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Comment on “Analysis of the superdeterministic invariant-set theory in a hidden-variable setting”

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In a recent paper (arXiv:2107.04761), Sen critiques a superdeterministic model of quantum physics, Invariant Set Theory, proposed by one of the authors. He concludes that superdeterminism is ‘unlikely to solve the puzzle posed by the Bell correlations’. He also claims that the model is neither local nor ψ -epistemic. We here detail multiple problems with Sen’s argument.

I. INTRODUCTION

Motivated by the fractal geometry of chaos, Invariant Set Theory [1] is a locally causal deterministic theory of quantum physics. It is based on the premise that the universe is a dynamical system evolving on a fractal set of measure zero in state space. This set is assumed to be the attractor of an unknown nonlinear deterministic dynamical law. The fractal geometry is therefore an algebraic replacement for the dynamical law. Since the set is an attractor, it is invariant under the action of the dynamical law, hence the name Invariant Set Theory, hereafter IST.

The essential assumption of IST is that the laws of physics at the most basic level describe the geometry of this invariant set. This means putative (mentally constructed, and hence conceivable) states which do not lie on the invariant set are, by construction, inconsistent with the laws of physics. Because of this, counterfactual states associated with worlds where some experiment might have been performed but wasn’t can lie off the invariant set. Indeed, in quantum mechanics this happens in Bell-type experiments: For these situations, IST predicts that counterfactual measurements on a prepared state (i.e., those that did not in fact happen) typically do not lie on the invariant set. For this reason IST formally violates the assumption of Statistical Independence in Bell’s Theorem, and the theory can be said to be superdeterministic [2, 3].

One of the characteristic features of IST is that ensembles of trajectories in state space correspond to a subset of the complex Hilbert-space of quantum theory. This arises because the fractal structure of the invariant set is homeomorphic to the set of p -adic integers ($p \in \mathbb{N}$ but not necessarily a prime number). This subset is defined by normalized vectors with complex coefficients, and thus similar to elements of the quantum mechanical Hilbert space. However, the squared amplitudes of these vectors are always rational numbers (i.e. of the form m/p , where $m \in \mathbb{N}_0$) and their phases are of the form $2\pi n/p$ (where $n \in \mathbb{Z}$). A Hilbert state cannot describe an ensemble of trajectories on the invariant set when these conditions are not met.

In his paper [4], Sen uses a definition of Statistical Independence which is ambiguous in its physical interpretation, and

the interpretation he chooses without considering the physics does not apply to IST. However, this confusion about what Statistical Independence physically means is omnipresent in the literature and certainly not specific to Sen’s criticism of IST. We will therefore address this general point in a separate paper [5]. In this present paper, we will instead focus on the shortcomings of Sen’s analysis of IST.

The most glaring shortcoming of Sen’s analysis is, confusingly, that he develops his own version of Invariant Set Theory, and then critiques it, for example, claiming it is neither local nor ψ -epistemic. However, Sen’s version is not an accurate representation of IST. In particular, he does not use the number-theoretic incompatibility of the rationality requirements for the phase and amplitude mentioned above. This incompatibility is central to the interpretation of IST. On top of this, Sen misinterprets some of the fundamental concepts underpinning IST, such as the notion of counterfactual indefiniteness. As a result, his conclusions do not apply to IST. The Sections below provide detailed criticisms on Sen’s version of Invariant Set Theory.

II. CLAIM THAT IST IS NON-LOCAL

In the usual formulation introduced by Bell, in a hidden-variable theory the measurement outcome is typically expressed as a function of some hidden variable λ and the measurement settings (in a Bell experiment these are described by the values of binary variables X and Y). In a superdeterministic theory, one cannot assume that λ can be varied independently of X and Y .

In IST, it is the requirement that states lie on the invariant set that prevents X and Y from being varied independently of λ . One can say it like this: in a superdeterministic hidden variables theory, the full specification of the prepared state already contains information about the measurement settings. The theory then trivially fulfils local causality because it is unnecessary to add the measurement settings a second time to calculate the measurement outcome. But the theory then violates statistical independence, simply because the measurement settings are correlated with the measurement settings.

It is a common mistake to not realize that the measurement settings effectively appear twice in any superdeterministic theory, once in the information that determines the evolution of

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the prepared state, and once as the setting itself. Needless to say, if one does not keep in mind that the measurement setting cannot be varied independently of the hidden variables, one arrives at the odd conclusion that the superdeterministic theory violates local causality. That, of course, makes no sense. The entire reason Bell had to even consider superdeterminism was that it's a way to restore local causality.

Concretely, Sen states that

“The outcome at the second wing $O_2(|\psi\rangle_{\text{singlet}}, \hat{B}(\hat{M}_1, \hat{b}, \hat{m}_1), \hat{C}(\hat{M}_2, \hat{c}, \hat{m}_2), k)$ depends on the exact measurement setting at the first wing. Therefore, our model is nonlocal.”

This is incorrect. Varying X , keeping Y and λ fixed, takes the state of the world off the invariant set: this counterfactual violates the rationality condition described in the Introduction. Hence there is no violation of local causality (and hence nonlocality). The model would be non-local if the exact measurement setting at the first wing provided additional information to calculate the outcome at the second wing. It does not because that information was already effectively in the hidden variables.

III. USE OF COUNTERFACTUALS

Sen constructs his own hidden variable version of IST. This includes versions of the rationality conditions in [1]. However, Sen never invokes the number-theoretic incompatibility between angles and their cosines in his paper. Why is this? The answer can be found in Section IV Part B b) where Sen states:

“Here we make use of the fact that the exact apparatus orientations are continuously fluctuating with time (see section II). If the ordering is changed, then the apparatuses will get used at different times than previously. Therefore, the exact apparatus orientations will also change.”

In IST, if we do a which-way experiment at some time t , it is certainly possible to do an interference experiment at some small time $t + \delta t$ later (say by removing the second half-silvered mirror of the interferometer). This can be explained as follows: since rational angles can be arbitrarily close to rational cosines, it is perfectly possible for $\phi(t)$ to be compatible with a which-way experiment and $\phi(t + \delta t)$ to be compatible with an interferometer experiment, where ϕ denotes the relative phase difference in the two arms of the interferometer.

Where the number-theoretic incompatibility bites is when we consider the possibility of performing a which-way experiment simultaneously with an interferometric experiment. Since this is not possible in reality, this can only mean that we are asking whether we have a theory which permits counterfactual experiments where everything (and hence ϕ) is held fixed, except for the measurement settings (the second half-silvered mirror in the case of the interferometer).

In case it seems overly metaphysical to be discussing such matter as counterfactual definiteness when assessing the nature of the laws of physics, there is an important quote from Bell himself that it worth repeating. Concerning his eponymous theorem he says: [6]

“I would insist here on the distinction between analysing various physical theories, on the one hand, and philosophising about the unique real world on the other hand. In this matter of causality, it is a great inconvenience that the real world is given to us once only. We cannot know what would have happened if something had been different. We cannot repeat an experiment changing just one variable the hand of the clock will have moved, and the moons of Jupiter. Physical theories are more amenable in this respect. We can calculate the consequences of changing free elements in the theory, be they only initial conditions, and so can explore the causal structure of the theory. I insist that [Bell's Theorem] is primarily an analysis of certain kinds of physical theory.”

Bell's quote makes clear that the matter of counterfactual definiteness is a vital ingredient of his theorem. By not fixing t when varying the order of Stern-Gerlach experiments (or of switching from a which-way to an interferometric experiment) Sen does not engage the vital number-theoretic incompatibility in Invariant Set Theory. In this way, his model is not a fair reflection of the explanatory power of Invariant Set Theory.

It is important to recognise that violating counterfactual definiteness in this way does not imply that the sub-ensembles of entangled particles in a Bell experiment are in any way statistically inequivalent.

By not engaging this vital number-theoretic incompatibility, Sen makes another mistake. In considering the sequential Stern-Gerlach experiment, Sen assumes that in IST it is necessary for the individual Stern-Gerlach devices to be precisely orthogonal. This is not the case at all! The relative orientation of the devices is effectively arbitrary within the constraints of the rationality assumption. The number-theoretic incompatibility that lies at the heart of IST does not assume orthogonal devices.

IV. MISUNDERSTANDING OF DISCRETENESS

Sen also makes an error in how he applies the discrete grid implicit to IST over the Bloch sphere, which then leads him to claim erroneously that “The dependence upon the past input \hat{p} arises from the method of rotation which ensures that the final exact orientation \hat{A} has the same orientation relative to \hat{a} as the initial exact orientation \hat{P} had relative to \hat{p} ,” (or more formally, $\delta\hat{a} = \delta\hat{p}$ in his model, where $\delta\hat{p} = \hat{P} - \hat{p}$, and $\delta\hat{A} = \hat{A} - \hat{p}$). This leads him to assume “In the single-particle case, the hidden variable \hat{P} and the experimentally set orientations \hat{p} and \hat{a} encoded the exact preparation setting \hat{A} corresponding to the eigenstate $|+\rangle_A$ ”.

This misunderstands how the discrete grid of valid points (formed by the intersection of the N lines of latitude and N lines of longitude allowed) forces eigenstates measured to the closest point to the eigenstate we'd expect for the given operator measured (e.g. eigenstates of \hat{P} instead of those of \hat{p}), but, depending on the idealised operator measured, the change between the ideal and experimentally-allowed eigenstate has no reason to have to be the same for different operators. Therefore, there is no reason $\delta\hat{A}$ need equal $\delta\hat{P}$, and so there are additional features required to encode the exact preparation setting \hat{A} corresponding to the eigenstate $|+\rangle_A$, than just hidden variable \hat{P} and the experimentally set orientations \hat{p} and \hat{a} .

Given this discreteness is a key part of the theory, allowing experimentally-testable differences from standard quantum mechanics to potentially be probed [7], Sen's misunderstanding of it further undermines the applicability of his model to analysing IST.

V. CLAIM OF CAUSAL INFLUENCE/ CONSPIRACY

Sen repeated brings up the vaguely-defined ideas of "causal influence" and "arbitrarily large correlations set up by initial conditions"; however, these concepts have been thoroughly dissected previously, and shown to be conflation of wide numbers of differing ideas, used commonly in the literature to attack superdeterminism for emotive, rather than rational, reasons. See [2] and [3] for a full discussion of why these ideas are not useful for discussing superdeterministic theories.

VI. CLAIM THAT IST IS ψ -ONTIC

Sen states

"The model is ψ -ontic [8, 9]. This can be noted by the fact that the individual outcomes depend on the bit-string representation of the quantum state. Given the bit string for a particular run, the exact quantum state prepared for that run can be inferred."

This statement is incorrect because the bit-string is not the ontic state of the model, it is an ensemble of ontic states. Each "bit" in the string is an ontic state, labelling a particular trajectory on the invariant set. To give a simple example, a string of

the type (*aaaabbbb*) would correspond to a Hilbert state that is an ensemble of 8 underlying ontic states, the first 4 of which end up in detector eigenstate $|a\rangle$ and the last 4 in eigenstate $|b\rangle$. The first bit of that string could belong to many other Hilbert states, hence the theory is ψ -epistemic according to the definition of [8].

While it is debatable whether this definition of ψ -epistemic is meaningful [10], IST fulfills it regardless.

We may note that Sen actually states correctly that the hidden variable λ is a function of the position on the string and not the string itself, he just fails to use that definition in his evaluation of the properties of the model.

VII. CLAIM "CONTINUOUSLY FLUCTUATING IN TIME"

As we mention above, Sen claims that the exact setting is

"...a fundamentally uncontrollable and unknowable quantity that is continuously fluctuating in time."

This quote yet again shows Sen's confusion about the basic ideas of IST. Nowhere in the theory is anything said about anything continuously fluctuating in time – the uncontrollability/unknowability is, as mentioned above, due to the fact that counterfactual options (such as those from taking the counterfactual as to what would happen if we had measured a conjugate variable) are not on the same fractal invariant set as the factual option which occurred, and so are not valid ways the world could be.

VIII. CONCLUSION

We have laid out numerous mistakes in Sen's recent criticism of Invariant Set Theory and maintain that the model is deterministic, locally causal and ψ -epistemic, and yet reproduces the predictions of quantum mechanics.

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