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Quantum Resource Theories

Operational Tasks and Information-Theoretic Quantities

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DOCTORAL THESIS

Quantum Resource Theories: Operational Tasks and Information-Theoretic Quantities

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in the

Quantum Information Theory group
Quantum Engineering Centre for Doctoral Training (QE-CDT)
Quantum Engineering Technology Labs (QET Labs)
School of Physics

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Declaration of Authorship

I, Andrés F. DUCUARA, declare that this thesis titled, "Quantum Resource Theories: Operational Tasks and Information-Theoretic Quantities" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- I have acknowledged all main sources of help.
- Work done in collaboration with others is indicated as such.

Signed: ____________________

Date: ____________________

"The most merciful thing in the world, I think, is the inability of the human mind to correlate all its contents. We live on a placid island of ignorance in the midst of black seas of infinity, and it was not meant that we should voyage far. The sciences, each straining in its own direction, have hitherto harmed us little; but some day the piecing together of dissociated knowledge will open up such terrifying vistas of reality, and of our frightful position therein, that we shall either go mad from the revelation or flee from the light into the peace and safety of a new dark age."

H. P. Lovecraft

UNIVERSITY OF BRISTOL

Abstract

School of Physics
Quantum Engineering Centre for Doctoral Training (QE-CDT)

Doctor of Philosophy

Quantum Resource Theories: Operational Tasks and Information-Theoretic Quantities

by Andrés F. DUCUARA

This Thesis deals with a resource-theoretic approach to Quantum Information Theory (QIT), or Quantum Resource Theories (QRTs) for short. In particular, it deals with the identification of *operational tasks* as well as their characterisation by means of *information-theoretic quantities*. These operational tasks can be thought of as exploiting (or harnessing) properties of quantum objects as *resources* and consequently, such properties become of relevant and practical interest. Whilst several different mathematical objects from QIT can be deemed as potential candidates in possession of useful resources for a QRT, this Thesis specifically deals with quantum *states*, quantum *measurements*, and quantum *channels* which are, arguably, amongst the most fundamental objects in QIT. Amongst the properties deemed as resources we have entanglement, incompatibility, Bell-nonlocality, EPR-steering amongst others. The main conceptual contributions of this Thesis are the following:

1. It characterises the operational tasks of quantum state exclusion (QSE) and quantum subchannel exclusion (QScE) in terms of the quantifier of *weight of resource*, and reports on the existence of a general correspondence between i) exclusion-based operational tasks and ii) weight-based resource quantifiers. This correspondence holds for general resources for states and measurements and therefore, evidencing a fundamental correspondence that extends across general QRTs.

2. It opens a multi-object paradigm for QRTs, by considering composite QRTs as well as multi-object operational tasks, where multiple quantum objects (states and measurements) can be used in conjunction for the benefit of multi-object operational tasks like multi-object subchannel discrimination and exclusion.

3. It introduces a QRT for Buscemi nonlocality, develops its operational significance, and derives relationships between such form of nonlocality and entanglement as well as non-classical teleportation.

4. It imports concepts from the theory of economics into QIT, specifically, it imports the concepts of *betting* and *risk-aversion* from Expected Utility Theory (EUT), and introduces operational tasks based on such concepts, as *quantum betting tasks* for short. It provides an operational interpretation of: i) the Rényi entropy, ii) Arimoto-Rényi conditional entropy and ii) Arimoto's mutual information, in terms of such quantum betting tasks.

Overall, the results presented in this Thesis can be seen as lying at the intersection of the three research fields of: i) quantum theory, ii) information theory, and iii) expected utility theory.

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I would now like to thank my office colleagues and collaborators: Patryk Lipka-Bartosik and Tom Purves. It was inspiring to see them judiciously working towards the completion of their own PhDs, and it was equally a pleasure to interact and interchange ideas with them as well. More generally, I would like to thank the quantum information theory group at the University of Bristol, for the incredibly thought-provoking seminars and lunch meetings; Noah Linden, Tony Short, Sandu Popescu, Ashley Montanaro, João Doriguello, Ryan Mann, Alex Moylett, Stephen Piddock, Changpeng Shao, Dominic Verdon, Benjamin Jones, and Jan Lukas Bosse. In particular, I would like to thank my Annual Progress Meeting (APM) panel: Ashley Montanaro and Sandu Popescu, for their guidance, valuable comments, and general advise on how to navigate the academic life.

I thank Cohort 4 (2017/18) of the Quantum Engineering Centre for Doctoral Training (QE-CDT), for all the amazing activities we did together as a cohort during the first year of training, the time in the office, the trip to California, the pubs, and so on: Jake Biele, Johnathan Frazer, Huili Hou, Friederike Jöhlinger, Ankur Khurana, Lana Mineh, David Payne, Ben Sayers, John Scott, Dominic Sulway, and Oliver Thomas. In particular, I thank Oli for introducing me to the city of Bristol and the UK in general. I also thank Ankur, for the amazing company in our trip through Eastern European countries. I'd like to thank my colleagues and flatmates: Ben Sayers, Lana Mineh, Dominic Sulway, and Sebastian Currie, for allowing me to live with them, which made my day-to-day life significantly easier. In particular, I thank Ben for all the fun times playing videogames.

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"I don't know half of you half as well as I should like; and I like less than half of you half as well as you deserve. "

Bilbo Baggins

Contents

Acronyms and Abbreviations

General acronyms	
QIT	**Q**uantum **I**nformation **T**heory
QRT	**Q**uantum **R**esource **T**heory
EUT	**E**xpected **U**tility **T**heory
PMF	**P**robability **M**ass **F**unction
POVM	**P**ositive **O**perator-**V**alued **M**easure
PVM	**P**rojection-**V**alued **M**easure
TP	**T**race **P**reserving
TNI	**T**race **N**on-**I**ncreasing
CP	**C**ompletely **P**ositive
CPTP	**C**ompletely **P**ositive **T**race **P**reserving
CPTNI	**C**ompletely **P**ositive **T**race **N**on-**I**ncreasing
CPP	**C**lassical **P**ost-**P**rocessing
BLP	**B**leuler-**L**apidoth-**P**fister
PBR	**P**usey-**B**arrett-**R**udolph
KL	**K**ullback-**L**eibler
CE	**C**ertainty **E**quivalent
ICE	**I**soelastic **C**ertainty **E**quivalent
RRA	**R**elative **R**isk-**A**version
CRRA	**C**onstant **R**elative **R**isk-**A**version
CR	**C**onditional **R**ényi
SDP	**S**emi-**D**efinite **P**rogramming
CP	**C**onic **P**rogramming
CCC	**C**losed **C**onvex **C**one
Acronyms for operational tasks	
QSD	**Q**uantum **S**tate **D**iscrimination
QSE	**Q**uantum **S**tate **E**xclusion
QSB	**Q**uantum **S**tate **B**etting
nQSD	**n**oisy **Q**uantum **S**tate **D**iscrimination
nQSE	**n**oisy **Q**uantum **S**tate **E**xclusion
nQSB	**n**oisy **Q**uantum **S**tate **B**etting
QCD	**Q**uantum **C**hannel **D**iscrimination
QCE	**Q**uantum **C**hannel **E**xclusion
QCB	**Q**uantum **C**hannel **B**etting
QScD	**Q**uantum **S**ub-**c**hannel **D**iscrimination
QScE	**Q**uantum **S**ub-**c**hannel **E**xclusion
QScB	**Q**uantum **S**ub-**c**hannel **B**etting
QHB	**Q**uantum **H**orse **B**etting
HB	**H**orse **B**etting

Abbreviations

l.h.s.	left hand side
r.h.s.	right hand side
s.t.	such that (or subject to)
a.k.a.	also known as
i. e.	id est (Latin for “that is”)
et al.	et alia (Latin for “and others”)
à la	French for "according to” or “in the manner of"

List of Symbols

General	
$\mathbb{N}, \mathbb{R}, \overline{\mathbb{R}}, \mathbb{C}$	Natural, real, extended real, and complex numbers
log	Logarithm in base 2
Linear algebra	
$\mathbb{H}_A$	Hilbert space of system A
$\mathcal{L}(A)$	Set of linear operators in $\mathbb{H}_A$
$Herm(A)$	Set of Hermitian operators in $\mathbb{H}_A$
$PSD(A)$	Set of positive-semidefinite operators in $\mathbb{H}_A$
$D(A)$	Set of density operators in $\mathbb{H}_A$
$\rho, \sigma, \gamma, ...$	Quantum states
$\mathbb{M}, \mathbb{M}', \mathbb{N}, \mathbb{N}', ...$	Quantum measurements
$X^\dagger$, X^T	The adjoint operator and transpose operator of X
$\mathcal{E}^\dagger$	The adjoint map of $\mathcal{E}$
$\mathbb{1}_A, \mathrm{id}_A$	Identity operator and identity map on A
Tr, Tr_A	Trace and partial trace over system A
$X \geq Y$	$X - Y \in PSD(A)$
Resource theories	
$F, \mathbb{F}, \mathcal{F}$	Sets of: free states, free measurements, and free channels
$\mathbb{UI}, \mathcal{C}$	Sets of: uninformative measurements, constant channels
$R_F(\rho)$	Generalised robustness of resource of a state ρ
$R_{\mathbb{F}}(\mathbb{M})$	Generalised robustness of resource of a measurement $\mathbb{M}$
$W_F(\rho)$	Weight of resource of a state ρ
$W_{\mathbb{F}}(\mathbb{M})$	Weight of resource of a measurement $\mathbb{M}$
w_R^{ICE}	Isoelastic certainty equivalent with risk-aversion R
Information theory	
$X, Y, G, ...$	Random variables
$p_X, q_G, ...$	Probability mass function (PMF)
$p_{XG}, p_{X\|G}$	Joint PMF, conditional PMF
$H_\alpha(X)$	Rényi entropy of order α
$H_\alpha(X, G)$	Rényi joint entropy of order α
$H_\alpha(X\|G)$	Arimoto-Rényi conditional entropy of order α
$I_\alpha(X; G)$	Arimoto-Rényi mutual information of order α
$C_\alpha(p_{G\|X})$	Rényi capacity of order α
$D_\alpha(\cdot\|\|\cdot)$	Rényi divergence of order α
$D_\alpha^{\mathrm{BLP}}(\cdot\|\| \cdot \|\cdot)$	BLP conditional Rényi divergence of order α
$D_\alpha^{\mathrm{C}}(\cdot\|\| \cdot \|\cdot)$	Csiszár conditional Rényi divergence of order α
$D_\alpha^{\mathrm{S}}(\cdot\|\| \cdot \|\cdot)$	Sibson conditional Rényi divergence of order α

Dedicated to Excelenia Osorio and Amanda Paz.

Chapter 1

Introduction

1.0.1 Quantum theory and quantum information theory

The theory of quantum mechanics, or *quantum theory*, is one of the most marvelous theoretical constructions made by humankind. From a conceptual point of view on the one hand, it provides a unifying and mathematically concise framework for understanding the universe at a fundamental level. Specifically, the *standard model* of particle physics, within the framework of quantum field theory, represents a milestone achievement for the natural sciences, which encompasses the catalogue of fundamental particles that make up pretty much everything known in the observable universe, as well as the way that these particles interact with each other via three out of the four known fundamental forces of nature. Whilst the standard model is still far away from being the ultimate theory of everything, it is the most complete account of the physical world that we have so far. From a pragmatic point of view on the other hand, quantum theory led to the so-called *first quantum revolution*, where devices exploiting the laws of quantum mechanical systems were invented, like the transistor and the laser, which consequently led to further developments and applications in areas of science like chemistry, biology, and medicine, but also in areas like engineering and computing, which in turn gave birth to the era of digital information, which has had a huge impact in the economy and culture of our society, and pretty much every aspect of modern human life.

Despite the unquestionable success of quantum theory, it is still unfortunately in possession of gaps and conceptual problems within its own landscape of operation. Although these conceptual problems were rapidly identified since the initial developments of the theory at the beginning of the 20th century, the general consensus within the community is that they still persist to this day. The first and blatantly evident problem is gravity not being covered within the standard model. Admittedly, this could well not be a problem of quantum theory in and of itself, but rather a matter of currently not correctly applying it to the quantisation of gravity, however, until this is fully resolved, it could well also be the case that quantum theory has to be extended/reformulated in a considerably manner, so to peacefully accommodate and coexist alongside gravity. A second and arguably more perturbing problem, is the existence of disagreements regarding the interpretation of quantum theory. Grossly oversimplifying the situation, this stems from the fact that quantum theory is intrinsically a probabilistic theory and, as such, it does not directly need to assume/guarantee the existence of a physical reality, independent of agents acting as observers and this, consequently, has led to the emergence of several different interpretations of the theory.

It was precisely trying to tackle these conceptual problems that a foundational line of research was born at the beginning of the 20th century, nowadays under the label of "quantum foundations", with the milestone Einstein-Podolsky-Rosen (EPR) article [77] in 1935, as well as the correspondence between Einstein and Bohr [105]. Later on, further seminal papers, like the de Broglie-Bohm theory in 1952 [34, 35], kept alive the discussion around the uneasiness regarding these fundamental aspects about the theory. In 1964, Bell introduced a concise theoretical framework where to address these types of questions [19] and, in particular, showed that quantum theory is a nonlocal theory [43]. Since then, great progress has been made in this direction, with experimental demonstrations of Bell-nonlocality [88, 11], as well as the development of additional theoretical results about what quantum theory can or cannot be (no-go theorems), like the Pusey-Barrett-Rudolph (PBR) theorem [181], Frauchiger-Renner theorem [87], amongst others [104, 61, 135]. The steady exploration of fundamental aspects of quantum theory, could potentially help to alleviate the tension within quantum theory and furthermore, shed a light into the unification problem.

Parallel to the progress on purely theoretical quantum foundations, scientists around the 70's and 80's started to think about these conceptual problems, and quantum theory in general, from an information-theoretic perspective and this, resulted in a new wave of practical consequences. Initially, figures like Benioff [20], Manin [147], Deutsch [67], and Feynman [85], addressed the fundamental ideas behind the concept of a *quantum computer*, ideas which would further evolve in subsequent decades and to consolidate into the fields of *quantum computing* and *quantum simulation*. In a similar vein, Wiesner [241], Bennett, Brassard [21], and Ekert [78], pioneered the ideas behind what would later be known as *quantum cryptography*. Moving on into the 90's, the discovery of further quantum algorithms like Bernstein-Vazirani [29], Deutsch-Josza [68], Simon [201], Grover, [98], Shor [198], error correction [197], amongst others, consolidated the field of quantum computing. Furthermore, information-theoretic protocols like entanglement swapping [253], superdense coding [22], teleportation [27], amongst others, established a basis for *quantum communications*. Overall, the amalgamation of all of these information-theoretic areas is nowadays known as *Quantum Information Theory (QIT)*, an interdisciplinary field of research lying at the intersection of physics, mathematics, computer science, and chemistry. Moreover, these interdisciplinary efforts have also involved huge experimental progress along the way, with milestone achievements like loophole-free experimental violations of Bell-inequalities [111], quantum supremacy [10], amongst others [132] and therefore, as a whole, the movement gave birth to what is nowadays known as the *second quantum revolution*, where the general goal is to conceive devices exploiting the properties of quantum-mechanical systems in a "fully quantum manner"[1], and construct devices like quantum processors, quantum computers, quantum sensors, as well as infrastructure connecting such devices in the form of quantum networks like a quantum internet and therefore, as a whole, a whole new ecosystem for these emergent quantum technologies.

Moving on now closer to the main topic of this Thesis, the general mantra behind the resource-theoretic approach to QIT, or *Quantum Resource Theories (QRTs)* for short [57], is precisely to fully embrace such a pragmatic paradigm; from trying to understand the conceptual problems of quantum theory, to a more practical perspective by importing ideas from classical information theory. Specifically, the main general idea is to think about properties of quantum systems as a *resource* for fuelling either algorithms or more generally information-theoretic protocols. Admittedly, this approach has subconsciously been addressed by the part of the community during the

[1]Here we distinguish that in the first quantum revolution devices exploited the laws of quantum theory in a "classical manner" (meaning doing classical computing), whilst in the second revolution, devices are exploiting quantum theory in a "quantum manner".

past few decades but, arguably, this pragmatic approach has only been embraced by the community after seminal resource-theoretic works in the 90's [25, 23], and more fully from the mid 2000's with the emergence of QRTs exploiting specific resources like Bell-nonlocality [17, 233], EPR-steering [89] and several more [57]. In summary, the types of questions being asked have evolved from *what really IS a quantum state?*, to more practical matters as *what quantum states, gates, channels, measurements are useful for?*[2]. Bearing this in mind, QRTs can then be seen as focusing on subjects like: identification and characterisation of operational tasks, development of resource quantifiers (and monotones) via either entropic-based, geometric-based, or witness-based measures, as well as no-go theorems, convertibility amongst objects and resources either in a single-shot, asymptotic, or catalytic manner, and many other subjects with similar *operational* flavour [57]. In this regard, this Thesis in particular introduces a new family of operational tasks which we will later refer to as *quantum betting tasks*, as well as witness-based resource monotones and quantifiers.

Coming back to QIT, and comparing it with its direct predecessors, one can argue that QIT is still a relatively young field and therefore, it is important to keep unveiling, exploiting, and strengthening the links between the theory of quantum and that of classical information theory. Even though this Thesis mostly dwells within QIT, it is worthwhile point out that information theory can still be approached from a purely classical point of view, which is a whole endeavour in its own right and therefore, it is important to be up to date with the developments in the area, as well as with the possible synergies between the classical and the quantum domains. This serves as a small introduction to address some of points in classical information theory which are going to be of relevance for this Thesis.

1.0.2 Classical information theory

The field of (classical) information theory, similarly to that of quantum theory, can also be dated back to the beginning of the 20th century and, together with the developments from the first quantum revolution in the form of transistors, processors, and alike, eventually led to the current era of digital information. Great efforts have been invested over the past century to mathematically formalise operational tasks such as computation and communication, as well as to describe/characterise such phenomena by means of *information-theoretic quantities*, like entropies and capacities. From a very oversimplified theoretical approach, classical information theory can then be thought of as applied probability theory, where the fundamental objects of study are such information-theoretic quantities. Classical information theory is still, after roughly a century, a very active and fruitful area of research.

Regarding information-theoretic quantities, the *Kullback-Leibler (KL) divergence* (also known as the Kullback-Leibler relative entropy) emerges as a central object of study [131]. The importance of this quantity is in part due to the fact that it acts as a parent quantity for many other quantities, such as the Shannon entropy, conditional entropy, conditional divergence, mutual information, and the channel capacity [63]. Within this classical framework, it has also proven fruitful to consider *Rényi-extensions* of these quantities [192]. In particular, there is a clear procedure for how

[2]It is worth pointing out here that, this change of perspective is not effectively solving any of the conceptual problems mentioned at the beginning, which we all want them to be solved but, in the meantime, and until we get a powerful insight or appropriate tools, we can still think in pragmatic terms, and try to further exploit the concepts and techniques of QIT to further help develop quantum technologies. Hopefully, one day we can have a full understanding of the true inner workings of the theory and consequently, of the devices that harness such quantum properties.

to define the Rényi-extensions of both Shannon entropy and KL-divergence, which are known as the Rényi entropy and the Rényi divergence, respectively [192, 229]. Interestingly however, there is yet no general consensus within the community as to what is the “proper" way to Rényi-extend other quantities. As a consequence of this, there are several different candidates for Rényi conditional entropies [84], Rényi conditional divergences [33], and Rényi mutual information measures [232]. The latter quantities are also known as measures of dependence [33] or α-mutual information measures [232]. In particular, important for this Thesis are the mutual information measures proposed by Sibson [199], Arimoto [8], Csiszár [64], as well as a recent proposal independently derived by Lapidoth-Pfister [133], and Tomamichel-Hayashi [223]. It is known that these mutual information measures (with the exception of Arimoto’s) can be derived from their respective conditional Rényi divergence [33] and therefore, this relationship can be addressed as a *mutual information-divergence* correspondence. Many of these information-theoretic measures are going to be relevant for the main findings reported in this Thesis.

Overall, similarly to quantum theory, information theory is also intrinsically a probabilistic theory and therefore, we see that probability theory is, at an abstract level, a common background framework between these two major research fields. We now address a third field which is also based on probability theory, and which is also going to be relevant in this Thesis, but that now emerges from the economic sciences.

1.0.3 Expected utility theory

A milestone for the economic sciences is the work of Neumann and Morgenstern in 1944 [157], where they laid out a mathematically rigorous foundation for the treatment of the behaviour of rational agents when dealing with good or services, in a general theory of games and economic behaviour. This general framework of operation is nowadays known as *expected utility theory* (EUT), and it has been the bread and butter, so to speak, for economists since the middle of the 20th century. In particular, central to the main results of this Thesis is the concept of *risk-aversion*; the behavioural tendency of rational agents to have a preference one way or another for guaranteed outcomes versus uncertain outcomes. This concept remains of great research interest in the economic sciences, with various Nobel prices having been awarded to the understanding and implications of this concept [14]

In general, the concept of *risk aversion* is a ubiquitous characteristic of rational agents and, as such, it naturally emerges as a subject of study in various different areas of knowledge such as: the economic sciences [76], biology and behavioural ecology [188, 254], and neuroscience [129, 83, 221]. Intuitively, a gambler spending money on bets with the hope of winning big, can be seen as an individual taking (potentially unnecessary) risks, in the eyes of a more conservative gambler. One of the challenges that economists have tackled, since roughly the second half of the previous century, is the incorporation of the concept of risk aversion into theoretical models describing the behaviour of rational agents, as well as its quantification, and exploitation of its descriptive power [76].

The concept of risk was first addressed within theoretical models by Bernoulli in 1738 (translated into English by Sommer in 1954) [28]. Later on, the theory of expected utility, formalised by von Neumann and Morgenstern in 1944 [157], provided a framework within which to address and incorporate behavioural tendencies like risk aversion. It was then further formalised, independently and within the theory of expected utility, by Arrow, Pratt, and Finetti in the 1950’s and 60’s [9, 179, 86] who, in particular, introduced measures for its quantification. The quest for further understanding and exploiting this concept has since remained of active research interest in the economic sciences [76]. Recently, an important step was taken in the work of Bleuler, Lapidoth and Pfister (BLP) in 2020 [33], where the concept of risk aversion was utilised within the realm of classical information theory. This Thesis takes this line of research as inspiration for proposing various operational tasks based on *betting* and *risk-aversion*, or *quantum betting tasks* for short, which unifies various other operational tasks in the literature, like discrimination and exclusion tasks, and admits a clear characterisation in terms of information-theoretic quantities.

The contents of this Thesis can be seen as lying at the intersection of the three research fields of: i) quantum theory, ii) information theory, and iii) utility theory,. We now briefly describe the way this Thesis is organised.

1.1 Organisation of this Thesis

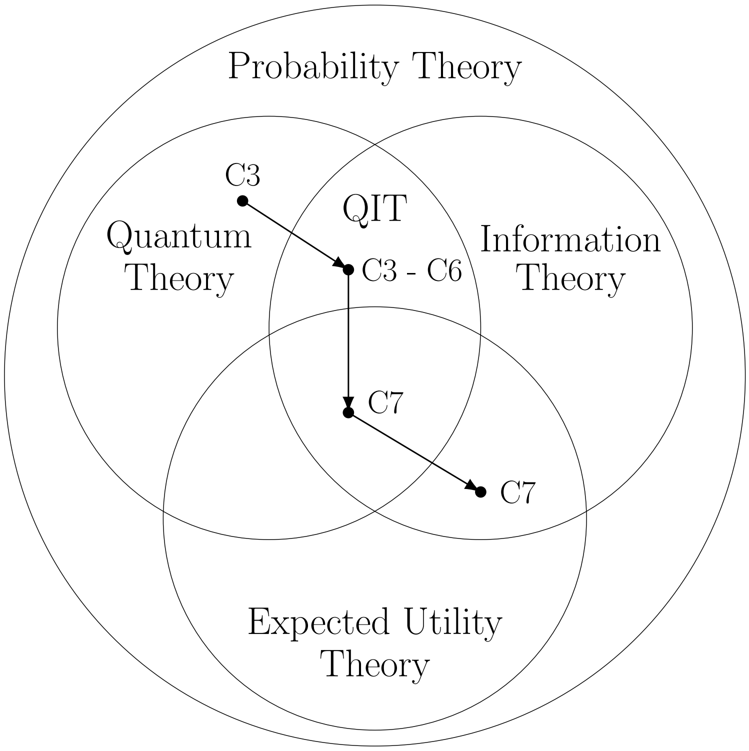

FIGURE 1.1: Main themes and a chronological road-map for the organisation of this Thesis. The main general themes are: quantum theory, information theory, and expected utility theory (EUT). These themes can conceptually be thought of as subsets of probability theory. In Chapter 2 (C2) we address the mathematical preliminaries for these three areas of knowledge. In Chapter 3 (C3) we start with motivations in quantum theory and quickly move to QIT. We continue working in QIT during Chapters 3, 4, 5, and 6 (C3 - C6). Finally, in Chapter 7 (C7) we import ideas from EUT and derive a first pack of results still within QIT. Finally, working with EUT and information theory alone, we also derive a result which is independent from quantum theory.

This Thesis is organised as follows. In Chapter 2 we start with background material, in the form of mathematical preliminaries on i) a resource-theoretic approach to quantum theory, ii) selected information-theoretic quantities in classical information theory as well as iii) selected notions in expected utility theory, the notion of risk-aversion in particular. The first results chapter is Chapter 3, in which we consider the QRT of measurement informativeness, and derive an operational interpretation for the resource quantifier of weight of informativeness, as well as a first connection to information-theoretic quantities. In Chapter 4 we consider general convex QRTs of states and general convex QRTs of measurements, and derive an operational interpretation for weight-based quantifiers in terms of exclusion games, as well as a more general connection to information-theoretic quantities. In Chapter 5 we keep on working within QIT and introduce multi-object operational tasks for composite QRTs where we are simultaneously interested in the resources provided by states and measurements. In Chapter 6 we address a QRT for Buscemi nonlocality, and connect it to the QRTs of entanglement and teleportation. In the final results chapter, Chapter 7, we develop a framework that incorporates ideas from expected utility theory (EUT), introduce quantum betting tasks, and provide an operational interpretation to Arimoto mutual information. We also derive a four-way correspondence for measurement informativeness between: quantum state betting, dependence measures, quantum Rényi divergences, and resource monotones. We also present here a result which lies in the intersection of information theory and expected utility theory, independently from quantum theory. Finally, In Chapter 8 we address general conclusions, perspectives, and avenues for future research. A chronological road-map of the organisation of this Thesis is in Figure 1.1, and alternative ways to read this Thesis are presented in Figure 1.2

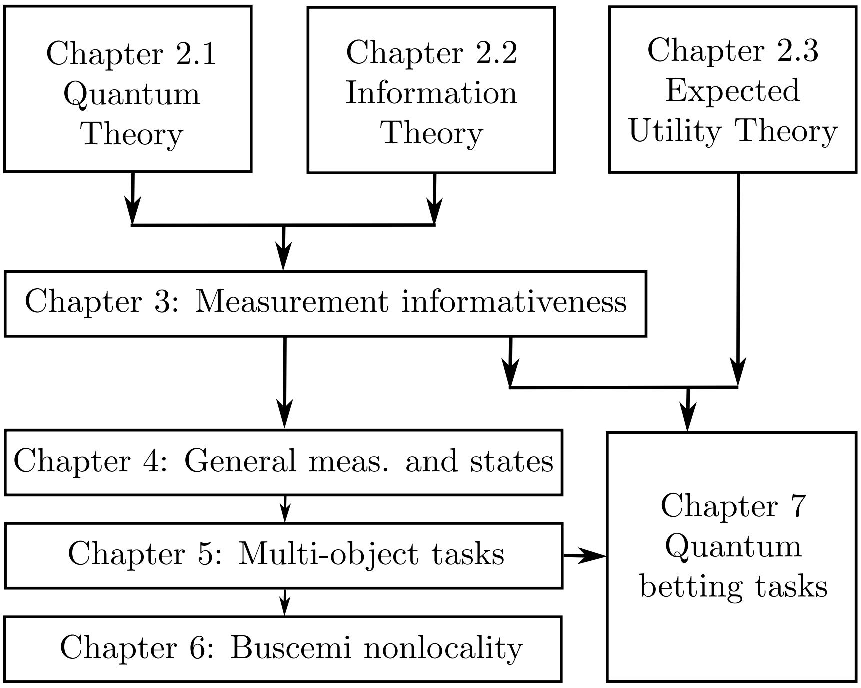

FIGURE 1.2: Paths on how to read this Thesis. Chapter 3 does not necessarily require Chapter 2.3. Similarly, Chapter 7 can be read independently from Chapter 6.

1.2 Publications and preprints

Publications and preprints developed during the PhD and addressed in this Thesis.

- Andrés F. Ducuara, Paul Skrzypczyk
  Weight of informativeness, state exclusion games and excludible information[3]
  https://arxiv.org/abs/1908.10347
  [Chapter 3]

- Andrés F. Ducuara, Paul Skrzypczyk
  Operational Interpretation of Weight-Based Resource Quantifiers in Convex Quantum Resource Theories
  https://arxiv.org/abs/1909.10486
  Phys. Rev. Lett. **125**, 110401, (2020)
  [Chapter 4]

- Andrés F. Ducuara, Patryk Lipka-Bartosik, Paul Skrzypczyk
  Multiobject operational tasks for convex quantum resource theories of state-measurement pairs
  https://arxiv.org/abs/2004.12898
  Phys. Rev. Research **2**, 033374, (2020)
  [Chapter 5]

- Patryk Lipka-Bartosik, Andrés F. Ducuara, Tom Purves, Paul Skrzypczyk
  Operational Significance of the Quantum Resource Theory of Buscemi Nonlocality
  https://arxiv.org/abs/2010.04585
  Phys. Rev. X. Quantum **2**, 020301, (2021)
  [Chapter 6]

- Andrés F. Ducuara, Paul Skrzypczyk
  Characterisation of quantum betting tasks in terms of Arimoto mutual information
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  [Chapter 7]

Publications and preprints developed during the PhD and *not* addressed here.

- Andrés F. Ducuara, Cristian E. Susa, John H. Reina
  Emergence of maximal hidden quantum correlations and its trade-off with the filtering probability in dissipative two-qubit systems
  https://arxiv.org/abs/2005.06339
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[3]The contents of this preprint are included in the second paper as a particular case.

Chapter 2

Mathematical preliminaries

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning. "

Eugene Paul Wigner

In this chapter we address the mathematical preliminaries behind the three themes of this thesis: i) quantum theory, ii) information theory and iii) expected utility theory. Before we start, we address some notation on standard probability theory which is the underlying framework behind these three themes. We consider random variables (RVs) $(X, Y, G, ...)$ on a finite alphabet $\mathcal{X}$, and the probability mass function (PMF), or probability distribution, of X represented as p_X satisfying: $p_X(x) \geq 0$, $\forall x \in \mathcal{X}$, and $\sum_{x \in \mathcal{X}} p_X(x) = 1$. For simplicity, we omit the alphabet when summing, and write $p_X(x)$ as $p(x)$ when evaluating. The support of p_X $\text{supp}(p_X) := \{x \,|\, p(x) > 0\}$, the cardinality of the support $|\text{supp}(p_X)|$, and the extended line of real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$. Similarly, joint PMFs p_{XG} satisfy $\sum_{x,g} p(x, g) = 1$ and conditional PMFs $p_{G|X}$ satisfy $\sum_x p(x|g) = 1$, $\forall g$. We now address quantum theory.

2.1 Background on Quantum Theory

In this section we address some mathematical preliminaries for quantum theory. The general mantra behind our approach is to focus on three aspects of the theory: i) *preparation* of physical systems, which is represented by the mathematical objects of quantum *states*, ii) *measurement*, represented by either projective or general measurements, and iii) *transformation* of physical systems, represented by the mathematical objects of quantum *channels*. This highlights the three fundamental quantum **objects** of: states, measurements, and channels. We start by addressing closed systems, followed by a more general density operator formalism, keeping a resource-theoretic approach to QIT by emphasising the **properties** of interest of each object. Important resources for this section includes: Wolf's lecture notes [246, 247], Preskill's lecture notes [180], Wilde's book [242], Watrous' book [238], Nielsen-Chuang's book [161], and Heinosaari-Ziman's book and lecture notes [109, 108].

2.1.1 Standard formalism

Quantum theory deals with *Hilbert spaces*. A Hilbert space is a *complete inner product space*. An *inner product space* is a structure $(\mathbb{H}, +, \cdot, \mathbb{K}, \langle\cdot|\cdot\rangle)$, where $(\mathbb{H}, +, \cdot, \mathbb{K})$ a *vector space* over a *field* $\mathbb{K}$, and $\langle\cdot|\cdot\rangle$ an *inner product*. Consider in this paragraph $x, y \in \mathbb{H}$. Taking into account that the inner product $\langle\cdot|\cdot\rangle$ induces a norm $||x|| := \langle x|x\rangle^{1/2}$, which in turn induces a metric $d(x,y) := ||x-y||$, the inner product space in question is *complete*, if the induced metric $d(\cdot,\cdot)$ is *complete* [246, 247]. The metric $d(\cdot,\cdot)$ also induces a *topology*, with the open balls defined as $B(x,\epsilon) := \{y|d(x,y) < \epsilon\}$, $\epsilon \geq 0$. For simplicity, we will denote complete inner product spaces (Hilbert spaces) $(\mathbb{H}, +, \cdot, \mathbb{K}, \langle\cdot|\cdot\rangle)$ by simply writing $\mathbb{H}$.

We now set restrictions on the Hilbert spaces that we work with, and establish the notation to be used in this thesis. First, we use complex Hilbert spaces, meaning that $\mathbb{K} = \mathbb{C}$. Second, we address finite-dimensional spaces and therefore, the Hilbert spaces are basically the Hilbert space $\mathbb{C}^d$ with dimension $d \geq 2$. The vectors of these Hilbert spaces are denoted by Greek letters inside the so-called "ket"-"Dirac" notation as $|\psi\rangle \in \mathbb{H}$. We write the standard orthonormal basis (orthogonal normalised basis) for these Hilbert spaces as $\{|i\rangle\}_{i=0}^{d-1}$. By *basis*, we will always mean one of these orthonormal bases. Third, we use the *standard* inner product $\langle\phi|\psi\rangle := \sum_{i=0}^{d-1} \phi_i^* \psi_i$, for $|\phi\rangle = \sum_{i=0}^{d-1} \phi_i |i\rangle$, and $|\psi\rangle = \sum_{j=0}^{d-1} \psi_j |j\rangle$, ϕ_i, ψ_j complex numbers and * complex conjugation. Throughout the text, $\mathbb{H}$ will mean one of these Hilbert spaces, and we will consider Hilbert spaces for different physical "systems" or "parties" and denote each system or party as $A, B, ...$, and their respective Hilbert spaces as $\mathbb{H}_A, \mathbb{H}_B$ and so on. We consider the set of linear operators from one Hilbert space $\mathbb{H}_A$ to another Hilbert space $\mathbb{H}_B$, as $\mathcal{L}(A,B)$, and denote $\mathcal{L}(A) := \mathcal{L}(A,A)$. In particular, we consider the linear operator $|\psi\rangle\langle\psi|$ as: $|\psi\rangle\langle\psi| (|\phi\rangle) := (\langle\psi|\phi\rangle) |\psi\rangle$. Let us now assign some of these mathematical objects to the physical concepts that we are interested in.

Some operators of interest and projective measurements

We now consider the set of linear operators $\mathcal{L}(A)$ for a Hilbert space $\mathbb{H}_A$, and define some special types of operators. Given a linear operator $A \in \mathcal{L}(A)$, its *adjoint operator* denoted as $A^\dagger$, is defined as the operator such that $(\langle\phi|A^\dagger)|\psi\rangle = \langle\phi|(A|\psi\rangle)$, $\forall |\phi\rangle, |\psi\rangle \in \mathbb{H}$. A linear operator A is called *Hermitian* when it is *self-adjoint*, meaning that $A^\dagger = A$. The set of all Hermitian operators is denoted as $Herm(A)$. A *positive semidefinite* operator A is such that $\langle\psi| A |\psi\rangle \geq 0$, $\forall |\psi\rangle \in \mathbb{H}$, this is written as $A \geq 0$. The set of all positive semidefinite operators is denoted as $PSD(A)$. One can check that $(\mathcal{L}(A), +, \cdot, \mathbb{C}, \langle\cdot|\cdot\rangle_{HS})$ forms a Hilbert space with with the so-called Hilbert-Schmidt inner product $\langle X|Y\rangle_{HS} := \mathrm{Tr}(X^\dagger Y)$. Given an Hermitian operator O, the *spectral decomposition theorem* establishes that O can be written as [246]:

$$O = \sum_{a=0}^{d-1} o_a P_a, \tag{2.1}$$

where $\{o_a\}$ are its eigenvalues, which are *real* numbers, and $P_a = |a\rangle\langle a|$ with $\{|a\rangle\}_{a=0}^{d-1}$ a basis. Hermitian operators are important because their eigenvalues are real eigenvalues and therefore they are suitable to represent values of *properties* of physical systems and so, they can be interpreted as physical *observables*. The set $\mathbb{M} = \{P_a\}$ is called a *projective* (or *von Neumann*) measurement, and the operators P_a are called *projections*. One can check that the set $\mathbb{M} = \{P_a\}$ satisfies the properties: i) positive semi-definite elements $P_a \geq 0, \ \forall a$, ii) completeness $\sum_{a=0}^{d-1} P_a = \mathbb{1}$, and iii) orthogonality $P_a P_{a'} = P_a \delta_{a,a'}, \ \forall a, a'$. With these elements in place, we now address the standard interpretation of quantum mechanics or Copenhagen interpretation.

Pure states

A *quantum (pure) state* or simply a *(pure) state* is a vector $|\psi\rangle \in \mathbb{C}^d$ such that it has norm one $\langle\psi|\psi\rangle = 1$ [180]. Considering a state written in an arbitrary basis; $|\psi\rangle := \sum_{i=0}^{d-1} \alpha_i |i\rangle$, $\alpha_i \in \mathbb{C}$, the normalisation condition is imposed because we want to interpret the coefficient $p(i) := |\alpha_i|^2$ as a probability distribution for the physical system to be represented by the vector $|i\rangle$, for which one can check that $\sum_i p(i) = \sum_{i=0}^{d-1} |\alpha_i|^2 = \langle\psi|\psi\rangle = 1$, and so we say that $|\psi\rangle$ is a *superposition* of the vectors $\{|i\rangle\}$. The normalisation condition also implies that global phases are irrelevant from a physical point of view. The meaning of *pure* is going to be made clear later on when more general *mixed states* are introduced, in short, a pure state represents a closed system. Having defined a quantum state representing a physical system, we now consider its interaction with a measurement process.

Born rule and post-measured states

Given an observable O^x (x acting as a counter $x = 1, 2, ...$), and an arbitrary state $|\psi\rangle$ written in the basis generated by O^x; $|\psi\rangle = \sum_a \psi_a |a\rangle$, this state has probability $|\psi_a|^2$ of having a value of o_a^x for the property represented by O^x [180]. This probability can also be written as:

$$p(a|x) := |\psi_a|^2 = \langle\psi| P_a^x |\psi\rangle = \mathrm{Tr}(P_a^x \psi), \tag{2.2}$$

with $\psi := |\psi\rangle\langle\psi|$. This rule then determines the conditional PMF $p(a|x)$ of obtaining an outcome a for a given measurement $\mathbb{M}^x = \{P_a^x\}$, a given state ρ, and it is known as the *Born Rule*. This is then a probabilistic interpretation of quantum mechanics. We can similarly address the expected value (average) of the observable as $E^x = \mathrm{Tr}(O^x\psi) = \mathrm{Tr}[\sum_{a=0}^{d-1} o_a^x P_a^x \psi] = \sum_{a=0}^{d-1} o_a^x p(a|x)$. Given the state $|\psi\rangle$ and the measurement $\mathbb{M}^x = \{P_a^x\}$, we can also talk about the state after the measurement, which is going to be one of the states $|\psi_a\rangle := P_a^x |\psi\rangle / ||P_a^x |\psi\rangle||$, with probability $||P_a^x |\psi\rangle|| = \mathrm{Tr}(P_a^x \psi P_a^x)$. After the the measurement process has taken place, the state of the system changes from ψ to one of the states $\{\psi_a\}$, in a probabilistic manner, and this is what it is commonly called as the reduction of the quantum state or the collapse of the wave function.

Unitary channels

We now want to consider transformations of states, and we naturally want for these transformations to keep the structure of the set of states invariant, meaning that it takes quantum states into quantum states. This therefore gives rise to unitary transformations, or unitary operators. An operator $U \in \mathcal{L}(A)$ is called *unitary* when it satisfies that $U^\dagger U = UU^\dagger = \mathbb{1}$.

So far, we have only considered *one* physical system. However, we might also consider the fact that a measurement process can be somehow representing another physical system (a measurement apparatus, an experimentalist, or more generally the rest of the universe), which should in principle also be represented as a *system*, and therefore, modelled with a Hilbert space too. Furthermore, the physical system of interest could also be composed of subsystems. These considerations lead us to consider multipartite systems.

Multipartite systems

A natural way to assign a Hilbert space to a multipartite system is by considering the *tensor product* of individual systems. Considering an n-partite system represented by $\mathbb{H}_{\otimes} = \bigotimes_{i=1}^{n} \mathbb{C}^{d_i}$ with $d = d_1 d_2 ... d_n$, where $\otimes$ stands for the tensor product, one can check that the structure $(\mathbb{H}_{\otimes}, +_{\otimes}, *_{\otimes}, \mathbb{C}, \langle \cdot | \cdot \rangle_{\otimes})$, with $+_{\otimes}, *_{\otimes}$ as element-wise extensions, and $\langle \psi_1 \otimes \psi_2 | \phi_1 \otimes \phi_2 \rangle_{\otimes} := \langle \psi_1 | \phi_1 \rangle \langle \psi_2 | \phi_2 \rangle$, is also a Hilbert space, and therefore, suitable and desirable to represent the n-partite system in question. Multipartite systems are going to be represented by the Hilbert space $\mathbb{H} = \bigotimes_{i=1}^{n} \mathbb{C}^{d_i}$, with different values for n and $\{d_i\}$ (we drop the subscript $_{\otimes}$ for simplicity). In particular, we are going to deal with *bipartite systems* which are going to be denoted as $\mathbb{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, (or $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$). Considering a basis for each system as $\{|i\rangle\}_{i=0}^{d_A-1}$, and $\{|j\rangle\}_{i=0}^{d_B-1}$, we write a bipartite basis as $\{|ij\rangle\}_{i,j=0}^{d_A-1,d_B-1}$, with $|ij\rangle := |i\rangle \otimes |j\rangle$. Let us now address a consequence of considering multipartite systems, in the so-called density operator formalism.

2.1.2 Density operator formalism

Considering multipartite systems, we now focus on the case when we are not interested in the whole system, but rather into a subsystem of it. Let us consider a bipartite system and rename the two parts as *system* (S) and *environment* (E). In doing so, we now address how this naturally leads to the generalisation of the concepts of states, projective measurements, and unitary transformations into *density operators*, *Positive Operator-Valued Measures* (POVM's), and *Completely-Positive Trace-Preserving* (CPTP) maps (or quantum channels), respectively. Let us address these generalisations.

Density operators

We start by introducing Hilbert spaces for system and environment as $\mathbb{H}_S = \mathbb{C}^{d_S}$ and $\mathbb{H}_E = \mathbb{C}^{d_E}$, respectively. The Hilbert space for the whole system is then $\mathbb{H} := \mathbb{C}^{d_S} \otimes \mathbb{C}^{d_E}$, and a general state for the whole system-environment is $|\psi_{SE}\rangle$. If we consider bases for the system and environment as $\{|i\rangle\}$,$\{|j\rangle\}$, respectively, we have that any state in $\mathbb{H}$ can be written as: $|\psi_{SE}\rangle = \sum_{ij} \alpha_{ij} |ij\rangle, \alpha_{ij} \in \mathbb{C}, \forall i, j$. However, we are now only interested in the system S. Inspired by (2.2), we then would like to find an operator, let us call it ρ_S, such that after local measurements on the system's side $\{P_a^x \otimes \mathbb{1}\}$, P_a^x a projection on $\mathbb{H}_S$, we get:

$$\langle \psi_{SE} | P_a^x \otimes \mathbb{1} | \psi_{SE} \rangle = \mathrm{Tr}\left[(P_a^x \otimes \mathbb{1}) \, \psi_{SE} \right] \stackrel{!}{=} \mathrm{Tr}(P_a^x \rho_S).$$

One can check that proposing $\rho_S := \sum_{ik} \beta_{ik} |i\rangle\langle k|$ with $\beta_{ik} := \sum_j \alpha_{ij} \alpha_{kj}^*$, and calculating $\mathrm{Tr}(P_a^x \rho_S)$, yields the desired probability [180]. The operator ρ_S is called *the reduced density operator* of $|\psi_{SE}\rangle$, and it can also alternatively be obtained by introducing the concept of *partial trace* with respect to the environment (Tr_E) from which we get $\mathrm{Tr}_E(\psi_{SE}) = \rho_S$ (Preskill's notes for full details [180]).

Analysing the density operator ρ_S, one can check that it satisfies the following two properties: i) $\mathrm{Tr}(\rho) = 1$, and ii) $\rho \geq 0$, and so the set of *density operators* is defined as the set of operators satisfying these two conditions and is denoted as $D(\mathbb{H})$, and so we have $D(\mathbb{H}) \subseteq PSD(\mathbb{H})$. Any density operator $\rho \in D(\mathbb{H})$ can be written as $\rho = \sum_i p_i \psi_i$ with $\{|\psi_i\rangle, p_i\}$ an *ensemble* of states, with $p_i > 0, \forall i$, and $\sum_i p_i = 1$ [180]. We call density operators of the form $\psi = |\psi\rangle\langle\psi|$ *pure* quantum states, and general density operators can then be seen as a statistical mixture of such pure states, and that is the reason why general density operators are also called *mixed states*.

From a resource-theoretic approach, quantum states can be thought of as in possession of various different valuable resources. For example, *entanglement* [25, 23], which means states which cannot be written in a separable form as $\rho^{AB} = \sum_i p_i \psi_i^A \otimes \psi_i^B$, and *coherence* [18, 245, 1], which means states which cannot be written in a incoherent form for a specific basis $\{i\}$ as $\rho^A = \sum_i p_i |i\rangle\langle i|$. Further properties of quantum states deemed as valuable resources include: purity [115], superposition [219, 2], athermality [124, 40, 116], magic [122], reference frames [94], asymmetry [148], non-Gaussianity [215], imaginarity [249, 250], amongst others [57]. We now consider general measurements.

General measurements (POVMs)

Similarly to the generalisation from *pure* to general *mixed* states, projective measurements can also be generalised. Explicitly, consider a projective measurement $\{P_a^{AB}\}$ in a bipartite system and a quantum state ρ^A. We then want to ask if this measurement can be thought of as being implemented as an operation being implement in subsystem A alone. Given any projective measurement $\{P_a^{AB}\}$ and any pure state ψ^B we want to find a set of operators $\{E_a^A\}$ such that $\forall a$:

$$p(a) = \mathrm{Tr}\left[\left(\rho^A \otimes \psi^B\right) P_a^{AB}\right] \stackrel{!}{=} \mathrm{Tr}(\rho^A E_a^A). \tag{2.3}$$

One can check that this can be achieved by considering $E_a^A := (\mathbb{1}^A \otimes \langle\psi|^B) P_a^{AB} (\mathbb{1}^A \otimes |\psi\rangle^B)$. One can also check that these operators satisfy $\{E_a^A\}$ satisfy the properties: i) positive semidefinite $E_a^A \geq 0, \forall a$ and ii) partition of the identity $\sum_a E_a^A = \mathbb{1}$. These operators do not necessarily need to satisfy the orthogonality condition. The set of operators $\{E_a^A\}$ is called a Positive Operator-Valued Measure (POVM). These operators can be seen as generalisations of the projections, where the orthogonality condition is dropped. The converse of the previous argument is also true, given any POVM $\mathbb{M} = \{E_a^A\}$ and a state ρ^A, there exists a bipartite projective measurement $\{P_a^{AB}\}$ and a quantum state ψ^B such that (2.3) holds (Wolf's notes for full details [246]).

We have that information of the state ρ and the measurement $M = \{E_a\}$ alone is not enough for describing the post-measured state (unlike the case for projective measurements). However, when we consider POVM elements of the form $E_a = F_a^\dagger F_a$, then it is possible to talk about the state after the measurement, which is given by $\rho_a = F_a \rho F_a^\dagger / \mathrm{Tr}(F_a \rho {F_a}^\dagger)$, with probability $p(a) = \mathrm{Tr}(F_a \rho F_a^\dagger)$. The set $\{F_a\}$ is called a *quantum instrument*, from which both the POVM and the post-measured state (provided ρ) can be recovered.

Properties of general measurements are also deemed as being valuable from a resource-theoretic perspective. QRTs of measurements address properties like: entanglement, coherence [162], non-projective simulability [165, 99], informativeness [204], amongst others [57]. Now that we have covered states and measurements, we now move on to address the generalisation of unitary transformations.

General channels

Consider two finite-dimensional Hilbert spaces $\mathbb{H}_A = \mathbb{C}^{d_A}$, $\mathbb{H}_B = \mathbb{C}^{d_B}$, and their respective sets of quantum states (or density operators) as $D(A)$ and $D(B)$. We address maps as objects $\mathcal{N}_{B\leftarrow A} : D(A) \rightarrow D(B)$. A *completely positive* (CP) map means that $\mathrm{id}_k \otimes \mathcal{N}_{B\leftarrow A}$ is a positive map for all $k \geq 0$, with the identity channel on an auxiliary k-dimensional Hilbert space. A *trace-preserving* (TP) map means that $\mathrm{Tr}[\mathcal{N}_{B\leftarrow A}(\rho_A)] = \mathrm{Tr}[\rho_A] = 1$, $\forall \rho_A \in D(A)$. A *trace nonincreasing* (TNI) map satisfies $0 \leq \mathrm{Tr}[\mathcal{N}_{B\leftarrow A}(\rho_A)] \leq \mathrm{Tr}[\rho_A] = 1$. A *quantum channel* is a *completely positive trace-preserving* (CPTP) map, and a *quantum subchannel* is a *completely positive trace-nonincreasing* (CPTNI) map. Considering the state $\psi = |\psi\rangle\langle\psi|$, we have that unitary operators act as a map $\Lambda_U : \psi \rightarrow U\psi U^\dagger$, and one can check that the map Λ_U is completely positive and trace-preserving. These types of maps are called *unitary channels*.

Similarly to the previous constructions for states and measurements, given a CPTP map $\mathcal{N}_{B\leftarrow A}$ there exists a unitary U_{AB} and a quantum state σ_B such that:

$$\mathcal{N}_{B\leftarrow A}(\rho_A) = \mathrm{Tr}_A \left[U_{AB} \left(\rho_A \otimes \sigma_B\right) U_{AB}^\dagger \right]. \tag{2.4}$$

This is sometimes called the environment representation of a quantum channel [246] and, similarly to the case for sates and measurements, it means that general quantum channels naturally emerge when analysing the effect on the subsystem, when an unitary channel acts on a larger Hilbert space. We now address the Choi-Jamiołkowski representation. We introduce the *unnormalised maximally entangled* (ME) state between two isomorphic Hilbert spaces $\mathbb{H}_A$ and $\mathbb{H}_{A'}$ as:

$$\Phi_{AA'} := |\Phi_{AA'}\rangle \langle \Phi_{AA'}| , \qquad |\Phi_{AA'}\rangle := \sum_{i=0}^{d-1} |ii_{AA'}\rangle . \tag{2.5}$$

We now also invoke the *Choi-Jamiołkowski* (CJ) isomorphism between maps (channels) and bipartite operators (states) (or *channel-state duality*) as follows. Given any channel $\mathcal{N}_{B\leftarrow A} : D(A) \rightarrow D(B)$ we define its associated CJ-operator (CJ-state) as:

$$J_{AB}^{\mathcal{N}} := (\mathrm{id}_A \otimes \mathcal{N}_{B\leftarrow A'})(\Phi_{AA'}) = \sum_{ij=0}^{d-1} |i\rangle\langle j|_A \otimes \mathcal{N}_{B\leftarrow A'}\left(|i\rangle\langle j|_{A'}\right), \tag{2.6}$$

with $\mathbb{H}_A$ and $\mathbb{H}_{A'}$ isomorphic Hilbert spaces. The bipartite operator $J_{AB}^{\mathcal{N}}$ is not directly a state, but we can define one as $\rho_{AB}^{\mathcal{N}} := (1/d_A) J_{AB}^{\mathcal{N}}$. We then see here that the ME state plays an important role in allowing the definition of the associated CJ-operator for a given channel. We also consider the other direction, meaning that given a bipartite operator (state) $J_{AB}^{\mathcal{N}}$ ($\rho_{AB}^{\mathcal{N}}$), we can define its CJ-map (CJ-channel) as:

$$\mathcal{N}_{B\leftarrow A}(X_A) := \mathrm{Tr}_A[J_{AB}^{\mathcal{N}}(X_A^T \otimes \mathbb{1}_B)]. \tag{2.7}$$

The properties of $\mathcal{N}_{B\leftarrow A}$ translate into properties of $J_{AB}^{\mathcal{N}}$ as follows:

- $\mathcal{N}_{B\leftarrow A}$ is CP map iff $J_{AB}^{\mathcal{N}} \geq 0$,
  (the CJ-operator is positive semidefinite).
- $\mathcal{N}_{B\leftarrow A}$ is TP map iff $J_{A}^{\mathcal{N}} := \mathrm{Tr}_B[J_{AB}^{\mathcal{N}}] = \mathbb{1}_A$,
  (the CJ-operator has maximally mixed marginal).
- $\mathcal{N}_{B\leftarrow A}$ is TNI map iff $J_{A}^{\mathcal{N}} := \mathrm{Tr}_B[J_{AB}^{\mathcal{N}}] \leq \mathbb{1}_A$.

This means that $\rho^{\mathcal{N}}_{AB} := (1/d_A)J^{\mathcal{N}}_{AB}$ is a state, with maximally mixed marginal in A as $\rho^{\mathcal{N}}_{A} = (1/d_A)\mathbb{1}_A$. We also consider the concept of dual map. $\mathcal{E}^{\dagger}$ is the adjoint map of $\mathcal{E}$, and is defined as the operator such that $\mathrm{Tr}(\mathcal{E}(X)Y) = \mathrm{Tr}(X\mathcal{E}^{\dagger}(Y))$, $\forall X \in \mathcal{L}(A)$, $\forall Y \in \mathcal{L}(B)$. Finally, from a resource-theoretic perspective, quantum channels are also deemed in possession of valuable resources like: non-constant channels, non-entanglement breaking, non-nonlocality breaking, and more generally resource non-destroying channels, amongst other properties [57]. We now address some additional objects of interest, which can be constructed by combining the previous objects of states, measurements, and channels.

On composite quantum objects and generalities of QRTs

We now briefly discuss composite objects. We can think about combining states, together with sets of measurements, and in order to generate new objects like steering assemblages and nonlocal boxes. Many of these composite objects have been explored under resource-theoretic lenses including: states [57], measurements [165, 162, 101], behaviours or boxes [233, 5], EPR-steering assemblages [89], teleportation assemblages [55] and channels [144, 141]. Properties of these objects that are deemed as resources include: entanglement [234], nonlocality [49], steering [172], asymmetry [174], coherence [155], informativeness [204], projective simulability [99], incompatibility [49, 144, 226, 48], teleportation [55], superposition [219], purity [210], magic [122], nongaussianity [215], nonmarkovianity [31, 236, 6], athermality [158], and reference frames [95], amongst others [57].

Having specified a set of objects and one of their properties to be treated as a resource, it is of interest to quantitatively specify the *amount* of resource contained within a given object. This can be accomplished by introducing appropriate measures known as *resource quantifiers* [57]. Two well-known families of these measures are the so-called *robustness-based* [234, 49, 172, 174, 155, 204, 49, 55, 138, 122] and *weight-based* [136, 79, 206, 49, 44] resource quantifiers. In this thesis we address these two particular measures for states and measurements. It turns out that these functions can be conveniently be written as optimisation problems, some of which we review in what follows.

Conic programming (CP) and semidefinite programming (SPD)

Conic programming (CP) is a sub-field of *convex optimisation* which includes other optimisation problems like *semidefinite, quadratic, and linear programming*. It is called *conic* programming because it deals with the optimisation of a convex function over the intersection of *closed convex cones* (CCC) and an *affine space*. This subsection follows Gärtner-Matoušek's book on general cone programs [90] and Johnston's PhD thesis [127].

In the context of QIT, we are dealing with finite-dimensional complex Hilbert spaces, and so these concepts can be addressed as follows. $\mathcal{K} \subseteq Herm(A)$ is a *cone* if $\lambda K \in \mathcal{K}$, $\forall K \in \mathcal{K}$ and $\forall \lambda \geq 0$. $\mathcal{K}$ is a *convex* set when it satisfies $pK_1 + (1-p)K_2 \in \mathcal{K}$, $\forall K_1, K_2 \in \mathcal{K}$, $\forall p \in [0,1]$. In the case of a cone $\mathcal{K}$, the convexity condition can be relaxed to $K_1 + K_2 \in \mathcal{K}$, $\forall K_1, K_2 \in \mathcal{K}$, this, because both $\lambda K_1 \in \mathcal{K}$ and $(1-\lambda)K_2 \in \mathcal{K}$. A *closed* set here is considered as being closed under the topology induced by the inner product of the Hilbert space. We also need the concept of the dual of a cone. The *dual* of a cone $\mathcal{K}$ is defined as $\mathcal{K}^{\circ} := \{O \in Herm(A) | \langle O|K\rangle_{HS} \geq 0, \forall K \in \mathcal{K}\}$. For any cone $\mathcal{K}$ we have $(\mathcal{K}^{\circ})^{\circ} = \overline{\mathrm{hull}(\mathcal{K})}$ (the closure of the convex hull), meaning that the dual cone of any cone is always a CCC, and so for any CCC we have $(\mathcal{K}^\circ)^\circ = \mathcal{K}$. With these tools at hand, we now address conic programs.

Definition 2.1. *(Conic program [90]) A conic program (CP) is a 5-tuple $(A, B, \mathcal{E}, \mathcal{K}, \mathcal{L})$, with $A \in Herm(A)$, $B \in Herm(B)$, $\mathcal{E} : \mathcal{L}(A) \to \mathcal{L}(B)$ a Hermiticity preserving linear map, and $\mathcal{K} \subseteq Herm(A)$, $\mathcal{L} \subseteq Herm(B)$ closed convex cones. Two optimisations associated to the 5-tuple $(A, B, \mathcal{E}, \mathcal{K}, \mathcal{L})$ are the primal conic program (CP) and the dual conic program (CP) given by:*

Primal CP

$$\begin{aligned} \text{maximise: } & \langle A|X\rangle_{HS} && (2.8a) \\ \text{subject to: } & B - \mathcal{E}(X) \in \mathcal{L}, && (2.8b) \\ & X \in \mathcal{K}. && (2.8c) \end{aligned}$$

Dual CP

$$\begin{aligned} \text{minimise: } & \langle B|Y\rangle_{HS} && (2.9a) \\ \text{subject to: } & \mathcal{E}^\dagger(Y) - A \in \mathcal{K}^\circ, && (2.9b) \\ & Y \in \mathcal{L}^\circ. && (2.9c) \end{aligned}$$

We can alternatively write the objective functions as $\langle A|X\rangle_{HS} = \mathrm{Tr}(A^\dagger X) = \mathrm{Tr}(AX)$ and $\langle B|Y\rangle_{HS} = \mathrm{Tr}(B^\dagger Y) = \mathrm{Tr}(BY)$. The variable X is called a "primal variable" or "primal feasible", and Y "dual variable" or "dual feasible". X^* denotes the "optimal primal variable/solution", and similarly Y^* as the "optimal dual variable/solution". If there are no feasible solutions we set the values $-\infty$ and $+\infty$, respectively.

We now address two particular cases of interest. First, we can consider that both cones are equal to the cone of positive semidefinite operators, $\mathcal{K} = \mathcal{L} = PSD$, in this case the conic program is called a semidefinite program (SDP), and it is usually addressed by the triple $(A, B, \mathcal{E})$. Second, we can consider the scenario when only one of the cones is the set of positive semidefinite operators, say $\mathcal{K} = PSD$, and so we have a conic program with only one general cone $\mathcal{L}$. In this thesis we are mostly going to deal with these latter types of conic problems. For the sake of consistency of notation with the literature, let us rename the cone as $\mathcal{L} = \mathcal{C}$ and rewrite the associated CP as follows.

Definition 2.2. *(One-cone conic program [127]) A one-cone conic program (CP) is a 4-tuple $(A, B, \mathcal{E}, \mathcal{C})$, with $A \in Herm(A)$, $B \in Herm(B)$, $\mathcal{E} : \mathcal{L}(A) \to \mathcal{L}(B)$ a Hermiticity preserving linear map, and $\mathcal{C} \subseteq Herm(A)$ a closed convex cone. Two optimisations of interest associated to the 4-tuple $(A, B, \mathcal{E}, \mathcal{C})$ are the primal CP and the dual CP. Taking into account that $PSD^\circ = PSD$ we can write these optimisations as follows:*

Primal CP

$$\begin{aligned} \text{maximise: } & \mathrm{Tr}(AX) && (2.10a) \\ \text{subject to: } & B - \mathcal{E}(X) \in \mathcal{C}, && (2.10b) \\ & X \geq 0. && (2.10c) \end{aligned}$$

Dual CP

$$\begin{aligned} \text{minimise: } & \mathrm{Tr}(BY) && (2.11a) \\ \text{subject to: } & \mathcal{E}^\dagger(Y) \geq A, && (2.11b) \\ & Y \in \mathcal{C}^\circ. && (2.11c) \end{aligned}$$

Working with these one-cone conic problems, we now further analyse some notions of interest. We now define the *primal and dual feasible sets* as:

$$\mathcal{A} := \{X \geq 0 | B - \mathcal{E}(X) \in \mathcal{C}\}, \qquad \mathcal{B} := \{Y \in \mathcal{C}^\circ | \mathcal{E}^\dagger(Y) \geq A\}. \tag{2.12}$$

Then optimal solutions are given by:

$$\alpha := \sup_{X \in \mathcal{A}} \{\mathrm{Tr}(AX)\}, \qquad \beta := \inf_{Y \in \mathcal{B}} \{\mathrm{Tr}(BY)\}. \tag{2.13}$$

The primal CP and the dual CP are related as $\alpha \leq \beta$, which is called *Weak duality*, and it represents the fact that for any CP $(A, B, \mathcal{E}, \mathcal{C})$ the primal CP is always less or equal than the dual CP because we have the chain of inequalities:

$$\mathrm{Tr}(AX) \leq \mathrm{Tr}(\mathcal{E}^\dagger(Y)X) = \mathrm{Tr}(Y\mathcal{E}(X)) \leq \mathrm{Tr}(YB). \tag{2.14}$$

In the first inequality we use (2.11b), the equality we use the dual map, in the second inequality we use (2.10b). Furthermore, it turns out that there are conditions, known as Slatter-type conditions [90, 127], for the primal solution and the dual solution to be equal, and this is called *strong duality*. These conditions are basically that either α (β) are finite, and that there exist at least a strictly feasible variable in the interior of the respective set $Y \in \mathrm{Int}(\mathcal{C}^\circ)$ ($X > 0$) [127]. As an example, we now consider the generalised robustness of resource, and see how it can be written as a one-cone conic program.

Example 1. *Consider the generalised robustness of resource of a state ρ, a convex closed cone (CCC), and the set of free states as F as the intersection of $\mathcal{F}$ and the set of trace-one operators as in* (2.15). *Consider now a conic program given by the 4-tuple $(A, B, \mathcal{E}, \mathcal{C})$ with: $A = \rho$, $B = \mathbb{1}$, $\mathcal{E} = id$ (so $\mathcal{E}^\dagger = id$), and $\mathcal{C}^\circ = \mathcal{F}$. The optimisation of the dual conic program is in* (2.16c), *and we can then see that the conic program is precisely $1 + R_{\mathrm{F}}(\rho)$, with the dual variable $Y = (1 + r)\sigma$.*

$$\begin{aligned} R_{\mathrm{F}}(\rho) = \min \quad & r && (2.15a)\\ \text{s.t.} \quad & \rho \leq (1+r)\sigma, && (2.15b)\\ & r \geq 0, && (2.15c)\\ & \sigma \in \mathbb{F}. && (2.15d) \end{aligned}$$

$$\begin{aligned} 1 + R_{\mathrm{F}}(\rho) = \mathit{min.} \quad & \mathrm{Tr}(BY) && (2.16a)\\ \mathit{s.\ t.} \quad & \mathcal{E}^\dagger(Y) \geq A, && (2.16b)\\ & Y \in \mathcal{C}^\circ. && (2.16c) \end{aligned}$$

Similarly, many other quantities in QIT can be expressed as conic programs (CPs) or semidefinite programs (SDPs). We now present a list of the information-theoretic quantities written as conic programs which we are going to use in this thesis.

Various information-theoretic quantities as conic programs

The generalised robustness of resource for states:

Primal CP

$$\begin{aligned} R_{\mathrm{F}}(\rho) = \min \quad & r && (2.17a)\\ \text{s.t.} \quad & \rho \leq (1+r)\sigma, && (2.17b)\\ & r \geq 0, && (2.17c)\\ & \sigma \in \mathrm{F}. && (2.17d) \end{aligned}$$

Dual CP

$$\begin{aligned} R_{\mathrm{F}}(\rho) = \max \quad & \mathrm{Tr}[X\rho] - 1 && (2.18a)\\ \text{s.t.} \quad & \mathrm{Tr}[X\sigma] \leq 1, && (2.18b)\\ & X \geq 0, && (2.18c)\\ & \sigma \in \mathrm{F}. && (2.18d) \end{aligned}$$

The weight of resource for states:

Primal CP

$$\begin{aligned} W_{\mathrm{F}}(\rho) = \min \quad & w && (2.19a)\\ \text{s.t.} \quad & \rho \geq (1-w)\sigma, && (2.19b)\\ & w \geq 0, && (2.19c)\\ & \sigma \in \mathrm{F}. && (2.19d) \end{aligned}$$

Dual CP

$$\begin{aligned} W_{\mathrm{F}}(\rho) = \max \quad & \mathrm{Tr}[(-Y)\rho] + 1 && (2.20a)\\ \text{s.t.} \quad & \mathrm{Tr}[Y\sigma] \geq 1, && (2.20b)\\ & Y \geq 0, && (2.20c)\\ & \sigma \in \mathrm{F}. && (2.20d) \end{aligned}$$

The generalised robustness of resource for measurements:

Primal CP

$$R_{\mathbb{F}}(\mathbb{M}) = \min \quad r \tag{2.21a}$$
$$\text{s.t. } M_x \leq (1+r)N_x, \ \forall x, \tag{2.21b}$$
$$r \geq 0, \tag{2.21c}$$
$$\mathbb{N} \in \mathbb{F}. \tag{2.21d}$$

Dual CP

$$\mathrm{R}_{\mathbb{F}}\,(\mathbb{M}) = \max_{\{Z_x\}} \sum_x \mathrm{Tr}[Z_x M_x] - 1, \tag{2.22a}$$
$$\text{s.t. } \sum_x \mathrm{Tr}[Z_x N_x] \leq 1, \tag{2.22b}$$
$$Z_x \geq 0, \quad \forall x, \tag{2.22c}$$
$$\mathbb{N} \in \mathbb{F}. \tag{2.22d}$$

The weight of resource for measurements:

Primal CP

$$W_{\mathbb{F}}(\mathbb{M}) = \min \quad w \tag{2.23a}$$
$$\text{s.t. } M_x \geq (1-w)N_x, \ \forall x, \tag{2.23b}$$
$$w \geq 0, \tag{2.23c}$$
$$\mathbb{N} \in \mathbb{F}. \tag{2.23d}$$

Dual CP

$$\mathrm{W}_{\mathbb{F}}\,(\mathbb{M}) = \max_{\{Y_x\}} \sum_x \mathrm{Tr}[(-Y_x)M_x] + 1, \tag{2.24a}$$
$$\text{s.t. } \sum_x \mathrm{Tr}[Y_x N_x] \geq 1, \tag{2.24b}$$
$$Y_x \geq 0, \quad \forall x, \tag{2.24c}$$
$$\mathbb{N} \in \mathbb{F}. \tag{2.24d}$$

The *trace norm* or Schatten 1-norm:

Primal CP

$$||X||_1 = \sup \ \mathrm{Tr}\,[MX] \tag{2.25a}$$
$$\text{s.t.} \quad -\mathbb{1} \leq M \leq \mathbb{1} \tag{2.25b}$$
$$M \in Herm. \tag{2.25c}$$

Dual CP

$$||X||_1 = \inf \ \mathrm{Tr}[X_1 + X_2] \tag{2.26a}$$
$$\text{s.t.} \quad X = X_1 - X_2, \tag{2.26b}$$
$$X_1 \geq 0, \quad X_2 \geq 0. \tag{2.26c}$$

2.2 Background on Information Theory

In this subsection we now address the information-theoretic quantities represented in Fig. 2.1 namely, the Rényi entropy, the Arimoto-Rényi conditional entropy, Arimoto's mutual information, the Rényi divergence, the conditional Rényi (CR) divergences of Sibson, Csiszár, and Bleuler-Lapidoth-Pfister, their respective mutual informations, and the Rényi channel capacity. We mostly follow these references: Cover & Thomas' book [63], Moser's lecture notes [152], and Pfister's PhD thesis [171].

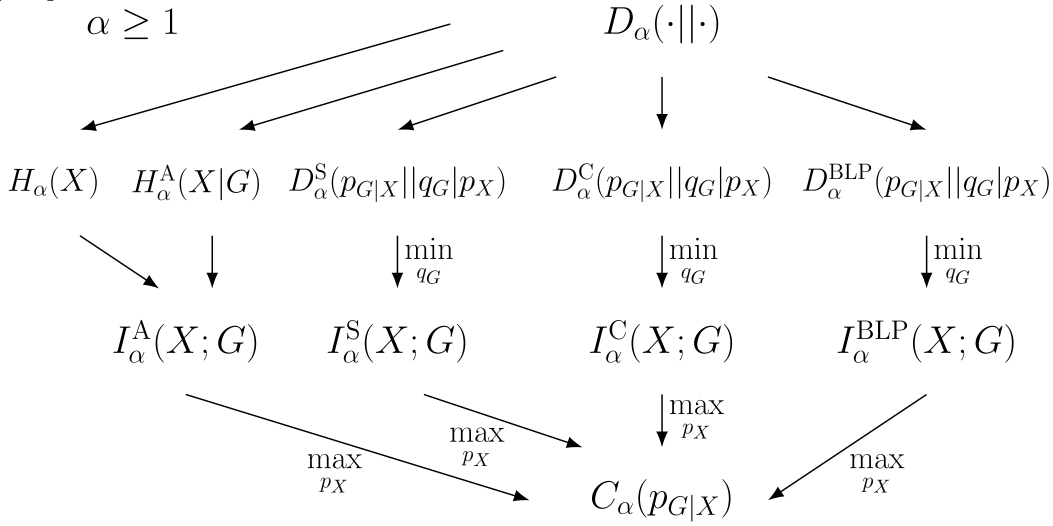

FIGURE 2.1: Hierarchical relationship between the Rényi divergence $D_\alpha(\cdot||\cdot)$, conditional Rényi divergences $D_\alpha^{\mathrm{V}}(\cdot|| \cdot |\cdot)$, mutual information measures $I_\alpha^{\mathrm{V}}(X;G)$, and the Rényi channel capacity $C_\alpha(p_{G|X})$ with $\alpha \geq 1$, and $\mathrm{V} \in \{\mathrm{S}, \mathrm{C}, \mathrm{BLP}\}$ a label specifying the measures of Sibson [199], Csiszár [64], and Bleuler-Lapidoth-Pfister [33]. The mutual information associated to the BLP-conditional-Rényi divergence was independently derived by Lapidoth-Pfister [133] and Tomamichel-Hayashi [223]. We particularly address the capacities generated by Sibson and Arimoto.

2.2.1 Rényi entropy

Definition 2.3. *(Rényi entropy [192]) The Rényi entropy of order $\alpha \in \overline{\mathbb{R}}$ of a PMF p_X is denoted by $H_\alpha(X)$. The orders $\alpha \in (-\infty, 0) \cup (0,1) \cup (1,\infty)$ are defined as:*

$$H_\alpha(X) := \frac{1}{1-\alpha} \log \left(\sum_x p(x)^\alpha \right). \tag{2.27}$$

The orders $\alpha \in \{0, 1, \infty, -\infty\}$ are defined by continuous extension of (2.27) as: $H_0(X) := \log|\mathrm{supp}(p_X)|$, $H_1(X) := H(X)$, with $H(X) := -\sum_x p(x)\log p(x)$ the Shannon entropy [63], $H_\infty(X) := -\log \max_x p(x) = -\log p_{\max}$, and $H_{-\infty}(X) := -\log \min_x p(x) = -\log p_{\min}$. The Rényi entropy is a function of the PMF p_X and therefore, one can alternatively write $H_\alpha(p_X)$. However, we keep the convention of writing $H_\alpha(X)$.

The Rényi entropy is mostly considered for positive orders, but it is also sometimes explored for negative values [230, 193, 227, 228]. In this thesis we use the whole spectrum $\alpha \in \overline{\mathbb{R}}$. We now consider the Arimoto-Rényi extension of the conditional entropy.

2.2.2 Arimoto-Rényi conditional entropy

Definition 2.4. *(Arimoto-Rényi conditional entropy [8]) The Arimoto-Rényi conditional entropy of order* $\alpha \in \overline{\mathbb{R}}$ *of a joint PMF* p_{XG} *is denoted as* $H_\alpha(X|G)$*. The orders* $\alpha \in (-\infty, 0) \cup (0,1) \cup (1,\infty)$ *are defined as:*

$$H_\alpha(X|G) := \frac{\alpha}{(1-\alpha)} \log \left[\sum_g \left(\sum_x p(x,g)^\alpha \right)^{\frac{1}{\alpha}} \right]. \tag{2.28}$$

The orders $\alpha \in \{0, 1, \infty, -\infty\}$ *are defined by continuous extension of (2.28) as:* $H_0(X|G) := \log \max_g |\mathrm{supp}(p_{X|G=g})|$, $H_1(X|G) := H(X|G)$*, with* $H(X|G) := -\sum_{x,g} p(x,g) \log p(x|g)$ *the conditional entropy [63],* $H_\infty(X|G) := -\log \sum_g \max_x p(x,g)$*, and* $H_{-\infty}(X|G) := -\log \sum_g \min_x p(x,g)$*. Arimoto-Rényi conditional entropy is a function of the joint PMF* p_{XG} *and therefore, one can alternatively write* $H_\alpha(p_{XG})$*. However, we keep the convention of writing* $H_\alpha(X|G)$*.*

We remark that there are alternative ways to extend the conditional entropy "à la" Rényi [84]. The Arimoto-Rényi conditional entropy is however, the only one (amongst five alternatives [84]) that simultaneously satisfy the following desirable properties for a conditional entropy [84]: i) monotonicity, ii) chain rule, iii) consistency with the Shannon entropy, and iv) consistency with the ∞ conditional entropy (also known as min-entropy). Consistency with the conditional entropy means that $\lim_{\alpha \to 1} H_\alpha(X|G) = H(X|G)$, and similarly for property iv). In this sense, one can think about the Arimoto-Rényi conditional entropy as the “most appropriate" Rényi-extension (if not the outright “proper" Rényi extension) of the conditional entropy. We now consider Arimoto's mutual information, and its associated Rényi channel capacity.

2.2.3 Arimoto's mutual information

Definition 2.5. *(Arimoto's mutual information [8]) Arimoto's mutual information of order* $\alpha \in \overline{\mathbb{R}}$ *of a joint PMF* p_{XG} *is given by:*

$$I_\alpha(X;G) := \mathrm{sgn}(\alpha) \left[H_\alpha(X) - H_\alpha(X|G) \right], \tag{2.29}$$

with the Rényi entropy (2.27) and the Arimoto-Rényi conditional entropy (2.28). The case $\alpha = 1$ *reduces to the standard mutual information [63]* $I_1(X;G) = I(X;G)$*, with* $I(X;G) := H(X) - H(X|G)$*. Arimoto's mutual information is a function of the joint PMF* p_{XG} *and therefore, one can alternatively write* $I_\alpha(p_{XG})$ *or* $I_\alpha(p_{G|X}p_X)$*, the latter taking into account that* $p_{XG} = p_{G|X}p_X$*. We use these three different notations interchangeably, so that we draw attention to the whole object* p_{XG} *or either* $p_{G|X}$ *or* p_X*.*

2.2.4 Arimoto-Rényi channel capacity

Definition 2.6. *(Rényi channel capacity [8, 64, 4, 154]) The Rényi channel capacity of order* $\alpha \in \overline{\mathbb{R}}$*, of a conditional PMF* $p_{G|X}$ *is given by:*

$$C_\alpha(p_{G|X}) := \max_{p_X} I_\alpha(p_{G|X}p_X) \tag{2.30}$$

with the maximisation over all PMFs p_X*, and Arimoto's mutual information (2.29). The case* $\alpha = 1$ *reduces to the standard channel capacity [63]* $C_1(p_{G|X}) = C(p_{G|X}) = \max_{p_X} I(X;G)$*.*

We remark that there are alternative candidates as Rényi-extensions of the mutual information [84, 232]. In particular, we highlight the mutual informations of: Sibson [199], Csiszár [64], and Bleuler-Lapidoth-Pfister [33], which we address as $I_\alpha^{\mathrm{V}}(X;G)$ with the label $\mathrm{V} \in \{\mathrm{S}, \mathrm{C}, \mathrm{BLP}\}$ representing each case. These mutual information measures are going to be useful, in particular, due to their connection to conditional Rényi divergences. We now extend these information-theoretic quantities to the quantum domain.

2.2.5 Rényi divergence

Definition 2.7. *(Rényi divergence [192, 229]) The Rényi divergence (R-divergence) of order $\alpha \in \overline{\mathbb{R}}$ of PMFs p_X and q_X is denoted as $D_\alpha(p_X||q_X)$. The orders $\alpha \in (-\infty, 0) \cup (0,1) \cup (1, \infty)$ are defined as:*

$$D_\alpha(p_X||q_X) := \frac{\operatorname{sgn}(\alpha)}{\alpha - 1} \log \left[\sum_x p(x)^\alpha q(x)^{1-\alpha} \right]. \tag{2.31}$$

The orders $\alpha \in \{1, 0, \infty, -\infty\}$ are defined define by continuous extension of (2.31) as:

$$D_1(p_X||q_X) := D(p_X||q_X), \tag{2.32}$$
$$D_0(p_X||q_X) := -\log \sum_{x \in \operatorname{supp}(p_X)} q(x), \tag{2.33}$$
$$D_\infty(p_X||q_X) := \log \max_x \frac{p(x)}{q(x)}, \tag{2.34}$$
$$D_{-\infty}(p_X||q_X) := -\log \min_x \frac{p(x)}{q(x)}. \tag{2.35}$$

with the standard Kullback-Leibler (KL) divergence given by $D(p_X||q_X) := \sum_x p(x) \log \frac{p(x)}{q(x)}$ [131, 63].

2.2.6 Conditional-Rényi (CR) divergences

Definition 2.8. *(Sibson's conditional-Rényi divergence [199]) The Sibson's conditional-Rényi divergence (S-CR-divergence) of order $\alpha \in \overline{\mathbb{R}}$ of PMFs $p_{X|G}$, $q_{X|G}$, and p_X is denoted as $D_\alpha^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X)$. The orders $\alpha \in (-\infty, 0) \cup (0,1) \cup (1, \infty)$ are defined as:*

$$D_\alpha^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) := \frac{\operatorname{sgn}(\alpha)}{\alpha - 1} \log \sum_x p(x) \sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}. \tag{2.36}$$

The orders $\alpha \in \{1, 0, \infty, -\infty\}$ are defined by continuous extension of (2.36) as:

$$D_1^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) := D(p_{G|X}||q_{G|X}|p_X), \tag{2.37}$$
$$D_0^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) := -\log \sum_{x \in \operatorname{supp}(p_X)} p(x) \sum_{g \in \operatorname{supp}(p_{G|X=x})} q(g|x), \tag{2.38}$$
$$D_\infty^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) := \log \max_{x \in \operatorname{supp}(p_X)} \max_g \frac{p(g|x)}{q(g|x)}, \tag{2.39}$$
$$D_{-\infty}^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) := -\log \min_{x \in \operatorname{supp}(p_X)} \min_g \frac{p(g|x)}{q(g|x)}, \tag{2.40}$$

with the conditional Rényi divergence given by $D\left(p_{G|X}||q_{G|X}|p_X\right) := D\left(p_{G|X}p_X||q_{G|X}p_X\right)$, *the latter being the standard KL-divergence [131, 63].*

Definition 2.9. *(Csiszár's conditional-Rényi divergence [64]) The Csiszár's conditional-Rényi divergence (C-CR-divergence) of order* $\alpha \in \overline{\mathbb{R}}$ *of PMFs* $p_{X|G}$, $q_{X|G}$, *and* p_X *is denoted as* $D^{\mathrm{C}}_{\alpha}(p_{G|X}||q_{G|X}|p_X)$. *The orders* $\alpha \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$ *are defined as:*

$$D^{\mathrm{C}}_{\alpha}(p_{G|X}||q_{G|X}|p_X) := \frac{\operatorname{sgn}(\alpha)}{\alpha - 1} \sum_x p(x) \log \left[\sum_g p(g|x)^{\alpha} q(g|x)^{1-\alpha} \right]. \tag{2.41}$$

The orders $\alpha \in \{1, 0, \infty, -\infty\}$ *are defined by continuous extension of* (2.41) *as:*

$$D^{\mathrm{C}}_{1}(p_{G|X}||q_{G|X}|p_X) := D(p_{G|X}||q_{G|X}|p_X), \tag{2.42}$$

$$D^{\mathrm{C}}_{0}(p_{G|X}||q_{G|X}|p_X) := -\sum_{x \in \operatorname{supp}(p_X)} p(x) \log \sum_{g \in \operatorname{supp}(p_{G|X=x})} q(g|x), \tag{2.43}$$

$$D^{\mathrm{C}}_{\infty}(p_{G|X}||q_{G|X}|p_X) := \sum_{x \in \operatorname{supp}(p_X)} p(x) \log \left[\max_g \frac{p(g|x)}{q(g|x)} \right], \tag{2.44}$$

$$D^{\mathrm{C}}_{-\infty}(p_{G|X}||q_{G|X}|p_X) := -\sum_{x \in \operatorname{supp}(p_X)} p(x) \log \left[\min_g \frac{p(g|x)}{q(g|x)} \right], \tag{2.45}$$

with the conditional Rényi divergence given by $D\left(p_{G|X}||q_{G|X}|p_X\right) := D\left(p_{G|X}p_X||q_{G|X}p_X\right)$, *the latter being the standard KL-divergence [131, 63].*

Definition 2.10. *(Bleuler-Lapidoth-Pfister conditional-Rényi divergence [33, 171]) The Bleuler-Lapidoth-Pfister conditional-Rényi divergence (BLP-CR-divergence) of order* $\alpha \in \overline{\mathbb{R}}$ *of PMFs* $p_{X|G}$, $q_{X|G}$, *and* p_X *is denoted as* $D^{\mathrm{BLP}}_{\alpha}(p_{G|X}||q_{G|X}|p_X)$. *The orders* $\alpha \in (-\infty, 0) \cup (0, 1) \cup (1, \infty)$ *are defined as:*

$$D^{\mathrm{BLP}}_{\alpha}(p_{G|X}||q_{G|X}|p_X) := \frac{|\alpha|}{\alpha - 1} \log \sum_x p(x) \left[\sum_g p(g|x)^{\alpha} q(g|x)^{1-\alpha} \right]^{\frac{1}{\alpha}}. \tag{2.46}$$

The orders $\alpha \in \{1, 0, \infty, -\infty\}$ *are defined by continuous extension of* (2.46) *as:*

$$D^{\mathrm{BLP}}_{1}(p_{G|X}||q_{G|X}|p_X) := D(p_{G|X}||q_{G|X}|p_X), \tag{2.47}$$

$$D^{\mathrm{BLP}}_{0}(p_{G|X}||q_{G|X}|p_X) := -\log \max_{x \in \operatorname{supp}(p_X)} \sum_{g \in \operatorname{supp}(p_{G|X=x})} q(g|x), \tag{2.48}$$

$$D^{\mathrm{BLP}}_{\infty}(p_{G|X}||q_{G|X}|p_X) := \log \sum_{x \in \operatorname{supp}(p_X)} p(x) \max_g \frac{p(g|x)}{q(g|x)}, \tag{2.49}$$

$$D^{\mathrm{BLP}}_{-\infty}(p_{G|X}||q_{G|X}|p_X) := -\log \sum_{x \in \operatorname{supp}(p_X)} p(x) \min_g \frac{p(g|x)}{q(g|x)}. \tag{2.50}$$

with the conditional Rényi divergence given by $D\left(p_{G|X}||q_{G|X}|p_X\right) := D\left(p_{G|X}p_X||q_{G|X}p_X\right)$, *the latter being the standard KL-divergence [131, 63].*

2.2.7 Relationship between the Rényi divergence and CR divergences

Remark 2.1. *([33, 171]) Relating conditional Rényi divergences to the Rényi divergence. For any conditional PMFs $p_{G|X}$, $q_{G|X}$, and any PMF p_X we have:*

$$D_\alpha^{\mathrm{S}}(p_{G|X}||q_{G|X}|p_X) = D_\alpha(p_{G|X}p_X||q_{G|X}p_X), \tag{2.51}$$

$$D_\alpha^{\mathrm{C}}(p_{G|X}||q_{G|X}|p_X) = \sum_x p(x) D_\alpha(p_{G|X=x}||q_{G|X=x}). \tag{2.52}$$

2.2.8 mutual informations

Definition 2.11. *(mutual information measures of: Sibson [199], Csiszár [64], and Bleuler-Lapidoth-Pfister [33]) The mutual information measures of Sibson, Csiszár, and Bleuler-Lapidoth-Pfister of order $\alpha \in \overline{\mathbb{R}}$ of a joint PMF p_{XG} are defined as:*

$$I_\alpha^{\mathrm{V}}(X;G) := \min_{q_G} D_\alpha^{\mathrm{V}}\left(p_{G|X}||q_G|p_X\right), \tag{2.53}$$

with the label $\mathrm{V} \in \{\mathrm{S}, \mathrm{C}, \mathrm{BLP}\}$ denoting each case, the minimisation being performed over all PMFs q_G, and $D_\alpha^{\mathrm{V}}(\cdot||\cdot|\cdot)$ the conditional Rényi (CR) divergences of: Sibson, Csiszár, and Bleuler-Lapidoth-Pfister, of order $\alpha \in \overline{\mathbb{R}}$, as defined previously. The case $\alpha = 1$ reduces, for all three cases, to the standard mutual information [63] $I_1^{\mathrm{V}}(X;G) = I(X;G)$. Similarly to Arimoto's measure, we also use the notation $I_\alpha^{\mathrm{V}}(p_{XG})$ and $I_\alpha^{\mathrm{V}}(p_{G|X}p_X)$ interchangeably.

2.2.9 Relationship between CR-divergences

Lemma 2.1. *Consider the conditional-Rényi divergences of Sibson, Csiszár, and Bleuler-Lapidoth-Pfister, then:*

$$\alpha \in [-\infty, 0],\ D_\alpha^{\mathrm{BLP}}(\cdots) \le D_\alpha^{\mathrm{C}}(\cdots) \le D_\alpha^{\mathrm{S}}(\cdots), \tag{2.54}$$

$$\alpha \in [0, 1],\ D_\alpha^{\mathrm{BLP}}(\cdots) \le D_\alpha^{\mathrm{S}}(\cdots) \le D_\alpha^{\mathrm{C}}(\cdots), \tag{2.55}$$

$$\alpha \in [1, \infty],\ D_\alpha^{\mathrm{C}}(\cdots) \le D_\alpha^{\mathrm{BLP}}(\cdots) \le D_\alpha^{\mathrm{S}}(\cdots), \tag{2.56}$$

Proof. The cases $\alpha \in [0, 1]$ and $\alpha \in [1, \infty]$ have already been proven in the literature [33]. A similar argument can be followed in order to prove the cases $\alpha \in [-\infty, 0]$. For completeness, we address it in what follows.
Part i) We start by proving that for $\alpha \in [-\infty, 0]$ we have $D_\alpha^{\mathrm{C}}(\cdot||\cdot|\cdot) \le D_\alpha^{\mathrm{S}}(\cdot||\cdot|\cdot)$. We prove it for $\alpha \in (-\infty, 0)$ and the extremes follow because of continuity. Starting from the Sibson's measure times the positive factor $(\alpha - 1)\,\mathrm{sgn}(\alpha)$ we get:

$$(\alpha - 1)\,\mathrm{sgn}(\alpha) D_\alpha^{\mathrm{S}}\left(p_{G|X}||q_G|p_X\right) = \log\left[\sum_x p(x) \sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}\right], \tag{2.57}$$

$$\ge \sum_x p(x) \log\left[\sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}\right], \tag{2.58}$$

$$= \mathrm{sgn}(\alpha)(\alpha - 1) D_\alpha^{\mathrm{C}}\left(p_{G|X}||q_G|p_X\right). \tag{2.59}$$

In the first equality we use the definition of the Sibson's conditional Rényi divergence (2.36). The inequality follows because of Jensen's inequality [126], and because $\log(\cdot)$ is a concave function. In the last equality we use the definition of the Csiszár's conditional Rényi divergence (2.41). Dividing both sides by $\mathrm{sgn}(\alpha)(\alpha - 1)$, which is positive because $\alpha \in (-\infty, 0)$, proves the claim.

Part ii) We now want to prove that for $\alpha \in [-\infty, 0]$, we have $D_\alpha^{\mathrm{BLP}}(\cdot||\cdot|\cdot) \le D_\alpha^{\mathrm{C}}(\cdot||\cdot|\cdot)$. Similarly, we prove it for cases $\alpha \in (-\infty, 0)$ with the extremes following because of continuity. Starting from Csiszár's measure:

$$D_\alpha^{\mathrm{C}}(p_{G|X}||q_G|p_X) = \frac{\mathrm{sgn}(\alpha)}{\alpha - 1}\sum_x p(x)\log\left[\sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}\right], \tag{2.60}$$

$$= \frac{|\alpha|}{\alpha - 1}\sum_x p(x)\log\left[\sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}\right]^{\frac{1}{\alpha}}, \tag{2.61}$$

$$\ge \frac{|\alpha|}{\alpha - 1}\log\left[\sum_x p(x)\left(\sum_g p(g|x)^\alpha q(g|x)^{1-\alpha}\right)^{\frac{1}{\alpha}}\right], \tag{2.62}$$

$$= D_\alpha^{\mathrm{BLP}}(p_{G|X}||q_G|p_X). \tag{2.63}$$

The first equality we use the definition of Csiszár's conditional Rényi divergence (2.41). In the second equality we multiply by one $1 = \frac{\alpha}{\alpha}$ and re-organise conveniently. The inequality follows because of Jensen's inequality [126], because $\log(\cdot)$ is a concave function, and because the coefficient $\frac{\mathrm{sgn}(\alpha)\alpha}{\alpha-1}$ is negative for $\alpha \in (-\infty, 0)$. In the last equality we use the definition of the Bleuler-Lapidoth-Pfister conditional Rényi divergence (2.46). □

2.2.10 Relationship between mutual information measures

Lemma 2.2. *Consider the mutual information measures of Sibson, Csiszár, and Bleuler-Lapidoth-Pfister, then:*

$$\alpha \in [-\infty, 0], \quad I_\alpha^{\mathrm{BLP}}(\cdot|\cdot) \le I_\alpha^{\mathrm{C}}(\cdot|\cdot) \le I_\alpha^{\mathrm{S}}(\cdot|\cdot), \tag{2.64}$$

$$\alpha \in [0, 1], \quad I_\alpha^{\mathrm{BLP}}(\cdot|\cdot) \le I_\alpha^{\mathrm{S}}(\cdot|\cdot) \le I_\alpha^{\mathrm{C}}(\cdot|\cdot), \tag{2.65}$$

$$\alpha \in [1, \infty], \quad I_\alpha^{\mathrm{C}}(\cdot|\cdot) \le I_\alpha^{\mathrm{BLP}}(\cdot|\cdot) \le I_\alpha^{\mathrm{S}}(\cdot|\cdot), \tag{2.66}$$

Proof. The cases $\alpha \in [0, 1]$ and $\alpha \in [1, \infty]$ were proven in [33], and they follow by considering the previous Lemma on the less or equal order between the conditional Rényi divergences and, by considering that the mutual information measures are defined in terms of the conditional Rényi divergences by minimising over p_X (2.53). The cases $\alpha \in [-\infty, 0]$ follow the same argument. □

2.2.11 Sibson-Arimoto-Rényi channel capacity

Having defined these mutual information measures, we now address the fact that some of them become equal when maximising over PMFs p_X, whilst keeping fixed the conditional PMF $p_{G|X}$.

Lemma 2.3. *(Rényi channel capacity [8, 64, 4]) The mutual information measures of Arimoto and Sibson of order $\alpha \in \overline{\mathbb{R}}$) become equal when maximised over p_X, and we refer to this quantity as the Rényi capacity of order α. The Rényi capacity of order $\alpha \in \overline{\mathbb{R}}$, of a conditional PMF $p_{G|X}$ is:*

$$C_\alpha(p_{G|X}) := \max_{p_X} I_\alpha^{\mathrm{V}}(p_{G|X}p_X), \tag{2.67}$$

with $\mathrm{V} \in \{\mathrm{A}, \mathrm{S}\}$*, the maximisation over all PMFs* p_X*, and the mutual information of Sibson as in* (2.53)*, and Arimoto's mutual information as in the main text. The case* $\alpha = 1$ *reduces to the standard channel capacity [63]* $C_1(p_{G|X}) = C(p_{G|X}) = \max_{p_X} I(X;G)$*.*

This Lemma, for the cases $\alpha \geq 0$, has been proven in different places in the literature [199, 64, 4]. For completeness, here we provide a proof for the cases $\alpha < 0$. We can understand this result as $C_\alpha(p_{G|X})$ being the Rényi capacity of the classical channel specified by the conditional PMF $p_{G|X}$, which simultaneously represents the mutual information measures of Arimoto and Sibson. On can similarly address Rényi capacities using the rest of mutual information measures, but using these two are enough for our purposes.

Proof. The cases for $\alpha \in [0, \infty)$ have been proven in different places in the literature [8, 64, 4]. We therefore only address here the interval $(-\infty, 0)$. Addressing Arimoto's measure for $\alpha \in (-\infty, 0)$:

$$\max_{p_X} I_\alpha^{\mathrm{A}}\left(p_{G|X} p_X\right) \overset{1}{=} \max_{p_X} \frac{|\alpha|}{\alpha - 1} \log \sum_g \left(\sum_x p(g|x)^\alpha \frac{p(x)^\alpha}{\sum_{x'} p(x')^\alpha} \right)^{\frac{1}{\alpha}}, \tag{2.68}$$

$$\overset{2}{=} \max_{r_X} \frac{|\alpha|}{\alpha - 1} \log \sum_g \left(\sum_x p(g|x)^\alpha r(x) \right)^{\frac{1}{\alpha}}. \tag{2.69}$$

In the first equality we replaced and reorganised the definition of Arimoto's mutual information (2.29). In the second equality we use the fact that both maximisations are equal, because from an optimal p_X^*, we can construct a feasible r_X as $r(x) := p^*(x)^\alpha / (\sum_{x'} p^*(x')^\alpha)$ and conversely, from an optimal r_X^*, we can construct a feasible p_X as $p(x) = r^*(x)^{\frac{1}{\alpha}} / (\sum_{x'} r^*(x')^{\frac{1}{\alpha}})$. We now relate the quantity in (2.69) to the quantity obtained from Sibson's. We now consider Sibson's CR-divergence and invoke the identity [64] $\forall p_{G|X}, q_G, p_X$:

$$D_\alpha^{\mathrm{S}}\left(p_{G|X} || q_G | p_X\right) = D_\alpha^{\mathrm{S}}\left(p_{G|X} || q_G^* | p_X\right) + D_\alpha\left(q_G^* | q_G\right), \tag{2.70}$$

with the PMF q_G^* given by:

$$q_G^*(g) := \frac{\left(\sum_x p(x) p(g|x)^\alpha\right)^{\frac{1}{\alpha}}}{\sum_g \left(\sum_x p(x) p(g|x)^\alpha\right)^{\frac{1}{\alpha}}}. \tag{2.71}$$

This identity can be checked by directly substituting (2.71) into the RHS of (2.70). We can now get an explicit expression for Sibson's mutual information, because minimising (2.70) over q_G is obtained for $q_G = q_G^*$, this, because the Rényi divergence is non-negative for $\alpha \in (-\infty, 0)$ [229]. We therefore get:

$$I_\alpha^{\mathrm{S}}\left(p_{G|X} p_X\right) = \min_{q_G} D_\alpha^{\mathrm{S}}\left(p_{G|X} || q_G | p_X\right), \tag{2.72}$$

$$= D_\alpha^{\mathrm{S}}\left(p_{G|X} || q_G^* | p_X\right), \tag{2.73}$$

$$= \frac{|\alpha|}{\alpha - 1} \log \sum_g \left(\sum_x p(x) p(g|x)^\alpha \right)^{\frac{1}{\alpha}}. \tag{2.74}$$

Maximising this quantity over p_X we get:

$$\max_{p_X} I_\alpha^{\mathrm{S}}\left(p_{G|X} p_X\right) = \max_{p_X} \frac{|\alpha|}{\alpha - 1} \log \sum_g \left(\sum_x p(x) p(g|x)^\alpha \right)^{\frac{1}{\alpha}}, \tag{2.75}$$

which is the same quantity than in (2.69) for Arimoto's measure. Altogether, we have that starting from either Sibson or Arimoto, we arrive to the same expression when maximising over p_X, as per equations (2.75) and (2.69). Consequently, the capacities they each define is the same, and thus proving the claim. □

2.2.12 Information-theoretic quantities in the quantum domain

We now move on to describe Arimoto's mutual information in this quantum setting, as well as the Rényi channel capacity.

Remark 2.2. *(Arimoto's mutual information in a quantum setting) We address Arimoto's dependence between two classical random variables encoded into quantum objects. Explicitly, the random variable X is encoded in an ensemble of states $\mathcal{E} = \{\rho_x, p(x)\}$ and therefore, we address it as $X_\mathcal{E}$. On the other hand, G is considered as the random variable obtained from a decoding measurement $\mathbb{D} = \{D_g = |g\rangle\langle g|\}$ and therefore, we address it as $G_\mathbb{D}$. We consider a conditional PMF as $p_{G|X}^{(\mathbb{M},\mathcal{S})}$, given by $p(g|x) := \mathrm{Tr}[D_g \Lambda_\mathbb{M}(\rho_x)]$, $\mathcal{S} := \{\rho_x\}$ a set of states, and the quantum-to-classical (measure-prepare) channel associated to the measurement $\mathbb{M}$ given by:*

$$\Lambda_\mathbb{M}(\sigma) := \sum_a \mathrm{Tr}[M_a \sigma] \, |a\rangle\langle a| \, , \tag{2.76}$$

with $\{|a\rangle\}$ an orthonormal basis. We effectively have $p(g|x) := \mathrm{Tr}[M_g \rho_x]$ and therefore we can think about the decoding variable $G_\mathbb{D}$ as $G_\mathbb{M}$. We are now interested in mutual information measures quantifying the dependence between variables $X_\mathcal{E}$ and $G_\mathbb{M}$, when encoded and decoded in the quantum setting described previously. We then consider the Arimoto's mutual information:

$$I_\alpha(X_\mathcal{E}; G_\mathbb{M}) := \mathrm{sgn}(\alpha) \left[H_\alpha(X_\mathcal{E}) - H_\alpha(X_\mathcal{E}|G_\mathbb{M}) \right], \tag{2.77}$$

with the standard Rényi entropy (2.27) and the Arimoto-Rényi conditional entropy (2.28) for the quantum conditional PMF described above.

Remark 2.3. *(Rényi capacity of a quantum conditional PMF) The Rényi capacity of order $\alpha \in \overline{\mathbb{R}}$ of a quantum conditional PMF $p_{G|X}^{(\mathbb{M},\mathcal{S})}$ is given by:*

$$C_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})}\right) := \max_{p_X} I_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X\right), \tag{2.78}$$

with the maximisation over all PMFs p_X.

The quantity we are interested in the quantum domain is the Rényi capacity of order α of a quantum-classical channel.

Definition 2.12. *(Rényi capacity of a quantum-classical channel) The Rényi capacity of order $\alpha \in \overline{\mathbb{R}}$ of a quantum-classical channel $\Lambda_\mathbb{M}$ associated to the measurement $\mathbb{M}$ is given by:*

$$C_\alpha(\Lambda_\mathbb{M}) := \max_{\mathcal{S}} C_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})}\right) = \max_{\mathcal{E}} I_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X\right), \tag{2.79}$$

with the maximisation over all sets of states $\mathcal{S} = \{\rho_x\}$ or over all ensembles $\mathcal{E} = \{\rho_x, p(x)\}$.

Mutual information measures in the quantum domain are defined via their Rényi conditional divergences counterparts as:

$$I_\alpha^{\mathrm{V}}(X_{\mathcal{E}}; G_{\mathbb{M}}) := \min_{q_G} D_\alpha^{\mathrm{V}}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right), \tag{2.80}$$

with the quantum conditional PMFs $p_{G|X}^{(\mathbb{M},\mathcal{E})}$ and $q_{G|X}^{(\mathbb{N},\mathcal{E})}$ given by $p(g|x) := \mathrm{Tr}(M_g \rho_x)$, $q(g|x) := \mathrm{Tr}(N_g \rho_x)$, respectively, the minimisation over all PMFs q_G, and the classical conditional Rényi divergences of: Sibson, Csiszár, and Bleuler-Lapidoth-Pfister, which we address with a label $\mathrm{V} \in \{\mathrm{S}, \mathrm{C}, \mathrm{BLP}\}$.

2.3 Background on Expected Utility Theory (EUT)

In this subsection we address the preliminary theoretical tools on expected utility theory. We start with the concept of risk in the theory of games and economic behaviour, a pair of games involving risk, and the quantities of certainty equivalent (CE) and the isoelastic certainty equivalent (ICE). References for this section are: the book by Eeckhoudt, Gollier, and Schlesinger [76], Parmigiani-Inoue-Lopes's book on decision theory [167], and Bonanno's book [36].

2.3.1 The concept of risk in the theory of games and economic behaviour

In expected utility theory [157], the level of 'satisfaction' of a rational agent, when *receiving* (obtaining, being awarded) a certain amount of wealth, or goods or services, is described by a utility function [157]. The utility function of a rational agent is a function $u : A \to \mathbb{R}$, with $A = \{a_i\}$ a the set of *alternatives* from which the rational agent can choose from. The set A is endowed with a binary relation $\preceq$. The utility function is asked to be a monotone for such a binary relation; if $a_1 \preceq a_2$ then $u(a_1) \leq u(a_2)$. We address the set of alternatives as representing *wealth* and therefore, it is enough to consider an interval of the real numbers.

We are going to consider two different types of situations. In the first case, the wealth will always be non-negative, and so we consider the interval being $A = \mathcal{I} = [0, w^M] \subseteq \mathbb{R}$, with $w^M > 0$ a maximal amount of wealth, and the standard binary relation $\leq$. Similarly, we also will also consider a situation where the wealth is *non-positive*, meaning we address a utility function taking negative arguments $w < 0$, with $\mathcal{I} = [-w^M, 0] \subseteq \mathbb{R}$, as the level of (dis)satisfaction when the rational agent has to *pay* an amount of money $|w|$ (or when the amount $|w|$ is taken away from him).

We note here that the utility function does not necessarily need to be positive (or negative), because it is only used to *compare* alternatives. The condition that the utility function is monotonic is the equivalent to it being an increasing function for both positive and negative wealth. Intuitively, this represents that the rational agent is interested in acquiring as much wealth as possible (for positive wealth), and losing the least amount of wealth as possible (for negative wealth). Additionally, the utility function is asked to be twice-differentiable, both for mathematical convenience and, because it is natural to assume that smooth changes in wealth imply smooth changes in the rational agent's satisfaction.

In order to address the concept of risk, we first need to introduce two games (or operational tasks), which involves a player Bob (the *Better* or *Gambler*, who we take to be a rational agent with a utility function u) and a referee Alice, who is in charge of the game. We are going to address two different games which we call here: i) *gain games* and ii) *loss games*.

2.3.2 A gain game and EUT

In a *gain game*, Alice (Referee) offers Bob (Gambler) the choice between two options: i) a fixed *guaranteed* amount of wealth $w^G \in [0, w^M]$ or ii) a *bet*. The bet consists of the following: Alice uses a random event distributed according to a probability mass function (PMF) p_W, (i.e. $\sum_{w\in\mathcal{I}} p_W(w) = 1$, $p_W(w) \geq 0$, $\forall w \in \mathcal{I}$, with W a random variable in the alphabet $\mathcal{I}$), in order to give Bob a reward. Specifically, Alice will reward Bob with an amount of wealth $w^B = w$, whenever the random event happens to be w, which happens with probability $p(w)$ (we drop the label W on $p_W(w)$ from now on). The choice facing Bob is therefore between a fixed guaranteed amount of wealth $w^G \in [0, w^M]$, or taking the bet and potentially earning more $w^B > w^G$, at the risk of earning less $w^B < w^G$.

Since the utility function $u(w)$ determines Bob's satisfaction when acquiring the amount wealth w, we will see below that it can be used to model his behaviour in this game, i.e. whether he chooses the first or second option. First, considering the bet (option ii) we can consider the *expected gain* of Bob at the end,

$$\mathbb{E}[W] = \sum_{w\in\mathcal{I}} p(w)w. \tag{2.81}$$

How satisfied Bob is with this expected amount of wealth is given by the utility of this value, i.e.

$$u\left(\mathbb{E}[W]\right) = u\left(\sum_{w\in\mathcal{I}} p(w)w\right). \tag{2.82}$$

Now, Bob's wealth at the end of the bet is a random variable, this means that his satisfaction will also be a random variable, with some uncertainty. We can also ask what Bob's expected satisfaction, i.e. *expected utility* will be at the end of the bet,

$$\mathbb{E}[u(W)] = \sum_{w\in\mathcal{I}} p(w)u(w). \tag{2.83}$$

This represents how satisfied Bob will be with the bet on average.

We can now introduce the first key concept, that of the *Certainty Equivalent (CE)*: it is the amount of (certain) wealth w^{CE} which Bob is as satisfied with as the average wealth he would gain from the bet. In other words, the amount of wealth which is as desirable as the bet itself. That is, it is the amount of wealth w^{CE} that satisfies

$$\mathbb{E}[u(W)] = u(w^{CE}). \tag{2.84}$$

It is crucial to note that the certainty equivalent wealth depends upon the utility function u and the PMF p_W, and therefore we interchangeably write it as $w^{CE}(u, p_W)$. We can now return to the original game, i.e. the choice between a fixed return w^G, or the average return $\mathbb{E}[W]$. The rational decision for Bob is to pick which of the two he is most satisfied with. We now see that if we set $w^G > w^{CE}$ then he will choose to take the guaranteed amount, if $w^G < w^{CE}$ he will choose the bet, and if $w^G = w^{CE}$ then in fact the two options are equivalent to Bob, and he can rationally pick either. That is, we see that the certainty equivalent w^{CE} sets the boundary between which option Bob will pick. Introducing the certainty equivalent moreover allows us to introduce the concept of Bob's *risk-aversion*. To do so, we will compare Bob's expected wealth, in relation to the certainty equivalent of the bet. There are only three possible scenarios,

$$w^{CE} < \mathbb{E}[W], \tag{2.85}$$

$$w^{CE} > \mathbb{E}[W], \tag{2.86}$$

$$w^{CE} = \mathbb{E}[W]. \tag{2.87}$$

In the first case (2.85), Alice can offer Bob an amount of wealth w^G that is larger than w^{CE} but less than $\mathbb{E}[W]$, $w^{CE} < w^G < \mathbb{E}[W]$ and Bob will rationally take this amount over accepting the bet, even though he will walk away with less wealth than the average he would have if he took the bet. In other words, Bob is *reluctant* to take the bet, and so we say that he is *risk-averse*.

In the second case (2.86), on the other hand, if Alice wants to make Bob walk away from the bet, and accept a fixed amount of wealth instead, she will have to offer him *more* than the expected gain. That is, Bob will only choose an amount w^G if $w^G > w^{CE} > \mathbb{E}[W]$. Here Bob is *risk-seeking*.

Finally, in the third case (2.87), Bob will take the bet if Alice offers him any w^G less than the expected gains from the bet, and will take the guaranteed amount w^G if it is larger. In this case, we say that Bob is *risk-neutral*, as Bob is essentially indifferent between the uncertain gains of the bet and the certain gains of the guaranteed return.

If we recall that by definition the utility function u is strictly increasing in the interval $\mathcal{I}$ (more wealth is also more satisfactory to Bob), then by applying u to the previous three equations, and using the definition of w^{CE} (2.84), we get

$$\mathbb{E}[u(W)] < u(\mathbb{E}[W]), \tag{2.88}$$

$$\mathbb{E}[u(W)] > u(\mathbb{E}[W]), \tag{2.89}$$

$$\mathbb{E}[u(W)] = u(\mathbb{E}[W]). \tag{2.90}$$

This is an important result, which shows that Bob's risk-aversion is characterised by the curvature of his utility function: Bob is risk-averse when his utility function is concave (2.88), risk-seeking when his utility function is convex (2.89), and risk-neutral when it is linear (2.90). This intuitively makes sense, since roughly speaking this corresponds to his satisfaction growing more slowly than wealth when he is risk-averse and his satisfaction growing faster than wealth when he is risk-seeking. We now move on to analyse the concept of risk in our second game.

2.3.3 A loss game and EUT

Let us now analyse a game which we call here a *loss game*. Similarly to the gain game from the previous section, in an loss game we have two agents, a Referee (Alice) and a Gambler (Bob), who has to make a payment to the Referee. In an loss game Bob is now asked to choose between two options: i) paying a fixed amount of wealth $|w^F|$, $w^F \in [-w^M, 0]$ or ii) a bet. Choosing the bet means Bob has to pay an amount of wealth according to the outcome of a PMF p_W. Similarly to the gain game, we address some quantities of interest: *expected debt* ($\mathbb{E}(W)$), *expected utility* ($\mathbb{E}[u(W)]$), and the *certainty equivalent (CE)* $w^{CE}(u, p_W)$, as the amount of wealth w^{CE} such that $u(w^{CE}) = \mathbb{E}[u(W)]$. We note the CE depends on the utility function u representing the Player, and the PMF p_W representing the bet. The CE is the amount of wealth that Bob pays to Alice, which generates the same level of (dis)satisfaction, had Bob opted for the bet instead. We also note here that both the expected debt and the certainty equivalent are now negative quantities.

We now analyse the meaning of the certainty equivalent in loss games, i.e., where Bob (the Gambler) has to choose between having to pay a certain fixed amount of wealth (fixed debt) $|w^F|$, or paying an average amount (average debt) $|\mathbb{E}[W]|$. The rational decision for Bob is to pick which of the two options he is *more satisfied* (equivalently, we could say least dissatisfied) with. We then see that if we set $w^F < w^{CE}$ he then will choose to take the bet, if $w^F > w^{CE}$ he will choose to pay the fixed amount, and if $w^F = w^{CE}$ he can rationally pick either. That is, we see that the certainty equivalent w^{CE} again sets here the boundary between which option Bob will pick in an loss game.

We now compare Bob's expected debt $\mathbb{E}[W]$ and the certainty equivalent of the bet w^{CE}. We have the three possible scenarios,

$$w^{CE} < \mathbb{E}[W], \quad \longleftrightarrow \quad |w^{CE}| > |\mathbb{E}[W]|, \tag{2.91}$$
$$w^{CE} > \mathbb{E}[W], \quad \longleftrightarrow \quad |w^{CE}| < |\mathbb{E}[W]|, \tag{2.92}$$
$$w^{CE} = \mathbb{E}[W], \quad \longleftrightarrow \quad |w^{CE}| = |\mathbb{E}[W]|. \tag{2.93}$$

In the first case (2.91), Alice can *request* from Bob a fixed amount of wealth $|w^F|$ as $w^{CE} < w^F < \mathbb{E}[W]$, which is equivalent to $|w^{CE}| > |w^F| > |\mathbb{E}[W]|$ and Bob will still *prefer* to pay this amount over opting for the bet, even though he will *potentially have to pay less* $|\mathbb{E}(W)|$, on average, had he opted for the bet. In other words, Bob is *reluctant* to take the bet, and so we see that he is *risk-averse*.

In the second case (2.92), if Alice wants to make Bob walk away from choosing the bet, and accept *paying* a fixed amount of wealth instead, she will have to offer him a deal where he has to pay *less* than the CE (and in turn less than the expected debt). In other words, in this case Bob is confident that the bet will allow him to pay less than the expected debt. That is, Bob will choose paying a fixed amount $|w^F|$ only if $w^F > w^{CE} > \mathbb{E}[W]$, which is equivalent to $|w^F| < |w^{CE}| < |\mathbb{E}[W]|$. Here Bob can then be considered as *risk-seeking*, because he is hopeful/optimistic about having the chance of paying less than the expected debt.

Taking into account the utility function is still an strictly increasing function for negative wealth, together with the definition of the certainty equivalent we get:

$$\mathbb{E}[u(W)] < u(\mathbb{E}[W]), \tag{2.94}$$
$$\mathbb{E}[u(W)] > u(\mathbb{E}[W]), \tag{2.95}$$
$$\mathbb{E}[u(W)] = u(\mathbb{E}[W]). \tag{2.96}$$

This means that in an loss game we can also characterise the risk tendencies of a Gambler in terms of the concavity/convexity/linearity of his utility function as: risk-averse (concavity (2.94)), risk-seeking (convexity (2.95)), risk-neutral (linear (2.96)). This characterisation of risk tendencies and the types of games are going to be useful later on when introducing more elaborate games in the form of operational tasks involving the discrimination or exclusion of quantum states. We now move on to the quantification of risk.

2.3.4 Quantifying risk tendencies

We can go one step further, and not only classify whether Bob (the Gambler) is risk-averse, risk-seeking, or risk-neutral, but moreover *quantify* how risk-averse he is. Let us start by addressing a *gain game*, which means we are interested in analysing Bob being represented by an utility function on positive wealth. Since Bob's attitude toward risk relates to the concavity/convexity/linearity of the utility function u, it is natural that the second derivative of the function is going to play a role. This, because u is concave on an interval if and only if its second derivative is non-positive on that interval. However, it is also desirable for measures representing risk to be invariant under *affine transformations* of the utility function, which in this context means that they are invariant under transformations of the form $u \to a + bu$, with $a, b \in \mathbb{R}$. This is because the actual values of utility aren't themselves physical, but only the *comparison* between values, and therefore rescaling or displacing the utility should not alter how risk-averse we quantify Bob to be. Given these requirements, a natural measure that emerges is the so-called Relative Risk Aversion (RRA) measure[1]:

$$RRA(w) := -w \frac{u^{''}(w)}{u'(w)}. \tag{2.97}$$

This measure assigns *positive* values for risk-averse players in a gain game (*concave* utility functions of positive wealth) because we have: i) $w > 0$, because we are considering the player *receiving* money ii) $u^{''}(w) < 0, \forall w$, because a risk-averse player in a gain game is represented by a concave function, and iii) $u^{'}(w) > 0$, because the utility function is a strictly increasing function. An analysis of signs then yields $RRA(w) > 0$.

Similarly, we now also analyse this measure of risk-aversion when Bob plays a *loss game*. A loss game is characterised by *negative wealth*, and we have already derived the fact that that a risk-averse Gambler is also characterised by a *concave* utility function. We now want to quantify the degree of risk-aversion of a Gambler playing the loss game, and therefore we then can proceed in a similar fashion as before, and define the risk-aversion measure RRA.

We now check that this measure assigns *negative* values for risk-averse players in a loss game (*concave* utility functions of negative wealth) because we have: i) $w < 0$ because we are considering the player *paying* money ii) $u^{''}(w) < 0, \forall w$, because a risk-averse player in a loss game is represented by a concave function, and iii) $u^{'}(w) > 0$, because the utility function is a strictly increasing function. An analysis of signs yields $RRA(w) < 0$. We can see that this is the opposite to what happens in gain games, where $RRA(w) > 0$ represents risk-averse players. We highlight this fact in Table 2.1, and present an analysis of the sign of the RRA measure for the two types of players (risk-averse or risk-seeking) and the two types of games (gain game or loss game).

2.3.5 Isoelastic Certainty Equivalent (ICE)

We now note that the RRA measure does not assign a global value for how risk averse Bob is, but allows this to depend upon the wealth w, i.e. Bob may be more or less risk averse depending on the wealth that is at stake. In order to remove this, it is usual to consider those utility functions where Bob's relative risk aversion is *constant*, independent of wealth. In this case, (2.97) can be solved assuming $RRA(w) = R$, which leads to the so-called *isoelastic utility function* for positive and negative wealth

[1] An additional benefit of this quantifier is that it is dimensionless, which is not satisfied by all quantifiers of risk-aversion

	Risk-averse player $u''(w) < 0$	Risk-seeking player $u''(w) > 0$
$w > 0$	$RRA(w) > 0$	$RRA(w) < 0$
$w < 0$	$RRA(w) < 0$	$RRA(w) > 0$

TABLE 2.1: Analysis of the sign of the quantity $RRA(w)$ for the different regimes being considered. We have that the utility function is always strictly increasing, meaning that $u'(w) > 0$, and therefore we then only need to analyse the signs of w and $u''(w)$. In particular, we have that risk-averse players are represented by positive RRA when dealing with positive wealth, and by negative RRA when dealing with negative wealth.

as:

$$u_R(w) := \begin{cases} \operatorname{sgn}(w) \frac{|w|^{1-R}-1}{1-R}, & \text{if } R \neq 1 \\ \operatorname{sgn}(w) \ln(|w|), & \text{if } R = 1 \end{cases}, \tag{2.98}$$

with the auxiliary “sign" function:

$$\operatorname{sgn}(w) := \begin{cases} 1, & w \geq 0; \\ -1, & w < 0. \end{cases} \tag{2.99}$$

The parameter R varies from minus to plus infinity, describing all possible risk tendencies of Bob, for either positive or negative wealth. For positive wealth for instance, R goes from maximally risk-seeking at $R = -\infty$, passing through risk-neutral at $R = 0$, to maximally risk-averse at $R = \infty$. In Fig. 2.2 we can see the behaviour of the isoelastic function for positive wealth and different values of R.

The certainty equivalent (2.84) for this setup can be calculated for either positive or negative wealth as:

$$w_R^{ICE} = u_R^{-1}\left(\mathbb{E}[u_R(W)]\right) = \left(\sum_{w \in \mathcal{I}} w^{1-R}\, p(w)\right)^{\frac{1}{1-R}}. \tag{2.100}$$

The certainty equivalent of the isoelastic function, or *isoelastic certainty equivalent* (ICE), is going to play an important role in this thesis. As we have already seen, the CE stands out as an important quantity because it: i) determines the choice of a Gambler when playing either a gain or loss game, helping to establish the characterisation of risk tendencies of said Gambler and ii) optimising the CE is equivalent to optimising the *expected utility*, given that the utility function is a strictly increasing function and that $u(w^{ICE}) = \mathbb{E}[u(W)]$. One may be tempted here to propose the *expected utility* function $\mathbb{E}[u(W)]$ as the figure of merit instead of the CE, but the expected utility unfortunately suffers from having the rather awkward set of units $[w]^{1-R}$, whilst the certainty equivalent on the other hand has simply units of wealth $[w]$ (\$, £, ...).

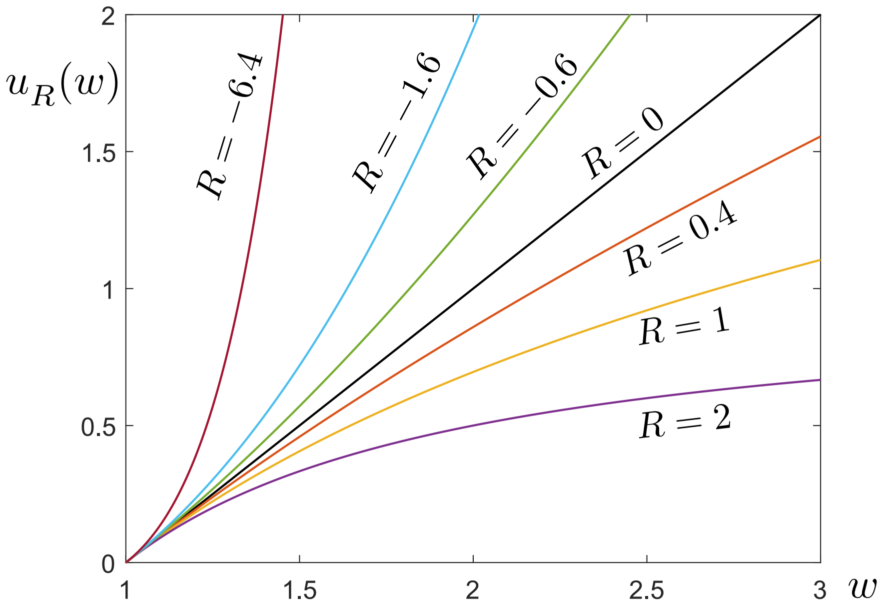

FIGURE 2.2: Isoelastic utility function $u_R(w)$ (2.98) as a function of positive wealth ($1 \leq w \leq 3$) for players with different risk tendencies (different values of R). The risk parameter R quantifies different types of risk tendencies: i) $R < 0$ risk-seeking players (convex) ii) $R = 0$ risk-neutral players (linear), and iii) $R > 0$ risk-averse players (concave). Risk-aversion for positive wealth then increases from $-\infty$ to ∞.

Chapter 3

Weight of informativeness, exclusion games, and excudible information

"You keep on learning and learning, and pretty soon you learn something no one has learned before."

Richard Feynman

In this chapter we consider the QRT of measurement informativeness and introduce a weight-based quantifier for informativeness. We show that this quantifier has operational significance from the perspective of quantum state exclusion (QSE) games, by showing that it precisely captures the advantage a measurement provides in minimising the error in this game. We furthermore introduce information theoretic quantities related to exclusion, in particular the notion of excludible information of a quantum channel, and show that for the case of quantum-to-classical channels it is determined precisely by the weight of informativeness. This establishes a three-way correspondence which sits in parallel to the correspondence in QRTs between robustness-based quantifiers, discrimination games, and accessible information [204]. This new correspondence between a weight-based quantifier and an exclusion-based task found here suggests that this is a generic correspondence that holds in the context of general QRTs.

3.1 Introduction and motivation

Quantum phenomena can be seen as a resource for fuelling quantum information protocols. In this regard, the framework of *Quantum Resource Theories* (QRTs) has been put forward in order to address these phenomena within a common unifying framework [57]. There are several QRTs of different quantum *objects* addressing different properties (of the object) as a *resource*. We can then broadly classify QRTs by first specifying the objects of the theory, followed by the property to be harnessed as a resource.

One of the main goals within the framework of QRTs is to define *resource quantifiers* for abstract QRTs, so that resources of different objects can be quantified and compared in a fair manner. There are different measures for quantifying resources, depending on the type of QRT being considered [57]. In particular, when considering *convex* QRTs, well-studied geometric quantifiers include the so-called *robustness-based* [234, 208, 172, 174, 155, 55, 138, 122] and *weight-based* [79, 136, 206, 182, 49, 44] quantifiers.

In addition to quantifying the amount of resource present in a quantum object, it is also of interest to develop practical applications in the form of *operational tasks* that explicitly take advantage of specific given resources, as well as to identify adequate resources and quantifiers characterising already existing operational tasks. In this regard, a general correspondence between robustness-based measures and *discrimination-based* operational tasks has recently been established: steering for subchannel discrimination [172], incompatibility for ensemble discrimination [203, 47, 151], coherence for unitary discrimination [174] and informativeness for state discrimination [204]. This correspondence initially considered for specific QRTs and resources, has been extended to QRT of states, measurements and channels with *arbitrary* resources [217, 212]. Furthermore, it turns out that when considering QRTs of measurements there exists an additional correspondence to single-shot information-theoretic quantities [204]. This three-way correspondence, initially considered for the resource of informativeness [204], has been extended to convex QRTs of measurements with *arbitrary* resources [212].

It is then natural to ask whether operational tasks can be devised in which, *weight-based* quantifiers play the relevant role. We conceptually address this motivation as a diagramme in Figure 3.1.

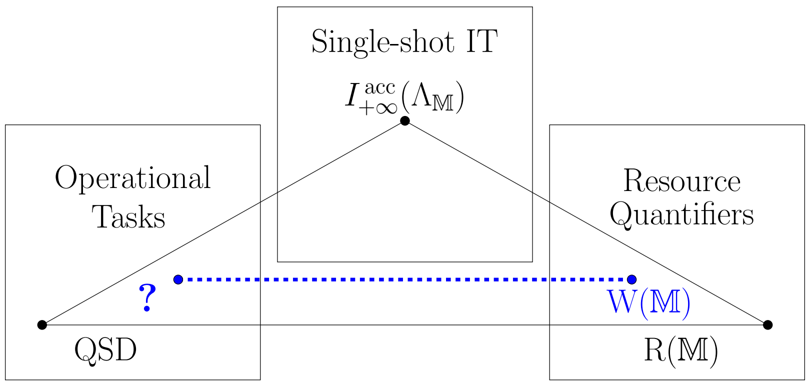

FIGURE 3.1: Motivation for the results presented in this chapter. The triangular correspondence (black) was proven in [204]. The starting question now is whether there exist operational tasks characterised by the quantifier of weight of informativeness.

Surprisingly, in this chapter we prove that one does not need to design any contrived operational task, but that there are natural operational tasks which are precisely characterised by these weight-based quantifiers, namely, the so-called *exclusion-based* operational tasks. Furthermore, we prove that these weight-based quantifiers for the QRTs of measurements also happen to satisfy a stronger three-way correspondence, establishing again a link to single-shot information-theoretic quantities.

This parallel three-way correspondence establishes that, in addition to robustness-based quantifiers, weight-based quantifiers also play a relevant role in the characterisation of operational tasks. We conjecture that the *weight-exclusion correspondence* found in this chapter holds for arbitrary QRTs of different objects beyond those of measurements. In the follow-up chapter, we support this conjecture by showing that this is the case for weight-based resource quantifiers in convex QRTs of states with arbitrary resources and therefore, providing an operational interpretation to these weight-based resource quantifiers as well.

3.2 Convex QRT of measurement informativeness

A general *resource theory* consists of: a set of *objects*, the identification of a *property* of these objects to be considered as a resource, and a consequent bipartition of the set of objects into *resourceful* and *free* objects. If the set of free objects forms a convex set, we say that we have a convex resource theory. In this section we focus on the convex QRT of quantum measurements with the resource of informativeness.

Definition 3.1. *(Convex QRT of measurement informativeness) Consider the set of Positive-Operator Valued Measures (POVMs) acting on a Hilbert space of dimension d. A POVM* $\mathbb{M}$ *is a collection of POVM elements* $\mathbb{M} = \{M_a\}$ *with* $a \in \{1, ..., o\}$ *satisfying* $M_a \geq 0$ $\forall a$ *and* $\sum_a M_a = \mathbb{1}$. *We now consider the resource of informativeness [204]. We say a measurement is uninformative when there exists a probability distribution* $q(a)$ *such that* $M_a = q(a)\mathbb{1}$, $\forall a$. *We say that the measurement is informative otherwise.*

One can check that the set of uninformative measurements forms a convex set and therefore, defines a convex QRT of measurements. It will be useful to introduce the notion of simulability of measurements.

Definition 3.2. *(Classical post-processing (CPP) or simulability of measurements [100]) We say that a measurement* $\mathbb{N} = \{N_x\}$, $x \in \{1, ..., k\}$ *is simulable by the measurement* $\mathbb{M} = \{M_a\}$, $a \in \{1, ..., o\}$ *when there exists a conditional probability distribution* $\{q(x|a)\}$ *such that* $\forall x$ *we have:*

$$N_x = \sum_a q(x|a) M_a. \tag{3.1}$$

One can check that the simulability of measurements defines a partial order for the set of measurements and therefore we use the notation $\mathbb{N} \preceq \mathbb{M}$, *meaning that* $\mathbb{N}$ *is simulable by* $\mathbb{M}$. *Simulability of the measurement* $\mathbb{N}$ *can be understood as a classical post-processing of the measurement* $\mathbb{M}$.

3.3 Quantum State Exclusion (QSE) games

We consider a game first formalised in [15] for analysing the Pusey-Barrett-Rudolph (PBR) theorem [181]. The property considered by PBR has been addressed under different names like antidistinguishability [107] or not-Post-Peierls compatibility (Post-Peierls incompatibility) [74, 53]. We adopt an operational approach here, so this property can be understood as considering that the game of state exclusion is won with probability one, or conclusive (perfect) state exclusion [15, 150]. The game of quantum state exclusion (QSE) has been explored under noisy channels [107], as well as its communication complexity properties [169, 145].

Operational Task 1. *(Quantum state exclusion (QSE) [15]) A referee has a collection of states* $\{\rho_x\}$, $x \in \{1, ..., k\}$, *and promises to send a player the state* ρ_x *with probability* $p(x)$. *The goal is for the player to output a guess* $g \in \{1, ..., k\}$ *of a state that was not sent. That is, the player succeeds at the game if* $g \neq x$ *and fails when* $g = x$. *A given quantum state exclusion game is fully specified by an ensemble* $\mathcal{E} = \{\rho_x, p(x)\}$.

This state exclusion game can be seen as being opposite to the game of state discrimination, in which the goal is to correctly identify the state that was sent. Since now the goal is to guess the state that was *not* sent, this game is referred to as *excluding*, rather than discriminating.

We are interested in quantum strategies for the player in this game using a fixed *resourceful* measurement $\mathbb{M}$, and how this compares to the best quantum strategy with *free* measurements (classical strategy). We will quantify how well the player does by the probability of error in excluding a state, which should be as small as possible.

Classical Protocol 1. *The best strategy for a classical player, one that is either unable to perform any quantum measurement, or allowed only to perform uninformative measurements, is easily seen to be to output the index of the least probable state. In this case, the minimal probability of error is:*

$$P_{\text{err}}^{\text{C}}(\mathcal{E}) = \min_{x} \; p(x). \tag{3.2}$$

Quantum Protocol 1. *On the other hand, we consider that the player has the ability to perform a single quantum measurement $\mathbb{M} = \{M_a\}$ with o outcomes. The player could nevertheless simulate a measurement $\mathbb{N} = \{N_x\}$ with k outcomes, according to (3.1), and use the measurement result as the guess of which state to exclude. The minimum probability of error following this strategy is then:*

$$P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}) = \min_{\mathbb{N} \preceq \mathbb{M}} \sum_{x} p(x) \operatorname{Tr}[N_x \rho_x], \tag{3.3}$$

with the minimisation being performed over all POVMs $\mathbb{N}$ that are simulable by $\mathbb{M}$ (3.1).

We are interested in comparing classical and quantum strategies for different games $\mathcal{E}$. In general, the player will have a smaller probability of error using a quantum strategy compared to a classical strategy, and hence $P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}) / P_{\text{err}}^{\text{C}}(\mathcal{E}) \leq 1$. We are interested in the optimal *advantage* that can be obtained by a fixed measurement $\mathbb{M}$ compared to the best classical strategy, over all games $\mathcal{E}$, i.e. in how small the ratio between quantum and classical error probabilities can be made. In the next section we will show that this is precisely characterised by the weight of informativeness.

3.4 Weight of informativeness

We now define a weight-based quantifier for informativeness. The idea is to geometrically quantify the amount of resource contained in an object. This quantifier was originally introduced in the so-called "EPR2 paper" [79] in the context of Bell-nonlocality and it was later independently rediscovered in [136] in the context of entanglement. This quantifier has several different names such as: part, content, cost and weight. In order to keep consistency with recent notation in the literature, we adopt *weight* in this Thesis.

Definition 3.3. *(Weight of informativeness) The weight of informativeness of a measurement $\mathbb{M} = \{M_a\}$ is given by:*

$$\operatorname{WoI}(\mathbb{M}) = \min_{\substack{w \geq 0 \\ \{q(a)\} \\ \mathbb{N}}} \left\{ w \,\middle|\, M_a = w N_a + (1 - w) q(a) \mathbb{1} \right\}, \tag{3.4}$$

where $\{q(a)\mathbb{1}\}$ *is an uninformative measurement and* $\mathbb{N} = \{N_a\}$ *is a general POVM,* $N_a \geq 0$, $\forall a$, $\sum_a N_a = \mathbb{1}$. *The weight quantifies the minimal amount with which some resourceful measurement* $\mathbb{N}$ *needs to be used in order to reproduce* $\mathbb{M}$. *Evaluating the WoI is a semi-definite program (SDP) [37] and hence it can be solved efficiently numerically.*

Lemma 3.1. *(Properties of* WoI*) The weight of informativeness (7.2) satisfies the following properties. (i)* Faithfulness*:* $\text{WoI}(\mathbb{M}) = 0 \leftrightarrow \mathbb{M} = \{M_a = q(a)\mathbb{1}\}$*. (ii)* Convexity*: given two measurements* $\mathbb{M}_1, \mathbb{M}_2$ *and* $p \in [0,1]$ *we have* $\text{WoI}\left(p\mathbb{M}_1 + (1-p)\mathbb{M}_2\right) \leq p\,\text{WoI}(\mathbb{M}_1) + (1-p)\text{WoI}(\mathbb{M}_2)$*. (iii)* Monotonicity *under measurement simulation:* $\mathbb{N} \preceq \mathbb{M} \rightarrow \text{WoI}(\mathbb{N}) \leq \text{WoI}(\mathbb{M})$*. (iv)* Explicit form $\text{WoI}(\mathbb{M}) = 1 - \sum_a \lambda_{\min}(M_a)$*, where* $\lambda_{\min}(\cdot)$ *is the smallest eigenvalue. (v)* Upper bounded by one*:* $0 \leq \text{WoI}(\mathbb{M}) \leq 1$, $\forall \mathbb{M}$.

Proof. We address the *optimal triple* associated to $\text{W}(\mathbb{M}) = w^*$ as $(w^*, q^*, \mathbb{N}^*)$ so that:

$$M_a = (1-w^*)q^*(a)\mathbb{1} + w^* N_a^*, \qquad \forall a. \tag{3.5}$$

Part (i). For the necessary condition we have that if $w^* = \text{WoI}(\mathbb{M}) = 0$, substituting this in (3.5), we have $M_a = q^*(a)\mathbb{1}$. For the sufficient condition we have that if $M_a = m(a)\mathbb{1}$, we are interested in triples $(w, q, \mathbb{N})$ allowing the decomposition $m(a)\mathbb{1} = (1-w)q(a)\mathbb{1} + wN_a$. We choose a trial function $q(a) := m(a)\ \forall a$ for which we have that $w = 0$, which is the minimum possible and so $w^* = w = 0$ with $q^*(a) = q(a)$.

Part (ii). Let us consider two measurements $\mathbb{M}_1 = \{M_{1a}\}, \mathbb{M}_2 = \{M_{2a}\}$ with respective quantities $\text{WoI}(\mathbb{M}_1), \text{WoI}(\mathbb{M}_2)$ and their associated optimal triples $(w_1^*, q_1^*, \mathbb{N}_1^*)$ and $(w_2^*, q_2^*, \mathbb{N}_2^*)$ satisfying:

$$\begin{aligned} M_{1a} &= (1-w_1^*)q_1^*(a)\mathbb{1} + w_1^* N_{1a}^*, \\ M_{2a} &= (1-w_2^*)q_2^*(a)\mathbb{1} + w_2^* N_{2a}^*. \end{aligned}$$

We now consider the quantities for $p \in [0,1]$:

$$\begin{aligned} &pM_{1a} + (1-p)M_{2a} \\ &= p\left[(1-w_1^*)q_1^*(a)\mathbb{1} + w_1^* N_{1a}^*\right] + (1-p)\left[(1-w_2^*)q_2^*(a)\mathbb{1} + w_2^* N_{2a}^*\right]. \end{aligned} \tag{3.6}$$

We now define the variables:

$$\begin{aligned} \tilde{w} &= pw_1^* + (1-p)w_2^*, \\ \tilde{q}(a) &= \frac{p(1-w_1^*)q_1^*(a) + (1-p)(1-w_2^*)q_2^*(a)}{1-\tilde{w}}, \\ \tilde{N}_a &= \frac{pw_1^* N_{1a}^* + (1-p)w_2^* N_{2a}^*,}{\tilde{w}}, \end{aligned}$$

and then we can rewrite (3.6) as:

$$pM_{1a} + (1-p)M_{2a} = (1-\tilde{w})\tilde{q}(a)\mathbb{1} + \tilde{w}\tilde{N}_a. \tag{3.7}$$

We now consider the quantity $\text{WoI}\left[p\mathbb{M}_1 + (1-p)\mathbb{M}_2\right]$ with associated optimal triple $(W^*, Q^*, \mathbb{N}^*)$ and therefore $\forall a$:

$$pM_{1a} + (1-p)M_{2a} = (1-W^*)Q^*(a)\mathbb{1} + W^* N_a^*. \tag{3.8}$$

Now comparing (4.30) with (3.7) we have that:

$$W^* \leq \tilde{w},$$

because W^* is the optimal, and therefore obtaining:

$$\mathrm{WoI}\left[p\mathbb{M}_1 + (1-p)\mathbb{M}_2\right] \leq p\,\mathrm{WoI}(\mathbb{M}_1) + (1-p)\mathrm{WoI}(\mathbb{M}_2).$$

Part (iii). Let us consider that $\mathbb{M}' \preceq \mathbb{M}$ which means:

$$M'_b = \sum_a p(b|a)M_a, \qquad \forall a. \tag{3.9}$$

We now consider the quantity $\mathrm{WoI}(\mathbb{M})$ and its associated optimal triple $(w^*, q^*, \mathbb{N}^*)$ then $\forall a$:

$$M_a = (1-w_1^*)q^*(a)\mathbb{1} + w_1^* N_a^*. \tag{3.10}$$

Substituting (3.10) in (3.9) we have:

$$\begin{aligned}
M'_b &= \sum_a p(b|a)M_a, \\
&= \sum_a p(b|a)\Big[(1-w_1^*)q^*(a)\mathbb{1} + w_1^* N_a^*\Big], \\
&= (1-w^*)\sum_a p(b|a)q^*(a)\mathbb{1} + w^*\sum_a p(b|a)N_a^*, \\
&= (1-w^*)\tilde{q}(b)\mathbb{1} + w^*\tilde{N}_b,
\end{aligned} \tag{3.11}$$

where in the last line we have defined the quantities $\tilde{q}(b) = \sum_a p(b|a)q^*(a)$ and $\tilde{N}_b = \sum_a p(b|a)N_a^*$. We now consider the quantity $\mathrm{WoI}(\mathbb{M}')$ and its associated optimal triple $(W^*, Q^*, \mathbb{M}^*)$. From (3.11) we have that w^* is a candidate for being W^* but we have that W^* is optimal and therefore $W^* \leq w^*$ which is equivalent to $\mathrm{WoI}(\mathbb{M}') \leq \mathrm{WoI}(\mathbb{M})$.

Part (iv) and (v). By definition we have that $\mathrm{WoI}(\mathbb{M}) \geq 0$ so we now check the upper bound. Let us start again with the weight of informativeness of a measurement $\mathbb{M} = \{M_a\}$. Renaming $\tilde{N}_a = wN_a$ and $\tilde{q}(a) = (1-w)q(a)$ we have that $\forall a$:

$$M_a - \tilde{q}(a)\mathbb{1} = \tilde{N}_a \geq 0. \tag{3.12}$$

Minimising w is equivalent to maximising $(1-w)$ and together with $\sum_a \tilde{q}(a) = 1-w$ we have:

$$1 - \mathrm{WoI}(\mathbb{M}) = \max_{w\geq 0}\,\{1-w\} = \max_{\tilde{q}} \sum_a \tilde{q}(a).$$

We can now explicitly define a primal SDP as:

$$\begin{aligned}
1 - \mathrm{WoI}(\mathbb{M}) = \max_{\tilde{q}} \sum_a \tilde{q}(a)&, \\
\text{s.t. } M_a - \tilde{q}(a)\mathbb{1} \geq 0&, \quad \forall a.
\end{aligned} \tag{3.13}$$

With the later inequality being the constraint (3.12). The constraint means that $M_a \geq \tilde{q}(a)\mathbb{1}$ and so $\max_{\tilde{q}} \sum_{a=1}^{o} \tilde{q}(a) = \sum_a \lambda_{\min}(M_a)$ with $\lambda_{\min}(M_a)$ the smallest eigenvalue of M_a and therefore:

$$\mathrm{WoI}(\mathbb{M}) = 1 - \sum_a \lambda_{\min}(M_a).$$

The operators M_a are POVM elements, $M_a \geq 0$, which means that $\lambda_{\min}(M_a) \geq 0$ and so $\mathrm{WoI}(\mathbb{M}) \leq 1$. The upper bound is achieved by any measurement such that all the POVM elements are non-full-rank. For example, a rank-1 (projective) measurement $\Pi = \{\Pi_a\}, \Pi_a \geq 0, \sum_a \Pi_a = \mathbb{1}, \Pi_a \Pi_b = \delta_{ab}\Pi_a$ has maximal weight of informativeness, since $\lambda_{\min}(\Pi_a) = 0\ \forall a$ and therefore $\mathrm{WoI}(\Pi) = 1$. □

These properties demonstrate that the weight of informativeness is good measure of measurement informativeness. We now show that it also has operational significance, by considering QSE games.

3.5 Main Results

3.5.1 Result 3.1. Weight of informativeness and QSE games

In this section we establish a first result relating the weight of informativeness of a measurement with its performance in the game of state exclusion.

Result 3.1. *Consider a state exclusion game in which the player is sent a state from the ensemble $\mathcal{E} = \{\rho_x, p(x)\}$. The optimal advantage offered by the measurement $\mathbb{M}$ over any classical strategy is given by:*

$$\min_{\mathcal{E}} \frac{P^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E}, \mathbb{M})}{P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E})} = 1 - \mathrm{WoI}(\mathbb{M}). \tag{3.14}$$

This shows that for all exclusion games the WoI bounds the decrease in error probability that can be obtained for any $\mathcal{E}$, and that there exists a game $\mathcal{E}^*$ where this decrease is given precisely by the WoI.

The proof consists of two parts. First we prove that the WoI lower bounds the advantage for all tasks $\mathcal{E}$. Then we prove that this lower bound can be achieved by extracting an optimal ensemble $\mathcal{E}^*$ out of the dual SDP formulation of the WoI.

First part: In this first part we prove that:

$$[1 - \mathrm{WoI}(\mathbb{M})] P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E}) \leq P^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E}, \mathbb{M}), \quad \forall \mathcal{E}, \mathbb{M}. \tag{3.15}$$

Proof. Let us start with the weight of informativeness of a measurement as given by (7.2). Consider that the minimum is achieved with the triple $(q^*, \mathbb{N}^*, w^*)$ so that $\forall a$:

$$M_a - (1 - w^*) q^*(a) \mathbb{1} = w^* N_a \geq 0,$$

which implies that

$$M_a \geq [1 - \mathrm{WoI}\,(\mathbb{M})]\, q^*(a) \mathbb{1}. \tag{3.16}$$

where we use the fact that $w^* = \mathrm{WoI}\,(\mathbb{M})$. We now address the probability of error in state exclusion:

$$\begin{aligned} P^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E},\mathbb{M}) &= \min_{\mathbb{M}\succeq\mathbb{N}} \sum_x \mathrm{Tr}\,(N_x\tilde{\rho}_x)\,, \\ &= \min_{\{p(x|a)\}} \sum_x \mathrm{Tr}\left\{\left[\sum_a p(x|a)M_a\right]\tilde{\rho}_x\right\}, \\ &\geq \min_{\{p(x|a)\}} \sum_x \mathrm{Tr}\left\{\left[\sum_a p(x|a)\left[(1-\mathrm{WoI}(\mathbb{M}))q(a)\mathbb{1}\right]\right]\tilde{\rho}_x\right\}, \\ &= \min_{\{p(x|a)\}} \sum_x\sum_a p(x)p(x|a)(1-\mathrm{WoI}(\mathbb{M}))q(a), \\ &= (1-\mathrm{WoI}(\mathbb{M}))\min_{\{p(x|a)\}} \sum_x\sum_a p(x)p(x|a)q(a). \end{aligned}$$

We use $\tilde{\rho}_x = p(x)\rho_x$. In the third line we used the inequality (3.16). We now use the fact that $p(x) \geq P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E}), \forall x$ and that $\sum_x p(x|a) = 1, \forall a$ and so we obtain:

$$\begin{aligned} P^{\mathrm{Q}}_{\mathrm{err}}(\mathcal{E},\mathbb{M}) &\geq (1-\mathrm{WoI}(\mathbb{M}))\min_{\{p(x|a)\}} \sum_x\sum_a P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E})p(x|a)q(a), \\ &= (1-\mathrm{WoI}(\mathbb{M}))P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E})\sum_a q(a), \\ &= (1-\mathrm{WoI}(\mathbb{M}))P^{\mathrm{C}}_{\mathrm{err}}(\mathcal{E}). \end{aligned}$$

□

Before proving the second part, let us consider the dual formulation of the primal SDP for the weight of informativeness [37]. We start by addressing the primal SDP for the weight of informativeness (3.13). We want to maximise the function $f = \sum_{a=1}^{o}\tilde{q}(a)$ under the constraints that $M_a - \tilde{q}(a)\mathbb{1} \geq 0\ \forall a$ which is equivalent to the constraint that $\forall\{\rho_a \geq 0\}\ \mathrm{Tr}\left[\rho_a(M_a - \tilde{q}(a)\mathbb{1})\right] \geq 0$ which implies that $\sum_a \mathrm{Tr}\left[\rho_a(M_a - \tilde{q}(a)\mathbb{1})\right] \geq 0$. We now write the Lagrangian function using this last constraint as:

$$L = \sum_a \tilde{q}(a) + \sum_a \mathrm{Tr}\{\rho_a[M_a - \tilde{q}(a)\mathbb{1}]\}. \tag{3.17}$$

Let us first note that by construction we have that:

$$L \geq \sum_a \tilde{q}(a). \tag{3.18}$$

We now rearrange (3.17) to get:

$$L = \sum_a \tilde{q}(a)[1-\mathrm{Tr}(\rho_a)] + \sum_a \mathrm{Tr}(\rho_a M_a).$$

Imposing the condition $1 - \mathrm{Tr}(\rho_a) = 0\ \forall a$ we have that:

$$L = \sum_a \mathrm{Tr}(\rho_a M_a).$$

Using this together with (3.18) we have:

$$L = \sum_a \mathrm{Tr}(\rho_a M_a) \geq \sum_a \tilde{q}(a).$$

Considering now maximising over $\{\tilde{q}\}$ we see that

$$L = \sum_a \mathrm{Tr}(\rho_a M_a) \geq \max_{\tilde{q}} \sum_a \tilde{q}(a) = 1 - \mathrm{WoI}(\mathbb{M}).$$

Furthermore, by minimising over $\{\rho_a\}$, and by *strong duality* [37], which guarantees the equality, we have:

$$\begin{aligned}\min_{\{\rho_a\}} L &= \min_{\{\rho_a\}} \sum_a \mathrm{Tr}(\rho_a M_a),\\ &= \max_{\tilde{q}} \sum_a \tilde{q}(a) = 1 - \mathrm{WoI}(\mathbb{M}).\end{aligned}$$

We then have the dual SDP of (3.13):

$$\begin{aligned}1 - \mathrm{WoI}(\mathbb{M}) = &\min_{\{\rho_a\}} \sum_a \mathrm{Tr}(\rho_a M_a),\\ &\text{s.t.}\rho_a \geq 0, \quad \mathrm{Tr}(\rho_a) = 1 \quad \forall a.\end{aligned} \tag{3.19}$$

This dual SDP is going to be useful in what follows.

Second part: (Achieving lower bound) In this second part we prove that $\forall \mathbb{M}$, $\exists \mathcal{E}^{\mathbb{M}}$ such that:

$$[1 - \mathrm{WoI}(\mathbb{M})] P_{\text{err}}^{\text{C}} \left(\mathcal{E}^{\mathbb{M}}\right) \geq P_{\text{err}}^{\text{Q}}(\mathcal{E}^{\mathbb{M}}, \mathbb{M}), \quad \forall \mathbb{M}. \tag{3.20}$$

Proof. We now claim that the optimal ensemble (for achieving the lower bound in (3.20)) is given by $\mathcal{E}^{\mathbb{M}} = \{\rho_a^{\mathbb{M}}, \frac{1}{o}\}$, $a = 1, ..., o$, $P_{\text{err}}^{\text{C}}\left(\mathcal{E}^{\mathbb{M}}\right) = \frac{1}{o}$ and $\{\rho_a^{\mathbb{M}}\}$ the set of operators coming from the dual SDP (3.19) for a given $\mathbb{M}$. The set $\{\rho_a^{\mathbb{M}}\}$ then satisfies:

$$1 - \mathrm{WoI}(\mathbb{M}) = \sum_a \mathrm{Tr}\left(\rho_a^{\mathbb{M}} M_a\right).$$

The probability of error in quantum state exclusion for the ensemble $\mathcal{E}^{\mathbb{M}}$ and the measurement $\mathbb{M}$ is then given by:

$$\begin{aligned}P_{\text{err}}^{\text{Q}}(\mathcal{E}^{\mathbb{M}}, \mathbb{M}) &= \min_{\mathbb{N} \prec \mathbb{M}} \sum_a \mathrm{Tr}\left(N_a \rho_a^{\mathbb{M}} \frac{1}{o}\right),\\ &= \min_{\mathbb{N} \prec \mathbb{M}} \frac{1}{o} \sum_a \mathrm{Tr}\left(N_a \rho_a^{\mathbb{M}}\right).\end{aligned}$$

Given the measurement $\mathbb{M}$, we now choose *not* to simulate any measurement $\mathbb{N}$ but to play with $\mathbb{M}$ instead so:

$$\begin{aligned}&\leq \frac{1}{o} \sum_a \mathrm{Tr}\left(M_a \rho_a^{\mathbb{M}}\right),\\ &= \frac{1}{o}[1 - \mathrm{WoI}(\mathbb{M})],\\ &= P_{\text{err}}^{\text{C}}\left(\mathcal{E}^{\mathbb{M}}\right)[1 - \mathrm{WoI}(\mathbb{M})].\end{aligned}$$

Putting together the inequalities (3.15) and (3.20) we obtain the claim in Result 1:

$$1 - \mathrm{WoI}(\mathbb{M}) = \min_{\mathcal{E}} \frac{P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M})}{P_{\text{err}}^{\text{C}}(\mathcal{E})}.$$

□

This establishes for the first time an operational interpretation of a weight-based quantifier, making a link to state exclusion, and thus establishing a connection between these two previously unrelated concepts.

3.5.2 Result 3.2. Connection to single-shot information theory

We now analyse the game of state exclusion from a different angle, of a communication task in information theory. Consider a hypothetical situation whereby a person needs to de-activate a bomb, by cutting an appropriate wire. The bomb will only explode if the blue wire is cut – if any wire is cut it will be deactivated. The person at the bomb doesn't know this, but is on the phone with a knowledgeable person, who tells them what to do. If the phoneline is noisy, what is the safest way to communicate this information? Instead of trying to faithfully communicate 'blue' (i.e. encoding which wire *not* to cut), a better coding strategy may be to communicate as the wire *to* cut, the wire which is least likely to be wrongly decoded as 'blue'.

Thus, in contrast to the usual communication problem, which is about faithfully identifying (or discriminating) information, the above example shows that there are communication problems where the goal is to *exclude* information. The ability of a channel to allow for faithful discrimination may be completely different from its ability to faithfully exclude and in general, different coding strategies should be employed.

Consider then a random variable X, distributed according to $p(x)$, for which an outcome should be successfully excluded, the error probability is $P_{\text{err}}(X) = \min_x p(x)$. The entropy associated with this error probability is the order minus-infinity Rényi entropy, $H_{-\infty}(X) = -\log P_{\text{err}}(X)$, which we shall call the 'exclusion entropy'. Consider a channel specified by the conditional probability distribution $p(y|x)$. The conditional error probability at the outcome of the channel is $P_{\text{err}}(X|Y) = \sum_y p(y) \min_x p(x|y)$ and the associated conditional exclusion entropy is $H_{-\infty}(X|Y) = -\log P_{\text{err}}(X|Y)$. The reduction in exclusion entropy is then associated to what we shall call the mutual exclusion information between X and Y, $I_{-\infty}(X;Y) = H_{-\infty}(X|Y) - H_{-\infty}(X)$.

We can now define the 'excludible' information of a quantum channel $\Lambda(\cdot)$ by optimising over all encodings, i.e. input ensembles $\mathcal{E} = \{p(x), \rho_x\}$, and all decodings, i.e. measurements $\mathbb{D} = \{D_g\}_g$:

Definition 3.4. *The single-shot excludible information of the quantum channel $\Lambda(\cdot)$ is:*

$$I_{-\infty}^{\text{exc}}(\Lambda) = \max_{\mathcal{E},\mathbb{D}} I_{-\infty}(X;G), \tag{3.21}$$

where $p(g|x) = \text{Tr}[\Lambda(\rho_x)D_g]$ is the conditional probability distribution of the outcome of the (decoding) measurement, applied to the output of the channel.

We now extend the above weight-exclusion correspondence to a three-way correspondence, by showing that the WoI is also related to the excludible information (3.21) of the quantum-to-classical channel $\Lambda_{\mathbb{M}}(\cdot)$ naturally associated to a measurement via

$$\Lambda_{\mathbb{M}}(\rho) = \sum_a |a\rangle\langle a| \, \text{Tr}[M_a \rho], \tag{3.22}$$

where $\{|a\rangle\}$ forms an arbitrary basis for the output Hilbert space of the channel.

Result 3.2. *The single-shot excludible information of a quantum-to-classical channel $\Lambda_{\mathbb{M}}$ of the form* (3.22) *is specified by the WoI and is given by:*

$$I_{-\infty}^{\mathrm{exc}}(\Lambda_{\mathbb{M}}) = -\log\left[1 - \mathrm{WoI}(\mathbb{M})\right]. \tag{3.23}$$

Proof. In this section we calculate the the single-shot excludible information, which we show is specified in terms of the weight of informativeness. In particular,

$$I_{-\infty}^{\mathrm{exc}}(\Lambda_{\mathbb{M}}) = \max_{\mathcal{E},\mathbb{D}} I_{-\infty}(X;G), \tag{3.24}$$

with the mutual exclusion information:

$$I_{-\infty}(X;G) = H_{-\infty}(X|G) - H_{-\infty}(X), \tag{3.25}$$

and the exclusion entropy and conditional exclusion entropy given by:

$$H_{-\infty}(X) = -\log \min_x p(x) = -\log P_{\mathrm{err}}^{\mathrm{C}}(\mathcal{E}), \tag{3.26}$$

$$H_{-\infty}(X|G) = -\log \sum_g \min_x p(x,g), \tag{3.27}$$

with $p(x,g) = p(x)p(g|x)$ and $p(g|x) = \mathrm{Tr}\left[\Lambda_{\mathbb{M}}(\rho_x)D_g\right] = \sum_a \mathrm{Tr}(M_a\rho_x)\langle a| D_g |a\rangle$. Choosing $D_g = |g\rangle\langle g|$ so that $\langle a| D_g |a\rangle = \delta_g^a$ and substituting we have:

$$\begin{aligned} H_{-\infty}(X|G) &= -\log \sum_g \min_x p(x) \sum_a \mathrm{Tr}(M_a\rho_x)\delta_g^a \\ &= -\log \sum_g \min_x p(x)\mathrm{Tr}(M_g\rho_x). \end{aligned} \tag{3.28}$$

Considering $f_g(x) = p(x)\mathrm{Tr}(M_g\rho_x)$ and using:

$$\min_x f_g(x) = \min_{\{p(x|g)\}} \sum_x p(x|g) f_g(x),$$

we have:

$$\begin{aligned} H_{-\infty}(X|G) &= -\log \sum_g \min_{\{p(x|g)\}} \sum_x p(x|g) f_g(x), \\ &= -\log \sum_g \min_{\{p(x|g)\}} \sum_x p(x|g)p(x)\mathrm{Tr}(M_g\rho_x). \end{aligned}$$

Denoting $\tilde{\rho}_x = p(x)\rho_x$, and re-arranging, this is equivalent to

$$\begin{aligned} H_{-\infty}(X|G) &= -\log \min_{\{p(x|g)\}} \sum_x \mathrm{Tr}\left[\left(\sum_g p(x|g)M_g\right)\tilde{\rho}_x\right], \\ &= -\log \min_{\mathbb{N}\prec\mathbb{M}} \sum_x \mathrm{Tr}(N_x\tilde{\rho}_x), \\ &= -\log P_{\mathrm{err}}^{\mathrm{Q}}(\mathcal{E},\mathbb{M}). \end{aligned} \tag{3.29}$$

Combining (3.29) and (3.26) with (3.25) we obtain:

$$I_{-\infty}(X;G) = \log\left[\frac{P_{\mathrm{err}}^{\mathrm{C}}(\mathcal{E})}{P_{\mathrm{err}}^{\mathrm{Q}}(\mathcal{E},\mathbb{M})}\right]. \tag{3.30}$$

Substituting now (3.30) into (3.24) we have:

$$
\begin{aligned}
I^{\text{exc}}_{-\infty}(\Lambda_{\mathbb{M}}) &= \max_{\mathcal{E},\mathbb{D}} I_{-\infty}(X;G), \\
&= \max_{\mathcal{E},\mathbb{D}} \log\left[\frac{P^{\text{C}}_{\text{err}}(\mathcal{E})}{P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M})}\right], \\
&= \max_{\mathcal{E},\mathbb{D}} -\log\left[\frac{P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M})}{P^{\text{C}}_{\text{err}}(\mathcal{E})}\right], \\
&= -\min_{\mathcal{E},\mathbb{D}} \log\left[\frac{P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M})}{P^{\text{C}}_{\text{err}}(\mathcal{E})}\right], \\
&= -\log\left[\min_{\mathcal{E},\mathbb{D}} \frac{P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M})}{P^{\text{C}}_{\text{err}}(\mathcal{E})}\right], \\
&= -\log\left[1-\text{WoI}(\mathbb{M})\right].
\end{aligned}
$$

In the last line we have used Result 1 (7.18). □

This result parallels the finding that robustness of informativeness is related to the single-shot accessible (rather than excludible) information of the associated channel, $I^{\text{acc}}_{+\infty}(\Lambda_{\mathbb{M}}) = \log\left[1+\text{RoI}(\mathbb{M})\right]$ (see [204] for definitions).

3.5.3 Result 3.3. Complete set of monotones

We have already seen that the simulability of measurements defines a partial order for the set of measurements (3.1). We now show that the probabilities of error at the state exclusion game are intimately connected to simulation, providing a complete set of monotones for the partial order.

Result 3.3. *Consider two measurements $\mathbb{M}$ and $\mathbb{N}$. The measurement $\mathbb{M}$ can simulate the measurement $\mathbb{N}$, $\mathbb{M} \succeq \mathbb{N}$, via (3.1), if and only if:*

$$
P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M}) \leq P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{N}), \quad \forall\, \mathcal{E} = \{p(x),\rho_x\}. \tag{3.31}
$$

That is, a measurement $\mathbb{M}$ can simulate a measurement $\mathbb{N}$ if and only if it is never worse in any state exclusion game $\mathcal{E}$.

First part: (necessary condition) Let us address the necessary condition:

$$
\mathbb{M} \succeq \mathbb{M}' \implies P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M}) \leq P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M}') \quad \forall \mathcal{E}.
$$

Proof. Let us consider the probability of error in state exclusion:

$$
\begin{aligned}
P^{\text{Q}}_{\text{err}}(\mathcal{E},\mathbb{M}') &= \min_{\mathbb{M}'\succeq\mathbb{N}'} \sum_x \operatorname{Tr}\left(N'_x\tilde{\rho}_x\right), \\
&= \min_{\{p(x|b)\}} \sum_x \operatorname{Tr} \sum_b p(x|b) M'_b \tilde{\rho}_x, \\
&= \min_{\{p(x|b)\}} \sum_x \operatorname{Tr} \sum_b p(x|b) \sum_a q(b|a) M_a \tilde{\rho}_x, \\
&= \min_{\{p(x|b)\}} \sum_x \operatorname{Tr} \left[\sum_a r(x|a) M_a\right] \tilde{\rho}_x.
\end{aligned}
$$

In the third line we have used the fact that $\mathbb{M} \succeq \mathbb{M}'$ which means that $M'_b = \sum_a q(b|a)M_a$, $\forall b$. We furthermore introduced the conditional probability $\{r(x|a)\}$ such that:

$$r(x|a) = \sum_b p(x|b)q(b|a).$$

This may not be the most general set of conditional probabilities, therefore

$$\begin{aligned} P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}') &\geq \min_{\{p(x|a)\}} \sum_x \text{Tr} \sum_a p(x|a) M_a \tilde{\rho}_x, \\ &= \min_{\mathbb{M} \succeq \mathbb{N}} \sum_x \text{Tr} \left(N_x \tilde{\rho}_x \right), \\ &= P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}), \end{aligned}$$

and therefore obtaining:

$$P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}') \geq P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}),$$

as required. □

Second part: (sufficient condition) We now address the sufficient condition:

$$\mathbb{M} \succeq \mathbb{M}' \impliedby P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}) \leq P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}') \quad \forall \mathcal{E}.$$

Proof. Let us start by assuming that the right-hand side is true. We now want to prove that $\mathbb{M} \succeq \mathbb{M}'$ which is equivalent to $\sum_a q(x|a)M_a = M'_x$. Let us continue by considering the inequality:

$$\begin{aligned} 0 &\geq P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}) - P_{\text{err}}^{\text{Q}}(\mathcal{E}, \mathbb{M}'), \quad \forall \mathcal{E} \\ &= \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x \text{Tr} \left(N_x \tilde{\rho}_x \right) - \min_{\mathbb{N}' \preceq \mathbb{M}'} \sum_x \text{Tr} \left(N'_x \tilde{\rho}_x \right), \\ &\geq \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x \text{Tr} \left(N_x \tilde{\rho}_x \right) - \sum_x \text{Tr} \left(M'_x \tilde{\rho}_x \right), \\ &= \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x \text{Tr} \left[\left(N_x - M'_x \right) \tilde{\rho}_x \right], \\ &= \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x \text{Tr} \left[\left(\sum_a p(x|a) M_a - M'_x \right) \tilde{\rho}_x \right]. \end{aligned} \tag{3.32}$$

In the third line we have chosen *not* to simulate any measurement $\mathbb{N}'$ but to keep $\mathbb{M}'$. Let us now define the operators and the magnitude:

$$\Delta_x(\mathbb{M}, \mathbb{M}') = \sum_a p(x|a) M_a - M'_x, \quad \forall x, \tag{3.33}$$

$$\Delta(\mathcal{E}, \mathbb{M}, \mathbb{M}') = \sum_{x=1}^{k} \text{Tr} \Big[\Delta_x(\mathbb{M}, \mathbb{M}') \tilde{\rho}_x \Big]. \tag{3.34}$$

Then the quantity in (3.32) becomes:

$$0 \geq \min_{\mathbb{N} \preceq \mathbb{M}} \Delta(\mathcal{E}, \mathbb{M}, \mathbb{M}').$$

This last equation is valid $\forall\mathcal{E}$ and therefore it is in particular, valid for the ensemble that maximises the magnitude:

$$
\begin{aligned}
0 &\geq \max_{\mathcal{E}} \min_{\mathbb{N}\preceq\mathbb{M}} \Delta(\mathcal{E},\mathbb{M},\mathbb{M}'),\\
0 &\geq \min_{\mathbb{N}\preceq\mathbb{M}} \max_{\mathcal{E}} \Delta(\mathcal{E},\mathbb{M},\mathbb{M}'),
\end{aligned} \tag{3.35}
$$

where we have used the minimax theorem to interchange orders. If $\Delta_x = \hat{0}\ \forall x$, we obtain the desired result that $\sum_a p(x|a)M_a = M'_x$. The idea now is to prove that if we assume otherwise, we obtain a contradiction. We then assume that:

$$
\Delta_x(\mathbb{M},\mathbb{M}') = \left(\sum_a p(x|a)M_a - M'_x\right) \neq \hat{0}, \quad \forall x. \tag{3.36}
$$

One can directly check that we also have:

$$
\sum_x \Delta_x(\mathbb{M},\mathbb{M}') = \hat{0}. \tag{3.37}
$$

It follows then that i) the operators $\{\Delta_x\}$ cannot *all* be positive, since this would be in contradiction to (3.37) ii) $\{\Delta_x\}$ cannot *all* be negative, since this also leads to a contradiction with (3.37) iii) $\{\Delta_x\}$ cannot *all* be the zero operator (by assumption (3.36)). Therefore, the set $\{\Delta_x\}$ has to contain *at least*: one positive and one negative operator. Let us consider the positive operator. There exists then at least one x, say x^*, such that $\Delta_{x^*} > \hat{0}$, which means that it has to have at least one positive eigenvalue $\lambda^{\text{pos}}_{x^*} > 0$ with eigenvector $|\lambda^{\text{pos}}_{x^*}\rangle$. Δ_{x^*} is a Hermitian operator and therefore is diagonalisable as $\Delta_{x^*} = \sum_i \lambda^i |\lambda^i_{x^*}\rangle\langle\lambda^i_{x^*}|$ with $\{|\lambda^i_{x^*}\rangle\}$ forming an orthonormal basis. Equivalently, we can write this as:

$$
\Delta_{x^*} = \lambda^{\text{pos}}_{x^*} |\lambda^{\text{pos}}_{x^*}\rangle\langle\lambda^{\text{pos}}_{x^*}| + \sum_{i\neq\text{pos}} \lambda^i_{x^*} \left|\lambda^i_{x^*}\rangle\langle\lambda^i_{x^*}\right|. \tag{3.38}
$$

We now consider an ensemble $\mathcal{E}^* = \{\delta^{x^*}_x, \rho_x\}$ with $\rho_{x^*} = |\lambda^{\text{pos}}_{x^*}\rangle\langle\lambda^{\text{pos}}_{x^*}|$, and the rest of states being arbitrary. With this ensemble we calculate the quantity in (3.34)

$$
\begin{aligned}
\Delta(\mathcal{E}^*,\mathbb{M},\mathbb{M}') &= \sum_x \mathrm{Tr}\left[\Delta_{x^*} |\lambda^{\text{pos}}_{x^*}\rangle\langle\lambda^{\text{pos}}_{x^*}|\, \delta^{x^*}_x\right],\\
&= \mathrm{Tr}\Big[\Delta_{x^*} |\lambda^{\text{pos}}_{x^*}\rangle\langle\lambda^{\text{pos}}_{x^*}|\Big],\\
&= \lambda^{\text{pos}}_{x^*} > 0.
\end{aligned}
$$

This is in contradiction with (3.35). This follows because from (3.35) we have $\Delta(\mathcal{E}^*) \leq \max_{\mathcal{E}} \Delta(\mathcal{E}) \leq 0$. Therefore, the assumption made in (3.36) is not true, which means that:

$$
\Delta_x(\mathbb{M},\mathbb{M}') = \sum_a p(x|a)M_a - M'_x = \hat{0}, \quad \forall x,
$$

from which we obtain

$$
M'_x = \sum_a p(x|a)M_a,
$$

or that $\mathbb{M}$ simulates $\mathbb{M}'$, $\mathbb{M} \succeq \mathbb{M}'$. □

This result shows then that the probabilities of error over all state exclusion games form a complete set of (decreasing) monotones for the partial order of measurement simulation. It is interesting to note that it was previously shown that the probability of succeeding in state discrimination also forms a complete set of (increasing) monotones for measurement simulation [204]. Hence, we now have a second, independent, complete set of monotones.

3.6 Summary of Results

In Figure 4.1 we have a diagrammatic representation of the three-way correspondence found in this chapter, depicted as the inner triangle. Explicitly, we prove that for convex QRTs of measurements with the resource of *informativeness*, the weight of informativeness quantifies both; the advantage of informative over uninformative measurements in the operational task of *state exclusion* [15], and a new type of single-shot information (of the quantum-classical channel induced by a measurement) associated to a novel communication problem.

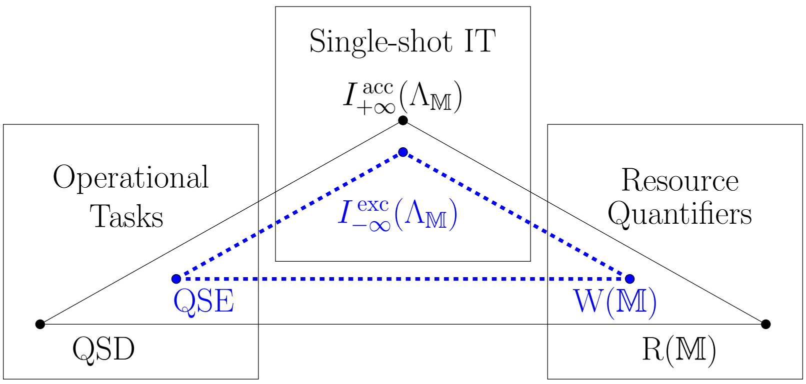

FIGURE 3.2: Summary of Results. Three-way correspondence between: operational tasks, resource quantifiers and single-shot information-theoretic quantities for the QRT of measurement informativeness. The outer three-way correspondence is linking [204]; quantum state discrimination (QSD), robustness of informativeness (RoI) and single-shot accessible information $I^{\mathrm{acc}}_{+\infty}(\Lambda_{\mathbb{M}})$. In this chapter, we derive a parallel three-way correspondence (inner triangle) linking: weight of informativeness (WoI), quantum state exclusion (QSE) and single-shot excludible information $I^{\mathrm{exc}}_{-\infty}(\Lambda_{\mathbb{M}})$. Definitions of these quantities in the main text.

3.7 Conclusions

In this section we have introduced a weight-based quantifier of measurement informativeness and shown that it has an operational interpretation as the biggest advantage that can be achieved in reducing the error probability in QSE games. We have furthermore introduced the notions of exclusion-entropy and excludible information associated to a communication task where the information being communicated is naturally related to *exclusion* rather than identification or discrimination, as is usually the case. We have shown that the weight of informativeness fully characterises the single-shot excludible information of the quantum-to-classical channel associated to a measurement, proving a three-way correspondence, in *parallel* to the one found for the robustness of informativeness [204]. Finally, we have shown that exclusion games also constitute a complete set of tasks for measurement simulation, with the error probability over all games forming a complete set of monotones.

Although we have focused here on the QRT of measurement informativeness, we conjecture that the insight we have found is in fact rather generic for arbitrary quantum resource theories. In particular, we conjecture that whenever a (generalised) robustness-based measure is related to a discrimination task, then a weight-based measure will be related to the corresponding exclusion task, when considering *arbitrary objects* and *arbitrary resources*. In Chapter 4 we provide support to this conjecture by proving that it holds true when considering convex QRTs of measurements and convex QRTs states with arbitrary resources [73].

Figure 4.1 raises the following fascinating question. Could there exist a more general three-way correspondence, whose extremes recover the cases for $\{+\infty, -\infty\}$? This of course necessarily requires the introduction of operational tasks which are more general than quantum state discrimination and exclusion, but yet recovering these two cases at the extremes $\{+\infty, -\infty\}$, respectively. This direction of research is further investigated in Chapter 7.

Chapter 4

Operational interpretation of weight-based resource quantifiers for general convex QRTs of measurements and states

"A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise. "

George Pólya

In this chapter we introduce the resource quantifier of weight of resource for convex quantum resource theories of states and measurements with *arbitrary* resources. We show that it captures the advantage that a resourceful measurement (state) offers over all possible free measurements (states), in the operational task of exclusion of states (subchannels). Furthermore, we introduce information-theoretic quantities related to exclusion for quantum channels, and find a connection between the weight of resource of a measurement, and the exclusion-type information of quantum-to-classical channels. The results found in this chapter apply, in particular, to the resource theory of entanglement, in which the weight of resource is part of the so-called best-separable approximation or Lewenstein-Sanpera decomposition, introduced in 1998 [136]. Consequently, the results found here provide an operational interpretation to this 21 year-old entanglement decomposition.

4.1 Introduction and motivation

In the previous chapter we proved that, for the QRT of measurement informativeness, there exists a parallel quantifier-task correspondence that connects the resource quantifier of weight of informativeness with the operational task of state exclusion, and it was conjectured that this holds true for convex QRTs of different objects, and with general resources. In this chapter we prove that this conjecture holds true in the context of general convex QRTs of measurements with arbitrary resources. Additionally, we prove that this conjecture also holds true in the context of convex QRTs of states also with arbitrary resources. Specifically, we consider the resource quantifier of *weight of resource* and prove that it quantifies the advantage that a resourceful state offers, when compared to all possible free states, in the operational task of *subchannel exclusion*. In particular, this result holds true when considering the resource of entanglement and therefore, provides an operational interpretation to the weight of entanglement, which is better known as the best separable approximation, or Lewenstein-Sanpera decomposition, introduced in 1998 [136].

The results presented here nicely complement the *weight-exclusion* correspondence found in the previous chapter within the QRT of measurement informativeness and therefore, support the conjecture made in there about the existence of such a correspondence for convex QRTs of arbitrary objects and arbitrary resources. Interestingly, we will furthermore show that it is possible to extend the full three-way correspondence found in the previous chapter, which now links weight and exclusion to the so-called 'excludible' information. Explicitly, we prove that for convex QRTs of measurements with arbitrary resources, the weight of resource also quantifies a single-shot excludible information-theoretic measure.

This chapter is divided in two main sections. In the first section we address convex QRTs of measurements with general resources, whilst in the second section we address convex QRTs of states with general resources.

4.2 Main results for general convex QRTs of measurements

4.2.1 Quantum state exclusion (QSE) games

In this section we address the operational task of quantum state exclusion and particularly the case of a player implementing a quantum protocol. We invoke here, for convenience, some of the definitions from the previous chapter.

Operational Task 2. *(Quantum State Exclusion [15]) A referee has a collection of states $\{\rho_x\}$, $x \in \{1,...,k\}$, and promises to send a player one of these states ρ_x with probability $p(x)$. The goal is for the player to output a guess $g \in \{1,...,k\}$ for a state that was not sent. That is, the player succeeds at the game if $g \neq x$ and fails when $g = x$. A given state exclusion game is fully specified by an ensemble $\mathcal{E} = \{\rho_x, p(x)\}$.*

Free Protocol 1. *The best strategy for a free player is to implement the best amongst all free measurements. In this case, the minimal probability of error is:*

$$\mathbb{P}^{\mathbb{F}}_{\text{err}}(\mathcal{E}) := \min_{\mathbb{N}\in\mathbb{F}} \sum_x p(x)\,\text{Tr}[N_x\rho_x], \tag{4.1}$$

Quantum Protocol 2. *We consider that the player performs a quantum measurement $\mathbb{M} = \{M_a\}$, $M_a \geq 0$, $\forall a$, $\sum_a M_a = \mathbb{1}$ with o outcomes and uses this to simulate a measurement [204] $\mathbb{N} = \{N_x\}$ with k outcomes as $N_x = \sum_a q(x|a)M_a$ in order to output the guess of which state to exclude. The probability of error following this strategy is [15]:*

$$P^{\text{QSE}}_{\text{err}}(\mathcal{E},\mathbb{M}) := \min_{\mathbb{N}\preceq\mathbb{M}} \sum_x p(x)\,\text{Tr}[N_x\rho_x], \tag{4.2}$$

with the minimisation being performed over all POVMs $\mathbb{N}$ that are simulable by $\mathbb{M}$ [204].

We are now naturally interested in minimising this probability of error by implementing an optimal POVM. If we consider a binary ensemble, we have that a QSE game is equivalent to a quantum state discrimination (QSD) game and therefore we have $P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) = P_{\text{err}}^{\text{QSD}}(\mathcal{E},\mathbb{M})$. Having this, we can then use the Holevo-Helstrom theorem [113, 110, 251] to address state exclusion games with *binary* ensembles.

Lemma 4.1. *(Holevo-Helstrom for state exclusion) The minimum probability of error over all possible POVMs in a state exclusion game with a binary ensemble* $\mathcal{E} = \{\rho_x, p(x)\}$ $x \in \{0,1\}$ *is given by:*

$$\min_{\mathbb{M}} P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) = \frac{1}{2}\Big(1 - \|\tilde{\rho}_0 - \tilde{\rho}_1\|_1\Big), \tag{4.3}$$

with $\tilde{\rho}_x = p(x)\rho_x$ *and the trace norm* $\|X\|_1 = \text{Tr}(\sqrt{X^\dagger X})$.

Proof. In a binary state exclusion game we have to exclude from a binary ensemble of states $\mathcal{E} = \{\rho_0, \rho_1, p(0), p(1)\}$ with $p(0) + p(1) = 1$ by using a general POVM $\mathbb{M} = \{M_0, M_1\}$, $M_0, M_1 \geq 0$, $M_0 + M_1 = \mathbb{1}$. The probability of error is then (4.2):

$$P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) = \text{Tr}(M_0\tilde{\rho}_0) + \text{Tr}(M_1\tilde{\rho}_1),$$

with $\tilde{\rho} = p(x)\rho$. We now define an operator T as $M_0 = \frac{\mathbb{1}-T}{2}$ and therefore it satisfies $-\mathbb{1} \leq T \leq \mathbb{1}$. We have:

$$P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) = \text{Tr}\left[\left(\frac{\mathbb{1}-T}{2}\right)\tilde{\rho}_0\right] + \text{Tr}\left[\left(\frac{\mathbb{1}+T}{2}\right)\tilde{\rho}_1\right].$$

Reorganising we get:

$$P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) = \frac{1}{2}\left(1 + \text{Tr}\left[T(\tilde{\rho}_1 - \tilde{\rho}_0)\right]\right).$$

Minimising over POVMs is equivalent to minimising over matrices T and then:

$$\begin{aligned}\min_{\mathbb{M}} P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}) &= \frac{1}{2}\left(1 + \min_{-\mathbb{1}\leq T\leq \mathbb{1}} \text{Tr}\left[T(\tilde{\rho}_1 - \tilde{\rho}_0)\right]\right),\\ &= \frac{1}{2}\left(1 - \max_{-\mathbb{1}\leq T\leq \mathbb{1}} \text{Tr}\left[T(\tilde{\rho}_0 - \tilde{\rho}_1)\right]\right).\end{aligned}$$

The last term is the trace norm (2.25) and therefore we get the statement in (4.3). □

This result then compares with the standard Holevo-Helstrom theorem for binary QSD [113, 110, 251] which is usually stated as the maximum probability of succeeding in a binary QSD game being given by $\max_{\mathbb{M}} P_{\text{succ}}^{\text{QSD}}(\mathcal{E},\mathbb{M}) = \frac{1}{2}(1 + \|\tilde{\rho}_0 - \tilde{\rho}_1\|_1)$. We have that for a binary ensemble, state exclusion is precisely the opposite to state discrimination. This however, does not directly scale when considering ensembles with more than two states, since k-state exclusion games can naturally be defined [15]. This Lemma is going to prove useful when addressing one of our main results.

4.2.2 Result 4.1. All resourceful measurements are useful in a QSE game

We first show a preliminary result, which formalises the intuition about resourceful measurements being useful for operational tasks.

Result 4.1. *For any resourceful measurement* $\mathbb{M} \notin \mathbb{F}$*, there exists a state exclusion game* $\mathcal{E}^{\mathbb{M}}$ *for which playing with measurement* $\mathbb{M}$ *has small error probability when compared with any free state (measurement). This statement is represented by the strict inequality:*

$$P_{\text{err}}^{\text{QSE}}(\mathcal{E}^{\mathbb{M}},\mathbb{M}) < \mathbb{P}_{\text{err}}^{\mathbb{F}}(\mathcal{E}^{\mathbb{M}}). \tag{4.4}$$

In order to prove this result, we will prove a slightly stronger result, which implies the simpler result stated. In particular, we will prove the following.

Result 4.1.A. Given a measurement $\mathbb{M}$ there exists an ensemble $\mathcal{E}^{\mathbb{M}}$ such that:

$$P_{\text{err}}^{\text{QSE}}(\mathcal{E}^{\mathbb{M}}, \mathbb{M}) \leq [1 - \text{W}_{\mathbb{F}}(\mathbb{M})] \mathbb{P}_{\text{err}}^{\mathbb{F}}(\mathcal{E}^{\mathbb{M}}). \tag{4.5}$$

This means that the any resourceful measurement $\mathbb{M}$ provides a strictly smaller error than any free measurement $\mathbb{N}$, at playing QSE with the ensemble $\mathcal{E}^{\mathbb{M}}$, since $1 - W_{\mathbb{F}}(\mathbb{M}) < 1$ for all resourceful measurements, and hence this result implies Result 4.1.

Proof. Consider the dual formulation of the weight of measurement (2.24), namely

$$\begin{aligned} 1 - \text{W}_{\mathbb{F}}(\mathbb{M}) = \quad \min_{\{Z_x\}} \quad & \sum_x \text{Tr}(M_x Z_x) \\ \text{s.t.} \quad & \sum_x \text{Tr}(N_x Z_x) \geq 1, \quad \forall \mathbb{N} \in \mathbb{F}, \\ & Z_x \geq 0, \qquad\qquad \forall x. \end{aligned}$$

Suppose we have solved the above problem using the optimal set of dual variables $\{Z_x^*\}$. We consider a *particular* ensemble of states $\mathcal{E}^* = \{p^*(x), \rho_x^*\}$:

$$\rho_x^* = \frac{1}{\text{Tr } Z_x^*} Z_x^*, \quad p^*(x) = \frac{1}{c} \text{Tr } Z_x^*, \quad c = \sum_x \text{Tr } Z_x^*.$$

This leads to:

$$\begin{aligned} P_{\text{err}}^{\text{QSE}}(\mathcal{E}^*, \mathbb{M}) &= \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x p^*(x) \, \text{Tr}\left(N_x \rho_x^*\right), \\ &\leq \sum_x p^*(x) \, \text{Tr}\left(M_x \rho_x^*\right), \\ &= \frac{1}{c} \sum_x \text{Tr}\left(M_x Z_x^*\right), \\ &= \frac{1}{c} [1 - \text{W}_{\mathbb{F}}(\mathbb{M})], \end{aligned} \tag{4.6}$$

where in the second line we applied the trivial simulation. On the other hand notice that we have:

$$\begin{aligned} \mathbb{P}_{\text{err}}^{\mathbb{F}}(\mathcal{E}^*) &= \min_{\mathbb{N} \in \mathbb{F}} \sum_x p^*(x) \, \text{Tr}\left(N_x \rho_x^*\right), \\ &= \frac{1}{c} \min_{\mathbb{N} \in \mathbb{F}} \sum_x \text{Tr}\left(N_x Z_x^*\right), \\ &\geq \frac{1}{c}, \end{aligned} \tag{4.7}$$

where we used the constraints from the dual formulation of the weight. Combining the bounds (4.6) and (4.7) proves the claim. □

This result shows that every resourceful measurement is better that any possible free measurement when playing a tailored exclusion game. We now explore how to *quantify* the performance of a resourceful object using exclusion games.

4.2.3 Result 4.2. Weight as the advantage in QSE games

We are now interested in *quantifying* the performance of a resourceful measurement in comparison to all free measurements when playing state exclusion games. A first main result of this section is the following:

Result 4.2. *For any measurement $\mathbb{M}$ we have:*

$$\min_{\mathcal{E}} \frac{P_{\text{err}}^{\text{QSE}}(\mathcal{E}, \mathbb{M})}{\mathbb{P}_{\text{err}}^{\mathbb{F}}(\mathcal{E})} = 1 - \text{W}_{\mathbb{F}}(\mathbb{M}), \tag{4.8}$$

with the minimisation over all ensembles of states $\mathcal{E} = \{\rho_x, p(x)\}$.

The proof of this statement consists of two parts. First we prove that the weight lower bounds the advantage for all tasks. We then also consider that this lower bound can be achieved, by extracting an optimal game out of the relevant dual formulation of the weight, as proved in Result 4.1.A.

Proof. Consider first the primal formulation of the optimisation problem for $\text{W}_{\mathbb{F}}(\mathbb{M})$. The constraint implies that for all measurements $\mathbb{M}$ we can lower-bound the POVM elements $\{M_a\}$ using $1 - \text{W}_{\mathbb{F}}(\mathbb{M})$ and an element of some free measurement, i.e.

$$M_a \geq [1 - \text{W}_{\mathbb{F}}(\mathbb{M})] N_a \tag{4.9}$$

where $\mathbb{N} = \{N_a\} \in \mathbb{F}$. This implies that for all ensembles of states $\mathcal{E}$ we have:

$$\begin{aligned} P_{\text{err}}^{\text{QSE}}(\mathcal{E}, \mathbb{M}) &= \min_{\mathbb{M}' \preceq \mathbb{M}} \sum_x p(x) \operatorname{Tr}\left(M'_x \rho_x\right) \\ &= \min_{q(x|a)} \sum_{x,a} p(x) q(x|a) \operatorname{Tr}\left(M_a \rho_x\right) \\ &\geq \min_{q(x|a)} [1 - \text{W}_{\mathbb{F}}(\mathbb{M})] \sum_{x,a} p(x) q(x|a) \operatorname{Tr}\left(N_a \rho_x\right) \\ &\geq [1 - \text{W}_{\mathbb{F}}(\mathbb{M})] \min_{\mathbb{N}' \in \mathbb{F}} \sum_x p(x) \operatorname{Tr}\left(N'_x \rho_x\right) \\ &= [1 - \text{W}_{\mathbb{F}}(\mathbb{M})] \mathbb{P}_{\text{err}}^{\mathbb{F}}(\mathcal{E}) \end{aligned} \tag{4.10}$$

where in the second line we used the definition of simulation, in the third line we used (4.9), in the fourth line we used the fact that a simulation of a free measurement is still a free measurement (due to convexity of the free set), and in the final line we used the definition of the classical error probability.

We then note that if we combine the bound (4.5), which holds for all ensembles $\mathcal{E}$, with (4.10), then this proves the claim. In particular, (4.10) shows that the bound in (4.5) is in fact achieved by the ensemble $\mathcal{E}^{\mathbb{M}}$. □

This theorem shows two things: that for all exclusion games the weight bounds the decrease in error probability that can be obtained; and that there exists a game where this decrease is given precisely by the weight. This theorem establishes for the first time an operational interpretation of weight-based quantifiers, making a link to exclusion tasks, and thus establishing a connection between these two previously unrelated concepts. It is also interesting to note that although the weight is discontinuous (unlike the generalised robustness), it still admits an operational interpretation.

4.2.4 Result 4.3. QRTs of measurements and information theory

We now introduce an exclusion-based quantity closely related to the accessible information of a channel, and show that it too relates to the weight of resource of a measurement. We are interested in the ability of a channel Λ to be useful for sending *exclusion-type information*. This is a type of information where identifying is not the relevant task, but excluding, e.g. the information of the statement 'do not cut the blue wire' in a bomb-defusing situation. Two possibilities for conveying this information are either to communicate the wire to be avoided, or to communicate a wire that should be cut. If there is a noisy communication channel, it could be advantageous to use one type of encoding over the other.

Formally, we assume that the information to be excluded is represented by a random variable X, with probability distribution $p(x)$. This is encoded into a quantum ensemble as $\mathcal{E} = \{\rho_x, p(x)\}$. The quantum state is sent through a channel Λ, and then an optimal decoding measurement $\mathbb{D} = \{D_g\}_g$ is performed, in order to make the best prediction for a value $x' \neq x$, which will always be $\arg\min_x p(x|g)$, i.e. the least likely value of x given the observed g, where $p(x,g) = p(x)\,\mathrm{Tr}[D_g\Lambda(\rho_x)]$. The error probability is $P_{\mathrm{err}}(X|G) = \min_{\mathbb{D}} \sum_g \min_x p(x,g)$ and the associated conditional entropy, which we call the 'exclusion conditional entropy' is

$$H_{-\infty}(X|G)_{\mathcal{E},\Lambda} = -\log P_{\mathrm{err}}(X|G), \tag{4.11}$$

which is the order minus-infinity conditional Rényi entropy, and where we have explicitly denoted the dependence on the quantum encoding $\mathcal{E}$ and the channel Λ.

We are now interested in comparing how different channels perform with the same quantum encoding. In particular, we are interested in how much larger the exclusion conditional entropy is for a given fixed channel Λ compared to a set of free channels $\mathcal{F}$ for sending the exclusion information stored in $\mathcal{E}$. Note that since the exclusion entropy is associated to an error probability, having a larger exclusion entropy signifies having a smaller average error probability. We thus define the gain in exclusion conditional entropy as

$$G^{\mathrm{exc}}_{-\infty}(\mathcal{E},\Lambda) = H_{-\infty}(X|G)_{\mathcal{E},\Lambda} - \max_{\Omega\in\mathcal{F}} H_{-\infty}(X|G)_{\mathcal{E},\Omega} \tag{4.12}$$

We think of this quantity as being a generalisation of the accessible information of a channel, in two ways: first we consider here exclusion-type information, instead of standard 'discrimination-type' information; second, we compare to a general set of free channels, rather than relative to a single free channel – the completely noisy channel. In the latter case, the second term would become simply $H_{-\infty}(X)_{\mathcal{E}} = -\log P_{\mathrm{err}}(X)$, the 'exclusion entropy' associated with the random variable X, and the definition would reduce to a mutual information-type quantity.

We now focus on quantum-to-classical channels which arise by the action of a measurement. In particular, to any measurement $\mathbb{M}$ we can define the associated channel $\Lambda_{\mathbb{M}}$ such that $\Lambda_{\mathbb{M}}(\rho) = \sum_a \mathrm{Tr}[M_a\rho]\,|a\rangle\langle a|$, where $\{|a\rangle\}$ forms an orthonormal basis, and records the measurement outcome. The conditional probability distribution that this channel leads to is $p(g|x) = \sum_a \mathrm{Tr}[M_a\rho_x]\,\langle a|\,D_g\,|a\rangle$.

We will then compare the fixed channel $\Lambda_{\mathbb{M}}$ associated with the measurement $\mathbb{M}$ with all of the channels $\Lambda_{\mathbb{N}}$ that can arise from a free measurement $\mathbb{N} \in \mathbb{F}$.

Remark 4.1. *An alternative way of introducing the quantity of interest in this section ($G^{\mathrm{exc}}_{-\infty}(\mathcal{E},\Lambda)$) is as follows. Consider a set of free measurements as $\mathbb{F}$, and a pair ensemble of states and measurement $(\mathcal{E},\mathbb{M})$,* Arimoto's gap on POVMs *of order $-\infty$ for such a* *pair is defined as:*

$$G_{-\infty}^{\mathbb{F}}(X;G)_{\mathcal{E},\mathbb{M}} := I_{-\infty}(X;G)_{\mathcal{E},\mathbb{M}} - \max_{\mathbb{N}\in\mathbb{F}} I_{-\infty}(X;G)_{\mathcal{E},\mathbb{N}}, \tag{4.13}$$

with $I_{-\infty}(X;G)$ Arimoto's mutual information of order $-\infty$. It can be checked that these two quantities relate as: $G_{-\infty}^{\text{exc}}(\mathcal{E},\Lambda_{\mathbb{M}}) = G_{-\infty}^{\mathbb{F}}(X;G)_{\mathcal{E},\mathbb{M}}$. The definition and notation addressed in this remark is going to be used in some of the following chapters, where it is going to be more convenient.

We find the following result:

Result 4.3. *The weight of resource of a measurement $\mathbb{M}$ quantifies the biggest gain in exclusion information of the associated measurement channel $\Lambda_{\mathbb{M}}$ relative to the set of free measurement channels $\mathcal{F} = \{\Lambda_{\mathbb{N}}|\mathbb{N}\in\mathbb{F}\}$*

$$\max_{\mathcal{E}} G_{-\infty}^{\text{exc}}(\mathcal{E},\Lambda_{\mathbb{M}}) = -\log\Big[1 - \mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big], \tag{4.14}$$

with the maximisation over all quantum encodings $\mathcal{E} = \{\rho_x, p(x)\}$ of X.

Proof. We want to calculate the the single-shot excludible information gain given by:

$$G_{-\infty}^{\text{exc}}(\mathcal{E},\Lambda) = H_{-\infty}(X|G)_{\mathcal{E},\Lambda} - \max_{\Omega\in\mathcal{F}} H_{-\infty}(X|G)_{\mathcal{E},\Omega} \tag{4.15}$$

with the exclusion conditional entropy given by

$$\begin{aligned} H_{-\infty}(X|G)_{\mathcal{E},\Lambda} &= -\log P_{\text{err}}(X|G), \\ &= -\log \min_{\mathbb{D}} \sum_g \min_x p(x,g), \\ &= -\log \min_{\mathbb{D}} \sum_g \min_x p(x)\,\mathrm{Tr}[D_g\Lambda(\rho_x)]. \end{aligned} \tag{4.16}$$

Writing $f_g(x) = p(x)\,\mathrm{Tr}[D_g\Lambda(\rho_x)]$ and using:

$$\min_x f_g(x) = \min_{\{p(x|g)\}} \sum_x p(x|g) f_g(x),$$

we can write:

$$\begin{aligned} H_{-\infty}(X|G)_{\mathcal{E},\Lambda_{\mathbb{M}}} &= -\log \min_{\{p(x|g)\},\mathbb{D}} \sum_{g,x} p(x|g)p(x)\mathrm{Tr}[D_g\Lambda_{\mathbb{M}}(\rho_x)], \\ &= -\log \min_{\{p(x|g)\},\mathbb{D}} \sum_{g,x,a} p(x|g)\,\mathrm{Tr}[M_a\rho_x]\langle a|D_g|a\rangle, \\ &= -\log \min_{\{p(x|g)\}} \sum_{g,x,a} p(x|g)\,\mathrm{Tr}[M_a\rho_x]\delta_{g,a}, \\ &= -\log \min_{\{p(x|g)\}} \sum_{x,a} p(x|a)\,\mathrm{Tr}[M_a\rho_x], \\ &= -\log \min_{\mathbb{N}\preceq\mathbb{M}} \sum_x p(x)\,\mathrm{Tr}[N_x\rho_x], \\ &= -\log P_{\text{err}}^{\text{QSE}}(\mathcal{E},\mathbb{M}), \end{aligned} \tag{4.17}$$

where we realise that $D_g = |g\rangle\langle g|$ is the optimal measurement, since any other measurement only constitutes a potential loss of information (from a to g, which will never be useful for the minimisation, so that $\langle a|\, D_g\, |a\rangle = \delta_{a,g}$.

Let us now write explicitly the optimised quantity in (4.15) for a measurement channel $\Lambda_{\mathbb{M}}$. Using (4.17) we obtain:

$$\begin{aligned} & H_{-\infty}(X|G)_{\mathcal{E},\Lambda_{\mathbb{M}}} - \max_{\Omega_{\mathbb{N}}\in\mathcal{F}} H_{-\infty}(X|G)_{\mathcal{E},\Omega_{\mathbb{N}}} \\ &= H_{-\infty}(X|G)_{\mathcal{E},\Lambda_{\mathbb{M}}} - \max_{\mathbb{N}\in\mathbb{F}} H_{-\infty}(X|G)_{\mathcal{E},\Omega_{\mathbb{N}}} \\ &= -\log P^{\rm QSE}_{\rm err}(\mathcal{E},\mathbb{M}) + \min_{\mathbb{N}\in\mathbb{F}} \log P^{\rm QSE}_{\rm err}(\mathcal{E},\mathbb{N}) \\ &= -\log \frac{P^{\rm QSE}_{\rm err}(\mathcal{E},\mathbb{M})}{\mathbb{P}^{\mathbb{F}}_{\rm err}(\mathcal{E}),}. \end{aligned}$$

If we now maximise over all ensembles of states $\mathcal{E}$ and use Result 4.2 we obtain the claim. □

This result, which mirrors the results found in [212], establishes, for the QRT of measurements with arbitrary resources, a new *three-way correspondence* between weight-based resource quantifiers, exclusion-based operational tasks, and single-shot information-theoretic quantities.

4.2.5 Summary of results for measurements

In Figure 4.1 we have a diagrammatic representation of the parallel three-way correspondence found in this chapter, depicted as the inner triangle. Explicitly, we prove that for convex QRTs of measurements, the weight of resource quantifies both; the advantage of resourceful over free measurements in the operational task of *state exclusion* [15], and a new type of single-shot accessible information (of the quantum-classical channel induced by a measurement) associated to a novel communication problem.

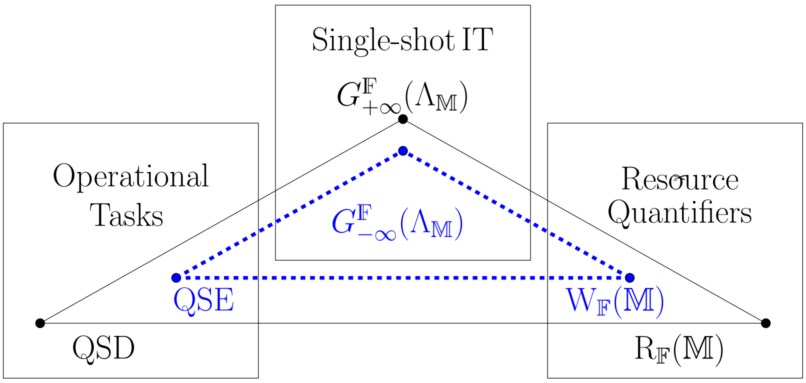

FIGURE 4.1: Three-way correspondence between: resource quantifiers, operational tasks, and single-shot information-theoretic quantities, for QRTs of measurements with arbitrary convex resources. The outer three-way correspondence is linking [204, 212]; quantum state discrimination (QSD), generalised robustness of resource $\mathrm{R}_{\mathbb{F}}(\mathbb{M})$, and the single-shot accessible information gain $G^{\rm acc}_{+\infty}(\Lambda_{\mathbb{M}})$. in this chapter, we derive a parallel three-way correspondence (inner triangle) linking: weight of resource $\mathrm{W}_{\mathbb{F}}(\mathbb{M})$, quantum state exclusion (QSE) and the single-shot excludible information gain $G^{\rm exc}_{-\infty}(\Lambda_{\mathbb{M}})$. Definitions of these quantities in the main text. The contribution of this chapter is the inner triangle (blue dotted lines).

4.3 Main results for general convex QRTs of states

We start by addressing convex QRTs of states with arbitrary resources.

Definition 4.1. *(Convex QRT of states [57]) Consider the set of quantum states in a Hilbert space of dimension d. Consider a property of these states defining a closed convex set which we will call the set of free states and denote as* F*. We say a state* $\rho \in \mathrm{F}$ *is a* free state*, and* $\rho \notin \mathrm{F}$ *is a* resourceful *state.*

There are numerous properties of quantum states considered as resources namely; entanglement, asymmetry, coherence, amongst many others [57]. We now want to quantify the amount of resource present in an state. We define a weight-based quantifier for an arbitrary resource.

Definition 4.2. *(Weight of resource of a state) Consider a convex QRT of states with an arbitrary resource. The weight of resource of a state is given by:*

$$\mathrm{W_F}\left(\rho\right) = \min_{\substack{r \geq 0 \\ \sigma \in \mathrm{F} \\ \rho_G}} \left\{ w \,\middle|\, \rho = w\rho_G + (1-w)\sigma \right\}. \tag{4.18}$$

This quantifies the minimal amount of a general state ρ_G *that has to be mixed with an arbitrary free state* σ*, in order to recover the state* ρ*.*

As a reminder, this quantifier was originally introduced in [79] in the context of nonlocality and independently rediscovered later on in [136] within the context of entanglement. It has been addressed under several different names such as: part, content, cost and weight. In this thesis we use the term *weight* in order to be consistent with recent literature. We now move on to operational tasks.

4.3.1 Quantum subchannel exclusion (QScE) games

In analogy to subchannel discrimination games [172, 217], we now define subchannel exclusion games as follows.

Operational Task 3. *(Quantum subchannel exclusion) The player sends a quantum state* ρ *to the referee who has a collection of subchannels* $\Psi = \{\Psi_x\}$*,* $x \in \{1, ..., k\}$*. The subchannels* Ψ_x *are completely-positive (CP) trace-nonincreasing linear maps such that* $\sum_x \Psi_x$ *forms a completely-positive trace-preserving (CPTP) linear map. The referee promises to apply one of these subchannels on the state* ρ *and the transformed state is then sent back to the player. The player then has access to the ensemble* $\mathcal{E}_\Psi = \{\rho_x, p(x)\}$ *with* $p(x) = \mathrm{Tr}[\Psi_x(\rho)]$*,* $\rho_x = \Psi_x(\rho)/p(x)$*. The goal is for the player to output a guess* $g \in \{1, ..., k\}$ *for a subchannel that did not take place. That is, the player succeeds at the game if* $g \neq x$ *and fails when* $g = x$*.*

This game can alternatively be seen as playing a quantum state exclusion game with the ensemble $\mathcal{E}_\Psi^\rho = \{\rho_x, p(x)\}$, in which the player has a certain level of control over the ensemble when proposing the state ρ. A particular case of subchannel exclusion is *quantum channel exclusion*, in which $\Psi = \Lambda = \{\Lambda_x, p(x)\}$ with $\{\Lambda_x\}$ being CPTP maps and $p(x)$ a probability distribution. We now consider a quantum protocol for the player to address this game.

Quantum Protocol 3. *Consider a subchannel exclusion game in which the player sends a state* ρ *to the referee who in turn, applies a subchannel from the collection* $\Psi = \{\Psi_x\}$ *with* $x \in \{1, ..., k\}$*. Having received the state back, the player now performs a quantum* *measurement* $\mathbb{M} = \{M_x\}$ *with* k *outcomes, and uses this to produce a guess of which subchannel to exclude. The probability of error in quantum subchannel exclusion following this protocol is given by:*

$$P_{\text{err}}^{\text{QScE}}(\Psi, \mathbb{M}, \rho) = \sum_x \text{Tr}\left[M_x \Psi_x(\rho)\right], \tag{4.19}$$

Similarly to state exclusion, we are interested in minimising this probability of error. We will be particularly interested in the performance of a resourceful state compared to the best free state when playing subchannel exclusion games.

4.3.2 Result 4.4. All resourceful states are useful in a QScE game

It has already been proven that any resourceful state is useful in a subchannel discrimination game [217]. This result addresses a *binary* discrimination game, and since we have already seen that $P_{\text{err}}^{\text{QSE}}(\mathcal{E}, \mathbb{M}) = P_{\text{err}}^{\text{QSD}}(\mathcal{E}, \mathbb{M})$ the result then follows. However, since we are now interested in the probability of error, we will write this in the context of state exclusion games as follows.

Result 4.4. *For any resourceful state* $\rho \notin \mathrm{F}$*, there exists a subchannel exclusion game* Ψ^ρ *for which playing with the state* ρ *generates fewer errors when compared with any free state:*

$$\min_{\mathbb{M}} P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{M}, \rho) < \min_{\mathbb{N}} \min_{\sigma \in \mathrm{F}} P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{N}, \sigma). \tag{4.20}$$

In the right-hand side the error probability is minimised over all possible free states and all measurements.

The proof of this result follows from the identification that for binary subchannel games we have $P_{\text{err}}^{\text{QSE}}(\mathcal{E}, \mathbb{M}) = P_{\text{err}}^{\text{QSD}}(\mathcal{E}, \mathbb{M})$ together with the exclusion version of the Holevo-Helstrom theorem addressed in the previous section. This result shows that every resourceful state is better that any possible free state when playing a tailored subchannel exclusion task, which turns out to always be binary.

Proof. We are going to consider a binary subchannel exclusion game $\Psi = \{\Psi^+, \Psi^-\}$. The probability of error in this binary subchannel exclusion problem is given by the exclusion (as opposed to the discrimination) version of the Holevo-Helstrom theorem which we derived in (4.3) so we have:

$$\frac{\min_{\mathbb{M}} P_{\text{err}}^{\text{QScE}}(\Psi, \mathbb{M}, \rho)}{\min_{\mathbb{N}} \min_{\sigma \in \mathrm{F}} P_{\text{err}}^{\text{QScE}}(\Psi, \mathbb{N}, \sigma)} = \frac{1 - \|(\Psi^+ - \Psi^-)(\rho)\|_1}{1 - \|(\Psi^+ - \Psi^-)(\sigma^*)\|_1}, \tag{4.21}$$

with σ^* the optimal free state. If we manage to construct subchannels such that:

$$\left\|(\Psi^+ - \Psi^-)(\sigma^*)\right\|_1 \leq \left\|(\Psi^+ - \Psi^-)(\rho)\right\|_1, \tag{4.22}$$

then the statement follows. Let us see how this can be done. Given any $\rho \notin \mathrm{F}$ and by the hyperplane separation theorem (or Hahn-Banach separation theorem) [75] we have that there exists a bounded self-adjoint operator W^ρ such that: i) $\text{Tr}(W^\rho \rho) < 0$ and ii) $\text{Tr}(W^\rho \sigma) \geq 0, \forall \sigma \in \mathrm{F}$. We now construct the operator $X^\rho = \mathbb{1} - \frac{W^\rho}{\|W^\rho\|_\infty}$ which has the properties i) $0 \leq X^\rho$, ii) $0 \leq \text{Tr}(X^\rho \sigma) \leq 1, \forall \sigma \in \mathrm{F}$ and iii) $1 < \text{Tr}(X^\rho \rho)$ and so we have the inequality:

$$0 \leq \text{Tr}(X^\rho \sigma) \leq 1 < \text{Tr}(X^\rho \rho), \quad \forall \sigma \in \mathrm{F}. \tag{4.23}$$

With this operator X^ρ we now construct an appropriate binary subchannel ensemble $\Psi^\rho = \left\{\frac{\Lambda_+^\rho}{2}, \frac{\Lambda_-^\rho}{2}\right\}$ as follows:

$$\Lambda_\pm^\rho(\eta) = \left(\frac{1}{2} \pm \frac{\mathrm{Tr}(X^\rho \eta)}{2\,\|X^\rho\|_\infty}\right)|0\rangle\langle 0| + \left(\frac{1}{2} \mp \frac{\mathrm{Tr}(X^\rho \eta)}{2\,\|X^\rho\|_\infty}\right)|1\rangle\langle 1|. \tag{4.24}$$

We can check that these operators are trace-preserving and therefore the subchannel game is well-defined. Now because of the way that the subchannels have been constructed we obtain:

$$\left\|(\Lambda_+^\rho - \Lambda_-^\rho)(\eta)\right\|_1 = \frac{2\,\mathrm{Tr}(X^\rho \eta)}{\|X^\rho\|_\infty}, \quad \forall \eta.$$

For any $\sigma \in \mathrm{F}$ we have:

$$\left\|(\Lambda_+^\rho - \Lambda_-^\rho)(\sigma)\right\|_1 = \frac{2\,\mathrm{Tr}(X^\rho \sigma)}{\|X^\rho\|_\infty} \leq \frac{2\,\mathrm{Tr}(X^\rho \rho)}{\|X^\rho\|_\infty} = \left\|(\Lambda_+^\rho - \Lambda_-^\rho)(\rho)\right\|_1.$$

The inequality follows from (4.23). We then obtain that the denominator in (4.21) is less than the numerator and so we obtain the claim in (4.20). □

4.3.3 Result 4.5. Weight of resource as the advantage in QScE games

We are now interested in quantifying the performance of a resourceful state in comparison to all free states when playing subchannel exclusion games.

Result 4.5. *Consider a subchannel exclusion game in which the player sends a quantum state ρ to the Referee, who in turns applies a subchannel from the ensemble $\Psi = \{\Psi_x\}$ with $x \in \{1, ..., k\}$ before sending the state back to the player. The player then implements a measurement $\mathbb{M} = \{M_x\}$ and produce the outcome guess $g \in \{1, ..., k\}$ representing the choice of a subchannel to be excluded. Then, the quantum-classical ratio of probability of error in subchannel exclusion is lower bounded by the weight of resource (4.18). Furthermore, there exists a subchannel ensemble Ψ^ρ and a measurement $\mathbb{M}^\rho$ for which the lower bound is tight as follows:*

$$1 - \mathrm{W}_\mathrm{F}(\rho) = \min_{\Psi,\mathbb{M}} \frac{P_\mathrm{err}^\mathrm{QScE}(\Psi, \mathbb{M}, \rho)}{\min\limits_{\sigma \in \mathrm{F}} P_\mathrm{err}^\mathrm{QScE}(\Psi, \mathbb{M}, \sigma)}. \tag{4.25}$$

Proof. Let us start by proving that for a given ρ, $1 - \mathrm{W}_\mathrm{F}(\rho)$ places a lower bound to the quantum-classical ratio in any subchannel exclusion game and any measurement, that this, for any tuple $(\mathbb{M}, \Psi)$. Given ρ and any tuple $(\Psi, \mathbb{M})$ we have:

$$\begin{aligned} P_\mathrm{err}^\mathrm{QScE}(\Psi, \mathbb{M}, \rho) &= \sum_x \mathrm{Tr}[M_x \Psi_x(\rho)], \\ &\geq \Big[1 - \mathrm{W}_\mathrm{F}(\rho)\Big] \sum_x \mathrm{Tr}[M_x \Psi_x(\sigma^*)], \\ &\geq \Big[1 - \mathrm{W}_\mathrm{F}(\rho)\Big] \min_{\sigma \in \mathrm{F}} \sum_x \mathrm{Tr}[M_x \Psi_x(\sigma)], \\ &= \Big[1 - \mathrm{W}_\mathrm{F}(\rho)\Big] \min_{\sigma \in \mathrm{F}} P_\mathrm{err}^\mathrm{QScE}(\Psi, \mathbb{M}, \sigma). \end{aligned} \tag{4.26}$$

In the first inequality we used (4.18) from which we get $\rho \geq [1 - \mathrm{W}_\mathrm{F}(\rho)]\sigma^*$ and since Ψ_x are linear maps we have $\Psi_x(\rho) \geq [1 - \mathrm{W}_\mathrm{F}(\rho)]\Psi_x(\sigma^*)$, $\forall x$. In the second inequality we allow ourselves to minimise over all free states. We now address how to achieve the lower bound. Consider a given ρ and let us construct an appropriate subchannel exclusion game $\Psi^\rho = \{\Psi_x^\rho\}$ and an appropriate measurement $\mathbb{M}^\rho = \{M_x^\rho\}$ achieving the lower bound. Let us start by considering the optimal operator coming out of the dual SDP formulation of the weight of resource (2.20) as Y^ρ. By spectral decomposition we have $Y^\rho = \sum_{i=1}^d y_i \left|e_i\right\rangle\!\left\langle e_i\right|$ with $y_i \in \mathbb{R}$ and $\{\left|e_i\right\rangle\}$ an orthonormal basis. We now consider a set of unitary matrices $\{U_x\}$, $x \in \{1, ..., d\}$ satisfying the property $\sum_x U_x \left|e_j\right\rangle\!\left\langle e_j\right| U_x^\dagger = \mathbb{1}$, $\forall j$. This can be done by defining, for instance, $U_y = \sum_{x=1}^d \left|e_{x+y}\right\rangle \left\langle e_x\right|$. We now define a subchannel game $\Psi^\rho = \{\Psi_x^\rho\}$ and a measurement $\mathbb{M}^\rho = \{M_x^\rho\}$ as:

$$\Psi_x^\rho(\cdot) = \frac{1}{d} U_x(\cdot)U_x^\dagger, \tag{4.27}$$

$$M_x^\rho = \frac{1}{\mathrm{Tr}(Y^\rho)} U_x Y^\rho U_x^\dagger. \tag{4.28}$$

One can check that the subchannels and the measurement are well defined. We can now check that the probability of error in subchannel exclusion for a state η is given by:

$$P_{\text{err}}^{\text{Q}}(\Psi^\rho, \mathbb{M}^\rho, \eta) = \frac{\mathrm{Tr}(Y^\rho \eta)}{\mathrm{Tr}(Y^\rho)}, \qquad \forall \eta.$$

The quantum-classical ratio then satisfies:

$$\begin{aligned} \frac{P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{M}^\rho, \rho)}{\min_{\sigma \in \mathrm{F}} P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{M}^\rho, \sigma)} &= \frac{\mathrm{Tr}(Y^\rho \rho)}{\mathrm{Tr}(Y^\rho \sigma^*)}, \\ &\leq \mathrm{Tr}(Y^\rho \rho), \qquad (4.29) \\ &= 1 - \mathrm{W}_{\mathrm{F}}(\rho). \qquad (4.30) \end{aligned}$$

The inequality follows because of (2.20b) and the last equality because of (2.20a). The inequality (4.30) together with (4.26) leads to:

$$\frac{P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{M}^\rho, \rho)}{\min_{\sigma \in \mathrm{F}} P_{\text{err}}^{\text{QScE}}(\Psi^\rho, \mathbb{M}^\rho, \sigma)} = 1 - \mathrm{W}_{\mathrm{F}}(\rho).$$

Therefore, for any given ρ we can find both a subchannel exclusion game $\Psi^\rho = \{\Psi_x^\rho\}$ and a measurement $\mathbb{M}^\rho = \{M_x^\rho\}$, given by (4.27) and (4.28) respectively, such that the the quantum-classical ratio achieves the lower bound and therefore obtaining the claim in (4.25). □

We remark that this result holds true for any property of a quantum state that defines a closed convex subset and therefore, it holds in particular for the weight of entanglement, a. k. a. the best separable approximation or Lewenstein-Sanpera decomposition [136], and for the weight of asymmetry [44].

Remark 4.2. *We note that this lower bound can also be achieved by quantum channel exclusion games.*

$$1 - \mathrm{W}_{\mathrm{F}}(\rho) = \min_{\Lambda, \mathbb{M}} \frac{P_{\text{err}}^{\text{QCE}}(\Lambda, \mathbb{M}, \rho)}{\min\limits_{\sigma \in \mathrm{F}} P_{\text{err}}^{\text{QCE}}(\Lambda, \mathbb{M}, \sigma)}, \tag{4.31}$$

with the minimisation over all measurements and all quantum channel games $\Lambda = \{\Lambda_x, p(x)\}$. The proof of this statement follows the same steps as for the case of subchannels, with the subtlety that now we define the set of channels as:

$$\Lambda_x^{\rho}(\cdot) := U_x(\cdot)U_x^{\dagger}, \quad p(x) := \frac{1}{d}, \quad \forall x. \tag{4.32}$$

We also note here that in the game the quantum and classical players are required to use the same measurement (4.25). In a different setting, we can alternatively ask for the measurements to be chosen independently. We now explore relaxing this measurement constraint.

4.3.4 Result 4.6. Quantum-classical ratio with independent measurements

We now consider a scenario in which the quantum and classical players implement independent measurements.

Result 4.6. *Consider a state ρ and the optimal dual variable $Y^{\rho} = \sum y_i \, |e_i\rangle\langle e_i|$ from the dual formulation of the weight of resource (2.20). If there exists a set of unitaries $\{U_x\}$ satisfying i) $\sum_x U_x|e_j\rangle\langle e_j|U_x^{\dagger} = \mathbb{1}, \forall j$ and ii) $U_i\sigma U_i^{\dagger} = U_j\sigma U_j^{\dagger}, \forall\sigma \in \mathrm{F}, \forall i, j$, then, the weight of the resource quantifies the advantage of the resourceful state ρ over all free states in subchannel exclusion with independent measurements as:*

$$1 - \mathrm{W}_{\mathrm{F}}(\rho) = \min_{\Psi} \frac{\min\limits_{\mathbb{M}} P_{\mathrm{err}}^{\mathrm{QScE}}(\Psi, \mathbb{M}, \rho)}{\min\limits_{\mathbb{N}} \min\limits_{\sigma\in\mathrm{F}} P_{\mathrm{err}}^{\mathrm{QScE}}(\Psi, \mathbb{N}, \sigma)}. \tag{4.33}$$

An example of a resource that satisfies the necessary conditions of Result 4.6 is coherence [217]. This stronger result thus holds for this particular resource. It would be interesting to know whether this also holds true for further additional resources not covered by these scenarios, including entanglement, but this is left for future research. We now address single-shot information-theoretic quantities that are also related to the weight of resource. In order to prove Result 4.6 we need the following Theorem 1 in [15].

Theorem 4.1. *(Necessary and sufficient conditions for optimality in state exclusion [15]) Consider a state exclusion game defined by the ensemble $\mathcal{E} = \{\rho_x, p(x)\}$. We now consider a measurement $\mathbb{M} = \{M_x\}$, and the operator:*

$$N = \sum_x \tilde{\rho}_x M_x. \tag{4.34}$$

The measurement $\mathbb{M} = \{M_x\}$ is an optimal measurement for playing quantum state exclusion with the ensemble $\mathcal{E} = \{\rho_x, p(x)\}$ if and only if the operator N satisfies the following two conditions:

$$i)\ N^{\dagger} = N, \tag{4.35}$$
$$ii)\ \tilde{\rho}_x - N \geq 0, \quad \forall x, \tag{4.36}$$

with $\tilde{\rho} = p(x)\rho_x$.

We now are ready to address Result 4.6. The proof of this result uses similar techniques to the discrimination case in [212]. The subtlety lies in that we now need to check the necessary and sufficient conditions for quantum state exclusion [15], as opposed to those for quantum state discrimination [212]. We explicitly write down the proof for completeness.

Proof. (of Result 4.6) Consider a state ρ and its associated operator from the dual of the weight of resource (2.20) written in spectral decomposition as $Y^\rho = \sum y_i |e_i\rangle\langle e_i|$. Consider the existence a set of unitaries $\{U_x\}$ satisfying the two conditions:

$$\sum_x U_x e_j U_x^\dagger = \mathbb{1}, \quad \forall j, \tag{4.37}$$

$$U_i \sigma U_i^\dagger = U_j \sigma U_j^\dagger, \quad \forall \sigma \in \mathrm{F}, \quad \forall i, j. \tag{4.38}$$

The lower bound can be proven similarly as in Result 4.5, so we only address here how to achieve the lower bound. Similarly to Result 4.5, we need to define an optimal subchannel game and a measurement. We are going to define them similarly to Result 4.5 and we will see that this measurement turns out to be optimal when considering free states, meaning that conditions (4.35) and (4.36) are satisfied. We now define the subchannels and measurement as:

$$\Psi_x^\rho(\cdot) := \frac{1}{d} U_x(\cdot) U_x^\dagger, \tag{4.39}$$

$$M_x^\rho := \frac{1}{\mathrm{Tr}(Y^\rho)} U_x Y^\rho U_x^\dagger, \tag{4.40}$$

with $\{U_x\}$ as described in the statement of Result 4.6. we now check that they satisfy the first optimality condition (4.35) when considering free states σ. We now look at the subchannel exclusion game as a state exclusion game with $\mathcal{E}_{\Psi^\rho}^{\sigma^*} = \{\frac{1}{d}, \sigma^x\}$ with $\sigma^x := U_x \sigma^* U_x^\dagger$, σ^* being the optimal free state for the subchannel exclusion game. We note that the assumption (4.38) translates now to $\sigma^i = \sigma^j\ \forall i, j$. We now want to argue that the measurement in (4.40) is optimal for this state exclusion game. We now calculate the operator N and check the first optimality condition (4.35). We have that:

$$M_i \left(\frac{1}{d}\sigma^i - \frac{1}{d}\sigma^j \right) M_j = 0 \quad \forall i, j.$$

because the quantity inside the parenthesis is always zero. These conditions imply that $N^\dagger = N$ as desired, let us see this. The previous is equivalent to:

$$\frac{1}{d} M_i \sigma^i M_j = \frac{1}{d} M_i \sigma^j M_j, \quad \forall i, j.$$

Summing over j we have:

$$\frac{1}{d} M_i \sigma^i = \frac{1}{d} M_i \sum_j \sigma^j M_j.$$

Summing now over i we have:

$$N^\dagger = \sum_i M_i \sigma^i \frac{1}{d} = \sum_j \frac{1}{d} \sigma^j M_j = N.$$

We take into account that M_i and ρ are positive operators so they are self-adjoint. Therefore we have that the first optimality condition (4.35) is satisfied. We now check the second optimality condition (4.36). We have:

$$\begin{aligned}\tilde{\rho}_x - N &= \frac{1}{d}\sigma^x - \sum_y \frac{1}{d}\sigma^y M_y, \\ &= \frac{1}{d}\sigma^x\left(\mathbb{1} - \sum_y M_y\right) = 0 \geq 0, \quad \forall x.\end{aligned}$$

In the second line we have used that $\sigma^x = \sigma^y$, $\forall x, y$ which is the assumption (4.38). Therefore the measurement (4.40) is an optimal measurement for quantum state exclusion and we obtain the statement in (4.33). □

4.4 Conclusions

In this chapter we have proven, in the context of convex QRTs of states, that *weight-based* resource quantifiers for *arbitrary resources* capture the advantage that a resourceful state has over all free states, in the operational task of *subchannel exclusion*. As a corollary of this result, we have shown that the best separable approximation or Lewenstein-Sanpera decomposition [136] quantifies the advantage that an entangled state has over all separable states, in the task of subchannel exclusion. To the best of our knowledge, this is the first operational interpretation that has been given to this entanglement quantifier. Going forward, it would be interesting to derive a version of our result that allows for independent measurements when comparing resourceful and free states, as was done in [217] for the robustness of a entanglement.

The results presented here also support the conjecture made in the previous chapter, stating that whenever there is an discrimination-based operational task where a robustness-based resource quantifier plays a relevant role, there is an exclusion-based operational task where a weight-based resource quantifier plays a relevant role as well. It would also be interesting to address this conjecture for other objects, such as steering assemblages or collections of incompatible measurements. All of these considerations are interesting in themselves, but we leave these for future research.

Furthermore, and beyond the weight-exclusion correspondence, we have provided a third connection to single-shot information-theoretic quantities for QRTs of measurements with arbitrary resources. In particular, we have shown that the weight of resource of a measurement is also closely related to an exclusion-version of the accessible information of quantum-to-classical channels.

The results presented in this chapter nicely fit within the endeavour of linking resource quantifiers to operational tasks in general convex QRTs. One can go even further and consider general probabilistic theories in which the discrimination-robustness correspondence has already been extended [212]. We believe that the results presented in this chapter can be extended to this regime as well, but we leave this for future research.

Chapter 5

Multi-object operational tasks for QRTs of state-measurement pairs

"The art and science of asking questions is the source of all knowledge."

Thomas Berger

The prevalent modus operandi within the framework of quantum resource theories has been to characterise and harness the resources within single objects, like either states, measurements, or channels, in what we can call *single-object* quantum resource theories. One can wonder, however, whether the resources contained within multiple different types of objects, now in a *multi-object* quantum resource theory, can simultaneously be exploited for the benefit of an operational task. In this chapter, we introduce examples of such multi-object operational tasks in the form of subchannel discrimination and subchannel exclusion games, in which the player harnesses the resources contained within the composite object of a state-measurement pair. We prove that for any state-measurement pair in which either of them is resourceful, there exist discrimination and exclusion games for which such a pair outperforms any possible free state-measurement pair. These results hold for arbitrary convex resources of states, and arbitrary convex resources of measurements where the set of free measurements is closed under classical post-processing. Furthermore, we prove that the advantage in these multi-object operational tasks is determined, in a multiplicative manner, by the resource quantifiers of: *generalised robustness of resource* of both state and measurement for discrimination games and *weight of resource* of both state and measurement for exclusion games [71].

5.1 Introduction and motivation

One common feature amongst most results dealing with quantum resource theories is that they address *single-object* operational tasks, meaning that a *single* object is thought of as the resourceful object, and the associated tasks are then exploiting the resource contained within such an individual object. One then can wonder, about the possibility of having operational tasks harnessing *two or more* different resources out of two, in principle different, objects. We refer to these tasks as *multi-object* tasks, and we can intuitively approach them from the following two general levels. In a first instance, one can consider a single QRT with two different resources, in which case it is natural to make the distinction of the resources being either: disjoint, intersecting or nested [207]. The case of QRTs of states with *disjoint* resources has been explored in the context of a first law for general QRTs [207], this, inspired by results from the thermodynamics of multiple conserved quantities [103, 252, 146]. In a second instance however, one can also consider a multi-object scenario in which a first QRT of certain objects with an arbitrary resource is being specified, followed by a second QRT with *different* objects with their respective arbitrary resource. We address this latter case by considering a multi-object scenario with two objects, one being a *state* and a second one being a *measurement* and therefore, the composite object of interest is now a state-measurement *pair*.

In this chapter we address composite QRTs made of convex QRTs of states with arbitrary resources and convex QRTs of measurements with arbitrary resources where the set of free measurements is closed under classical post-processing (CPP). Taking into account that the set of free measurements is closed under CPP for many important resources for measurements like: entanglement, coherence and informativeness, the results found in this chapter naturally apply to all of these instances. Explicitly, we introduce *multi-object* operational tasks in the form of subchannel discrimination and subchannel exclusion games in which, a state-measurement pair is being deemed as the composite object of the theory, as opposed to the state (or the measurement) alone. Interestingly, we find that *any* resourceful state-measurement pair offers an advantage, over *all* possible free pairs, when performing at particular multi-object tasks. Furthermore, we find that this advantage can be *quantified*, in a multiplicative manner, by the amount of resource contained within each object, here measured by the resource quantifiers of *generalised robustness* and *weight*, for discrimination and exclusion games respectively. Moreover, these quantifiers also find operational significance in an multi-object encoding-decoding communication task involving the state-measurement pair. We believe that the results found in this chapter open the door for the exploration of multi-object operational tasks in general convex QRTs with different objects beyond states and measurements.

5.2 Composite convex QRTs and multi-object operational tasks

We start by addressing convex QRTs of states and measurements with arbitrary resources.

Definition 5.1. *(Composite convex QRTs of states and measurements) We say that a state-measurement pair* $(\rho, \mathbb{M})$ *is:* fully free *when both state and measurement are free,* partially resourceful *when either is resourceful, and* fully resourceful *when both are resourceful.*

We now recall the concept of classical post-processing, introduced in the previous chapters.

Definition 5.2. *(Classical post-processing (CPP)) We say that a measurement* $\mathbb{N} = \{N_x\}$, $x \in \{1, ..., k\}$ *is* simulable by *the measurement* $\mathbb{M} = \{M_a\}$, $a \in \{1, ..., o\}$ *when there exists a conditional probability distribution* $\{q(x|a)\}$ *such that* $N_x = \sum_{a=1}^{o} q(x|a) M_a$, $\forall x \in \{1, ..., k\}$ *[100]. One can check that the simulability of measurements defines a partial order for the set of measurements and therefore we use the notation* $\mathbb{N} \preceq \mathbb{M}$*, meaning that* $\mathbb{N}$ *is simulable by* $\mathbb{M}$*. We refer to this property as simulability of measurements or classical post-processing (CPP).*

Intuitively, a set of free measurements is closed under CPP when there is no physical significance to the specific measurement *label*. For example, when labelling an outcome 0 or 1 does not signify anything. An example where this does not hold is in thermodynamics, where the labels on an energy measurement have physical significance (labelling particular energies) and relabelling is not automatically free, unless the relationship between the label and the energy is also accordingly updated.

We can check that the set of free measurements is closed under CPP for QRTs of measurements with the resources of: entanglement, coherence and informativeness. The sets of free measurements for these resources (separable, coherent, uninformative) are defined by specifying the POVM elements respectively as [162, 204]: $M_x^S = \sum_k M_{x,k}^A \otimes M_{x,k}^B$, $M_x^C = \sum_j p(x|j) \, |j\rangle\langle j|$, $\{|j\rangle\}$ a basis of the Hilbert space being considered, and $M_x^U = p(x)\mathbb{1}$, p_X a probability distribution. Since any CPP operation always maps measurements into measurements and can never generate entanglement, coherence nor increase purity, all of these exemplary sets of measurements remain closed under CPP. We will then be addressing, from now on, convex QRTs of measurements with its free set being closed under CPP. We now introduce multi-object operational tasks in the form of subchannel discrimination/exclusion, which are meant to be played with state-measurement pairs.

Definition 5.3. *(Multi-object subchannel discrimination/exclusion games) Consider a player with access to a state-measurement pair $(\rho, \mathbb{M})$. The player sends the state ρ to the referee who is in possession of a collection of subchannels $\Psi = \{\Psi_x\}$, $x \in \{1, ..., k\}$. The subchannels $\{\Psi_x\}$ are completely-positive (CP) trace-nonincreasing maps, such that $\sum_x \Psi_x$ forms a completely-positive trace-preserving (CPTP) map. The referee promises to apply one of these subchannels on the state ρ and the transformed state is then sent back to the player. The player then effectively has access to the ensemble $\mathcal{E}_\Psi^\rho = \{\rho_x, p(x)\}$ with $p(x) = \mathrm{Tr}[\Psi_x(\rho)]$, $\rho_x = \Psi_x(\rho)/p(x)$. In a* subchannel discrimination *game, the goal is for the player to output a guess $g \in \{1, ..., k\}$ for the subchannel that was applied, the player succeeds at the game if $g = x$ and fails when $g \neq x$. In a* subchannel exclusion *game on the other hand, the goal is for the player to output a guess $g \in \{1, ..., k\}$ for a subchannel that was not applied, that is, the player succeeds at the game if $g \neq x$ and fails when $g = x$. In order to generate a guess, the player proceeds to implement the measurement $\mathbb{M} = \{M_a\}$ on the received state and classically post-process the measurement outcome a to produce an output guess g, according to a probability distribution $\{p(g|a)\}$, for playing either a discrimination or an exclusion game. The probability of success at subchannel discrimination and the probability of error at subchannel exclusion are given by:*

$$P_{\text{succ}}^{\text{D}}(\Psi, \rho, \mathbb{M}) = \max_{\{p(g|a)\}} \sum_{x,a,g} \delta_{x,g} \, p(g|a) \, p(a|x) \, p(x), \tag{5.1}$$

$$P_{\text{err}}^{\text{E}}(\Psi, \rho, \mathbb{M}) = \min_{\{p(g|a)\}} \sum_{x,a,g} \delta_{x,g} \, p(g|a) \, p(a|x) \, p(x), \tag{5.2}$$

with $p(a|x) = \mathrm{Tr}[M_a \rho_x]$ and the maximisation (minimisation) over all classical post-processings of the measurements outputs $p(g|a)$. A subchannel discrimination/exclusion game is specified by the collection of subchannels $\Psi = \{\Psi_x\}$.

A key point to remark, is that the object of interest is now the state-measurement pair $(\rho, \mathbb{M})$, as opposed to the state (or measurement) alone. We now proceed to establish a first result comparing the performance of a *fully resourceful* state-measurement pair against all *fully free* pairs when addressing a particular game.

5.3 Result 5.1: Any fully resourceful state-measurement pair is useful for QScD and QScE games

Result 5.1. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given a* fully resourceful *state-measurement pair* $(\rho, \mathbb{M})$, *meaning that we have both a resourceful state* $\rho \notin \mathrm{F}$ *and a resourceful measurement* $\mathbb{M} \notin \mathbb{F}$, *then, there exist subchannel games* $\Psi_D^{(\rho,\mathbb{M})}$ *and* $\Psi_E^{(\rho,\mathbb{M})}$ *such that:*

$$\max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} P_{\mathrm{succ}}^{\mathrm{D}}(\Psi_D^{(\rho,\mathbb{M})}, \sigma, \mathbb{N}) < P_{\mathrm{succ}}^{\mathrm{D}}(\Psi_D^{(\rho,\mathbb{M})}, \rho, \mathbb{M}), \tag{5.3}$$

$$P_{\mathrm{err}}^{\mathrm{E}}(\Psi_E^{(\rho,\mathbb{M})}, \rho, \mathbb{M}) < \min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}\in\mathbb{F}} P_{\mathrm{err}}^{\mathrm{E}}(\Psi_E^{(\rho,\mathbb{M})}, \sigma, \mathbb{N}). \tag{5.4}$$

These two strict *inequalities mean that the state-measurement pair* $(\rho, \mathbb{M})$ *provides* strictly *larger (smaller) advantage (error) than* all *fully free state-measurement pairs, when playing the subchannel discrimination (exclusion) game specified by* $\Psi_D^{(\rho,\mathbb{M})}$ *(*$\Psi_E^{(\rho,\mathbb{M})}$*).*

The proof of this result relies on the hyperplane separation theorem [189] as well as on a method first used in the context of quantum steering [172], for "completing" a set of subchannels, from which one can extract suitable operators in order to construct the tailored subchannel games $\Psi_D^{(\rho,\mathbb{M})}$ and $\Psi_E^{(\rho,\mathbb{M})}$, for which playing with the pair $(\rho, \mathbb{M})$ is optimal.

5.4 Proof of Result 5.1

In order to prove this result we start by rewriting the figures of merit in a more compact form, we then extract some useful operators using the hyperplane separation theorem and define a particular classical post-processing (CPP) operation. With this in place, we proceed to address the discrimination case followed by the exclusion case.

5.4.1 Rewriting the figures of merit

We start by rewriting the probability of success (error) in multi-object discrimination (exclusion) games in a more compact form. Given a multi-object discrimination game $\Psi = \{\Psi_x(\cdot)\}$ and a state-measurement pair $(\rho, \mathbb{M})$, the probability of success can be written as:

$$\begin{aligned}
P_{\mathrm{succ}}^{\mathrm{D}}(\Psi, \rho, \mathbb{M}) &= \max_{\{p(g|a)\}} \sum_{x,a,g} \delta_{x,g}\, p(g|a)\, p(a|x)\, p(x), \\
&= \max_{\{p(g|a)\}} \sum_{x,a,g} \delta_{x,g} p(g|a) \operatorname{Tr}\left[M_a \frac{\Psi_x(\rho)}{\operatorname{Tr}[\Psi_x(\rho)]} \right] p(x), \\
&= \max_{\{p(g|a)\}} \sum_{x,a,g} \delta_{x,g} p(g|a) \operatorname{Tr}[M_a \Psi_x(\rho)], \\
&= \max_{\{p(g|a)\}} \sum_{x} \operatorname{Tr}\left\{ \left[\sum_a \left(\sum_g p(g|a) \delta_{x,g} \right) M_a \right] \Psi_x(\rho) \right\},
\end{aligned}$$

$$= \max_{\{p(x|a)\}} \sum_x \mathrm{Tr}\left\{\left[\sum_a p(x|a) M_a\right] \Psi_x(\rho)\right\},$$
$$= \max_{\mathbb{N} \preceq \mathbb{M}} \sum_x \mathrm{Tr}\left[N_x \Psi_x(\rho)\right],$$

where in the third line we used $p(x) = \mathrm{Tr}[\Psi_x(\rho)]$, and in the last line the maximisation is over all measurements simulable by $\mathbb{M}$. Similarly, for the multi-object exclusion case we get:

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi, \rho, \mathbb{M}) = \min_{\mathbb{N} \preceq \mathbb{M}} \sum_x \mathrm{Tr}\left[N_x \Psi_x(\rho)\right].$$

5.4.2 Some useful operators

Given any *fully resourceful* state-measurement pair $(\rho, \mathbb{M})$, meaning that $\rho \notin \mathrm{F}$ and $\mathbb{M} = \{M_x\} \notin \mathbb{F}$, $x \in \{1, ..., k\}$ and using the hyperplane separation theorem [189], we have that there exist positive semidefinite operators Z^ρ and $\{Z^{\mathbb{M}}_x\}$, $x \in \{1, ..., k\}$ such that:

$$\mathrm{Tr}(Z^\rho \rho) > 1, \quad \sum_x \mathrm{Tr}(Z^{\mathbb{M}}_x M_x) > 1, \tag{5.5}$$
$$\mathrm{Tr}(Z^\rho \sigma) \leq 1, \quad \sum_x \mathrm{Tr}(Z^{\mathbb{M}}_x N_x) \leq 1, \quad \forall \sigma \in \mathrm{F}, \mathbb{N} \in \mathbb{F}. \tag{5.6}$$

Similarly, there exist positive semidefinite operators Y^ρ and $\{Y^{\mathbb{M}}_x\}$, $x \in \{1, ..., k\}$ such that:

$$\mathrm{Tr}(Y^\rho \rho) < 1, \quad \sum_x \mathrm{Tr}(Y^{\mathbb{M}}_x M_x) < 1, \tag{5.7}$$
$$\mathrm{Tr}(Y^\rho \sigma) \geq 1, \quad \sum_x \mathrm{Tr}(Y^{\mathbb{M}}_x N_x) \geq 1, \quad \forall \sigma \in \mathrm{F}, \mathbb{N} \in \mathbb{F}. \tag{5.8}$$

These sets of operators are going to be useful when constructing the subchannel games for discrimination and exclusion.

5.4.3 Particular CPP operation

Given an arbitrary measurement $\mathbb{N} = \{N_a\}$ with $a \in \{1, ..., k+n\}$, n and k integers, we then construct a measurement $\tilde{\mathbb{N}} = \{\tilde{N}_x\}$ with k elements as:

$$\tilde{N}_x := N_x, \qquad x \in \{1, ..., k-1\},$$
$$\tilde{N}_k := N_k + \sum_{y=k+1}^{k+n} N_y. \tag{5.9}$$

We can check that this is a well-defined measurement and that the operation taking $\mathbb{N}$ into $\tilde{\mathbb{N}}$ is a CPP operation on the initial measurement $\mathbb{N}$. This corresponds to a coarse graining of measurement outcomes, such that any outcome of $\mathbb{N}$ greater or equal than k is declared as outcome k.

5.4.4 Discrimination case

Result 5.1A. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given any* fully resourceful *state-measurement pair* $(\rho, \mathbb{M})$*, meaning that we have a resourceful state* $\rho \notin \mathrm{F}$ *and a resourceful measurement* $\mathbb{M} \notin \mathbb{F}$*, then, there exists a subchannel game* $\Psi^{(\rho,\mathbb{M})}$ *such that:*

$$\max_{\sigma\in \mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M})}, \sigma, \mathbb{N}) < P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M})}, \rho, \mathbb{M}), \tag{5.10}$$

with the left side being maximised over all possible free states and free measurements.

Proof. We start by considering a *fully resourceful* state-measurement pair $(\rho, \mathbb{M})$. Using the hyperplane separation theorem [189], there exist positive semidefinite operators Z^ρ and $\{Z^{\mathbb{M}}_x\}$, $x \in \{1, ..., k\}$ satisfying the conditions (5.5) and (5.6). We now define the set of maps $\{\Phi^{(\rho,\mathbb{M})}_x(\cdot)\}$ such that for any state η we have:

$$\begin{aligned} \Phi^{(\rho,\mathbb{M})}_x(\eta) &:= \alpha \operatorname{Tr}(Z^\rho \eta) Z^{\mathbb{M}}_x, \\ \alpha &:= \frac{1}{\|Z^\rho\|_1 \operatorname{Tr}(Z^{\mathbb{M}})}, \qquad Z^{\mathbb{M}} := \sum_{x=1}^{k} Z^{\mathbb{M}}_x, \end{aligned} \tag{5.11}$$

with $\|X\|_1 = \operatorname{Tr}(\sqrt{X^\dagger X})$ the trace norm. Note that α is in general a function of ρ and $\mathbb{M}$, since Z^ρ and $Z^{\mathbb{M}}$ depend on ρ and $\mathbb{M}$, respectively. We can check that these maps are completely-positive and linear, and that they satisfy that $\forall \eta$:

$$F(\eta) := \operatorname{Tr}\left[\sum_{x=1}^{k} \Phi^{(\rho,\mathbb{M})}_x(\eta)\right] = \frac{\operatorname{Tr}(Z^\rho \eta)}{\|Z^\rho\|_1} \leq 1.$$

The inequality follows from the variational characterisation of the trace norm, establishing that $\|X\|_1 = \max_{-\mathbb{1}\leq M\leq \mathbb{1}}\{\operatorname{Tr}(XM)\}$ for any Hermitian operator X [242]. We can also write $F(\eta) = \alpha \operatorname{Tr}(Z^\rho \eta) \operatorname{Tr}(Z^{\mathbb{M}})$. The set of maps $\{\Phi^{(\rho,\mathbb{M})}_x(\cdot)\}$ then add up to a completely positive trace-nonincreasing linear map $\Phi^{(\rho,\mathbb{M})}(\cdot) := \sum_x \Phi^{(\rho,\mathbb{M})}_x(\cdot)$. We can then *complete* this set to be a set of subchannels by adding an extra subchannel $\Psi^{(\rho,\mathbb{M})}_{k+1}(\cdot) := \Lambda(\cdot) - \Phi^{(\rho,\mathbb{M})}(\cdot)$, with Λ being an arbitrary CPTP map such that $\Psi^{(\rho,\mathbb{M})}_{k+1}(\cdot) \geq 0$ (take the identity channel for instance). Therefore, with this construction we obtain a well-defined set of *subchannels* with $k+1$ elements. We now proceed to define a family of sets of subchannels in the following manner. Given a state-measurement pair $(\rho, \mathbb{M})$, $\mathbb{M} = \{M_x\}$, $x \in \{1, ..., k\}$, and an integer $n \geq 1$, we define the family of sets of subchannels given by $\Psi^{(\rho,\mathbb{M},n)} = \{\Psi^{(\rho,\mathbb{M},n)}_y(\cdot)\}$, $y \in \{1, ..., k+n\}$ with:

$$\Psi^{(\rho,\mathbb{M},n)}_y(\eta) := \begin{cases} \alpha \operatorname{Tr}[Z^\rho \eta] Z^{\mathbb{M}}_y, & y = 1, ..., k \\ \frac{1}{n}[1 - F(\eta)]\xi, & y = k+1, ..., k+n \end{cases} \tag{5.12}$$

with ξ begin an arbitrary quantum state $\xi \geq 0$, $\operatorname{Tr}(\xi) = 1$. We can check that this is a well-defined set of subchannels, because they now add up to a CPTP linear map:

$$\operatorname{Tr}\left[\sum_{y=1}^{k+n} \Psi^{(\rho,\mathbb{M},n)}_y(\eta)\right] = 1, \qquad \forall n, \forall \eta.$$

We now analyse the multi-object subchannel discrimination game given by $\Psi^{(\rho,\mathbb{M},n)}$. The probability of success of a player using the state-measurement pair $(\rho,\mathbb{M})$ is given by:

$$\begin{aligned} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)}\rho,\mathbb{M}) &= \max_{\mathbb{N}\preceq\mathbb{M}} \sum_{y=1}^{k+n} \mathrm{Tr}[N_y \Psi_y^{(\rho,\mathbb{M},n)}(\rho)] \\ &\geq \sum_{x=1}^{k} \mathrm{Tr}[M_x \Psi_x^{(\rho,\mathbb{M},n)}(\rho)] \\ &= \alpha\,\mathrm{Tr}[Z^\rho \rho] \sum_{x=1}^{k} \mathrm{Tr}\left[M_x Z_x^{\mathbb{M}}\right]. \end{aligned} \tag{5.13}$$

The inequality follows because we have chosen to simulate a particular measurement, i.e. $N_y = M_y$ for $y \leq k$ and $N_y = 0$ for $y > k$. In the last equality we have replaced the subchannel discrimination game with (5.12). Now, because of the conditions in (5.5), we have the *strict* inequality:

$$P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)},\rho,\mathbb{M}) > \alpha. \tag{5.14}$$

We now analyse the best *fully free* player:

$$\max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)},\sigma,\mathbb{N}) = \max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} \max_{\tilde{\mathbb{N}}\preceq\mathbb{N}} \sum_{x=1}^{k+n} \mathrm{Tr}\left[\tilde{N}_x \Psi_x^{(\rho,\mathbb{M},n)}(\sigma)\right].$$

We are considering QRTs of measurements closed under CPP and therefore, CPP is redundant here and we have:

$$\max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)},\sigma,\mathbb{N}) = \max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} \sum_{x=1}^{k+n} \mathrm{Tr}\left[N_x \Psi_x^{(\rho,\mathbb{M},n)}(\sigma)\right].$$

Let us now consider, without loss of generality, that these two maximisations are being achieved by the *fully free* pair $(\sigma^*,\mathbb{N}^*)$. We then have:

$$\begin{aligned} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)},\sigma^*,\mathbb{N}^*) &= \sum_{x=1}^{k+n} \mathrm{Tr}\left[N_x^* \Psi_x^{(\rho,\mathbb{N},n)}(\sigma^*)\right], & (5.15)\\ &= \alpha\,\mathrm{Tr}[Z^\rho \sigma^*] \sum_{y=1}^{k} \mathrm{Tr}\left[N_y^* Z_y^{\mathbb{M}}\right] & (5.16)\\ &+ \frac{1}{n}[1-F(\sigma^*)] \sum_{y=k+1}^{k+n} \mathrm{Tr}\left[N_y^* \xi\right]. & (5.17) \end{aligned}$$

In the second equality we have replaced the subchannel game (5.12). The first term can be upper bounded as:

$$\sum_{y=1}^{k} \mathrm{Tr}\left[N_y^* Z_y^{\mathbb{M}}\right] \leq \sum_{y=1}^{k} \mathrm{Tr}\left[\tilde{N}_y^* Z_y^{\mathbb{M}}\right] \leq 1,$$

with the measurement $\tilde{\mathbb{N}}^*$ defined in (5.9). The first inequality follows from the definition of the measurement $\tilde{\mathbb{N}}^*$. In the second inequality we use the fact that $\tilde{\mathbb{N}}^*$ is a free measurement (because it was constructed from a free measurement $\mathbb{N}^*$ and a CPP operation) and therefore we can use the conditions in (5.6). We now also use the fact that $1 - F(\eta) \leq 1, \forall \eta$, then equation (5.17) becomes:

$$P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)}, \sigma, \mathbb{N}) \leq \alpha + \frac{1}{n} \sum_{y=k+1}^{k+n} \mathrm{Tr}\left[N^*_y \xi\right].$$

The second term can be upper bounded as:

$$\sum_{y=k+1}^{k+n} \mathrm{Tr}[N^*_y \xi] \leq \sum_{y=1}^{k+n} \mathrm{Tr}[N^*_y \xi] = \mathrm{Tr}\left[\left(\sum_{y=1}^{k+n} N^*_y\right) \xi\right] = 1.$$

The inequality follows because we have added positive terms and the equality follows from $\mathbb{N}^*$ being a measurement $\sum_{y=1}^{k+n} \tilde{N}_y = \mathbb{1}$ and ξ being a state. We then get:

$$P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)}, \sigma, \mathbb{N}) \leq \alpha + \frac{1}{n}.$$

We now choose the subchannel game given by $\Psi^{(\rho,\mathbb{M},n\to\infty)} := \lim_{n\to\infty} \Psi^{(\rho,\mathbb{M},n)}$ and therefore we get:

$$P^{\mathrm{D}}_{\mathrm{succ}}\left(\Psi^{(\rho,\mathbb{M},n\to\infty)}, \sigma, \mathbb{N}\right) \leq \alpha. \tag{5.18}$$

Finally, equations (5.14) and (5.18) together imply that:

$$\max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n\to\infty)}, \sigma, \mathbb{N}) < P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n\to\infty)}, \rho, \mathbb{M}),$$

as desired. □

5.4.5 Exclusion case

Result 5.1B. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given any* fully resourceful *state-measurement pair, meaning that we have a resourceful state $\rho \notin \mathrm{F}$ and a resourceful measurement $\mathbb{M} \notin \mathbb{F}$, then, there exist a subchannel game $\Psi^{(\rho,\mathbb{M})}$ such that:*

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})}, \rho, \mathbb{M}) < \min_{\sigma\in\mathrm{F}} \min_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})}, \sigma, \mathbb{N}), \tag{5.19}$$

with minimisation over all possible free states and measurements.

Proof. This proof is closely related to the discrimination proof, but the subchannel game has to be constructed differently. We start by considering a *fully resourceful* state-measurement pair $(\rho, \mathbb{M})$. Using the hyperplane separation theorem [189], there exist positive semidefinite operators Y^ρ and $\{Y^{\mathbb{M}}_x\}$, $x \in \{1, ..., k\}$ satisfying the conditions (5.7) and (5.8). We now define the set of maps $\{\Phi^{(\rho,\mathbb{M})}_x(\cdot)\}$ with:

$$\Phi^{(\rho,\mathbb{M})}_x(\eta) := \beta \,\mathrm{Tr}(Y^\rho \eta) Y^{\mathbb{M}}_x,$$

$$\beta := \frac{1}{2\left\|Y^\rho\right\|_1 \mathrm{Tr}(Y^{\mathbb{M}})}, \qquad Y^{\mathbb{M}} := \sum_{x=1}^{k} Y^{\mathbb{M}}_x. \tag{5.20}$$

with $\|X\|_1 = \mathrm{Tr}(\sqrt{X^\dagger X})$ the trace norm. Note that β is in general a function of ρ and $\mathbb{M}$, since Y^ρ and $Y^{\mathbb{M}}$ depend on ρ and $\mathbb{M}$, respectively. As before, these operators are completely-positive linear maps and they now satisfy that $\forall \eta$:

$$G(\eta) := \mathrm{Tr}\left[\sum_{x=1}^{k} \Phi_x^{(\rho,\mathbb{M})}(\eta)\right] = \frac{\mathrm{Tr}(Y^\rho \eta)}{2\,\|Y^\rho\|_1} \leq \frac{1}{2},$$

which can also be written as:

$$G(\eta) = \beta\,\mathrm{Tr}(Y^\rho \eta)\,\mathrm{Tr}(Y^{\mathbb{M}}). \tag{5.21}$$

The set of maps $\{\Phi_x^{(\rho,\mathbb{M})}(\cdot)\}$ then add up to a completely positive trace-nonincreasing linear map $\Phi^{(\rho,\mathbb{M})}(\cdot) := \sum_x \Phi_x^{(\rho,\mathbb{M})}(\cdot)$. We can then *complete* this set to be a set of *subchannels* by adding an extra subchannel $\Psi_{k+1}^{(\rho,\mathbb{M})}(\cdot) := \Lambda(\cdot) - \Phi^{(\rho,\mathbb{M})}(\cdot)$, with Λ being an arbitrary CPTP map such that it $\Psi_{k+1}^{(\rho,\mathbb{M})}(\cdot) \geq 0$ (take the identity channel for instance). Therefore, with this construction we obtain a well-defined set of subchannels with $k+1$ elements. We now proceed to define a set of subchannels in the following manner. Given a state-measurement pair $(\rho, \mathbb{M})$, $\mathbb{M} = \{M_x\}$, $x \in \{1, ..., k\}$, we define the set of subchannels given by $\Psi^{(\rho,\mathbb{M})} = \{\Psi_y^{(\rho,\mathbb{M})}(\cdot)\}$, $y \in \{1, ..., k+n\}$ with:

$$\Psi_y^{(\rho,\mathbb{M})}(\eta) := \begin{cases} \beta\,\mathrm{Tr}[Y^\rho \eta] Y_y^{\mathbb{M}}, & y = 1, ..., k \\ [1 - G(\eta)]\xi^{\mathbb{M}}, & y = k+1 \end{cases} \tag{5.22}$$

with the quantum state:

$$\xi^{\mathbb{M}} := \frac{\sum_{x=1}^{k} p(x) Y_x^{\mathbb{M}}.}{\sum_{x=1}^{k} p(x)\,\mathrm{Tr}\left(Y_x^{\mathbb{M}}\right)}. \tag{5.23}$$

$\{p(x)\}$ being an arbitrary probability distribution. We can also check that this is a well-defined set of subchannels, i. e., they add up to a CPTP linear map:

$$\mathrm{Tr}\left[\sum_{y=1}^{k+1} \Psi_y^{(\rho,\mathbb{M})}(\eta)\right] = 1, \quad \forall \eta.$$

We remark here that, unlike the discrimination case, we are not generating a family of sets of subchannels, but only a specific one. We now analyse the multi-object subchannel exclusion game given by $\Psi^{(\rho,\mathbb{M})}$ and the probability of error of a player using the state-measurement pair $(\rho, \mathbb{M})$ which is given by:

$$\begin{aligned} P_{\mathrm{err}}^{\mathrm{E}}(\Psi^{(\rho,\mathbb{M})}\rho, \mathbb{M}) &= \min_{\mathbb{N} \preceq \mathbb{M}} \sum_{y=1}^{k+n} \mathrm{Tr}[N_y \Psi_y^{(\rho,\mathbb{M})}(\rho)] \\ &\leq \sum_{x=1}^{k} \mathrm{Tr}[M_x \Psi_x^{(\rho,\mathbb{M})}(\rho)] \\ &= \beta\,\mathrm{Tr}[Y^\rho \rho] \sum_{x=1}^{k} \mathrm{Tr}\left[M_x Y_x^{\mathbb{M}}\right]. \end{aligned} \tag{5.24}$$

The inequality follows because we have chosen to simulate a particular measurement, i.e. $N_y = M_y$ for $y \leq k$ and $N_y = 0$ for $y > k$. In the last equality we have replaced the subchannel exclusion game with (5.22). Now, because of (5.7) and (5.8), we have the *strict* inequality:

$$P_{\text{err}}^{\text{E}}(\Psi^{(\rho,\mathbb{M})},\rho,\mathbb{M}) < \beta. \tag{5.25}$$

As before, we now analyse the best *fully free* player:

$$\begin{aligned}\min_{\substack{\sigma\in\text{F}\\ \mathbb{N}\in\mathbb{F}}} P_{\text{err}}^{\text{E}}(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N}) &= \min_{\substack{\sigma\in\text{F}\\ \mathbb{N}\in\mathbb{F}\\ \tilde{\mathbb{N}}\preceq\mathbb{N}}} \sum_{x=1}^{k+1} \text{Tr}\left[\tilde{N}_x\Psi_x^{(\rho,\mathbb{M})}(\sigma)\right]\\ &= \min_{\substack{\sigma\in\text{F}\\ \mathbb{N}\in\mathbb{F}}} \sum_{x=1}^{k+1} \text{Tr}\left[N_x\Psi_x^{(\rho,\mathbb{M})}(\sigma)\right],\end{aligned}$$

where the equality follows because CPP is redundant. Let us now consider, without loss of generality, that these two minimisations are achieved by the *fully free* pair $(\sigma^*,\mathbb{N}^*)$. We then have:

$$\begin{aligned}P_{\text{err}}^{\text{E}}(\Psi^{(\rho,\mathbb{M})},\sigma^*,\mathbb{N}^*) =& \sum_{x=1}^{k+1} \text{Tr}[N_x^*\Psi_x^{(\rho,\mathbb{N})}(\sigma^*)]\\ =&\beta\,\text{Tr}[Y^\rho\sigma] \sum_{y=1}^{k} \text{Tr}\left[N_y^* Y_y^{\mathbb{M}}\right]\\ &+[1-G(\sigma^*)]\,\text{Tr}[N_{k+1}^*\xi^{\mathbb{M}}].\end{aligned}$$

We now add and subtract a convenient term as:

$$\begin{aligned}P_{\text{err}}^{\text{E}}(\Psi^{(\rho,\mathbb{M})},\sigma^*,\mathbb{N}^*) =& \,\beta\,\text{Tr}[Y^\rho\sigma] \sum_{x=1}^{k} \text{Tr}\left[N_x^* Y_x^{\mathbb{M}}\right]\\ &+\beta\,\text{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\text{Tr}(N_{k+1}^* Y_x^{\mathbb{M}})\\ &-\beta\,\text{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\text{Tr}(N_{k+1}^* Y_x^{\mathbb{M}})\\ &+[1-G(\sigma^*)]\,\text{Tr}\left[N_{k+1}^*\xi^{\mathbb{M}}\right].\end{aligned}$$

We now define a measurement given by $\tilde{\mathbb{N}} = \{\tilde{N}_x^*\}$ with $\tilde{N}_x^* = N_x^* + p(x)N_{k+1}^*$, and $p(x)$ being the probability distribution from (5.23), and we can reorganise this as:

$$\begin{aligned}P_{\text{err}}^{\text{E}}(\Psi^{(\rho,\mathbb{M})},\sigma^*,\mathbb{N}^*) =& \,\beta\,\text{Tr}[Y^\rho\sigma] \sum_{y=1}^{k} \text{Tr}\left[\tilde{N}_y^* Y_y^{\mathbb{M}}\right]\\ &+[1-G(\sigma^*)]\,\text{Tr}\left[N_{k+1}^*\xi^{\mathbb{M}}\right]\\ &-\beta\,\text{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\text{Tr}(N_{k+1}^* Y_x^{\mathbb{M}}).\end{aligned}$$

The first term is lower bounded by β by using the conditions in (5.8) and therefore:

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M},n)},\sigma^*,\mathbb{N}^*) \tag{5.26}$$

$$\geq \beta + [1-G(\sigma^*)]\,\mathrm{Tr}\left[N^*_{k+1}\xi^{\mathbb{M}}\right] - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(N^*_{k+1}Y^{\mathbb{M}}_x). \tag{5.27}$$

We now prove that the remaining term (last two lines) is always greater than or equal to zero. We start by rewriting this term as:

$$\begin{aligned}&[1-G(\sigma^*)]\,\mathrm{Tr}\left[N^*_{k+1}\xi^{\mathbb{M}}\right] - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(N^*_{k+1}Y^{\mathbb{M}}_x)\\ &= \mathrm{Tr}\left\{N^*_{k+1}\left[(1-G(\sigma^*))\,\xi^{\mathbb{M}} - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x\right]\right\}.\end{aligned} \tag{5.28}$$

We have $N^*_{k+1}\geq 0$ and therefore we now only need to prove that the operator inside the square brackets is positive semidefinite. We rewrite this operator as:

$$\begin{aligned}&[1-G(\sigma^*)]\,\xi^{\mathbb{M}} - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x\\ &= [1-G(\sigma^*)]\,\frac{\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x}{\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(Y^{\mathbb{M}}_x)} - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x,\end{aligned}$$

where we used (5.23) to substitute for $\xi^{\mathbb{M}}$. We now multiply by the positive term $\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(Y^{\mathbb{M}}_x)$ and obtain:

$$[1-G(\sigma^*)]\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\left(\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(Y^{\mathbb{M}}_x)\right)\left(\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x\right).$$

We now factorise the positive semidefinite operator $\sum_{x=1}^{k} p(x)Y^{\mathbb{M}}_x$ and analyse the coefficient as follows:

$$1-G(\sigma^*) - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(Y^{\mathbb{M}}_x) \tag{5.29}$$

$$= 1-\beta\,\mathrm{Tr}(Y^\rho\sigma^*)\,\mathrm{Tr}(Y^{\mathbb{M}}) - \beta\,\mathrm{Tr}(Y^\rho\sigma^*)\sum_{x=1}^{k} p(x)\,\mathrm{Tr}(Y^{\mathbb{M}}_x), \tag{5.30}$$

$$\geq 1-2\beta\,\mathrm{Tr}(Y^\rho\sigma^*)\,\mathrm{Tr}(Y^{\mathbb{M}}) = 1-\frac{\mathrm{Tr}(Y^\rho\sigma^*)}{\|Y^\rho\|_1}\geq 0. \tag{5.31}$$

In the first equality we replaced $G(\sigma^*)$ using (5.21). The first inequality follows because we are subtracting a larger quantity. In the second equality we substituted β (5.20). The second inequality follows because $\frac{\mathrm{Tr}(Y^\rho\eta)}{\|Y^\rho\|_1}\leq 1, \forall\eta$. Coming back to (5.27) we then have:

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\sigma^*,\mathbb{N}^*)\geq\beta. \tag{5.32}$$

Putting together (5.25) and (5.32) we obtain:

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\rho,\mathbb{M}) < \min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N}),$$

as desired. □

We would now like to *quantify* this advantage by specifying how large this gap can be. In order to do this, we need to define a suitable resource quantifier for the composite objects of state-measurement pairs. A natural starting point is to quantify the amount of resource contained within the individual objects of interest, states and measurements.

5.5 Result 5.2: Resource quantifiers and multi-object games

It turns out that it is enough to quantify the resources contained within the individual objects, as we will see in what follows. We now establish a connection between robustness-based (weight-based) resource quantifiers for states and measurements and multi-object subchannel discrimination (exclusion) games.

Result 5.2. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given* any *state-measurement pair* $(\rho, \mathbb{M})$ *we have:*

$$\max_{\Psi} \frac{P^{\mathrm{D}}_{\mathrm{succ}}(\Psi, \rho, \mathbb{M})}{\max\limits_{\sigma \in \mathrm{F}} \max\limits_{\mathbb{N} \in \mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi, \sigma, \mathbb{N})} = \Big[1 + \mathrm{R}_{\mathrm{F}}(\rho)\Big] \Big[1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big], \tag{5.33}$$

$$\min_{\Psi} \frac{P^{\mathrm{E}}_{\mathrm{err}}(\Psi, \mathbb{M}, \rho)}{\min\limits_{\sigma \in \mathrm{F}} \min\limits_{\mathbb{N} \in \mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi, \sigma, \mathbb{N})} = \Big[1 - \mathrm{W}_{\mathrm{F}}(\rho)\Big] \Big[1 - \mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big], \tag{5.34}$$

with the maximisation (minimisation) over all subchannel games.

The first thing we can notice is, that by considering a *fully resourceful* state- measurement pair $(\rho, \mathbb{M})$, one recovers the strict inequalities in (5.3) and (5.4). Additionally, we can also see that by considering now a *partially resourceful* pair $(\rho, \mathbb{M})$, meaning that either the state or the measurement is resourceful, we still get an advantage. This may seem counter-intuitive at first sight, as using a resourceless measurement should not allow the player to obtain any advantage, even with the most resourceful state. However, as we explicitly showed, there still exists a game which allows the player to utilise the advantage arising in such a partially-resourceful scenario. The resolution to this apparent paradox is based on the crucial difference between channel and subchannel discrimination/exclusion tasks. In particular, in a subchannel discrimination/exclusion game, a resourceful state has the additional ability to "influence" the ensemble of states from which the player needs to discriminate/exclude, since $\mathcal{E}^{\rho}_{\Psi} = \{\rho_x, p(x)\}$ with $p(x) = \mathrm{Tr}[\Psi_x(\rho)]$, $\rho_x = \Psi_x(\rho)/p(x)$ and therefore, this leads to suitable ensembles, even for resourceless measurements. Similarly, having access to a resourceful measurement provides better guessing strategies, even for ensembles generated by resourceless states. Finally, for a fixed *fully free* pair, there exists a game for which the pair is still optimal amongst all free pairs. Therefore, the ratios considered in Result 5.2 are comparing the performance of *any* pair against *all* fully free pairs.

It is illustrative to compare these results with their *single-object* counterparts [217, 212]. When considering subchannel games being played with a state alone, and allowing maximisations over arbitrary measurements, the advantage becomes $[1 + \mathrm{R}_{\mathrm{F}}(\rho)]$ [217]. In the multi-object scenario considered here however, we get $[1 +$

$\mathrm{R}_{\mathrm{F}}(\rho)][1+\mathrm{R}_{\mathbb{M}}(\mathbb{M})]$ instead, which can be larger, whenever $\mathbb{M}$ is resourceful. A similar analysis can be made for the weight-exclusion case [212]. This increment can be conceptually understood by the fact that we are now addressing a composite object and therefore, it is natural that each object contributes to the overall advantage. Nevertheless, it is still surprising that the advantage can be quantified in this elegant multiplicative manner.

It is also interesting to note that this result applies to convex QRTs of states with *arbitrary resources* and convex QRTs of measurements with *arbitrary resources* closed under CPP and therefore it covers, as particular instances, several important resources for both states and measurements. It would be interesting to explore whether these results still hold when CPP is dropped or, on the other hand, if a counterexample can be found. We leave this however for future research.

5.6 Proof of Result 5.2

We divide this result in two parts. In the first part we prove the upper bound for discrimination and the lower bound for exclusion. In the second part, we show how to achieve these bounds.

5.6.1 Upper bound for multi-object discrimination and lower bound for multi-object exclusion

We start by proving that for *any* state-measurement pair $(\rho, \mathbb{M})$, the product $[1+\mathrm{R}_{\mathrm{F}}(\rho)][1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})]$ places an upper bound on the advantage ratio in any subchannel game Ψ.

Proof. Given *any* subchannel game Ψ and *any* state-measurement pair $(\rho, \mathbb{M})$ we have:

$$
\begin{aligned}
&P^{\mathrm{D}}_{\mathrm{succ}}(\Psi, \rho, \mathbb{M}) \\
&= \max_{\mathbb{N}\preceq\mathbb{M}} \sum_x \mathrm{Tr}[N_x \Psi_x(\rho)] \\
&\leq \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big] \max_{\mathbb{N}\preceq\mathbb{M}} \sum_x \mathrm{Tr}[N_x \Psi_x(\sigma^*)], \\
&\leq \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big] \max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\preceq\mathbb{M}} \sum_x \mathrm{Tr}[N_x \Psi_x(\sigma)], \\
&= \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big] \max_{\sigma\in\mathrm{F}} \max_{\{q(x|a)\}} \sum_x \mathrm{Tr}\left[\left(\sum_a q(x|a) M_a\right) \Psi_x(\sigma)\right], \\
&\leq \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big] \max_{\sigma\in\mathrm{F}} \max_{\{q(x|a)\}} \sum_x \mathrm{Tr}\left[\left(\sum_a q(x|a) \tilde{N}^*_a\right) \Psi_x(\sigma)\right], \\
&= \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big] \max_{\sigma\in\mathrm{F}} \max_{\tilde{\tilde{\mathbb{N}}}\preceq\tilde{\mathbb{N}}^*} \sum_x \mathrm{Tr}\left[\tilde{\tilde{N}}_x \Psi_x(\sigma)\right], \\
&\leq \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big] \max_{\sigma\in\mathrm{F}} \max_{\tilde{\mathbb{N}}\in\mathbb{F}} \max_{\tilde{\tilde{\mathbb{N}}}\preceq\tilde{\mathbb{N}}} \sum_x \mathrm{Tr}\left[\tilde{\tilde{N}}_x \Psi_x(\sigma)\right], \\
&= \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big] \max_{\sigma\in\mathrm{F}} \max_{\tilde{\mathbb{N}}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi, \sigma, \tilde{\mathbb{N}}).
\end{aligned} \tag{5.35}
$$

In the first inequality we use the definition of the generalised robustness from which we get $\rho \leq [1+\mathrm{R}_{\mathrm{F}}(\rho)]\sigma^*$ and since Ψ_x are linear maps we have $\Psi_x(\rho) \leq [1+$

$\mathrm{R}_{\mathrm{F}}(\rho)]\Psi_x(\sigma^*)$, $\forall x$. In the second inequality we allow ourselves to maximise over all free states. In the third inequality, we use the definition of the generalised robustness from which we get $M_a \leq [1+\mathrm{R}_{\mathbb{M}}(\mathbb{M})]\tilde{N}_a^*$, $\forall a$. In the fourth inequality we allow ourselves to maximise over all free measurements. □

The proof for the lower bound for multi-object subchannel exclusion follows similar arguments.

5.6.2 Achieving upper bound for discrimination and lower bound for exclusion

Result 5.2A. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given* any *state-measurement pair $(\rho, \mathbb{M})$ we have:*

$$\max_{\Psi} \frac{P^{\mathrm{D}}_{\mathrm{succ}}(\Psi,\rho,\mathbb{M})}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi,\sigma,\mathbb{N})} = \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.36}$$

Proof. Given *any* state-measurement pair $(\rho, \mathbb{M})$, we want to find a suitable subchannel game Ψ so that we achieve the upper bound in (5.35). We start by noting that the primal SDPs of the generalised robustness for states and measurements can be seen as refined versions of the hyperplane separation theorem, from which we can extract positive semidefinite operators Z^ρ, $\{Z_x^{\mathbb{M}}\}$, $x \in \{1,...,k\}$ satisfying the conditions (5.5) and (5.6). Therefore, the construction of the set of subchannels from the previous section (5.11) applies here as well. We then continue from (5.13) which can now be written as:

$$P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n)},\rho,\mathbb{M}) \geq \alpha\,\mathrm{Tr}[Z^\rho\rho]\sum_{y=1}^{k}\mathrm{Tr}\Big[M_y Z_y^{\mathbb{M}}\Big] \tag{5.37}$$

$$= \alpha\Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.38}$$

The equality follows from the dual SDPs of the generalised robustness for states and measurements as per (2.18a) and (2.22a). We now analyse the *fully free* player. Similarly, we now choose the subchannel game given by $\Psi^{(\rho,\mathbb{M},n\to\infty)}$ (5.11) and invoking (5.18) we have:

$$\max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n\to\infty)},\sigma,\mathbb{N}) \leq \alpha. \tag{5.39}$$

We now analyse the ratio of interest with this particular subchannel game and have:

$$\frac{P^{\mathrm{D}}_{\mathrm{succ}}\left(\Psi^{(\rho,\mathbb{M},n\to\infty)},\rho,\mathbb{M}\right)}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}\left(\Psi^{(\rho,\mathbb{M},n\to\infty)},\sigma,\mathbb{N}\right)} \geq \frac{\alpha\Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big]}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}\left(\Psi^{(\rho,\mathbb{M},n\to\infty)},\sigma,\mathbb{N}\right)} \tag{5.40}$$

$$\geq \frac{\alpha\Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big]}{\alpha} \tag{5.41}$$

$$= \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.42}$$

In the first inequality we used (5.38) whilst in the second we used (5.39). Putting together (5.42) and (5.35) we obtain:

$$\frac{P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n\to\infty)},\rho,\mathbb{M})}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Psi^{(\rho,\mathbb{M},n\to\infty)},\sigma,\mathbb{N})} = \Big[1+\mathrm{R}_{\mathrm{F}}(\rho)\Big]\Big[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big].$$

as desired. □

Result 5.2B. *Consider a convex QRT of states with an arbitrary resource and a convex QRT of measurements with an arbitrary resource closed under CPP. Given* any *state-measurement pair* $(\rho,\mathbb{M})$ *we have:*

$$\min_{\Psi}\frac{P^{\mathrm{E}}_{\mathrm{err}}(\Psi,\rho,\mathbb{M})}{\min\limits_{\sigma\in\mathrm{F}}\min\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi,\sigma,\mathbb{N})} = \Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.43}$$

Proof. This proof follows a similar logic to that of the generalised robustness, and we write down for completeness. Given *any* state-measurement pair $(\rho,\mathbb{M})$, we want to find a suitable subchannel game Ψ so that we achieve the lower bound in (5.43). The construction of the set of subchannels form the previous section applies here as well. We then continue from (5.24) which can now be rewritten as:

$$P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})}\rho,\mathbb{M}) \leq \beta\,\mathrm{Tr}[Y^{\rho}\rho]\sum_{y=1}^{k}\mathrm{Tr}\left[M_y Y^{\mathbb{M}}_y\right], \tag{5.44}$$

$$= \beta\Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.45}$$

The equality follows from (2.20a) and (2.24a). We now analyse the *fully free* player and invoke (5.32) which reads:

$$\min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N}) \geq \beta. \tag{5.46}$$

We now analyse the ratio of interest with this particular subchannel game and have:

$$\frac{P^{\mathrm{E}}_{\mathrm{err}}\left(\Psi^{(\rho,\mathbb{M})},\rho,\mathbb{M}\right)}{\min\limits_{\sigma\in\mathrm{F}}\min\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\left(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N}\right)} \leq \frac{\beta\Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big]}{\min\limits_{\sigma\in\mathrm{F}}\min\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}\left(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N}\right)} \tag{5.47}$$

$$\leq \frac{\beta\Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big]}{\beta} \tag{5.48}$$

$$= \Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big]. \tag{5.49}$$

In the first inequality we used (5.45) whilst in the second we used (5.46). Putting together (5.49) and the lower bound in (5.43) we obtain:

$$\frac{P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\rho,\mathbb{M})}{\min\limits_{\sigma\in\mathrm{F}}\min\limits_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi^{(\rho,\mathbb{M})},\sigma,\mathbb{N})} = \Big[1-\mathrm{W}_{\mathrm{F}}(\rho)\Big]\Big[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big].$$

as desired. □

We now address multi-object single-shot information-theoretic quantities in the context of an encoding-decoding communication task.

5.7 Result 5.3: Connection to single-shot information theory

Consider a state-measurement pair $(\rho, \mathbb{M})$ and the following communication task. Suppose we have an ensemble of subchannels $\Lambda = \{\Lambda_x\}$ which add up to a completely positive and trace preserving map. Our goal is to encode the information about which of these subchannels has been applied in a classical random variable X. We do so by applying one of these subchannels to the state ρ, resulting in the ensemble of states $\mathcal{E}^{(\Lambda,\rho)} = \{\sigma_x^{(\Lambda,\rho)}, p(x)\}$ with $\sigma_x^{(\Lambda,\rho)} = \Lambda_x(\rho)$. We refer to the classical random variable X encoded in such a way as $X_{\Lambda,\rho}$. We then consider a decoding scheme using the measurement $\mathbb{M} = \{M_g\}$ with its outcomes representing a (guess) classical random variable G. Similarly, we refer to such a decoded variable as $G_{\mathbb{M}}$. We then have that this encoding-decoding scheme depends on the state-measurement pair $(\rho, \mathbb{M})$. A well studied figure of merit for communication tasks is the so-called *accessible information* [242]. Additionally, it has recently been introduced a complementary figure of merit which has been coined the *excludible information*, as in previous chapters, for its natural connection to exclusion tasks [217, 212]. These quantities depend on the plus (minus) infinity mutual information (respectively), which are given by:

$$I_{\pm\infty}(X_{\Lambda,\rho} : G_{\mathbb{M}}) = \pm\Big[H_{\pm\infty}(X_{\Lambda,\rho}) - H_{\pm\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}})\Big],$$

with the order plus and minus infinity entropies $H_{+\infty}(X_{\Lambda,\rho}) = -\log\{\max_x p(x)\}$ and $H_{-\infty}(X_{\Lambda,\rho}) = -\log\{\min_x p(x)\}$, the order plus and minus infinity conditional entropies $H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log\Big\{\sum_g \max_x \ p(x,g)\Big\}$ and $H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log\Big\{\sum_g \min_x \ p(x,g)\Big\}$, with $p(x,g) = p(g|x)p(x)$ and $p(g|x) = \mathrm{Tr}[M_g \Lambda_x(\rho)]$. The $\pm\infty$ mutual information quantifies the amount of the respective type of information (accessible or excludible) that can be conveyed by the state-measurement pair and the ensemble of channels at play. These measures are usually functions of the channel but we consider them here as functions of the pair $(\rho, \mathbb{M})$ instead.

Definition 5.4. *Consider a set of free states* F*, a set of free measurements* $\mathbb{F}$*, and a triple* $(\Lambda, \mathbb{M}, \rho)$*, then,* Arimoto's gap on state-measurement pairs *of order* $\pm\infty$ *for such a triple is given by:*

$$G^{\mathrm{F},\mathbb{F}}_{\pm\infty}(X;G)_{\Lambda,\mathbb{M},\rho} := I_{\pm\infty}(X;G)_{\Lambda,\mathbb{M},\rho} - \max_{\sigma\in\mathrm{F}} \max_{\mathbb{N}\in\mathbb{F}} I_{\pm\infty}(X;G)_{\Lambda,\mathbb{N},\sigma}. \tag{5.50}$$

We now address these quantities for a state-measurement pair in comparison to *all* fully free pairs.

Result 5.3. *Consider a state-measurement pair* $(\rho, \mathbb{M})$*. The maximum gap between the plus (minus) infinity mutual information between this pair and* all *fully free state-measurement pairs is upper bounded as:*

$$\max_{\Lambda} G^{\mathrm{F},\mathbb{F}}_{+\infty}(X;G)_{\Lambda,\mathbb{M},\rho} \leq \log\Big[1 + \mathrm{R}_{\mathrm{F}}(\rho)\Big] + \log\Big[1 + \mathrm{R}_{\mathbb{F}}(\mathbb{M})\Big], \tag{5.51}$$

$$\max_{\Lambda} G^{\mathrm{F},\mathbb{F}}_{-\infty}(X;G)_{\Lambda,\mathbb{M},\rho} \leq -\log\Big[1 - \mathrm{W}_{\mathrm{F}}(\rho)\Big] - \log\Big[1 - \mathrm{W}_{\mathbb{F}}(\mathbb{M})\Big], \tag{5.52}$$

with the maximisation over all ensembles of channels.

This result means that the resource quantifiers place upper bounds for these quantities. It would be interesting to see whether they can be saturated.

5.8 Proof of Result 5.3

Result 5.3A. *The maximum gap between the order plus-infinity mutual information of* any *state-measurement pair* $(\rho, \mathbb{M})$ *when compared to the best* fully free *state-measurement pair is upper bounded as:*

$$\max_{\Lambda} G^{\mathrm{F},\mathbb{F}}_{+\infty}(X;G)_{\Lambda,\mathbb{M},\rho} \leq \log\left[1+\mathrm{R}_{\mathrm{F}}(\rho)\right] + \log\left[1+\mathrm{R}_{\mathbb{F}}(\mathbb{M})\right], \tag{5.53}$$

with the maximisation over all ensembles of channels.

Proof. The plus-infinity mutual information between classical random variables $X_{\Lambda,\rho}$ and $G_{\mathbb{M}}$ is given by [186]:

$$I_{+\infty}(X_{\Lambda,\rho} \colon G_{\mathbb{M}}) = + \left[H_{+\infty}(X_{\Lambda,\rho}) - H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}})\right],$$

with $H_{+\infty}(X_{\Lambda,\rho}) = -\log(\max_x p(x))$, $H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log(\sum_g \max_x p(g,x))$ with $p(g,x) = p(g|x)p(x)$. We have $p(g|x) = \mathrm{Tr}(M_g\Lambda_x(\rho))$ and $H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log\sum_g \max_x \mathrm{Tr}[M_g\Lambda_x(\rho)]p(x)$. Considering $f_g(x) = \mathrm{Tr}[M_g\Lambda_x(\rho)]p(x)$ and using:

$$\max_x f_g(x) = \max_{\{p(x|g)\}} \sum_x p(x|g) f_g(x), \tag{5.54}$$

we have:

$$\begin{aligned}
H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) &= -\log\sum_g \max_{\{p(x|g)\}} \sum_x p(x|g) f_g(x),\\
&= -\log\sum_g \max_{\{p(x|g)\}} \sum_x p(x|g)\mathrm{Tr}[M_g\Lambda_x(\rho)]p(x),\\
&= -\log \max_{\{p(x|g)\}} \sum_x \mathrm{Tr}\left[\left(\sum_g p(x|g)M_g\right)\Lambda_x(\rho)\right] p(x),\\
&= -\log \max_{\mathbb{N}\prec\mathbb{M}} \sum_x \mathrm{Tr}[N_x\Lambda_x(\rho)]p(x),\\
&= -\log P^{\mathrm{D}}_{\mathrm{succ}}(\Lambda,\mathbb{M},\rho).
\end{aligned} \tag{5.55}$$

We then have the following expression:

$$\begin{aligned}
&I_{+\infty}(X_{\Lambda,\rho} \colon G_{\mathbb{M}}) - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} I_{+\infty}(X_{\Lambda,\sigma} \colon G_{\mathbb{N}})\\
&= -H_{+\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} -H_{+\infty}(X_{\Lambda,\sigma}|G_{\mathbb{N}}),\\
&= \log\left[P^{\mathrm{D}}_{\mathrm{succ}}(\Lambda,\mathbb{M},\rho)\right] - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} \log\left[P^{\mathrm{D}}_{\mathrm{succ}}(\Lambda,\mathbb{N},\sigma)\right],\\
&= \log\left\{\frac{P^{\mathrm{D}}_{\mathrm{succ}}(\Lambda,\mathbb{M},\rho)}{\max_{\mathbb{N}\in\mathbb{F}}\max_{\sigma\in\mathrm{F}} P^{\mathrm{D}}_{\mathrm{succ}}(\Lambda,\mathbb{N},\sigma)}\right\}.
\end{aligned}$$

We now maximise over all ensembles of channels and using Result 5.2 we obtain the claim in (5.53). □

Result 5.3B. *The maximum gap between the order minus-infinity mutual information of* any *state-measurement pair* $(\rho, \mathbb{M})$ *when compared to the best* fully free *state-measurement pair is upper bounded as:*

$$\max_{\Lambda} G^{\mathrm{F},\mathbb{F}}_{-\infty}(X;G)_{\Lambda,\mathbb{M},\rho} \leq -\log\left[1-\mathrm{W}_{\mathrm{F}}(\rho)\right] - \log\left[1-\mathrm{W}_{\mathbb{F}}(\mathbb{M})\right], \tag{5.56}$$

with the maximisation over all ensembles of channels.

Proof. The minus-infinity mutual information between classical random variables $X_{\Lambda,\rho}$ and $G_{\mathbb{M}}$ is given by [217, 212]:

$$I_{-\infty}(X_{\Lambda,\rho} \colon G_{\mathbb{M}}) = -\left[H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) - H_{-\infty}(X_{\Psi})\right],$$

with $H_{-\infty}(X_{\Lambda,\rho}) = -\log(\min_x p(x))$, $H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log\sum_g \min_x p(g,x)$, $p(g,x) = p(g|x)p(x)$. Using $p(g|x) = \mathrm{Tr}[M_g\Lambda_x(\rho)]$ then $H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) = -\log\sum_g \min_x \mathrm{Tr}[M_g\Lambda_x(\rho)]p(x)$. Considering $f_g(x) = \mathrm{Tr}[M_g\Lambda_x(\rho)]p(x)$ and using:

$$\min_x f_g(x) = \min_{\{p(x|g)\}} \sum_x p(x|g)f_g(x), \tag{5.57}$$

we have:

$$\begin{aligned}
H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) &= -\log\sum_g \min_{\{p(x|g)\}} \sum_x p(x|g)f_g(x),\\
&= -\log\sum_g \min_{\{p(x|g)\}} \sum_x p(x|g)\mathrm{Tr}[M_g\Lambda_x(\rho)]p(x),\\
&= -\log \min_{\{p(x|g)\}} \sum_x \mathrm{Tr}\left[\left(\sum_g p(x|g)M_g\right)\Lambda_x(\rho)\right]p(x),\\
&= -\log \min_{\mathbb{N}\prec\mathbb{M}} \sum_x \mathrm{Tr}[N_x\Lambda_x(\rho)]p(x),\\
&= -\log P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{M},\rho).
\end{aligned} \tag{5.58}$$

We then have the following expression:

$$\begin{aligned}
&I_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} I_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{N}})\\
&= H_{-\infty}(X_{\Lambda,\rho}|G_{\mathbb{M}}) - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} H_{-\infty}(X_{\Lambda,\sigma}|G_{\mathbb{N}}),\\
&= -\log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{M},\rho)\right] - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} -\log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{N},\sigma)\right],\\
&= -\log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{M},\rho)\right] + \min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}\in\mathbb{F}} \log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{N},\sigma)\right],\\
&= -\left\{\log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{M},\rho)\right] - \min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}\in\mathbb{F}} \log\left[P^{\mathrm{E}}_{\mathrm{err}}(\Lambda,\mathbb{N},\sigma)\right]\right\},\\
&= -\log\left\{\frac{P^{\mathrm{Q}}_{\mathrm{err}}(\Psi,\mathbb{M},\rho)}{\min_{\sigma\in\mathrm{F}}\min_{\mathbb{N}\in\mathbb{F}} P^{\mathrm{E}}_{\mathrm{err}}(\Psi,\mathbb{N},\sigma),}\right\}.
\end{aligned}$$

We now maximise over all ensembles of channels and using Result 5.2 we obtain the claim in (5.56). □

5.9 Conclusions

In this chapter we have introduced *multi-object* operational tasks in which the composite objects of interest are state-measurement pairs. The results found in this article hold for convex QRTs of states with arbitrary resources and convex QRTs of measurements closed under CPP. In particular, we have shown that *any* resourceful pair is useful for multi-object subchannel discrimination and exclusion games, when compared to the best possible strategy using *fully free* state-measurement pairs.

Furthermore, we have found that this advantage can be quantified, in a *multiplicative manner*, by the quantifiers of generalised robustness and weight of the state and the measurement, for discrimination and exclusion respectively. This means that the advantage is always possible whenever *at least* one of the pair is a resource. This is a consequence of the fact that in our case the resources do not interact with each other (i.e. the set of free objects is the set of all state-measurement pairs in which both of them are free, i.e. the total free set is $\mathrm{F} \times \mathbb{F}$). This leads to a natural open question: can we find relevant information-theoretic tasks in situations in which the set of free objects is more complicated, i.e. allows for a nontrivial interplay between the constituent resources? This could be achieved by considering a superset of $\mathrm{F} \times \mathbb{F}$ as the free set and quantifying the quantum advantage in this new case. It would be interesting to see whether this alteration can provide any new insights for other information-theoretic tasks and quantifiers.

Moreover, the objects which we jointly studied (state and measurement) are used in subchannel discrimination and exclusion tasks in a way which does not allow them to interfere with each other. It would be interesting to study objects which can influence one another and find tasks exploiting these interactions. For example, one could consider the pair: state and quantum instrument and study the advantage which they provide in tasks with multiple number of guesses. We believe that this may lead to new insights related to the information-disturbance trade-off purely from a resource theoretic perspective.

Our results also provide support, now in the multi-object regime, to the conjecture made in [217], about the existence of a weight-exclusion correspondence whenever there is a robustness-discrimination one. We have also introduced a communication task in which the log-robustness and the log-weight place upper bounds for information-theoretic quantities.

Finally, we believe that this chapter opens the door for exploring *multi-object* operational tasks in general QRTs of *arbitrary composite objects* with *arbitrary resources*, beyond those of states and measurements, as well as tasks for pairs of the same type of objects but exploiting different resources, and whether the distinction between the resources being disjoint, intersecting, and nested plays any major role.

Chapter 6

Quantum resource theory of Buscemi nonlocality

"A mathematician who can only generalise is like a monkey who can only climb up a tree, and a mathematician who can only specialise is like a monkey who can only climb down a tree. In fact neither the up monkey nor the down monkey is a viable creature. A real monkey must find food and escape his enemies and so must be able to incessantly climb up and down. A real mathematician must be able to generalise and specialise. "

George Pólya

In the previous chapter we addressed composite QRTs of state-measurement pairs, and explored how they can be used in conjunction for the benefit of multi-object subchannel discrimination/exclusion games. In this chapter we will still continue thinking about *multi-object* QRTs, though in a slightly different manner. Whilst state-measurement pairs can be seen as a composite object made out of two "independent objects", we now want to think about a composite object with its constituents having a little more structure. The composite object of study in this chapter is a *distributed measurement*, which is going to be constructed out of a triple: one bipartite state, and two bipartite measurements. In this chapter we will develop a QRT for these objects (distributed measurements) and one of their properties which has been coined as *Buscemi nonlocality*.

In 2012 F. Buscemi [45] extended the standard notion of a Bell experiment by allowing Alice and Bob to be asked quantum, instead of classical, questions. This gives rise to a broader notion of nonlocality, one which can be observed for every entangled state, and which we referred here to as *Buscemi nonlocality*. In this chapter we propose the generalised robustness of Buscemi nonlocality as a geometric quantifier measuring the ability of a given state and local measurements to produce Buscemi nonlocal correlations and prove the following results. First, we show that any distributed measurement which can demonstrate Buscemi nonlocal correlations provides strictly better performance than any distributed measurement which does not use entanglement in the task of distributed state discrimination, and that this advantage is quantified by the generalised robustness, thus establishing its operational significance. Second, we prove a quantitative relationship between: Buscemi nonlocality, the ability to perform nonclassical teleportation, and entanglement. In particular, we show that the maximal amount of Buscemi nonlocality that can be generated using a given state is precisely equal to its entanglement content. Using these relationships we propose new discrimination tasks for which nonclassical teleportation and entanglement lead to an advantage over their classical counterparts. Third, we interpret Buscemi nonlocality from the perspective of information theory and show that it is related to a single-shot capacity of a quantum-to-classical bipartite channel.

6.1 Introduction and motivation

Quantum entanglement is one of the most characteristic features of quantum theory [121]. During the early years of its development, however, it was recognised mainly as a bizarre property which distinguished it from classical physics. It was due to the discovery of Bell nonlocality [42] and subsequent development of Bell inequalities which allowed this distinction to be formulated quantitatively and to verify the predictions of quantum theory in an experimentally feasible setting.

Bell nonlocality is today perceived as a phenomenon in its own right and can be defined and tested irrespectively of the underlying theory. In simple terms Bell nonlocality refers to the situation when correlations shared between spatially separated parties cannot be explained as arising from a shared classical resource. The concept of Bell nonlocality is perhaps best understood in terms of a Bell experiment, which is sometimes also called a “no-signalling game”. In such a game, a referee distributes two physical systems to two spatially separated players, Alice (A) and Bob (B). Upon receiving their systems, each player is asked a question from a pre-arranged set of questions, labelled x for Alice and y for Bob. Depending on which of the questions was asked, Alice measures her system locally and obtains an outcome a. Similarly, based on his own question, Bob measures his share of the system and obtains b. The data produced from the experiment can be described using a conditional probability distribution $p(a, b|x, y)$, that is the probability of producing outcomes a and b given the choice of measurements labelled by x and y.

Entanglement and standard Bell-nonlocality are two quantum properties which are deemed as major resources for quantum technologies and yet, the relationship between them is still not yet fully understood. It is well known that entanglement is necessary for observing Bell-nonlocality and, in a similar vein, in a series of papers between 1991-92 by Gisin [91], Gisin-Peres [92], Popescu-Rohrlich [178], it was further proven that these two properties can be thought of as being "equivalent" for arbitrary pure states, since any pure-entangled state can be used to violate Bell-inequalities [178]. The conclusions of these three papers are nowadays colloquially addressed as "Gisin’s theorem". The relationship between entanglement and Bell-nonlocality starts to get less clear when considering mixed states. In 1989 R. Werner [240] proved that there exist mixed entangled states which cannot violate Bell-inequalities for projective measurements. Moving forward, in 2002 J. Barrett [16] proved a stronger statement by showing that there exist mixed entangled states which cannot violate Bell-inequalities for general POVMs. These types of states are known in the literature as "entangled-local" states [13], and whilst they may be deemed as "useless" for nonlocality-related applications, scientists over the year have still explored more exotic scenarios where these entangled-local states can still hopefully be used. These more exotic modified scenarios include: hidden Bell-nonlocality, superactivation of nonlocality, activation via networks, Buscemi nonlocality, amongst others [43]. The focus of this chapter is the scenario known as Buscemi nonlocality.

In 2012 [45] F. Buscemi generalised Bell's original experiment by allowing the referee to ask "quantum questions". This amounts to replacing the original set of classical (and therefore mutually orthogonal) questions, which could be encoded in states as $\{|x\rangle\}$, with a set of quantum states $\{|\omega_x\rangle\}$ which need not be orthogonal. The correlation data $p(a, b|\omega_x, \omega_y)$ obtained in this modified experiment, dubbed *semi-quantum non-signalling games*, differs significantly from its archetypical counterpart [45]. Perhaps the most striking consequence is that the new experiment is powerful enough to reveal the nonlocality of any entangled quantum state, even the nonlocality which would be hidden under a standard Bell test [45].

In this chapter we propose interpreting the correlation data obtained in a semi-quantum non-signalling game as an indicator of a this type of nonlocality which we refer to as *Buscemi nonlocality*. In order to formalise this notion we utilise the framework of Quantum Resource Theories (QRTs) [118, 58]. This is a set of tools and techniques developed to systematically quantify different properties of quantum systems. QRTs can be classified in terms of *objects* and *resources* studied in a given theory. Classification of QRTs with respect to the object lead to the resource theories of states [58], measurements [205, 66, 73, 102, 163, 166], channels [220, 143, 142, 243], and boxes [248, 195, 194, 190]. On the other hand, classifying QRTs with respect to the type of the studied resource leads to the resource theories of pure [160] and mixed-state entanglement [235], coherence [156], purity [120, 211], athermality [125, 40, 41, 117, 159, 114], nonlocality [50], asymmetry [175], measurement incompatibility [46], teleportation [56, 51], magic [123], nonmarkovianity [32, 237, 7] or nongaussianity [216], amongst many more. Its worth mentioning that although many QRTs use essentially the same mathematical formalism, their physical implications can be genuinely different. Hence the wide applicability of the framework to otherwise unrelated problems is a truly surprising aspect of Nature.

In this chapter, we focus on the quantum resource theory of Buscemi nonlocality, which is an instance of the resource theory from [194, 190]. The natural object relevant for this theory is a generalised measurement (POVM) performed by spatially-separated parties that do not communicate (distributed measurement). We investigate a geometric measure that quantifies the amount of Buscemi nonlocality contained within a given distributed measurement termed Robustness of Buscemi Nonlocality (RoBN). We then address Buscemi nonlocality as a property of states, by considering the maximal amount of Buscemi nonlocality that can be obtained using a given state by any local set of measurements on Alice's and Bob's side.

As a first and main result we show that Buscemi nonlocality has operational significance, by finding an operational task for which Buscemi nonlocality is a natural resource. This can be seen as akin to several seminal results in the field of quantum information which showed the operational character of coherence [156], entanglement [216], steering [173] or Bell nonlocality [3] in terms of experimentally relevant information-processing tasks. Moreover, our task gives rise to a complete family of monotones for this resource theory, i.e provides a sufficient and necessary characterisation of Buscemi nonlocality contained in a distributed measurement. Consequently, the average probability of guessing in these family of tasks can be interpreted as a simple and complete set of "Buscemi inequalities" which characterise nonlocality of distributed measurements, in analogy with the celebrated Bell inequalities characterising nonlocality of states [43].

The second main result concerns how Buscemi nonlocality relates to other types of nonclassical phenomena studied in the literature: nonclassical teleportation [51] and entanglement [121]. We show that the maximal value of RoBN which can be achieved when Bob (Alice) is allowed to use any measurement is precisely the so-called robustness of teleportation (RoT) of a teleportation channel from Alice (Bob) to Bob (Alice) [51]. On the other hand, optimising RoBN over all local measurements for both parties leads to the robustness of entanglement of the state shared by Alice and Bob [208]. This result, despite its clarifying character being of independent interest, leads to new operational tasks for which both nonclassical teleportation and entanglement are natural resources. These quantitative relationships further expand the results presented in [50], [216] and [139] by proposing new discrimination tasks for which both entanglement and nonclassical teleportation provide advantage over their classical (i.e. separable) counterparts.

As third and final main result we interpret Buscemi nonlocality from the perspective of single-shot quantum information theory. We show that Buscemi nonlocality, when viewed as a property of a communication channel between the sender (the Referee) and receiver (Alice and Bob), quantifies the maximal amount of information that can be sent reliably when the channel is used only once (the so-called single-shot capacity of a quantum channel). This establishes an important link between Buscemi nonlocality and quantum communication.

This chapter is organised as follows. In Sec. 6.2 we cover the relevant formalism, remind the idea of characterising nonlocality in terms of non-signalling games and recall the robustness quantifier of Buscemi nonlocality (RoBN). In Sec. 6.2.2 we find its operational interpretation in terms of the advantage in the task of distributed state discrimination (DSD). In Sec. 6.3.1 we explore the relationship between Buscemi nonlocality and the concepts of nonclassical teleportation and entanglement. Finally, in Sec. 6.3.2 we describe a tangential view on RoBN from the perspective of single-shot information theory. We conclude with Sec. 6.4 where we summarise our findings and highlight several open questions.

6.2 Framework

In what follows we will denote a local bipartite measurement on Alice's side (system AA') with $\mathbb{M}^{\mathrm{A}} = \{M_a^{\mathrm{AA}'}\}$, where each $M_a^{\mathrm{AA}'}$ is a positive semi-definite operator that adds up to the identity (POVM). Similarly we will use $\mathbb{M}^{\mathrm{AB}}$ to indicate that the measurement is non-local, i.e. we will treat systems labelled with different letters, e.g. A and B , as two spatially separated parties. We are interested in the most general type of measurement that can be performed in this bipartite scenario without the aid of classical or quantum communication. This can be realised by (*i*) allowing Alice and Bob to apply arbitrary bipartite measurements in their labs, denoted respectively $\mathbb{M}^{\mathrm{A}} = \{M_a^{\mathrm{AA}'}\}$ and $\mathbb{M}^{\mathrm{B}} = \{M_b^{\mathrm{B}'\mathrm{B}}\}$, where $a \in \{1, \ldots, o_{\mathrm{A}}\}$ and $b \in \{1, \ldots, o_{\mathrm{B}}\}$ denote Alice's and Bob's outcomes and (*ii*) allowing the two parties to share a quantum state $\rho^{\mathrm{A}'\mathrm{B}'}$. In this way Alice and Bob can store and share all types of classical information (e.g. classical memory or measurement strategy), as well as quantum information (i.e. shared entanglement). We denote such a measurement with $\mathbb{M}^{\mathrm{AB}} = \{M_{ab}^{\mathrm{AB}}\}$, where the corresponding POVM elements are of the following general form:

$$M_{ab}^{\mathrm{AB}} = \mathrm{Tr}_{\mathrm{A}'\mathrm{B}'}\left[\left(M_a^{\mathrm{AA}'} \otimes M_b^{\mathrm{B}'\mathrm{B}}\right)\left(\mathbb{1}^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right]. \tag{6.1}$$

Since the sets of all quantum states and quantum measurements are both convex sets, it follows that the set of measurements of the form (6.1) is also a convex set. We will refer to measurements of the form (6.1) as *distributed measurements* and denote

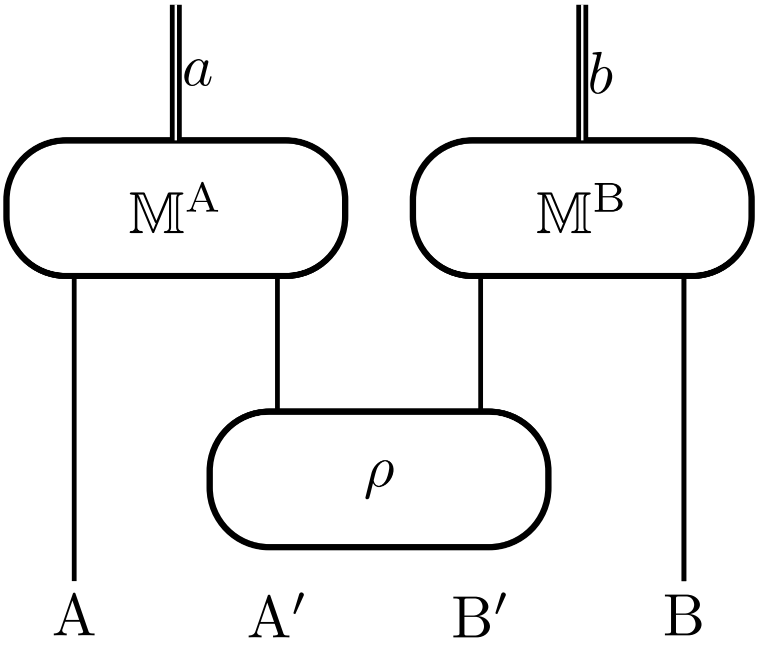

FIGURE 6.1: A schematic diagram of a distributed measurement $\mathbb{M}^{\mathrm{AB}}$ composed of local measurements for Alice $\mathbb{M}^{\mathrm{A}} = \{M_a^{\mathrm{AA}'}\}$, for Bob $\mathbb{M}^{\mathrm{B}} = \{M_b^{\mathrm{B}'\mathrm{B}}\}$ and a state $\rho^{\mathrm{A}'\mathrm{B}'}$ shared between them. This is the most general type of measurement which Alice and Bob can perform in a distributed scenario which does not allow for communication.

the set of all such measurements with $\mathcal{R}_{\mathrm{BN}}$. These measurements are the main (*resourceful*) objects of the resource theory we consider here. Whenever the elements of measurement $\mathbb{M}^{\mathrm{AB}}$ can be written as in (6.1) for some choice of shared state and local measurements we will write $\mathbb{M}^{\mathrm{AB}} \in \mathcal{R}_{\mathrm{BN}}$. Later in Sec. 6.2.2 we will formally define the set of *free measurements* of this resource theory, which turn out to be distributed measurements with a separable shared state. Fig. 6.1 illustrates a distributed measurement and describes the relationship between different subsystems. This type of objects appear naturally in a wide range of contexts when studying non-local effects in an MDI setting [45, 52, 54, 191].

We now specify the most general class of operations that the separated parties in A and B can perform, without communicating, to improve the properties of their distributed measurement $\mathbb{M}^{\mathrm{AB}} = \{M_{ij}^{\mathrm{AB}}\}$, where indices $i \in \{1, \dots o_{\mathrm{A}}\}$ and $j \in \{1, \dots o_{\mathrm{B}}\}$ describe measurements outcomes. This is done as a generalisation of the simulation of measurements introduced in previous chapters. The free operations for the QRT of Buscemi nonlocality can be addressed within the framework of Local Operations and Shared Randomness (LOSR) [248, 195, 194, 190]. There, Alice and Bob are allowed to share any amount of classical memory described by a random variable λ. Formally this is specified by providing a probability distribution $p(\lambda)$ which is available to both parties. Moreover, before measuring their systems both parties are allowed to locally perform any completely positive and trace-preserving map, potentially conditioned on the value of the shared memory, i.e. we allow for applying $\mathcal{E}_\lambda$ on Alice's and $\mathcal{N}_\lambda$ on Bob's side. Finally, the parties are allowed to post-process their measurement outcomes using arbitrary classical channels $p(a|i, \lambda)$ and $p(b|j, \lambda)$ to produce their final guesses. This procedure leads to the most general type of LOSR operation that can be performed on a measurement of the form (6.1)

[194, 190]. In what follows we will refer to this as *quantum simulation*:

Definition 6.1. *(Quantum simulation) A quantum simulation of a bipartite measurement* $\mathbb{M} = \{M_{ij}\}$ *with a subroutine:*

$$\mathcal{S} = \{p(\lambda), p(a|i,\lambda), p(b|j,\lambda), \mathcal{E}_\lambda, \mathcal{N}_\lambda\} \tag{6.2}$$

is a transformation which maps the POVM elements of $\mathbb{M}$ *into:*

$$M'_{ab} = \sum_{i,j,\lambda} p(\lambda)p(a|i,\lambda)p(b|j,\lambda)(\mathcal{E}^\dagger_\lambda \otimes \mathcal{N}^\dagger_\lambda)[M_{ij}], \tag{6.3}$$

where $\mathcal{E}^\dagger$ *denotes the (unique) dual map to* $\mathcal{E}$*.*

In other words, any action that can be performed by Alice and Bob in their labs without access to communication can be described by some quantum simulation subroutine.

Quantum simulation induces a natural preorder on the set of all bipartite measurements. Formally, a preorder is an ordering relation that is reflexive ($a \succ a$) and transitive ($a \succ b$) and ($b \succ c$) implies ($a \succ c$). Here the preorder induced by quantum simulation will be denoted with $\succ_q$, i.e. $\mathbb{M} \succ_q \mathbb{M}'$ if and only if there there exists a subroutine $\mathcal{S}$ which allows $\mathbb{M}$ to simulate $\mathbb{M}'$, i.e. for the two measurements $\mathbb{M}$ and $\mathbb{M}'$, condition (6.3) in Definition 6.1 holds. The notion of simulation will turn out to be relevant for the operational tasks introduced later on.

6.2.1 Nonlocality from the perspective of no-signalling games

Bell nonlocality can be best understood from the perspective of no-signalling games, which also provides an intuitive understanding of Bell inequalities. Such games have been extensively studied in computer science for a long time, where they are a special instance of *interactive proof systems* [60].

The standard scenario of a no-signalling game involves two cooperating players (Alice and Bob) who play the game against a third party, the referee. The referee chooses a question $x \in \mathcal{X}$ for Alice and $y \in \mathcal{Y}$ for Bob according to some probability distribution $p(x,y) : \mathcal{X} \times \mathcal{Y} \to [0,1]$, where $\mathcal{X}$ and $\mathcal{Y}$ denote finite sets of questions. Without communicating, and therefore, without knowing what question the other player was asked, Alice (Bob) returns an answer $a \in \mathcal{A}$ ($b \in \mathcal{B}$) from a finite set of possible answers $\mathcal{A}$ ($\mathcal{B}$). Based on the questions asked and the received answers, the referee determines whether the players win or lose the game, according to a pre-arranged set of rules. Such rules are typically expressed using a function $V : \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y} \to [0,1]$, where $V(a,b,x,y) = 1$ if and only if Alice and Bob win the game by answering a and b for questions x and y.

Alice and Bob know the rules of the game, that is, they know the function V and the distribution of questions $p(x,y)$. Before the game starts they can agree on any strategy which provides them with the best chances of winning. However, once the game starts, they are not allowed to communicate any more. In the classical setting any strategy they can possibly devise can be encoded in a classical memory system, represented by a shared random variable λ and a probability distribution $p(\lambda)$. In the more general quantum case, any possible strategy can be described by a shared quantum state ρ and a choice of local measurements.

In order to relate the above game setting with Bell inequalities note that the referee's questions x and y can be thought of as labels for different measurement settings. Similarly, the answers correspond to the outcomes of local measurements.

Any measurement strategy (be it classical or quantum) leads to a conditional probability $p(a,b|x,y)$ which describes when Alice and Bob give answers a and b for questions x and y, respectively. In the language of Bell inequalities $p(a,b|x,y)$ determine the probability that Alice and Bob obtain measurement outcomes a and b when performing the measurements labelled by x and y. The average probability that Alice and Bob win, maximised over all possible strategies, can be written as:

$$p^{V}_{\text{guess}}(\mathcal{G},\mathbb{M}) = \sum_{a,b,x,y} p(x,y)p(a,b|x,y)V(a,b,x,y), \tag{6.4}$$

where $\mathcal{G} = \{p(x,y), V\}$ defines the game and the conditional probabilities $p(a,b|x,y)$ are related to the local measurements $\{M^{\mathrm{A}}_{a|x}\}$ for Alice and $\{M^{\mathrm{B}}_{b|y}\}$ for Bob, via the Born rule:

$$p(a,b|x,y) = \mathrm{Tr}\left[\left(M^{\mathrm{A}'}_{a|x} \otimes M^{\mathrm{B}'}_{b|y}\right)\rho^{\mathrm{A}'\mathrm{B}'}\right]. \tag{6.5}$$

With this in mind, Bell inequalities can be thought of as upper bounds on the average guessing probability $p_{\text{guess}}(\mathcal{G},\mathbb{M})$ with which Alice and Bob can win a nonlocal game $\mathcal{G}$ using a classical strategy (i.e. when $\rho^{\mathrm{A}'\mathrm{B}'}$ is a separable state), optimised over all local measurements $\{M^{\mathrm{A}'}_{a|x}\}$ and $\{M^{\mathrm{B}'}_{b|y}\}$. A violation of a Bell inequality corresponds to the situation when there is a quantum strategy which uses an entangled shared state and outperforms the best classical strategy in a particular game $\mathcal{G}$.

Importantly, there are entangled states which can never violate any Bell inequality [240, 16, 12]. In the language of no-signalling games this means that there are states $\rho^{\mathrm{A}'\mathrm{B}'}$ which, although entangled, can never outperform the best classical strategy. However, in [45] Buscemi showed that when we modify the rules of the no-signalling game and allow the referee to ask *quantum* instead of classical questions, then all entangled states can outperform the best classical strategy in some nonlocal game, or equivalently, violate the corresponding Bell inequality.

Before going into the details, let us note that "asking classical questions" can also be mathematically modelled by sending states from a collection of orthogonal states from a fixed basis, e.g. $\{|x\rangle\}$ such that $\sum_x |x\rangle\langle x| = \mathbb{1}$ and $\langle x|x'\rangle = \delta_{x,x'}$ and similarly for $\{|y\rangle\}$. Such states are perfectly distinguishable and hence Alice and Bob, after receiving their questions, may choose their measurements unambiguously. This can be viewed as giving Alice and Bob the ability to perform controlled bipartite measurements $\mathbb{M}^{\mathrm{AA}'} = \{M^{\mathrm{AA}'}_a\}$ and $\mathbb{M}^{\mathrm{B}'\mathrm{B}} = \{M^{\mathrm{B}'\mathrm{B}}_b\}$ with the POVM elements:

$$M^{\mathrm{AA}'}_a = \sum_x |x\rangle\langle x|^{A} \otimes M^{\mathrm{A}'}_{a|x}, \tag{6.6}$$

$$M^{\mathrm{B}'\mathrm{B}}_b = \sum_y M^{\mathrm{B}'}_{b|y} \otimes |y\rangle\langle y|^{\mathrm{B}}. \tag{6.7}$$

If Alice and Bob share a quantum state $\rho^{\mathrm{A}'\mathrm{B}'}$ then effectively they have access to a distributed measurement $\mathbb{M}^{\mathrm{AB}}$ of the form (6.1). This measurement is then applied to the "questions" they receive, which we denote here with $\omega^{\mathrm{A}}_x = |x\rangle\langle x|^{\mathrm{A}}$ for Alice and $\omega^{\mathrm{B}}_y = |y\rangle\langle y|^{\mathrm{B}}$ for Bob. Therefore their behaviour $p(a,b|\omega_x,\omega_y)$ can be written as:

$$p(a,b|\omega_x,\omega_y) := \mathrm{Tr}\left[M_{ab}^{\mathrm{AB}}\left(\omega_x^{\mathrm{A}} \otimes \omega_y^{\mathrm{B}}\right)\right], \tag{6.8}$$

$$= \mathrm{Tr}\left[M_{ab}^{\mathrm{AB}}\left(|x\rangle\langle x|^{\mathrm{A}} \otimes |y\rangle\langle y|^{\mathrm{B}}\right)\right], \tag{6.9}$$

$$= \mathrm{Tr}\left[\left(M_{a|x}^{\mathrm{A}'} \otimes M_{b|y}^{\mathrm{B}'}\right)\rho^{\mathrm{A}'\mathrm{B}'}\right], \tag{6.10}$$

$$= p(a,b|x,y). \tag{6.11}$$

With this in mind we can now formalise the process of asking "quantum questions". This happens precisely when the states sent by the referee are chosen from an arbitrary collection of states $\{\omega_x\}$. Crucially, these states need not be distinguishable and so each of them can be in a superposition of different orthogonal states.

Notice, however, that using quantum states as inputs to the distributed measurement $\mathbb{M}^{\mathrm{AB}}$ with local measurements of the form (6.6) and (6.7) can only lead to a probabilistic version of the standard no-signalling game, i.e. Alice and Bob randomise their choices of measurements according to the respective overlaps $p(x'|x) = \langle x'|\omega_x|x'\rangle$ and $p(y'|y) = \langle y'|\omega_y|y'\rangle$. Thus, in order to use the power of asking genuinely quantum questions, one needs to allow for arbitrary bipartite local measurements on both sides. This leads to the general form of a distributed measurement (6.1) with the local POVM elements $\{M_a^{\mathrm{AA}'}\}$ and $\{M_b^{\mathrm{BB}'}\}$ being now fully general bipartite measurements, and therefore a Buscemi behaviour is of the form:

$$p(a,b|\omega_x,\omega_y) = \mathrm{Tr}\left[\left(M_a^{\mathrm{AA}'} \otimes M_b^{\mathrm{B}'\mathrm{B}}\right)\left(\omega_x^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \omega_y^{\mathrm{B}}\right)\right]. \tag{6.12}$$

The above extension of a no-signalling game leads to a novel type of nonlocality which was noticed for the first time in [45]. Here we will refer to this type of nonclassical correlations as *Buscemi nonlocality*. In this language the main result of [45] states that all entangled states are Buscemi nonlocal.

In what follows we present a consistent way of quantifying Buscemi nonlocality. First we define a proxy quantity which quantifies how much Buscemi nonlocality can be evidenced using a fixed distributed measurement. This provides a natural quantifier for the resource theory of Buscemi nonlocality of distributed measurements, which is our main focus here. Optimising this quantity over all choices of local measurements for Alice and Bob gives rise to quantity which measures the maximal degree of Buscemi nonlocality which can ever be obtained using a given quantum state.

6.2.2 Quantitative measure of Buscemi nonlocality

The fact that Alice and Bob may share entanglement in (6.1) and use it to perform a measurement means that the measurement is inherently nonlocal and can lead to interesting correlations, even when measured on completely independent systems. Our central question then is how to quantify this nonlocality present in a bipartite measurement. To build a valid reference point we first consider the case when the measurement does not lead to any type of quantum correlations. This means that the behaviour $p(a,b|\omega_x,\omega_y) = \mathrm{Tr}[M_{ab}^{\mathrm{AB}}(\omega_x^{\mathrm{A}} \otimes \omega_y^{\mathrm{B}})]$ results from the measurement $\{M_{ab}^{\mathrm{AB}}\}$ formed using a separable shared state $\rho^{\mathrm{A}'\mathrm{B}'} \in \mathsf{SEP}$, where SEP denotes the set of all separable operators. Any separable state can be written as:

$$\rho^{A'B'} = \sum_{\lambda} p(\lambda)\, \rho_{\lambda}^{A'} \otimes \rho_{\lambda}^{B'}, \tag{6.13}$$

where $p(\lambda)$ is a classical probability distribution corresponding to a shared random variable λ and $\{\rho_{\lambda}^{A'}\}$ and $\{\rho_{\lambda}^{B'}\}$ are collections of local quantum states. The associated distributed measurement from Eq. (6.1) takes the form:

$$M_{ab}^{AB} = \sum_{\lambda} p(\lambda)\, M_{a|\lambda}^{A} \otimes M_{b|\lambda}^{B}, \tag{6.14}$$

where we denoted $M_{a|\lambda}^{A} := \mathrm{Tr}_{A'}[M_a^{AA'}(\mathbb{1}^A \otimes \rho_{\lambda}^{A'})]$ for Alice and $M_{b|\lambda}^{B} := \mathrm{Tr}_{B'}[M_b^{B'B}(\rho_{\lambda}^{B'} \otimes \mathbb{1}^B)]$ for Bob. This is the most general classical measurement scheme which can be realised if Alice and Bob have access only to classical randomness λ and the ability to locally prepare quantum states in their labs. The set of all measurements that can be written as in (6.14) will be denoted by $\mathcal{F}_{\mathrm{BN}}$. These measurements are the most natural candidates for free objects in the resource theory of Buscemi nonlocality. Notice that measurements from this set have POVM elements that are all separable (SEP) and admit a quantum realisation ($\mathcal{R}_{\mathrm{BN}}$), i.e can be written as in (6.1) for some choice of local measurements and shared state. Such measurements can never demonstrate Buscemi nonlocality, regardless of the state being measured.

In order to better understand the difference between the sets $\mathcal{R}_{\mathrm{BN}}$ (all distributed measurements) and $\mathcal{F}_{\mathrm{BN}}$ (free distributed measurements), let us consider the following simple example:

Example 2. *Let Alice and Bob share a two-qubit Werner state:*

$$\rho^{A'B'} = p\,\phi_+^{A'B'} + (1-p)\frac{\mathbb{1}^{A'B'}}{4}, \tag{6.15}$$

where $p \in [0,1]$, the state $\phi_+ = |\phi_+\rangle\langle\phi_+|$ and $|\phi_+\rangle := \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$ is a maximally-entangled state. It is widely known that the Werner state (6.15) is separable for all $p \le 1/3$. Let $\{U_a\}$ for $a = 1, \ldots, 4$, be a set of Pauli operators. Consider a measurement $\mathbb{M}^A = \{M_a^{A'A}\}$ with elements:

$$M_a^{A'A} = (U_a^{A'} \otimes \mathbb{1}^A)\phi_+^{A'A}(U_a^{A'} \otimes \mathbb{1}^A)^{\dagger}, \tag{6.16}$$

Defining an analogous measurement for Bob $\mathbb{M}^B = \{M_b^{BB'}\}$ and using the definition (6.1) allows us to write the distributed measurement $\mathbb{M}^{AB} = \{M_{ab}^{AB}\}$ for Alice and Bob as:

$$M_{ab}^{AB} = (U_a^{A'} \otimes U_b^{B'})\rho^{A'B'}(U_a^{A'} \otimes U_b^{B'})^{\dagger} \tag{6.17}$$

$$= p\,\phi_{ab}^{A'B'} + (1-p)\frac{\mathbb{1}^{A'B'}}{4}, \tag{6.18}$$

where we labelled $\phi_{ab}^{A'B'} := (U_a^{A'} \otimes U_b^{B'\dagger})\phi_+^{A'B'}(U_a^{A'} \otimes U_b^{B'\dagger})^{\dagger}$. Clearly, $\phi_{ab}^{A'B'}$ is again a maximally-entangled state and therefore each POVM element of $\mathbb{M}^{AB}$ is a Werner state, up to local unitaries. Since entanglement is preserved under local unitary operations, all elements of the distributed measurement $\mathbb{M}^{AB}$ are entangled operators for $p > 1/3$.

Therefore we can conclude that for $p \le 1/3$ the distributed measurement $\mathbb{M}^{AB}$ can be written as in (6.14), which by definition means that $\mathbb{M}^{AB} \in \mathcal{F}_{\mathrm{BN}}$. Moreover, for $p > 1/3$ we know that each $M_{ab}^{AB} \notin$ SEP and therefore $\mathbb{M}^{AB} \notin \mathcal{F}_{\mathrm{BN}}$. This implies that this *distributed measurement is a resourceful measurement in the resource theory of Buscemi nonlocality.* □

A natural question at this point is: given an arbitrary bipartite measurement $\mathbb{M}^{\mathrm{AB}} \in \mathcal{R}_{\mathrm{BN}}$, how can its nonlocal properties be quantified, in particular its ability to generate Buscemi nonlocality? For this purpose it is useful to define the following quantity:

Definition 6.2. *(Robustness of Buscemi Nonlocality [54]) The robustness of Buscemi nonlocality (RoBN) of a distributed measurement $\mathbb{M}^{\mathrm{AB}} = \{M_{ab}^{\mathrm{AB}}\}$ is the solution to the following optimisation problem:*

$$\begin{aligned} \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \min \quad & r \\ \text{s.t.} \quad & M_{ab}^{\mathrm{AB}} + rN_{ab}^{\mathrm{AB}} = (1+r)O_{ab}^{\mathrm{AB}} \quad \forall\, a, b, \\ & \{O_{ab}^{\mathrm{AB}}\} \in \mathcal{F}_{\mathrm{BN}}, \quad \{N_{ab}^{\mathrm{AB}}\} \in \mathcal{R}_{\mathrm{BN}}. \end{aligned} \tag{6.19}$$

Although this may not seem obvious at first sight, the above is a convex optimisation problem and hence can be efficiently solved numerically [38, 239, 97] (see Appendix A.1 for details). Moreover, due to the duality of convex optimisation problems the dual formulation of the above has several nice properties which will be useful for our purposes. Robustness-based quantifiers were introduced in [235, 209] as entanglement quantifiers and since then successfully applied in a wide range of QRTs. The above variant is closely related to the MDI-nonlocality robustness introduced in [54] at the level of probabilities (6.12). In particular, the two quantities are equivalent when the sets of input states $\{\omega_x\}$ and $\{\omega_y\}$ are tomographically-complete, meaning that they form a basis for their respective Hilbert spaces. It is also worth mentioning that the quantity defined in Def. 6.2 is not a particular case of the robustness defined for general convex resource theories of measurements [164, 213]. In particular, in Def. 6.2 the optimisation is over all measurements $\{N_{ab}^{\mathrm{AB}}\}$ and $\{O_{ab}^{\mathrm{AB}}\}$ which have a quantum realisation in the no-signalling scenario, whereas the quantifiers considered in [164] allow for arbitrary measurements (in particular also those which require communication). In other words, the above general approach is valid only for measurements performed in a single location, whereas here we are explicitly interested in a distributed, multipartite scenario. Hence our robustness measure is a genuinely different quantity than the generalised robustness of measurements studied in the above papers.

In Appendix A.1 we derived the dual formulation of the RoBN, which will be used to study its operational characterisation. Furthermore, we note that RoBN possesses three natural properties which one would expect from a reasonable measure of nonlocality, i.e:

(*i*) It is *faithful*, meaning that it vanishes if and only if the measurement is classical, i.e:

$$\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = 0 \iff \mathbb{M}^{\mathrm{AB}} \in \mathcal{F}_{\mathrm{BN}}. \tag{6.20}$$

(*ii*) It is *convex*, meaning that having access to two distributed measurements $\mathbb{M}_1^{\mathrm{AB}}$ and $\mathbb{M}_2^{\mathrm{AB}}$ one cannot obtain a better one by using them probabilistically, i.e for $\mathbb{M}^{\mathrm{AB}} = p\,\mathbb{M}_1 + (1-p)\,\mathbb{M}_2$ with $0 \le p \le 1$, we have:

$$\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) \le p\,\mathbf{R}_{\mathrm{BN}}(\mathbb{M}_1^{\mathrm{AB}}) + (1-p)\,\mathbf{R}_{\mathrm{BN}}(\mathbb{M}_2^{\mathrm{AB}}). \tag{6.21}$$

(*iii*) It is *monotonic* (non-increasing) under all quantum simulations. That is, if $\mathbb{N}^{AB}$ can be simulated by $\mathbb{M}^{AB}$ using some quantum simulation strategy (6.2) then

$$\mathbf{R}_{BN}(\mathbb{N}^{AB}) \leq \mathbf{R}_{BN}(\mathbb{M}^{AB}). \tag{6.22}$$

These properties were proven in [190] for a more general class of objects. For completeness, we given an independent proof in Appendix A.2.

Finally, we introduce a quantity which measures how much Buscemi nonlocality can be generated by using a fixed shared state. In this way we define the *robustness of Buscemi nonlocality* of a state ρ_{AB} as:

$$\mathbf{R}_{BN}(\rho_{AB}) := \max_{\mathbb{M}^{A},\, \mathbb{M}^{B}} \mathbf{R}_{BN}(\mathbb{M}^{AB}), \tag{6.23}$$

where the optimisation ranges over all local measurements on Alice's and Bob's side, $\mathbb{M}^{AB}$ is a distributed measurement of the form (6.1) and $\mathbf{R}_{BN}(\mathbb{M}^{AB})$ is the robustness quantifier defined in (6.19). In this way the quantity from Eq. (6.23) is only a function of the *shared state*, rather than the whole *distributed measurement*. It quantifies the maximal "amount" of nonlocality of the corresponding behaviour $\{p(a,b|\omega_x,\omega_y)\}$ that can be generated using a fixed $\rho^{A'B'}$, arbitrary local measurements $\mathbb{M}^{A}$, $\mathbb{M}^{B}$ and arbitrary input states $\{\omega_x\}$, $\{\omega_y\}$.

6.3 Main results

6.3.1 Operational characterisation of RoBN

In the previous subsection we introduced a measure of Buscemi nonlocality quantifying how "close" a given measurement is to that which would arise from using only local measurements and shared randomness, i.e. a measurement of the form (6.14). In what follows we will show that RoBN quantifies the advantage offered by a fixed distributed measurement over all classical measurements in a special type of a state discrimination task relevant in the distributed scenario.

Let us now consider a task which is a special case of the no-signalling game described in Sec. 6.2.1. In this case we choose the function $V(a,b,x,y) = \delta_{ax}\delta_{by}$. This means that Alice and Bob win if they both manage to guess the values of $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ which were supplied to them by the referee. This is a variation of the standard state discrimination task in which a single player has to guess the realisation of a single random variable x, as addressed in previous chapters. Interestingly, due to the assumption that the players cannot communicate, distributed state discrimination cannot be reduced to the standard state discrimination task.

Operational Task 4. *(Distributed state discrimination (DSD)) The task consists of the following steps:*

1. *The referee chooses a bipartite state from the ensemble $\{p(x,y), \sigma_{xy}\}$ according to $p(x,y)$ and distributes it among parties by sending one part of it to Alice and the other part to Bob.*

2. *After receiving their systems, Alice and Bob can preprocess them using arbitrary channels $\{\mathcal{E}^{A}_{\lambda}\}$ and $\{\mathcal{N}^{B}_{\lambda}\}$, potentially conditioned on a shared randomness λ.*

3. *Alice and Bob apply* fixed *local measurements* $\mathbb{M}^{\mathrm{AA}'} = \{M_i^{\mathrm{AA}'}\}$ *and* $\mathbb{M}^{\mathrm{B}'\mathrm{B}} = \{M_j^{\mathrm{AA}'}\}$ *to their shares of the state* σ_{xy} *and a part of the shared state* $\rho^{\mathrm{A}'\mathrm{B}'}$*. They obtain outcomes* i *and* j *respectively, which they can postprocess to produce their guesses* a *and* b*.*

4. *Alice and Bob communicate their guesses* a *and* b *to the referee and win the game if they* both *correctly guess, i.e. when* $a = x$ *and* $b = y$*.*

Notice that the second and the third step can be also formulated as allowing Alice and Bob apply any *quantum simulation* (6.3) to their distributed measurement $\mathbb{M}^{\mathrm{AB}} \in \mathcal{R}_{\mathrm{BN}}$. Hence the two players are effectively simulating a distributed measurement, denoted by $\mathbb{N}^{\mathrm{AB}} \prec \mathbb{M}^{\mathrm{AB}}$. The average probability of discriminating states in this discrimination game as specified by $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ can be expressed as:

$$p_{\text{guess}}^{\text{DSD}}(\mathcal{G}, \mathbb{M}^{\mathrm{AB}}) = \max_{\mathbb{N}^{\mathrm{AB}} \prec_q \mathbb{M}^{\mathrm{AB}}} \sum_{a,b,x,y} p(x,y)\, \mathrm{Tr}\left[N_{ab}\sigma_{xy}\right] \delta_{xa}\, \delta_{yb}, \tag{6.24}$$

where the optimization ranges over all measurements $\mathbb{N}^{\mathrm{AB}} = \{N_{ab}\}$ which can be quantum-simulated using $\mathbb{M}^{\mathrm{AB}}$.

Let us now consider two different situations: (*i*) a classical scenario in which the distributed measurement performed by Alice and Bob is classical, i.e. $\mathbb{M}^{\mathrm{AB}} \in \mathcal{F}_{\mathrm{BN}}$, and (*ii*) a quantum scenario in which the measurement performed by Alice and Bob is genuinely quantum, i.e it cannot be written as in (6.14).

In the classical case (*i*) the optimal average probability of guessing which state from the ensemble $\{p(x,y), \sigma_{xy}\}$ was provided can be expressed as:

$$p_{\text{guess}}^{\text{DSD}}(\mathcal{G}) = \max_{\mathbb{N}^{\mathrm{AB}} \in \mathcal{F}_{\mathrm{BN}}} p_{\text{guess}}^{\text{DSD}}(\mathcal{G}, \mathbb{N}^{\mathrm{AB}}), \tag{6.25}$$

Note that the above optimisation has to be performed over the convex set of measurements of the form (6.14), which is a subset of all separable measurements.

In the quantum case (*ii*) the above score can be further improved by exploiting Buscemi nonlocality contained in an entangled state which forms the distributed measurement $\mathbb{M}^{\mathrm{AB}}$. The maximal amount by which quantum score outperforms classical one can be quantified by studying the ratio:

$$\max_{\mathcal{G}} \frac{p_{\text{guess}}^{\text{DSD}}(\mathcal{G}, \mathbb{M}^{\mathrm{AB}})}{p_{\text{guess}}^{\text{DSD}}(\mathcal{G})}. \tag{6.26}$$

In Appendix A.3 we show that the maximal advantage which Alice and Bob can achieve when using $\mathbb{M}^{\mathrm{AB}} \in \mathcal{R}_{\mathrm{BN}}$ over the best classical distributed measurement is precisely equal to the robustness of Buscemi nonlocality defined in (6.19). Formally, we have the following relation:

Result 6.1. *Let* $\mathbb{M}^{\mathrm{AB}} = \{M_{ab}^{\mathrm{AB}}\}$ *be a distributed measurement and* $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ *be an ensemble of bipartite states. Then :*

$$\max_{\mathcal{G}} \frac{p_{\text{guess}}^{\text{DSD}}(\mathcal{G}, \mathbb{M}^{\mathrm{AB}})}{p_{\text{guess}}^{\text{DSD}}(\mathcal{G})} = 1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}). \tag{6.27}$$

This provides a direct operational meaning for Buscemi nonlocality. The proof of Result 6.1 consists of three parts. First we use the primal formulation of the problem (6.19) to show that the advantage from (6.27) is always upper-bounded by the

RoBN. Secondly, we identify a set of properties which characterize all distributed measurements and add them to the optimization problem (6.19) as superfluous constraints. Finally, using this characterization we obtain a dual formulation of the problem which, after some simplifications, allows us to extract the optimal ensemble of states $\{p(x,y), \sigma_{xy}\}$ which achieves the optimum in (6.27). The full proof of this result is in Appendix A.3.

The task of distributed state discrimination is a particular instance of a no-signalling game. In this respect we can further consider an advantage (6.26), with the average score $p_{\text{succ}}(\mathcal{G}, \mathbb{M}^{\text{AB}})$ given by (6.4), and optimize it over all ensembles $\mathcal{G}$ *and* scoring functions $V(a,b,x,y)$. This would allow us to find the largest possible advantage which can be achieved in any possible nonsignalling game. In this way Result 6.1 naturally leads to the following corollary:

Corollary 6.1. *Let $\mathbb{M}^{\text{AB}}$ and $\mathcal{G}$ be defined as above and let $V(a,b,x,y) : \mathcal{A} \times \mathcal{B} \times \mathcal{X} \times \mathcal{Y} \to [0,1]$. Then:*

$$\max_{V,\mathcal{G}} \frac{p^V_{guess}(\mathcal{G}, \mathbb{M}^{AB})}{\max_{\widetilde{\mathbb{N}}^{AB}} p^V_{guess}(\mathcal{G}, \widetilde{\mathbb{N}}^{AB})} = 1 + \mathbf{R}_{\text{BN}}(\mathbb{M}^{AB}), \tag{6.28}$$

where the maximisation in the denominator is over all distributed measurements $\mathbb{N}^{AB}$ that use a separable shared state $\sigma^{A'B'} \in \mathsf{SEP}$. In this way we can also interpret RoBN as a quantifier of the Buscemi nonlocality contained within a given distributed measurement.

6.3.2 Connecting Buscemi nonlocality with other notions of nonclassicality

In this section we show that Buscemi nonlocality can be viewed as a type of nonlocality which is strictly stronger than two other well-known notions of nonlocal correlations: entanglement [121] and nonclassical teleportation [51].

It is worth mentioning that the authors of [194] also studied the relationship between Buscemi nonlocality, nonclassical teleportation and entanglement by studying a partial order between objects representing these resources: distributed measurements for Buscemi nonlocality. teleportation instruments for nonclassical teleportation and bipartite states for entanglement. Here we address an analogous problem using a more direct approach: we relate robustness quantifiers of these resource theories and find a direct and simple relationship between them.

Recall that a distributed measurement is composed of two local bipartite measurements and a shared state. This setting is very similar to the teleportation protocol in which Alice locally measures an input state provided by the referee and a part of an entangled state which she shares with Bob. Since the resource used in the teleportation task is effectively "contained" in the resource which is used in the task of distributed state discrimination, it is natural to ask if we can see some connection between these two tasks. In particular, how is the ability of performing nonclassical teleportation related to the ability of demonstrating Buscemi nonlocality? Furthermore, since teleportation is intrinsically related with entanglement [51], also Buscemi nonlocality should be quantitatively related to the entanglement content of a state. In the next section we will show that in fact these three notions of nonclassical correlations are inherently connected and all describe different types of nonlocality.

Buscemi nonlocality and nonclassical teleportation

Quantum teleportation is one of the most important and thought-provoking discoveries in the whole quantum information theory. In the ideal version of the teleportation protocol proposed by Bennett et. al. in [26] two players, Alice and Bob, share a maximally entangled state. A third party, the referee, gives Alice an unknown quantum state. She then performs a Bell-state measurement on that system and her share of the entangled state and communicates her measurement result to Bob. With this new information Bob applies an appropriate correcting unitary to his share of the entangled state, transforming it into the state which was initially given to Alice. This protocol can be naturally generalised to more realistic scenarios in which the shared entangled state and measurements performed by Alice are arbitrary.

Teleportation experiment can be also viewed as a way of testing nonlocality of a pair of objects: a state and measurement. In particular, the "teleportation resource" in that case is the teleportation channel or, more precisely, a collection of subchannels which form a *teleportation instrument* constructed using the shared state and Alice's measurement. Recall that an instrument $\mathbb{E} = \{\mathcal{E}_a\}$ for $\{a = 1, \ldots, o_{\mathrm{A}}\}$ is a collection of o_{A} completely positive and trace non-increasing linear maps $\mathcal{E}_a$, so-called subchannels, such that $\sum_{a=1}^{o_{\mathrm{A}}} \mathcal{E}_a$ is a channel. It was recently shown that the nonlocality present in a teleportation instrument can be exploited in several quantum-information theoretic tasks [139]. In order to relate nonclassical teleportation with Buscemi nonlocality we first formally introduce the notion of a teleportation instrument.

Definition 6.3. *(Teleportation instrument) A teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$ from Alice to Bob is a collection of subchannels $\{\Lambda_a^{\mathrm{A}\to\mathrm{B}'}\}$ defined as:*

$$\Lambda_a^{\mathrm{A}\to\mathrm{B}'}[\omega^{\mathrm{A}}] = \mathrm{Tr}_{\mathrm{AA}'}\left[\left(M_a^{\mathrm{AA}'} \otimes \mathbb{1}^{\mathrm{B}'}\right)\left(\omega^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'}\right)\right]. \tag{6.29}$$

The above notion fully captures the type of channel obtained during the generalised teleportation experiment. For some applications it may be easier to work with states rather than subchannels. In that case for a collection of input states $\{\omega_x^{\mathrm{A}}\}$ one can consider the so-called teleportation assemblages (teleportages) $\{\tau_{a|x}^{\mathrm{B}'}\}$, where the elements of the assemblage are given by $\tau_{a|x}^{\mathrm{B}'} := \Lambda_a^{\mathrm{A}\to\mathrm{B}'}[\omega_x^{\mathrm{A}}]$.

Notice that any teleportation instrument satisfies its own 'no-signalling' constraint, which now reads: $\sum_i \Lambda_i^{\mathrm{A}\to\mathrm{B}'}[\omega^{\mathrm{A}}] = \mathrm{Tr}_{\mathrm{A}'}[\rho^{\mathrm{A}'\mathrm{B}'}]$ for all input states ω^{A}. In fact, it can also be shown that teleportation instruments are the most general type of no-signalling instruments acting between two parties [139]. A teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$ is said to be classical (or free) if it describes a teleportation experiment performed using a separable shared state. We can find a general form of a classical teleportation instrument by taking $\rho^{\mathrm{A}'\mathrm{B}'} = \sum_\lambda p_\lambda\, \rho_\lambda^{\mathrm{A}'} \otimes \rho_\lambda^{\mathrm{B}'}$. The associated (classical) teleportation instrument reads:

$$\begin{aligned}\Lambda_a^c(\omega_x) &= \sum_\lambda p_\lambda\, \mathrm{Tr}_{\mathrm{AA}'}\left[\left(M_a^{\mathrm{AA}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\left(\omega_x^{\mathrm{A}} \otimes \rho_\lambda^{\mathrm{A}'} \otimes \rho_\lambda^{\mathrm{B}'}\right)\right] \\ &= \sum_\lambda p_\lambda\, p(a|x,\lambda)\, \rho_\lambda^{\mathrm{B}'},\end{aligned} \tag{6.30}$$

where $p(a|x,\lambda) = \mathrm{Tr}[M_a^{\mathrm{AA}'}(\omega_x^{\mathrm{A}} \otimes \rho_\lambda^{\mathrm{A}'})]$. This is the most general classical teleportation scheme which can be realised if Alice and Bob have access only to classical randomness λ and the ability to locally prepare quantum states in their labs. In what follows we will denote the set of all instruments which can be written as in

(6.30) by $\mathcal{F}_{\mathrm{T}}$. If a teleportation instrument cannot be written in this way, we will refer to it as "nonclassical" and denote the set of all such instruments with $\mathcal{R}_{\mathrm{T}}$. The quantity which quantitatively measures the amount of nonclassicality associated with a given teleportation instrument is called Robustness of Teleportation (RoT) [51]. For a teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'} = \{\Lambda_a^{\mathrm{A}\to\mathrm{B}'}\}$ it is defined as:

$$\begin{aligned}
\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'}) = \min_{r,\{\Gamma_a^{\mathrm{A}\to\mathrm{B}'}\},\{\Omega_a^{\mathrm{A}\to\mathrm{B}'}\}} \quad & r \\
\text{s.t.} \quad & \Lambda_a^{\mathrm{A}\to\mathrm{B}'} + r\,\Omega_a^{\mathrm{A}\to\mathrm{B}'} = (1+r)\,\Gamma_a^{\mathrm{A}\to\mathrm{B}'} \;\forall a, \\
& \{\Gamma_a^{\mathrm{A}\to\mathrm{B}'}\} \in \mathcal{F}_{\mathrm{T}}, \quad \{\Omega_a^{\mathrm{A}\to\mathrm{B}'}\} \in \mathcal{R}_{\mathrm{T}}.
\end{aligned} \tag{6.31}$$

It turns out that the above is also a convex optimisation problem which can be seen by formulating the constraints using the Choi-Jamiołkowski isomorphism (see Appendix A.4 for details). With the above notation we can now address our next result which relates Buscemi nonlocality with nonclassical teleportation.

Result 6.2. *Let $\mathbb{M}^{\mathrm{AB}}$ be a distributed measurement composed of local bipartite measurements $\mathbb{M}^{\mathrm{A}}$ and $\mathbb{M}^{\mathrm{B}}$ and a shared state $\rho^{\mathrm{A}'\mathrm{B}'}$. Then:*

$$\max_{\mathbb{M}^{\mathrm{B}}} \quad \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'}), \tag{6.32}$$

where the optimisation is over all local measurements $\mathbb{M}^{\mathrm{B}} = \{M_b^{\mathrm{B}'\mathrm{B}}\}$ for Bob. An analogous result holds for a teleportation instrument $\Lambda^{\mathrm{B}\to\mathrm{A}'}$ if we instead optimise the LHS of Eq. (6.32) over all local measurements for Alice.

The proof of this result is in Appendix A.4. Let us now use this result to show a new operational interpretation of the above teleportation quantifier.

Consider a task involving two players, Alice and Bob, who have access to a teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$ connecting their labs. Let the referee be in possession of an ensemble of bipartite quantum states $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$. Just as before, the players may discuss on their strategy before the game begins. This means that they may use a shared classical memory λ with a corresponding distribution $p(\lambda)$ and conditioning on it Alice may apply one of the channels $\{\mathcal{E}_\lambda^{\mathrm{A}}\}$ to the input of the teleportation instrument and Bob may apply $\{\mathcal{N}_\lambda^{\mathrm{B}'}\}$ to the output. The crucial difference here between the standard teleportation protocol is that Bob *does not* know Alice's measurement outcome and so his correction cannot depend on it. The task posed between Alice and Bob is the following:

Operational Task 5. *(Teleportation-assisted state discrimination (TSD)) The task consists of the following steps:*

1. *The referee chooses a bipartite state from the ensemble $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ according to $p(x,y)$ and distributes it among parties by sending one part of it to Alice and the other part to Bob.*

2. *Alice sends her part of the state to Bob using a teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$. She is also allowed to pre-process her part of the state conditioned on the classical randomness λ using a collection of channels $\{\mathcal{N}_\lambda^{\mathrm{A}}\}$. Based on the outcome of the teleportation instrument i and potentially λ she produces a guess a via $p(a|i,\lambda)$.*

3. *Bob applies a correction $\{\mathcal{E}_\lambda^{\mathrm{B}'}\}$ conditioned on the value of a shared random variable λ to the teleported state he received from Alice. He then measures both parts of the system using an* arbitrary *measurement $\mathbb{M}^{\mathrm{B}} = \{M_b^{\mathrm{BB}'}\}$ and produces a guess b.*

4. *Alice and Bob win the game if they both simultaneously guess x and y.*

The average probability of guessing in the above discrimination task can be expressed as:

$$p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}, \Lambda^{\mathrm{A}\to\mathrm{B}'}) = \max_{\mathbb{M}^{\mathrm{B}}} \max_{\Phi \prec_q \Lambda} \sum_{a,b,x,y} p(x,y) \operatorname{Tr}\left[M_b^{\mathrm{B}'\mathrm{B}} \left(\Phi_a^{\mathrm{A}\to\mathrm{B}'} \otimes \mathrm{id}^{\mathrm{B}} \right) \sigma_{xy}^{\mathrm{AB}} \right] \delta_{xa}\delta_{yb} \quad (6.33)$$

where the optimization ranges over all measurements $\mathbb{M}^{\mathrm{B}} = \{M_b^{\mathrm{B}'\mathrm{B}}\}$ on Bob's side and all teleportation instruments $\Phi^{\mathrm{A}\to\mathrm{B}'} = \{\Phi_a^{\mathrm{A}\to\mathrm{B}'}\}$ which can be quantum-simulated using the instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'} = \{\Lambda_i^{\mathrm{A}\to\mathrm{B}'}\}$. The elements of such a simulated instrument are of the form:

$$\Phi_a^{\mathrm{A}\to\mathrm{B}'}(\cdot) = \sum_{i,\lambda} p(\lambda) p(a|i,\lambda) \circ \mathcal{N}_\lambda^{\mathrm{A}} \circ \Lambda_i^{\mathrm{A}\to\mathrm{B}'} \circ \mathcal{E}_\lambda^{\mathrm{B}'}(\cdot) \quad (6.34)$$

for some choice of local channels $\{\mathcal{E}_\lambda^{\mathrm{B}'}\}$, $\{\mathcal{N}_\lambda^{\mathrm{A}}\}$ and probabilities $p(a|i,\lambda)$ and $p(\lambda)$.

The optimal average probability of guessing that can be achieved using only classical resources (i.e. a separable shared state, meaning that the teleportation instrument is classical) can be written as:

$$p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}) = \max_{\mathbb{F}^{\mathrm{A}\to\mathrm{B}'} \in \mathcal{F}_{\mathrm{T}}} p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{F}^{\mathrm{A}\to\mathrm{B}'}), \quad (6.35)$$

where $\mathbb{F}^{\mathrm{A}\to\mathrm{B}'}$ stands for a classical teleportation instrument from Alice to Bob. The maximal advantage which can be offered by any resourceful teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$ in the task of TSD is precisely equal to the quantifier of nonclassical teleportation defined in (6.31). This is captured by the following result:

Result 6.3. *Let $\Lambda^{\mathrm{A}\to\mathrm{B}'} = \{\Lambda_a^{\mathrm{A}\to\mathrm{B}'}\}$ be a teleportation instrument from Alice to Bob and let $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ be an ensemble of bipartite states. Then the following holds:*

$$\max_{\mathcal{G}} \frac{p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}, {}^{\mathrm{A}\to\mathrm{B}})}{p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G})} = 1 + \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}}). \quad (6.36)$$

Proof. Consider maximizing both sides of Eq. (6.27) over all measurements $\mathbb{M}^{\mathrm{B}}$ on Bob's side. Due to Result 6.2, the right-hand side of Eq. (6.27) is equal to $1 + \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$. On the other hand, notice that we can interchange maximisation over $\mathcal{G}$ with maximisation over $\mathbb{M}^{\mathrm{B}}$. Since $p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G})$ does not depend on $\mathbb{M}^{\mathrm{B}}$, the left-hand side of Eq. (6.27) becomes:

$$\max_{\mathcal{G}} \frac{\max_{\mathbb{M}^{\mathrm{B}}} p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{M}^{\mathrm{AB}})}{p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G})} = \max_{\mathcal{G}} \frac{p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}, \Lambda^{\mathrm{A}\to\mathrm{B}'})}{p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G})} \quad (6.37)$$

$$= \max_{\mathcal{G}} \frac{p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G}, \Lambda^{\mathrm{A}\to\mathrm{B}'})}{p^{\mathrm{TSD}}_{\mathrm{guess}}(\mathcal{G})}, \quad (6.38)$$

where the last equality follows since:

$$p_{\text{guess}}^{\text{DSD}}(\mathcal{G}) = \max_{\mathbb{F}^{AB}\in\mathcal{F}_{BN}} p_{\text{guess}}^{\text{DSD}}(\mathcal{G},\mathbb{F}^{AB}) \tag{6.39}$$

$$= \max_{\mathbb{F}^{A\to B'}\in\mathcal{F}_{T}} \max_{\mathbb{M}^{B}} p_{\text{guess}}^{\text{TSD}}(\mathcal{G},\mathbb{F}^{A\to B'}) \tag{6.40}$$

$$= p_{\text{guess}}^{\text{TSD}}(\mathcal{G}). \tag{6.41}$$

This completes the proof. □

Buscemi nonlocality and entanglement

Let us now explore the link between Buscemi nonlocality, which we defined as a property of a bipartite state and local measurements, and entanglement (a property of the state only). Among the large variety of known entanglement quantifiers [177, 121, 24, 168, 231, 183, 196, 106], we are going to choose the one which most naturally relates to the RoBN — the so-called generalised Robustness of Entanglement (RoE), denoted here with $\mathbf{R}_{\text{E}}(\rho)$. This entanglement quantifier was considered for the first time in [235] and generalised in [209] and since then proved to be useful in several different contexts, e.g. in proving that all entangled states can demonstrate nonclassical teleportation [50], in exploring the connection between entanglement and permutation symmetry [184] or in studying the effects of local decoherence on multi-party entanglement [200]. This quantifier also has two interesting operational interpretations: it quantifies the maximal advantage that can be achieved in a bipartite subchannel discrimination task [218] and the maximal advantage in the task of local subchannel discrimination with a quantum memory [139]. It is defined in terms of the following convex optimisation problem:

$$\begin{aligned}
\mathbf{R}_{\text{E}}(\rho^{AB}) = \min_{r,\eta^{AB},\sigma^{AB}} \quad & r \\
\text{s.t.} \quad & \rho^{AB} + r\,\eta^{AB} = (1+r)\sigma^{AB} \\
& \eta^{AB} \geq 0, \quad \operatorname{Tr}\eta^{AB} = 1 \\
& \sigma^{AB} \in \mathsf{SEP}, \quad \operatorname{Tr}\sigma^{AB} = 1.
\end{aligned} \tag{6.42}$$

Using this definition we can now address our next result which relates Buscemi nonlocality with entanglement.

Result 6.4. *Let $\mathbb{M}^{AB}$ be a distributed measurement composed of local measurements $\mathbb{M}^{A}$ and $\mathbb{M}^{B}$ and a shared state $\rho^{A'B'}$. Then:*

$$\max_{\mathbb{M}^{A},\mathbb{M}^{B}} \mathbf{R}_{\text{BN}}(\mathbb{M}^{AB}) = \mathbf{R}_{\text{E}}(\rho^{A'B'}), \tag{6.43}$$

where the optimization is over all local measurements for Alice $\mathbb{M}^{A} = \{M_a^{AA'}\}$ and for Bob $\mathbb{M}^{B} = \{M_b^{B'B}\}$.

The proof of this result is in Appendix A.5. Notice that the above relationship allows us to directly infer that the maximal amount of Buscemi nonlocality that can ever be generated using a given state, defined in (6.23), is precisely equal to its entanglement content. Therefore we may write:

$$\mathbf{R}_{\text{BN}}(\rho_{AB}) = \mathbf{R}_{\text{E}}(\rho^{AB}). \tag{6.44}$$

The relationship (6.43) along with Result 6.1 also allows to find a new operational interpretation of the RoE. Consider again the task of DSD with the relaxation that Alice and Bob may now apply arbitrary local measurements in their labs. The goal for Alice and Bob remains the same: to guess which state from the ensemble $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ was prepared, under the assumption that no communication is allowed. In this way the task posed between Alice and Bob is the following:

Operational Task 6. *(Entanglement-assisted state discrimination (ESD)) The task consists of the following steps:*

1. *The referee chooses a bipartite state from the ensemble $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ according to $p(x,y)$ and distributes it among parties by sending one part of it to Alice and the other part to Bob.*
2. *Alice and Bob apply* arbitrary *local measurements $\mathbb{M}^{\mathrm{A}}$ and $\mathbb{M}^{\mathrm{B}}$ to the states they received and their part of the shared state $\rho^{\mathrm{A'B'}}$ and receive outcomes a and b, respectively.*
3. *Alice and Bob win the game if they both guess which state was provided, i.e. guess both x and y.*

The average probability of guessing in this task can be expressed as:

$$p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}, \rho^{\mathrm{A'B'}}) = \max_{\mathbb{M}^{\mathrm{A}}, \mathbb{M}^{\mathrm{B}}} \sum_{a,b,x,y} p(x,y) \operatorname{Tr}\left[M^{\mathrm{AB}}_{ab} \sigma^{\mathrm{AB}}_{xy}\right] \delta_{xa}\delta_{yb}, \tag{6.45}$$

where the optimization ranges over all measurements $\mathbb{M}^{\mathrm{A}} = \{M^{\mathrm{AA'}}_a\}$ on Alice's and $\mathbb{M}^{\mathrm{B}} = \{M^{\mathrm{B'B}}_b\}$ on Bob's side with measurement M^{AB}_{ab} of the form (6.1).

The best average probability of guessing in the classical scenario (i.e. when the shared state is separable) is given by:

$$\begin{aligned} p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}) &= \max_{\sigma^{\mathrm{A'B'}} \in \mathsf{SEP}} p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}, \sigma^{\mathrm{A'B'}}) \\ &= \max_{\mathbb{N}^{\mathrm{AB}} \in \mathcal{F}_{\mathrm{BN}}} p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{N}^{\mathrm{AB}}) \\ &= p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}). \end{aligned} \tag{6.46}$$

The maximal advantage which can be offered by an entangled state $\rho^{\mathrm{A'B'}}$ in the ESD task can be quantified using the RoE. This is the content of our next result:

Result 6.5. *Let $\rho^{\mathrm{A'B'}}$ be a bipartite state shared between Alice and Bob and let $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ be an ensemble of bipartite states. Then the following holds:*

$$\max_{\mathcal{G}} \frac{p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}, \rho^{\mathrm{A'B'}})}{p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G})} = 1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A'B'}}). \tag{6.47}$$

Proof. The proof of this result proceeds similarly to the case of nonclassical teleportation. Let us maximise both sides of (6.27) over all measurements on Alice's and Bob's side, i.e. over all $\mathbb{M}^{\mathrm{A}}$ and $\mathbb{M}^{\mathrm{B}}$. Due to Result Result 6.4, the right-hand side of (6.27) is equal to $1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A'B'}})$. On the other hand, due to (6.46) we can write the left-hand side of (6.27) as:

$$\max_{\mathcal{G}} \frac{\max_{\mathbb{M}^{\mathrm{A}}, \mathbb{M}^{\mathrm{B}}} p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{M}^{\mathrm{AB}})}{p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G})} = \max_{\mathcal{G}} \frac{p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G}, \rho^{\mathrm{A'B'}})}{p^{\mathrm{ESD}}_{\mathrm{guess}}(\mathcal{G})}. \tag{6.48}$$

This completes the proof. □

Finally, let us note that entanglement-assisted state discrimination is a particular instance of a no-signalling game in which we fix $V(a,b,x,y) = \delta_{xa}\delta_{by}$ and allow for optimising over local measurements. This exactly corresponds to the average score studied in Ref. [45]. Using this realisation we can now consider the maximal advantage in the task of entanglement-assisted state discrimination (6.47) and optimise it not only over ensembles $\mathcal{G}$, but also over all predicates $V(a,b,x,y)$, in a manner exactly similar as in the case of Corollary 6.1. This therefore yields the largest possible advantage that can be achieved in *any* no-signalling game. In this way Result 6.5 naturally leads to the following corollary:

Corollary 6.2. *Let $\mathbb{M}^{\mathrm{AB}}$ and $\mathcal{G}$ be defined as before and let $V(a,b,x,y) : \mathcal{A}\times\mathcal{B}\times\mathcal{X}\times\mathcal{Y}\to[0,1]$. Then:*

$$\max_{V,\mathcal{G}} \frac{p^{V}_{guess}(\mathcal{G},\mathbb{M}^{AB})}{\max_{\sigma\in\mathsf{SEP}}\max_{\mathbb{N}^{A},\mathbb{N}^{B}} p^{V}_{guess}(\mathcal{G},\mathbb{N}^{AB})} = 1+\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{AB}), \tag{6.49}$$

where $\mathbb{N}^{\mathrm{AB}} = \{N^{\mathrm{AB}}_{ab}\}$ with the POVM elements defined as

$$N_{ab} := \mathrm{Tr}_{\mathrm{A'B'}}\left[\left(N_a^{\mathrm{AA'}}\otimes N_b^{\mathrm{B'B}}\right)\left(\mathbb{1}^{\mathrm{A}}\otimes\sigma^{\mathrm{A'B'}}\otimes\mathbb{1}^{\mathrm{B}}\right)\right]. \tag{6.50}$$

In this way we can now interpret RoE as a quantifier of the Buscemi nonlocality contained within a given *state*. This not only re-derives the main result of Ref. [45], but also makes it significantly stronger; the RoE can now be seen as the quantifier of the maximal advantage in any no-signalling game, therefore providing a completely new interpretation for this well-known entanglement quantifier.

Complete sets of monotones for quantum simulation

We finish this section by showing that the average guessing probability in the task of DSD completely describes the preorder induced by quantum simulation on distributed measurements $\mathbb{M}^{\mathrm{AB}}$. Formally this means that the average guessing probability $p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G},\mathbb{M}^{\mathrm{AB}})$ when viewed as a function of $\mathcal{G}$ forming a complete set of monotones for quantum simulation of $\mathbb{M}^{\mathrm{AB}}$. This is captured by the following result:

Result 6.6. *Any distributed measurement $\mathbb{M}^{\mathrm{AB}}$ can quantum-simulate another measurement $\mathbb{N}^{\mathrm{AB}}$ if and only if for all ensembles $\mathcal{G} = \{p(x,y),\sigma_{xy}\}$ the following holds:*

$$p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G},\mathbb{M}^{\mathrm{AB}}) \geq p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G},\mathbb{N}^{\mathrm{AB}}). \tag{6.51}$$

In other words, quantum simulation (or LOSR) can never improve the discrimination ability of any distributed measurement. The proof of this result is in Appendix A.6.

6.3.3 RoBN as a quantifier in single-shot information theory

We now address another way of interpreting RoBN from the point of view of single-shot quantum information theory. In particular, in Appendix A.7 we show that RoBN also quantifies the entanglement-assisted min-accessible information of a quan tum-to-classical bipartite channel (i.e. a channel with quantum inputs and classical outputs). This connection parallels analogous results from the previous chapters which correspond to single party quantum-to-classical channels [205, 213].

We start by noticing that any distributed measurement $\mathbb{M}^{\mathrm{AB}}$ can be seen as an entanglement-assisted quantum-to-classical channel:

$$\mathcal{N}^{\mathrm{AB}\to\mathrm{XY}}[\omega^{\mathrm{A}} \otimes \omega^{\mathrm{B}}] = \sum_{a,b} p(a,b|\omega_x,\omega_y)\, |a\rangle\langle a|^{\mathrm{X}} \otimes |b\rangle\langle b|^{\mathrm{Y}}, \tag{6.52}$$

with $p(a,b|\omega_x,\omega_y)$ as in (6.12). In quantum information theory the standard quantifier of the maximal amount of classical information that can be reliably sent through a quantum channel is the accessible information which is defined for an arbitrary quantum channel $\mathcal{R}$ as:

$$I^{\mathrm{acc}}(\mathcal{R}) = \max_{\mathcal{E},\mathcal{D}} I(X:G), \tag{6.53}$$

where $\mathcal{E} = \{p(x),\sigma_x\}$ is an ensemble of states which encode classical random variable X distributed according to $p(x)$, $\mathcal{D} = \{D_g\}$ is the decoding POVM which produces an outcome g with probability $p(g|x) := \mathrm{Tr}[D_g \cdot \mathcal{R}[\sigma_x]]$ and $I(X;G) = H(X) - H(X|G)$ is the mutual information of the distribution $p(x,g) := p(x)p(g|x)$. In the single-shot case a more relevant quantity is the *min-accessible information* $I^{\mathrm{acc}}_{\mathrm{min}}(\mathcal{R})$ which is defined as [59]:

$$I^{\mathrm{acc}}_{+\infty}(\mathcal{R}) = \max_{\mathcal{E},\mathcal{D}} \left[H_{+\infty}(X) - H_{+\infty}(X|G)\right], \tag{6.54}$$

where the optimization ranges over the same encodings and decodings as before and single-shot entropies are given by [187]:

$$H_{+\infty}(X) = -\log \max_x p(x), \tag{6.55}$$

$$H_{+\infty}(X|G) = -\log\left[\sum_g \max_x p(x,g)\right], \tag{6.56}$$

Let us now consider an encoding of a bipartite random variable $X \times Y$, i.e $\mathcal{E} = \{p(x,y),\sigma_{xy}\}$ and the associated decoding $\mathcal{D} = \{D_g\}$ for $g = 1,\ldots,|X|\cdot|Y|$. In Appendix A.7 we show that for this particular setting RoBN quantifies the min-accessible information of the channel $\mathcal{N}^{\mathrm{AB}\to\mathrm{XY}}$. Formally, we have the following result:

Result 6.7. *Let $\mathcal{N}^{\mathrm{AB}\to\mathrm{XY}}$ be a quantum-to-classical channel of the form (6.52). Then the following holds:*

$$I^{\mathrm{acc}}_{+\infty}(\mathcal{N}^{\mathrm{AB}\to\mathrm{XY}}) = \log[1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}})] \tag{6.57}$$

The proof of this result is in Appendix A.7. The above result provides an alternative way of interpreting RoBN as the maximal amount of min-mutual information that can be obtained between the input and output of the channel (6.52) when using it only once.

6.4 Conclusions

In this chapter we have studied the notion of Buscemi nonlocality when it is formalised as a quantum resource theory of distributed measurements. This formulation allowed us to establish a direct operational interpretation of Buscemi nonlocality in terms of a practical information-theoretic task called distributed state discrimination (Result 6.1). We have shown that the average guessing probability in this task provides a complete set of monotones for the partial order of distributed measurements induced by quantum simulation (Result 6.6). This also gives rise to a simple and *complete* family of "Buscemi inequalities" which quantify nonlocal properties of distributed measurements.

This operational link was derived using a geometric quantity measuring the strength of nonlocal correlations generated using a given distributed measurement (RoBN). By connecting this quantifier with other measures of nonlocality we inferred a quantitative relationship between distributed measurements, nonclassical teleportation and quantum entanglement, a realisation which we believe to be of an independent interest. In particular, we have shown that the robustness of Buscemi nonlocality optimised over all local measurements for one party is equal to the robustness of nonclassical teleportation (Result 6.2). Similarly, optimising RoBN over local measurements for both parties gives the robustness of entanglement (Result 6.4). This naturally leads to new operational interpretations for both of these quantifiers, in terms of appropriately tailored state discrimination tasks of: teleportation-assisted state discrimination (Result 6.3), and entanglement-assisted state discrimination (Result 6.5).

We have also shown that the maximal amount of nonlocality that can ever be generated using a fixed bipartite state, is directly proportional to its entanglement content. The entanglement content in this case is characterised by the robustness of entanglement, a widely-known entanglement quantifier with direct operational significance. Importantly, this not only re-derives the main result of Ref. [45], but also makes it significantly stronger; the generalised robustness of entanglement can now be seen as the quantifier of the maximal advantage in any no-signalling game (Corollary 6.1 and Corollary 6.2)

As our last result we have interpreted Buscemi nonlocality from the perspective of single-shot quantum information theory (Result 6.7). In particular, we have shown that Buscemi nonlocality, when viewed as a property of a communication channel between the sender (the Referee) and receiver (Alice and Bob), quantifies the maximal amount of information that can be sent reliably when the channel is used only once (the so-called single-shot capacity of a quantum channel). We have shown that the RoBN can be viewed as the maximal single-shot capacity offered by a bipartite quantum-to-classical channel. This establishes an important link between Buscemi nonlocality and the single-shot theory of quantum communication.

Finally, we emphasise that while we focused exclusively on quantifying Buscemi nonlocality using a robustness-based measure, our results can be easily extended to address the so-called *weight-based* resource quantifiers [80, 137]. These geometric measures find their operational meaning in the so-called exclusion tasks [73, 224], as we explored in the previous chapters. Consequently, the resource quantifiers of: weight of Buscemi nonlocality, weight of nonclassical teleportation, and the weight of entanglement, are quantifiers characterising: distributed state exclusion (DSE), teleportation-assisted state exclusion (TSE), and entanglement-assisted state exclusion (ESE), respectively.

We believe that the results presented in this chapter will shed new light on the complex structure of different types of nonclassical effects observed in Nature, as well as on their practical relevance for physically-motivated tasks.

This chapter also provides an example of a multiobject quantum resource theory which cannot be reduced to a theory of either measurements, states, channels, or state-measurement pairs [72]. This also means that the composite objects we study here constitute genuine multiobject quantum resources. It is an interesting open question to see if one can find additional examples of multiobject resource theories which address such irreducible resources. This is in sharp contrast to a recently introduced multiobject resource theory of state-measurement pairs, where the resources independently contribute to the benefit of the operational task of discrimination and exclusion of subchannels [72].

One of the standard questions addressed by quantum resource theories is determining when and at what rate a large number of copies of one resource can be converted into another. The fact that multiobject QRTs cannot be seen as resource theories of constituent objects leads a natural question of whether this can be used to improve the existing asymptotic protocols. For example, in the resource theory of nonclassical teleportation one can ask whether n uses of teleportation instrument can lead to a better teleportation than using n copies of the shared state. Similarly we can ask whether access to n uses of a distributed measurement can be in advantageous over using bipartite measurements and n copies of the shared state.

Chapter 7

Characterisation of quantum betting tasks in terms of Arimoto mutual information

"If you are receptive and humble, mathematics will lead you by the hand. Again and again, when I have been at a loss how to proceed, I have just had to wait until I have felt the mathematics led me by the hand. It has led me along an unexpected path, a path where new vistas open up, a path leading to new territory, where one can set up a base of operations, from which one can survey the surroundings and plan future progress. "

Paul A. M. Dirac

This chapter is the spiritual sequel of chapter 3 on the weight of informativeness. One of the main questions that were left open there, was about the apparently minor remark that, given the two triangular correspondences (between operational tasks, information-theoretic quantities, and resource quantifiers), one at $+\infty$ and the other at $-\infty$, one can naturally speculate and hope for something reasonable to populate the in-between. This was the starting point of the research contained in this chapter. It turns out that there are indeed reasonable things in-between those two extreme cases, and the main goal of this chapter is to provide such construction. Surprisingly, the answer to our problems came by importing ideas from the economic sciences, specifically the concepts of *betting* and *risk-aversion*, which allowed us to introduce a family of new operational tasks which we have coined as *quantum betting tasks*.

In this chapter, we introduce operational quantum tasks based on betting with risk-aversion – or quantum betting tasks for short – inspired by standard quantum state discrimination and classical horse betting with risk-aversion and side information. In particular, we introduce the operational tasks of quantum state betting (QSB), noisy quantum state betting (nQSB), and quantum channel betting (QCB) played by gamblers with different risk tendencies. We prove that the advantage that informative measurements (non-constant channels) provide in QSB (nQSB) is exactly characterised by Arimoto's α-mutual information, with the order α determining the risk aversion of the gambler. More generally, we show that Arimoto-type information-theoretic quantities characterise the advantage that resourceful objects offer at playing quantum betting tasks when compared to resourceless objects, for general quantum resource theories (QRTs) of measurements, channels, states, and state-measurement pairs, with arbitrary resources. In limiting cases, we show that QSB (QCB) recovers the known tasks of quantum state (channel) discrimination when $\alpha \to \infty$, and quantum state (channel) exclusion when $\alpha \to -\infty$. Inspired by these connections, we also introduce new quantum Rényi divergences for measurements, and derive a new family of resource monotones for the QRT of measurement informativeness. This family of resource monotones recovers in the same limiting cases as above, the generalised robustness and the weight of informativeness. Altogether, these results establish a broad and continuous family of four-way correspondences between operational tasks, mutual information measures, quantum Rényi divergences, and resource monotones, that can be seen to generalise two limiting correspondences that were recently discovered for the QRT of measurement informativeness.

7.1 Introduction and motivation

The field of quantum information theory (QIT) was born out of the union of the theory of quantum mechanics and the classical theory of information [161]. This union also happened to kickstart what it is nowadays known as the (ongoing) second quantum revolution which, roughly speaking, aims at the development of quantum technologies [70, 176]. Compared with its direct predecessors however, QIT is still a relatively young field and therefore, it is important to keep unveiling, exploiting, and strengthening the links between quantum theory and classical information theory.

In this direction, the framework of quantum resource theories (QRTs) has emerged as a fruitful approach to quantum theory [119, 57]. A central subject of study within QRTs is that of *resource quantifiers* [119, 57]. Two well-known families of these measures are the so-called *robustness-based* [234, 208, 49, 172, 174, 155, 204, 55, 138, 122, 140, 185, 81] and *weight-based* [79, 136, 206, 44] resource quantifiers. Importantly, these quantities have been shown to be linked to operational tasks and therefore, this establishes a type of quantifier-task correspondence. Explicitly, robustness-based quantifiers are linked to discrimination-based operational tasks [172, 217, 203, 174, 204, 212, 214], whilst weight-based resource quantifiers are linked to exclusion-based operational tasks [73, 225]. A resource quantifier is a particular case of a more general quantity known as a *resource monotone* [93] and therefore, this correspondence can alternatively be addressed as a *monotone-task* correspondence.

From a different direction, in classical information theory, the *Kullback-Leibler (KL) divergence* (also known as the Kullback-Leibler relative entropy) emerges as a central object of study [131]. The importance of this quantity is in part due to the fact that it acts as a parent quantity for many other quantities, such as the Shannon entropy, conditional entropy, conditional divergence, mutual information, and the channel capacity [63]. Within this classical framework, it has also proven fruitful to consider *Rényi-extensions* of these quantities [192]. In particular, there is a clear procedure for how to define the Rényi-extensions of both Shannon entropy and KL-divergence, which are known as the Rényi entropy and the Rényi divergence, respectively [192, 229]. Interestingly however, there is yet no consensus within the community as to what is the “proper" way to Rényi-extend other quantities. As a consequence of this, there are several different candidates for Rényi conditional entropies [84], Rényi conditional divergences [33], and Rényi mutual information measures [232]. The latter quantities are also known as measures of dependence [33] or α-mutual information measures [232], and we address them here as (Rényi) *dependence* measures or *mutual information measures*. In particular, we highlight the mutual information measures proposed by Sibson [199], Arimoto [8], Csiszár [64], as well as one recent proposal, independently derived by Lapidoth-Pfister [133], and Tomamichel-Hayashi [223]. It is known that these mutual information measures (with the exception of Arimoto's) can be derived from their respective conditional Rényi divergence [33] and therefore, we address this relationship as a *mutual information-divergence* correspondence.

The links between the two worlds of QRTs and classical information theory are now beginning to be understood to run much deeper than just the *monotone-task* and *mutual information-divergence* correspondences from above. In fact, they are intimately connected via a more general *four-way* monotone-task-mutual information-divergence correspondence, which holds true in particular for the QRT of measurement informativeness (a QRT where the resource is a measurement's ability to extract information encoded in a state) [204]. Explicitly, the robustness-discrimination correspondence [204, 212] is furthermore connected to the information-theoretic quantity known as the *accessible information* [242] which can, in turn, be written in terms of mutual information measures. In a similar manner the weight-exclusion correspondence [73, 225] is linked to the *excludible information* [73, 71], which can also be written in terms of mutual information measures. Even though it was not explicitly stated in any of the above references the fourth corner in terms of "Rényi divergences", it is nowadays a well known fact within the community, first noted by Datta, that the generalised robustness is related to the Rényi divergence of order ∞ (also called the max quantum divergence) [65], with a similar case happening for the weight and the divergence of order $-\infty$ [73]. These two apparently "minor" remarks raise the following fascinating question: Could there exist a whole spectrum of connections between mutual information measures, Rényi divergences, resource monotones, and operational tasks, with only the two extreme ends at $\pm\infty$ currently being uncovered? [73].

In this chapter we start by providing a positive answer to this question, by implementing insights from the theory of games and economic behaviour [157]. This latter theory, in short, encompasses many of the theoretical tools currently used in the economic sciences. In particular, we invoke here the so-called *expected utility theory* [157] and more specifically, we borrow the concept of *risk-aversion*; the behavioural tendency of rational agents to have a preference one way or another for guaranteed outcomes versus uncertain outcomes. This concept remains of great research interest in the economic sciences, with various Nobel prices having been awarded to its understanding [14].

In general, the concept of *risk aversion* is a ubiquitous characteristic of rational agents and, as such, it naturally emerges as a subject of study in various different areas of knowledge such as: the economic sciences [76], biology and behavioral ecology [188, 254], and neuroscience [129, 83, 221]. In short, it addresses the behavioural tendencies of rational agents when faced with uncertain events. Intuitively, a gambler spending money on bets with the hope of winning big, can be seen as an individual taking (potentially unnecessary) risks, in the eyes of a more conservative gambler. One of the challenges that economists have tackled, since roughly the second half of the previous century, is the incorporation of the concept of risk aversion into theoretical models describing the behaviour of rational agents, as well as its quantification, and exploitation of its descriptive power [76].

The concept risk was first addressed within theoretical models by Bernoulli in 1738 (translated into English by Sommer in 1954) [28]. Later on, the theory of expected utility, formalised by von Neumann and Morgenstern in 1944 [157], provided a framework within which to address and incorporate behavioural tendencies like risk aversion. It was then further formalised, independently and within the theory of expected utility, by Arrow, Pratt, and Finetti in the 1950's and 60's [9, 179, 86] who, in particular, introduced measures for its quantification. The quest for further understanding and exploiting this concept has since remained of active research interest in the economic sciences [76]. Recently, an important step was taken in the work of Bleuler, Lapidoth and Pfister (BLP) in 2020 [33], where the concept of risk aversion was implemented within the realm of classical information theory, as part of the operational tasks of horse betting games with risk and side information.

In this chapter, inspired by the concepts of *betting*, *risk aversion*, the tasks introduced by BLP [33], as well as by standard quantum state discrimination, we introduce operational *quantum betting tasks*. Surprisingly, we find that these tasks turn out to provide the correct approach for solving the conundrum regarding the four-way correspondence for QRTs described above. Specifically, we find that the concept of risk aversion allows us to define operational quantum tasks which can be viewed as a generalisation of discrimination and exclusion.

We start by exploring the QRT of measurement informativeness, and find that Arimoto's α-mutual information exactly quantifies the advantage provided by informative measurements when playing one of these quantum betting tasks which we call quantum state betting (QSB). We then explore general QRTs of measurements with arbitrary resources, and similarly derive Arimoto-type information-theoretic measures which quantify the advantage provided by resourceful measurements. Specifically, we find that the concept of *Arimoto's gap*, an information-theoretic quantity which generalises Arimoto's mutual information, characterises QSB games when comparing a resourceful gambler with gamblers with access only to free resources.

In addition to QRTs of measurements, we also explore QRTs of other objects. First, we explore the QRT of non-constant channels. In this scenario we introduce the tasks of noisy quantum state betting (nQSB), and find appropriate Arimoto-type quantities which characterise the performance gain of resourceful objects over resourceless objects in these tasks. Furthermore, we extend these results to QRTs of channels with arbitrary resources, and similarly characterise the advantage provided by resourceful channels in comparison to the best resourceless alternatives.

We also explore the concept of *betting* and *risk-aversion* for tasks beyond QSB and nQSB games, by introducing quantum channel betting (QCB) tasks. We first address these tasks for general single-object QRTs of states with arbitrary resources. In this regime we find that, similarly to the case of QSB and nQSB, there exist Arimoto-type information-theoretic quantities which characterise the performance of resourceful gamblers over resourceless gamblers. We further extend these results to multi-object QRTs of state-measurement pairs. These results therefore altogether highlight that *betting* and *risk-aversion* are powerful and useful concepts that naturally emerge in general QRTs with arbitrary resources, objects, as well as different tasks.

Finally, we report additional results for the QRT of measurement informativeness, by deriving a continuous four-way correspondence between operational tasks, mutual information measures, Rényi divergences, and resource monotones, which generalise correspondences recently found in the literature [204, 73].

We believe that the concepts of *betting* and *risk-aversion* have the potential to positively impact our understanding of the framework of resource theories as well as our understanding of the theory of quantum information more generally.

This chapter is organised as follows. In Sec. 7.1.1 we describe the QRT of measurement informativeness and the QRT of non-constant channels. In Sec. 7.1.2 we address further Arimoto-type information-theoretic quantities for general QRTs of measurements, channels, states, and state-measurement pairs with arbitrary resources. Our main results sections start in Sec. 7.2, where we introduce operational quantum tasks based on betting with risk-aversion, or quantum betting tasks for short, and introduce various tasks as follows: quantum state betting (QSB) in Sec. 7.2.1, 7.2.2, 7.2.3, noisy quantum state betting (nQSB) in Sec. 7.2.4, and quantum channel betting (QCB) in Sec. 7.2.5. In Sec. 7.3 we address the characterisation of quantum betting tasks in terms of Arimoto-type information-theoretic quantities. In Sec. 7.3.1 we relate QSB games to Arimoto's mutual information, for the QRT of measurement informativeness. In Sec. 7.3.2 we characterise noisy QSB (nQSB) games in terms of a noisy Arimoto mutual information, for the QRT of non-constant channels. In Sec. 7.3.3 we characterise QSB and nQSB games in terms of Arimoto-type quantities, for general QRTs of measurements and channels with general resources. In Sec. 7.3.4 we characterise QCB games in terms of Arimoto-type measures for single-object QRTs of states with arbitrary resources as well as multi-object QRTs of state-measurement pairs with arbitrary resources. In Sec. 7.3.5 we characterise horse betting games in terms of the Arimoto's mutual information in the classical regime, without invoking quantum theory. In Sec. 7.3.6 and 7.3.7 we address quantum Rényi divergences and resource monotones, and derive a four-way correspondence for the QRT of measurement informativeness. We finish in Sec. 7.4 with conclusions, open questions, perspectives, and avenues for future research.

7.1.1 The quantum resource theories of measurement informativeness and non-constant channels

The framework of quantum resource theories (QRTs) has proven a fruitful approach towards quantum theory [119, 57]. In this chapter we particularly deal with convex QRTs of measurements, channels. We start with the QRT of measurement informativeness [204].

Definition 7.1. *(QRT of measurement informativeness [204]) Consider the set of Positive-Operator Valued Measures (POVMs) acting on a Hilbert space of dimension d. A POVM* $\mathbb{M}$ *is a collection of POVM elements* $\mathbb{M} = \{M_a\}$ *with* $a \in \{1, ..., o\}$ *satisfying* $M_a \geq 0\ \forall a$ *and* $\sum_a M_a = \mathbb{1}$. *We now consider the resource of informativeness [204]. We say a measurement* $\mathbb{N}$ *is uninformative when there exists a PMF* q_A *such that* $N_a = q(a)\mathbb{1}$, $\forall a$. *We say that the measurement is informative otherwise, and denote the set of all uninformative measurements as* UI.

The set of uninformative measurements forms a convex set and therefore, defines a convex QRT of measurements. We now introduce the notion of simulability of measurements, which is also called classical post-processing (CPP).

Definition 7.2. *(Simulability of measurements [100, 204]) A measurement* $\mathbb{N} = \{N_x\}$, $x \in \{1, ..., k\}$ *is simulable by the measurement* $\mathbb{M} = \{M_a\}$, $a \in \{1, ..., o\}$ *when there exists a conditional PMF* $q_{X|A}$ *such that:* $N_x = \sum_a q(x|a) M_a$, $\forall x$. *The simulability of measurements defines a partial order for the set of measurements which we denote as* $\mathbb{N} \preceq \mathbb{M}$, *meaning that* $\mathbb{N}$ *is simulable by* $\mathbb{M}$. *Simulability of the measurement* $\mathbb{N}$ *can alternatively be understood as a classical post-processing of the measurement* $\mathbb{M}$.

Two quantifiers for informativeness are the following.

Definition 7.3. *(Generalised robustness and weight of informativeness) The generalised robustness [208, 204] and the weight [79, 73] of informativeness of a measurement* $\mathbb{M}$ *are* *given by:*

$$\mathrm{R}\left(\mathbb{M}\right) := \min_{\substack{r\geq 0\\ \mathbb{N}\in \mathrm{UI}\\ \mathbb{M}^G}} \left\{ r \,\middle|\, M_a + rM_a^G = (1+r)N_a \right\}, \tag{7.1}$$

$$\mathrm{W}\left(\mathbb{M}\right) := \min_{\substack{w\geq 0\\ \mathbb{N}\in \mathrm{UI}\\ \mathbb{M}^G}} \left\{ w \,\middle|\, M_a = wM_a^G + (1-w)N_a \right\}. \tag{7.2}$$

The generalised robustness quantifies the minimum amount of a general measurement $\mathbb{M}^G$ that has to be added to $\mathbb{M}$ such that we get an uninformative measurement $\mathbb{N}$. The weight on the other hand, quantifies the minimum amount of a general measurement $\mathbb{M}^G$ that has to be used for recovering the measurement $\mathbb{M}$.

These resource quantifiers are going to be useful later on. We now introduce the QRT of non-constant channels.

Definition 7.4. *(QRT of non-constant channels) Consider the set of completely-positive trace-preserving (CPTP) maps acting on a Hilbert space of dimension d. We now consider the resource of non-constant channels. We say that a channel $\mathcal{N}(\cdot)$ is constant, when there exist a state $\rho_{\mathcal{N}}$ such that $\mathcal{N}(\rho) = \rho_{\mathcal{N}}$, $\forall \rho \in D(\mathbb{H})$. We say that a channel is non-constant otherwise, and denote the set of all constant channels as $\mathcal{C}$.*

We now consider information-theoretic quantities for various general QRTs.

7.1.2 Arimoto-type information-theoretic quantities for general QRTs of measurements, channels, states, and state-measurement pairs

We now address a generalisation of Arimoto's α-mutual information to the concept of Arimoto's gap for general resources of measurements, channels, states, and state-measurement pairs. In order to introduce the concept of Arimoto's gap, let us first fix some notation. In this subsection we consider general QRTs with arbitrary resources, meaning that we address a set of free measurements as $\mathbb{F}$, and a set of free channels as $\mathcal{F}$, which are usually assumed to be convex and closed sets [217, 212, 73]. We now introduce the concept of *Arimoto's gap*, which is defined in terms of the standard Arimoto's α-mutual information, and for which we introduce here two variants as follows.

Definition 7.5. *(Arimoto's gap for measurements and channels [212, 71]) Consider a set of free measurements as $\mathbb{F}$, and a pair $(\mathcal{E}, \mathbb{M})$,* Arimoto's gap on POVMs *of order $\alpha \in \overline{\mathbb{R}}$ for such a pair is given by:*

$$G_\alpha^{\mathbb{F}}(X;G)_{\mathcal{E},\mathbb{M}} := I_\alpha(X;G)_{\mathcal{E},\mathbb{M}} - \max_{\mathbb{N}\in\mathbb{F}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N}}. \tag{7.3}$$

Similarly, consider a set of free channels $\mathcal{F}$ and a triple $(\mathcal{E}, \mathbb{M}, \mathcal{N})$, Arimoto's gap on channels *of order $\alpha \in \overline{\mathbb{R}}$ for such a triple is given by:*

$$G_\alpha^{\mathcal{F}}(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}} := I_\alpha(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}} - \max_{\widetilde{\mathcal{N}}\in\mathcal{F}} \max_{\mathbb{N}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N},\widetilde{\mathcal{N}}}. \tag{7.4}$$

Similarly to the previous section, we also address a more refined quantity as:

$$G_\alpha^{\mathcal{F}}(X;G)_{\mathcal{E},\mathcal{N}} := \max_{\mathbb{M}} G_\alpha^{\mathcal{F}}(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}}. \tag{7.5}$$

These quantities are information-theoretic in nature, being defined in terms of Arimoto's α-mutual information. We can think about them as the maximum *gap*, in terms of the Arimoto's α-mutual information, between the free set $\mathbb{F}$ $(\mathcal{F})$ and the fixed object of interest $\mathbb{M}$ $(\mathcal{N})$. These two measures can be thought of as generalisations of Arimoto's noisy α-mutual information and Arimoto's α-mutual information, respectively. This can be checked by setting $(\mathbb{F} = \mathbb{UI})$ and $(\mathcal{F} = \mathcal{C})$, for which we get:

$$G_\alpha^{\mathbb{UI}}(X;G)_{\mathcal{E},\mathbb{M}} = I_\alpha(X;G)_{\mathcal{E},\mathbb{M}}, \tag{7.6}$$

$$G_\alpha^{\mathcal{C}}(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}} = I_\alpha(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}}. \tag{7.7}$$

This is because uninformative measurements achieve $p(g|x) = \mathrm{Tr}[M_g\rho_x] = p(g)\,\mathrm{Tr}[\rho_x] = p(g)$, and similarly for constant channels $p(g|x) = \mathrm{Tr}[M_g\widetilde{\mathcal{N}}(\rho)] = \mathrm{Tr}[M_g\rho_{\widetilde{\mathcal{N}}}] = p(g)$, meaning that random variables G and X are independent from each other in both cases and therefore

$$\max_{\mathbb{N}\in\mathbb{UI}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N}} = \max_{\widetilde{\mathcal{N}}\in\mathcal{C}} \max_{\mathbb{N}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N},\widetilde{\mathcal{N}}} = 0. \tag{7.8}$$

Inspired by these information-theoretic quantities for measurements and channels, we now also consider Arimoto-type gaps for states as well as for a hybrid scenario with state-measurements pairs. Similarly for the case of measurements and channels, we address a set of free states as F, which is usually assumed to be convex and closed [217, 212]. We now define two variants of the concept of Arimoto's gap for QRTs of states as well as for QRTs of state-measurement pairs.

Definition 7.6. *(Arimoto's gap for states and for state-measurement pairs) Consider a set of free states* F*, and a triple* $(\Lambda, \mathbb{M}, \rho)$*, then,* Arimoto's gap on states *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a triple is given by:*

$$G_\alpha^{\mathrm{F}}(X;G)_{\Lambda,\mathbb{M},\rho} := I_\alpha(X;G)_{\Lambda,\mathbb{M},\rho} - \max_{\sigma\in\mathrm{F}} I_\alpha(X;G)_{\Lambda,\mathbb{M},\sigma}. \tag{7.9}$$

Similarly, consider a set of free states F*, a set of free measurements* $\mathbb{F}$*, and a triple* $(\Lambda, \mathbb{M}, \rho)$*, then,* Arimoto's gap on state-measurement pairs *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a triple is given by:*

$$G_\alpha^{\mathrm{F},\mathbb{F}}(X;G)_{\Lambda,\mathbb{M},\rho} := I_\alpha(X;G)_{\Lambda,\mathbb{M},\rho} - \max_{\substack{\sigma\in\mathrm{F}\\ \mathbb{N}\in\mathbb{F}}} I_\alpha(X;G)_{\Lambda,\mathbb{N},\sigma}. \tag{7.10}$$

Similarly to the previous variants on Arimoto's gaps, we have that these information-theoretic measures can be understood as quantifying the maximum *gap*, in terms of the standard Arimoto's α-mutual information, between the set of free objects and a fixed triple $(\Lambda, \mathbb{M}, \rho)$. The first variant was first introduced in [212] whilst the second multi-object variant was first introduced in [71].

Here we finish with the preliminary concepts and theoretical tools needed to describe our main results which we do next.

7.2 Quantum betting tasks with risk aversion

We now introduce the main new operational tasks that we consider in this chapter. We start by describing quantum betting tasks being played by gamblers with different risk tendencies. This is inspired by both standard quantum state discrimination and horse betting games in classical information theory.

Horse betting (HB) games were first introduced by Kelly in 1956 [128], a modern introduction can be found, for instance, in Cover & Thomas [63], as well as in the lectures notes by Moser [152]. Recently, Bleuler, Lapidoth, and Pfister generalised HB games in order to include a factor $\beta = 1 - R$ [33], representing the risk-aversion of the Gambler (Bob) playing these games, with standard HB games being recovered by setting $\beta = 0$, corresponding to $R = 1$, i.e. a risk-averse Bob.

Inspired by this, here we introduce three types of quantum betting tasks. First, we introduce quantum state betting (QSB) games. Specifically, we will introduce two variants of QSB games in the form of quantum state discrimination (QSD) with risk, and quantum state exclusion (QSE) with risk. We will then introduce the central figure of merit for QSB games – the isoelastic certainty equivalent (ICE), and show how it generalises the quantification of standard quantum state discrimination and exclusion. We then introduce important variants of this first game. In particular, we introduce noisy quantum state betting (nQSB) games and quantum channel betting (QCB) games, which generalises both quantum channel discrimination and exclusion. The tasks considered in this section, and the way they relate to each other is depicted in Figure 7.1.

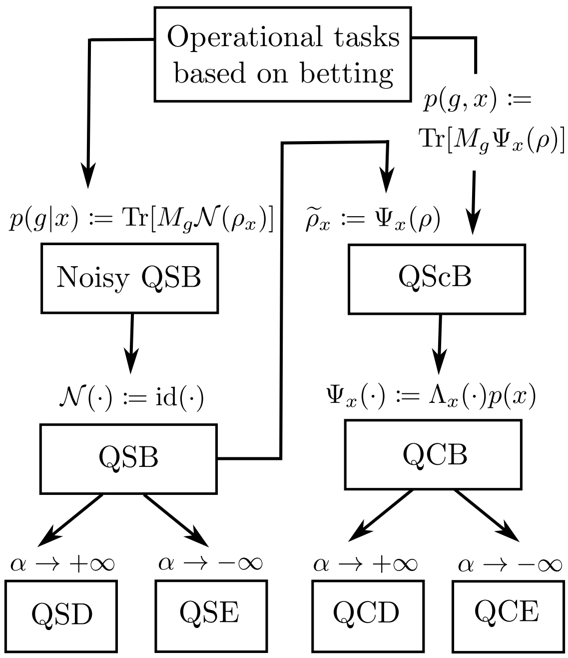

FIGURE 7.1: Operational tasks based on betting and risk-aversion. Quantum state betting (QSB), quantum subchannel betting (QScB), quantum channel betting (QCB), quantum state discrimination/exclusion (QSD/QSE), quantum channel discrimination/exclusion (QCD/QCE). $A \to B$ means that the task A is more general than B.

7.2.1 Quantum state betting (QSB) games

Consider two rational agents, a Referee (Alice) and a Gambler (Bob). Alice is in possession of an ensemble of quantum states $\mathcal{E} = \{\rho_x, p(x)\}$, $x \in \{1, ..., K\}$, and is going to send one of these states to Bob, say ρ_x. We address here a *quantum state*, or *state* for short, as a positive semidefinite ($\rho_x \geq 0$) and trace one ($\mathrm{Tr}(\rho_x) = 1$) operator in an finite-dimensional Hilbert space.

As above, we will consider two different classes of state betting games, gain games, and loss games. In a gain game, Alice offers Bob odds $o(x)$, which is a positive function ($o(x) > 0, \forall x$) but not necessarily a PMF, such that if Bob places a unit bet on the state being ρ_x, and this is the correct state, then Alice will pay out $o(x)$ to Bob. In a loss game, on the contrary, we take the 'odds' to be *negative*, $o(x) < 0$, for all x, such that if Bob places a unit bet on ρ_x, then he will have to pay out to Alice an amount $|o(x)|$.[1]

In order to decide how to place his bets, Bob is allowed to first perform a *quantum measurement* on the state given to him by Alice. In general, this will be a positive operator-valued measure (POVM), $\mathbb{M} = \{M_g\}$, $M_g \geq 0\, \forall g$, $\sum_g M_g = \mathbb{1}$, which will allow him to (hopefully) extract some useful information from the state.

Let us assume that Bob measures the state he receives from Alice using a measurement $\mathbb{M} = \{M_g\}$, producing a measurement result g, with probability given by the Born rule, $p(g|x) = \mathrm{Tr}[M_g \rho_x]$. Bob will then use this result to decide on his *betting strategy*. We assume that he bets all of his wealth, and divides this in some way amongst all the possible options $x \in \{1, ...K\}$. That is, Bob's strategy is a PMF $b_{X|G}$, such that Bob bets the proportion $b(x|g)$ of his wealth on state x being the sent state, when his measurement outcome was g.[2] We note that Bob's overall strategy is then defined by the pair $(b_{X|G}, \mathbb{M})$. We also note that the PMF p_X from the ensemble of states together with the conditional PMF $p_{G|X}$ from the measurement implemented by Bob, defines the joint PMF $p_{XG} := p_{G|X} p_X$.

Therefore, when the quantum state was ρ_x, and Bob obtained the measurement outcome g, he bet the proportion of his wealth $b(x|g)$ on the actual state, and hence Alice either pays out $w(x, g) = o(x)b(x|g)$ in the case of a gain game, or Bob has to pay Alice the amount $|w(x, g)|$ (i.e. he loses $|w(x, g)|$) in a loss game. We can view gain games as a generalisation of *state discrimination*. Here, since Bob is winning money, it is advantageous, in general, for him to correctly *identify* the state that was sent. On the other hand, we see that loss games can be viewed as a generalisation of *state exclusion*, since now in order to minimise his losses, it is useful for Bob to be able to *avoid* or *exclude* the state that was sent.

Finally, we note that the settings of the game are specified by the pair $(o_X, \mathcal{E})$. It is important to stress that by assumption Bob is fully aware of the settings of the game, meaning that the pair $(o_X, \mathcal{E})$ is known to him prior to playing the game, and therefore he can use this knowledge in order to select an optimal betting strategy $b_{X|G}$.

7.2.2 Figure of merit for quantum state betting games

Given these two variants of QSB games, we now want analyse the behaviour of different types of Gamblers (represented by different utility functions), according to their risk tendencies. We will consider quantities of interest like in the previous sections such as: expected wealth, expected utility, and similar. In particular, we model Gamblers with utility functions displaying constant relative risk aversion (CRRA) and therefore, the utility functions we consider are isoelastic functions $u_R(w)$ (2.98). The figure of merit we are interested in is then the *isoelastic certainty equivalent (ICE)*

[1] That is, similarly to in thermodynamics, we take the sign of the odds to signify whether this is a gain or a loss for Bob.

[2] Note that for loss games, Bob can end up having to pay out *more* than the wealth he bet (similarly to how in a game gain Bob can walk away with more wealth than he started with).

w_R^{ICE} with $R \in \overline{\mathbb{R}}$. For risk $R \in (-\infty, 1) \cup (1, \infty)$, this quantity is given by:

$$\begin{aligned} w_R^{ICE}(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}) &= u_R^{-1}\left(\mathbb{E}_{p_{XG}}\left[u_R(w_{XG})\right]\right), \\ &= \left[\sum_{g,x} \left[b(x|g)o(x)\right]^{1-R} p(g|x)p(x)\right]^{\frac{1}{1-R}}. \end{aligned} \tag{7.11}$$

The cases $R \in \{1, \infty, -\infty\}$ are defined by continuous extension of (7.11). In summary, the game is specified by the pair $(o_X, \mathcal{E})$, the behavioural tendency of Bob is represented by the utility function $u_R(w_{XG})$ with a fixed $R \in \overline{\mathbb{R}}$, the overall strategy of Bob is specified by the pair $(b_{X|G}, \mathbb{M})$, and the figure of merit here considered is the isoelastic certainty equivalent (ICE) (2.100). We can alternatively address these operational tasks as horse betting games with risk and quantum side information, or quantum horse betting (QHB) games for short, and we describe this in more detail later on.

Bob is in charge of the measurement and the betting strategy $(b_{X|G}, \mathbb{M})$, so in particular, for a fixed measurement $\mathbb{M}$, Bob is interested in maximising the ICE (maximising gains in a gain game, and minimising losses in a loss game) so we are going to be interested in the following quantity:

$$\max_{b_{X|G}} w_R^{ICE}\left(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}\right),$$

for a fixed QSB game $(o_X, \mathcal{E})$ with either positive or negative odds, and Bob's risk tendencies being fixed, and specified by an isoelastic utility function u_R.

7.2.3 Quantum state betting games generalise discrimination and exclusion games

We will now show that quantum state betting games with risk can indeed be seen as generalisations of standard quantum state discrimination and exclusion games. We can see this by considering a risk-neutral ($R = 0$) Bob playing a gain game (positive odds) which are constant: $o^c(x) := C$, $C > 0$, $\forall x$, in which case we find that the quantity of interest becomes:

$$\begin{aligned} \max_{b_{X|G}} w_0^{ICE}(b_{X|G}, \mathbb{M}, o_X^c, \mathcal{E}) &= C \max_{b_{X|G}} \sum_{g,x} b(x|g)p(g|x)p(x), \\ &= C\, P_{\text{succ}}^{\text{QSD}}(\mathcal{E}, \mathbb{M}). \end{aligned} \tag{7.12}$$

For more details on standard quantum state discrimination games we refer to [204, 212]. Therefore, standard quantum state discrimination can be seen as as special instance of quantum state betting games with constant odds, and played by a risk-neutral player. Similarly, for a loss game, with negative constant odds $o^{-c}(x) := -C$, $C > 0$, $\forall x$:

$$\begin{aligned} \max_{b_{X|G}} w_0^{ICE}(b_{X|G}, \mathbb{M}, o_X^{-c}, \mathcal{E}) &= C \max_{b_{X|G}} - \sum_{g,x} b(x|g)p(g|x)p(x), \\ &= -C\, P_{\text{err}}^{\text{QSE}}(\mathcal{E}, \mathbb{M}). \end{aligned} \tag{7.13}$$

For more details on standard quantum state exclusion games we refer to [73, 225]. Therefore, standard quantum state exclusion can be seen as a quantum state betting game constant negative odds, again played by a risk-neutral player.

7.2.4 Noisy quantum state betting (nQSB) games

We now introduce noisy quantum state betting (nQSB) games. We first note that standard QSB games (from the previous section) are implicitly assuming that the states that Alice (referee) sends to Bob (player) are perfectly transmitted, meaning that they are not affected by undesired interactions due to the environment. This is an idealised situation, and a more realistic scenario including such effects can be addressed by considering a completely-positive trace-preserving (CPTP) map (or quantum channel) $\mathcal{N}$, so that the probability of obtaining an outcome g after receiving the state ρ_x is now given by $p(g|x) = \mathrm{Tr}[M_g \mathcal{N}(\rho_x)]$. We refer to this more general and realistic scenario as *noisy* QSB (nQSB) games.

Definition 7.7. *(Noisy quantum state betting games) The isoelastic certainty equivalent (ICE) for a* noisy quantum state betting *(nQSB) game is given by:*

$$w_R^{\mathrm{nQSB}}(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}, \mathcal{N}) := w_R^{ICE}(b_{X|G}, \mathcal{N}^\dagger(\mathbb{M}), o_X, \mathcal{E}) \tag{7.14}$$

with $p(g|x) = \mathrm{Tr}[\mathcal{N}^\dagger(M_g)\,\rho_x] = \mathrm{Tr}[M_g \mathcal{N}(\rho_x)]$, $\mathcal{N}(\cdot)$ a completely-positive trace-preserving (CPTP) map, $\mathbb{M} = \{M_g\}$ a POVM, and the POVM $\mathcal{N}^\dagger(\mathbb{M}) := \{\mathcal{N}^\dagger(M_g)\}$. The cases $R \in \{1, \infty, -\infty\}$ are defined by continuous extension of (7.14).

We note that we recover standard QSB games by considering a noiseless scenario $\mathcal{N}(\cdot) = \mathrm{id}(\cdot)$. Whilst noisy QSB games can be seen as noiseless QSB games by considering the POVM $\mathcal{N}^\dagger(\mathbb{M}) := \{\mathcal{N}^\dagger(M_g)\}$, it is still important from a physical point to view to make the distinction between both noisy and noiseless scenarios. Later on we will see how this is relevant for the resource theory of non-constant channels.

7.2.5 Quantum channel betting (QCB) games

In this subsection we introduce quantum channel betting (QCB) games. Taking inspiration from the previous QSB games, where Bob (player) is asked to bet on an ensemble of states, we now consider Bob being asked to bet instead on a set of channels $\Lambda = \{\Lambda_x\}$, distributed according to a PMF p_X. In this scenario, Bob is in possession of a quantum state ρ, which he would consequently send to Alice (referee). Alice then proceeds to generate the ensemble $\{\Lambda_x(\rho), p(x)\}$, and send back one of these states to Bob. Bob then proceeds to measure the received state with a fixed POVM $\mathbb{M} = \{M_g\}$, and use the extracted information g in order to place a bet $b_{X|G}$ and effectively play the game. Following a similar logic to the case for QSB games, we can formalise and derive a figure of merit for QCB games in terms of the isoelastic certainty equivalent as follows.

Definition 7.8. *(Quantum channel betting) The isoelastic certainty equivalent (ICE) for a* quantum channel betting (QCB) *game is given by:*

$$w_R^{\mathrm{QCB}}(b_{X|G}, o_X, p_X, \Lambda, \rho, \mathbb{M}) := w_R^{ICE}(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}_{\Lambda,\rho}) \tag{7.15}$$

with $p(g|x) = \mathrm{Tr}[M_g \Lambda_x(\rho)]$, $\Lambda = \{\Lambda_x(\cdot)\}$ a set of completely-positive trace-preserving (CPTP) maps, $\mathbb{M} = \{M_g\}$ a POVM, and $\mathcal{E}_{\Lambda,\rho} := \{\Lambda_x(\rho), p(x)\}$. The cases $R \in \{1, \infty, -\infty\}$ are defined by continuous extension of (7.15).

First, we note here that these tasks can be further extended to *quantum subchannel betting (QScB)* games where we address a set of subchannels $\Psi = \{\Psi_x(\cdot)\}$, or set of completely-positive trace-nonincresing (CPTNI) maps, with $p(x, g) = \mathrm{Tr}[M_g \Psi_x(\rho)]$.

Second, whilst QCB can be seen as noiseless QSB games with the ensemble given by $\mathcal{E}_{\Lambda,\rho} := \{\Lambda_x(\rho), p(x)\}$, it is still important to distinguish these two cases from a physical point of view, this, because in a QCB game Bob (player) is now allowed to have an influence on the ensemble of states as $\mathcal{E} = \mathcal{E}_{\Lambda,\rho}$. Third, we can see that QCB games generalise standard channel discrimination and standard channel exclusion as follows. Consider a risk-neutral ($R = 0$) Bob playing a gain game (positive odds) which are constant: $o^c(x) := C$, $C > 0$, $\forall x$, in which case we find that the ICE becomes:

$$\max_{b_{X|G}} w_0^{\mathrm{QCB}}(b_{X|G}, o^c_X, p_X, \Lambda, \rho, \mathbb{M}) = C \max_{b_{X|G}} \sum_{g,x} b(x|g) \; \mathrm{Tr}[M_g \Lambda_x(\rho)] \, p(x),$$
$$= C \, P^{\mathrm{QCD}}_{\mathrm{succ}}(\Lambda, \rho, \mathbb{M}), \tag{7.16}$$

with $\Lambda = \{\Lambda_x(\cdot)\}$ a set CPTP maps, $\mathbb{M} = \{M_g\}$ a POVM. Therefore, standard quantum channel discrimination can be seen as as special instance of quantum sub-channel betting games with constant odds, and played by a risk-neutral player. For more details on standard quantum channel discrimination (QCD) games we refer the reader to [217, 212]. Similarly, for a loss game, with negative constant odds $o^{-c}(x) := -C, C > 0, \forall x$:

$$\max_{b_{X|G}} w_0^{\mathrm{QCB}}(b_{X|G}, o^{-c}_X, p_X, \Lambda, \rho, \mathbb{M}) = C \max_{b_{X|G}} - \sum_{g,x} b(x|g) \; \mathrm{Tr}[M_g \Lambda_x(\rho)] p(x),$$
$$= -C \, P^{\mathrm{QCE}}_{\mathrm{err}}(\Lambda, \rho, \mathbb{M}), \tag{7.17}$$

with $\Lambda = \{\Lambda_x(\cdot)\}$ a set of CPTP maps, $\mathbb{M} = \{M_g\}$ a POVM. Therefore, standard quantum channel exclusion can be seen as a quantum channel betting game with constant negative odds, again played by a risk-neutral gambler. For more details on standard quantum channel exclusion (QCE) games we refer the reader to [73, 225]. We now proceed to address our main results.

7.3 Main results

We are now ready to present the main results of this chapter.

7.3.1 Result 7.1. Arimoto's α-mutual information and quantum state betting games

The main motivation now is to compare the performance of two gamblers via the maximised isoelastic certainty equivalent (ICE) $\max_{b_{X|G}} w_R^{ICE}\left(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}\right)$. Specifically, we want to compare: i) a general gambler using a fixed measurement $\mathbb{M}$ with ii) the best uninformative gambler, meaning a gambler who can implement any uninformative measurement $\mathbb{N} \in \mathrm{UI}$, or equivalently, a gambler described by the quantity $\max_{\mathbb{N}\in\mathrm{UI}} \max_{b_{X|G}} w_R^{ICE}\left(b_{X|G}, \mathbb{N}, o_X, \mathcal{E}\right)$. We have the following main result.

Result 7.1. *Consider the a QSB game defined by the pair $(o_X^{\mathrm{sgn}(\alpha)c}, \mathcal{E})$ with constant odds as $o^{\mathrm{sgn}(\alpha)c}(x) := \mathrm{sgn}(\alpha)C$, $C > 0$, $\forall x$, and an ensemble of states $\mathcal{E} = \{\rho_x, p(x)\}$. Consider a Gambler playing this game using a fixed measurement $\mathbb{M}$ in comparison to a Gambler being allowed to implement any uninformative measurement $\mathbb{N} \in \mathrm{UI}$. Consider both Gamblers with the same attitude to risk, meaning that they are represented by isoelastic functions $u_R(W)$ with the risk parametrised as $R(\alpha) := 1/\alpha$. Each Gambler is allowed to play the game with the optimal betting strategies, meaning they can each propose a betting strategy* *independently from each other. Remembering that the Gamblers are interested in maximising the isoelastic certainty equivalent (ICE), we have the following relationship:*

$$I_\alpha(X;G)_{\mathcal{E},\mathbb{M}} = \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w^{ICE}_{1/\alpha}\left(b_{X|G}, \mathbb{M}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}\right)}{\max\limits_{\mathbb{N}\in \mathrm{UI}} \max\limits_{b_{X|G}} w^{ICE}_{1/\alpha}\left(b_{X|G}, \mathbb{N}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}\right)} \right]. \tag{7.18}$$

This shows that Arimoto's α-mutual information quantifies the ratio of the isoelastic certainty equivalent with risk $R(\alpha) := 1/\alpha$ of the game defined by $(o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E})$, when the QSB game is played with the best betting strategy, and when we compare a Gambler implementing a fixed measurement $\mathbb{M}$ against a Gambler using any uninformative measurement $\mathbb{N} \in$ UI.

The full proof of Result 7.1 is in Appendix B.1. We now analyse two cases of particular interest ($\alpha \in \{\infty, -\infty\}$), as the following corollaries.

Corollary 7.1. *In the case $\alpha \to \infty$ we recover the result found in [204]. Explicitly, we have:*

$$C_\infty(\Lambda_\mathbb{M}) = \max_{\mathcal{E}} I_\infty(X;G)_{\mathcal{E},\mathbb{M}} = \log \left[\max_{\mathcal{E}} \frac{P^{\mathrm{QSD}}_{\mathrm{succ}}(\mathcal{E},\mathbb{M})}{\max_{\mathbb{N}\in\mathrm{UI}} P^{\mathrm{QSD}}_{\mathrm{succ}}(\mathcal{E},\mathbb{N})} \right], \tag{7.19}$$

where $P^{\mathrm{QSD}}_{\mathrm{succ}}(\mathcal{E},\mathbb{M})$ is the probability of success in the quantum state discrimination (QSD) game defined by $\mathcal{E}$, with the Gambler using the measurement $\mathbb{M}$, given explicitly by:

$$P^{\mathrm{QSD}}_{\mathrm{succ}}(\mathcal{E},\mathbb{M}) := \max_{q_{G|A}} \sum_{g,a,x} \delta^g_x\, q(g|a)\, p(a|x)\, p(x), \tag{7.20}$$

with $p(a|x) := \mathrm{Tr}[M_a\rho_x]$, and the maximisation over all classical post-processing $q_{G|A}$. We remark that the Rényi capacity of order ∞ has also been called as the accessible min-information of a channel, and denoted as $I^{\mathrm{acc}}_\infty(\Lambda_\mathbb{M})$ [204, 242]. This shows that quantum state betting with risk (QSB$_{R(\alpha)}$) becomes equivalent to quantum state discrimination (QSD) when $\alpha \to \infty$.

Corollary 7.2. *In the case $\alpha \to -\infty$ we recover the result found in [73]. Explicitly:*

$$C_{-\infty}(\Lambda_\mathbb{M}) = \max_{\mathcal{E}} I_{-\infty}(X;G)_{\mathcal{E},\mathbb{M}} = -\log \left[\min_{\mathcal{E}} \frac{P^{\mathrm{QSE}}_{\mathrm{err}}(\mathcal{E},\mathbb{M})}{\min_{\mathbb{N}\in\mathrm{UI}} P^{\mathrm{QSE}}_{\mathrm{err}}(\mathcal{E},\mathbb{N})} \right], \tag{7.21}$$

where $P^{\mathrm{QSE}}_{\mathrm{err}}(\mathcal{E},\mathbb{M})$ is the probability of error in the quantum state exclusion (QSE) game defined by $\mathcal{E}$, with the Gambler using the measurement $\mathbb{M}$ explicitly given by:

$$P^{\mathrm{QSE}}_{\mathrm{err}}(\mathcal{E},\mathbb{M}) := \min_{q_{G|A}} \sum_{g,a,x} \delta^g_x\, q(g|a)\, p(a|x)\, p(x). \tag{7.22}$$

with $p(a|x) := \mathrm{Tr}[M_a\rho_x]$, and the minimisation being performed over all classical post-processing $q_{G|A}$. We remark that the Rényi capacity of order $-\infty$ has also been called the excludible information of a channel, and denoted as $I^{\mathrm{exc}}_{-\infty}(\Lambda_\mathbb{M})$ [73, 71]. This shows that quantum state betting with risk (QSB$_{R(\alpha)}$) becomes equivalent to quantum state exclusion (QSE) when $\alpha \to -\infty$.

In Appendix B.2 we provide further details on these two corollaries.

Result 7.1 establishes a connection between Arimoto's α-mutual information and QSB games, which recovers two known cases at $\alpha \in \{\infty, -\infty\}$ [204, 73]. We emphasise that the right hand side of (7.18) is a completely *operational* quantity, which represents the advantage that an informative measurement provides when being used as a resource for QSB games, whilst the left hand side is the raw *information-theoretic* mutual information measure proposed by Arimoto and consequently, this result provides an operational interpretation of Arimoto's α-mutual information in the quantum domain.

Furthermore, it shows that the Rényi parameter can be interpreted as characterising the risk tendency of the Gamblers as $R = 1/\alpha$. It is also interesting to note that this works for all ensembles $\mathcal{E} = \{\rho_x, p(x)\}$, all measurements $\mathbb{M} = \{M_g\}$, as well as for the whole range of the Rényi parameter $\alpha \in \overline{\mathbb{R}}$, including negative values. We summarise the interpretation of this result in Fig. 7.2.

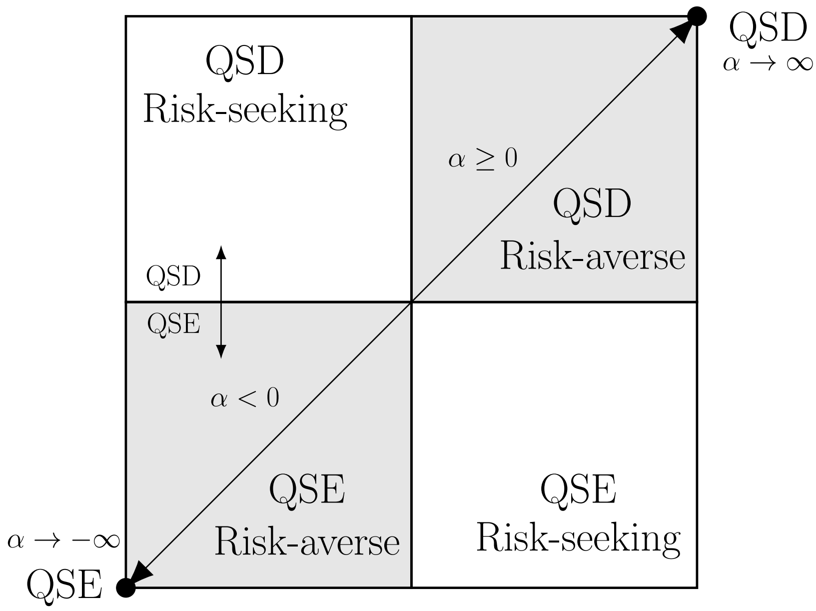

FIGURE 7.2: Possible scenarios for quantum state discrimination (QSD) and quantum state exclusion (QSE) games being played by Gamblers with different risk tendencies: risk-averse, risk-seeking, or risk-neutral, with the risk being parametrised as $R(\alpha) = 1/\alpha$. Result 7.1 establishes that Arimoto's mutual information quantifies the shaded region for $\alpha \in \overline{\mathbb{R}}$, meaning that it characterises risk-averse Gamblers playing either QSD ($\alpha \geq 0$) and QSE games ($\alpha < 0$). The left bottom corner ($\alpha \to -\infty$) and the top-right corner ($\alpha \to \infty$) represent a risk-neutral Gambler $R = 0$ playing either standard exclusion or discrimination games, respectively. This means that standard QSD games can be understood as a risk-neutral Gambler playing QSD games with risk. Similarly, standard QSE games can be understood as a risk-neutral Gambler playing QSE games with risk. The middle point at $\alpha \to 0$ represents the transition between a maximally risk-averse Gambler playing QSD games and a maximally risk-averse Gambler playing QSE games.

We also highlight here that Result 7.1 lies at the intersection of three major fields: quantum theory, information theory, and the theory of games and economic behaviour. We believe that this result has the potential to spark further cross-fertilisation of ideas between these three major areas of knowledge, with only these particular examples currently being unfolded. We now address the characterisation of additional tasks based on betting and risk-aversion.

7.3.2 Result 7.2. Arimoto's mutual information and noisy quantum state betting games

We now naturally would like to address a characterisation for nQSB games in the same vein that their standard counterpart. Intuitively, we are now addressing a general quantum channel $\mathcal{N}(\cdot)$ as a new ingredient, and that Bob is still in charge of the decoding measurement $\mathbb{M}$. From the noiseless scenario, we understand that Arimoto-like quantities are giving account for the amount of *side information* being conveyed to Bob. When we consider Bob using a fixed measurement, a decisive factor that naturally emerges is the resource of *informativeness*, because this resources defines the frontier for the cases when side information can or cannot be transmitted. In noisy QSB games the other hand, with a general channel $\mathcal{N}(\cdot)$, the same reasoning leads to consider the resource of *non-constant channels*, this, because they will effectively *destroy* the side information carried by the state since $p(g|x) = \mathrm{Tr}[N_g \mathcal{N}'(\rho_x)] = \mathrm{Tr}(N_g \rho_{\mathcal{N}'}) = p(g)$, for all constant channels $\mathcal{N}'$, and for all measurements $\mathbb{N}$. The following result confirms this intuition, and consequently characterises nQSB games.

Result 7.2. *Consider a nQSB game defined by the pair $(o_X^{\mathrm{sgn}(\alpha)c}, \mathcal{E})$ with constant odds as $o^{\mathrm{sgn}(\alpha)c}(x) := \mathrm{sgn}(\alpha)C$, $C > 0$, $\forall x$, an ensemble of states $\mathcal{E} = \{\rho_x, p(x)\}$. Consider a Gambler playing this game being able to implement any measurement $\mathbb{M}$, and having access to a fixed channel $\mathcal{N}$. We want to compare this first Gambler against a second Gambler also being allowed to implement any measurement $\mathbb{N}$, but now having access only to constant channels $\mathcal{N}' \in \mathcal{C}$. Consider both Gamblers with the same attitude to risk, meaning that they are represented by isoelastic functions $u_R(W)$ with the risk parametrised as $R(\alpha) := 1/\alpha$. Each Gambler is allowed to play the game with optimal betting strategies, meaning they can each propose a betting strategy $b_{X|G}$ independently from each other. Remembering that the Gamblers are interested in maximising the isoelastic certainty equivalent (ICE), we have:*

$$I_\alpha(X;G)_{\mathcal{E},\mathcal{N}} = \mathrm{sgn}(\alpha) \log \left[\frac{\max\limits_{\mathbb{M}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{nQSB}} \left(b_{X|G}, \mathbb{M}, o_X^{\mathrm{sgn}(\alpha)c}, \mathcal{E}, \mathcal{N} \right)}{\max\limits_{\mathcal{N}' \in \mathcal{C}} \max\limits_{\mathbb{N}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{nQSB}} \left(b_{X|G}, \mathbb{N}, o_X^{\mathrm{sgn}(\alpha)c}, \mathcal{E}, \mathcal{N}' \right)} \right]. \tag{7.23}$$

This means that Arimoto's noisy mutual information quantifies the ratio of the ICE with risk $R(\alpha) := 1/\alpha$ of the nQSB game defined by $(o_X^{\mathrm{sgn}(\alpha)c}, \mathcal{E})$, when the nQSB games are being played with the best betting strategy, and when we compare a Gambler implementing a fixed channel $\mathcal{N}$ against a Gambler using any constant channel $\mathcal{N}' \in \mathcal{C}$.

The proof of this result follows a similar argument than that of Result 7.1. We have seen that two natural resources have emerged, or equivalently, two sets of free objects: i) the set of uninformative measurements and ii) the set of constant channels. We then wonder whether the results so far presented are unavoidably linked to these particular resources or, on the other hand, whether they are particular cases of a more general underlying structure governing the relationship between information-theoretic quantities and operational tasks for general QRTs. We address such a question in the next subsection, where we address an extension of these results to general QRTs of measurements and channels with arbitrary resources.

7.3.3 Result 7.3. QSB and noisy QSB games for general QRTs of measurements and channels

We have seen that both uninformative measurements and non-constant channels are related to Arimoto's mutual information, and we now want to address general resources. In order to do this we can expect to need quantities which are more general than Arimoto's mutual information. We now consider the *Arimoto's gaps* introduced in the previous sections, and provide operational characterisations for these information-theoretic quantities in terms of QSB and nQSB games as follows.

Result 7.3. *Consider a set of free measurements as* $\mathbb{F}$ *and a couple* $(\mathcal{E}, \mathbb{M})$*, then,* Arimoto's gap on POVMs *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a couple can be written as:*

$$G_\alpha^{\mathbb{F}}(X;G)_{\mathcal{E},\mathbb{M}} = \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\text{QSB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}, \mathbb{M}\right)}{\max\limits_{\mathbb{N}\in\mathbb{F}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\text{QSB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}, \mathbb{N}\right)} \right]. \tag{7.24}$$

Similarly, consider a set of free channels $\mathcal{F}$ *and a triple* $(\mathcal{E}, \mathbb{M}, \mathcal{N})$*, then,* Arimoto's gap on channels *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a triple can be written as:*

$$G_\alpha^{\mathcal{F}}(X;G)_{\mathcal{E},\mathcal{N}} = \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{\mathbb{M}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\text{nQSB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}, \mathbb{M}, \mathcal{N}\right)}{\max\limits_{\widetilde{\mathcal{N}}\in\mathcal{F}} \max\limits_{\mathbb{N}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\text{nQSB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \mathcal{E}, \mathbb{N}, \widetilde{\mathcal{N}}\right)} \right].$$

This means that Arimoto-type gaps quantify the usefulness of a given measurement (channel) $\mathbb{M}$ $(\mathcal{N})$ *when playing QSB (nQSB) games, in comparison with the best free measurements (channels)* $\mathbb{N} \in \mathbb{F}$ $(\widetilde{\mathcal{N}} \in \mathcal{F})$*.*

The proof of Result 7.3 follows a similar logic to that of Result 7.1 but, for completeness, we present its proof in Appendix B.3. It is interesting to note the level of generality of this result. This results holds true for any $\alpha \in \overline{\mathbb{R}}$, any ensemble $\mathcal{E}$, any measurement $\mathbb{M}$, any channel $\mathcal{N}$, as well as any reasonable and physically motivated choices of sets of free measurements $\mathbb{F}$ and free channels $\mathcal{F}$. In particular, by specifying the sets of free objects we can recover some of the previous results as corollaries.

Corollary 7.3. *Imposing the set of free measurements to be the set of uninformative measurements in* (7.24) $(\mathbb{F} = \mathbb{UI})$*, we recover Result 7.1* (7.18)*. Similarly, imposing the set of free channels to be the set of constant channels in* (7.25) $(\mathcal{F} = \mathcal{C})$*, we recover Result 7.2* (7.23)*.*

We have so far addressed QSB games and more generally nQSB games. The main idea behind these operational tasks is the inclusion of the concept of *betting*, which is represented by the constant relative risk aversion (CRRA) coefficient R, and which is ultimately related to the Rényi parameter as $R = 1/\alpha$. We now address the fact that the concept of *betting* is an useful and powerful concept that allows for the generalisation of additional operational tasks. In particular, we now address the characterisation of quantum channel betting (QCB) games.

7.3.4 Result 7.4. QCB games and QRTs of states and state-measurement pairs

Similarly to the case for noisy quantum state betting (nQSB) games, we would now like to characterise quantum subchannel betting (QCB) games in terms of information-theoretic quantities. We now provide an operational interpretation for Arimoto-type quantities for QRTs of states and hybrid multi-object scenarios, in terms of quantum channel betting (QCB) games.

Result 7.4. *Consider a set of free states as* F *and a triple* $(\Lambda, \mathbb{M}, \rho)$*, then,* Arimoto's gap on states *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a triple can be written as:*

$$G_\alpha^{\mathrm{F}}(X;G)_{\Lambda,\mathbb{M},\rho} = \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \Lambda, \rho, \mathbb{M}\right)}{\max\limits_{\sigma \in \mathrm{F}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \Lambda, \sigma, \mathbb{M}\right)} \right]. \tag{7.25}$$

Similarly, consider a set of free states F*, a set of free measurements* $\mathbb{F}$*, and a triple* $(\Lambda, \mathbb{M}, \rho)$*, then,* Arimoto's gap on state-measurement pairs *of order* $\alpha \in \overline{\mathbb{R}}$ *for such a triple can be written as:*

$$G_\alpha^{\mathrm{F},\mathbb{F}}(X;G)_{\Lambda,\mathbb{M},\rho} = \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \Lambda, \rho, \mathbb{M}\right)}{\max\limits_{\substack{\sigma \in \mathrm{F} \\ \mathbb{N} \in \mathbb{F}}} \max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}\left(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, \Lambda, \sigma, \mathbb{N}\right)} \right]. \tag{7.26}$$

These two statements mean that Arimoto's gap quantifies the usefulness of resourceful objects when compared to gamblers only having access to free objects.

The proof of this result follows a similar logic than that of Result 7.1 but, for completeness, we present its proof in Appendix B.4. Similarly to the case for QSB and nQSB games, we have that quantum channel betting (QCB) games can also be characterised by means of Arimoto-type information-theoretic quantities, for single-object QRTs of states with arbitrary resources, but also for more exotic scenarios as the case of multi-object QRTs of state-measurement pairs. The second statement generalises some of the multi-object results presented in [71], which considered the cases for $\alpha \in \{+\infty, -\infty\}$, and so this result generalises this to the whole extended line of real numbers $\alpha \in \overline{\mathbb{R}}$.

7.3.5 Result 7.5. Arimoto's mutual information and horse betting games in the classical regime

We now consider operational tasks based on betting and risk-aversion in the form of horse betting (HB) games with risk and side information, without making reference to quantum theory, and derive a result interpreting Arimoto's mutual information as quantifying the advantage provided by side information when playing such horse betting games.

We consider here the Gambler now having access to a random variable G, which is potentially correlated with the outcome of the 'horse race' X and therefore, the Gambler can try to use this for her/his advantage. This means that these horse betting games are defined by the pair (o_X, p_{GX}), and the Gambler is in charge of proposing the betting strategy $b_{X|G}$. We highlight here that this contrasts the case of

QSB games, because there the Gambler could in principle be in charge of intervening in the conditional PMF $p_{G|X}$, as the Gambler had access to a measurement and $p_{G|X} = \mathrm{Tr}(M_g \rho_x)$, whilst here on the other hand, $p_{GX} = p_{G|X} p_X$ is a given, and the Gambler cannot in principle influence the PMF $p_{G|X}$. However, the figure of merit is still the isoelastic certainty equivalent for risk $R \in (-\infty, 1) \cup (1, \infty)$ which is now written as:

$$w_R^{ICE}(b_{X|G}, o_X, p_{XG}) := \left[\sum_{g,x} \left[b(x|g) o(x) \right]^{1-R} p(x,g) \right]^{\frac{1}{1-R}}. \tag{7.27}$$

The cases $R \in \{1, \infty, -\infty\}$ are defined again by continuous extension of (7.27). A HB game is then specified by the pair (o_X, p_{GX}), and the Gambler plays this game with a betting strategy $b_{X|G}$.

Horse betting games were characterised by Bleuler, Lapidoth, and Pfister (BLP), in terms of the BLP-CR divergence [33] (see Appendix B.1 for more details on this). We now modify these tasks in order to consider both gain games (when the odds are positive) and loss games (when the odds are negative), and relate Arimoto's mutual information to HB games with the following result, which can be derived in a similar manner as the previous ones.

Result 7.5. *Consider a horse betting game defined by the pair $(o_X^{\mathrm{sgn}(\alpha)c}, p_{XG})$ with constant odds as $o^{\mathrm{sgn}(\alpha)c}(x) := \mathrm{sgn}(\alpha) C$, $C > 0$, $\forall x$, and a joint PMF p_{XG}. Consider a Gambler playing this game having access to the side information G, against a Gambler without access to any side information. Consider both Gamblers with the same attitude to risk, meaning they are represented by isoelastic functions $u_R(w)$ with the risk parametrised as $R(\alpha) := 1/\alpha$. The Gamblers are allowed to play these games with the optimal betting strategies, which they can each choose independently from each other. Remembering that the Gamblers are interested in maximising the isoelastic certainty equivalent (ICE), we have the following relationship:*

$$I_\alpha(X;G) = \mathrm{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{ICE}(b_{X|G}, o_X^{\mathrm{sgn}(\alpha)c}, p_{XG})}{\max\limits_{b_X} w_{1/\alpha}^{ICE}(b_X, o_X^{\mathrm{sgn}(\alpha)c}, p_X)} \right]. \tag{7.28}$$

This means that Arimoto's mutual information quantifies the ratio of the isoelastic certainty equivalent with risk $R(\alpha) := 1/\alpha$ of the games defined by $(o_X^{\mathrm{sgn}(\alpha)c}, p_{XG})$, when each HB game is played with the best betting strategy, and we compare the performance of a first Gambler who makes use of the side information G, against a second gambler which has no access to side information.

We emphasise that this result is purely "classical", as it does not invoke any elements from quantum theory. This result also complements a previous relationship between HB games and the BLP-CR divergence [33]. Here on the other hand we characterise instead *the ratio* between the two HB scenarios, where we compare a first gambler with access to side information against a second gambler having no access to side information. We now address a particular known case as the following corollary.

Corollary 7.4. *In the case $\alpha = 1$, which means HB games with risk aversion given by $R = 1$, we get:*

$$I(X;G) = \max_{b_{X|G}} U_0(b_{X|G}, o^c_X, p_{XG}) - \max_{b_X} U_0(b_X, o^c_X, p_X), \tag{7.29}$$

with $I_1(X;G) = I(X;G)$ the standard mutual information, and $U_0 := \log w_0^{ICE}$ the logarithm of the isoelastic certainty equivalent. This is a particular case of a relationship known to hold for all odds $o(x)$ [63, 152].

We now come back to the QRT of measurement informativeness, and explore further connections between Arimoto's mutual information, QSB games, and additional information-theoretic quantities in the form of quantum Rényi divergences and resource monotones.

7.3.6 Result 7.6. Quantum Rényi divergences

Considering that the KL-divergence is of central importance in classical information theory, it is natural to consider quantum-extensions of such quantity. There are many ways to define quantum Rényi divergences [222, 170, 153, 244, 149, 82, 69], with most of the effort being concentrated on divergences as a functions of quantum states. Recently however, divergences and entropies for additional objects like channels and measurements have been started to be explored [62, 134, 96]. We are now interested in addressing quantum Rényi divergences for measurements. The approach we take here takes inspiration from both: measured Rényi divergences for states [30, 112, 69], as well as Rényi conditional divergences in the classical domain [199, 64, 33]. Explicitly, we invoke the measures for Rényi conditional divergences, and use them to define measured Rényi divergences for measurements.

Definition 7.9. *(Measured quantum Rényi divergence of Sibson) The measured Rényi divergence of Sibson of order $\alpha \in \overline{\mathbb{R}}$ and a set of states $\mathcal{S} = \{\rho_x\}$ of two measurements $\mathbb{M} = \{M_g\}$ and $\mathbb{N} = \{N_g\}$ is given by:*

$$D^{\mathcal{S}}_\alpha(\mathbb{M}||\mathbb{N}) := \max_{p_X} D_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} \middle|\middle| q^{(\mathbb{N},\mathcal{S})}_{G|X} \middle| p_X\right). \tag{7.30}$$

with the maximisation over all PMFs p_X, and the conditional PMFs $p^{(\mathbb{M},\mathcal{S})}_{G|X}$ and $q^{(\mathbb{N},\mathcal{S})}_{G|X}$ given by $p(g|x) := \mathrm{Tr}(M_g\rho_x)$, $q(g|x) := \mathrm{Tr}(N_g\rho_x)$, respectively, and $D(\cdot||\cdot|\cdot)$ the conditional Rényi divergence of Sibson [199].

We now use this measured Rényi divergence in order to define a distance measure with respect to a free set of interest, the set of uninformative measurements in this case.

Definition 7.10. *(Measurement informativeness measure of Sibson) The measurement informativeness measure of Sibson of order $\alpha \in \overline{\mathbb{R}}$ and set of states $\mathcal{S}$ of a measurement $\mathbb{M}$ is given by:*

$$E^{\mathcal{S}}_\alpha(\mathbb{M}) := \min_{\mathbb{N}\in \mathrm{UI}} D^{\mathcal{S}}_\alpha(\mathbb{M}||\mathbb{N}), \tag{7.31}$$

with the minimisation over all uninformative measurements.

Interestingly, it turns out that this quantity becomes equal to a quantity which we have already introduced.

Result 7.6. *The informativeness measure of Sibson is equal to the Rényi capacity of order $\alpha \in \overline{\mathbb{R}}$ of the measurement $\mathbb{M}$ as:*

$$E_{\alpha}^{\mathcal{S}}(\mathbb{M}) = C_{\alpha}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})}\right), \tag{7.32}$$

with the quantum-classical channel associated to the measurement $\mathbb{M}$ (2.76).

The proof of this result is in Appendix B.5. This result establishes a connection between Rényi mutual information (which are used to define the Rényi channel capacity) and quantum Rényi divergences of measurements (which are used to define the measurement informativeness measure). We now consider the quantity $E_{\alpha}(\mathbb{M}) := \max_{\mathcal{S}} E_{\alpha}^{\mathcal{S}}(\mathbb{M}) = C_{\alpha}(\Lambda_{\mathbb{M}})$ and analyse the particular cases of $\alpha \in \{\infty, -\infty\}$.

Corollary 7.5. *The measurement informativeness measure of Sibson recovers the generalised robustness and the weight of resource at the extremes $\alpha \in \{\infty, -\infty\}$ as:*

$$E_{\infty}(\mathbb{M}) = \log\left[1 + \mathrm{R}(\mathbb{M})\right], \tag{7.33}$$
$$E_{-\infty}(\mathbb{M}) = -\log\left[1 - \mathrm{W}(\mathbb{M})\right], \tag{7.34}$$

with the generalised robustness of informativeness (7.1) [204], and the weight of informativeness (7.2) [73].

This result follows from the fact that the Rényi channel capacity becomes the accessible min-information and the excludible information at the extremes $\alpha \in \{\infty, -\infty\}$, together with the results from [204] and [73]. Result 7.6 therefore establishes a connection between Rényi mutual informations and quantum Rényi divergences of measurements. Inspired by these results, we now proceed to propose a family of resource monotones.

7.3.7 Result 7.7. Resource monotones

Resource quantifiers are special cases of resource monotones, which are central objects of study within QRTs [57, 93]. Two common families of resource monotones are the so-called *robustness-based* [234, 49, 172, 174, 155, 204, 49, 55, 138, 122] and *weight-based* [136, 79, 206, 49, 44] resource monotones. Inspired by the previous results, we now define measures which turn out to be monotones for the order induced by the simulability of measurements and furthermore, that this new family of monotones recover, at its extremes, the generalised robustness and the weight of informativeness.

Definition 7.11. *(α-measure of informativeness) The α-measure of informativeness of order $\alpha \in \overline{\mathbb{R}}$ of a measurement $\mathbb{M}$ is given by:*

$$\mathrm{M}_{\alpha}(\mathbb{M}) := \mathrm{sgn}(\alpha) 2^{\mathrm{sgn}(\alpha) E_{\alpha}(\mathbb{M})} - \mathrm{sgn}(\alpha), \tag{7.35}$$

with $E_{\alpha}(\mathbb{M}) := \max_{\mathcal{S}} E_{\alpha}^{\mathcal{S}}(\mathbb{M})$ and the measurement informativeness measure defined in (7.31).

The motivation behind the proposal of this resource measure is because: i) it recovers the generalised robustness and the weight of resource as $\mathrm{M}_{\infty}(\mathbb{M}) = \mathrm{R}(\mathbb{M})$ and $\mathrm{M}_{-\infty}(\mathbb{M}) = \mathrm{W}(\mathbb{M})$ and, ii) it allows the following operational characterisation.

Remark 7.1. *They α-measure of informativeness of order $\alpha \in \overline{\mathbb{R}}$ of a measurement $\mathbb{M}$ characterises the performance of the measurement $\mathbb{M}$, when compared to the performance of* *all possible uninformative measurements, when playing the same QSB game as:*

$$\max_{\mathcal{E}} \frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{ICE}\left(b_{X|G}, \mathbb{N}, o_X^c, \mathcal{E}\right)}{\max\limits_{\mathbb{N}\in\mathrm{UI}} \max\limits_{b_{X|G}} w_{1/\alpha}^{ICE}\left(b_{X|G}, \mathbb{N}, o_X^c, \mathcal{E}\right)} = 1 + \mathrm{M}_\alpha(\mathbb{M}), \tag{7.36}$$

$$\min_{\mathcal{E}} \frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{ICE}\left(b_{X|G}, \mathbb{N}, o_X^{-c}, \mathcal{E}\right)}{\max\limits_{\mathbb{N}\in\mathrm{UI}} \max\limits_{b_{X|G}} w_{1/\alpha}^{ICE}\left(b_{X|G}, \mathbb{N}, o_X^{-c}, \mathcal{E}\right)} = 1 - \mathrm{M}_\alpha(\mathbb{M}), \tag{7.37}$$

for $\alpha \geq 0$ and $\alpha < 0$, respectively. These two equalities follow directly from the definitions and the previous results.

This result is akin to the connections between generalised robustness characterising discrimination games, and the weight of resource characterising exclusion games. We now also have that the α-measure of informativeness defines a resource monotone for the simulability of measurements.

Result 7.7. *(The α-measure of informativeness is a resource monotone) The α-measure of informativeness (7.35) defines a resource monotone for the simulability of measurements, meaning that it satisfies the following properties. (i) Faithfulness: $\mathrm{M}_\alpha(\mathbb{M}) = 0 \leftrightarrow \mathbb{M} = \{M_a = q(a)\mathbb{1}\}$ and (ii) Monotonicity under measurement simulation: $\mathbb{N} \preceq \mathbb{M} \rightarrow \mathrm{M}_\alpha(\mathbb{N}) \leq \mathrm{M}_\alpha(\mathbb{M})$.*

The proof of this result is in Appendix B.6. It would be interesting to find a geometric interpretation of this measure, in a similar manner that its two extremes admit a geometric interpretation as in (7.1) and (7.2), as well as to explore additional properties, like convexity, in order to talk about it being a resource quantifier. It would also be interesting to explore additional monotones, in particular, whether the isoelastic certainty equivalent forms a complete set of monotones for the simulability of measurements, this, given that this holds for the two extremes at plus and minus infinity.

Altogether, the above results establish a four-way task-mutual information-divergence-monotone correspondence for the QRT of measurement informativeness, by means of a risk aversion factor parametrised by the Rényi parameter α as $R(\alpha) = 1/\alpha$, as qualitatively depicted in Fig. 7.3.

7.4 Conclusions

In this chapter, we have proposed that using the ideas of betting, risk-aversion, and utility theory are a powerful way of extending the well studied tasks of quantum state discrimination and quantum state exclusion. We have used this to introduce various quantum operational tasks based on betting and risk-aversion, or *quantum betting tasks*. In particular, we have shown that this places two recently discovered four-way correspondences [204, 73] into a much broader continuous family of correspondences. For the first time, this shows that there exist deep connections between operational state identification tasks, mutual information measures, Rényi divergences, and resource monotones.

The seven main results in this manuscript are the following. First, we relate Arimoto's α-mutual information (in the quantum domain) to the quantum state betting games with risk, for the QRT of measurement informativeness. As corollaries

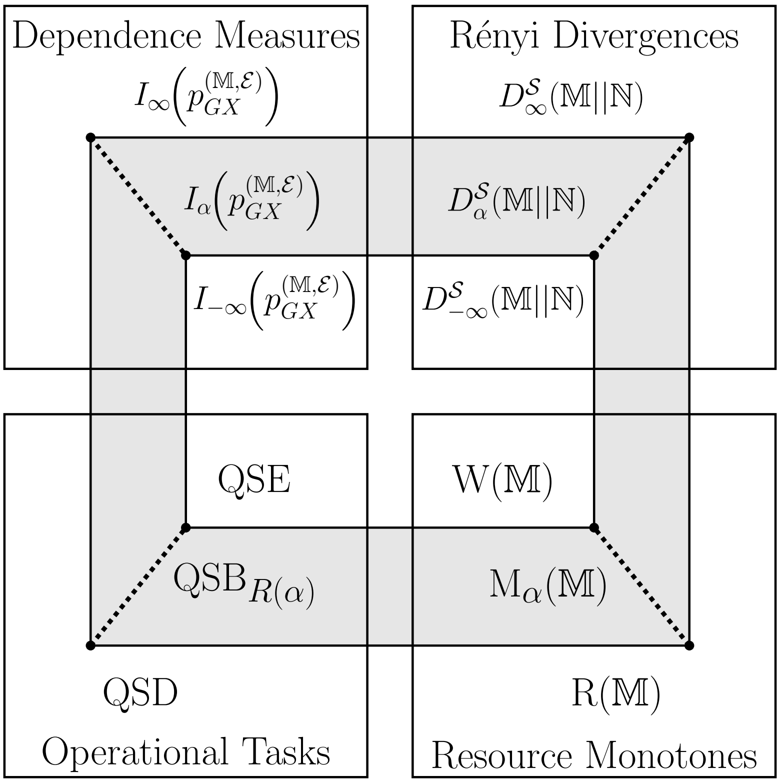

FIGURE 7.3: A four-way correspondence for the QRT of measurement informativeness. The correspondence is parametrised by the Rényi parameter $\alpha \in \mathbb{R} \cup \{\infty, -\infty\}$. The outer rectangle represents $\alpha = \infty$, the inner rectangle represents $\alpha = -\infty$, and the shaded region in-between represents the values $\alpha \in \mathbb{R}$. This four-way correspondence links: operational tasks, dependence measures, Rényi divergences, and resource monotones. The operational task is quantum state betting (QSB) played by a Gambler with risk aversion $R(\alpha) = 1/\alpha$. This task generalises quantum state discrimination (QSD) (recovered when $\alpha \to \infty$), and quantum state exclusion (QSE) (recovered when $\alpha \to -\infty$). $I_\alpha(p_{GX}^{(\mathbb{M},\mathcal{E})})$ is Arimoto's dependence measure, from which we recover the accessible information $I_\infty^{\mathrm{acc}}(\Lambda_{\mathbb{M}})$ when $\alpha \to \infty$, and the excludible information $I_{-\infty}^{\mathrm{exc}}(\Lambda_{\mathbb{M}})$ when $\alpha \to -\infty$, with $\Lambda_{\mathbb{M}}$ the measure-prepare channel of the measurement $\mathbb{M}$. We introduce $D_\alpha^{\mathcal{S}}(\mathbb{M}||\mathbb{N})$, the quantum Rényi divergence of two measurements $\mathbb{M}$ and $\mathbb{N}$ for a given set of states $\mathcal{S} = \{\rho_x\}$. We also introduce $\mathrm{M}_\alpha(\mathbb{M})$, a new family of resource monotones, which generalise the robustness of informativeness $\mathrm{R}(\mathbb{M})$ (when $\alpha \to \infty$) and the weight of informativeness $\mathrm{W}(\mathbb{M})$ (when $\alpha \to -\infty$). The outer rectangle was uncovered in [204], whilst the inner rectangle was first uncovered in [73]. The final set of results of this chapter is to fill the shaded region, and connect these two correspondences for all $\alpha \in \mathbb{R}$.

of this result, we recover the previous two known relationships relating: i) the accessible information to quantum state discrimination, and ii) the excludible information to quantum state exclusion. Second, we characterise nQSB games for the QRT of non-constant channels in terms of Arimoto's mutual information. Third, we consider a generalisation of the two previous scenarios to general QRTs of measurements with arbitrary resources (beyond that of informativeness) and QRTs of channels with general resources (beyond that of non-constant channels), and relate

QSB/nQSB games to Arimoto-type measures. Fourth, we address quantum channel betting (QCB) games, and consider this task for QRTs of states with arbitrary resources, as well as an hybrid scenario in a multi-object regime, addressing QRTs of state-measurements pairs, in which states and measurements are simultaneously considered in possession of valuable resources. Fifth, we relate Arimoto's mutual information to horse betting (HB) games with side information in the classical regime, without invoking quantum theory. This result can be seen as giving a very clean operational interpretation of Arimoto's mutual information, showing that it exactly quantifies the advantage provided by *side information*, and that the Rényi parameter can be understood operationally as quantifying the risk aversion of a gambler. Sixth, using the insights from the results on the QRT of measurement informativeness, we derive new quantum measured Rényi divergences for measurements. Seventh, we introduce resource monotones for the order generated by the simulability of measurements, which additionally recover the resource monotones of *generalised robustness*, as well as the *weight* of informativeness. Finally, results 1, 6, and 7 are elegantly connected via a four-way correspondence, which substantially extended the two correspondences previously uncovered [204, 73], which we now understand to be the two extremes of a continuous spectrum.

We believe our results are the start of a much broader and deeper investigation into the use of betting, risk-aversion, utility theory, and other ideas from economics, to obtain a broader unified understanding of many topics in quantum information theory. Our results raise many questions and open up various avenues for future research, a number of which we briefly describe below.

7.5 Open problems, perspectives, and avenues for future research

1. An exciting broad possibility, is to explore more generally the concept of risk aversion in quantum information theory. This is a concept which we are just starting to understand and incorporate into the theory of information and therefore, we believe this is an exciting avenue of research which could have far-reaching implications when considered for additional operational tasks, like Bell-nonlocal games, and interactive proof systems.

2. Similarly, the scenario here considered represents the convergence of three major research fields: i) quantum theory, ii) information theory and iii) the theory of games and economic behaviour. Specifically, we borrowed the concept of risk aversion from the economic sciences in order to solve an open problem in quantum information theory. We believe that this is just an example of the benefits that can be obtained from considering the cross-fertilisation of ideas between these three major current research fields. Consequently, it would be interesting to keep importing further concepts (in addition to risk aversion), as well as to explore the other direction, i. e., whether quantum information theory can provide insights into the theory of games and economic behaviour. We believe this can be a fruitful approach for future research. In particular, horse betting games are a particular family of a larger family of tasks which are related to the investment in portfolios [63], and it therefore would be interesting to explore quantum versions of the operational tasks that emerge in these scenarios.

3. The set of connections we have established here are by means of the Rényi entropies, and we have seen that the parameter α is intimately linked to the risk aversion of a gambler. It is interesting to speculate whether other types of connections might be possible. For example, Brandao [39] previously found a family of entanglement witnesses that encompassed both the generalised robustness and the weight of entanglement. We do not know if this is intimately related with our findings here, or whether our insights might shed further light, e.g. operational significance, on these entanglement witnesses and their generalisations.

4. We were led to introduce new measured quantum Rényi divergences for measurements. We believe that they should find relevance and application in settings far removed from the specific setting we considered here. It would also be interesting to further explore their relevance in other other areas within quantum information theory.

5. We have also introduced new resource monotones, for which we do not yet have a full understanding. In particular, unlike numerous other monotones, these do not yet have an obvious geometric interpretation. It would be interesting to develop such ideas further.

6. It would be interesting to explore additional monotones, in particular, whether the isoelastic certainty equivalent $w^{ICE}_{R(\alpha)}$ forms (for all α) a complete set of monotones for the order induced by the simulability of measurements, this, given that this is the case for the two extremes at $\alpha \in \{\infty, -\infty\}$ [204, 73].

7. We point out that we have used information-theoretic quantities with the Rényi parameter α taking both positive and negative values. Whilst negative values have been explored in the literature, it is fair to say that they have not been the main focus of attention. Here we have proven that information-theoretic quantities with negative orders posses a descriptive power *different* from their positive counterparts and therefore, it would be interesting to explore their usefulness in other information-theoretic scenarios.

Chapter 8

Conclusions and perspectives

This thesis deals with a resource-theoretic framework to quantum information theory, or quantum resource theories (QRTs) for short. In particular, it focuses on the identification of operational tasks and their characterisation in terms of information-theoretic quantities. The research is organised in results chapters from chapters four to seven (C4-C7), so we now address conclusions for each chapter as well as some perspectives from each one.

In chapter 4 we introduced the resource quantifier of *weight of resource* for general QRTs of states and measurements, each with arbitrary closed and convex resources. We proved that this measure quantifies the advantage that potentially resourceful obejcts offer, when compared against the best possible free objects, when such obejcts are used to play exclusion-based operational tasks. For QRTs of measurements the relevant operational task is quantum state exclusion, whilst for QRTs of states the relevant task is quantum subchannel exclusion. Furthermore, we proved that the weight of resource also characterises the information-theoretic quantity of Arimoto's gap of order $-\infty$ (a generalisation of Arimoto's mutual information) for general QRTs of measurements with arbitrary resources.

Various open questions for future research were identified in chapter 4. First, it would be interesting to determine whether the ratio of quantum subchannel exclusion games with *independent measurements* can still be characterised by means of the weight of resource, ideally for general resources, but at least for the case of entanglement. Second, it would also be interesting to explore whether there could exist an *information-theoretic corner for general QRTs of states*, in a similar vein as the case for QRTs of measurements.

In chapter 5 we introduced and developed a multi-object paradigm for composite QRTs. Specifically, we introduced the operational tasks of multi-object subchannel discrimination/exclusion, where the objects of interest are state-measurement pairs, and which was consequently explored for general QRTs of states and measurements, each with arbitrary closed convex resources. We proved that the advantage that a resourceful state-measurement pair offers at playing these tasks, when compared with the best fully free pairs, is completely characterised by the generalised robustness and the weight of resource, respectively, with both cases resulting in an elegant multiplicative characterisation. In this multi-object regime, the generalised robustness and weight place upper bounds for the information-theoretic quantity of Arimoto's gap of order $+\infty$ and $-\infty$, respectively.

Some ways to further develop the multi-object approach is as follows. First, it would be interesting to develop *additional multi-object tasks* beyond subchannel discrimination/exclusion. One potential candidate is to explore quantum ensemble discrimination in a multi-object manner [203], with the objects of interest being one state and one *set* of measurements [203]. Second, it would be interesting to explore the tightness of the upper bounds derived for Arimoto's gap.

In chapter 6 we introduce a QRT of Buscemi nonlocality. We derived an operational significance for this property in terms of discrimination games and, quantitatively related this property to both entanglement and non-classical teleportation.

It would be interesting to explore further quantum objects à la Buscemi. For instance, one natural next step is to develop a theory and further uses/consequences of an object which we can call a *Buscemi set of measurements*. Explicitly, given a bipartite POVM $\{M_a^{AB}\}$ and a set of states $\{\omega_x^B\}$, a Buscemi set of measurements will be the set of POVMs with elements given by $M^A_{a|\omega_x} := \mathrm{Tr}_B[M_a^{AB}(\mathbb{1}^A \otimes \omega_x^B)]$. It could be worthwhile to explore the sort of things these objects can do and, particularly, the type of things which cannot be achieved by their standard counterparts, i e., standard sets of measurements $\{M_{a|x}\}$.

In chapter 7 we imported ideas from the theory of games and economic behaviour and introduce operational tasks based on the concepts of *betting* and *risk-aversion*, or *quantum betting tasks* for short. We prove that these tasks generalise both discrimination and exclusion tasks, and prove that they can be characterised by information-theoretic quantities based on *Arimoto's mutual information*. In particular, we introduce the quantum betting tasks of: quantum state betting (QSB), noisy quantum state betting (nQSB), and quantum subchannel betting (QScB). We analysed these tasks from the point of view of general QRTs of measurements, channels, states, and state-measurement pairs, each with arbitrary resources. In the fully classical case in particular, we derived a clean and clear operational interpretation for Arimoto's mutual information. In short, it quantifies the usefulness of side information when this is available to a gambler playing the operational tasks known as horse betting. Finally, for the specific QRT of measurement informativeness, we derived a four-way correspondence between operational tasks, information-theoretic quantities, resource monotones, and Rényi divergences.

In general terms, it would be interesting to further explore more generally the concept of *betting* and *risk-aversion* in quantum information theory. It would also be interesting to keep importing ideas and concepts form economics into quantum information, as well as the other way around (from quantum info to economics). It would also be interesting to explore whether the four-way correspondence found for the specific QRT of measurement informativeness can be lifted to additional resources beyond informativeness, as well as additional objects beyond measurements.

Appendix A

Proofs of results on the QRT of Buscemi nonlocality

A.1 Equivalent formulation for the Robustness of Buscemi Nonlocality (RoBN)

By definition RoBN is a conic program. This means that we can use the tools of convex optimization theory to find its dual and from that obtain useful information about the primal problem. We will assume a knowledge of the tools of conic programming, and direct the interested reader to [38]. Let us start from the formulation given in the main text and substitute $\widetilde{N}_{ab}^{\mathrm{AB}} = rN_{ab}^{\mathrm{AB}}$ and $\widetilde{O}_{ab}^{\mathrm{AB}} = (1+r)O_{ab}^{\mathrm{AB}}$. After this substitution the primal problem can be written as:

$$\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \min \quad r \tag{A.1}$$

$$\text{s.t.} \quad M_{ab}^{\mathrm{AB}} + \widetilde{N}_{ab}^{\mathrm{AB}} = \widetilde{O}_{ab}^{\mathrm{AB}} \qquad \forall\, a, b, \tag{A.2}$$

$$\{\widetilde{O}_{ab}^{\mathrm{AB}}\} \in \mathcal{F}_{\mathrm{BN}}, \quad \{\widetilde{N}_{ab}^{\mathrm{AB}}\} \in \mathcal{R}_{\mathrm{BN}}, \tag{A.3}$$

where the optimization is performed over r, $\{\widetilde{N}_{ab}^{\mathrm{AB}}\}$ and $\{\widetilde{O}_{ab}^{\mathrm{AB}}\}$. Notice that any collection of operators inside $\mathcal{R}_{\mathrm{BN}}$ or $\mathcal{F}_{\mathrm{BN}} = \mathcal{F}_{\mathrm{SEP}} \cap \mathcal{R}_{\mathrm{BN}}$ satisfies its own "no-signalling" constraint which can be easily deduced from the definition of the set $\mathcal{R}_{\mathrm{BN}}$. Moreover, any operator in $\mathcal{F}_{\mathrm{BN}}$ is separable. In this way for any $\{X_{ab}^{\mathrm{AB}}\} \in \mathcal{F}_{\mathrm{BN}}$ we can write:

$$\sum_a X_{ab}^{\mathrm{AB}} = \mathbb{1}^{\mathrm{A}} \otimes X_b^{\mathrm{B}} \quad \forall\, b, \qquad \sum_b X_{ab}^{\mathrm{AB}} = X_a^{\mathrm{A}} \otimes \mathbb{1}^{\mathrm{B}} \quad \forall\, a, \qquad X_{ab}^{\mathrm{AB}} \in \mathsf{SEP}, \tag{A.4}$$

$$\sum_b X_b^{\mathrm{B}} = \mathbb{1}^{\mathrm{B}}, \qquad \sum_a X_a^{\mathrm{A}} = \mathbb{1}^{\mathrm{A}}. \tag{A.5}$$

Now we are going to add a family of such redundant constraints to our optimization problem. Note that we can always do that since adding constraints which are automatically satisfied by any operator in the feasible set does not change the optimal value of the program. Moreover, we can also relax the constraint (A.2) to an inequality $M_{ab}^{\mathrm{AB}} + \widetilde{N}_{ab}^{\mathrm{AB}} \leq \widetilde{O}^{\mathrm{AB}}$ without changing the optimal value of the conic program. To see why this is the case suppose we have solved the relaxed problem using variables r^{rel}, $\{\widetilde{N}_{ab}^{\mathrm{AB,rel}}\}$, $\{\widetilde{O}_{ab}^{\mathrm{AB,rel}}\}$ and $X_{ab}^{\mathrm{AB,rel}} \geq 0$ and such that for all a and b we have: $M_{ab}^{\mathrm{AB}} + \widetilde{N}_{ab}^{\mathrm{AB,rel}} = \widetilde{O}_{ab}^{\mathrm{AB,rel}} - X_{ab}^{\mathrm{AB,rel}}$. Then the optimal value of the relaxed program becomes:

$$\mathbf{R}^{\mathrm{rel}}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = -1 + \frac{1}{d^2}\sum_{ab} \operatorname{Tr} \widetilde{O}^{\mathrm{AB,rel}}_{a}, \tag{A.6}$$

$$= -1 + \frac{1}{d^2}\sum_{ab} \operatorname{Tr}\left[M^{\mathrm{AB}}_{ab} + \widetilde{N}^{\mathrm{AB,rel}}_{ab} + X^{\mathrm{AB,rel}}_{ab}\right], \tag{A.7}$$

$$\geq -1 + \frac{1}{d^2}\sum_{ab} \operatorname{Tr}\left[M^{\mathrm{AB}}_{ab} + \widetilde{N}^{\mathrm{AB,rel}}_{ab}\right], \tag{A.8}$$

$$\geq -1 + \frac{1}{d^2}\sum_{ab} \operatorname{Tr}\left[M^{\mathrm{AB}}_{ab} + \widetilde{N}'_{ab}\right], \tag{A.9}$$

$$\geq -1 + \frac{1}{d^2}\sum_{ab} \operatorname{Tr}\left[M^{\mathrm{AB}}_{ab} + \widetilde{N}^{\mathrm{AB}}_{ab}\right], \tag{A.10}$$

$$= \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}). \tag{A.11}$$

where $\{\widetilde{N}'_{ab}\}$ is a set of dual variables feasible for our initial problem A.1. In this way the conic program defining RoBN becomes:

$$\mathbf{R}_{\mathrm{BN}}(\mathbb{M}) = \min \quad r \tag{A.12}$$

$$\text{s.t.} \quad M^{\mathrm{AB}}_{ab} + \widetilde{N}^{\mathrm{AB}}_{ab} \leq \widetilde{O}^{\mathrm{AB}}_{ab} \quad \forall\, a, b, \tag{A.13}$$

$$\sum_{a} \widetilde{O}^{\mathrm{AB}}_{ab} = \mathbb{1}^{\mathrm{A}} \otimes \widetilde{O}^{\mathrm{B}}_{b} \quad \forall\, b, \qquad \sum_{b} \widetilde{O}^{\mathrm{B}}_{b} = (1+r)\mathbb{1}^{\mathrm{B}}, \tag{A.14}$$

$$\sum_{b} \widetilde{O}^{\mathrm{AB}}_{ab} = \widetilde{O}^{\mathrm{A}}_{a} \otimes \mathbb{1}^{\mathrm{B}} \quad \forall\, a, \qquad \sum_{a} \widetilde{O}^{\mathrm{A}}_{a} = (1+r)\mathbb{1}^{\mathrm{A}}, \tag{A.15}$$

$$\{\widetilde{O}^{\mathrm{AB}}_{ab}\} \in \mathcal{F}_{\mathrm{BN}} \quad \forall\, a, b, \qquad O^{\mathrm{AB}}_{ab} \in \mathsf{SEP} \quad \forall\, a, b, \tag{A.16}$$

$$\{\widetilde{N}^{\mathrm{AB}}_{ab}\} \in \mathcal{R}_{\mathrm{BN}} \quad \forall\, a, b, \tag{A.17}$$

where the minimization is performed over r, $\{\widetilde{O}^{\mathrm{AB}}_{ab}\}$, $\{\widetilde{O}^{\mathrm{A}}_{a}\}$, $\{\widetilde{O}^{\mathrm{B}}_{b}\}$ and $\{\widetilde{N}^{\mathrm{AB}}_{ab}\}$.

In what follows we will denote a dual cone to $\mathcal{R}$ using $\mathcal{R}^*$, that is $\mathcal{R}^* := \{X \mid \operatorname{Tr} XQ \geq 0 \text{ for all } Q \in \mathcal{R}\}$. We will now write the dual formulation of the above problem. To do so we first write the associated Lagrangian using dual Hermitian variables associated with a corresponding set of constraints: $\{A^{\mathrm{AB}}_{ab}\}$ such that $A^{\mathrm{AB}}_{ab} \geq 0$ for all a, b, $\{B^{\mathrm{AB}}_{b}\}$, $\{C^{\mathrm{AB}}_{a}\}$, $D^{\mathrm{A}} \geq 0$, $E^{\mathrm{B}} \geq 0$, $\{F^{\mathrm{AB}}_{ab}\} \in \mathcal{F}^*_{\mathrm{BN}}$ meaning that $\sum_{ab} \operatorname{Tr}[F^{\mathrm{AB}}_{ab} X^{\mathrm{AB}}_{ab}] \geq 0$ for all $\{X^{\mathrm{AB}}_{ab}\} \in \mathcal{F}_{\mathrm{BN}}$, $G^{\mathrm{AB}}_{ab} \in \mathcal{F}^*_{\mathrm{SEP}}$ for all a, b, meaning that $\operatorname{Tr}[G^{\mathrm{AB}}_{ab} X^{\mathrm{AB}}] \geq 0$ for all a, b and all separable operators $X^{\mathrm{AB}} \in \mathcal{F}_{\mathrm{SEP}}$ and, finally, $\{H^{\mathrm{AB}}_{ab}\} \in \mathcal{R}^*_{\mathrm{BN}}$. With this the

Lagrangian function of the conic program (A.12—A.16) becomes:

$$\mathcal{L} = r + \sum_{ab} \mathrm{Tr}\, A^{\mathrm{AB}}_{ab} \left[M^{\mathrm{AB}}_{ab} + \widetilde{N}^{\mathrm{AB}}_{ab} - \widetilde{O}^{\mathrm{AB}}_{ab} \right] + \sum_{b} \mathrm{Tr}\, B^{\mathrm{AB}}_{b} \left[\sum_{a} \widetilde{O}^{\mathrm{AB}}_{ab} - \mathbb{1}^{\mathrm{A}} \otimes \widetilde{O}^{\mathrm{B}}_{b} \right] \tag{A.18}$$

$$+ \sum_{a} \mathrm{Tr}\, C^{\mathrm{AB}}_{a} \left[\sum_{b} \widetilde{O}^{\mathrm{AB}}_{ab} - \widetilde{O}^{\mathrm{A}}_{a} \otimes \mathbb{1}^{\mathrm{B}} \right] + \mathrm{Tr}\, D^{\mathrm{A}} \left[\sum_{a} \widetilde{O}^{\mathrm{A}}_{a} - (1+r)\mathbb{1}^{\mathrm{A}} \right] \tag{A.19}$$

$$+ \mathrm{Tr}\, E^{\mathrm{B}} \left[\sum_{b} \widetilde{O}^{\mathrm{B}}_{b} - (1+r)\mathbb{1}^{\mathrm{B}} \right] - \sum_{a,b} \mathrm{Tr} \left[F^{\mathrm{AB}}_{ab} \widetilde{O}^{\mathrm{AB}}_{ab} \right] \tag{A.20}$$

$$- \sum_{a,b} \mathrm{Tr} \left[G^{\mathrm{AB}}_{ab} \widetilde{O}^{\mathrm{AB}}_{ab} \right] - \sum_{a,b} \mathrm{Tr} \left[H^{\mathrm{AB}}_{ab} \widetilde{N}^{\mathrm{AB}}_{ab} \right] \tag{A.21}$$

$$= r \cdot \left[1 - \mathrm{Tr}\, D^{\mathrm{A}} - \mathrm{Tr}\, E^{\mathrm{B}} \right] + \sum_{a,b} \mathrm{Tr}\, \widetilde{N}_{ab} \left[A^{\mathrm{AB}}_{ab} - H^{\mathrm{AB}}_{ab} \right] \tag{A.22}$$

$$+ \sum_{a,b} \mathrm{Tr}\, \widetilde{O}_{ab} \left[-A^{\mathrm{AB}}_{ab} + B^{\mathrm{AB}}_{b} + C^{\mathrm{AB}}_{a} - F^{\mathrm{AB}}_{ab} - G^{\mathrm{AB}}_{ab} \right] \tag{A.23}$$

$$+ \sum_{a} \mathrm{Tr}\, O^{\mathrm{A}}_{a} \left[D^{\mathrm{A}} - C^{\mathrm{A}}_{a} \right] + \sum_{b} \mathrm{Tr}\, O^{\mathrm{B}}_{b} \left[E^{\mathrm{B}} - B^{\mathrm{B}}_{b} \right] + \sum_{ab} \mathrm{Tr} \left[A^{\mathrm{AB}}_{ab} M^{\mathrm{AB}}_{ab} \right] \tag{A.24}$$

$$- \mathrm{Tr}\, D^{\mathrm{A}} - \mathrm{Tr}\, E^{\mathrm{B}}. \tag{A.25}$$

By demanding that the terms in the square brackets which appear along with the dual variables vanish we can ensure $\mathcal{L} \leq r$. This leads to the following (dual) conic program:

$$\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \max \quad \sum_{ab} \mathrm{Tr} \left[A^{\mathrm{AB}}_{ab} M^{\mathrm{AB}}_{ab} \right] - 1 \tag{A.26}$$

$$\text{s.t.} \quad C^{\mathrm{AB}}_{a} + B^{\mathrm{AB}}_{b} = A^{\mathrm{AB}}_{ab} + F^{\mathrm{AB}}_{ab} + G^{\mathrm{AB}}_{ab} \quad \forall\, a, b, \tag{A.27}$$

$$A^{\mathrm{AB}}_{ab} = H^{\mathrm{AB}}_{ab} \quad \forall\, a, b, \quad C^{\mathrm{A}}_{a} = D^{\mathrm{A}} \quad \forall\, a, \quad B^{\mathrm{B}}_{b} = E^{\mathrm{B}} \quad \forall\, b, \tag{A.28}$$

$$A^{\mathrm{AB}}_{ab} \geq 0 \quad \forall\, a, b, \quad \{H^{\mathrm{AB}}_{ab}\} \in \mathcal{R}^{*}_{\mathrm{BN}}, \quad \{F^{\mathrm{AB}}_{ab}\} \in \mathcal{F}^{*}_{\mathrm{BN}}, \tag{A.29}$$

$$\mathrm{Tr}\, D^{\mathrm{A}} + \mathrm{Tr}\, E^{\mathrm{B}} = 1.$$

Notice now that the set $\mathcal{F}_{\mathrm{BN}} \in \mathcal{F}_{\mathrm{SEP}}$, which implies that the dual sets satisfy $\mathcal{F}^{*}_{\mathrm{SEP}} \in \mathcal{F}^{*}_{\mathrm{BN}}$. Hence without loss of generality we can assume $G^{\mathrm{AB}}_{ab} = 0$ for all a and b. In this way we can express the above program in the following way:

$$1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \max \quad \sum_{ab} \mathrm{Tr} \left[A^{\mathrm{AB}}_{ab} M^{\mathrm{AB}}_{ab} \right] \tag{A.30}$$

$$\text{s.t.} \quad C^{\mathrm{AB}}_{a} + B^{\mathrm{AB}}_{b} - A^{\mathrm{AB}}_{ab} = F^{\mathrm{AB}}_{ab} \in \mathcal{F}^{*}_{\mathrm{BN}} \quad \forall\, a, b, \tag{A.31}$$

$$C^{\mathrm{A}}_{a} = D^{\mathrm{A}} \quad \forall\, a, \qquad C^{\mathrm{A}}_{a}, D^{\mathrm{A}} \geq 0 \quad \forall\, a, \tag{A.32}$$

$$A^{\mathrm{AB}}_{ab} \geq 0 \quad \forall\, a, b, \quad \mathrm{Tr}\, D^{\mathrm{A}} + \mathrm{Tr}\, E^{\mathrm{B}} = 1. \tag{A.33}$$

Using both primal (A.1) and dual (A.30) formulations we can now describe some basic properties of the RoBN.

A.2 Basic properties of the RoBN

Here we prove the three basic properties of RoBN highlighted in the main text.

Faithfulness If $\mathbb{M}^{AB} \in \mathcal{F}_{BN}$ then we can always choose a feasible $r = 0$ in the primal form (A.1). Since the solution is always non-negative, $r = 0$ is also optimal.

Convexity Let $\{N^1_{ab}, O^1_{ab}\}$ be optimal primal variables for $\mathbf{R}_{BN}(\mathbb{M}^1)$ and similarly let $\{N^2_{ab}, O^2_{ab}\}$ be primal-optimal for $\mathbf{R}_{BN}(\mathbb{M}^2)$. Define $\mathbb{M}' = \{M'_{ab}\}$ as a convex combination of the two measurements, that is $M'_{ab} = p\, M^1_{ab} + (1-p)\, M^2_{ab}$ for each a and b. We can construct a set of feasible variables for $\mathbf{R}_{BN}(\mathbb{M}')$ in the following way: $N'_{ab} = p\, N^1_{ab} + (1-p)\, N^2_{ab}$ and $O'_{ab} = p\, O^1_{ab} + (1-p)\, O^2_{ab}$. Substituting N'_{ab} and O'_{ab} into the constraints of the primal form for $\mathbf{R}_{BN}(\mathbb{M})$ shows that this choice is feasible. In this way we obtain an upper bound on $\mathbf{R}_{BN}(\mathbb{M}')$:

$$\begin{aligned}\mathbf{R}_{BN}(\mathbb{M}') \leq \operatorname{Tr}\sum_{a,b} N'_{ab} &= p \cdot \operatorname{Tr}\sum_{a,b} N^1_{ab} + (1-p) \cdot \operatorname{Tr}\sum_{a,b} N^2_{ab} \\ &= p \cdot \mathcal{R}(\mathbb{M}^1) + (1-p) \cdot \mathcal{R}(\mathbb{M}^2). \end{aligned} \tag{A.34}$$

Monotonicity Let us start with the assumption that there is a subroutine:

$$\mathcal{S} = \{p(\lambda), p(a|i,\lambda), p(b|j,\lambda), \mathcal{E}_\lambda, \mathcal{N}_\lambda\}$$

which allows to simulate $\mathbb{M}'$ using $\mathbb{M}$, i.e. $\mathbb{M} \succ_q \mathbb{M}'$. This means that the POVM elements $\{M_{ab}\}$ of $\mathbb{M}$ can be mapped into:

$$M'_{ab} = \sum_{i,j,\lambda} p(\lambda) p(a|i,\lambda) p(b|j,\lambda) (\mathcal{E}^\dagger_\lambda \otimes \mathcal{N}^\dagger_\lambda)[M_{ij}].$$

Suppose now that we solved the dual problem for $\mathbf{R}_{BN}(\mathbb{M}')$ using the optimal dual variables $\{A'_{ab}\}$, $\{B'_b\}$,$\{C'_a\}$, D', E' and $\{F'_{ab}\}$. Using these we construct an educated guess for $\mathbf{R}_{BN}(\mathbb{M})$ in the following way:

$$A^*_{ij} = \sum_{a,b,\lambda} p(\lambda)\, p(a|i,\lambda) p(b|j,\lambda)\ (\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)\, [A'_{ab}], \tag{A.35}$$

$$B^*_j = \sum_{b,\lambda} p(\lambda)\, p(b|j,\lambda)\ (\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)\, [B'_b], \tag{A.36}$$

$$C^*_i = \sum_{a,\lambda} p(\lambda)\, p(a|i,\lambda)\ (\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)\, [C'_a], \tag{A.37}$$

$$D^* = \sum_{\lambda} p(\lambda)\, \mathcal{E}_\lambda[D'], \tag{A.38}$$

$$E^* = \sum_{\lambda} p(\lambda)\, \mathcal{N}_\lambda[E'], \tag{A.39}$$

$$F^*_{ij} = \sum_{a,b,\lambda} p(\lambda)\, p(a|i,\lambda) p(b|j,\lambda)\ (\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)\, [F'_{ab}]. \tag{A.40}$$

It can be verified that the above choice of variables is feasible for the dual problem (A.30). In particular, notice that by construction we have $C^*_i + B^*_j - A^*_{ij} = F^*_{ij}$ for all i, j since the primed dual variables satisfy the constraints of (A.30). Furthermore, since $\operatorname{Tr}_B(\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)[X^{AB}] = \mathcal{E}_\lambda[X^A]$ we can infer that $\operatorname{Tr}_B C^*_i = D^*$ and $\operatorname{Tr}_A B^*_j = E^*$. Moreover, as separable maps preserve both positivity and separability we also have that $A^*_{ij} \geq 0$ for all i, j and $\{F^*_{ij}\} \in \mathcal{F}^*_{BN}$. Using the proposed set of dual variables we find the following lower bound:

$$1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}) \geq \sum_{i,j} \mathrm{Tr}[M_{ij} A^*_{ij}] \tag{A.41}$$
$$= \sum_{a,b,i,j,\lambda} p(\lambda)\, p(a|i,\lambda) p(b|j,\lambda)\ \mathrm{Tr}[M_{ij} \cdot (\mathcal{E}_\lambda \otimes \mathcal{N}_\lambda)\, [A_{ab}]] \tag{A.42}$$
$$= \sum_{a,b,i,j,\lambda} p(\lambda)\, p(a|i,\lambda) p(b|j,\lambda)\ \mathrm{Tr}[\left(\mathcal{E}^\dagger_\lambda \otimes \mathcal{N}^\dagger_\lambda\right) [M_{ij}] \cdot A_{ab}] \tag{A.43}$$
$$= \sum_{a,b} \mathrm{Tr}[M_{ab} A_{ab}] \tag{A.44}$$
$$= 1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}'). \tag{A.45}$$

This proves that RoBN is monotonic under quantum simulation.

A.3 Proof of Result 6.1

In this section we prove that RoBN can be seen as a quantifier of the advantage a given distributed measurement provides in the task of distributed state discrimination. To simplify notation in this section we shall omit subsystem labels whenever it is clear from the context. Let us recall that the average guessing probability in the task of distributed state discrimination using a distributed measurement $\mathbb{M}$ can be expressed as:

$$p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{M}) = \max_{\mathbb{N} \prec_q \mathbb{M}} \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[N_{ab}\, \sigma_{xy}\right] \delta_{xa} \delta_{yb}, \tag{A.46}$$

where the optimization ranges over all measurements $\mathbb{N} = \{N_{ab}\}$ which can be quantum-simulated using $\mathbb{M} = \{M_{ij}\}$, where

$$M_{ij} = \mathrm{Tr}_{\mathrm{AB}} \left[\left(M_i^{\mathrm{A'A}} \otimes M_j^{\mathrm{BB'}} \right) \left(\mathbb{1}^{\mathrm{A'}} \otimes \rho^{\mathrm{AB}} \otimes \mathbb{1}^{\mathrm{B'}} \right) \right] \tag{A.47}$$

is a distributed measurement and $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ is an ensemble of bipartite states. Suppose that we have solved the dual problem for RoBN (A.30) using the set of dual variables $\{A_{ab}\}$, $\{C_a\}$, $\{B_b\}$, D, E and $\{G_{ab}\}$. Notice also that due to the constraints in (A.30) the matrix A_{ab} is positive semi-definite for all values of a and b. Let us now consider a particular game setting $\mathcal{G}^* = \{p^*(x,y), \sigma^*_{xy}\}$ defined in the following way:

$$C = \sum_{x,y} \mathrm{Tr}\, A_{xy}, \qquad p^*(x,y) = \frac{\mathrm{Tr}\, A_{xy}}{C}, \qquad \sigma^*_{xy} = \frac{A_{xy}}{\mathrm{Tr}\, A_{xy}}, \tag{A.48}$$

where $x = 1, \ldots, o_{\mathrm{A}}$, $y = 1, \ldots, o_{\mathrm{B}}$ and o_{A}, o_{B} are the numbers of outcomes of local measurements performed by A and B. The best average guessing probability which can be achieved in the game $\mathcal{G}^*$ using a distributed measurement $\mathbb{M}$ is given by:

$$p_{\text{guess}}^{\text{DSD}}(\mathcal{G}^*, \mathbb{M}) = \max_{\mathbb{N} \prec_q \mathbb{M}} \sum_{a,b,x,y} p^*(x,y) \operatorname{Tr}\left[N_{ab}\, \sigma^*_{xy}\right] \delta_{xa}\delta_{yb} \tag{A.49}$$

$$\geq \sum_{x,y} \frac{\operatorname{Tr} A_{xy}}{C} \cdot \operatorname{Tr}\left[M_{xy} \frac{A_{xy}}{\operatorname{Tr} A_{xy}}\right] \tag{A.50}$$

$$= \frac{1}{C} \sum_{x,y} \operatorname{Tr}\left[M_{xy} A_{xy}\right] \tag{A.51}$$

$$= \frac{1}{C}\left[1 + \mathbf{R}_{\text{BN}}(\mathbb{M}^{\text{AB}})\right], \tag{A.52}$$

where the inequality in the second line we follows from choosing a particular subroutine $\mathcal{S}$ with $p(\lambda) = 1/|\lambda|$, $p(a|i,\lambda) = \delta_{ai}$, $p(b|j,\lambda) = \delta_{bj}$ and $\mathcal{E}_\lambda = \mathcal{N}_\lambda = \text{id}$. Let us now look at the corresponding classical (i.e. without access to entanglement) probability of guessing:

$$p_{\text{guess}}^{\text{DSD}}(\mathcal{G}^*)$$

$$= \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} p_{\text{guess}}^{\text{DSD}}(\mathcal{G}^*, \mathbb{N})$$

$$= \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} \sum_{x,y} p^*(x,y) \operatorname{Tr}\left[N_{xy}\, \sigma^*_{xy}\right] \tag{A.53}$$

$$= \frac{1}{C} \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} \sum_{x,y} \operatorname{Tr}\left[N_{xy} A_{xy}\right] \tag{A.54}$$

$$= \frac{1}{C} \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} \sum_{x,y} \operatorname{Tr}\left[N_{xy}(C_x + B_y - F_{xy})\right] \tag{A.55}$$

$$= \frac{1}{C} \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} \left(\sum_x \operatorname{Tr}\left[(N_x \otimes \mathbb{1})C_x\right] + \sum_y \operatorname{Tr}\left[(\mathbb{1} \otimes N_y)B_y\right] - \sum_{x,y} \operatorname{Tr}\left[N_{xy}F_{xy}\right] \right) \tag{A.56}$$

$$\leq \frac{1}{C} \max_{\mathbb{N} \in \mathcal{F}_{\text{BN}}} \left(\sum_x \operatorname{Tr}\left[N_x D\right] + \sum_y \operatorname{Tr}\left[N_y E\right] \right) \tag{A.57}$$

$$= \frac{1}{C}\left(\operatorname{Tr} D + \operatorname{Tr} E\right) \tag{A.58}$$

$$= \frac{1}{C}, \tag{A.59}$$

where the inequality follows since for all $\mathbb{N} \in \mathcal{F}_{\text{BN}}$ we have $\sum_{xy} \operatorname{Tr}[N_{xy}F_{xy}] \geq 0$. Combining bounds (A.52) and (A.59) leads to:

$$\max_{\mathcal{G}} \frac{p_{\text{guess}}^{\text{DSD}}(\mathcal{G}, \mathbb{M})}{p_{\text{class}}^{\text{DSD}}(\mathcal{G})} \geq \frac{p_{\text{guess}}^{\text{DSD}}(\mathcal{G}^*, \mathbb{M})}{p_{\text{class}}^{\text{DSD}}(\mathcal{G}^*)} \geq 1 + \mathbf{R}_{\text{BN}}(\mathbb{M}). \tag{A.60}$$

In order to prove the upper bound notice that the first line of constraints in the primal formulation for RoBN (A.1) implies:

$$\forall\, a, b \qquad M'_{ab} = \widetilde{O}'_{ab} - \widetilde{N}'_{ab}, \tag{A.61}$$

where $\widetilde{O}'_{ab} = [1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M})]\, O'_{ab}$ for all a, b and $\{O'_{ab}\} \in \mathcal{F}_{\mathrm{BN}}$. This allows to write:

$$
\begin{aligned}
& p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{M}) \\
&= \max_{\mathbb{M}' \prec_q \mathbb{M}} \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[M'_{ab}\, \sigma_{xy}\right] \delta_{xa}\delta_{yb} && \text{(A.62)} \\
&= \max_{\mathbb{M}' \prec_q \mathbb{M}} \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[(\widetilde{O}'_{ab} - \widetilde{N}'_{ab})\sigma_{xy}\right] \delta_{xa}\delta_{yb} && \text{(A.63)} \\
&\leq \max_{\mathbb{M}' \prec_q \mathbb{M}} \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[\widetilde{O}'_{ab}\sigma_{xy}\right] \delta_{xa}\delta_{yb} && \text{(A.64)} \\
&= \max_{\mathbb{M}' \prec_q \mathbb{M}} \left[1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}')\right] \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[O'_{ab}\sigma_{xy}\right] \delta_{xa}\delta_{yb} && \text{(A.65)} \\
&\leq \left(\max_{\mathbb{M}' \prec_q \mathbb{M}} \left[1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}')\right] \right) \left(\max_{\{O_{ab}\} \in \mathcal{F}_{\mathrm{BN}}} \sum_{a,b,x,y} p(x,y)\ \mathrm{Tr}\left[O_{ab}\sigma_{xy}\right] \delta_{xa}\delta_{yb} \right) && \text{(A.66)} \\
&\leq \left[1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M})\right] p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}), && \text{(A.67)}
\end{aligned}
$$

where the last inequality follows from the monotonicity of RoBN under quantum simulation. Combining bounds (A.60) and (A.67) yields:

$$
\max_{\mathcal{G}} \frac{p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G}, \mathbb{M})}{p^{\mathrm{DSD}}_{\mathrm{guess}}(\mathcal{G})} = 1 + \mathbf{R}_{\mathrm{BN}}(\mathbb{M}). \tag{A.68}
$$

A.4 Proof of Result 6.2

Before proving the result we recall the primal and dual formulation of the RoT quantifier. Let $\Lambda = \{\Lambda_a\}$ be a teleportation instrument whose elements are defined as:

$$
\Lambda_a^{\mathrm{A} \to \mathrm{B}'}[\omega] := \mathrm{Tr}_{\mathrm{AA}'}\left[\left(M_a^{\mathrm{AA}'} \otimes \mathbb{1}^{\mathrm{B}} \right) \left(\omega^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \right) \right], \tag{A.69}
$$

for some measurement $M_a^{\mathrm{AA}'}$ and a shared state $\rho^{\mathrm{A}'\mathrm{B}'}$. We denote the set of Choi-Jamiolkowski states corresponding to this of these subchannels with $\{J_a^{\mathrm{VB}'}\}$, i.e. each $J_a^{\mathrm{VB}'} := (\mathrm{id}^{\mathrm{V}} \otimes \Lambda_a^{\mathrm{A}\to\mathrm{B}'})[\phi_+^{\mathrm{VA}}]$ with system V isomorphic to A. With these definitions RoT for a teleportation instrument $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$ can be written as:

$$
\begin{aligned}
\min \quad & \mathrm{Tr}\, \widetilde{\sigma}^{\mathrm{B}'}, & \iff \quad \max \quad & \sum_a \mathrm{Tr}\left[A_a^{\mathrm{VB}'} J_a^{\mathrm{VB}'} \right] - 1, \\
\text{s.t.} \quad & J_a^{\mathrm{VB}'} \leq F_a^{\mathrm{VB}'} \quad \forall a, & \text{s.t.} \quad & B^{\mathrm{VB}'} - A_a^{\mathrm{VB}'} = W_a^{\mathrm{VB}'} \in \mathcal{F}^*_{\mathrm{SEP}} \quad \forall a, \\
& \sum_a F_a^{\mathrm{VB}'} = \frac{\mathbb{1}^{\mathrm{V}}}{d} \otimes \widetilde{\sigma}^{\mathrm{B}'}, & & B^{\mathrm{B}'} = \mathbb{1}^{\mathrm{B}'}, \quad A_a^{\mathrm{VB}'} \geq 0 \quad \forall a. \\
& F_a^{\mathrm{VB}'} \in \mathcal{F}_{\mathrm{SEP}} \quad \forall a, \quad \widetilde{\sigma}^{\mathrm{B}'} \geq 0. & &
\end{aligned}
\tag{A.70}
$$

Let us now proceed with the proof of Result 6.2.

Proof. As before, the proof consists of two steps. First we will show that $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$ lower bounds $\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}})$ for a particular choice of local measurement $\mathbb{M}^{\mathrm{B}'\mathrm{B}}$. Then we will show that for any choice of local measurements on Bob's side $\mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}})$ is never larger than the teleportation quantifier $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$.

Let $A_a^{\mathrm{VB}'} \geq 0$, $W_a^{\mathrm{VB}'} \in \mathcal{F}_{\mathrm{SEP}}^*$ and $B^{\mathrm{VB}'}$ be optimal dual variables for $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$. Let $\{U_b^{\mathrm{B}}\}$ for $b \in \{1,\ldots,d^2\}$ be a set of Pauli operators with respect to a basis $\{|i\rangle^{\mathrm{B}}\}$. Consider the following measurement with $o_{\mathrm{B}} = d^2$ outcomes:

$$M_b^{\mathrm{B}'\mathrm{B}} = (\mathrm{id}^{\mathrm{B}'} \otimes \mathcal{U}_b^{\mathrm{B}})[\phi_+^{\mathrm{B}'\mathrm{B}}], \tag{A.71}$$

where $\mathcal{U}_b[\cdot] := U_b(\cdot)U_b^\dagger$. We are interested in the lower bound for $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})$. Let us choose a set of dual variables in (A.30) inspired by the optimal dual variables for $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{V}\to\mathrm{B}'})$:

$$A_{ab}^{\mathrm{AB}} = (\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(A_a^{\mathrm{AB}})^T], \tag{A.72}$$
$$F_{ab}^{\mathrm{AB}} = (\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(W_a^{\mathrm{AB}})^T], \tag{A.73}$$
$$B_b^{\mathrm{AB}} = \frac{1}{d}(\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(B^{\mathrm{AB}})^T], \tag{A.74}$$
$$C_a^{\mathrm{AB}} = 0, \qquad D^{\mathrm{B}} = \frac{1}{d}\mathbb{1}^{\mathrm{B}}, \qquad E^{\mathrm{A}} = 0. \tag{A.75}$$

It can be verified by direct substitution that the above choice is feasible. In particular, the above choice for $\{F_{ab}^{\mathrm{AB}}\}$ is feasible as $\mathcal{F}_{\mathrm{SEP}}^* \in \mathcal{F}_{\mathrm{BN}}^*$ and both sets are invariant under local unitaries. This leads to the following chain of inequalities:

$$
\begin{aligned}
&1 + \max_{\mathbb{M}^{\mathrm{B}}} \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) \\
&\geq \sum_{ab} \mathrm{Tr}[A_{ab}^{\mathrm{AB}} M_{ab}^{\mathrm{AB}}] && \text{(A.76)}\\
&= \sum_{ab} \mathrm{Tr}\left[(\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(A_a^{\mathrm{AB}})^T] \cdot \mathrm{Tr}_{\mathrm{A}'\mathrm{B}'}\left[(M_a^{\mathrm{AA}'} \otimes M_b^{\mathrm{B}'\mathrm{B}})(\mathbb{1}^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}})\right]\right] && \text{(A.77)}\\
&= \sum_{ab} \mathrm{Tr}\left[(\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(A_a^{\mathrm{AB}})^T] \cdot \mathrm{Tr}_{\mathrm{A}'\mathrm{B}'}\left[(M_a^{\mathrm{AA}'} \otimes (\mathrm{id}^{\mathrm{B}'} \otimes \mathcal{U}_b^{\mathrm{B}})[\phi_+^{\mathrm{B}'\mathrm{B}}])(\mathbb{1}^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}})\right]\right] \\
& && \text{(A.78)}\\
&= \sum_{ab} \mathrm{Tr}\left[\mathbb{1}^{\mathrm{A}'\mathrm{B}'} \otimes (\mathrm{id}^{\mathrm{A}} \otimes (\mathcal{U}_b^\dagger)^{\mathrm{B}})[(A_a^{\mathrm{AB}})^T] \cdot \left[(M_a^{\mathrm{AA}'} \otimes (\mathrm{id}^{\mathrm{B}'} \otimes \mathcal{U}_b^{\mathrm{B}})[\phi_+^{\mathrm{B}'\mathrm{B}}])(\mathbb{1}^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}})\right]\right] \\
& && \text{(A.79)}\\
&= \sum_{ab} \mathrm{Tr}\left[\left(\mathbb{1}^{\mathrm{A}'\mathrm{B}'} \otimes (A_a^{\mathrm{AB}})^T\right)\left(M_a^{\mathrm{AA}'} \otimes \phi_+^{\mathrm{B}'\mathrm{B}}\right)\left(\mathbb{1}^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right] && \text{(A.80)}\\
&= \frac{1}{d^2}\sum_{ab} \mathrm{Tr}\left[A_a^{\mathrm{VB}'} \cdot \mathrm{Tr}_{\mathrm{AA}'}\left[(\mathbb{1}^{\mathrm{V}} \otimes M_a^{\mathrm{AA}'} \otimes \mathbb{1}^{\mathrm{B}'})(\phi_+^{\mathrm{VA}} \otimes \rho^{\mathrm{A}'\mathrm{B}'})\right]\right] && \text{(A.81)}\\
&= \sum_{a} \mathrm{Tr}[A_a^{\mathrm{VB}'} J_a^{\mathrm{VB}'}] && \text{(A.82)}\\
&= 1 + \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{V}\to\mathrm{B}'}). && \text{(A.83)}
\end{aligned}
$$

We now prove the upper bound. Notice that for any distributed measurement $\mathbb{M}^{\mathrm{AB}}$ we can construct $\mathbb{M}^{\mathrm{VB}} := \{M_{ab}^{\mathrm{VB}}\}$ such that $M_{ab}^{\mathrm{VB}} := d\ \mathrm{Tr}_{\mathrm{A}}[(\mathbb{1}^{\mathrm{V}} \otimes M_{ab}^{\mathrm{AB}})(\phi_+^{\mathrm{VA}} \otimes \mathbb{1}^{\mathrm{B}})]$. This in turn can be written as:

$$M_{ab}^{\mathrm{VB}} := d\ \mathrm{Tr}_{\mathrm{AA}'\mathrm{B}'}\left[\left(\mathbb{1}^{\mathrm{A}} \otimes M_a^{\mathrm{AA}'} \otimes M_b^{\mathrm{B}'\mathrm{B}}\right)\left(\phi_+^{\mathrm{VA}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right] \tag{A.84}$$
$$= d\ \mathrm{Tr}_{\mathrm{B}'}\left[\left(\mathbb{1}^{\mathrm{V}} \otimes M_b^{\mathrm{B}'\mathrm{B}}\right)\left(J_a^{\mathrm{VB}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right]. \tag{A.85}$$

Note that we can always write $J_a^{\mathrm{VB}'} \leq [1 + \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]F_a^{\mathrm{VB}'}$, where $\{F_a^{\mathrm{VB}'}\}$ are Choi-Jamiolkowski operators of some classical teleportation instrument. This allows us to further rewrite (A.85) as:

$$M_{ab}^{\mathrm{VB}} \leq d\,[1+\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]\ \mathrm{Tr}_{\mathrm{A}}[(\mathbb{1}^{\mathrm{V}}\otimes M_{ab}^{\mathrm{AB}})(\phi_{+}^{\mathrm{VA}}\otimes\mathbb{1}^{\mathrm{B}})] = [1+\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]\,O_{ab}^{\mathrm{VB}}. \tag{A.86}$$

Where $\{O_{ab}^{\mathrm{VB}}\}$ is a free distributed measurement. Hence also $M_{ab}^{\mathrm{AB}} \leq [1+\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]$ O_{ab}^{AB} for some free distributed measurement $\{O_{ab}^{\mathrm{AB}}\}$. This finally allows us to write:

$$\max_{\mathbb{M}^{\mathrm{B}}} \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) \leq [1+\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]\ \max_{\mathbb{M}^{\mathrm{B}}} \sum_{ab} \mathrm{Tr}[A_{ab}^{\mathrm{AB}}O_{ab}^{\mathrm{AB}}] \leq [1+\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})]. \tag{A.87}$$

This proves the lemma. □

A.5 Proof of Result 6.4

Let us recall that the conic program formulation of RoE is given by:

$$\begin{aligned}
\mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A'B'}}) = \min \quad & \mathrm{Tr}\,\widetilde{\sigma}^{\mathrm{A'B'}}, & \iff \qquad \max \quad & \sum_a \mathrm{Tr}\left[A^{\mathrm{A'B'}}\rho^{\mathrm{A'B'}}\right] - 1,\\
\text{s.t.} \quad & \rho^{\mathrm{A'B'}} \leq \widetilde{\sigma}^{\mathrm{A'B'}} & \text{s.t.} \quad & \mathbb{1}^{\mathrm{A'B'}} - A^{\mathrm{A'B'}} = W^{\mathrm{A'B'}} \in \mathcal{F}_{\mathrm{SEP}}^{*},\\
& \widetilde{\sigma}^{\mathrm{A'B'}} \in \mathcal{F}_{\mathrm{SEP}}. & & A^{\mathrm{A'B'}} \geq 0.
\end{aligned} \tag{A.88}$$

The proof is based on three parts. First we use Result 6.2 to connect RoBN with RoT. Then we essentially parallel the steps taken in the proof of Result 6.2 to link RoT with RoE. It is worth mentioning that the link between RoT and RoE has already been obtained some time ago in [51]. Here for convenience we state an independent proof.

Proof. Let us begin by noting that Result 6.2 implies:

$$\max_{\mathbb{M}^{\mathrm{A}},\mathbb{M}^{\mathrm{B}}} \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}}) = \max_{\mathbb{M}^{\mathrm{A}}}\left[\max_{\mathbb{M}^{\mathrm{B}}} \mathbf{R}_{\mathrm{BN}}(\mathbb{M}^{\mathrm{AB}})\right] = \max_{\mathbb{M}^{\mathrm{A}}} \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'}). \tag{A.89}$$

Let $A^{\mathrm{A'B'}} \geq 0$, $W^{\mathrm{A'B'}} \in \mathcal{F}_{\mathrm{SEP}}^{*}$ be optimal dual variables for $\mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A'B'}})$. Let $\{U_a^{\mathrm{A'}}\}$ for $a \in \{1,\ldots,d^2\}$ be a set of Pauli operators with respect to a basis $\{|i\rangle^{\mathrm{A'}}\}$. Consider the following measurement with $o_{\mathrm{A}} = d^2$ outcomes:

$$M_a^{\mathrm{AA'}} = (\mathrm{id}^{\mathrm{A}} \otimes \mathcal{U}_a^{\mathrm{A'}})[\phi_{+}^{\mathrm{AA'}}]. \tag{A.90}$$

We are interested in the lower bound for $\mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A'B'}})$, let us construct a set of (potentially sub-optimal) dual variables in the maximization (A.88) using the optimal set of dual variables for $\mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{V}\to\mathrm{B}'})$, i.e.:

$$A_a^{\mathrm{VB'}} = ((\mathcal{U}_a^{\dagger})^{\mathrm{V}} \otimes \mathrm{id}^{\mathrm{B'}})[A^{\mathrm{VB'}}], \quad W_a^{\mathrm{VB'}} = ((\mathcal{U}_a^{\dagger})^{\mathrm{V}} \otimes \mathrm{id}^{\mathrm{B'}})[(W^{\mathrm{VB'}}], \quad B^{\mathrm{VB'}} = \frac{1}{d}\mathbb{1}^{\mathrm{VB'}}. \tag{A.91}$$

It can be verified by direct substitution that the above choice is feasible. This leads to the following chain of inequalities:

$$
\begin{aligned}
&1 + \max_{\mathbb{M}^{\mathrm{A}}} \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'}) \\
&\geq \sum_a \mathrm{Tr}[A_a^{\mathrm{VB}'} J_a^{\mathrm{VB}'}] && \text{(A.92)} \\
&= \sum_a \mathrm{Tr}\left[((\mathcal{U}_a^\dagger)^{\mathrm{V}} \otimes \mathrm{id}^{\mathrm{B}'})[A_a^{\mathrm{VB}'}] \cdot \mathrm{Tr}_{\mathrm{AA}'}\left[(\mathbb{1}^{\mathrm{V}} \otimes M_a^{\mathrm{AA}'} \otimes \mathbb{1}^{\mathrm{B}'})(\phi_+^{\mathrm{VA}} \otimes \rho^{\mathrm{A}'\mathrm{B}'})\right]\right] && \text{(A.93)} \\
&= \sum_a \mathrm{Tr}\left[((\mathcal{U}_a^\dagger)^{\mathrm{V}} \otimes \mathrm{id}^{\mathrm{B}'})[A_a^{\mathrm{VB}'}] \cdot \mathrm{Tr}_{\mathrm{AA}'}\left[(\mathbb{1}^{\mathrm{V}} \otimes (\mathrm{id}^{\mathrm{A}} \otimes \mathcal{U}_a^{\mathrm{A}'})[\phi_+^{\mathrm{AA}'}] \otimes \mathbb{1}^{\mathrm{B}'})(\phi_+^{\mathrm{VA}} \otimes \rho^{\mathrm{A}'\mathrm{B}'})\right]\right] \\
& && \text{(A.94)} \\
&= \sum_a \mathrm{Tr}\left[\mathbb{1}^{\mathrm{AA}'} \otimes ((\mathcal{U}_a^\dagger)^{\mathrm{V}} \otimes \mathrm{id}^{\mathrm{B}'})[A_a^{\mathrm{VB}'}] \cdot (\mathbb{1}^{\mathrm{V}} \otimes (\mathrm{id}^{\mathrm{A}} \otimes \mathcal{U}_a^{\mathrm{A}'})[\phi_+^{\mathrm{AA}'}] \otimes \mathbb{1}^{\mathrm{B}'})(\phi_+^{\mathrm{VA}} \otimes \rho^{\mathrm{A}'\mathrm{B}'})\right] \\
& && \text{(A.95)} \\
&= \frac{1}{d^2} \sum_a \mathrm{Tr}[((\mathcal{U}_a^\dagger)^{\mathrm{A}'} \otimes \mathrm{id}^{\mathrm{B}'}) A^{\mathrm{A}'\mathrm{B}'} \cdot (\mathcal{U}_a^{\mathrm{A}'} \otimes \mathrm{id}^{\mathrm{B}'}) \rho^{\mathrm{A}'\mathrm{B}'}] && \text{(A.96)} \\
&= \mathrm{Tr}[A^{\mathrm{A}'\mathrm{B}'} \rho^{\mathrm{A}'\mathrm{B}'}] && \text{(A.97)} \\
&= 1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A}'\mathrm{B}'}). && \text{(A.98)}
\end{aligned}
$$

We now prove the upper bound. Notice that any teleportation instrument $\Lambda^{\mathrm{A}\to\mathrm{B}'}$ expressed using Choi-Jamiolkowski operators $\{J_a^{\mathrm{VB}'}\}$ satisfies:

$$
\begin{aligned}
J_a^{\mathrm{VB}'} &:= \mathrm{Tr}_{\mathrm{VA}}\left[\left(M_a^{\mathrm{VA}} \otimes \mathbb{1}^{\mathrm{B}'}\right)\left(\phi_+^{\mathrm{A}} \otimes \rho^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right] && \text{(A.99)} \\
&\leq [1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A}'\mathrm{B}'})]\, \mathrm{Tr}_{\mathrm{VA}}\left[\left(M_a^{\mathrm{VA}} \otimes \mathbb{1}^{\mathrm{B}'}\right)\left(\phi_+^{\mathrm{A}} \otimes \sigma^{\mathrm{A}'\mathrm{B}'} \otimes \mathbb{1}^{\mathrm{B}}\right)\right] && \text{(A.100)} \\
&= [1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A}'\mathrm{B}'})] O_a^{\mathrm{VB}'}, && \text{(A.101)}
\end{aligned}
$$

for some state $\sigma^{\mathrm{A}'\mathrm{B}'} \in \mathcal{F}_{\mathrm{SEP}}$ and corresponding (classical) teleportation operators $\{O_a^{\mathrm{VB}'}\}$. In this way we can write:

$$
\begin{aligned}
\max_{\mathbb{M}^{\mathrm{A}}} [1 + \mathbf{R}_{\mathrm{T}}(\Lambda^{\mathrm{A}\to\mathrm{B}'})] &= \max_{\mathbb{M}^{\mathrm{A}}} \max_{\{A_a^{\mathrm{VB}'}\}} \sum_a \mathrm{Tr}[A_a^{\mathrm{VB}'} J_a^{\mathrm{VB}'}] && \text{(A.102)} \\
&\leq [1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A}'\mathrm{B}'})] \sum_a \mathrm{Tr}[A_a^{\mathrm{VB}'} O_a^{\mathrm{VB}'}] && \text{(A.103)} \\
&\leq [1 + \mathbf{R}_{\mathrm{E}}(\rho^{\mathrm{A}'\mathrm{B}'})]. && \text{(A.104)}
\end{aligned}
$$

This proves the claim. □

A.6 Proof of Result 6.6

In this section, unless explicitly specified, all bipartite operators act on subsystems A and B. We begin by assuming that a distributed measurement $\mathbb{M}$ can be used to simulate $\mathbb{M}^*$, that is $\mathbb{M} \succ_q \mathbb{M}^*$. We have:

$$
\begin{aligned}
p_{\mathrm{guess}}^{\mathrm{DSD}}(\mathcal{G}, \mathbb{M}) &= \max_{\mathbb{M} \succ_q \mathbb{M}'} \sum_{a,b} p(a,b)\ \mathrm{Tr}\left[M'_{ab}\, \sigma_{ab}\right] && \text{(A.105)} \\
&\geq \max_{\mathbb{M}^* \succ_q \mathbb{M}'} \sum_{a,b} p(a,b)\ \mathrm{Tr}\left[M'_{ab}\, \sigma_{ab}\right] && \text{(A.106)} \\
&= p_{\mathrm{guess}}^{\mathrm{DSD}}(\mathcal{G}, \mathbb{M}^*), && \text{(A.107)}
\end{aligned}
$$

since the set $\{\mathbb{M}'|\mathbb{M}^* \succ_q \mathbb{M}'\}$ is a subset of $\{\mathbb{M}'|\mathbb{M} \succ_q \mathbb{M}'\}$. Now we are going to assume that $p_{\text{guess}}^{\text{DSD}}(\mathcal{G},\mathbb{M}) \geq p_{\text{guess}}^{\text{DSD}}(\mathcal{G},\mathbb{M}^*)$ holds for all games $\mathcal{G} = \{p(x,y), \sigma_{xy}\}$ and show show that there always exist a subroutine $\mathcal{S}$ which allows to simulate $\mathbb{M}^*$ using $\mathbb{M}$. We thus have:

$$\forall\, \mathcal{G} \qquad \max_{\mathbb{M}' \prec_q \mathbb{M}} \sum_{a,b} p(a,b) \operatorname{Tr}\left[M'_{ab}\, \sigma_{ab}\right] - \max_{\mathbb{M}'' \prec_q \mathbb{M}^*} \sum_{a,b} p(a,b) \operatorname{Tr}\left[M''_{ab}\, \sigma_{ab}\right] \geq 0. \tag{A.108}$$

Let us now choose a particular subroutine in the second maximization, i.e.: $\mathcal{S}^* = \{p(\lambda) = \delta_{\lambda 0},\, p(a|i,\lambda) = \delta_{ai},\, p(b|j,\lambda) = \delta_{bj},\, U_\lambda = V_\lambda = \mathbb{1}\}$. In this way (A.108) implies:

$$\forall\, \mathcal{G} \qquad \max_{\mathbb{M}' \prec_q \mathbb{M}} \sum_{a,b} p(a,b) \operatorname{Tr}\left[(M'_{ab} - M^*_{ab})\, \sigma_{ab}\right] \geq 0. \tag{A.109}$$

Let us denote $\Delta_{ab} := M'_{ab} - M^*_{ab}$. Since both M'_{ab} and M^*_{ab} are measurements we have that $\sum_{a,b} \Delta_{ab} = 0$. This also means that only one of the two situations can hold: either (i) $\Delta_{ab} = 0$ for all a, b or (ii) there exists at least one Δ_{ab} with at least one negative eigenvalue.

We will now show by contradiction that (ii) cannot be true. Let us assume that (ii) holds and label the negative eigenvalue with $\lambda_{a^*b^*}$ and the associated eigenvector with $|\lambda_{a^*b^*}\rangle$. Then, since (A.109) holds for all games $\mathcal{G}$, it also holds for a particular game $\mathcal{G}^* = \{p(a,b) = \delta_{aa^*}\delta_{bb^*}, \sigma_{ab} = |\lambda_{a^*b^*}\rangle\langle\lambda_{a^*b^*}|\}$. Hence (A.109) implies:

$$\langle \lambda_{a^*b^*}|\, \Delta_{a^*b^*}\, |\lambda_{a^*b^*}\rangle = \lambda_{a^*b^*} < 0, \tag{A.110}$$

which is a contradiction. Hence we infer that (ii) cannot be true and the only possibility is that each operator Δ_{ab} is identically zero. This means that:

$$M^*_{ab} = M'_{ab} := \sum_{i,j,\lambda} p(\lambda) p(a|i,\lambda) p(b|j,\lambda) (U^\dagger_\lambda \otimes V^\dagger_\lambda) M_{ij} (U_\lambda \otimes V_\lambda), \tag{A.111}$$

i.e. $\mathbb{M}^*$ can be simulated using $\mathbb{M}$.

A.7 Proof of Result 6.7

The accessible min-information $I^{\text{acc}}_{+\infty}(\mathcal{N})$ of a channel $\mathcal{N}$ is defined as [243]:

$$I^{\text{acc}}_{+\infty}(\mathcal{N}) = \max_{\mathcal{E},\mathcal{D}} \left[H_{+\infty}(X) - H_{+\infty}(X|G)\right], \tag{A.112}$$

where the optimization is over all encodings $\mathcal{E} = \{p(x), \sigma_x\}$ and decodings $\mathcal{D} = \{D_g\}$ and the min-entropies are defined as:

$$H_{+\infty}(X) = -\log \max_x p(x), \tag{A.113}$$

$$H_{+\infty}(X|G) = -\log\left[\sum_g \max_x p(x,g)\right], \tag{A.114}$$

and $p(x,g)$ is the probability distribution induced by channel $\mathcal{N}$, i.e.:

$$p(x,g) = p(x)p(g|x) = p(x) \operatorname{Tr}\left[\mathcal{N}[\sigma_x] D_g\right]. \tag{A.115}$$

Consider now encoding a bipartite random variable $X \times Y$ in an ensemble of bipartite quantum states, i.e.: $\mathcal{E} = \{p(x,y), \sigma_{xy}^{\mathrm{AB}}\}$ and $\mathcal{D} = \{D_g^{\mathrm{A'B'}}\}$ for $g = 1, \ldots, o_{\mathrm{A}} \cdot o_{\mathrm{B}}$. Moreover, consider the channel $\mathcal{N} = \mathcal{N}^{\mathrm{AB}\to\mathrm{A'B'}}$ to be a quantum-to-classical measurement channel, which can be written as:

$$\mathcal{N}^{\mathrm{AB}\to\mathrm{A'B'}}(\rho^{\mathrm{AB}}) = \sum_{a,b} \mathrm{Tr}[M_{ab}^{\mathrm{AB}}\rho^{\mathrm{AB}}]\, |a\rangle\langle a|_{\mathrm{A'}} \otimes |b\rangle\langle b|_{\mathrm{B'}}\,, \tag{A.116}$$

where $\mathbb{M} = \{M_{ab}^{\mathrm{AB}}\}$ is a distributed measurement. We have:

$$\begin{aligned}
&I_{+\infty}^{\mathrm{acc}}(\mathcal{N}^{\mathrm{AB}\to\mathrm{A'B'}}) \\
&= \max_{\mathcal{E},\mathcal{D}} \log\left[\sum_g \max_{x,y} p(x,y)\,\mathrm{Tr}[\mathcal{N}^{\mathrm{AB}\to\mathrm{A'B'}}[\sigma_{xy}^{\mathrm{AB}}]D_g^{\mathrm{A'B'}}]\right] - \log\max_{a,b} p(a,b) \qquad \text{(A.117)}\\
&= \max_{\mathcal{E},\mathcal{D}} \log\left[\sum_g \sum_{a,b} \max_{x,y} p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}]\,\mathrm{Tr}[D_g^{\mathrm{A'B'}}\,|a\rangle\langle a|_{\mathrm{A'}}\otimes|b\rangle\langle b|_{\mathrm{B'}}]\right]\\
&- \log\max_{a,b} p(a,b) \qquad \text{(A.118)}\\
&= \log\left[\sum_{a,b} \max_{\mathcal{E}} \max_{x,y} p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}]\right] - \log\max_{a,b} p(a,b). \qquad \text{(A.119)}
\end{aligned}$$

Notice now that we can always express the optimization over (x,y) as:

$$\begin{aligned}
&\max_{x,y} p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}]\\
&= \max_{p(x|a)} \max_{p(y|b)} \sum_{x,y} p(x|a)p(y|b)p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}] \qquad \text{(A.120)}\\
&= \max_{p(\lambda)} \max_{p(x|a,\lambda)} \max_{p(y|b,\lambda)} \sum_{x,y,\lambda} p(x|a,\lambda)p(y|b,\lambda)p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}] \qquad \text{(A.121)}
\end{aligned}$$

Notice further that if we carry out the optimisation of the above expression over $\mathcal{E}$ we can additionally write:

$$\begin{aligned}
&\max_{\mathcal{E}} \max_{x,y} p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}]\\
&= \max_{\mathcal{E}} \max_{p(\lambda)} \max_{p(x|a,\lambda)} \max_{p(y|b,\lambda)} \sum_{x,y,\lambda} p(x|a,\lambda)p(y|b,\lambda)p(x,y)\,\mathrm{Tr}[M_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}] \qquad \text{(A.122)}\\
&= \max_{\mathcal{E}} \max_{\{\mathcal{E}_\lambda\},\{\mathcal{F}_\lambda\}} \max_{p(\lambda)} \max_{p(x|a,\lambda)} \max_{p(y|b,\lambda)} \sum_{x,y,\lambda} p(x|a,\lambda)p(y|b,\lambda)p(x,y)\times \qquad \text{(A.123)}\\
&\qquad \mathrm{Tr}[M_{ab}^{\mathrm{AB}}(\mathcal{E}_\lambda^{\mathrm{A}}\otimes\mathcal{F}_\lambda^{\mathrm{B}})(\sigma_{xy}^{\mathrm{AB}})] \qquad \text{(A.124)}\\
&= \max_{\mathcal{E}} \max_{\mathbb{N}\prec\mathbb{M}} \sum_x p(x,y)\,\mathrm{Tr}\left[N_{ab}^{\mathrm{AB}}\sigma_{xy}^{\mathrm{AB}}\right]. \qquad \text{(A.125)}
\end{aligned}$$

Hence we can further continue from (A.119) and write:

$$
\begin{aligned}
&I_{\min}^{\text{acc}}(\mathcal{N}^{\text{AB}\to\text{A}'\text{B}'}) \\
&= \log\left[\sum_{a,b}\max_{\mathcal{E}}\max_{\mathbb{N}\prec\mathbb{M}} p(a,b)\,\text{Tr}\left[N_{ab}^{\text{AB}}\sigma_{ab}^{\text{AB}}\right]\right] - \log\max_{a,b} p(a,b) && \text{(A.126)}\\
&= \max_{\mathcal{E}}\ \log\left[\max_{\mathbb{N}\prec\mathbb{M}}\sum_{a,b} p(a,b)\,\text{Tr}\left[M_{ab}^{\text{AB}}\sigma_{ab}^{\text{AB}}\right]\right] - \max_{a,b} p(a,b) && \text{(A.127)}\\
&= \max_{\mathcal{E}}\log\left[p_{\text{guess}}^{\text{DSD}}(\mathcal{G},\mathbb{M}^{\text{AB}})\right] - \log\left[p_{\text{guess}}^{\text{DSD}}(\mathcal{G})\right] && \text{(A.128)}\\
&= \log\left[\max_{\mathcal{E}}\frac{p_{\text{guess}}^{\text{DSD}}(\mathcal{G},\mathbb{M}^{\text{AB}})}{p_{\text{guess}}(\mathcal{G})}\right] && \text{(A.129)}\\
&= \log\left[1+\mathbf{R}_{\text{BN}}(\mathbb{M}^{\text{AB}})\right]. && \text{(A.130)}
\end{aligned}
$$

Appendix B

Proofs of results on quantum betting tasks

B.1 Proof of Result 7.1

We start by mentioning that the tasks which are of interest to us are quantum state betting (QSB) games, but that from an operational point of view, they are equivalent to “horse betting games with risk and quantum side information", or quantum horse betting (QHB) games for short. Given this equivalence, in this appendix we would address QSB games as QHB or HB games only.

In order to prove Result 1 we need two Theorems on horse betting (HB) with risk: one for HB games *without* side information, and other for HB games *with* side information. These two Theorems depend on the of Rényi divergence and the BLP conditional Rényi divergence.

B.1.1 Preliminary steps

We start by addressing a simplified notation.

$$w_R^{ICE}(b_{X|G}, o_X, p_{XG}) := w_R^{ICE}(b_{X|G}, \mathbb{M}, o_X, \mathcal{E}), \tag{B.1}$$

with $p(x,g) = p(g|x)p(x)$, $p(g|x) = \mathrm{Tr}[M_g \rho_x]$. We also notice that that optimising over uninformative measurements $\mathbb{N} \in \mathrm{UI}$, meaning $N_g = p(g)\mathbb{1}$, $\forall g$, is equivalent to a horse betting game with risk but *without* side information because $p(g|x) = \mathrm{Tr}(N_g \rho_x) = p(g)\,\mathrm{Tr}(\mathbb{1}\rho_x) = p(g)$ and then:

$$
\begin{aligned}
&\max_{b_{X|G}} \max_{\mathbb{N}\in\mathrm{UI}} w_R^{ICE}(b_{X|G}, o_X, p_{XG}) \\
&= \max_{b_{X|G}} \max_{\mathbb{N}\in\mathrm{UI}} \left[\sum_{g,x} \left[b(x|g) o(x) \right]^{1-R} p(g|x) p(x) \right]^{\frac{1}{1-R}}, && \text{(B.2)} \\
&= \max_{b_{X|G}} \max_{p_G} \left[\sum_{g,x} \left[b(x|g) o(x) \right]^{1-R} p(g) p(x) \right]^{\frac{1}{1-R}}, && \text{(B.3)} \\
&= \max_{b_{X|G}} \left[\sum_{x} \left(\max_{p_G} \sum_{g} b(x|g)^{1-R} p(g) \right) o(x)^{1-R} p(x) \right]^{\frac{1}{1-R}}, && \text{(B.4)} \\
&= \max_{b_{X|G}} \left[\sum_{x} \left(\max_{g} b(x|g)^{1-R} \right) o(x)^{1-R} p(x) \right]^{\frac{1}{1-R}}, && \text{(B.5)}
\end{aligned}
$$

$$= \max_{b_X} \left[\sum_x \left[b(x) o(x) \right]^{1-R} p(x) \right]^{\frac{1}{1-R}}, \tag{B.6}$$

$$= \max_{b_X} w_R^{ICE}(b_X, o_X, p_X). \tag{B.7}$$

This defines a HB game *without* side information, meaning without the random variable G. We now define the auxiliary function of *the logarithm of the isoelastic certainty equivalent* as:

$$U_R(b_{X|G}, o_X, p_{XG}) := \operatorname{sgn}(o) \log \left| w_R^{ICE}(b_{X|G}, o_X, p_{XG}) \right|, \tag{B.8}$$

and similarly without side information as:

$$U_R(b_X, o_X, p_X) := \operatorname{sgn}(o) \log \left| w_R^{ICE}(b_X, o_X, p_X) \right|, \tag{B.9}$$

with $\operatorname{sgn}(o)$ as a shorthand for the sign of the odds $o(x)$, $\forall x$. We also highlight here that we are interested in the strategy that achieves:

$$\max_{b_X} w_R^{ICE}(b_X, o_X, p_X), \tag{B.10}$$

and we can see that this is equivalent to finding the best strategy for the auxiliary optimisation:

$$\max_{b_X} U_R(b_X, o_X, p_X). \tag{B.11}$$

B.1.2 Horse betting games with risk

We now present two results on horse betting games. We remark here that we invoke these results, in contrast with the original presentation in [33], with the following modifications in the notation: i) the original version involves a parameter β, here instead we directly use the risk aversion parameter R, taking into account that these two parameters are related as $\beta = 1 - R$, ii) we have defined the Rényi divergence as a non-negative quantity, for all $\alpha \in \overline{\mathbb{R}}$, even for negative values of alpha, and this explains the appearance of the term $\operatorname{sgn}(R)$, iii) we allow for the odds and consequently the wealth to be negative, and this explains the appearance of the term $\operatorname{sgn}(o)$. We now address a result that characterises this task in terms of the R-divergence.

Theorem B.1. *(Bleuler-Lapidoth-Pfister [33, 171]) Consider a HB game with risk defined by the triple (o_X, p_X, R), and a Gambler playing this game with a betting strategy b_X. The logarithm of the isoelastic certainty equivalent is characterised by the R-divergence $D_\alpha(\cdot||\cdot)$ as:*

$$\begin{aligned} U_R(b_X, o_X, p_X) &= \operatorname{sgn}(o) \log |c^o| && \text{(B.12)} \\ &+ \operatorname{sgn}(o) \operatorname{sgn}(R)\, D_{1/R}(p_X || r_X^o) && \text{(B.13)} \\ &- \operatorname{sgn}(o) \operatorname{sgn}(R)\, D_R(h_X^{(R,o,p)} || b_X), && \text{(B.14)} \end{aligned}$$

with the parameter and the PMF:

$$c^o := \left(\sum_x \frac{1}{o(x)}\right)^{-1}, \quad r^o(x) := \frac{c^o}{o(x)}, \tag{B.15}$$

and the PMF:

$$h^{(R,o,p)}(x) := \frac{p(x)^{\frac{1}{R}} o(x)^{\frac{1-R}{R}}}{\sum_{x'} p(x')^{\frac{1}{R}} o(x')^{\frac{1-R}{R}}}. \tag{B.16}$$

Note that the quantities r^o_X and $h^{(R,o,p)}_X$ define valid PMFs even for negative odds ($o(x) < 0$, $\forall x$).

We are particularly interested in the best possible betting strategy for a given game (o_X, p_X) and fixed R, so we have the following two corollaries.

Corollary B.1. *(Bleuler-Lapidoth-Pfister [33, 171]) Consider a classical horse discrimination (HD) game (o^+_X, meaning $\operatorname{sgn}(o) = 1$) being played by a risk-averse Gambler ($R \geq 0$, meaning $\operatorname{sgn}(R) = 1$). We then want to maximise the logarithm of the isoelastic certainty equivalent over all possible betting strategies. The gambler plays optimally when choosing $b^*(x) = h^{(R,o,p)}(x)$ and then:*

$$\begin{aligned}\max_{b_X} U_R(b_X, o^+_X, p_X) &= U_R(b^*_X, o^+_X, p_X),\\ &= \log|c^o| + D_{1/R}(p_X||r^o_X).\end{aligned} \tag{B.17}$$

This is because the Rényi divergence $D_R(\cdot||\cdot)$ is non-negative $\forall R \in \overline{\mathbb{R}}$.

Corollary B.2. *Consider a classical horse exclusion (HE) game (o^-_X, meaning $\operatorname{sgn}(o) = -1$) being played by a risk-averse Gambler ($R < 0$, meaning $\operatorname{sgn}(R) = -1$). We then want to maximise the logarithm of the isoelastic certainty equivalent over all possible betting strategies. The gambler plays optimally when choosing $b^*(x) = h^{(R,o,p)}(x)$ and then:*

$$\begin{aligned}\max_{b_X} U_R(b_X, o^-_X, p_X) &= U_R(b^*_X, o^-_X, p_X),\\ &= -\log|c^o| + D_{1/R}(p_X||r^o_X).\end{aligned} \tag{B.18}$$

This is because the Rényi divergence $D_R(\cdot||\cdot)$ is non-negative $\forall R \in \overline{\mathbb{R}}$.

B.1.3 Horse betting with risk and side information

We now address a result that characterises this task in terms of the BLP-CR-divergence and the R-divergence.

Theorem B.2. *(Bleuler-Lapidoth-Pfister [33, 171]) Consider a HB game with risk and side information defined by the triple (o_X, p_{XG}, R), and a Gambler playing this game with a betting strategy $b_{X|G}$. The utility function of log-wealth is characterised by the the BLP-CR-divergence $D^{\mathrm{BLP}}_\alpha(\cdot||\cdot|\cdot)$ and R-divergence $D_\alpha(\cdot||\cdot)$ as:*

$$U_R(b_{X|G}, o_X, p_{XG}) = \operatorname{sgn}(o)\log|c^o| \tag{B.19}$$
$$+ \operatorname{sgn}(o)\operatorname{sgn}(R) D^{\mathrm{BLP}}_{1/R}\left(p_{X|G}||r^o_X|p_G\right) \tag{B.20}$$
$$- \operatorname{sgn}(o)\operatorname{sgn}(R) D_R\left(h^{(R,o,p)}_{X|G} h^{(R,o,p)}_G \middle|\middle| b_{X|G}\, h^{(R,o,p)}_G\right), \tag{B.21}$$

with the parameter and the PMF:

$$c^o := \left(\sum_x \frac{1}{o(x)}\right)^{-1}, \quad r^o(x) := \frac{c^o}{o(x)}, \tag{B.22}$$

and the conditional PMF and PMF:

$$h^{(R,o,p)}(x|g) := \frac{p(x|g)^{\frac{1}{R}} o(x)^{\frac{1-R}{R}}}{\sum_{x'} p(x'|g)^{\frac{1}{R}} o(x')^{\frac{1-R}{R}}}, \tag{B.23}$$

$$h^{(R,o,p)}(g) := \frac{p(g)\left[\sum_{x'} p(x'|g)^{\frac{1}{R}} o(x')^{\frac{1-R}{R}}\right]^R}{\sum_{g'} p(g')\left[\sum_{x'} p(x'|g')^{\frac{1}{R}} o(x')^{\frac{1-R}{R}}\right]^R}. \tag{B.24}$$

Note that the quantities r^o_X, $h^{(R,o,p)}_{X|G}$, $h^{(R,o,p)}_G$ define valid PMFs even for negative odds ($o(x) < 0, \forall x$).

We are particularly interested in the best possible betting strategy $b_{X|G}$ for a given game (o_X, p_{XG}) and fixed R, so we have the following two corollaries.

Corollary B.3. *(Bleuler-Lapidoth-Pfister [33, 171]) Consider a horse discrimination (HD) game (o^+_X, meaning $\mathrm{sgn}(o) = 1$) being played by a risk-averse Gambler ($R > 0$, meaning $\mathrm{sgn}(R) = 1$) with access to side information. We then want to maximise the logarithm of the isoelastic certainty equivalent over all possible betting strategies. The Gambler plays optimally when choosing $b^*(x|g) = h^{(R,o,p)}(x|g)$ and then:*

$$\begin{aligned}\max_{b_{X|G}} U_R(b_{X|G}, o^+_X, p_{XG}) &= U_R(b^*_{X|G}, o^+_X, p_{XG}),\\ &= \log|c^o| + D^{\mathrm{BLP}}_{1/R}(p_{X|G}||r^o_X|p_G),\end{aligned} \tag{B.25}$$

with the BLP-CR-divergence $D^{\mathrm{BLP}}_\alpha(\cdot||\cdot|\cdot)$. This is because the Rényi divergence $D_R(\cdot||\cdot)$ is non-negative $\forall R \in \overline{\mathbb{R}}$.

Corollary B.4. *Consider a classical horse exclusion (HE) game (o^-_X) being played by a risk-averse Gambler ($R < 0$) with access to side information. We then want to maximise the logarithm of the isoelastic certainty equivalent over all possible betting strategies. The Gambler plays optimally when choosing $b^*(x|g) = h^{(R,o,p)}(x|g)$ and then:*

$$\begin{aligned}\max_{b_{X|G}} U_R(b_{X|G}, o^-_X, p_{XG}) &= U_R(b^*_{X|G}, o^-_X, p_{XG}),\\ &= -\log|c^o| + D^{\mathrm{BLP}}_{1/R}(p_{X|G}||r^o_X|p_G),\end{aligned} \tag{B.26}$$

with the BLP-CR-divergence $D^{\mathrm{BLP}}_\alpha(\cdot||\cdot|\cdot)$. This is because the Rényi divergence $D_R(\cdot||\cdot)$ is non-negative $\forall R \in \overline{\mathbb{R}}$.

B.1.4 Proving Result 1

In order to prove Result 1 we need two Lemmas. Let us start by rewriting the Rényi entropy in a more convenient form:

$$H_\alpha(X) = -\log\left[p_\alpha\left(X\right)\right], \tag{B.27}$$

$$p_\alpha\left(X\right) := \left(\sum_x p(x)^\alpha\right)^{\frac{1}{(\alpha-1)}}. \tag{B.28}$$

We are now ready to establish a first Lemma.

Lemma B.1. *(Operational interpretation of the Rényi entropy) Consider a PMF p_X, the Rényi probability of order $\alpha \in \overline{\mathbb{R}}$ can be written as:*

$$\operatorname{sgn}(\alpha)\, C\, p_\alpha(X) = \max_{b_X} w^{ICE}_{1/\alpha}(b_X, o_X^{\operatorname{sgn}(\alpha)c}, p_X), \tag{B.29}$$

with the maximisation over all possible betting strategies b_X, and constant odds $o^{\operatorname{sgn}(\alpha)c}(x) :=$ $\operatorname{sgn}(\alpha)C$, $C>0$, $\forall x$.

Proof. We start by considering a HB game with constant odds $o^{\operatorname{sgn}(\alpha)}(x) := \operatorname{sgn}(\alpha)C$, $C>0$, $\forall x$, and consider a risk-aversion coefficient parametrised as $R(\alpha) := 1/\alpha$. We first notice that the best strategy for the Gambler is given by (B.16):

$$b^*(x) = \frac{p(x)^\alpha}{\sum_{x'} p(x')^\alpha}. \tag{B.30}$$

Considering now the isoelastic certainty equivalent and replacing the constant odds and the best strategy we get:

$$\begin{aligned}
w^{ICE}_{1/\alpha}(b^*_X, o_X^{\operatorname{sgn}(\alpha)c}, p_X) &= \left[\sum_x p(x)\left[b^*(x) o^{\operatorname{sgn}(\alpha)c}(x)\right]^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.31)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_x p(x)\left[b^*(x)\right]^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.32)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_x p(x)\left[\frac{p(x)^\alpha}{\sum_{x'} p(x')^\alpha}\right]^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.33)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_x p(x)\frac{p(x)^{\alpha-1}}{\left[\sum_{x'} p(x')^\alpha\right]^{\frac{\alpha-1}{\alpha}}}\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.34)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_x \frac{p(x)^{\alpha}}{\left[\sum_{x'} p(x'|g)^\alpha\right]^{\frac{\alpha-1}{\alpha}}}\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.35)}\\
&= \operatorname{sgn}(\alpha)C\frac{1}{\sum_{x'} p(x')^\alpha}\left[\sum_x p(x)^\alpha\right]^{\frac{\alpha}{\alpha-1}}, && \text{(B.36)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_x p(x)^\alpha\right]^{\frac{1}{\alpha-1}}, && \text{(B.37)}\\
&= \operatorname{sgn}(\alpha)Cp_\alpha(X), && \text{(B.38)}
\end{aligned}$$

and therefore proving the claim. □

We now move on to rewrite the Arimoto-Rényi conditional entropy in a more convenient form:

$$H_\alpha(X|G) = -\log\left[p_\alpha\left(X|G\right)\right], \tag{B.39}$$

$$p_\alpha\left(X|G\right) := \left(\sum_g \left(\sum_x p(x,g)^\alpha\right)^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{(\alpha-1)}}. \tag{B.40}$$

We are now ready to establish a second Lemma.

Lemma B.2. *(Operational interpretation of the Arimoto-Rényi conditional entropy) Consider a joint PMF p_{XG}, the Arimoto-Rényi conditional entropy of order $\alpha \in \overline{\mathbb{R}}$ can be written as:*

$$\operatorname{sgn}(\alpha)\, C\, p_\alpha(X|G) = \max_{b_{X|G}} w_{1/\alpha}^{ICE}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, p_{XG}), \tag{B.41}$$

with the maximisation over all possible betting strategies $b_{X|G}$, and constant odds $o^{\operatorname{sgn}(\alpha)c}(x) :=$ $\operatorname{sgn}(\alpha)C$, $C > 0$, $\forall x$.

Proof. We start by considering a HB game with constant odds $o^{\operatorname{sgn}(\alpha)}(x) := \operatorname{sgn}(\alpha)C$, $C > 0$, $\forall x$, and consider a risk-aversion coefficient parametrised as $R(\alpha) := 1/\alpha$. We now notice that the best strategy for the Gambler with access to side information is given by (B.23):

$$\begin{aligned}
b^*(x|g) &= g^{(R,o,p)}(x|g), &\text{(B.42)}\\
&= \frac{p(x|g)^{\frac{1}{R}} o^{\operatorname{sgn}(\alpha)c}(x)^{\frac{1-R}{R}}}{\sum_{x'} p(x'|g)^{\frac{1}{R}} o^{\operatorname{sgn}(\alpha)c}(x')^{\frac{1-R}{R}}}, &\text{(B.43)}\\
&= \frac{p(x|g)^{\frac{1}{R}} (\operatorname{sgn}(\alpha)C)^{\frac{1-R}{R}}}{\sum_{x'} p(x'|g)^{\frac{1}{R}} (\operatorname{sgn}(\alpha)C)^{\frac{1-R}{R}}}, &\text{(B.44)}\\
&= \frac{p(x|g)^{\frac{1}{R}}}{\sum_{x'} p(x'|g)^{\frac{1}{R}}}, &\text{(B.45)}\\
&= \frac{p(x|g)^{\alpha}}{\sum_{x'} p(x'|g)^{\alpha}}. &\text{(B.46)}
\end{aligned}$$

Considering now the isoelastic certainty equivalent and replacing the constant odds and the best strategy we get:

$$\begin{aligned}
w_{1/\alpha}^{ICE}(b^*_{X|G}, o_X^{\operatorname{sgn}(\alpha)c}, p_{XG}) &= \left[\sum_{x,g} p(x,g)\left[b^*(x|g) o^{\operatorname{sgn}(\alpha)c}(x)\right]^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}, &\text{(B.47)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_{x,g} p(x,g)\left[\frac{p(x|g)^\alpha}{\sum_{x'} p(x'|g)^\alpha}\right]^{\frac{\alpha-1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}, &\text{(B.48)}\\
&= \operatorname{sgn}(\alpha)C\left[\sum_{x,g} p(x,g)\frac{p(x|g)^{\alpha-1}}{\left[\sum_{x'} p(x'|g)^\alpha\right]^{\frac{\alpha-1}{\alpha}}}\right]^{\frac{\alpha}{\alpha-1}}. &\text{(B.49)}
\end{aligned}$$

Using $p(x,g) = p(x|g)p(g)$ and reorganising:

$$w^{ICE}_{1/\alpha}(b^{*}_{X|G}, o_X^{\mathrm{sgn}(\alpha)c}, p_{XG}) = \mathrm{sgn}(\alpha) C \left[\sum_{x,g} p(g) \frac{p(x|g)^\alpha}{\left[\sum_{x'} p(x'|g)^\alpha\right]^{\frac{\alpha-1}{\alpha}}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{B.50}$$

$$= \mathrm{sgn}(\alpha) C \left[\sum_{g} p(g) \frac{\sum_x p(x|g)^\alpha}{\left[\sum_{x'} p(x'|g)^\alpha\right]^{\frac{\alpha-1}{\alpha}}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{B.51}$$

$$= \mathrm{sgn}(\alpha) C \left[\sum_{g} p(g) \left[\sum_x p(x|g)^\alpha \right]^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{B.52}$$

$$= \mathrm{sgn}(\alpha) C \left[\sum_{g} \left[\sum_x p(x|g)^\alpha p(g)^\alpha \right]^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{B.53}$$

$$= \mathrm{sgn}(\alpha) C \left[\sum_{g} \left[\sum_x p(x,g)^\alpha \right]^{\frac{1}{\alpha}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{B.54}$$

$$= \mathrm{sgn}(\alpha) C p_\alpha(X|G), \tag{B.55}$$

and therefore proving the claim. □

We are now ready to prove Result 1.

Proof. (of Result 1) Consider the Arimoto's mutual information of order $\alpha \in \overline{\mathbb{R}}$, we have the following chain of equalities:

$$I_\alpha(X;G) = \mathrm{sgn}(\alpha)[H_\alpha(X) - H_\alpha(X|G)], \tag{B.56}$$

$$= \mathrm{sgn}(\alpha) \log \left[\frac{p_\alpha(X|G)}{p_\alpha(X)} \right], \tag{B.57}$$

$$= \mathrm{sgn}(\alpha) \log \left[\frac{\mathrm{sgn}(\alpha)\, C\, p_\alpha(X|G)}{\mathrm{sgn}(\alpha)\, C\, p_\alpha(X)} \right], \tag{B.58}$$

$$= \mathrm{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w^{ICE}_{1/\alpha}(b_{X|G}, o_X^{\mathrm{sgn}(\alpha)}, p_{XG})}{\max\limits_{b_X} w^{ICE}_{1/\alpha}(b_X, o_X^{\mathrm{sgn}(\alpha)}, p_X)} \right]. \tag{B.59}$$

The first equality is the definition of the Arimoto's mutual information. The second equality comes from replacing the Rényi entropy and the Arimoto-Rényi conditional entropy. The third inequality we have multiplied and divided by $\mathrm{sgn}(\alpha)\, C$. The fourth and last equality follows from invoking Lemma B.1 and Lemma B.2. This proves the claim. □

B.2 Proof of Corollaries 2 and 3

Proof. (of Corollary 2) In the case $\alpha \to \infty$, we have:

$$\max_{\mathcal{E}} I_\infty(X;G)_{\mathcal{E},\mathbb{M}} = \log \left[\max_{\mathcal{E}} \frac{\max\limits_{b_{X|G}} w^{ICE}_0(b_{X|G}, \mathbb{M}, o_X^c, \mathcal{E})}{\max\limits_{\mathbb{N}\in\mathrm{UI}} \max\limits_{b_{X|G}} w^{ICE}_0(b_{X|G}, \mathbb{N}, o_X^c, \mathcal{E})} \right]. \tag{B.60}$$

To prove the claim, it is enough to prove:

$$\max_{b_{X|G}} w_0^{ICE}(b_{X|G}, \mathbb{M}, o_X^c, \mathcal{E}) = CP_{\text{succ}}^{\text{QSD}}(\mathcal{E}, \mathbb{M}). \tag{B.61}$$

We have already shown this in the main document, but we can also double check it from Lemma B.2 from which we have that for $\alpha \geq 0$:

$$\max_{b_{X|G}} w_{1/\alpha}^{ICE}(b_{X|G}, \mathbb{M}, o_X^c, \mathcal{E}) = Cp_\alpha(X|G), \tag{B.62}$$

$$= C\left[\sum_g \left[\sum_x p(x,g)^\alpha\right]^{\frac{1}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}. \tag{B.63}$$

Considering now $\alpha \to \infty$ we have:

$$\max_{b_{X|G}} w_0^{ICE}(b_{X|G}, \mathbb{M}, o_X^c, \mathcal{E}) = C\sum_g \max_x p(x,g). \tag{B.64}$$

Further analysing this quantity we have:

$$\sum_g \max_x p(x,g) = \sum_g \max_{q(x|g)} \sum_x q(x|g)p(x,g), \tag{B.65}$$

$$= \max_{q(x|g)} \sum_{g,x} q(x|g)p(g|x)p(x), \tag{B.66}$$

$$= \max_{q(x|g)} \sum_{g,x} \left[\sum_a \delta_x^a q(a|g)\right] p(g|x)p(x), \tag{B.67}$$

$$= \max_{q(a|g)} \sum_{a,g,x} \delta_x^a q(a|g)p(g|x)p(x), \tag{B.68}$$

$$= P_{\text{succ}}^{\text{QSD}}(\mathcal{E}, \mathbb{M}). \tag{B.69}$$

In the first line we use the identity:

$$\max_{q(x)} \sum_x q(x)f(x) = \max_x f(x). \tag{B.70}$$

This proves the claim. □

Proof. (of Corollary 3) The proof of Corollary 3 follows a similar argument than that of Corollary 2. □

B.3 Proof of Result 7.3 on noisy quantum state betting (nQSB) games

The proof of this result is similar to that of result 1, and we write below for completeness. We start with the case for QRTs of measurements with general resources.

Proof. (of first part) Consider the Arimoto's gap of order $\alpha \in \overline{\mathbb{R}}$, we have the following chain of equalities:

$$\begin{aligned}
G_\alpha^{\mathbb{F}}(X;G)_{\mathcal{E},\mathbb{M}} &= I_\alpha(X;G)_{\mathcal{E},\mathbb{M}} - \max_{\mathbb{N}\in\mathbb{F}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N}}, && \text{(B.71)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{M})}{p_\alpha(X)}\right] - \max_{\mathbb{N}\in\mathbb{F}} \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{N})}{p_\alpha(X)}\right], && \text{(B.72)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{M})}{\max\limits_{\sigma\in\mathrm{F}} p_\alpha(X_\mathcal{E};G_\mathbb{N})}\right], && \text{(B.73)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\operatorname{sgn}(\alpha)\, C\, p_\alpha(X_\mathcal{E}|G_\mathbb{M})}{\max\limits_{\mathbb{N}\in\mathbb{F}} \operatorname{sgn}(\alpha)\, C\, p_\alpha(X_\mathcal{E};G_\mathbb{N})}\right], && \text{(B.74)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QSB}}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \mathcal{E}, \mathbb{M})}{\max\limits_{\mathbb{N}\in\mathbb{F}}\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QSB}}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \mathcal{E}, \mathbb{N})}\right]. && \text{(B.75)}
\end{aligned}$$

The first equality is the definition of the Arimoto's gap for a fixed couple $(\mathcal{E},\mathbb{M})$. The second equality comes from replacing the Rényi entropy and the Arimoto-Rényi conditional entropy. In the third equality we reorganised the expression. In the fourth equality we have multiplied and divided by $\operatorname{sgn}(\alpha)\, C$. The fifth and last equality follows from invoking Lemma B.2. This proves the claim. □

We now consider the case for QRTs of channels with arbitrary resources.

Proof. (of second part) Consider the Arimoto's gap of order $\alpha \in \overline{\mathbb{R}}$, we have the following chain of equalities:

$$\begin{aligned}
&G_\alpha^{\mathcal{F}}(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}}\\
&= I_\alpha(X;G)_{\mathcal{E},\mathbb{M},\mathcal{N}} - \max_{\widetilde{\mathcal{N}}\in\mathcal{F}}\max_{\mathbb{N}} I_\alpha(X;G)_{\mathcal{E},\mathbb{N},\widetilde{\mathcal{N}}}, && \text{(B.76)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{M})_\mathcal{N}}{p_\alpha(X)}\right] - \max_{\widetilde{\mathcal{N}}\in\mathcal{F}}\max_{\mathbb{N}} \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{N})_{\widetilde{\mathcal{N}}}}{p_\alpha(X)}\right], && \text{(B.77)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\mathcal{E}|G_\mathbb{M})_\mathcal{N}}{\max\limits_{\widetilde{\mathcal{N}}\in\mathcal{F}}\max\limits_{\mathbb{N}} p_\alpha(X_\mathcal{E}|G_\mathbb{N})_{\widetilde{\mathcal{N}}}}\right], && \text{(B.78)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\operatorname{sgn}(\alpha)\, C\, p_\alpha(X_\mathcal{E}|G_\mathbb{M})_\mathcal{N}}{\max\limits_{\widetilde{\mathcal{N}}\in\mathcal{F}}\max\limits_{\mathbb{N}} \operatorname{sgn}(\alpha)\, C\, p_\alpha(X_\mathcal{E}|G_\mathbb{N})_{\widetilde{\mathcal{N}}}}\right], && \text{(B.79)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QSB}}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \mathcal{E}, \mathbb{M}, \mathcal{N})}{\max\limits_{\widetilde{\mathcal{N}}\in\mathcal{F}}\max\limits_{\mathbb{N}}\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QSB}}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \mathcal{E}, \mathbb{N}, \widetilde{\mathcal{N}})}\right]. && \text{(B.80)}
\end{aligned}$$

The first equality is the definition of the Arimoto's gap for a fixed triple $(\mathcal{E},\mathbb{M},\mathcal{N})$. The second equality comes from replacing the Rényi entropy and the Arimoto-Rényi conditional entropy. In the third equality we reorganised the expression. In the fourth equality we have multiplied and divided by $\operatorname{sgn}(\alpha)\, C$. The fifth and last equality follows from invoking Lemma B.2. This proves the claim. □

B.4 Proof of Result 7.4 on quantum channel betting (QCB) games

The proof of this result similar to that of result 1, and we write below for completeness. We start with the case for QRTs of states with arbitrary resources.

Proof. (of first part) Consider the Arimoto's gap of order $\alpha \in \overline{\mathbb{R}}$, we have the following chain of equalities:

$$
\begin{aligned}
&G_\alpha^{\mathrm{F}}(X;G)_{\Lambda,\mathbb{M},\rho}\\
&= I_\alpha(X;G)_{\Lambda,\mathbb{M},\rho} - \max_{\sigma\in\mathrm{F}} I_\alpha(X;G)_{\Lambda,\mathbb{M},\sigma}, && \text{(B.81)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{p_\alpha(X)}\right] - \max_{\sigma\in\mathrm{F}}\operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda;G_\mathbb{M})_\sigma}{p_\alpha(X)}\right], && \text{(B.82)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{\max\limits_{\sigma\in\mathrm{F}} p_\alpha(X_\Lambda|G_\mathbb{M})_\sigma}\right], && \text{(B.83)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\operatorname{sgn}(\alpha)\,C\,p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{\max\limits_{\sigma\in\mathrm{F}}\operatorname{sgn}(\alpha)\,C\,p_\alpha(X_\Lambda|G_\mathbb{M})_\sigma}\right], && \text{(B.84)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}(b_{X|G},o_X^{\operatorname{sgn}(\alpha)},\Lambda,\rho,\mathbb{M})}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{b_{X|G}} w_{1/\alpha}^{\mathrm{QCB}}(b_{X|G},o_X^{\operatorname{sgn}(\alpha)},\Lambda,\sigma,\mathbb{M})}\right]. && \text{(B.85)}
\end{aligned}
$$

The first equality is the definition of the Arimoto's gap for a fixed triple $(\Lambda,\rho,\mathbb{M})$. The second equality comes from replacing the Rényi entropy and the Arimoto-Rényi conditional entropy. In the third equality we reorganised the expression. In the fourth equality we have multiplied and divided by $\operatorname{sgn}(\alpha)\,C$. The fifth and last equality follows from invoking Lemma B.2. This proves the claim. □

We now consider the case for multi-object QRTs of state-measurement pairs.

Proof. (of second part) Consider the Arimoto's gap of order $\alpha \in \overline{\mathbb{R}}$, we have the following chain of equalities:

$$
\begin{aligned}
&G_\alpha^{\mathrm{F},\mathbb{F}}(X;G)_{\Lambda,\mathbb{M},\rho}\\
&= I_\alpha(X;G)_{\Lambda,\mathbb{M},\rho} - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}} I_\alpha(X;G)_{\Lambda,\mathbb{N},\sigma}, && \text{(B.86)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{p_\alpha(X)}\right] - \max_{\sigma\in\mathrm{F}}\max_{\mathbb{N}\in\mathbb{F}}\operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda|G_\mathbb{N})_\sigma}{p_\alpha(X)}\right], && \text{(B.87)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}} p_\alpha(X_\Lambda|G_\mathbb{N})_\sigma}\right], && \text{(B.88)}\\
&= \operatorname{sgn}(\alpha)\log\left[\frac{\operatorname{sgn}(\alpha)\,C\,p_\alpha(X_\Lambda|G_\mathbb{M})_\rho}{\max\limits_{\sigma\in\mathrm{F}}\max\limits_{\mathbb{N}\in\mathbb{F}}\operatorname{sgn}(\alpha)\,C\,p_\alpha(X_\Lambda;G_\mathbb{N})_\sigma}\right], && \text{(B.89)}
\end{aligned}
$$

$$= \operatorname{sgn}(\alpha) \log \left[\frac{\max\limits_{b_{X|G}} w^{\mathrm{QCB}}_{1/\alpha}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \Lambda, \rho, \mathbb{M})}{\max\limits_{\sigma \in \mathrm{F}} \max\limits_{\mathbb{N} \in \mathbb{F}} \max\limits_{b_{X|G}} w^{\mathrm{QCB}}_{1/\alpha}(b_{X|G}, o_X^{\operatorname{sgn}(\alpha)}, \Lambda, \sigma, \mathbb{N})} \right]. \tag{B.90}$$

The first equality is the definition of the Arimoto's gap for a fixed triple $(\Lambda, \rho, \mathbb{M})$. The second equality comes from replacing the Rényi entropy and the Arimoto-Rényi conditional entropy. In the third equality we reorganised the expression. In the fourth equality we have multiplied and divided by $\operatorname{sgn}(\alpha)\, C$. The fifth and last equality follows from invoking Lemma B.2. This proves the claim. □

B.5 Proof of Result 7.6 on Rényi divergences

Proof. (of Result 7.6) For $\alpha > 1$ we have:

$$E^{\mathcal{S}}_\alpha(\mathbb{M}) \overset{1}{=} \min_{\mathbb{N} \in \mathrm{UI}} D^{\mathcal{S}}_\alpha(\mathbb{M}||\mathbb{N}), \tag{B.91}$$

$$\overset{2}{=} \min_{\mathbb{N} \in \mathrm{UI}} \max_{p_X} D^{\mathrm{S}}_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} \middle|\middle| q^{(\mathbb{N},\mathcal{S})}_{G|X} \middle| p_X \right), \tag{B.92}$$

$$\overset{3}{=} \min_{q_G} \max_{p_X} D^{\mathrm{S}}_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} \middle|\middle| q_G \middle| p_X \right), \tag{B.93}$$

$$\overset{4}{=} \min_{q_G} \max_{p_X} \frac{1}{\alpha - 1} \log \left[\sum_x p(x) \sum_g p(g|x)^\alpha q(g)^{1-\alpha} \right], \tag{B.94}$$

$$\overset{5}{=} \frac{1}{\alpha - 1} \log \left[\min_{q_G} \max_{p_X} \sum_x p(x) \sum_g p(g|x)^\alpha q(g)^{1-\alpha} \right], \tag{B.95}$$

$$\overset{6}{=} \frac{1}{\alpha - 1} \log \left[\min_{q_G} \max_{p_X} f^{\mathrm{S}}_\alpha(q_G, p_X) \right], \tag{B.96}$$

$$\overset{7}{=} \frac{1}{\alpha - 1} \log \left[\max_{p_X} \min_{q_G} f^{\mathrm{S}}_\alpha(q_G, p_X) \right], \tag{B.97}$$

$$\overset{8}{=} \max_{p_X} \min_{q_G} \frac{1}{\alpha - 1} \log \left[f^{\mathrm{S}}_\alpha(q_G, p_X) \right], \tag{B.98}$$

$$\overset{9}{=} \max_{p_X} \min_{q_G} D^{\mathrm{S}}_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} \middle|\middle| q_G \middle| p_X \right), \tag{B.99}$$

$$\overset{10}{=} \max_{p_X} I^{\mathrm{S}}_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} p_X \right), \tag{B.100}$$

$$\overset{11}{=} C_\alpha\left(p^{(\mathbb{M},\mathcal{S})}_{G|X} \right). \tag{B.101}$$

In the first equality we use the definition of $E_{\alpha,\mathcal{S}}(\mathbb{M})$. In the second equality we replace $D^{\mathcal{S}}_\alpha(\mathbb{M}||\mathbb{N})$. In the third equality we notice that minimising over uninformative measurements is equivalent to minimising over PMFs q_G. In the fourth equality we replace the Sibson's CR-divergence. In the fifth equality we move the optimisation inside $\log(\cdot)$ because the term $\alpha - 1$ is positive and because $\log(\cdot)$ is an increasing function. In the sixth equality we introduce the function:

$$f^{\mathrm{S}}_\alpha(q_G, p_X) := \sum_x p(x) \sum_g p(g|x)^\alpha q(g)^{1-\alpha}. \tag{B.102}$$

In the seventh equality we use Sion's minimax theorem [202, 130] because the function $f^{\mathrm{S}}_\alpha(q_G, p_X)$ is being optimised over convex and compact sets, and because it is a convex-concave function. Specifically, the function $f_\alpha^{\mathrm{S}}(q_G, p_X)$ is convex in g_G because the function $f(q) = q^{1-\alpha}$ with $\alpha > 1$ and positive values of q, is convex, and because the sum of convex functions is convex. The function $f_\alpha^{\mathrm{S}}(q_G, p_X)$ is concave in p_X because it is linear in p_X. In the eight equality we take the maximisation out of $\log(\cdot)$ because $\alpha - 1$ is positive and because $\log(\cdot)$ is an increasing function. In the ninth equality we use the definition of Sibson's CR-divergence. In the tenth equality we use the definition of Sibson's mutual information. In the eleventh and final equality we use Lemma 3. The cases for $0 < \alpha < 1$ and $\alpha < 0$ follow a similar argument, taking into account the sign of $\alpha - 1$, and the convexity/concavity of the function $f(q) = q^{1-\alpha}$. $\square$

B.6 Proof of Result 7.7 on resource monotones

Proof. (of Result 7.7) It is straightforward to check that $M_\alpha(\mathbb{M})$ is a resource monotone (meaning that it satisfies i) faithfulness and ii) monotonicity) if and only if $E_\alpha(\mathbb{M})$ is a resource monotone. We now prove these properties for $E_\alpha(\mathbb{M})$. In short, we will expand this function in terms of the Rényi divergence, and exploit the properties of this function.

Part i) Faithfulness. Consider $\mathbb{M} \in \mathrm{UI}$, and let us see that this implies $E_\alpha(\mathbb{M}) = 0$ with $\alpha \geq 0$:

$$E_\alpha(\mathbb{M}) \overset{1}{=} \max_{\mathcal{S}} \min_{q_G} \max_{p_X} D_\alpha^{\mathrm{S}}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right), \tag{B.103}$$

$$\overset{2}{=} \max_{\mathcal{S}} \min_{q_G} \max_{p_X} D_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X \middle|\middle| q_G\, p_X\right), \tag{B.104}$$

$$\overset{3}{=} \max_{\mathcal{S}} \min_{q_G} \max_{p_X} D_\alpha\left(p_G\, p_X \middle|\middle| q_G\, p_X\right), \tag{B.105}$$

$$\overset{4}{=} \max_{\mathcal{S}} \max_{p_X} \min_{q_G} D_\alpha\left(p_G\, p_X \middle|\middle| q_G\, p_X\right), \tag{B.106}$$

$$\overset{5}{\leq} \max_{\mathcal{S}} \max_{p_X} D_\alpha\left(p_G\, p_X \middle|\middle| p_G\, p_X\right), \tag{B.107}$$

$$\overset{6}{=} \max_{\mathcal{S}} \max_{p_X} 0 = 0. \tag{B.108}$$

In the first equality we use the definition of the measure. In the second equality we write Sibson's mutual information in terms of the Rényi divergence. In the third equality we use the assumption that $\mathbb{M} \in \mathrm{UI}$. In the fourth equality we use Sion's minimax theorem [202, 130], using the same arguments as in Result 2. In the fifth inequality we use that $q_G = p_G$ is a feasible option. In the sixth equality we invoke the property of the Rényi divergence which reads $D_\alpha(p_X || q_X) = 0$ if and only if $q_X = p_X$. This chain means that $E_\alpha(\mathbb{M}) \leq 0$, and remembering that that $E_\alpha(\mathbb{M})$ is non-negative (being an optimisation over the Rényi divergence which is itself non-negative) implies $E_\alpha(\mathbb{M}) = 0$ as desired.

Consider now that $\mathbb{M}$ achieves $E_\alpha(\mathbb{M}) = 0$, and let us prove that $\mathbb{M} \in \mathrm{UI}$.

$$\begin{aligned}
0 &\overset{1}{=} E_\alpha(\mathbb{M}) && \text{(B.109)}\\
&\overset{2}{=} \max_{\mathcal{S}} \min_{q_G} \max_{p_X} D_\alpha^{\mathrm{S}}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right), && \text{(B.110)}\\
&\overset{3}{=} \max_{\mathcal{S}} \max_{p_X} \min_{q_G} D_\alpha^{\mathrm{S}}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right), && \text{(B.111)}\\
&\overset{4}{=} \max_{\mathcal{S}} \max_{p_X} \min_{q_G} D_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X \middle|\middle| q_G\, p_X\right), && \text{(B.112)}\\
&\overset{5}{=} \max_{\mathcal{S}} \max_{p_X} D_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X \middle|\middle| q_G^*\, p_X\right). && \text{(B.113)}
\end{aligned}$$

The first equality is the assumption. In the second equality we invoke the definition of the measure. In the third equality we use Sion's minimax theorem [202, 130] as per Result 2. In the fourth equality we expand Sibson's CR-divergence in terms of the Rényi divergence. In the fifth equality we denote the optimal PMF as q_G^*. We now notice that the latter equality implies:

$$D_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} p_X \middle|\middle| q_G^*\, p_X\right) = 0, \tag{B.114}$$

from which we get that $p_{G|X}^{(\mathbb{M},\mathcal{S})} = q_G^*$. This means that $p(g|x) = q(g)$, $\forall g, x$, or that $\mathrm{Tr}[M_g \rho_x] = \mathrm{Tr}[q(g)\mathbb{1}\rho_x]$, $\mathrm{Tr}[(M_g - q(g)\mathbb{1})\rho_x] = 0$, $\forall g, x$ which implies $M_g = q(g)\mathbb{1}$, $\forall g$, or that $\mathbb{M} \in \mathrm{UI}$ as desired.

Part ii) Monotonicity for the order induced by the simulability of measurements. Given two measurements $\mathbb{N} = \{N_g\}$, $\mathbb{M} = \{M_y\}$ such that $\mathbb{N} \preceq \mathbb{M}$, we now show that this implies $E_\alpha(\mathbb{N}) \leq E_\alpha(\mathbb{M})$. Let us consider that $\mathbb{N} \preceq \mathbb{M}$, meaning that $\forall g$ and some $s_{G|Y}$ we have:

$$N_g = \sum_y s(g|y) M_y. \tag{B.115}$$

This implies that for any set of states $\mathcal{S} = \{\rho_x\}$:

$$r(g|x) := \mathrm{Tr}[N_g \rho_x] = \sum_y s(g|y) p(y|x), \tag{B.116}$$

with $p(y|x) = \mathrm{Tr}[M_y \rho_x]$. We now invoke the data processing inequality for the Rényi divergence [229] and get:

$$D_\alpha\left(r_{G|X}^{(\mathbb{N},\mathcal{S})}\, p_X \middle|\middle| q_G\, p_X\right) \leq D_\alpha\left(p_{G|X}^{(\mathbb{M},\mathcal{S})}\, p_X \middle|\middle| q_G\, p_X\right), \tag{B.117}$$

with arbitrary PMFs p_X and p_G. Recognising that these quantities are the Sibson's CR-divergence leads to:

$$D_\alpha^{\mathrm{S}}\left(r_{G|X}^{(\mathbb{N},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right) \leq D_\alpha^{\mathrm{S}}\left(p_{G|X}^{(\mathbb{M},\mathcal{S})} \middle|\middle| q_G \middle| p_X\right). \tag{B.118}$$

We now perform the optimisations $\max_{\mathcal{S}}$, $\min_{q_G}$, $\max_{p_X}$ on both sides and get:

$$E_\alpha(\mathbb{N}) \leq E_\alpha(\mathbb{M}). \tag{B.119}$$

This finishes the proof for the cases $\alpha \geq 0$. The cases $\alpha < 0$ follow a similar argument. □

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