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Abstract

In the stochastic frontier model we extend the multivariate probability statements of Horrace (2005) to calculate the conditional probability that a firm is any particular efficiency rank in the sample. From this we construct the conditional expected efficiency rank for each firm. Compared to the traditional ranked efficiency point estimates, firm-level conditional expected ranks are more informative about the degree of uncertainty of the ranking. The conditional expect ranks may be useful for empiricists. A Monte Carlo study and an empirical example are provided.

Key Words: Efficiency estimation, Order statistics, Multivariate inference, Multiplicity

JEL Codes: C12, C16, C44, D24

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1. Introduction

Given a sample of firm-level data, parametric stochastic frontier models specify production output (or cost) as the sum of a linear response function and an additively composed error, consisting of a two-sided error, representing noise, and a one-sided error, representing inefficiency. See, for example, Aigner, Lovell and Schmidt (1977), Battese and Coelli (1988), Battese and Coelli (1992), and Greene (2005). It is very often assumed that the two-sided error is normally distributed and the one-sided error is truncated normal or exponential. If so, the distribution of inefficiency *conditional* on the composed error is truncated normal. Given these conditional inefficiency distributions (one for each firm), a common empirical question is how does one assess relative inefficiency in the sample? There are essentially two approaches. The first approach is to calculate the mean of each conditional inefficiency distribution, using the value of the regression residual for each firm in the conditioning argument. See Jondrow et al. (1982) for the cross-sectional case and Battese and Coelli (1988) for the panel data case. These conditional means (evaluated at the residual values) can be ordered across firms, and a sample-wide view of inefficiency is inferred from the order statistic. In particular, the firm with the smallest conditional mean may be deemed efficient relative to the rest in the sample. A second approach is to use the conditional inefficiency distributions to calculate the probability that each firm is best (has lowest inefficiency), conditional on the (joint) composed errors. See Horrace (2005). These conditional *efficiency probabilities* can be evaluated at the values of the (joint) regression residuals to provide an alternative view of (in)efficiency in the sample, and, in particular, the firm with the largest efficiency probability may be deemed the most efficient. The first approach is a *marginal* approach in that each conditional mean is derived from a *single* conditional inefficiency distribution. The second approach is *simultaneous* in that each conditional efficiency probability is derived from *all* the conditional distributions, jointly. In this sense the conditional probabilities contain information from the efficiency rank statistic that the conditional means do not provide. In the parlance of the *multiple comparisons* and *ranking and selection* literatures (e.g., Bechhofer, 1954; Dunnet, 1955; Gupta, 1956, 1955), the conditional efficiency probabilities account for the "multiplicity" in the rank statistic (e.g., firm 1 is better than firm 2 *and* firm 3 *and*…).

This paper extends the conditional probability statements of Horrace (2005) to calculate not just the conditional probability that each firm is *best* (lowest inefficiency), but also the conditional probabilities that each firm is *any* efficiency rank (best, 2^{nd} best, …, 2^{nd} worst, worst) in the sample. The suite of conditional probabilities provide a complete picture of efficiency in the sample and is informative. To see this, let the sample consist of n firms and let the unconditional distribution of efficiency be the same for each firm (a common assumption). Then, the *unconditional* probability that any firm is a particular efficiency rank is simply $1/n$, an uninteresting result. That is, the unconditional probability of any particular efficiency rank can be characterized by a discrete uniform distribution across firms. Once we condition on the sample data (on the regression residuals), the shape of this distribution across firms becomes less uniform (more informative). It is in this sense that the proposed conditional efficiency probabilities are empirically useful. In fact, our simulations show that when the variance of the one-sided error is small relative to that of the two-sided error (a noisy experiment), the conditional probabilities are close to the unconditional result, $1/n$. As noise decreases, the probability weights of being a particular efficiency rank shift across firms, so the distribution becomes more informative.

Given the suite of conditional efficiency rank probabilities (a partition of the event space that firm *i* is efficiency rank *r*), it is a simple matter to calculate the expected rank for each firm, conditional on the composed errors, evaluated at the residual values. These conditional expected ranks are also useful. Like the *unconditional* efficiency rank probabilities, the unconditional expect rank for each firm is constant across firms. For example, if $n = 5$ and if the unconditional distribution of inefficiency is (again) identical across firms, then the unconditional expected rank for each firm is (1 + … + 5)/5 = 3, an uninteresting result. The *conditional* expected rank, however, varies across firms, and this variability informs our understanding of the efficiency rankings. Continuing the example, if the firm with the highest efficiency score has a conditional expected rank of 1.2 (1 being the best and 5 being the worst), we are much more confident that it is the best firm in the sample than if it has a conditional expected rank of 2.2, and the conditional expected rank of 1.2 is certainly more informative than its unconditional expected rank of 3. Not surprisingly, the informativeness of the conditional expected rank is increasing in the signal to noise ratio in our simulations. Continuing the example, the conditional expected rank of 1.2 for the firm with highest inefficiency score might be from a less noisy experiment than the 2.2 result. In a *very* noisy experiment the same conditional expected rank might be close to 3, the unconditional result. Our simulations also reveal interesting relationships between the skew of the one-sided error and the distribution of the conditional expected ranks across firms.

This paper is organized as follows. The next section presents the parametric frontier model, the conditional efficiency rank probabilities, and the conditional expected rank measure. The model allows for unbalanced panels, a case which has not been treated extensively in previous work on efficiency probabilities. In section 3, a Monte Carlo study demonstrates how the empirical distribution of conditional efficiency rank probabilities and the conditional expected ranks vary with a) the amount of noise in the experiment and b) the skew of the unconditional inefficiency distributions. Section 4 presents an empirical application to vessel efficiency in the US North Atlantic Herring fleet, and section 5 concludes.

2. Conditional Inefficiency Rank Probabilities For Parametric Frontiers

We consider the parametric stochastic frontier model for an unbalanced panel of firms:

$$y_{it} = \alpha + x_{it}\beta - u_i + v_{it}, \quad i = 1,...,n, \; t = 1,...,T_i . \qquad (1)$$

Here, y_{it} is the observed logarithm of output of the i^{th} firm in the t^{th} period, the x_{it} are observed production inputs, the $u_i \geq 0$ are *iid* unobserved errors representing unobserved inefficiency, and the v_{it} are *iid* unobserved errors that cause the efficiency frontier to be stochastic. We assume that the distribution of v_{it} is $N(0,\sigma_v^2)$ and distribution of u_i is the truncation below zero of a $N(\mu,\sigma_u^2)$ random variate.[1] Other distributions for u_i have been considered (e.g., Greene, 1990), but are beyond the scope of what follows. We also require that x_{it}, u_i and v_{it} be independent. Since y_{it} is in log points, firm-level technical efficiency is defined as $TE_i = \exp(-u_i)$. Maximum likelihood estimation of the model's parameters ($\hat{\alpha}$, $\hat{\beta}$, $\hat{\mu}$, $\hat{\sigma}_u^2$ and $\hat{\sigma}_v^2$) is consistent (as $n \to \infty$ or as $T_i \to \infty$).

The model in (1) is fairly flexible. It can represent both Cobb-Douglas and trans-log specifications, and it can be recast as a cost, revenue or profit function. Generalizations for time-varying u_i are plentiful. For example, see Kumbhakar (1990), Cornwell, Schmidt, and Sickles (1990), Battese and Coelli (1992), Lee and Schmidt (1993), Cuesta (2000), Han, Orea and Schmidt (2005), Lee (2005), and Ahn, Lee and Schmidt (2007). Our empirical example in

[1] A fixed effect model is also considered in Schmidt and Sickles (1984).

section 4 involves a more flexible form than in (1), where the marginal products are allowed to vary across groups of firms; the model is estimated using the El-Gamal and Grether (1995, 2000) estimation classification algorithm.

Based on our assumptions in (1), Battese and Coelli (1988) show that the distribution of u_i conditional on the composed error $\varepsilon_{it} = v_{it} - u_i$ is the truncation below zero of a $N(\mu_{*i}, \sigma_{*i}^2)$ random variate with,

$$\mu_{*i} = (-\sigma_u^2 T_i \bar{\varepsilon}_i + \mu\sigma_v^2) / (T_i\sigma_u^2 + \sigma_v^2), \quad \bar{\varepsilon}_i = T_i^{-1}\sum_{t=1}^{T_i}\varepsilon_{it} \text{ and } \sigma_{*i}^2 = \sigma_u^2\sigma_v^2 / (T_i\sigma_u^2 + \sigma_v^2).$$

That is, the conditional density function of u_i is:

$$f(u \mid \varepsilon_i) = \frac{(2\pi\sigma_{*i}^2)^{-1/2}}{1-\Phi(-\mu_{*i}/\sigma_{*i})}\exp\left\{-\frac{(u-\mu_{*i})^2}{2\sigma_{*i}^2}\right\} \text{ with } \varepsilon_i = [\varepsilon_{i1} \quad \ldots \quad \varepsilon_{iT_i}]'.$$

Then the conditional distribution function is:

$$F(u \mid \varepsilon_i) = \int_0^u f(u \mid \varepsilon_i)\, du = \frac{\Phi(\{u-\mu_{*i}\}/\sigma_*) - \Phi(-\mu_{*i}/\sigma_{*i})}{1-\Phi(\mu_{*i}/\sigma_{*i})},$$

where Φ is the cumulative distribution function of a standard normal random variate. Then the conditional mean of u_i is:

$$E(u \mid \varepsilon_i) = \mu_{*i} + \sigma_{*i}\frac{\phi(-\mu_{*i}/\sigma_{*i})}{\Phi(\mu_{*i}/\sigma_{*i})},$$

with ϕ the density of a standard normal random variate.

In principle, population efficiency ranking is in terms of u_i. That is, $u_{[1]} \le u_{[2]} \le \ldots \le u_{[n]}$, so that firm [1] is most efficient in the population, and firm [n] is least efficient. However, u_i is unobserved and cannot be directly estimated, so what is often done is to calculate the vector of residuals $e_i = [e_{i1} \ \ldots \ e_{iT_i}]'$ with $e_{it} = y_{it} - \hat{\alpha} - x_{it}\hat{\beta}$, and estimate inefficiency as

$\hat{u}_i = E(u \mid \varepsilon_i = e_i)$, the conditional mean evaluated at $\varepsilon_i = e_i$ (with $\hat{\mu} = \mu$, $\sigma_u^2 = \hat{\sigma}_u^2$ and $\sigma_v^2 = \hat{\sigma}_v^2$) for each firm. Empirical exercises often include a rank ordering of the $\hat{u}_i$, which serves as a predictor of the ordered u_i conditional on ε_i. If $T_i = T$, so that $\sigma_{*i}^2 = \sigma_*^2$, then the firm rankings based on $\hat{u}_i$ will be identical to the rankings based on $\overline{e}_i$. However, in the case of unbalanced panels, it is possible that the rankings will not be identical, because σ_{*i}^2 causes $\hat{u}_i$ is no longer be monotonic in $\overline{e}_i$.

Based on the distribution of u_i conditional on ε_i, Horrace (2005) calculates the conditional probabilities that firm i is most efficient in the sample, $\Pr(i = [1] \mid \varepsilon_1, ..., \varepsilon_n)$, and least efficient in the sample $\Pr(i = [n] \mid \varepsilon_1, ..., \varepsilon_n)$. The conditional efficiency probabilities are predicted by evaluating them at $\varepsilon_i = e_i$, $i = 1, ..., n$. We generalize those results to calculate the conditional probability that firm i is *any* efficiency rank, *r*, in the sample, $\Pr(i = [r] \mid \varepsilon_1, ..., \varepsilon_n)$. This conditional rank probability is the sum of the probabilities of all possible events where u_i is rank *r* (and there may be many), however it is not necessary to calculate all possible event permutations to determine it. Instead we can start from events in which a particular set of firms are more efficient than i and the remaining firms are less efficient without regard for the rankings of the firms within each set. Consider one such event with firm *i* at rank *r*, and define subsets $N_i^-(r) = \{j : u_j < u_i\}$ and $N_i^+(r) = \{j : u_j > u_i\}$, conditional on $\varepsilon_1, ..., \varepsilon_n$. The conditional probability of this event is,

$$\int_0^\infty f(u \mid \varepsilon_i) \prod_{j \in N_i^-(r)} F(u \mid \varepsilon_j) \prod_{k \in N_i^+(r)} [1 - F(u \mid \varepsilon_k)]\, du .$$

For any rank, except $r = 1$ or $r = n$, there are multiple combinations of ranked firms above and below i which yield the same rank. In fact, there are ${}_{n-1}C_{r-1} = \frac{(n-1)!}{(n-r)!(r-1)!}$ combinations that have firm i at rank r out of n firms. Accordingly, we index the set of firms above and below i for each different combination that produces the same rank as $N_i^{l-}(r)$ and $N_i^{l+}(r)$, $l = 1,..., {}_{n-1}C_{r-1}$. Then the conditional efficiency rank probability for rank r out of n is,

$$\Pr(i = [r] \mid \varepsilon_1,...,\varepsilon_n) = \sum_{l=1}^{{}_{n-1}C_{r-1}} \int_0^{\infty} f(u \mid \varepsilon_i) \prod_{j \in N_i^{l-}(r)} F(u \mid \varepsilon_j) \prod_{k \in N_i^{l+}(r)} [1 - F(u \mid \varepsilon_k)]\, du\,, \quad (2)$$

$i = 1,...,n$, $r = 1,...,n$. When $r = 1$ or $r = n$ these reduce the conditional efficiency probabilities of Horrace (2005). The n^2 probabilities in (2) can be predicted by evaluating them at $\varepsilon_i = e_i$, $i = 1,...,n$ (with $\hat{\mu} = \mu$, $\sigma_u^2 = \hat{\sigma}_u^2$ and $\sigma_v^2 = \hat{\sigma}_v^2$). It is not difficult to generate computer algorithms for efficient calculation of these probabilities. When n is large, numerical calculation of the probabilities may be difficult, but they could certainly be estimated using resampling techniques.

If we substitute the unconditional density function, $f(u)$, and distribution function, $F(u)$, for the conditional density function and distributions (respectively) in (2), then it is clear that $\Pr(i = [r]) = \Pr(j = [r])$ with $\sum_i \Pr(i = [r]) = 1$, so that $\Pr(i = [r]) = 1/n$. This argument hinges on the unconditional draws of u (over i) being identically distribted. Obviously, if the unconditional distribution of the u_i varies over i (e.g., Battese and Coelli, 1995), then the unconditional $\Pr(i = [r])$ would *not* equal $1/n$ in general, and would be a function of the parameters of the underlying unconditional distributions. Also, if σ_v^2 is large relative to σ_u^2, then realizations of ε_i contain relatively little information about u_i, so that the conditional distribution of u_i is close to its unconditional distribution. Therefore, when σ_v^2 is large relative to σ_u^2, the probabilities in (2) will be close to $1/n$.

Of course reporting the n^2 probabilities in (2) in an empirical exercise may be impractical. However, much of the pertinent information contained in the $r = 1,...,n$ conditional rank probabilities for a firm can be summarized with its conditional expected rank statistic,

$$\rho_i = \sum_{r=1}^{n} r \Pr(i = [r] \mid \varepsilon_1,...,\varepsilon_n) \in [1,n], \tag{3}$$

$i = 1,...,n$. This measure is an alternative way to characterize efficiency ranks that accounts for multiplicity in the rank statistic through the probabilities in (2). Again, it can be predicted by evaluating ε_i at the values of e_i for every firm. It also responds to the relative magnitudes of the signal (σ_u^2) and noise (σ_v^2) in the same way as the probabilities in (2). In a particularly noisy setting, the conditional rank probabilities in (2) are approximately equal to $1/n$, and the conditional expected rank will be approximately equal across firms. In this sense (2) and (3) provide information on one source of uncertainty in the efficiency ranks that the conditional means, $E(u \mid \varepsilon_i)$, do not.[2] Of course, the conditional means, the conditional rank probabilities, and the conditional expected rank are all different measures, so comparisons of their abilities to serve as substitutes should not be overstated.

All of the different characterizations of inefficiency (and their relative rankings) are evaluated at $\varepsilon_i = e_i$. Therefore, they all ignore estimation error, which (of course) is asymptotically negligible. Nonetheless, it may be important in finite samples. For the conditional means, there are ways to address the issue. Simar and Wilson (2009) and Wheat, Smith and

[2] One could supplement the conditional means with the conditional prediction intervals of Horrace and Schmidt (1996), to judge how much the marginal distributions overlap. The degree of overlap may correspond to the extent to which the conditional probabilities and expected ranks are close to their *unconditional* counterparts, but this might be highly subjective and (perhaps) lead to an inaccurate assessment of the nature of efficiency in the population.

Greene (2013) recommend resampling techniques to incorporate estimation error into confidence intervals on technical inefficiency. Resampling techniques could certainly be employed to assess the effects of estimation error on the conditional expected ranks. The procedure to do so would be straightforward, but this is not the focus of the evaluation presented below.

3. Monte Carlo Study

We use a series of simulations to demonstrate properties of the conditional expected rank statistic. For simplicity, we always set $T_i = 1$. As equation 2 shows, the conditional rank probabilities for each firm depend on the conditional distributions, $f(u \mid \varepsilon_i)$, which themselves depend on three parameters μ, σ_u^2 and σ_v^2. First, we follow standard simulation practice for stochastic frontier models (e.g., Olson, Schmidt and Waldman, 1980), and explore how statistical noise, σ_v^2, affects the empirical distribution of the conditional rank probabilities in (2) and the conditional expected ranks in (3). To this end we consider $\sigma_v^2 = \{0.01,\ 0.1,\ 1,\ 10\}$ for fixed $V(u)$. The point is that increasing noise should degrade the efficiency rank probabilities' ability to accurately detect the true rank of any firm, so that the conditional expected ranks are increasingly uninformative. Second, Feng and Horrace (2012) show that the skew of the inefficiency distribution can also confound our detection of firm ranks at different ends of the order statistic in different ways.[3] If the inefficiency distribution is "mostly stars" having many firms in the left tail ($u_i \cong 0$ with high probability), then it is difficult to differentiate the individual ranks of these highly efficient firms (low [r] firms). Conversely, if the inefficiency distribution is "mostly dogs" having fewer firms in the left tail ($u_i \cong 0$ with low probability),

[3] Feng and Horrace are only concerned with detecting the best firm. We want to detect the rank of all firms.

then it is easier to differentiate the individual ranks of these highly efficient firms.[4] The amount of relative mass in one tail of a distribution affects the skew of the distribution. Therefore, our second interest is in seeing the effects of distributional skew (for a fixed variance) on the conditional rank probabilities and the conditional expected ranks. To do this we select values μ and σ_u^2 that hold the variance constant at $V(u) = 0.36$ (the variance of a standard normal random variable truncated at zero) but produce skewnesses of 0.5, 1.0, and 1.5 respectively.[5] These values are listed in Table 1.

The different combinations of parameters (μ , σ_u^2 , σ_v^2) yield a total of 12 separate exercises (four exercises for each of three skew levels). In each exercise we use a total of 5,000 replications. We use a modest number of firms, $n = 5$, to reduce the computational burden in (2) and simplify exposition.[6] We ignore the frontier specification and simulate the model: $\varepsilon_i = v_i - u_i$, so we are implicitly assuming that the production function is known. Our interest is not to understand how well the stochastic frontier model in (1) can be estimated, for this is widely known (e.g., Olson, Schmidt and Waldman, 1980). It is simply to demonstrate the empirical utility of the proposed conditional rank probabilities and the conditional expected rank statistic, and to examine their responses to changes in noise and skew.

Results are shown in Figures 1, 2, and 3 for skew equal 0.5 (low), 1.0 (medium), and 1.5 (high), respectively. We couch our discussion on the effects of changes in σ_v^2 in terms of Figure 1 (low-skew, $Skew(u) = 0.5$), but it could equally apply to Figures 2 and 3. To achieve $Skew(u) = 0.5$ while holding $V(u) = 0.36$, Table 1 shows that we select $\mu = 0.89$, $\sigma_u^2 = 0.52$ for

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[4] The nomenclature "mostly stars and dogs" is due to Qian and Sickles (2008).
[5] The skew of a truncated normal is necessarily positive. We use the "standardized" definition of skewness where the 3rd central moment is divided by the third power of the standard deviation.
[6] Again, the probabilities in (2) could be easily simulated for large *n*, but for the purposes of illustration, small *n* is sufficient.

the results in Figure 1. The figure contains four panels, corresponding to each of four different values of noise, $\sigma_v^2 = \{0.01,\ 0.1,\ 1,\ 10\}$. Each panel is read similarly. Consider the upper-left panel of Figure 1 where $\sigma_v^2 = 0.01$. For each of 5 firms we have the average of the conditional rank probabilities computed in each replication: probability of rank 1 (dark blue), probability of rank 2 (red), probability of rank 3 (green), probability of rank 4 (purple), and probability of rank 5 (light blue). The population ranks are assigned in each replication such that $i = [i]$. That is, by design firm 1 is 1st most efficient in the sample, firm 2 is 2nd most efficient in the sample, …, and firm 5 is least efficient in the sample for any of our 5,000 Monte Carlo draws. These firm numbers, *i*, are along the x-axis, and the average conditional rank probabilities, $\Pr(i = [r] \mid \varepsilon_1, ..., \varepsilon_n)$, are along the y-axis on the graph. Consider $\Pr(i = [1] \mid \varepsilon_1, ..., \varepsilon_n)$, the dark blue series. Based on $\sigma_v^2 = 0.01$, the probability that firm 1 is rank 1, $\Pr(1 = [1] \varepsilon_1, ..., \varepsilon_n)$, is 0.820; the probability that firm 2 is rank 1, $\Pr(2 = [1] \mid \varepsilon_1, ..., \varepsilon_n)$, is 0.151; the probability that firm 3 is rank 1, $\Pr(3 = [1] \mid \varepsilon_1, ..., \varepsilon_n)$, is 0.026; and $\Pr(4 = [1] \mid \varepsilon_1, ..., \varepsilon_n) = \Pr(5 = [1] \mid \varepsilon_1, ..., \varepsilon_n) = 0$. The probabilities for the other series are read from the graph similarly. For example, for $\sigma_v^2 = 0.01$ we have: $\Pr(2 = [2] \mid \varepsilon_1, ..., \varepsilon_n) = 0.673$ (red), $\Pr(3 = [3] \mid \varepsilon_1, ..., \varepsilon_n) = 0.662$ (green), $\Pr(4 = [4] \mid \varepsilon_1, ..., \varepsilon_n) = 0.716$ (purple), and $\Pr(5 = [5] \mid \varepsilon_1, ..., \varepsilon_n) = 0.867$ (light blue). For the lowest noise experiment ($\sigma_v^2 = 0.01$), we see that the analysis is better at correctly detecting firms with higher true [*r*] than firms with lower true [*r*] (compare $\Pr(1 = [1] \mid \varepsilon_1, ..., \varepsilon_n)$ < $\Pr(5 = [5] \mid \varepsilon_1, ..., \varepsilon_n)$ and $\Pr(2 = [2] \mid \varepsilon_1, ..., \varepsilon_n)$ < $\Pr(4 = [4] \mid \varepsilon_1, ..., \varepsilon_n)$), and this is always the case for our simulations (regardless of skew), because the distribution of *u* will always have a thinner right tail (where $[r]$ is large) than left tail (where $[r]$ is small) as the skew of a truncated normal is always positive. However, the low skew (0.50) of the Figure 1 simulations means that the distribution is relatively symmetric, so we shall see that differences in the ability of the conditional rank probabilities to accurately detect high and low ranked firms will become even more stark as we increase the skew (and increase uncertainly over which firms have lower true [*r*]). See Figures 4, 5 and 6 for a typical empirical inefficiency distribution for each of our three levels of skew: 0.5, 1.0 and 1.5, respectively. Each figure is a kernel density plot using a Gaussian kernel, a Silverman-type bandwidth selection rule, and no boundary-bias correction.

Continuing with the low-skew results of Figure 1, as we increase $\sigma_v^2 = \{0.01,\ 0.1,\ 1,\ 10\}$, the empirical distribution of the efficiency probabilities become more uniform (and less informative). However, we also see in the four panels of Figure 1 that it is always the case that $\Pr(1=[1]\,|\,\varepsilon_1,...,\varepsilon_n) < \Pr(5=[5]\,|\,\varepsilon_1,...,\varepsilon_n)$, even in the noisiest ($\sigma_v^2 = 10$) panel. Both of these empirical phenomena remain as we increase the skew (asymmetry) of the distribution of *u* to 1.0 and to 1.5 in Figures 2 and 3, respectively (while holding $V(u)$ constant). Looking across the figures we see the effect. Consider the lowest noise panel (upper left panel) in Figures 1, 2 and 3. As the skew increases across Figure 1, 2 and 3, $\Pr(1=[1]\,|\,\varepsilon_1,...,\varepsilon_n)$ is decreasing (0.820, 0.754 and 0.699, respectively), while $\Pr(5=[5]\,|\,\varepsilon_1,...,\varepsilon_n)$ is slightly increasing (0.867, 0.873 and 0.879, respectively). In the words of Qian and Sickles (2008), when the conditional distribution of *u* has "fewer stars" (low skew of Figure 4) it is easier to detect stars, $\Pr(1=[1]\,|\,\varepsilon_1,...,\varepsilon_n) = 0.82\,0$, than when there are "mostly stars" (high skew of Figure 6), $\Pr(1=[1]\,|\,\varepsilon_1,...,\varepsilon_n) = 0.69\,6$. These are inferential insights that the conditional means, $E(u\,|\,\varepsilon_i)$, would not uncover. These are also manifest in the conditional expected ranks which we now consider.

Once the conditional rank probabilities are calculated for each firm at each rank, calculation of the conditional expected ranks of equation 3 is straight-forward. The distributions of conditional expected rank (for each simulation run in Figures 1, 2 or 3) are contained in Tables 2, 3 and 4 (respectively). The utility of the conditional expected ranks is immediately obvious. First, the extent to which noise affects $\Pr(i=[r]\,|\,\varepsilon_1,...,\varepsilon_n)$ is clear. Consider the first panel ($\sigma_v^2 = 0.01$) of Table 2. The difference between the true rank of firm 1 (first column) and the average conditional expected rank (second column) is relatively small $(1-1.21=-0.21)$, but this difference is increasing in magnitude as the we read down the panels and the level of noise increases: $1-1.71=-0.72$, $1-2.66=-1.66$, and $1-2.96=-1.96$. These qualitative results are true for all firms (true rank) and for all levels of skew (Tables 2 thru 4). Obviously, as noise increases the conditional expected ranks are moving toward the *unconditional* expected rank, 3 (bottom panel in Table 2), which reflects the nearly uniform distribution of the conditional efficiency probabilities (bottom panel of Figure 1). Also, the response of the quantiles of the expected ranks (columns with the heading "Quantiles") to increasing noise is clear: noise tends to push extreme quantiles (and their surrounding probability mass) to the center of the empirical distribution of the conditional expected ranks. Second, the effect of skew is clear across the tables. Consider firms 1 and 5 in the first (low noise) panels of Tables 2 – 4. For firm 1, the difference between its true rank [1] and the average conditional expected rank is increasing in magnitude (-0.21, -0.30, -0.39) as skew increases (0.5, 1.0, and 1.5) across Tables 2, 3 and 4, respectively, while the same differences for firm 5 are non-increasing in magnitude across the tables (0.16, 0.15, 0.15). Again, this reflects the fact that as skew increases (and there are relatively more stars in the inefficiency distribution) it is harder to detect "stars" in the left tail of the inefficiency distribution than to detect "dogs" in the right tail. Third, the conditional expected ranks are a convenient normalization of relative efficiency. Notice that the normalization is pegged to both ends of the true order statistic (1 and 5), such that $\rho_i \in [1, n]$. Compare this to the traditional predictor of $TE_i = \exp(-u_i)$,

$$\widehat{TE}_i = E[\exp(-u) \mid \varepsilon_i] = \frac{1 - \Phi\left[\sigma_* - \left(\frac{\mu_{*i}}{\sigma_*}\right)\right]}{1 - \Phi\left(\frac{-\mu_{*i}}{\sigma_*}\right)} \exp\left\{-\mu_{*i} + \frac{1}{2}\sigma_*^2\right\}, \tag{4}$$

evaluated at $\varepsilon_i = e_i$. (See Jondrow, Lovell, Materov and Schmidt, 1982.) This absolute predictor normalizes efficiency predictions to the unit interval, $\widehat{TE}_i \in (0,1)$. Therefore, linear renormalizations of expected rank, like $1-(1-\rho_i)/n$, can be thought of as alternatives to the $\widehat{TE}_i$ normalization. However, the former is measured on a relative (within sample) scale, while the latter is measured on an absolute (out of sample) scale.

4. Empirical Example

To illustrate our results on expected ranks we revisit the empirical exercise in Flores-Lagunes, Horrace and Schnier (2007), who estimate a stochastic production frontier for an unbalanced panel for $n = 39$ vessels from the US North Atlantic Herring fleet (2000-2003). They specify a heterogeneous production function and use the El-Gamal and Grether estimation classification algorithm (El-Gamal and Grether, 1995, 2000) to classify the fleet into three production tiers. See Flores-Lagunes, Horrace and Schnier (2007) for a complete discussion of the data, the production function and the estimation algorithm.[7] Suffice it to say that vessel output is total catch (tons) and inputs are things like vessel size (tons), hours at sea, and crew size. The estimation yields μ_{*i} and σ_{*i}^2 for each vessels. That is, each vessel's conditional inefficiency distribution is a $N(\mu_{*i}, \sigma_{*i}^2)$ truncated at zero.

The North Atlantic Herring fleet consists of two technologies: trawlers and "purse seiners." While in motion, trawling vessels drag large nets to take catch. A purse seine is a large net that is dropped toward the ocean floor while the vessel is at rest. The gear encircles catch as it is hauled back up to the boat. Vessels use only one of these technology (there are costs to refitting vessels with the different gear types). The El-Gamal and Grether estimation classification algorithm stratifies the fleet into three production tiers, where each tier has separate marginal product estimates (estimates of α and β in equation 1). The first and second tiers consist exclusively of trawlers and the third tier consists of a mix of trawlers and purse seiners. Efficiency is characterized within (and not across) each production tier.

The estimates of μ_{*i} and σ_{*i}^2 for the five most efficient vessels in each production tier (tier, 1, tier, 2 and tier 3) are reproduced in the second and third columns of Tables 5, 6, and 7, respectively. The first column contains the unique vessel numbers from the Flores-Lagunes, Horrace and Schnier analysis. The fourth column contains traditional technical efficiency predictors $\widehat{TE}_i = E[\exp(-u) \mid \varepsilon_i]$ from (4) evaluated at $\varepsilon_i = e_i$, and the results in Tables 5-7 are ranked on this value. The last column contains the conditional expected ranks, ρ_i, in (3) for the five most efficient vessels in each tier. The results are compelling. Starting with Table 5, we see that the conditional expected ranks only range in value from 2.272 (vessel 14) to 3.479 (vessel

[7] The results of the estimation are not reproduced here to focus attention on the different characterization of efficiency ranks and the importance of the proposed conditional expected rank statistic.

2), indicating a fairly noisy analysis. Had this been a particularly precise empirical exercise the range would be closer to 1 to 5. One cannot infer this noisiness directly from the $\widehat{TE}_i$, but it is reassuring to see that the predictor only ranges from 0.846 (vessel 14) to 0.928 (vessel 2), as well. Also the vessel rankings based on $\widehat{TE}_i$ match those based on ρ_i in Table 5. However, Table 6 tells a slightly different story. The range of the conditional expected ranks is tighter than in Table 5 and only ranges from 2.535 (vessel 21) to 3.533 (vessel 7), so the analysis is more noisy, however, the ranks based on $\widehat{TE}_i$ are different than those based on ρ_i. In particular, the ranks of vessel 13 and 12 are reversed, and it is clear why this is the case: the truncated normal distributions (upon which they are based) are vastly different in shape even though the means are approximately the same. That is, $E[\exp(-u) \mid \varepsilon_{12} = e_{12}] \cong E[\exp(-u) \mid \varepsilon_{13} = e_{13}] = 0.863$, however the means and variance of the distribution before truncation are extremely different. Compare $\mu_{*13} = 0.135$ to $\mu_{*12} = -0.518$ and $\sigma^2_{*13} = 0.009$ to $\sigma^2_{*12} = 0.125$. Vessel 12 has more mass near zero in the distribution of $f(u \mid \varepsilon_i)$, which is better captured by the expected rank. This underscores the danger of using the conditional means alone to make inferences on ranked technical efficiency scores: they simply do not capture the multiplicity that underlies the ranking.

Table 7 tells an even more nuanced story. The range of the expected ranks are wider: from 1.395 (vessel 3) to 4.151 (vessel 34), so this is the most precise rank statistic of the three, yet there is still some switching in the ranks based on ρ_i. In particular the ranks of vessels 33 and 16, and of 30 and 34 are switched. Notice that the differences in $\widehat{TE}_i$ for these vessel pairs are not that large, so it is really not surprising that the additional information provided by the conditional rank probabilities might switch the ranking. (This was even more so the case for vessels 13 and 12 in Table 6.) However, it underscores the importance of taking into account multiplicity and noise in any ranking exercise.

Table 7 also includes an additional column with the heading "Trimmed ρ_i." Sometimes empiricists will calculate ranked $\widehat{TE}_i$ and, in an ad hoc manner, determine that firms with the highest values are "super-efficient." Super-efficient firms are then dropped from the sample, and the remaining efficiency scores are discussed without re-estimating the production function. Obviously this procedure has no effect on the individual efficiency scores, $\widehat{TE}_i$. It does, however, have implications for the conditional expected rank. In the last column of Table 7, we trim the most efficient vessel (vessel 3) based on its efficiency score, $\widehat{TE}_3 = 0.971$. The rationale is that the distance between its score and the second most efficient vessel, $\widehat{TE}_{33} = .934$, is the largest among the most and second most efficient vessels across Tables 5 – 7. In doing so, we have deemed vessel 3 to be "super-efficient." Based on the $\widehat{TE}_i$ scores among the remaining vessels, all the (implied) ranks move up by 1. Vessel 33's rank moves from 2 to 1, vessel 16's rank moves from 3 to 2, etc.. The $\widehat{TE}_i$ scores are *marginal* predictors of efficiency, so all the changes in (implied) ranks are uniform. By contrast, the conditional rank probabilities and, consequently, the conditional expected ranks account for ranking multiplicity and are, therefore, affected in a non-uniform way by dropping a firm (or firms) from the rank statistic. This can be seen in the last two columns of Table 7. The improvement in conditional expected ranks are 0.855, 0.796, 1.000, and 0.954 for vessels 33, 16, 30 and 34, respectively. Consequently, the ordering of the vessels based on ρ_i and "trimmed ρ_i" are different. The ordering based on ρ_i is 16, 33, 34 and 30 (best to worst), and the ordering based on "trimmed ρ_i" is 33, 16, 34 and 30. This switching of the expected ranks of vessels 33 and 16 underscores the fallacy of trimming "super-efficient" firms without a statistical basis that takes into account noise and the multiplicity implied by the order statistic.

5. Conclusions

We extend the current literature on ranked efficiency scores by defining and proposing the use of conditional efficiency rank probabilities and conditional expected efficiency ranks as a means to provide improved insight into efficiency score rankings. Although our model was fairly restrictive, our results can be more broadly applied than this might indicate. Indeed, there is a broad class of parametric models that yield conditional efficiency distributions that are truncated normal and to which our results directly apply. Additionally, even if the resulting conditional distributions are *not* truncated normals, our results (and the results of Horrace, 2005) can be adapted to these cases.

We demonstrated nuances of the proposed measures with a Monte Carlo Study. The conditional expected ranks responded in predictable ways to the inherent noisiness of a statistical exercise and to the skewness of the underlying efficiency distribution. While it is generally ignored in empirical applications of the stochastic frontier model, skew is a *very* important moment to consider in drawing conclusions on ranked efficiency predictors. Our empirical example based on fishing vessels underscores the importance of taking into account multiplicity and noise in any ranking exercise, and the empirical relevance of the conditional rank probabilities and the conditional expected ranks is made clear. We also demonstrated that *ad hoc* trimming of “super-efficient” firms can lead to incorrect inference on the implied ranks of the remaining firms. It may not be wise to uniformly shift the remaining firms up in the efficiency order statistic.

One potential area of future research is that the OLS residuals are necessarily correlated, so while the conditional inefficiency distribution based on the true regression errors are independent, these distributions based on the residuals are technically not so. It would be interesting to see if analytic solutions were forthcoming and the correlation of the residual could be estimated or approximated. It may also be worthwhile to considering resampling techniques to calculate conditional expected rank statistics, so large *n* will not be problematic, and to estimate confidence intervals for the conditional expected ranks, so that the usual assumption $\hat{\beta} = \beta$ can be relaxed.

It may also be fruitful to explore higher moments of the conditional rank distribution for each firm. We have discussed the conditional expectation of the distribution, but it may be worthwhile to consider the conditional variance of the rank of each firm. Calculating the variance, and any higher moments, would be a straightforward exercise based on the conditional rank probabilities that we have presented. One might speculate that firms with high conditional probabilities of being best and worst would have higher conditional variance of their rank distribution than those with high probability of being in the center of the efficiency rank statistic. The best and worst firms will have more weight in one tail of their conditional rank distributions than firms with higher probability at the median efficiency ranks. However, this remains to be seen.

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Table 1: Simulation Parameters and the Resulting Truncated Moments

Underlying $N(\mu, \sigma_u^2)$		Moments of the Truncated Distribution			Range of
Mean(μ)	Variance(σ_u^2)	Mean	Variance	Skewness	(σ_v^2)
0.89	0.52	1.04	0.36	0.50	0.01 to 10
0.00	1.00	0.80	0.36	1.00	0.01 to 10
-3.00	2.83	0.67	0.36	1.50	0.01 to 10

Table 2. Average and Quantiles of Conditional Expected Rank, Low Skew, Skew(u) = 0.50.

$\sigma_v^2 = 0.01$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.21	1.00	1.00	1.04	1.30	1.70
2	2.05	1.53	1.85	2.00	2.23	2.64
3	2.98	2.36	2.77	3.00	3.19	3.55
4	3.91	3.37	3.76	3.99	4.06	4.37
5	4.84	4.45	4.83	4.99	5.00	5.00

$\sigma_v^2 = 0.1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.72	1.12	1.29	1.58	1.99	2.54
2	2.31	1.50	1.81	2.20	2.73	3.24
3	2.93	1.99	2.43	2.93	3.44	3.87
4	3.63	2.64	3.17	3.71	4.13	4.49
5	4.41	3.56	4.12	4.59	4.88	4.98

$\sigma_v^2 = 1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.66	2.10	2.32	2.61	2.95	3.27
2	2.82	2.23	2.48	2.78	3.13	3.45
3	2.97	2.37	2.61	2.96	3.30	3.60
4	3.15	2.51	2.79	3.13	3.50	3.84
5	3.40	2.71	3.02	3.38	3.77	4.11

$\sigma_v^2 = 10$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.96	2.75	2.84	2.95	3.07	3.17
2	2.98	2.77	2.87	2.98	3.09	3.19
3	3.00	2.79	2.88	3.00	3.11	3.21
4	3.02	2.80	2.90	3.02	3.13	3.24
5	3.04	2.82	2.93	3.04	3.15	3.26

Table 3. Average and Quantiles of Conditional Expected Rank, Medium Skew, Skew(u) = 1.0.

$\sigma_v^2 = 0.01$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.30	1.00	1.01	1.14	1.48	1.83
2	2.04	1.44	1.75	2.00	2.26	2.67
3	2.94	2.26	2.70	2.99	3.17	3.58
4	3.87	3.26	3.73	3.99	4.03	4.34
5	4.85	4.43	4.85	5.00	5.00	5.00

$\sigma_v^2 = 0.1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.86	1.20	1.41	1.74	2.19	2.65
2	2.30	1.53	1.81	2.18	2.72	3.21
3	2.86	1.94	2.34	2.83	3.34	3.83
4	3.55	2.50	3.06	3.65	4.07	4.47
5	4.43	3.48	4.13	4.67	4.93	4.99

$\sigma_v^2 = 1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.65	2.01	2.25	2.60	2.98	3.34
2	2.78	2.12	2.39	2.73	3.14	3.52
3	2.95	2.26	2.53	2.90	3.32	3.70
4	3.14	2.39	2.70	3.12	3.55	3.94
5	3.48	2.65	3.03	3.49	3.95	4.33

$\sigma_v^2 = 10$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.95	2.67	2.79	2.95	3.10	3.24
2	2.97	2.69	2.82	2.97	3.12	3.26
3	3.00	2.72	2.84	2.99	3.14	3.29
4	3.02	2.73	2.85	3.01	3.17	3.32
5	3.06	2.76	2.90	3.06	3.22	3.36

Table 4. Average and Quantiles of Conditional Expected Rank, High Skew, Skew(u) = 1.5.

$\sigma_v^2 = 0.01$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.39	1.00	1.04	1.26	1.60	1.95
2	2.04	1.43	1.70	2.00	2.30	2.73
3	2.88	2.14	2.58	2.97	3.14	3.55
4	3.84	3.20	3.67	3.98	4.01	4.31
5	4.85	4.46	4.87	5.00	5.00	5.00

$\sigma_v^2 = 0.1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	1.96	1.27	1.50	1.85	2.32	2.80
2	2.32	1.54	1.82	2.22	2.71	3.25
3	2.79	1.90	2.26	2.75	3.27	3.76
4	3.48	2.44	2.95	3.56	4.03	4.42
5	4.44	3.45	4.13	4.69	4.95	5.00

$\sigma_v^2 = 1$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.63	1.90	2.17	2.55	3.00	3.48
2	2.76	2.00	2.29	2.70	3.18	3.62
3	2.90	2.12	2.41	2.85	3.33	3.82
4	3.14	2.29	2.63	3.11	3.62	4.06
5	3.57	2.60	3.03	3.58	4.13	4.55

$\sigma_v^2 = 10$		Quantiles				
True Rank	**Average**	**10%**	**25%**	**50%**	**75%**	**90%**
1	2.93	2.51	2.69	2.91	3.15	3.39
2	2.96	2.52	2.72	2.94	3.19	3.42
3	2.98	2.56	2.73	2.97	3.21	3.44
4	3.02	2.59	2.78	3.01	3.26	3.48
5	3.10	2.64	2.85	3.09	3.34	3.58

Table 5. Heterogeneous Vessel Efficiency Results Tier 1, Sorted on $\widehat{TE}_i$

Vessel i	σ_i^{*2}	μ_i^*	$\widehat{TE}_i$	ρ_i
2	0.001	0.074	0.928	2.272
39	0.013	0.03	0.905	2.698
11	0.011	0.094	0.881	3.157
5	0.032	0.04	0.859	3.394
14	0.064	-0.077	0.846	3.479

Table 6. Heterogeneous Vessel Efficiency Results Tier 2, Sorted on $\widehat{TE}_i$

Vessel i	σ_i^{*2}	μ_i^*	$\widehat{TE}_i$	ρ_i
21	0.011	0.041	0.907	2.535
19	0.039	-0.182	0.903	2.547
13	0.009	0.135	0.864	3.318
12	0.125	-0.518	0.863	3.067
7	0.345	-1.129	0.814	3.533

Table 7. Heterogeneous Vessel Efficiency Results Tier 3, Sorted on $\widehat{TE}_i$

Vessel i	σ_i^{*2}	μ_i^*	$\widehat{TE}_i$	ρ_i	Trimmed ρ_i
3	0.001	0.001	0.971	1.395	----
33	0.002	0.065	0.934	2.528	1.673
16	0.008	0.011	0.930	2.500	1.704
30	0.001	0.193	0.825	4.425	3.425
34	0.023	0.173	0.817	4.151	3.197