Polygon Pi

by Iris Coles, Alf Coles, Tracy Helliwell, Julian Brown

This brief article is an account of an emerging mathematical exploration, based on extending standard definitions (the author order signals the chronological order of engagement).

Iris had been set a homework to memorise π to as many decimal places as she could (she got to around 20). In a car with Alf, conversation turned to what π means, and Alf offered that it is the number of times the diameter of a circle can fit around its circumference. Iris then asked, "Does that work for other shapes?". Alf recognised this as a question he had not asked himself before.

The extent of Alf and Iris's short car journey exploration was to imagine a square and if the "diameter" of a square is measured from the middle of one face to the opposite face, then that diameter fits around the perimeter 4 times. We had just defined "pi" for a square to be 4. (We will use π for the concept we know and "pi" for this new idea.)

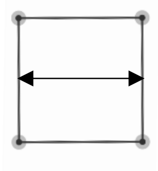

Figure 1: The "diameter" of a square, which would wrap around its perimeter 4 times

The next day, Alf met Tracy and Julian online at work and relayed the events above, asking if either of them had explored ideas around the extension of "pi" to polygons, which they said they hadn't.

Tracy wanted the diameter to go corner to corner, not face to face. And when we thought about an equilateral triangle, she suggested we think about a polygon's "radius" instead of diameter. If you imagine a "centre" point of any polygon, the radius goes from there to any corner.

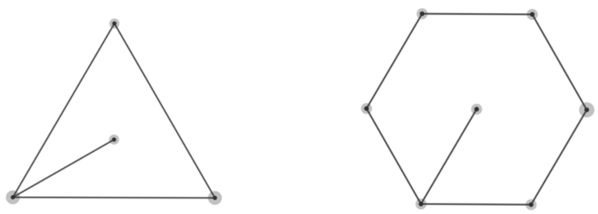

Figure 2: A regular triangle and hexagon, with "centre" and "radius"

We were encouraged in this interpretation by remembering that the "centres" we were using would allow us to draw a circle going through all the corners (Figure 3), something we recalled being labelled the "circumcircle" of a polygon.

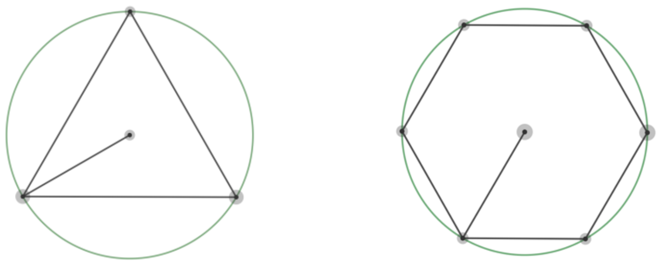

Figure 3: The circumcircle of a triangle and hexagon

This was another short conversation. Julian went away wanting to generalise how many times the radius (x2) would fit around the perimeter of any regular polygon. The next day, he wrote to Tracy and Alf, saying:

> "With the image of the circumcircle of regular polygons, I ended up with a series of polygons inscribed in a unit circle and thinking about the perimeter of these polygons. The ratio of the perimeter to the circle diameter (which is the same as twice our polygon "radius") approaches pi from below quite quickly...".

On the axes below (created by Julian on Desmos), that ratio is plotted on the y-axis, against the number of sides of the polygon on the x-axis (Figure 4).

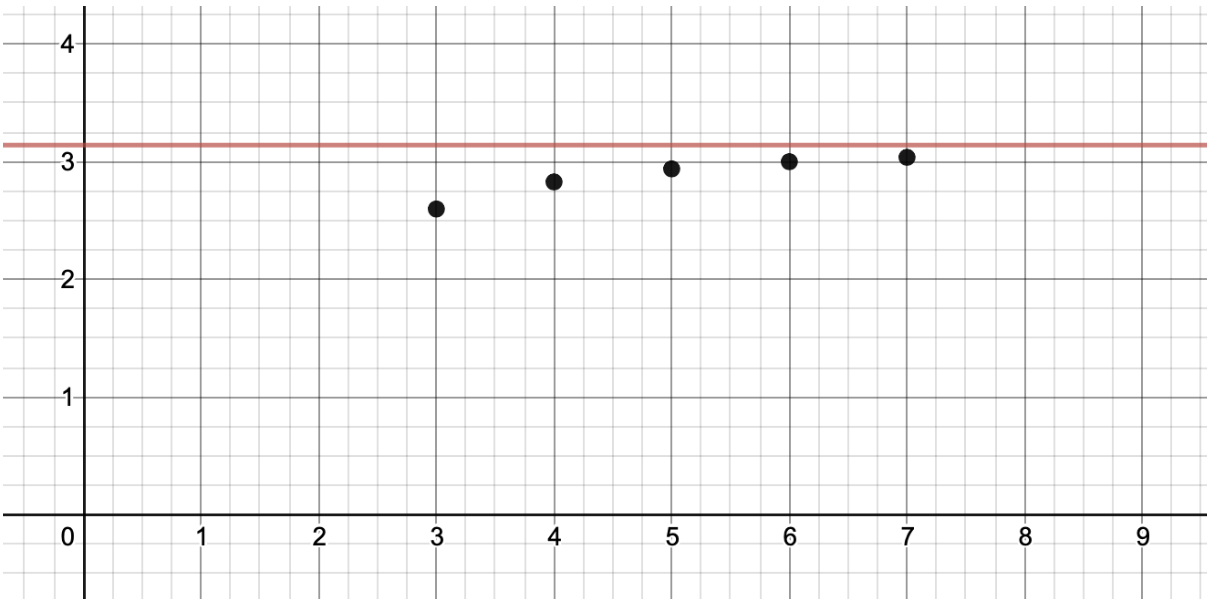

Figure 4: Values of "pi" for different regular polygons

How satisfying to see the values of different polyon's pi converging on the π we know!

We now saw that the way Iris and Alf had originally been thinking could lead to a definition of "radius" as going from centre to the middle of a face, and this has its own circle associated with it.

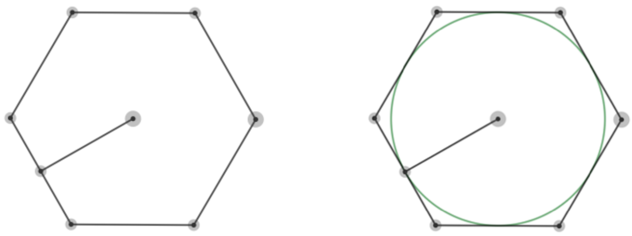

Figure 5: A different “radius” of a hexagon, and associated “in-circle”

Alf then created some equivalent values to Julian’s, for how this second definition of polygon pi changes for different sided polygons (see Figure 6).

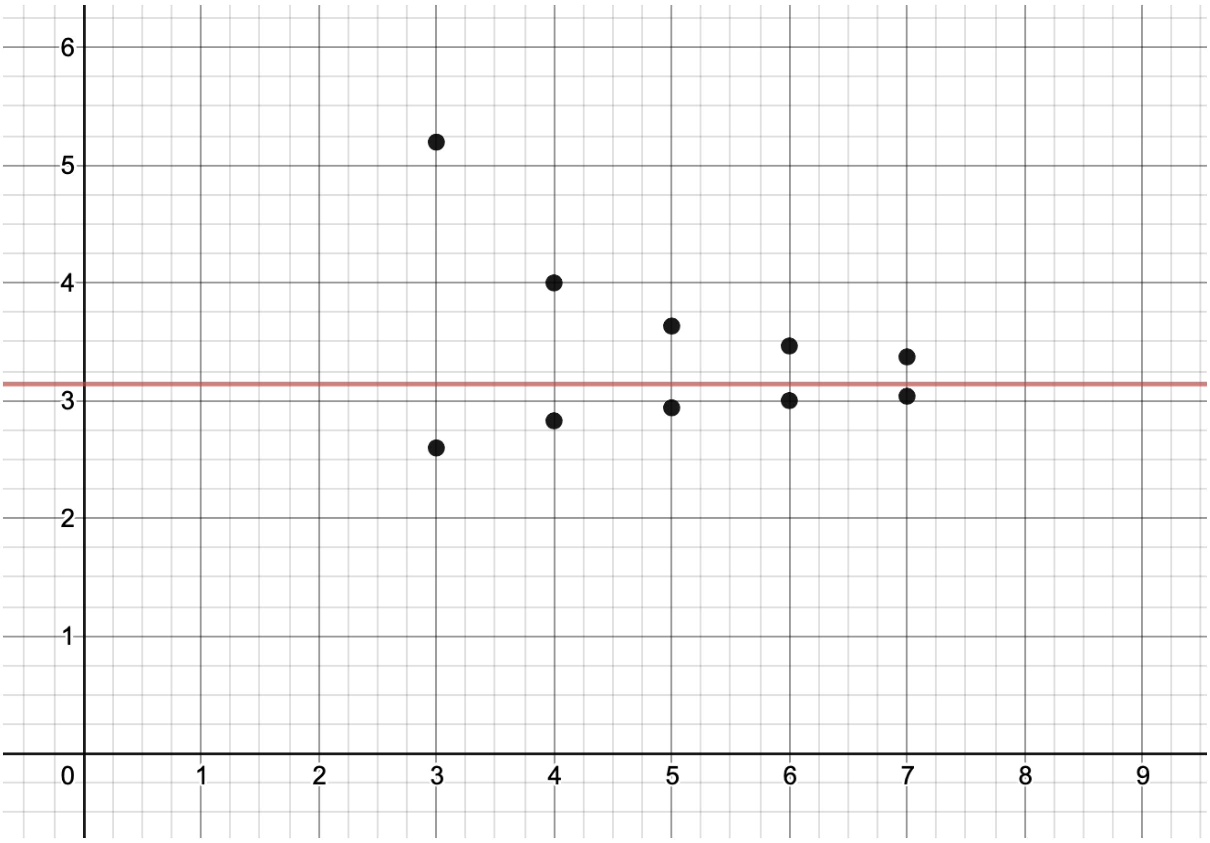

Figure 6: Different values of “pi” for regular polygons, converging on π

We could now see that what we had in effect been doing was estimating the value of the circumference of our green circles, either using polygons drawn inside the circle, or drawn outside. And this was work we remembered doing before (less satisfying to realise this!). What felt different, however, was a new way of interpreting what we were doing, defining a radius of a polygon and calculating its own value of “pi”, as the ratio between twice its “radius” and its perimeter.

In the history of mathematics, major breakthroughs have come through finding definitions that open up new territories and help make sense of known results. And these definitions have been human decisions. Openness to re-thinking definitions, and their consequences, is one attribute we associate with thinking mathematically and recognise enjoying, as well as feeling that engaging in such work with students can help unmask mathematics from its claims to eternal truth. There are usually very good reasons for us having the mathematical definitions we do, for example, the way operations of multiplication and division are defined (although, historically, these were not at all obvious for vectors, for example, especially in dimensions beyond 3). The effectiveness of our standard definitions does not stop it being possible to explore alternatives, if only to realise why our definitions are as they are. Indeed, their very effectiveness might warrant exploration of alternatives, to gain an appreciation of definitional decisions. We could envisage rich discussions in response to questions such as “How should we add fractions? What would we want the answer to tell us?” (put in any operation you want), or, “How shall we define a power that is a fraction/negative?” and then working with the consequences of a range of possibilities.

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