V} \in C(0, T/\varepsilon, H^{s-3})$ .

In the shallow-water small-amplitude regime specified above ( $\mu \ll 1, \varepsilon = \mathcal{O}(\mu)$ ), one can derive the Peregrine system. For one-dimensional surfaces and flat bottoms, these equations couple the free surface elevation  $\zeta$  to the vertically averaged horizontal component of the velocity, and can be written as

$$\begin{cases} \xi_t + [(1 + \varepsilon\xi)u]_x = 0, \\ u_t + \xi_x + \varepsilon uu_x = \frac{\mu}{3} u_{xxt}. \end{cases} \tag{9}$$

Based on results proved in [5, 18], the authors of [6] formulate the following result showing that this system is globally well posed.

**Theorem 3.** Suppose  $s \geq 1$ , and initial data  $(\xi_0, u_0) \in H^s \times H^{s+1}$  are given with the additional assumption that  $\inf \xi > -1$ . Then there is a unique solution  $(\xi, u)$  of the system (9) which for any  $\mathcal{T} > 0$  lies in  $C(0, \mathcal{T}, H^s(\mathbb{R})) \times C(0, \mathcal{T}, H^{s+1}(\mathbb{R}))$ , and such that  $\xi(x, 0) = \xi_0(x)$  and  $u(x, 0) = u_0(x)$ . Moreover the solution depends continuously on the initial data in the norm of

$$C(0, \mathcal{T}, H^s(\mathbb{R})) \times C(0, \mathcal{T}, H^{s+1}(\mathbb{R})).$$

Proving convergence of the unknown quantities in the model system to the unknowns in the full water-wave problem requires consistency and stability. Stability of the Peregrine system was proved in [16, Proposition 6.5]. Restricting this result to a flat bottom and to one space dimension yields the following Theorem 4.

**Theorem 4.** If a pair of functions  $(\tilde{\xi}, \tilde{u})$  exists, such that

$$\begin{cases} \tilde{\xi}_t + [(1 + \varepsilon\tilde{\xi})\tilde{u}]_x = r, \\ \tilde{u}_t - \frac{\mu}{3} \tilde{u}_{xxt} + \tilde{\xi}_x + \varepsilon \tilde{u} \tilde{u}_x = R, \end{cases}$$

then

$$\|[(\tilde{\xi}, \tilde{u}) - (\xi, u)](\cdot, t)\|_{H^s \times H^s_\mu} \leq C \left( \|[(\tilde{\xi}, \tilde{u}) - (\xi, u)](\cdot, 0)\|_{H^s \times H^s_\mu} + t \|(r, R)\|_{L^\infty} \right), \tag{10}$$

where the norm  $\|\cdot\|_{H^s_\mu}$  is defined by

$$\|f\|_{H^s_\mu}^2 = \|f\|_{H^s}^2 + \mu \|f_x\|_{H^s}^2.$$

Next following the procedure laid out in [7, 9, 16], we define consistency based on system (9).

**Definition 5.** A family of function pairs  $(\zeta^{\varepsilon, \mu}, v^{\varepsilon, \mu})$  is consistent with (9) if for all  $\varepsilon > 0$ , we have

$$\begin{cases} \zeta_t^{\varepsilon, \mu} + [(1 + \varepsilon\zeta^{\varepsilon, \mu})v^{\varepsilon, \mu}]_x = \varepsilon^2 r^{\varepsilon, \mu}, \\ v_t^{\varepsilon, \mu} - \frac{\mu}{3} v_{xxt}^{\varepsilon, \mu} + \zeta_x^{\varepsilon, \mu} + \varepsilon v^{\varepsilon, \mu} v_x^{\varepsilon, \mu} = \varepsilon^2 R^{\varepsilon, \mu}, \end{cases}$$

with  $(r^{\varepsilon, \mu}, R^{\varepsilon, \mu})$  bounded in  $L^\infty(0, T/\varepsilon, H^s(\mathbb{R}) \times H^s(\mathbb{R}))$ .

Clearly, if we are able to find a family of function pairs consistent with the Peregrine system, then by the stability result, these functions will converge towards the unknowns of the Peregrine system. It turns out that both the ZCS equations and the KdV equation can be shown to be consistent with the Peregrine system. From [16, Corollary 5.20] (with flat bottom), we have the following result.

**Theorem 6.** *The water-wave equations are consistent with the Peregrine system. Indeed for a solution  $\zeta, \Phi$  of the water wave problem, we can define  $\phi$  and  $\bar{V}$  as explained above, and we have*

$$\begin{cases} \zeta_t + [(1 + \varepsilon\zeta) \bar{V}]_x = 0, \\ \bar{V}_t - \frac{\mu}{3} \bar{V}_{xxt} + \zeta_x + \varepsilon \bar{V} \bar{V}_x = \mu^2 R, \end{cases}$$

with  $\|R(\cdot, t)\|_{H^s}$  bounded for  $t \in [0, T/\varepsilon]$ .

Now following the proof of [16, Corollary 6.23], one may put all these theorems together to obtain the following result.

**Theorem 7.** *Suppose initial data  $(\zeta_0, \Phi_0) \in H^N(\mathbb{R}) \times \dot{H}^{N+1}(\mathbb{R})$  are given for a large enough Sobolev index  $N$ . Defining initial data for (9) by  $\xi_0 = \zeta_0$  and  $u_0 = \frac{1}{1+\varepsilon\zeta_0} \int_{-1}^{\varepsilon\zeta_0} \partial_x \phi(x, z, 0) dz$ , there exists a constant  $C$  depending only on  $N$ , such that the estimate*

$$\|(\zeta, \bar{V}) - (\xi, u)\|_{L^\infty} \leq C\mu^2 t \tag{11}$$

holds for the solutions of (7) and (9) and with  $\bar{V}$  defined by (8).

Next we turn to the uni-directional KdV model. Existence, uniqueness and continuous dependence on the initial data follow from the results proved in [8], and are by now classical.

**Theorem 8.** *For the KdV equation with initial data in  $H^s$ , where  $s \geq 2$ , there is a unique solution  $\eta \in C(0, \mathcal{T}, H^s)$  for any  $\mathcal{T} > 0$ , and the solution depends continuously on the initial data.*

Note also that it was proved in [11] that  $\eta \in C^1(0, T/\varepsilon, H^{s-3})$ . In order to prove consistency in the sense of Definition 5, we need to define an appropriate velocity. Following [11], we define

$$v_{KdV} = \eta - \frac{\varepsilon}{4} \eta^2 - \frac{\mu}{6} \eta_{xt}. \tag{12}$$

Now given initial data  $(\xi_0, u_0) \in H^s \times H^{s+1}$  and the solution  $(\xi, u) \in C(0, \mathcal{T}, H^s \times H^{s+1})$  of the system (9) we define initial data for (1) by  $\eta_0 = \xi_0$ , and using the solution  $\eta \in C(0, \mathcal{T}, H^s)$  guaranteed by Theorem 8 we define  $v_{KdV}$  by (12). Then following the proof laid out in [11], we can obtain the following estimate.

**Theorem 9.** *With the above provisos, we have the estimate*

$$\|(\eta, v_{KdV}) - (\xi, u)\|_{L^\infty} \leq C\mu^2 t. \tag{13}$$

Thus it becomes clear that since the Peregrine system approximates the full water wave problem, so do solutions of the KdV equation.

### 3. Convergence of momentum density

We now give a proof of Theorem 1.

**Proof.** First of all, from (10) and (13) and the triangle inequality we have

$$\|\bar{V} - v_{KdV}\|_{L^\infty} \leq C\mu^2 t \quad \text{and} \quad \|\zeta - \eta\|_{L^\infty} \leq C\mu^2 t. \tag{14}$$

Next we have

$$\begin{aligned} \int_{-1}^{\varepsilon\zeta} \phi_x dz - I &= \int_{-1}^{\varepsilon\zeta} \phi_x dz - \frac{1}{1+\varepsilon\zeta} \int_{-1}^{\varepsilon\zeta} \phi_x dz + \frac{1}{1+\varepsilon\zeta} \int_{-1}^{\varepsilon\zeta} \partial_x \phi dz - \left( \eta - \frac{\varepsilon}{4} \eta^2 - \varepsilon \frac{1}{6} \eta_{xt} \right) \\ &\quad + \left( \eta - \frac{\varepsilon}{4} \eta^2 - \varepsilon \frac{1}{6} \eta_{xt} \right) - \left( \eta + \frac{3\varepsilon}{4} \eta^2 + \varepsilon \frac{1}{6} \eta_{xt} \right) \\ &= \left( 1 - \frac{1}{1+\varepsilon\zeta} \right) \int_{-1}^{\varepsilon\zeta} \phi_x dz + \bar{V} - v_{KdV} - \varepsilon \left( \frac{1}{4} + \frac{3}{4} \right) \eta^2 - \frac{\mu}{6} (\eta_{xt} + \eta_{xx}). \end{aligned}$$

Now observe that equation (1) together with the fact that solutions of (1) are bounded in  $H^s(\mathbb{R})$  for all time (cf. [8]) can be used to show that  $\|\eta_{xx}(\cdot, t) + \eta_{xt}(\cdot, t)\|_{L^\infty} \leq C\mu$ . Using this estimate together with the triangle inequality, and the fact that  $\|\bar{V}(\cdot, t) - v_{KdV}(\cdot, t)\|_{L^\infty} \leq C\mu^2 t$  from (14) leads to

$$\left\| \int_{-1}^{\varepsilon\zeta(\cdot, t)} \partial_x \phi(\cdot, z, t) dz - I(\cdot, t) \right\|_{L^\infty} \leq \varepsilon \|\zeta(\cdot, t) \bar{V}(\cdot, t) - \eta^2(\cdot, t)\|_{L^\infty} + C\mu^2(1+t).$$

Finally, notice that

$$\begin{aligned} \zeta \bar{V} - \eta^2 &= \zeta^2 - \zeta^2 + \zeta \bar{V} - \eta^2 \\ &= (\zeta + \eta)(\zeta - \eta) + \zeta(-\zeta + \bar{V}) \\ &= (\zeta + \eta)(\zeta - \eta) + \zeta(\bar{V} - v_{KdV} + v_{KdV} - \eta + \eta - \zeta). \end{aligned}$$

The function  $\zeta$  is bounded in  $C(0, T/\varepsilon, H^s)$ ,  $\bar{V}$  is bounded in  $C(0, T/\varepsilon, H^{s-3})$ ,  $\eta$  is bounded in  $C(0, \mathcal{T}, H^s)$  and  $v_{KdV}$  is bounded in  $C(0, T/\varepsilon, H^{s-2})$ , so that we have

$$\left\| \int_{-1}^{\varepsilon\zeta(\cdot, t)} \partial_x \phi(\cdot, z, t) dz - I(\cdot, t) \right\|_{L^\infty} \leq \varepsilon C \{ \|\zeta - \eta\|_{L^\infty} + \|\bar{V} - v_{KdV}\|_{L^\infty} + \|v_{KdV} - \eta\|_{L^\infty} + \|\eta - \zeta\|_{L^\infty} \}.$$

Finally using (14), and the definition of  $v_{KdV}$  (12) along with the fact that  $\eta \in C^1(0, T/\varepsilon, H^{s-3})$  (cf. [11]) proves the required result.  $\square$

## References

- [1] M. J. Ablowitz, H. Segur, "On the evolution of packets of water waves", *J. Fluid Mech.* **92** (1979), p. 691-715.
- [2] A. Ali, H. Kalisch, "Energy balance for undular bores", *C. R. Méc. Acad. Sci. Paris* **338** (2010), no. 2, p. 67-70.
- [3] ———, "Mechanical balance laws for Boussinesq models of surface water waves", *J. Nonlinear Sci.* **22** (2012), no. 3, p. 371-398.
- [4] ———, "On the formulation of mass, momentum and energy conservation in the KdV equation", *Acta Appl. Math.* **133** (2014), no. 1, p. 113-131.
- [5] C. J. Amick, "Regularity and uniqueness of solutions to the Boussinesq system of equations", *J. Differ. Equations* **54** (1984), no. 2, p. 231-247.
- [6] J. L. Bona, M. Chen, J.-C. Saut, "Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II: The nonlinear theory", *Nonlinearity* **17** (2004), no. 3, p. 925-952.
- [7] J. L. Bona, T. Colin, D. Lannes, "Long wave approximations for water waves", *Arch. Ration. Mech. Anal.* **178** (2005), no. 3, p. 373-410.
- [8] J. L. Bona, R. B. Smith, "The initial value problem for the Korteweg-de Vries equation", *Proc. R. Soc. Lond., Ser. A* **278** (1975), no. 1287, p. 555-601.
- [9] A. Constantin, D. Lannes, "The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations", *Arch. Ration. Mech. Anal.* **192** (2009), no. 1, p. 165-186.
- [10] W. Craig, "An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits", *Commun. Partial Differ. Equations* **10** (1985), no. 8, p. 787-1003.
- [11] S. Israwi, "Variable depth KdV equations and generalizations to more nonlinear regimes", *ESAIM, Math. Model. Numer. Anal.* **44** (2010), no. 2, p. 347-370.
- [12] S. Israwi, H. Kalisch, "Approximate conservation laws in the KdV equation", *Physics Letters A* **383** (2019), no. 9, p. 854-858.
- [13] A. Karczewska, P. Rozmej, E. Infeld, "Energy invariant for shallow-water waves and the Korteweg-de Vries equation: Doubts about the invariance of energy", *Phys. Rev. E* **92** (2015), no. 5, article no. 053202 (15 pages).

- [14] A. Karczewska, P. Rozmej, E. Infeld, G. Rowlands, “Adiabatic invariants of the extended KdV equation”, *Phys. Lett.* **381** (2017), no. 4, p. 270-275.
- [15] D. Lannes, “Well-posedness of the water-waves equations”, *J. Am. Math. Soc.* **18** (2005), no. 3, p. 605-654.
- [16] ———, *The Water Waves Problem. Mathematical analysis and asymptotics*, Mathematical Surveys and Monographs, vol. 188, American Mathematical Society, 2013.
- [17] G. Schneider, C. E. Wayne, “The long-wave limit for the water wave problem. I. The case of zero surface tension”, *Commun. Pure Appl. Math.* **53** (2000), no. 12, p. 1475-1535.
- [18] M. E. Schonbek, “Existence of solutions for the Boussinesq system of equations”, *J. Differ. Equations* **42** (1981), p. 325-352.
- [19] S. Wu, “Well-posedness in Sobolev spaces of the full water wave problem in 2-D”, *Invent. Math.* **130** (1997), no. 1, p. 39-72.