Theta liftings on double covers of orthogonal groups

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We study the generalized theta lifting between the double covers of split special orthogonal groups, which uses the non-minimal theta representations constructed by Bump, Friedberg and Ginzburg. We focus on the theta liftings of non-generic representations and make a conjecture that gives an upper bound of the first non-zero occurrence of the liftings, depending only on the unipotent orbit. We prove both global and local results that support the conjecture.

Contents

Citations to previous work

This thesis paper is based on [19].

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1 Introduction

The classical Jacobi theta function has been understood through the theory of the Weil representation defined on the metaplectic double cover of the symplectic group. See [26], [25] and [28]. Motivated by these works, the theory of automorphic representations on metaplectic covers of reductive groups has been much developed over the past half centry (for example, [18], [11] and [5]).

One of the most important applications of the Weil representation is the construction of the classical theta correspondence. The construction ultilizes the fact that the Weil representation is a minimal representation. Globally, this means that its Fourier coefficients associated to all non-trivial unipotent orbits except for the minimal one are zero. However, there may not exist any such minimal representations for certain groups (for example, see [27]). In those cases, one may seek representations that are small enough to enable an analogous theta correspondence.

Let F be a number field containing the group of fourth roots of unity, with the ring of adeles $\mathbb{A}$. In [5], Bump-Friedberg-Ginzburg constructed the global theta representation Θ_m on the double cover $\widetilde{\mathrm{SO}}_m(\mathbb{A})$ of the split odd orthogonal group $\mathrm{SO}_m(\mathbb{A})$ as the residues of certain metaplectic Eisenstein series. In contrast to the Weil representation, such a theta representation is only small in the sense that its Fourier coefficients attached to most unipotent orbits vanish. Nonetheless, the same authors show in [6] that one can still construct a non-minimal theta correspondence by using such a small automorphic representation.

Suppose $(\pi, \mathcal{V})$ is an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Let $\mathrm{SO}_{2k'}$ be a split even special orthogonal group. The natural embedding $\mathrm{SO}_{2k'}(\mathbb{A}) \times \mathrm{SO}_{2k+1}(\mathbb{A}) \hookrightarrow \mathrm{SO}_{2k+2k'+1}(\mathbb{A})$ is covered by an embedding of $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A}) \times \widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ into $\widetilde{\mathrm{SO}}_{2k+2k'+1}(\mathbb{A})$. Consider the theta representation $\Theta_{2k+2k'+1}$

on the metaplectic double cover $\widetilde{\mathrm{SO}}_{2k+2k'+1}(\mathbb{A})$. For any $\varphi \in \mathcal{V}$ and $\theta_{2k+2k'+1}$ a function in the representation space of $\Theta_{2k+2k'+1}$, Bump-Friedberg-Ginzburg defined a function on $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$ (see equation (2) of [6]) via the integral

$$f(h) = \int\limits_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi(g)\bar{\theta}_{2k+2k'+1}(h,g)dg. \tag{1.1}$$

Functions of the form $f(h)$ generate a genuine automorphic representation $\Theta_{2k+2k'+1}(\pi)$ on the cover $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$.

By fixing the representation π and using the theta representations $\widetilde{\mathrm{SO}}_{2k+2k'+1}(\mathbb{A})$ with varying k', one obtains a tower of liftings of the representation π to the groups $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$. According to [6], for a fixed genuine cuspidal automorphic representation π on $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$, one has the following:

1. As an automorphic representation of $\widetilde{\mathrm{SO}}_{8k}(\mathbb{A})$, $\Theta_{10k+1}(\pi) \neq 0$.

2. If $\Theta_{2k+2k'+1}(\pi) = 0$, then $\Theta_{2k+2k'-1}(\pi) = 0$.

In view of these, it is natural to ask when the first non-zero lifting occurs along the tower. In [6], Bump-Friedberg-Ginzburg show that if $\Theta_{4k+5}(\pi)$ is generic as an automorphic representation of $\widetilde{\mathrm{SO}}_{2k+4}(\mathbb{A})$, then the representation π of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ must be generic as well. They also make a conjecture that a generic representation π of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ should lift to a generic representation of $\widetilde{\mathrm{SO}}_{2k+4}(\mathbb{A})$.

However, little is known if the representations are not generic, i.e. not supported on the maximal unipotent orbit. In this thesis, we make a general conjecture on when the lift of a given automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ is nonzero, depending only on the unipotent orbit that the representation is supported on. Recall that unipotent orbits are parametrized by partition of integers. We require that the representation is supported on a unipotent orbit whose corresponding partition consists of only odd integers. This condition implies that the attached unipotent subgroup $V_{2,\mathcal{O}}$ defined in Section 3 below is the unipotent radical of a parabolic subgroup.

Conjecture 1.1. *Let $(\pi, \mathcal{V})$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose π is supported on the unipotent orbit*

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p})$$

with $n_1 > n_2 > \cdots > n_p \geqslant 0$ and $r_i > 0$ for all i. Let $l = r_1 + r_2 + \cdots + r_p$ be the length of the partition corresponding to $\mathcal{O}$. Then π lifts nontrivially to an automorphic representation $\Theta_{4k+2l+3}(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$ which is supported on the unipotent orbit

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1)).$$

Conjecture 1.1 gives an upper bound of the first non-zero occurrence of the theta lifting. In the generic case where $\mathcal{O} = (2k+1)$, Conjecture 1.1 agrees with the conjecture made in [6] that is mentioned above. In Proposition 4.10, we show that this conjecture is consistent with the "dimension equation" described in [12], [13] and [10], which proposes dimension constraints on when the first non-zero lifting may occur. We remark that not every orbit of SO_{2k+1} has all odd parts, but we do not know what to expect when there are even parts in the partition.

In this thesis, we prove the following theorem which gives evidence towards the above conjecture.

Theorem 1.2. *Let $(\pi, \mathcal{V})$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose the theta lifting $\Theta_{4k+2l+3}(\pi)$, as a representation of*

$\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$, *has a non-zero Fourier coefficient associated with the unipotent orbit*

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1)) .$$

Then the representation π has a non-zero Fourier coefficient associated with the unipotent orbit

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p}) .$$

In the generic case, Theorem 1.2 agrees with the result proved in [6] and mentioned above.

Moveover, we establish a local counterpart of Theorem 1.2 (which is new even in the generic case). In the local setting, we turn our attention to the category of genuine admissible representations of the double covers of the split special orthogonal groups over a non-archimedean local field F. In [5], the local theta representation of the double cover of a split odd orthogonal group $\widetilde{\mathrm{SO}}_{2k+1}(F)$ is constructed as the image of an intertwining operator. Fourier coefficients as the global analytic tool are replaced by the twisted Jacquet modules. We prove the following result:

Theorem 1.3. *Let $(\pi, \mathcal{V})$ be an irreducible genuine admissible representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. Suppose there exists an irreducible admissible representation $\Theta(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+2l+2}(F)$ such that, as representations of the group $\widetilde{\mathrm{SO}}_{2k+1}(F) \times \widetilde{\mathrm{SO}}_{2k+2l+2}(F)$,*

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times\widetilde{\mathrm{SO}}_{2k+2l+2}}(\Theta_{4k+2l+3}, \pi\otimes\Theta(\pi)) \neq 0. \tag{1.2}$$

Furthermore, suppose there exists a non-trivial character $\psi_{\mathcal{O}'}$ (explicitly defined in Section 6) associated with the unipotent orbit $\mathcal{O}'$ such that the corresponding twisted Jacquet module of $\Theta(\pi)$ is non-zero. Then there exists a non-trivial character associated with the unipotent orbit $\mathcal{O}$ such that the corresponding twisted Jacquet module *of π is also non-zero.*

In the case of the classical symplectic-orthogonal theta liftings based on the Weil representation, Ginzburg-Gurevich [14] give both upper and lower bounds for the first non-zero occurrence in the theta tower. These bounds can be parametrized by the partition corresponding to the unipotent orbit that supports the cuspidal automorphic representation of $\mathrm{Sp}_{2k}(\mathbb{A})$.

The liftings considered here are related to the extension of Langlands functoriality to covering groups as follows. According to [23], [29] and [24], one can define the dual group of a metaplectic group. In the case of the metaplectic double cover of SO_m, we have that

$$^L\widetilde{\mathrm{SO}}_m^0 \cong \mathrm{SO}_m(\mathbb{C}).$$

This suggests that there should be a lifting of genuine automorphic representations from $\widetilde{\mathrm{SO}}_{2k+1}$ to $\widetilde{\mathrm{SO}}_{2k'}$ corresponding to the inclusion of $\mathrm{SO}_{2k+1}(\mathbb{C})$ into $\mathrm{SO}_{2k'}(\mathbb{C})$ with $k' > k$.

The study of non-generic cuspidal automorphic representations is an important part of understanding the automorphic discrete spectrum. Jiang [15] proposed a conjecture that relates Arthur parameters to the maximal unipotent orbit that supports an automorphic representation. In Section 13 of [20], Leslie conjectured an extension of Arthur parameters to the metapletic groups. In view of these works, the results of this thesis are conjecturally related to the question of how Arthur parameters behave under the non-minimal theta liftings introduced in [6].

This thseis is organized as follows: After setting up the basic notations, we briefly recall the construction of the metaplectic double cover of the split orthogonal groups in Section 2. In Section 3, we review the definition of the unipotent orbits and recall the construction of the Fourier coefficients and the twisted Jacquet modules associated with a unipotent orbit and a generic character. These are the global and local tools for proving the respective main theorems. In Section 4, we briefly recall the construction of both the local and global theta representations of the metaplectic double cover of SO_{2k+1}. We then prove an invariance property of the theta representations which is crucial for the proof of the main theorem. We also establish the compatibility of Conjecture 1.1 with the dimension equation. In Section 5, we prove the global main theorem Theorem 1.2. Lastly, the local theory is treated in Section 6.

2 Preliminaries

In this thesis, we let F be either a number field with ring of adeles $\mathbb{A}$ or a non-archimedean local field with the ring of integers $\mathcal{O}_F$. For the latter, we require that the characteristic of its residue field is not 2. Fix an algebraic closure $\bar{F}$ of F, and denote by

$$\mu_4 = \{x \in \bar{F} : x^4 = 1\}$$

the group of all forth roots of unity in $\bar{F}$. Throughout this thesis, we assume that F contains μ_4. We fix a choice of non-trivial additive character $\psi : F \setminus \mathbb{A} \to \mathbb{C}^\times$ when F is a number field, or $\psi : F \to \mathbb{C}^\times$ which is unramified when F is a non-archimedean local field.

2.1 Split orthogonal groups

For any positive integer m, let $\mathrm{SO}_m(F)$ denote the split special orthogonal group consisting of $g \in \mathrm{SL}_m$ such that $gJ_mg^T = J_m$, where g^T is the transpose of g and

$$J_m = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \in \mathrm{Mat}_{m\times m}(F).$$

The matrix J_m corresponds to a non-degenerate bilinear form on F^m, where $\mathrm{SO}_m(F)$ is the group of isometries upon fixing a basis. In this way, we have the maximal split torus $T_m \subset \mathrm{SO}_m$ consisting of diagonal elements

$$\mathrm{diag}(t_1, \cdots, t_n, 1, t_n^{-1}, \cdots, t_1^{-1}), \quad t_i \in F^\times \tag{2.1}$$

if $m = 2n+1$, or

$$\mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}), \quad t_i \in F^\times \tag{2.2}$$

if $m = 2n$. We can take the ordering of the roots so that the positive roots correspond to upper triangular matrices. If we denote by α_i ($1 \leqslant i \leqslant n$) the positive simple roots with respect to the usual order in the standard Borel subgroup B_m of upper triangular matrices, and $e_{i,j}$ be the $m \times m$ matrix with value one on the (i,j)-th entry and zero elsewhere, then we let the corresponding one-parameter root subgroup be $r \to x_{\alpha_i}(r)$, where

$$x_{\alpha_i}(r) = \exp\left(r(e_{i,i+1} - e_{n-i,n-i+1})\right)$$

if m is odd, and

$$x_{\alpha_i}(r) = \begin{cases} \exp\left(r(e_{i,i+1} - e_{n-i,n-i+1})\right) & 1 \leqslant i < n \\ \exp\left(r(e_{n-1,n+1} - e_{n,n+2})\right) & i = n \end{cases}$$

if m is even.

We fix an embedding of any two orthogonal groups SO_{2k+1} and $\mathrm{SO}_{2k'}$ into $\mathrm{SO}_{2k+2k'+1}$ by

$$\iota(h,g) = \begin{pmatrix} a & 0 & b \\ 0 & g & 0 \\ c & 0 & d \end{pmatrix} \in \mathrm{SO}_{2k+2k'+1}, \quad g \in \mathrm{SO}_{2k+1}, h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SO}_{2k'}. \tag{2.3}$$

2.2 Central extensions

Let G be a group and A be an abelian group. A group $\widetilde{G}$ is called a central extension of G by A if it satisfies the short exact sequence

$$1 \to A \xrightarrow{i} \widetilde{G} \xrightarrow{\mathbf{p}} G \to 1 \tag{2.4}$$

and $i(A) \subset Z(\widetilde{G})$ where $Z(\widetilde{G})$ is the center of $\widetilde{G}$.

Two central extensions $\widetilde{G}_1$ and $\widetilde{G}_2$ of G by A are equivalent if there exists a homomorphism $\phi : \widetilde{G}_1 \to \widetilde{G}_2$ that induces the identity maps on G and A and the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{i_1} & \widetilde{G}_1 & \xrightarrow{\mathbf{p}_1} & G & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & A & \xrightarrow{i_2} & \widetilde{G}_2 & \xrightarrow{\mathbf{p}_2} & G & \longrightarrow & 1. \end{array} \tag{2.5}$$

The equivalence classes of central extensions $\mathrm{CExt}(G, A)$ of G by A are completely determined by $H^2(G, A)$ where A is considered as a trivial G-module. Given a 2-cocycle $\sigma : G \times G \to A$, there is a central extension $\widetilde{G}$ with elements in the set $G \times A$ and its group operation defined by

$$(g, a) \cdot (g', a') = (gg', aa'\sigma(g, g')).$$

In this case, we have that $i(a) = (1, a)$ and $\mathbf{p}\left((g, a)\right) = g$. It is easy to verify that two 2-cocycles σ_1 amd σ_2 give equivalent central extensions $\widetilde{G}_1$ and $\widetilde{G}_2$ if and only if they are different by a 2-coboundary. This gives an injective map

$$\Phi : H^2(G, A) \to \mathrm{CExt}(G, A). \tag{2.6}$$

To see this is surjective, let $\widetilde{G}$ be a central extension of G by A that satisfies (2.4).

Suppose $s : G \to \mathbf{p}^{-1}(G)$ is a section of $\mathbf{p}$. Then

$$\sigma : G \times G \to A : \sigma(g, g') = s(g)s(g')s(gg')^{-1} \tag{2.7}$$

defines a 2-cocycle in $H^2(G, A)$. The two covers $\Phi(\sigma)$ and $\widetilde{G}$ are isomorphic via the map $(g, a) \mapsto s(g)i(a)$.

For any subgroup $H \subset G$, we say that H is split in $\widetilde{G}$ by the central extension (2.4) if there exists a section $s : H \to \mathbf{p}^{-1}(H)$ that is also a homomorphism.

2.3 The local double cover $\widetilde{\mathrm{SO}}_m(F)$

Suppose F is a non-archimedean local field. Let $(\ ,\ )_4 : F^\times \times F^\times \to \mu_4$ be the 4-th order Hilbert symbol. Let $\widetilde{\mathrm{SL}}_m(F)$ be the metaplectic 4-fold cover defined by Matsumoto in [21] using the Steinberg symbol corresponding to $(\ ,\ )_4^{-1}$. This covering group is a central extension of $\mathrm{SL}_m(F)$ by μ_4 that satisfies the short exact sequence

$$1 \to \mu_4 \xrightarrow{i} \widetilde{\mathrm{SL}}_m(F) \xrightarrow{\mathbf{p}} \mathrm{SL}_m(F) \to 1. \tag{2.8}$$

In [1], Banks-Levy-Sepanski gave an explicit choice of section $\mathfrak{s}$ of $\mathrm{SL}_m(F)$ and the corresponding 2-cocycle which we denote by σ_m. This 2-cocycle has the advantage that it has the "block compatibility property" (See Section 2 of [1]), which implies that the restriction of σ_m on diagonal elements is given by

$$\sigma_m(\mathrm{diag}(t_1, t_2, \cdots, t_m), \mathrm{diag}(t'_1, t'_2, \cdots, t'_m)) = \prod_{1 \leqslant i < j \leqslant m} (t_i, t'_j)_4. \tag{2.9}$$

Pulling back the image of $\mathrm{SO}_m(F)$ in $\mathrm{SL}_m(F)$, we obtain a central extension $\widetilde{\mathrm{SO}}_m(F)$ of $\mathrm{SO}_m(F)$, with the corresponding 2-cocycle as the restriction of σ_m to $\mathrm{SO}_m(F)$. For any $t, t' \in T_m$ of the form (2.1) or (2.2) depending on the parity of m, we have

$$\sigma_m(t,t') = \prod_{i=1}^{n} (t_i, t'_i)_4^{-1}. \tag{2.10}$$

Let $SN : \mathrm{SO}_m(F) \to F^\times/(F^\times)^2$ be the spinor norm homomorphism, with the kernel denoted by $\mathrm{SO}_m^{(2)}(F)$. From [4] and [5], one has that

$$\sigma_m^2(g,g') = (SN(g), SN(g'))_2\,, \tag{2.11}$$

where $(\ ,\ )_2$ is the quadratic Hilbert symbol and the equality is up to a coboundary. This shows that σ_m^2 is trivial on the large subgroup $\mathrm{SO}_m^{(2)}(F)$. As a result, the cover $\widetilde{\mathrm{SO}}_m(F)$ performs almost like a double cover. We call this cover $\widetilde{\mathrm{SO}}_m(F)$ the metaplectic double cover of $\mathrm{SO}_m(F)$. We have the following short exact sequence.

$$1 \to \mu_4 \xrightarrow{i} \widetilde{\mathrm{SO}}_m(F) \xrightarrow{\mathbf{p}} \mathrm{SO}_m(F) \to 1. \tag{2.12}$$

Fix an embedding $\mu_4 \hookrightarrow \mathbb{C}^\times$ and identify μ_4 with its image in $\mathbb{C}^\times$. We say that a representation ρ of any subgroup of $\widetilde{\mathrm{SO}}_m(F)$ is genuine if $\rho(i(\varepsilon)g) = \varepsilon\rho(g)$ for any $\varepsilon \in \mu_4$.

By Section 4 of [23], any upper triangular unipotent subgroup of $\mathrm{SO}_m(F)$ is split in $\widetilde{\mathrm{SO}}_m(F)$ via the trivial section. For any non-archimedean F with residue characteristic not equal to 2, the hyperspecial maximal compact subgroup $K_m = \mathrm{SO}_m(\mathcal{O}_F)$ is split in $\widetilde{\mathrm{SO}}_m(F)$ via certain section κ. The choice of κ may not be unique. We fix a choice of such κ by following [18]

2.4 The global double cover $\widetilde{\mathrm{SO}}_m(\mathbb{A})$

Let F be a number field with its ring of adeles $\mathbb{A}$. For each ν of F, we have the local metaplectic double cover $\widetilde{\mathrm{SO}}_{2k+1}(F_\nu)$ defined in the previous subsection satisfying the short exact sequence

$$1 \to \mu_4 \xrightarrow{i_\nu} \widetilde{\mathrm{SO}}_m(F_\nu) \xrightarrow{\mathbf{p}_\nu} \mathrm{SO}_m(F_\nu) \to 1. \tag{2.13}$$

Note that when ν is archimedean, $F_\nu = \mathbb{C}$ since F contains μ_4. In this case, the metaplectic cover is split.

Let K_ν be a maximal compact subgroup of $\mathrm{SO}_m(F_\nu)$ which is split in the local metaplectic cover $\widetilde{\mathrm{SO}}_m(F_\nu)$. By the previous subsection, for almost all the places ν, we may pick $K_\nu = \mathrm{SO}_m(\mathcal{O}_\nu)$ where $\mathcal{O}_\nu$ is the ring of integer of the local field F_ν. We identify the embedded image of $\mu_4 \hookrightarrow \mathbb{C}^\times$ over different places ν. Let S_0 be the finite set of places which contains the archimedean place and those places with residue characteristic equal to 2. For any $S \supset S_0$, define

$$\widehat{\mathrm{SO}}_m(\mathbb{A})_S = \prod_{\nu \in S} \widetilde{\mathrm{SO}}_m(F_\nu) \prod_{\nu \notin S} \kappa_\nu(K_\nu), \tag{2.14}$$

where κ_ν is the local section fixed in the previous subsection. Let $\widehat{\mu_4}$ be the subgroup of $\widehat{\mathrm{SO}}_m(\mathbb{A})$ generated by elements of the form $i_{\nu_1}(\xi) i_{\nu_2}^{-1}(\xi)$ with $\nu_1, \nu_2 \in S$ and $\xi \in \mu_4$. Then we define the global metaplectic double cover via the direct limit

$$\widetilde{\mathrm{SO}}_m(\mathbb{A}) = \varinjlim \widehat{\mu_4} \setminus \widehat{\mathrm{SO}}_m(\mathbb{A})_S. \tag{2.15}$$

We have the short exact sequence

$$1 \to \mu_4 \xrightarrow{i} \widetilde{\mathrm{SO}}_m(\mathbb{A}) \xrightarrow{\mathbf{p}} \mathrm{SO}_m(\mathbb{A}) \to 1. \tag{2.16}$$

Note that if we let $\prod'_\nu \widetilde{\mathrm{SO}}_m(F_\nu)$ be the restricted product with respect to $\kappa_\nu(K_\nu)$, and $\mu_4^* = \{(i_\nu(\xi_\nu))_\nu \in \prod'_\nu \widetilde{\mathrm{SO}}_m(F_\nu) : \prod_\nu \xi_\nu = 1\}$, then $\widetilde{\mathrm{SO}}_m(\mathbb{A}) = \mu_4^* \setminus \prod'_\nu \widetilde{\mathrm{SO}}_m(F_\nu)$.

Similar to the local setting, any upper triangular unipotent subgroup of $\mathrm{SO}_m(\mathbb{A})$ is split canonically over $\widetilde{\mathrm{SO}}_m(\mathbb{A})$. Recall that for each place ν, there is the local section $\mathfrak{s}_\nu$ given by [1]. The product $\mathfrak{s} = \prod_\nu \mathfrak{s}_\nu$ is a section when restricted to $\mathrm{SO}_m(F)$. It follows that $\mathrm{SO}_m(F)$ is split in $\widetilde{\mathrm{SO}}_m(\mathbb{A})$ via this section (see [17] for example).

3 Fourier coefficients associated to a unipotent orbit

In this section, we explain how to associate Fourier coefficients with a unipotent orbit. In the global case, these Fourier coefficients are integrations over certain unipotent subgroups, while the local counterparts are the twisted Jacquet modules with respect to certain unipotent subgroups. In either case, the argument proceeds regardless of whether the group is linear or metaplectic. Therefore, we only illustrate this association in the non-metaplectic setup so as to simplify the notations.

3.1 Unipotent orbits

The standard reference for unipotent orbits is [22]. Let F be any field (not necessarily containing all 4th roots of unity), with a fixed algebraic closure $\bar{F}$. Unipotent orbits of the group SO_{2k+1} are parametrized by partitions of the integer $2k+1$ with the restriction that each even number occurs with even multiplicity. For an orbit $\mathcal{O}$ corresponding to the partition $(p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s})$ where $p_i > p_{i+1}$ and $r_i > 0$ for all i, we write

$$\mathcal{O} = (p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}).$$

Define the length of the partition to be $l = r_1 + r_2 + \cdots + r_s$.

Suppose $\mathcal{O}_1 = (p_1p_2\cdots p_r)$ and $\mathcal{O}_2 = (q_1q_2\cdots q_s)$. We impose a partial order by $\mathcal{O}_1 \geqslant \mathcal{O}_2$ if $p_1 + \cdots + p_i \geqslant q_1 + \cdots + q_i$ for all $1 \leqslant i \leqslant s$.

Let $\mathcal{O} = (p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s})$. For each p_i, we associate r_i copies of the diagonal matrix

$$h_{p_i}(t) = \mathrm{diag}(t^{p_i-1}, t^{p_i-3}, \cdots, t^{3-p_i}, t^{1-p_i}), \quad t \in F^\times.$$

We obtain a one parameter torus element $h_{\mathcal{O}}(t)$ with non-increasing powers of t

along the diagonal after combining and rearranging all the $h_{p_i}(t)$'s. For example, if $\mathcal{O} = (3^2 1)$, then

$$h_{\mathcal{O}}(t) = \mathrm{diag}(t^2, t^2, 1, 1, 1, t^{-2}, t^{-2}).$$

The conjugation action of $h_{\mathcal{O}}(t)$ on the unipotent radical U of the upper triangular Borel subgroup B of SO_{2k+1} induces a filtration on U:

$$I_{2k+1} \subset \cdots \subset V_{2,\mathcal{O}} \subset V_{1,\mathcal{O}} \subset V_{0,\mathcal{O}} = U,$$

where each $V_{i,\mathcal{O}}$ is the subgroup of U generated by

$$\{x_\alpha(r) \in U : h_{\mathcal{O}}(t) x_\alpha(r) h_{\mathcal{O}}(t)^{-1} = x_\alpha(t^j r) \quad \text{for some} \quad j \geqslant i\}.$$

Let

$$M(\mathcal{O}) := T \cdot \{x_{\pm\alpha}(r) : h_{\mathcal{O}}(t) x_\alpha(r) h_{\mathcal{O}}(t)^{-1} = x_\alpha(r)\}.$$

Then $P(\mathcal{O}) = M(\mathcal{O}) V_{1,\mathcal{O}}$ is a standard maximal parabolic subgroup of SO_{2k+1}.

In general, $V_{2,\mathcal{O}}$ is not the unipotent radical of a parabolic subgroup. However, if $\mathcal{O}$ is odd, then $V_{2,\mathcal{O}}$ is the unipotent radical of the parabolic subgroup $P(\mathcal{O})$. We say that a unipotent orbit $\mathcal{O} = (p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s})$ is odd if all the integers p_i are odd. In this case, we have $V_{1,\mathcal{O}} = V_{2,\mathcal{O}}$.

Let $V_{2,\mathcal{O}}^{(1)}$ be the commutator subgroup of $V_{2,\mathcal{O}}$. The Levi subgroup $M(\mathcal{O})$ acts by conjugation on the maximal abelian quotient $V_{2,\mathcal{O}}/V_{2,\mathcal{O}}^{(1)}$. Over the algebraic closure $\bar{F}$, there is a dense open orbit under this action. Pick a representative u_0 of this orbit, and set $M^{u_0}(\mathcal{O})(\bar{F})$ to be its stabilizer. It follows from the general theory (for example, see [8]) that the connected component of $M^{u_0}(\mathcal{O})$ is a reductive group.

3.2 Generic characters

Suppose F is a number field with ring of adeles $\mathbb{A}$. Let $L_{2,\mathcal{O}} = V_{2,\mathcal{O}}/V_{2,\mathcal{O}}^{(1)}$ be the maximal abelian quotient of $V_{2,\mathcal{O}}$. The action of $M(\mathcal{O})$ on $L_{2,\mathcal{O}}$ induces an action of $M(\mathcal{O})(F)$ on the character group

$$\widehat{L_{2,\mathcal{O}}(F) \setminus L_{2,\mathcal{O}}}(\mathbb{A}) \cong L_{2,\mathcal{O}}(F).$$

We say that a character $\psi_{\mathcal{O}} : L_{2,\mathcal{O}}(F) \setminus L_{2,\mathcal{O}}(\mathbb{A}) \to \mathbb{C}^\times$ is a generic character if the connected component of its stabilizer in $M(\mathcal{O})(F)$ is the group of rational points of a reductive group of type $M^{u_0}(\mathcal{O})$. We extend any such character trivially to $V_{2,\mathcal{O}}(F) \setminus V_{2,\mathcal{O}}(\mathbb{A})$. There may exist infinitely many $M(\mathcal{O})$-conjugacy classes of such generic characters for a specific unipotent orbit $\mathcal{O}$.

Example 3.1. Let $\mathcal{O}$ be the unipotent orbit corresponding to the partition (3^21) in SO_7. We have

$$V_{2,\mathcal{O}} = \left\{ \begin{pmatrix} I_2 & X & Y \\ & I_3 & X^* \\ & & I_2 \end{pmatrix} \in \mathrm{SO}_7 : \; X^* := -J_3 X^T J_2, \, Y^T J_2 + J_2 Y = 0 \right\}.$$

A choice of generic character is $\psi_{\mathcal{O}} : V_{2,\mathcal{O}}(F) \setminus V_{2,\mathcal{O}}(\mathbb{A}) \to \mathbb{C}^\times$ given by

$$\psi_{\mathcal{O}}(v) = \psi(v_{1,3} + v_{2,5}).$$

By Pontryagin dualilty, we may identify each character with an element in $L_{2,\mathcal{O}}(F) \cong$

$\mathrm{Mat}_{2\times 3}(F)$. The above generic character $\psi_{\mathcal{O}}$ corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The rank of the matrix and the fact that its row space is not totally isotropic are invariant under the action of $M(\mathcal{O})(F) \cong \mathrm{GL}_2(F) \times \mathrm{SO}_3(F)$ on $\mathrm{Mat}_{2\times 3}(F)$. Any matrix in $\mathrm{Mat}_{2\times 3}(F)$ with full rank and non-totally-isotropic row space corresponds to a generic character.

3.3 Global Fourier coefficients

Let $(\pi, \mathcal{V})$ be an automorphic representation of $\mathrm{SO}_{2k+1}(\mathbb{A})$. We define the Fourier coefficients of π associated with a unipotent orbit $\mathcal{O}$ and a generic character by the following:

Definition 3.2. Let $\psi_{\mathcal{O}} : V_{2,\mathcal{O}}(F) \setminus V_{2,\mathcal{O}}(\mathbb{A}) \to \mathbb{C}^\times$ be a generic character associated with a unipotent orbit $\mathcal{O}$ in SO_{2k+1}. For an automorphic function $\varphi \in (\pi, \mathcal{V})$, the Fourier coefficient of φ with respect to $\psi_{\mathcal{O}}$ is

$$F_{\psi_{\mathcal{O}}}(\varphi)(g) = \int\limits_{[V_{2,\mathcal{O}}]} \varphi(ug)\psi_{\mathcal{O}}(u)\, du. \tag{3.1}$$

Henceforth, we use $[K]$ to denote $K(F) \setminus K(\mathbb{A})$ for any group K. We say that the orbit $\mathcal{O}$ supports π if there exists some $\varphi \in \mathcal{V}$ and generic character $\psi_{\mathcal{O}}$ such that the above integral is non-zero. Otherwise, we say that $\mathcal{O}$ does not support the representation $(\pi, \mathcal{V})$.

Example 3.3. This is the motivating example. Suppose the unipotent orbit is $\mathcal{O} = (2k+1)$ in SO_{2k+1}. Then $V_{2,\mathcal{O}} = U$ is the maximal unipotent subgroup of upper triangular matrices. The Gelfand-Graev character $\psi_{U,a} : U(F) \setminus U(\mathbb{A}) \to \mathbb{C}^\times$ defined by

$$\psi_{U,a}(v) = \psi(v_{1,2} + v_{2,3} + \cdots + v_{k,k+1} + av_{k,k+2}), \quad a \in F^\times \tag{3.2}$$

is a generic character. For a generic automorphic representation π of $\mathrm{SO}_{2k+1}(\mathbb{A})$, its Fourier coefficients associated with $\mathcal{O}$ are the Whittaker coefficients.

Remark 3.4. For a fixed generic character $\psi_{\mathcal{O}}$, suppose the Fourier coefficients of π associated to $\mathcal{O}$ and $\psi_{\mathcal{O}}$ is identically zero, i.e. $F_{\psi_{\mathcal{O}}}(\varphi)(g) = 0$ for all $\varphi \in \pi$ and $g \in \mathrm{SO}_{2k+1}(\mathbb{A})$. For any $\psi'_{\mathcal{O}}$ that lies in the same conjugacy class as $\psi_{\mathcal{O}}$, there exists some $m \in M(\mathcal{O})(F)$ such that $\psi'_{\mathcal{O}}(u) = \psi_{\mathcal{O}}(mum^{-1})$ for any $u \in \mathrm{SO}_{2k+1}(\mathbb{A})$. By the automorphicity of φ, this implies that

$$F_{\psi'_{\mathcal{O}}}(\varphi_\pi)(g) = F_{\psi_{\mathcal{O}}}(\varphi_\pi)(mg) = 0.$$

This result can be generalized to any conjugacy class of characters (not necessarily generic) under the action of a discrete subgroup.

3.4 Twisted Jacquet modules

Suppose now F is a non-archimedean local field. Let N be a unipotent subgroup of $\mathrm{SO}_{2k+1}(F)$, with $\psi_N : N \to \mathbb{C}^\times$ a character on N. Let $(\pi, \mathcal{V})$ be a smooth representaion of $\mathrm{SO}_{2k+1}(F)$. Suppose there exists a subgroup $M \subset \mathrm{SO}_{2k+1}(F)$ which normalizes N and stablizes the character ψ_N. Consider the subspace $\mathcal{V}(N, \psi_N)$ of $\mathcal{V}$ generated by vectors of the form $\{\pi(u)v - \psi_N(u)v \,|\, v \in \mathcal{V}, u \in N\}$. The twisted Jacquet module of π with respect to ψ_N is defined by $J_{N,\psi_N}(\pi) = \mathcal{V}/\mathcal{V}(N, \psi_N)$. The subgroup M acts smoothly on $J_{N,\psi_N}(\pi)$. If ψ_N is trivial, we denote it by $J_N(\pi)$ and call it the Jacquet module of π with respect to N. This defines an exact functor between the categories of smooth representations of the two groups

$$J_{N,\psi_N} : Rep(\mathrm{SO}_{2k+1}(F)) \to Rep(M).$$

Recall that for a unipotent orbit $\mathcal{O}$ in SO_{2k+1}, the Levi subgroup $M(\mathcal{O})(F)$ acts on $L_{2,\mathcal{O}}(F)$, and hence on the character group

$$\widehat{L_{2,\mathcal{O}}(F)} \cong L_{2,\mathcal{O}}(F).$$

Again, we only look at those generic characters whose connected component of the stablizer under this action is of the same Cartan type as $M^{\mu_0}(\mathcal{O})(\bar{F})$.

Definition 3.5. Let $(\pi, \mathcal{V})$ be an admissible representation of $\mathrm{SO}_{2k+1}(F)$. The twisted Jacquet module of π associated to a unipotent orbit $\mathcal{O}$ and a generic character $\psi_{\mathcal{O}} : V_{2,\mathcal{O}}(F) \to \mathbb{C}^{\times}$ is given by

$$J_{V_{2,\mathcal{O}},\psi_{\mathcal{O}}}(\pi). \tag{3.3}$$

We say that the unipotent orbit $\mathcal{O}$ supports π if there exists some generic character $\psi_{\mathcal{O}}$ such that (3.3) is non-zero.

3.5 Wavefront sets

We have the following definition that applies to both global and local situations.

Definition 3.6. Let $(\pi, \mathcal{V})$ be either an automorphic representation of $\mathrm{SO}_{2k+1}(\mathbb{A})$ (where $\mathbb{A}$ is the ring of adeles of a number field F) or an admissible representation of $\mathrm{SO}_{2k+1}(F)$ (where F is a non-archimedean local field). The wavefront set $\mathcal{O}(\pi)$ of π is the set of unipotent orbits of SO_{2k+1} such that $\mathcal{O} \in \mathcal{O}(\pi)$ if and only if $\mathcal{O}$ supports π and $\mathcal{O}'$ does not support π for any $\mathcal{O}' > \mathcal{O}$.

Remark 3.7. In all the known cases, the set $\mathcal{O}(\pi)$ is singleton for any irreducible automorphic representation π of a split reductive group $G(\mathbb{A})$. We do not assume that this is necessarily true, though we may write $\mathcal{O}(\pi) = \mathcal{O}_0$ when we mean that the wavefront set of π consists of one single unipotent orbit $\mathcal{O}_0$.

4 Theta representations and the tower of theta liftings

In this section, we review the construction of the theta representations in both local and global settings. Locally, the theta representation is defined as the irreducible Langlands quotient of the principle series representation attached to an exceptional character. Globally, it is realized as the residue of certain Eisenstein series.

The key property of the theta representations is that their wavefront sets contain only relatively small unipotent orbits. This allows one to generalize the idea of theta liftings by using theta representations as the integral kernel. In particular, we will talk about tower of theta liftings and end our discussion by stating a conjecture giving an upper bound of the first non-zero lifting along one particular tower of theta liftings.

4.1 Principal series representations

Suppose F is a non-archimedean local field. Let $B(F) = T(F)U(F)$ be the standard Borel Subgroup with $T(F)$ the maximal split torus and $U(F)$ the maximal unipotent subgroup of upper triangular matrices.

Note that $\widetilde{T}(F) = \mathbf{p}^{-1}(T(F))$, the inverse image of $T(F)$ in $\widetilde{\mathrm{SO}}_{2k+1}(F)$, is a Heisenberg group instead of an abelian group. By the Stone-von Neumann theorem, the genuine irreducible representations of $\widetilde{T}(F)$ are parameterized by genuine characters on its center $Z(\widetilde{T}(F))$ in the following way: For any genuine central character χ, we may extend it to a character χ' on any maximal abelian subgroup $A \subset \widetilde{T}(F)$ containing $Z(\widetilde{T}(F))$. The induced representation $i(\chi) := \mathrm{ind}_A^{\widetilde{T}(F)} \chi'$ is irreducible, and independent of the choices of A and χ'. Moreover, the map $\chi \mapsto i(\chi)$ is a bijection between genuine representations of $Z(\widetilde{T}(F))$ and $\widetilde{T}(F)$. See Section 5 of [23] for more details.

Extend $i(\chi)$ to a representation of $\widetilde{B}(F) = \widetilde{T}(F)U(F)$ by allowing the trivial action of $U(F)$. Let δ_B be the modular character of $B(F)$. We have the principal series representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$ defined by $\mathrm{Ind}(\chi) := \mathrm{Ind}_{\widetilde{B}(F)}^{\widetilde{\mathrm{SO}}_{2k+1}(F)}\left(\delta_B^{1/2} i(\chi)\right)$.

Following the notations of [5], we have $Z(\widetilde{T}(F)) \cong T_Z(F) \times \mu_4$, where $T_Z(F) \subset T(F)$ is the subgroup of diagonal matrices with each entry a fourth power. Let $\mathbf{s} = (s_1, \cdots, s_k) \in \mathbb{C}^k$ and consider the character

$$\chi_{\mathbf{s}} : T(F) \to \mathbb{C}^\times \tag{4.1}$$

$$\mathrm{diag}(t_1, \cdots, t_k, 1, t_k^{-1}, \cdots, t_1^{-1}) \mapsto \prod_{i=1}^{k} |t_i|^{s_i}. \tag{4.2}$$

Regarding $\chi_{\mathbf{s}}$ as a character on $T_Z(F)$ by restriction, and hence a character on $Z(\widetilde{T}(F))$, the resulting parabolically induced representation $\mathrm{Ind}(\chi_{\mathbf{s}})$ is irreducible when $\mathbf{s}$ is in the general position. These are the unramified principal series representations.

4.2 Local theta representations

We continue to assume that F is a non-archimedean local field. For every positive root α of SO_{2k+1}, there is a standard embedding $\tau_\alpha : \mathrm{SL}_2 \to \mathrm{SO}_{2k+1}$ which restricts to $x_{\pm\alpha}$ on the upper and lower triangular matrices in SL_2. We say that a root α is metaplectic if the pullback via τ_α in $\widetilde{\mathrm{SO}}_{2k+1}(F)$ is non-trivial. In SO_{2k+1}, the long roots are the metaplectic ones. Set

$$n(\alpha) = \begin{cases} 1 & \alpha \text{ is non-metaplectic} \\ 2 & \alpha \text{ is metaplectic.} \end{cases}$$

For any character χ, we define

$$\chi_\alpha(t) = \chi\left(\tau_\alpha\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}^{n(\alpha)}\right). \tag{4.3}$$

Following [5] or [18], $\mathrm{Ind}(\chi)$ is irreducible if $\chi_\alpha \neq |\cdot|^{\pm 1}$ for any positive root α. On the other hand, if $\chi_\alpha = |\cdot|$ for all positive simple roots, then $\mathrm{Ind}(\chi)$ is reducible. In this case, we call χ an exceptional character.

Any ω in the Weyl group W of SO_{2k+1} acts on $T(F)$ by conjugation, and therefore it acts on any character χ of $T(F)$. Thus, one can define the intertwining operator $M_\omega : \mathrm{Ind}(\chi) \to \mathrm{Ind}({}^\omega\chi)$ by

$$M_\omega(f)(g) = \int\limits_{(U(F)\cap\omega U(F)\omega^{-1})\backslash U(F)} f(\omega^{-1}ug)du \tag{4.4}$$

whenever the integral is convergent. Following [9], this integral has meromorphic continuation for general χ.

Let $\mathbf{s}_\theta = (k/2, (k-1)/2, \cdots, 1/2) \in \mathbb{C}^k$. A direct computation shows that $\chi_\theta := \chi_{\mathbf{s}_\theta}$ is exceptional, and therefore $\mathrm{Ind}(\chi_\theta)$ is reducible. Let ω_0 be the longest Weyl group element in SO_{2k+1}, where ${}^{\omega_0}\chi_\theta = \chi_\theta^{-1}$. Then we have the following result:

Theorem 4.1. *The intertwinning operator $M_{\omega_0} : \mathrm{Ind}(\chi_\theta) \to \mathrm{Ind}(\chi_\theta^{-1})$ has an irreducible and self-contragredient image, which is isomorphic to the unique irreducible quotient of* $\mathrm{Ind}(\chi_\theta)$.

We call the representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$ on this irreducible Langlands quotient the local theta representation, denoted by Θ_{2k+1}.

Proof. This is Theorem 2.2 of [5]. $\square$

The theta representation Θ_{2k+1} of the double cover $\widetilde{\mathrm{SO}}_{2k+1}(F)$ is a small representation in the terminology of [5]. It agrees with the minimal representation when $k=2$ or 3. When $k=4$ or 5, $\mathcal{O}(\Theta_{2k+1})$ is the singleton set containing the next smallest unipotent orbit. In [5], Bump-Friedberg-Ginzburg gave the exact description of the wavefront set of Θ_{2k+1} as follows:

Proposition 4.2. *Let Θ_{2k+1} be the theta representation of the double cover $\widetilde{\mathrm{SO}}_{2k+1}(F)$. Let n be a positive integer such that $n=\lfloor k/2 \rfloor$. Then*

$$\mathcal{O}(\Theta_{2k+1}) = \begin{cases} (2^{2n}1) & \text{if } k=2n, \\ (2^{2n}1^3) & \text{if } k=2n+1. \end{cases} \tag{4.5}$$

Proof. See either [5] or Section 2 of [17]. □

Let r be an integer such that $1 \leqslant r \leqslant k$. Suppose $P_r = M_r U_r$ is a maximal standard parabolic subgroup of SO_{2k+1} with the indicated Levi decomposition, where $M_r = \mathrm{GL}_r \times \mathrm{SO}_{2k-2r+1}$ and U_r is the unipotent radical. By the block compatibility of the 2-cocycle $\sigma_{2k+1} \in H^2(\mathrm{SO}_{2k+1}(F), \mu_4)$ we adapt from [1] (See Section 2.3), for any $(g,h),(g',h') \in \mathrm{GL}_r(F) \times \mathrm{SO}_{2k-2r+1}(F)$, we have

$$\sigma_{2k+1}\left((g,h),(g',h')\right) = \sigma_{\mathrm{GL}_r}(g,g')^2 \left(\det g, \det g'\right)_4 \sigma_{2k-2r+1}(h,h'), \tag{4.6}$$

where σ_{GL_r} is the 2-cocycle in $H^2(\mathrm{GL}_r(F), \mu_4)$ defined in [1]. Following [18], the twist of the square of the cocycle σ_{GL_r} by the 4-th order Hilbert symbol pre-composed with the determinant corresponds to a metaplectic double cover of $\mathrm{GL}_r(F)$. Thus, the pullback of the Levi subgroup $M_r(F)$ in $\widetilde{\mathrm{SO}}_{2k+1}(F)$ is the direct product amalgamated at μ_4 given by

$$\widetilde{M}_r(F) = \widetilde{\mathrm{GL}}_r(F) \times_{\mu_4} \widetilde{\mathrm{SO}}_{2k-2r+1}(F),$$

where $\widetilde{\mathrm{GL}}_r(F)$ is the metaplectic double cover of $\mathrm{GL}_r(F)$.

Let χ_{GL_r} be the character on the diagonal torus of $\mathrm{GL}_r(F)$ given by

$$\mathrm{diag}(t_1, \cdots, t_r) \mapsto \prod_{i=1}^{r} |t_i|^{\frac{r-i}{2}}. \tag{4.7}$$

This is an exceptional character in the sense of [18] and [7]. Therefore, we can form the exceptional representation Θ_{GL_r} of the double cover $\widetilde{\mathrm{GL}}_r(F)$ following their construction. In this way, we have the following:

Proposition 4.3. *Let Θ_{2k+1} be the local theta representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. Considered as a representation of $\widetilde{\mathrm{GL}}_r \times \widetilde{\mathrm{SO}}_{2k-2r+1}(F)$, the Jacquet module of Θ_{2k+1} with respect to U_r is ismorphic to $\Theta_{\mathrm{GL}_r} \otimes \Theta_{2k-2r+1}$. In the case when $r = k$, $\Theta_{2k-2r+1}$ is the trivial representation.*

Proof. This is Theorem 2.3 of [5], or Proposition 1 of [6]. □

4.3 Global theta representations

Suppose now F is a number field, with its ring of adeles $\mathbb{A}$. The global theta representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ is given by the residues of an Eisenstein series associated to an exceptional character. We follow [5] for the construction.

Let $\chi_{\mathbf{s}} : T(F) \setminus T(\mathbb{A}) \to \mathbb{C}^\times$ given by

$$\mathrm{diag}(t_1, \cdots, t_k, 1, t_k^{-1}, \cdots, t_1^{-1}) \mapsto \prod_{i=1}^{k} |t_i|^{s_i} \tag{4.8}$$

be the character attached to the complex parameter $\mathbf{s} = (s_1, s_2, \cdots, s_k) \in \mathbb{C}^k$. Similar to the local setting in Section 4.1, we can form the induced representation $\mathrm{Ind}(\chi_{\mathbf{s}}) := \mathrm{Ind}_{\widetilde{B}(\mathbb{A})}^{\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})} \delta_B^{1/2} i(\chi_{\mathbf{s}})$ by first inducing χ_s from $T_Z(\mathbb{A})$ to $\widetilde{T}(\mathbb{A})$.

Let $K = \prod_\nu K_\nu$ be a maximal compact subgroup where $K_\nu = \mathrm{SO}_m(\mathcal{O}_{F_\nu})$ for any non-archimedean F_ν with odd residue characteristic. Denote its inverse image in $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ by $\widetilde{K}$. There is a section $\mathbf{s} \mapsto f_{\mathbf{s}} \in \mathrm{Ind}(\chi_{\mathbf{s}})$ such that the restriction of $f_{\mathbf{s}}$ to $\widetilde{K}$ is independent of $\mathbf{s}$.

For any $f_{\mathbf{s}} \in \mathrm{Ind}(\chi_{\mathbf{s}})$, let the Eisenstein series associated to $f_{\mathbf{s}}$ be

$$E(g, f_{\mathbf{s}}) = \sum_{\gamma \in B(F) \backslash \mathrm{SO}_{2k+1}(F)} f_{\mathbf{s}}(\gamma g), \quad g \in \widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A}). \tag{4.9}$$

This sum is absolutely convergent in a suitable cone and has meromorphic continuation to all $\mathbf{s} \in \mathbb{C}^k$.

Recall that $\mathbf{s}_\theta = (k/2, (k-1)/2, \cdots, 1/2)$ corresponds to an exceptional character χ_θ in the previous section. In the global setting, the Eisenstein series (4.9) has a pole at $\mathbf{s}_\theta$. We let

$$\theta_f = \mathrm{res}_{\mathbf{s}=\mathbf{s}_\theta} E(g, f_{\mathbf{s}}). \tag{4.10}$$

This is an automorphic form on $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ that is square integrable. We call the irreducible representation on the subspace of $L^2(\mathrm{SO}_{2k+1} \backslash \widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A}))$ spanned by such θ_f the global metaplectic theta representation Θ_{2k+1} of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$.

Let $P_r = M_r U_r$ be the maximal parabolic subgroup of SO_{2k+1} with Levi subgroup $M_r = \mathrm{GL}_r \times \mathrm{SO}_{2k-2r+1}$. Similar to the local context, the pullback of $M_r(\mathbb{A})$ in $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ is given by

$$\widetilde{M}_r(\mathbb{A}) = \widetilde{\mathrm{GL}}_r(\mathbb{A}) \times_{\mu_4} \widetilde{\mathrm{SO}}_{2k-2r+1}(\mathbb{A}),$$

where $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ is the global metaplectic double cover of $\mathrm{GL}_r(\mathbb{A})$.

Define the global exceptional character on the torus of $\mathrm{GL}_r(\mathbb{A})$ by

$$\mathrm{diag}(t_1, \cdots, t_r) \mapsto \prod_{i=1}^{r} |t_i|^{\frac{r-i}{2}},$$

with the corresponding global exceptional representation Θ_{GL_r} of $\widetilde{\mathrm{GL}}_r(\mathbb{A})$ (See [7] and [18]).

Proposition 4.4. *Let θ be a function in the theta representation Θ_{2k+1} of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Considered as a function of $(g, h) \in \widetilde{\mathrm{GL}}_r(\mathbb{A}) \times \widetilde{\mathrm{SO}}_{2k-2r+1}(\mathbb{A})$, the integral*

$$\int_{U_r(F)\backslash U_r(\mathbb{A})} \theta(u(g, h))\, du$$

is in the space of the automorphic representation $\Theta_{\mathrm{GL}_r} \otimes \Theta_{2k-2r+1}$.

Proof. This is the global version of Proposition 4.3. The case $r = 1$ is proved in [5]. The general case is proved in Theorem 1.2 of [16]. □

The global theta representation Θ_{2k+1} has the same smallness property given by 4.11. We have the following description of its wavefront set.

Proposition 4.5. *Let Θ_{2k+1} be the global theta representation of the metaplectic double cover $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Then*

$$\mathcal{O}(\Theta_{2k+1}) = \begin{cases} (2^{2n}1) & \text{if } k = 2n, \\ (2^{2n}1^3) & \text{if } k = 2n+1. \end{cases} \tag{4.11}$$

Proof. See Theorem 4.2(i) of [5], or Proposition 2 of [6]. □

For the following discussion, we need new notations for the unipotent subgroups. If $r < k/4$ is a positive integer, we denote $H_{r,2k+1}$ the unipotent radical of the maximal parabolic subgroup of SO_{2k+1} whose Levi part is $\mathrm{GL}_r \times \mathrm{SO}_{2k-2r+1}$. In other words, $H_{r,2k+1}$ consists of upper triangular matrices of the form

$$\left\{ \begin{pmatrix} I_r & x & * \\ & I_{2k-2r+1} & x^* \\ & & I_r \end{pmatrix} \in \mathrm{SO}_{2k+1} : \quad x^* = -J_{2k-2r+1} x^T J_r \right\},$$

where $*$ denotes the entries in the correponding positions determined by the condition that the matrix is orthogonal. Similarly, let $H_{r,2k-2r+1}$ be the unipotent radical of the standard maximal parabolic subgroup of $\mathrm{SO}_{2k-2r+1}$ with Levi subgroup $\mathrm{GL}_r \times \mathrm{SO}_{2k-4r+1}$. Via the embedding (2.3), we identify $H_{r,2k-2r+1}$ with its image in SO_{2k+1}. Hence, $H_{r,2k-2r+1}$ consists of matrices of the form

$$\left\{ \begin{pmatrix} I_r & & & & \\ & I_r & y & * & \\ & & I_{2k-4r+1} & y^* & \\ & & & I_r & \\ & & & & I_r \end{pmatrix} \in \mathrm{SO}_{2k+1} : \quad y^* = -J_{2k-4r+1} y^T J_r \right\}.$$

For any $u = (u_{i,j}) \in H_{r,2k+1}(\mathbb{A})$, we define the character $\psi_1 : H_{r,2k+1}(F) \setminus H_{r,2k+1}(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\psi_1(u) = \psi(\sum_{j=1}^{r} u_{j,j+r}).$$

Proposition 4.6. *Fix a function $\theta \in \Theta_{2k+1}$. The integral*

$$f(g) = \int\limits_{H_{r,2k+1}(F)\setminus H_{r,2k+1}(\mathbb{A})} \theta(ug)\psi_1(u)\,du \tag{4.12}$$

is left invariant by $H_{r,2k-2r+1}(\mathbb{A})$. That is, $f(g) = f(vg)$ for any $v \in H_{r,2k-2r+1}(\mathbb{A})$.

Proof. Via the embedding (2.3), the center $Z(H_{r,2k-2r+1})$ consists of matrices of the form

$$\left\{\begin{pmatrix} I_r &&&& \\ & I_r && z & \\ && I_{2k-4r+1} && \\ &&& I_r & \\ &&&& I_r \end{pmatrix} : \quad z \in \mathrm{Mat}_{r\times r}, \quad z^T J_r + J_r z = 0 \right\}.$$

We first expand (4.12) against $Z(R_{r,2k-2r+1})(F)\backslash Z(R_{r,2k-2r+1})(\mathbb{A})$. Embed the group $\mathrm{GL}_r(F)$ into $\mathrm{SO}_{2k+1}(F)$ via

$$h \hookrightarrow \mathrm{diag}(I_r, h, I_{2k-4r+1}, h^*, I_r), \quad \forall h \in \mathrm{GL}_r(F).$$

The $\mathrm{GL}_r(F)$-action on $Z(H_{r,2k-2r+1})(F) \setminus Z(H_{r,2k-2r+1})(\mathbb{A})$ induces an action on its character group, which may be identified with $Z(H_{r,2k-2r+1})(F)$. This action preserves the rank of the matrices in $Z(H_{r,2k-2r+1})(F) \cong \big\{z \in \mathrm{Mat}_{r\times r},\ z^T J_r + J_r z = 0\big\}$.

The rank of any $z \in Z(H_{r,2k-2r+1})(F)$ must be even. Suppose $\mathrm{Rank}(z) = 2q$, where $0 < 2q \leqslant r$. We may choose a representative of this z-orbit by

$$\begin{pmatrix} 0 & z_q \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{r\times r}(F), \tag{4.13}$$

where

$$z_q = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_q, -\lambda_q, \cdots, -\lambda_2, -\lambda_1) \in \mathrm{Mat}_{2q\times 2q}(F), \lambda_i \in F^\times \quad \forall i = 1, 2, \cdots, q.$$

This gives the corresponding character on $[Z(H_{r,2k-2r+1})]$ by:

$$\psi_{Z,q}(v) = \psi(\lambda_1 v_{1,r-2q+1} + \lambda_2 v_{2,r-2q+2} + \cdots + \lambda_q v_{q,r-q}),$$

where

$$v \in Z(H_{s,2k-2s+1})(\mathbb{A}) \cong \mathrm{Mat}_{r\times r}(\mathbb{A}).$$

The contribution of $\psi_{Z,q}$ in the expansion is given by the integral

$$\int\limits_{[H_{r,2k+1}]} \int\limits_{[Z(H_{r,2k-2r+1})]} \theta(uvg)\psi_1(u)\psi_{Z,q}(v)\, dv\, du. \tag{4.14}$$

Notice that the character $\psi_1\psi_{Z,q}$ coincides with a generic character associated to the unipotent orbit $\mathcal{O}_q = (4^{2q}3^{r-2q}1^{2k-2q-3r+1})$ (or $(4^r 1^{2k-4r+1})$ in the case $r = 2q$). Moreover, we have $V_{2,\mathcal{O}_q} \subseteq H_{r,2k+1}Z(H_{r,2k-2r+1})$. As a result, the integral (4.14) contains an inner integral that is a Fourier coefficient of θ with respect to the unipotent orbit $\mathcal{O}_q$. The integral (4.14) is then identically zero because any such Fourier coefficient of θ is zero by Proposition 4.5. By Remark 3.4, every character that lies in the same orbit as $\psi_{Z,q}$ has zero contribution. Thus, only the integral corresponding to the trivial character contributes, and (4.12) is equal to

$$f(g) = \int\limits_{[H_{r,2k+1}Z(H_{r,2k-2r+1})]} \theta(ug)\psi_1(u)\, du. \tag{4.15}$$

As the center $Z(H_{r,2k-2r+1})$ is now contained in the domain of integration, we can further expand (4.15) against

$$[H_{r,2k-2r+1}/Z(H_{r,2k-2r+1})] \cong \mathrm{Mat}_{r\times(2k-4r+1)}(F) \setminus \mathrm{Mat}_{r\times(2k-4r+1)}(\mathbb{A}). \tag{4.16}$$

Embed the group $\mathrm{GL}_r(F) \times \mathrm{SO}_{2k-4r+1}(F)$ into $\mathrm{SO}_{2k+1}(F)$ via

$$(h_1, h_2) \hookrightarrow \mathrm{diag}(I_r, h_1, h_2, h_1^*, I_r), \quad (h_1, h_2) \in \mathrm{GL}_r(F) \times \mathrm{SO}_{2k-4r+1}(F).$$

It acts by conjugation on the quotient $[H_{r,2k-2r+1}/Z(H_{r,2k-2r+1})]$, which induces an action on the character group of the latter. We identify this character group with $\mathrm{Mat}_{r\times(2k-4r+1)}(F)$. For any $\xi \in \mathrm{Mat}_{r\times(2k-4r+1)}(F)$ with $\mathrm{Rank}(\xi) = q$, $1 \leqslant q \leqslant r$, if any of its row vectors is non-isotropic, then the corresponding contribution is given by the integral

$$\int_{[H_{r,2k+1}]} \int_{[H_{r,2k-2r+1}]} \theta(uvg)\psi_1(u)\psi_\xi(v)\, dv\, du. \tag{4.17}$$

The product of the characters ψ_1 and ψ_ξ is a generic character associated to the unipotent orbit $(5^q 3^{r-q} 1^{2k-3r-2q+1})$. Thus, the integral (4.17) contains a Fourier coefficient of Θ_{2k+1} associated to the unipotent orbit $(5^q 3^{r-q} 1^{2k-3r-2q+1})$, which is identically zero by Proposition 4.5. Under the conjugation action by $\mathrm{GL}_r(F) \times \mathrm{SO}_{2k-4r+1}(F)$, any $\xi \in \mathrm{Mat}_{r\times(2k-4r+1)}(F)$ whose row space is not totally isotropic lies in the same orbit as some non-isotropic ones with the same rank. The same argument shows that the corresponding contribution is zero. Therefore, the only possible non-zero contributions are from those $\xi \in \mathrm{Mat}_{r\times(2k-4r+1)}(F)$ whose row space is totally isotropic. We may pick the representatives of these orbits to be

$$z_q = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{r\times(2k-4r+1)}(F), q = 0, 1, \cdots, r.$$

For any $z_q \in \mathrm{Mat}_{r\times(2k-4r+1)}(F)$, the correponding contribution in the expansion is

$$\int_{[H_{r,2k+1}]} \int_{[H_{r,2k-2r+1}]} \theta(uvzg)\psi_1(u)\psi_{2,q}(v)\, dv\, du, \tag{4.18}$$

where the character $\psi_{2,q}$ corresponding to z_q is the trivial character when $q = 0$ and is given by

$$\psi_{2,q}(u) = \psi(u_{1,1} + u_{2,2} + \cdots + u_{q,q}), \quad \forall u \in \mathrm{Mat}_{r\times(2k-4r+1)}(\mathbb{A})$$

when $q > 0$ via the identification (4.16). We claim that the only non-zero contribution is when $q = 0$. Suppose on the contrary that $q > 0$. We further expand (4.18) against $[Z(H_{r,2k-4r+1})]$ and follow by another expansion against the abelian quotient $[H_{r,2k-4r+1}/Z(H_{r,2k-4r+1})]$. Here, $H_{r,2k-4r+1}$ is similarly defined as the unipotent subgroup of SO_{2k+1} consisting of matrices of the form

$$\left\{ \begin{pmatrix} I_{2r} &&&&\\ & I_r & y & * &\\ && I_{2k-6r+1} & y^* &\\ &&& I_r &\\ &&&& I_{2r} \end{pmatrix}, \quad y \in \mathrm{Mat}_{r\times(2k-6r+1)} \right\}.$$

For any character ψ^* on $[H_{r,2k-4r+1}/Z(H_{r,2k-4r+1})]$, the corresponding contribution is

$$\int_{[H_{r,2k+1}]} \int_{[H_{r,2k-2r+1}]} \int_{[H_{r,2k-4r+1}]} \theta(uvwg)\psi_1(u)\psi_{2,q}(v)\psi^*(w)\, dw\, dv\, du. \tag{4.19}$$

Any character ψ^* corresponding to a matrix in $\mathrm{Mat}_{r\times(2k-6r+1)}(F)$ that contains non-isotropic row vectors in $F^{2k-6r+1}$ again contributes zero.

We claim that the constant term is also zero. If we denote

$$H_r = H_{r,2k+1} H_{r,2k-2r+1} H_{r,2k-4r+1},$$

then the constant term is of the form

$$\int\limits_{[H_r]} \theta(ug)\psi_{1,q}(u)\, du. \tag{4.20}$$

Here, if $u = u_1 u_2 u_3$ with $u_1 \in H_{r,2k+1}(\mathbb{A}), u_2 \in H_{r,2k-2r+1}(\mathbb{A})$ and $u_3 \in H_{r,2k-4r+1}(\mathbb{A})$, then

$$\psi_{1,q}(u) = \psi_1(u_1)\psi_{2,q}(u_2).$$

The integral (4.20) is some Fourier coefficients of the constant term of Θ_{2k+1} with respect to the maximal parabolic subgroup whose Levi part is $\mathrm{GL}_{3r} \times \mathrm{SO}_{2k-6r+1}$. By Proposition 4.4, we may regard this integral as a function in the representation $\Theta_{\mathrm{GL}_{3r}} \otimes \Theta_{2k-6r+1}$ of $\widetilde{\mathrm{GL}}_{3r}(\mathbb{A}) \times \widetilde{\mathrm{SO}}_{2k-6r+1}(\mathbb{A})$. According to [7], the corresponding Fourier coefficient of $\Theta_{\mathrm{GL}_{3r}}$ is the semi-Whittaker coefficient associated with the partition $\Lambda = (3^q 2^{r-q} 1^{r-q})$. Following [7], let $P_\Lambda = M_\Lambda U_\Lambda$ be the standard parabolic subgroup of GL_{3r} with the Levi subgroup $M_\Lambda \cong \mathrm{GL}_3^q \times \mathrm{GL}_2^{r-q} \times \mathrm{GL}_1^{r-q}$ and U_Λ the unipotent radical. Let U be the unipotent radical of the standard Borel subgroup of GL_{3r} and $\psi_\Lambda : U(F) \setminus U(\mathbb{A}) \to \mathbb{C}^\times$ be the Gelfand-Graev character such that it acts non-trivially as ψ on each one-parameter subgroup corresponding to a simple positive root contained in M_Λ and acts trivially otherwise. Then the Λ-semi-Whittaker coefficient of any function $\theta_{\mathrm{GL}_{3r}}$ in $\Theta_{\mathrm{GL}_{3r}}$ is given by

$$\int\limits_{U(F)\backslash U(\mathbb{A})} \theta_{\mathrm{GL}_{3r}}(ug)\psi_\Lambda(u)\, du. \tag{4.21}$$

By Proposition 4.1 of [7], any such semi-Whittaker coefficient is identically zero.

Therefore, we only need to consider those terms corresponding to characters on $[H_{r,2k-4r+1}]$ represented by

$$\psi_{3,q'}(u) = \psi(y_{1,1} + y_{2,2} + \cdots + y_{q',q'}), \tag{4.22}$$

with $1 < q' \leqslant r$ and

$$u = \begin{pmatrix} I_{2r} & & & & \\ & I_r & y & * & \\ & & I_{2k-6r+1} & y^* & \\ & & & I_r & \\ & & & & I_{2r} \end{pmatrix} \in H_{r,2k-4r+1}(\mathbb{A}), y = (y_{i,j}) \in \mathrm{Mat}_{r\times(2k-6r+1)}(\mathbb{A}). \tag{4.23}$$

We repeat this argument by further expanding the integral (4.22) against smaller unipotent subgroups $H_{r,2k-2m+1}$ with $m = 3, 4, \cdots$. For each time, the only non-zero contribution is coming from those characters correponding to some totally isotropic matrices. In the end, we either obtain a Fourier coefficient of Θ_{2k+1} associated to unipotent orbit that is either larger than or incomparable to $\mathcal{O}(\Theta_{2k+1})$, or we obtain some semi-Whittaker coefficient on the double cover $\widetilde{\mathrm{GL}}_{nr}$ corresponding to a partition of the integer nr (with $n > 3$) that contains an integer greater than 2. The former is identically zero by Proposition 4.5, while the latter is also zero by Proposition 4.1 of [7]. This shows that the only contribution of the expansion of (4.15) against $[H_{r,2k-2r+1}/Z(H_{r,2k-2r+1})]$ is the constant term, which completes the proof. $\square$

For intergers $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n$, we further denote $H(r_1, r_2, \cdots, r_{n-1}; r_n)$ the unipotent radical of the parabolic subgroup of SO_{2k+1} with Levi subgroup

$$L(r_1, r_2, \cdots, r_{n-1}; r_n) = \mathrm{GL}_{r_1}^2 \times \mathrm{GL}_{r_2}^2 \times \cdots \times \mathrm{GL}_{r_{n-1}}^2 \times \mathrm{GL}_{r_n} \times \mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})-2r_n+1}\,.$$

We also let

$$L = \mathrm{GL}_{2r_1} \times \mathrm{GL}_{2r_2} \times \cdots \times \mathrm{GL}_{2r_{n-1}} \times \mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})+1}\,.$$

Denote the diagonal embedding by $\iota^* : L(r_1, r_2, \cdots, r_{n-1}; r_n) \subset L \hookrightarrow \mathrm{SO}_{2k+1}$.

For each $i = 1, 2, \cdots, n-1$, let H_{r_i} be the unipotent radical of the standard Siegel parabolic subgroup of GL_{2r_i} whose Levi part is $\mathrm{GL}_{r_i} \times \mathrm{GL}_{r_i}$. Also, let H_{r_n} be the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})+1}$ with Levi subgroup $\mathrm{GL}_{r_n} \times \mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})-2r_n+1}$. Then we define

$$H^0 := H_{r_1} \times H_{r_2} \times \cdots \times H_{r_{n-1}} \times H_{r_n} \subseteq H(r_1, r_2, \cdots, r_{n-1}; r_n).$$

We now define a character on $[H^0]$ and extend it trivially to $[H(r_1, r_2, \cdots, r_{n-1}; r_n)]$. For any $u_i \in H_{r_i}(\mathbb{A})$ with $i = 1, 2, \cdots, n-1$, we may write

$$u_i = \begin{pmatrix} I_{r_i} & x_i \\ & I_{r_i} \end{pmatrix}, \quad x_i \in \mathrm{Mat}_{r_i \times r_i}(\mathbb{A}).$$

Define $\psi_{r_i} : H_{r_i}(F) \setminus H_{r_i}(\mathbb{A}) \to \mathbb{C}^\times$ by

$$\psi_{r_i}(u_i) = \psi(\mathrm{Tr}(x_i)).$$

For the factor H_{r_n}, define the character by

$$\psi_{r_n}(u) = \psi(\sum_{j=1}^{r_n} u_{j,j+r_n}), \quad u \in H_{r_n}(\mathbb{A}).$$

Pulling back each ψ_{r_i} via the projection of H^0 onto the corresponding factor and taking the product afterwards, we obtain $\psi_n := \prod_{i=1}^n \psi_{r_i} : [H^0] \to \mathbb{C}^\times$. Extend ψ_n trivially to $[H(r_1, r_2, \cdots, r_{n-1}; r_n)]$. Furthermore, let H_n be the unipotent radical of the standard maximal parabolic subgroup of $\mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})-2r_n+1}$ with Levi subgroup given by

$$\mathrm{GL}_{r_n} \times \mathrm{SO}_{2k-4(r_1+\cdots+r_{n-1})-2r_n+1}\,.$$

Embed H_n into SO_{2k+1} via (2.3) and still denote its image by H_n. Proposition 4.6 admits a straightforward corollary.

Corollary 4.7. *The function*

$$f(g) = \int_{[H(r_1,r_2,\cdots,r_{n-1};r_n)]} \theta(ug)\psi_n(u)\,du \tag{4.24}$$

is left invariant under $H_n(\mathbb{A})$.

Proof. Apply Proposition 4.4 with $r = 2(r_1+r_2+\cdots+r_{n-1})$. Then apply Proposition 4.6 to the theta representation of the smaller orthogonal group. □

4.4 Tower of the theta liftings

Let $(\pi, \mathcal{V})$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose $\mathrm{SO}_{2k'}$ is a split even orthogonal group. By identifying $\mathrm{SO}_{2k'} \times \mathrm{SO}_{2k+1}$ with its embedded image in $\mathrm{SO}_{2k+2k'+1}$ via (2.3), we consider functions on $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$ of the form

$$f(h) = \int\limits_{[\mathrm{SO}_{2k+1}]} \varphi(g)\bar{\theta}_{2k+2k'+1}(h,g)\,dg, \tag{4.25}$$

where φ is any function in $\mathcal{V}$ and $\theta_{2k+2k'+1}$ is any function in the representation space of $\Theta_{2k+2k'+1}$. This integral defines a map from the irreducible cuspidal genuine automorphic representation π of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ to a genuine automorphic representation $\Theta_{2k+2k'+1}(\pi)$ of $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$.

By fixing the representation π of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ and varying the theta representations $\Theta_{2k+2k'+1}$ of $\widetilde{\mathrm{SO}}_{2k+2k'+1}(\mathbb{A})$ with increasing k', we obtain a tower of liftings of representations of $\widetilde{\mathrm{SO}}_{2k'}(\mathbb{A})$:

$$\begin{tikzcd}
 & \Theta_{2k+2k'+3}(\pi) \\
 & \Theta_{2k+2k'+1}(\pi) \\
 & \vdots \\
\pi \arrow[uuur] \arrow[uur] \arrow[r] & \Theta_{2k+3}(\pi).
\end{tikzcd}$$

In [6], Bump-Friedberg-Ginzburg show that if $\Theta_{2k+2k'+1}(\pi) = 0$, then $\Theta_{2k+2k'-1}(\pi) = 0$. It is also proved in [6] that any genuine cuspidal automorphic representation π of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ lifts nontrivially to an automorphic representation of $\widetilde{\mathrm{SO}}_{8k}(\mathbb{A})$. This raises the question of when the first non-zero theta lifting occurs along the tower for a fixed π. In the case when π is generic, a conjecture in [6] states that π should lift non-trivially to an automorphic representation of $\widetilde{\mathrm{SO}}_{2k+4}(\mathbb{A})$. The same authors also proved the following result:

Theorem 4.8. *Let π be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. If the representation $\Theta_{4k+5}(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+4}(\mathbb{A})$ is generic, then the representation π is also generic.*

On the other hand, there is not much known yet for the theta liftings when π is non-generic. Motivated by the generic case, we make the following more general conjecture:

Conjecture 4.9. *Let $(\pi, \mathcal{V})$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose*

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p}) \in \mathcal{O}(\pi)$$

with $n_1 > n_2 > \cdots > n_p \geqslant 0$ and $r_i > 0$ for all i. Then π lifts nontrivially to an automorphic representation $\Theta(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$ such that

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1)) \in \mathcal{O}\left(\Theta(\pi)\right).$$

Recall that $l = r_1 + r_2 + \cdots + r_p$ is the length of the partition corresponding to $\mathcal{O}$. If π is an irreducible cuspidal generic automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$, then $\mathcal{O}(\pi) = (2k+1)$. Conjecture 4.9 predicts that it lifts to an automorphic representation $\Theta(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+4}(\mathbb{A})$ with $\mathcal{O}(\Theta(\pi)) = ((2k+3)(1))$, which agrees with the conjecture proposed in [6].

Recall that the Gelfand-Kirillov dimension of a representation ρ (see [10]) is given by

$$\dim(\rho) = \frac{1}{2}\dim(\mathcal{O}(\rho)) = \dim(V_{2,\mathcal{O}(\rho)}) + \frac{1}{2}\dim(V_{1,\mathcal{O}(\rho)}/V_{2,\mathcal{O}(\rho)}).$$

Proposition 4.10. *Suppose $(\pi, \mathcal{V})$ is an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$ with $\mathcal{O}(\pi) = \mathcal{O}$ such that its theta lifting $\Theta(\pi)$ on $\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$ has $\mathcal{O}\left(\Theta(\pi)\right) = \mathcal{O}'$. Then*

$$\dim(\mathrm{SO}_{2k+1}) + \dim(\Theta(\pi)) = \dim(\pi) + \dim(\Theta_{4k+2l+3}). \tag{4.26}$$

Before we verify (4.26), we need to fix some notations. Suppose, as in Conjecture 4.9,

$$\mathcal{O} = \left((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p}\right), \quad n_1 > n_2 > \cdots > n_p \geqslant 0.$$

Denote

$$s_i := \sum_{j=1}^{i} r_j, \quad i = 1, 2, \cdots, p. \tag{4.27}$$

In particular, $l = s_p$ is the length of the partition corresponding to $\mathcal{O}$. The fact that $\mathcal{O}$ is odd implies that $V_{1,\mathcal{O}} = V_{2,\mathcal{O}}$, and both of these are the unipotent radical of the parabolic subgroup $P_{\mathcal{O}}$ whose Levi part is

$$M(\mathcal{O}) = \mathrm{GL}_{s_1}^{n_1-n_2} \times \mathrm{GL}_{s_2}^{n_2-n_3} \times \cdots \times \mathrm{GL}_{s_{p-1}}^{n_{p-1}-n_p} \times \mathrm{GL}_{l}^{n_p} \times \mathrm{SO}_l\,.$$

For simplicity, we denote $U_{\mathcal{O}} = V_{1,\mathcal{O}} = V_{2,\mathcal{O}}$.

Likewise, we denote by $V_{\mathcal{O}'}$ the unipotent subgroup $V_{1,\mathcal{O}'} = V_{2,\mathcal{O}'}$ associated to the unipotent orbit

$$\mathcal{O}' = \left((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1)\right)$$

in $\mathrm{SO}_{2k+2l+2}$. It is the unipotent radical of some maximal parabolic subgroup whose corresponding Levi subgroup is

$$M(\mathcal{O}') = \mathrm{GL}_{s_1}^{n_1-n_2} \times \mathrm{GL}_{s_2}^{n_2-n_3} \times \cdots \times \mathrm{GL}_{s_{p-1}}^{n_{p-1}-n_p} \times \mathrm{GL}_{l}^{n_p+1} \times \mathrm{SO}_{l+1}\,.$$

Proof. Observe that $\mathcal{O}(\Theta_{4k+2l+3}) = (2^{2k+l+1}1)$ since l is odd, and $\dim(\Theta_{4k+2l+3}) = \frac{(2k+l+1)^2}{2}$. Also, $\dim(\mathrm{SO}_{2k+1}) = 2k^2+k$. Although the dimension of the representations $\Theta_{4k+2l+3}(\pi)$ and π may vary, it suffices to check that the difference between the dimensions agrees with that between $\dim(\Theta_{4k+2l+3})$ and $\dim(\mathrm{SO}_{2k+1})$.

Notice that the difference between $\dim(\pi)$ and the dimension of the unipotent radical of the Borel subgroup of SO_{2k+1} is related to that between $\dim(\Theta_{4k+2l+3}(\pi))$ and the dimension of the unipotent radical of the Borel subgroup of $\mathrm{SO}_{2k+2l+2}$. The former is precisely the dimension of the unipotent radical of the Borel subgroup of $M(\mathcal{O})$. We denote this dimension by $t + \frac{(l-1)^2}{4}$, where $\frac{(l-1)^2}{4}$ and t are the dimensions of the maximal unipotent subgroup of the factor SO_l and the remaining Levi factors respectively. Hence, we obtain that $\dim(\pi) = k^2 - t - \frac{(l-1)^2}{4}$. Similarly, we have $\dim(\Theta_{4k+2l+3}(\pi)) = (k+l+1)(k+l) - t - \frac{l^2-1}{4} - \frac{(l-1)(l)}{2}$. Therefore, the difference is exactly

$$\frac{l^2+1}{2} + 2kl + k + l = \frac{(2k+l+1)^2}{2} - (2k^2+k) = \dim(\Theta_{4k+2l+3}) - \dim(\mathrm{SO}_{2k+1}). \quad (4.28)$$

□

Proposition 4.10 shows that Conjecture 4.9 agrees with the dimension equation proposed in [12]. The general philosophy of the dimension equation is that the sum of the dimensions of the representations is equal to the sum of the dimensions of the groups in the domain of integration in a global unipotent integral. In our case, this is given by equation (4.26). Refer to [12], [13] and [10] for more details on dimension equations.

5 Global theory

In this section, we will state and prove our main result in the global setting.

5.1 Choice of generic characters

We follow the notations in and after Conjecture 4.9. Suppose $(\pi, \mathcal{V})$ is an irreducible genuine cuspidal automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Let $\mathcal{O}$ be the unipotent orbit of SO_{2k+1} such that

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2} \cdots (2n_p+1)^{r_p}) .$$

Let $l = r_1 + r_2 + \cdots + r_p$ be the length of the partition corresponding to $\mathcal{O}$. Suppose $\Theta_{4k+2l+3}(\pi)$ is the automorphic representation of $\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$ obtained by the theta lifting from π via integrating functions in $\mathcal{V}$ against the theta integral kernel $\Theta_{4k+2l+3}$ in the form of (4.25). As we are only concerned about a fixed theta lifting in this section, we suppress the subscript and simply let $\Theta(\pi) = \Theta_{4k+2l+3}(\pi)$. We denote by $\mathcal{O}'$ the unipotent orbit of the group $\mathrm{SO}_{2k+2l+2}$ associated to the partition

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2} \cdots (2n_p+3)^{r_p}(1)) .$$

Recall that $V_{\mathcal{O}'} = V_{1,\mathcal{O}'} = V_{2,\mathcal{O}'}$ is the unipotent radical of the parabolic subgroup of $\mathrm{SO}_{2k+2l+2}$ whose Levi subgroup is

$$M(\mathcal{O}') = \mathrm{GL}_{s_1}^{n_1-n_2} \times \mathrm{GL}_{s_2}^{n_2-n_3} \times \cdots \times \mathrm{GL}_{s_{p-1}}^{n_{p-1}-n_p} \times \mathrm{GL}_{l}^{n_p+1} \times \mathrm{SO}_{l+1} .$$

Notice that

$$V_{\mathcal{O}'}/V_{\mathcal{O}'}^{(1)} \cong \left(\bigoplus_{j=1}^{p-1} \mathrm{Mat}_{s_j \times s_{j+1}}\right) \oplus \left(\bigoplus_{j=1}^{p-1} \mathrm{Mat}_{s_j \times s_j}^{n_j - n_{j+1} - 1}\right) \oplus \mathrm{Mat}_{l\times l}^{n_p} \oplus \mathrm{Mat}_{l\times(l+1)}. \quad (5.1)$$

We first define a character on $[V_{\mathcal{O}'}/V_{\mathcal{O}'}^{(1)}]$ by specifying it on each of the components, and then extend it trivially to a character on $[V_{\mathcal{O}'}]$. For any abelian group $\mathrm{Mat}_{i\times j}$, we may identify the character group of $[\mathrm{Mat}_{i\times j}]$ with $\mathrm{Mat}_{i\times j}(F)$. Recall

$$s_i := \sum_{j=1}^{i} r_j, \quad i = 1, 2, \cdots, p. \quad (5.2)$$

Let

$$I_{s_j, s_{j+1}} = \begin{pmatrix} I_{s_j} & 0 \end{pmatrix} \in \mathrm{Mat}_{s_j \times s_{j+1}}(F), \quad j = 1, 2, \cdots, p-1,$$

and

$$I_{l,a} = \begin{pmatrix} I_{\frac{l-1}{2}} & & & \\ & \frac{1}{2} & -a & \\ & & & -I_{\frac{l-1}{2}} \end{pmatrix} \in \mathrm{Mat}_{l\times(l+1)}(F), \quad a \in F^\times.$$

Consider the following characters each defined on the respective component in (5.1):

$$\begin{cases} \psi_{s_j \times s_j}(v) = \psi(\mathrm{Tr}\, v) & \text{if } v \in \mathrm{Mat}_{s_j \times s_j}(\mathbb{A}), j = 1, 2, \cdots, p, \\ \psi_{s_j \times s_{j+1}}(v) = \psi(\mathrm{Tr}(I_{s_j, s_{j+1}} v^T)) & \text{if } v \in \mathrm{Mat}_{s_j \times s_{j+1}}(\mathbb{A}), j = 1, 2, \cdots, p-1, \\ \psi_{l\times(l+1)}(v) = \psi(\mathrm{Tr}(I_{l,a} v^T)) & \text{if } v \in \mathrm{Mat}_{l\times(l+1)}(\mathbb{A}). \end{cases} \quad (5.3)$$

Pulling back each of these characters via the projection map onto the respective component and taking the product afterwards, we obtain a character on $[V_{\mathcal{O}'}/V_{\mathcal{O}'}^{(1)}]$. Extend it to a character on $[V_{\mathcal{O}'}]$ and denote the resulting character by $\psi_{a, V_{\mathcal{O}'}}$. This is a generic character attached to the unipotent orbit $\mathcal{O}'$.

For any automorphic function in $\Theta(\pi)$ of the form

$$f(h) = \int\limits_{\mathrm{SO}_{2k+1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \varphi(g)\overline{\theta_{4k+2l+3}(h,g)}\, dg,$$

we let

$$F_{\psi_{a,V_{\mathcal{O}'}}}(f)(h) = \int\limits_{[\mathrm{SO}_{2k+1}]} \int\limits_{[V_{\mathcal{O}'}]} \varphi(g)\overline{\theta_{4k+2l+3}(vh,g)}\psi_{a,V_{\mathcal{O}'}}(v)\, dv\, dg. \tag{5.4}$$

The maximal split torus of $\mathrm{SO}_{4k+2l+3}(F)$ normalizes $V_{\mathcal{O}'}$. Conjugating the variable v in the inner integration of (5.4) by $\tau = \mathrm{diag}(t, \cdots, t, 1, t^{-1}, \cdots, t^{-1})), t \in F^\times$ leaves the integral unchanged. The automorphicity of $\theta_{4k+2l+3}$ implies that it is left invariant by τ. After a change of variables by $v \mapsto v\tau^{-1}$, we see that the Fourier coefficient depends only on the square class of a in $F^\times$. When a is a square, the connected component of the stabilizer of $\psi_{a,V_{\mathcal{O}'}}$ in $M(\mathcal{O}')(F)$ is split. In this case, we call $\psi_{a,V_{\mathcal{O}'}}$ a split generic character, and denote it by $\psi_{\mathcal{O}'}$.

On the other hand, recall $U_{\mathcal{O}} = V_{1,\mathcal{O}} = V_{2,\mathcal{O}}$ is the unipotent radical of the maximal parabolic subgroup $P_{\mathcal{O}}$ of SO_{2k+1} with the corresponding Levi part

$$M(\mathcal{O}) = \mathrm{GL}_{s_1}^{n_1-n_2} \times \mathrm{GL}_{s_2}^{n_2-n_3} \times \cdots \times \mathrm{GL}_{s_{p-1}}^{n_{p-1}-n_p} \times \mathrm{GL}_l^{n_p} \times \mathrm{SO}_l\,.$$

In order to define a generic character on $[U_{\mathcal{O}}]$, it suffices to specify the respective character on each of the components of the maximal abelian quotient

$$U_{\mathcal{O}}/U_{\mathcal{O}}^{(1)} \cong \left(\bigoplus_{j=1}^{p-1} \mathrm{Mat}_{s_j \times s_{j+1}}\right) \oplus \left(\bigoplus_{j=1}^{p-1} \mathrm{Mat}_{s_j \times s_j}^{n_j-n_{j+1}-1}\right) \oplus \mathrm{Mat}_{l\times l}^{n_p-1}\,. \tag{5.5}$$

We define these characters by

$$\begin{cases} \psi_{s_j\times s_j}(v)=\psi(\mathrm{Tr}\, v) & \text{if } v\in \mathrm{Mat}_{s_j\times s_j}(\mathbb{A}), j=1,2,\cdots,p, \\ \psi_{s_j\times s_{j+1}}(v)=\psi(\mathrm{Tr}(I_{s_j,s_{j+1}}v^T)) & \text{if } v\in \mathrm{Mat}_{s_j\times s_{j+1}}(\mathbb{A}), j=1,2,\cdots,p-1. \end{cases} \tag{5.6}$$

This gives a generic character $\psi_{\mathcal{O}} : [U_{\mathcal{O}}] \to \mathbb{C}^\times$.

5.2 The global main theorem

Theorem 5.1. *Let $(\pi, \mathcal{V})$ be an irreducible cuspidal genuine automorphic representation of $\widetilde{\mathrm{SO}}_{2k+1}(\mathbb{A})$. Suppose the theta lifting $\Theta(\pi)$, as a representation of $\widetilde{\mathrm{SO}}_{2k+2l+2}(\mathbb{A})$, has a non-zero Fourier coefficient with respect to a split generic character associated with the unipotent orbit*

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1)) .$$

Then the representation π has a non-zero Fourier coefficient with respect to some generic character associated with the unipotent orbit

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p}) .$$

Proof. Throughout the proof, we identify any subgroup of SO_{2k+1} or $\mathrm{SO}_{2k+2l+2}$ with its embedded image in $\mathrm{SO}_{4k+2l+3}$ via 2.3. The Fourier coefficient of $\Theta(\pi)$ with respect to a generic character $\psi_{a,V_{\mathcal{O}'}}$ depends only on the square class of a. Therefore, we may assume there exists data such that the following integral is non-vanishing:

$$F_{\psi_{\mathcal{O}'}}(f)(1) = \int_{[\mathrm{SO}_{2k+1}]}\int_{[V_{\mathcal{O}'}]} \varphi(g)\overline{\theta}_{4k+2l+3}(v,g)\psi_{\mathcal{O}'}(v)\, dv\, dg, \tag{5.7}$$

where $\psi_{\mathcal{O}'} = \psi_{1,V_{\mathcal{O}'}}$.

Denote by $R_{s_1} = R_{s_1,4k+2l+3}$ the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{4k+2l+3}$ with Levi subgroup $\mathrm{GL}_{s_1} \times \mathrm{SO}_{4k+2l-2s_1+3}$. Notice that $V_{s_1} = V_{\mathcal{O}'} \cap R_{s_1}$ is non-trivial. The quotient $V_{s_1} \setminus R_{s_1}$ may be identified with the subgroup of matrices of the form

$$H_{s_1} := \left\{ \begin{pmatrix} I_{s_1} & & x & & * \\ & I_{k+l-s_1+1} & & & \\ & & I_{2k+1} & & x^* \\ & & & I_{k+l-s_1+1} & \\ & & & & I_{s_1} \end{pmatrix} \in \mathrm{SO}_{4k+2l+3} : x \in \mathrm{Mat}_{s_1 \times (2k+1)} \right\}.$$

Although H_{s_1} is not abelian, it is a Heisenberg group with $Z(H_{s_1})$ corresponding to matrices of the form

$$\left\{ \begin{pmatrix} I_{s_1} & 0 & z \\ & I_{4k+2l-2s_1+3} & 0 \\ & & I_{s_1} \end{pmatrix} : z^T J_{s_1} + J_{s_1} z = 0 \right\}.$$

Notice that the center $Z(H_{s_1})(\mathbb{A}) \subset V_{\mathcal{O}'}(\mathbb{A})$ is included in the domain of integration in (5.7). As a result, we expand the integral (5.7) against the abelian quotient $[(H_{s_1}/Z(H_{s_1}))] \cong [\mathrm{Mat}_{s_1 \times (2k+1)}]$.

We may identify the character group of $[(H_{s_1}/Z(H_{s_1}))]$ with $\mathrm{Mat}_{s_1 \times (2k+1)}(F)$. The action of the diagonally embedded subgroup $\mathrm{GL}_{s_1}(F) \times \mathrm{SO}_{2k+1}(F)$ on the character group can be realized as its conjugation action on $\mathrm{Mat}_{s_1 \times (2k+1)}(F)$. This action preserves the rank and the fact whether the row space is totally isotropic. For any $\xi \in \mathrm{Mat}_{s_1 \times (2k+1)}(F)$ whose row space is not totally isotropic, it must lie on the same orbit as some ξ' that contains some non-isotropic row vectors in F^{2k+1}. Thus, we may classify the orbits by their representatives ξ given as follows:

(i) ξ is the zero matrix.

(ii) $\mathrm{Rank}(\xi) = q$ with $1 \leqslant q \leqslant s_1$ and ξ contains some non-isotropic vector in F^{2k+1}.

(iii) $\mathrm{Rank}(\xi) = q$ with $1 \leqslant q \leqslant s_1$ and the row space of ξ is totally isotropic.

For any character ψ_ξ identified with some $\xi \in \mathrm{Mat}_{s_1 \times (2k+1)}(F)$, its contribution in the Fourier expansion is given by the integral

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V_{\mathcal{O}'}]} \int_{[H_{s_1}/Z(H_{s_1})]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v,g)\right)\psi_{\mathcal{O}'}(v)\psi_\xi(u)\, du\, dv\, dg. \tag{5.8}$$

By Remark 3.4, the contribution of every character in the same orbit is zero as long as the integral contribution of one representative is identically zero.

We will show that the only non-zero contribution in the Fourier expansion is from the orbit represented by some $\xi \in \mathrm{Mat}_{s_1 \times (2k+1)}(F)$ with maximal rank and totally-isotropic row space.

First, we look at the contribution from the orbit corresponding to the zero matrix. This is the constant term of the Fourier expansion given by

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V_{\mathcal{O}'}]} \int_{[H_{s_1}/Z(H_{s_1})]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v,g)\right)\psi_{\mathcal{O}'}(v)\, du\, dv\, dg. \tag{5.9}$$

Write $v = v_{s_1}v^1$ where $v_{s_1} \in V_{s_1}(\mathbb{A})$ and $v^1 \in V^1_{\mathcal{O}'}(\mathbb{A}) := V_{\mathcal{O}'}(\mathbb{A}) \cap \mathrm{SO}_{2k+2l-2s_1+2}(\mathbb{A})$. Combining the two variables u and v_{s_1}, we obtain

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1}]} \varphi(g)\overline{\theta_{4k+2l+3}\left(u(v^1,g)\right)}\psi_1(u)\psi_{\mathcal{O}'}(v^1)\, du\, dv^1\, dg, \tag{5.10}$$

where ψ_1 is the character on $[R_{s_1}]$ given by

$$\psi_1(u) = \psi(u_{1,s+1} + u_{2,s+2} + \cdots + u_{s+2s}). \tag{5.11}$$

Applying Proposition 4.6, we see that (5.10) is equal to the integral

$$\int\limits_{[\mathrm{SO}_{2k+1}]} \int\limits_{[V^1_{\mathcal{O}'}]} \int\limits_{[R_{s_1^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v^1,g)\right)\psi_1(u)\psi_{\mathcal{O}'}(v^1)\,du\,dv^1\,dg, \tag{5.12}$$

where we extend ψ_1 trivially to $[R_{s_1^2}]$ and $R_{s_1^2} = R_{s_1,4k+2l+3}R_{s_1,4k+2l-2s_1+3}$ is the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{4k+2l+3}$ with Levi subgroup

$$\mathrm{GL}_{s_1} \times \mathrm{GL}_{s_1} \times \mathrm{SO}_{4k+2l-4s_1+3}\,.$$

Notice that $R_{s_1^2} \cap V^1_{\mathcal{O}'}$ is non-trivial. Let β be the root inside $\mathrm{SO}_{2k+2l+2}$ such that

$$\beta = \begin{cases} \sum_{j=1}^{s_2} \alpha_{s_1+j} & \text{if } n_1 - n_2 = 1 \\ \sum_{j=1}^{s_1} \alpha_{s_1+j} & \text{if } n_1 - n_2 > 1, \end{cases} \tag{5.13}$$

where we recall that α_i's are the positive simple roots of $\mathrm{SO}_{2k+2l+2}$. By construction, the one parameter subgroup $\{x_\beta(r) : r \in \mathbb{A}\}$ associated to β is in the intersection $R_{s_1^2}(\mathbb{A}) \cap V^1_{\mathcal{O}'}(\mathbb{A})$, and $\psi_{\mathcal{O}'}$ is non-trivial on $x_\beta(r)$. We may write (5.12) as

$$\int\limits_{[\mathrm{SO}_{2k+1}]} \int\limits_{V^1_{\mathcal{O}'}(F)x_\beta(\mathbb{A})\backslash V^1_{\mathcal{O}'}(\mathbb{A})} \int\limits_{[R_{s_1^2}]} \int\limits_{\mathbb{A}/F} \varphi(g)\overline{\theta}_{4k+2l+3}\left(ux_\beta(r)(v^1,g)\right)\psi_1(u)\psi(r)\psi_{\mathcal{O}'}(v^1)\,dr\,du\,dv^1\,dg$$
$$= \left(\int\limits_{\mathbb{A}/F} \psi(r)\,dr\right) \int\limits_{[\mathrm{SO}_{2k+1}]} \int\limits_{V^1_{\mathcal{O}'}(F)x_\beta(\mathbb{A})\backslash V^1_{\mathcal{O}'}(\mathbb{A})} \int\limits_{[R_{s_1^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v^1,g)\right)\psi_1(u)\psi_{\mathcal{O}'}(v^1)\,du\,dv^1\,dg.$$

This is zero since

$$\int_{\mathbb{A}/F} \psi(r)\, dr = 0$$

for the non-trivial character ψ.

Second, we look at the contributions from the second type of orbits. For any of such orbits, it suffices to check on the contribution of a representative ξ that contains some non-isotropic row vector in F^{2k+1}, which is given by

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v^1, g)\right) \psi_{1,\xi}(u)\psi_{\mathcal{O}'}(v^1)\, du\, dv^1\, dg. \tag{5.14}$$

Here, for any $u = u_1 u_2 \in R_{s_1}(\mathbb{A})$ with $u_1 \in V_{s_1}(\mathbb{A}), u_2 \in H_{s_1}(\mathbb{A})$, $\psi_{1,\xi}(u) = \psi_1(u_1)\psi_\xi(u_2)$ is a generic character associated with the unipotent orbit corresponding to the partition $(3^{s_1}1^{4k+2l+3-3s_1})$. Hence, (5.14) contains a Fourier coefficient of $\theta_{4k+2l+3}$ associated to the unipotent orbit $\mathcal{O}_\xi = (3^{s_1}1^{4k+2l+3-3s_1})$, which is zero by Proposition 4.5.

Thus, the only possible non-zero contributions are from orbits that belong to the last type. We may classify these orbits by the representatives given by

$$\xi_q = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{s_1\times(2k+1)}(F), \quad q = 1, 2, \cdots, s_1. \tag{5.15}$$

For a given ξ_q, the contribution of the corresponding character $\psi_{\xi_q} : [(H_{s_1}/Z(H_{s_1}))] \to \mathbb{C}^\times$ is

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u(v^1, g)\right) \psi_{1,\xi_q}(u)\psi_{\mathcal{O}'}(v^1)\, du\, dv^1\, dg, \tag{5.16}$$

where $\psi_{1,\xi_q} = \psi_1\psi_{\xi_q}$ is the character on $[R_{s_1}/Z(R_{s_1})]$ that corresponds to the matrix

$$\begin{pmatrix} I_{s_1} & 0_{s_1\times(k+l+1-2s_1)} & \xi & 0_{s_1\times(k+l-s_1+1)} \end{pmatrix} \in \mathrm{Mat}_{s_1\times(4k+2l-2s_1+3)}(F).$$

Let $z_q \in \mathrm{SO}_{4k+2l+3}(F)$ be the unipotent element

$$z_q = \begin{pmatrix} I_{s_1} & & & & \\ & \mu_q & & & \\ & & I_{2k+1-2q} & & \\ & & & \mu_q^* & \\ & & & & I_{s_1} \end{pmatrix}, \mu_q = \begin{pmatrix} I_q & & -I_q \\ & I_{k+l+1-s_1-q} & \\ & & I_q \end{pmatrix}.$$

Performing a change of variables by $u \mapsto uz_q$, (5.16) is equal to

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(uz_q(v^1,g)\right)\psi'_{1,\xi_q}(u)\psi_{\mathcal{O}'}(v^1)\,du\,dv^1\,dg, \tag{5.17}$$

where ψ'_{1,ξ_q} is the character on $[(H_{s_1}/Z(H_{s_1}))]$ that corresponds to the matrix

$$\begin{pmatrix} I^*_{s_1-q} & 0_{s_1\times(k+l+1-2s_1)} & \xi & 0_{s_1\times(k+l-s+1)} \end{pmatrix} \in \mathrm{Mat}_{s_1\times(4k+2l-2s_1+3)}(F),$$

$$I^*_{s_1-q} = \begin{pmatrix} 0 & 0 \\ 0 & I_{s_1-q} \end{pmatrix} \in \mathrm{Mat}_{s_1\times s_1}(F).$$

Let ω_q be the Weyl group element given by

$$\omega_q = \begin{pmatrix} I_{s_1} & & & & \\ & \nu_q & & & \\ & & I_{2k+1-2q} & & \\ & & & \nu_q^{-1} & \\ & & & & I_{s_1} \end{pmatrix} \in \mathrm{SO}_{4k+2l+3}(F),$$

with

$$\nu_q = \begin{pmatrix} & & & I_q & \\ & I_{s_1-q} & & & \\ I_q & & & & \\ & & I_{k+l+q+1-3s_1} & & \\ & & & & I_{s_1-q} \end{pmatrix}.$$

The conjugation action of ω_q stabilizes $[R_{s_1}]$. Therefore, we change the variable $u \mapsto \omega_q^{-1} u \omega_q$ and use the automorphicity of $\theta_{4k+2l+3}$ to obtain

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_q z_q(v^1, g)\right) \psi_1(u)\psi_{\mathcal{O}'}(v^1)\, du\, dv^1\, dg. \tag{5.18}$$

Apply Proposition 4.6 to the integral (5.18) to replace the integration on $[R_{s_1}]$ by $[R_{s_1^2}]$. We obtain

$$\int_{[\mathrm{SO}_{2k+1}]} \int_{[V^1_{\mathcal{O}'}]} \int_{[R_{s_1^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_q z_q(v^1, g)\right) \psi_1(u)\psi_{\mathcal{O}'}(v^1)\, du\, dv^1\, dg. \tag{5.19}$$

This integral is similar to (5.12) except for the presence of the Weyl group element ω_q. However, the same argument implies that the contribution is zero as long as $R_{s_1^2} \cap \omega_q V^1_{\mathcal{O}'} \omega_q^{-1}$ is non-trivial. This happens only when $\mathrm{Rank}(\xi_q) < s_1$. Hence, we conclude that the only non-zero contributions of the Fourier expansion are from characters ψ_ξ on $[(H_{s_1}/Z(H_{s_1}))]$ corresponding to some $\xi \in \mathrm{Mat}_{s_1\times(2k+1)}(F)$ with rank s_1 and totally isotropic row space. Notice that $\mathrm{SO}_{2k+1}(F)$ acts transitively from the right on such matrices. If we take

$$\xi_{s_1} = \begin{pmatrix} I_{s_1} & 0 \end{pmatrix} \in \mathrm{Mat}_{s_1\times(2k+1)}(F)$$

as a representative, then we conclude that the Fourier coefficient (5.7) is equal to

$$\int\limits_{[\mathrm{SO}_{2k+1}]} \int\limits_{[V^1_{\mathcal{O}'}]} \int\limits_{[R_{s_1^2}]} \sum_{\gamma\in P^0_{s_1}(F)\backslash \mathrm{SO}_{2k+1}(F)} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_{s_1}z_{s_1}\gamma(v^1,g)\right)\psi_1(u)\psi_{\mathcal{O}}(v^1)\,du\,dv^1\,dg. \tag{5.20}$$

Here, $P_{s_1} = P_{s_1,2k+1}$ is the standard maximal parabolic subgroup of SO_{2k+1} with Levi part $\mathrm{GL}_{s_1}\times\mathrm{SO}_{2k-2s_1+1}$. The upper zero indicates that we omit the GL_{s_1} factor. In fact, $P^0_{s_1}(F)$ is the stabilizer of $\psi_{\xi_{s_1}}$ under the $\mathrm{SO}_{2k+1}(F)$ action. As any $\gamma \in P^0_{s_1}(F)\setminus\mathrm{SO}_{2k+1}(F)$ commutes with any $v^1\in V^1_{\mathcal{O}'}(\mathbb{A})$, we can combine the summation with the integration in (5.20) to rewrite it as

$$\int\limits_{P^0_{s_1}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{[V^1_{\mathcal{O}'}]} \int\limits_{[R_{s_1^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_{s_1}z_{s_1}(v^1,g)\right)\psi_1(u)\psi_{\mathcal{O}'}(v^1)\,du\,dv^1\,dg. \tag{5.21}$$

The next step of the proof is to apply the same argument in the previous step repeatedly to smaller unipotent subgroups. To proceed, we assume that $n_1-n_2=1$ only for notational simplicity. The argument works in great generalities that it does not depend on the assumption. Consider the unipotent subgroup $R_{s_2} = R_{s_2,4k+2l-4s_1+3}\subset \mathrm{SO}_{4k+2l+3}$, which is the unipotent radical of the standard maximal parabolic subgroup whose Levi part is $\mathrm{GL}^2_{s_1}\times\mathrm{GL}_{s_2}\times\mathrm{SO}_{4k+2l-4s_1-2s_2+3}$. The group R_{s_2} consists of matrices of the form

$$\left\{ \begin{pmatrix} I_{2s_1} & & & & \\ & I_{s_2} & x & * & \\ & & I_{4k+2l-4s_1-2s_2+3} & x^* & \\ & & & I_{s_2} & \\ & & & & I_{2s_1} \end{pmatrix} \in \mathrm{SO}_{4k+2l+3} : x \in \mathrm{Mat}_{s_2\times(4k+2l-4s_1-2s_2+3)} \right\}. \tag{5.22}$$

We set $V_{s_2} = R_{s_2} \cap (\omega_{s_1} z_{s_1} V^1_{\mathcal{O}'}(\omega_{s_1} z_{s_1})^{-1})$, and the quotient $H_{s_2} = V_{s_2} \setminus R_{s_2}$. As the center $[Z(H_{s_2})] \subset [V_{s_2}]$ is included in the domain of integration in (5.21), we continue to expand (5.21) against $[H_{s_2}/Z(H_{s_2})]$. We check on the contributions from each type of the characters on $[H_{s_2}/Z(H_{s_2})]$ under the action of $\mathrm{GL}_{s_2}(F)\times \mathrm{SO}_{4k+2l-4s_1-2s_2+3}(F)$. By the same argument, it follows that the only contribution is from the orbit of characters represented by $\psi_{\xi_{s_2}}$ corresponding to the matrix

$$\xi_{s_2} = \begin{pmatrix} I_{s_2} & 0 \end{pmatrix} \in \mathrm{Mat}_{s_2\times(4k+2l-4s_1-2s_2+3)}(F).$$

We continue the same argument repeatedly with the assumption that $n_i - n_{i+1} = 1$ for all $i = 1, 2, \cdots, p-1$ and $n_p = 1$ for notational simplicity. We deduce that (5.21) equals to

$$\int_{P^0_{\mathcal{O}}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{[V^p_{\mathcal{O}'}]} \int_{[R_{s_p^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\prod_{i=0}^{p-1}\omega_{s_{p-i}} z_{s_{p-i}}(v^p, g)\right)\psi_p(u)\psi_{\mathcal{O}'}(v^p)\,du\,dv^p\,dg. \tag{5.23}$$

We explain the notations here. First, $P_{\mathcal{O}} = M(\mathcal{O})U_{\mathcal{O}}$ is the maximal parabolic subgroup of SO_{2k+1} with the corresponding Levi decomposition. The upper zero indicates that we omit all the GL-factors in the Levi factor $M(\mathcal{O})$. That is $P^0_{\mathcal{O}} \cong \mathrm{SO}_l\, U_{\mathcal{O}}$. Next, $V^p_{\mathcal{O}'} = V_{\mathcal{O}'} \cap \mathrm{SO}_{3l+1}$ with $V^p_{\mathcal{O}'}/(V^p_{\mathcal{O}'})^{(1)} \cong \mathrm{Mat}_{l\times(l+1)}$. Also, $R_{s_p^2}$ is the unipotent radical of the standard maximal parabolic subgroup of $\mathrm{SO}_{4k+2l+3}$ with Levi part given by

$$\mathrm{GL}_{s_1}^{2(n_1-n_2)} \times \mathrm{GL}_{s_2}^{2(n_2-n_3)} \times \cdots \times \mathrm{GL}_{s_{p-1}}^{2(n_{p-1}-n_p)} \times \mathrm{GL}_{l}^{2n_p} \times \mathrm{SO}_{4l+1}\,.$$

The character $\psi_p : [R_{s_p^2}] \to \mathbb{C}^\times$ is the product of ψ_1 and the characters corresponding to the non-zero contribution in each of the repeated steps. It is given by (similar to ψ_n in Corollary 4.7)

$$\psi_p(u) = \psi\left(\sum_{i=1}^{s_1} u_{j,s_1+j} + \sum_{j=1}^{s_2} u_{2s_1+j,2s_1+s_2+j} + \cdots + \sum_{j=1}^{s_p} u_{2(s_1+\cdots+s_{p-1})+j,2(s_1+\cdots+s_{p-1})+s_p+j}\right).$$

The term $\prod_{i=0}^{p-1} \omega_{s_{p-i}} z_{s_{p-i}}$ is the product of $\omega_{s_1} z_{s_1}$ and the corresponding Weyl group elements and unipotent elements produced during each of the repeated steps. To be more precise, for each $1 \leqslant j \leqslant p$, we have

$$\omega_{s_j} = \begin{pmatrix} I_{2(s_1+\cdots+s_{j-1})} & & & & \\ & \nu_{s_j} & & & \\ & & I_{2k-2(s_1+\cdots+s_j)+1} & & \\ & & & \nu_{s_j}^{-1} & \\ & & & & I_{2(s_1+\cdots+s_{j-1})} \end{pmatrix},$$

$$\nu_{s_j} = \begin{pmatrix} I_{s_j} & & \\ & & I_{s_j} \\ & I_{k+l+1-2(s_1+\cdots+s_{j-1})-s_j} & \end{pmatrix},$$

and

$$z_{s_j} = \begin{pmatrix} I_{2(s_1+\cdots+s_{j-1})} &&&& \\ & \mu_{s_j} &&& \\ && I_{2k-2(s_1+\cdots+s_j)+1} && \\ &&& \mu_{s_j}^* & \\ &&&& I_{2(s_1+\cdots+s_{j-1})} \end{pmatrix},$$

$$\mu_{s_j} = \begin{pmatrix} I_{s_j} &&& \\ & I_{s_j} && -I_{s_j} \\ && I_{k+l+1-2(s_1+\cdots+s_j)} & \\ &&& I_{s_j} \end{pmatrix}.$$

Notice that we can conjugate all the z_j's to the right of all the ω_j's and rewrite (5.23) as

$$\int\limits_{P^0_{\mathcal{O}}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{[V^p_{\mathcal{O}'}]} \int\limits_{[R_{s_p^2}]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_l z_l(v^p, g)\right)\psi_p(u)\psi_{\mathcal{O}'}(v^p)\, du\, dv^p\, dg, \qquad (5.24)$$

with $\omega_l = \omega_{s_{p-1}}\omega_{s_{p-2}}\cdots\omega_{s_1}$ and

$$z_l = \begin{pmatrix} I_{s_1} &&&& \\ & \mu_l &&& \\ && I_l && \\ &&& \mu_l^{-1} & \\ &&&& I_{s_1} \end{pmatrix}, \mu_l = \begin{pmatrix} I_{s_1+\cdots+s_{p-1}} && -I_{s_1+\cdots+s_{p-1}} \\ & I_{k+l-2(s_1+\cdots+s_{p-1})+1} & \\ && I_{s_1+\cdots+s_{p-1}} \end{pmatrix}.$$

Now we proceed to the last step. Let $R_p \subset \mathrm{SO}_{4l+1}$ be the unipotent radical of the standard maximal parabolic subgroup of SO_{4l+1} with Levi part $\mathrm{GL}_l \times \mathrm{SO}_{2l+1}$. In terms of matrices,

$$R_p = \left\{ \begin{pmatrix} I_{2(s_1+\cdots+s_{p-1})} & & & & \\ & I_l & x & * & \\ & & I_{2l+1} & x^* & \\ & & & I_l & \\ & & & & I_{2(s_1+\cdots+s_{p-1})} \end{pmatrix} \in \mathrm{SO}_{4k+2l+3} : x \in \mathrm{Mat}_{l\times(2l+1)} \right\}.$$

The subgroup R_p is again a Heisenberg group with the maximal abelian quotient $R_p/R_p^{(1)} \cong \mathrm{Mat}_{l\times(2l+1)}$. On the other hand, for any $y = \begin{pmatrix} y_1 & y_2 \end{pmatrix} \in V_{\mathcal{O}'}^p/(V_{\mathcal{O}'}^p)^{(1)} \cong \mathrm{Mat}_{l\times(l+1)}$ where $y_1, y_2 \in \mathrm{Mat}_{l\times(\frac{l+1}{2})}$, we have

$$\omega_l z_l y (\omega_l z_l)^{-1} = \begin{pmatrix} y_1 & 0_{l\times l} & y_2 \end{pmatrix} \in R_p/R_p^{(1)}.$$

Let $H_p = (\omega_l z_l V^p (\omega_l z_l)^{-1}) \setminus R_p$, which is also a Heisenberg group. As the center $[Z(H_p)] \subset [(\omega_l z_l V^p (\omega_l z_l)^{-1})]$ is included in the domain of integration in the integral (5.24), we further expand (5.24) against $[(H_p/Z(H_p))]$.

We may identify the character group of $[(H_p/Z(H_p))]$ with

$$\left(H_p/Z(H_p)\right)(F) \cong \mathrm{Mat}_{l\times l}(F).$$

In the expansion, we conjugate elements in $[V_{\mathcal{O}'}^p]$ to the left inside the function $\theta_{4k+2l+3}$ and combine this domain of integration with $[(H_p/Z(H_p))]$. Denote the resulting domain of integration by $[R_p]$. Let $R = R_{s_p^2} R_p$, which is the unipotent subgroup that coincides with $V_{2,\mathcal{O}^2}$ where $\mathcal{O}^2$ is the unipotent orbit in $\mathrm{SO}_{4k+2l+3}$ corresponding to the partition

$$\left((4n_1+3)^{r_1}(4n_2+3)^{r_2}\cdots(4n_p+3)^{r_p}(1)^{l+1}\right).$$

As a result, (5.24) is equal to

$$\int_{P^0_{\mathcal{O}}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{[R]} \sum_{\xi \in \mathrm{Mat}_{l\times l}(F)} \varphi(g)\overline{\theta_{4k+2l+3}\left(u\omega_l z_l(1,g)\right)}\psi_{\mathcal{O}',p,\xi}(u)\,du\,dg. \tag{5.25}$$

Here, we combine all the characters involved to get the character $\psi_{\mathcal{O}',p,\xi}$ on $[R]$. For any $x = x_1x_2x_3 \in R(\mathbb{A})$ with $x_1 \in R_{s_p^2}(\mathbb{A}), x_2 \in V^p_{\mathcal{O}'}(\mathbb{A})$ and $x_3 \in H_p(\mathbb{A})/(Z(H_p))(\mathbb{A})$,

$$\psi_{\mathcal{O}',p,\xi}(x) = \psi_p(x_1)\psi_{\mathcal{O}'}(x_2)\psi_\xi(x_3).$$

By Proposition 4.4, the inner integral of (5.25) factors as the product of an integral of a theta function in the theta representation Θ_{GL} of the double cover $\widetilde{\mathrm{GL}}_{2(s_1+\cdots+s_{p-1})}(\mathbb{A})$ with respect to the character ψ_p and an integral of a theta function in Θ_{4l+1} of $\widetilde{\mathrm{SO}}_{4l+1}(\mathbb{A})$ with respect to the character $\psi_{\mathcal{O}'}\psi_\xi$. When the character $\psi_{\mathcal{O}'}\psi_\xi$ is generic with respect to the unipotent orbit associated with the partition (3^l1^{l+1}), the second integral is a Fourier coefficient of the theta representation Θ_{4l+1} with respect to the unipotent orbit associated to (3^l1^{l+1}). By Proposition 4.5, such an integral must be zero. Thus, the non-zero contributions in the expansion (5.25) come from those ψ_ξ corresponding to $\xi \in \mathrm{Mat}_{l\times l}(F)$ such that $\psi_{\mathcal{O}'}\psi_\xi$ is not generic.

To proceed, let us fix a

$$\xi = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix} \in \mathrm{Mat}_{l\times l}(F),$$

with row vectors $\lambda_1, \cdots, \lambda_l \in F^l$. By construction, the character $\psi_\eta := \psi_{\mathcal{O}'}\psi_\xi$ corresponds to the matrix

$$\eta = \begin{pmatrix} 1 & & & & \lambda_1 & & & & \\ & \ddots & & & \vdots & & & & \\ & & 1 & & \lambda_{\frac{l-1}{2}} & & & & \\ & & & \frac{1}{2} & \lambda_{\frac{l+1}{2}} & -1 & & & \\ & & & & \lambda_{\frac{l+3}{2}} & & -1 & & \\ & & & & \vdots & & & \ddots & \\ & & & & \lambda_l & & & & -1 \end{pmatrix} \in \mathrm{Mat}_{l\times(2l+1)}(F).$$

Notice that ψ_η is generic as long as the row space of η is not totally isotropic with respect to the non-degenerate bilinear form on F^{2l+1} defined by J_{2l+1}. As a result, the contribution of ψ_η is non-zero only if the row space of η is totally isotropic and of rank l. That is, the row space of η is a maximal totally isotropic subspace of F^{2l+1}. Consequently, we only need to consider ξ with row vectors $\lambda_1, \lambda_2, \cdots \lambda_l$ satisfying the following conditions:

1. Each λ_i except $\lambda_{\frac{l+1}{2}}$ is non-zero isotropic.

2. Each pair λ_i and λ_{l+1-i} for $i = 1, 2, \cdots, \frac{l-1}{2}$ is a hyperbolic pair, i.e.

$$(\lambda_i, \lambda_{l+1-i}) = 1,\ i = 1, 2, \cdots, \frac{l-1}{2}.$$

   All these hyperbolic pairs are mutually orthogonal.

3. The vector $\lambda_{\frac{l+1}{2}}$ is non-isotropic with $(\lambda_{\frac{l+1}{2}}, \lambda_{\frac{l+1}{2}}) = 1$.

The group $\mathrm{SO}_l(F)$ acts transitively from the right on the set of ξ's and we pick the identity matrix $I_l \in \mathrm{Mat}_{l\times l}(F)$ as a representative. As a result, the integral (5.25) is equal to

$$\int\limits_{P^0_{\mathcal{O}}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{[R]} \sum_{\gamma\in \mathrm{SO}_{2l+1}(F)} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\gamma\omega_l z_l(1,g)\right)\psi_{\mathcal{O}',p,I_l}(u)\,du\,dg. \tag{5.26}$$

Since γ commutes with the product $\omega_l z_l$, we can conjugate it to the right and collapse the summation with the outer integration to obtain

$$\int\limits_{U_{\mathcal{O}}(F)\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int\limits_{[R]} \varphi(g)\overline{\theta}_{4k+2l+3}\left(u\omega_l z_l(1,g)\right)\psi_{\mathcal{O}',p,I_l}(u)\,du\,dg. \tag{5.27}$$

We further decompose the domain of integration of the outer integral as

$$U_{\mathcal{O}}(F)\setminus \mathrm{SO}_{2k+1}(\mathbb{A}) = \left(U_{\mathcal{O}}(F)\setminus U_{\mathcal{O}}(\mathbb{A})\right)\left(U_{\mathcal{O}}(\mathbb{A})\setminus \mathrm{SO}_{2k+1}(\mathbb{A})\right).$$

We obtain

$$\int\limits_{U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})} \int\limits_{U_{\mathcal{O}}(\mathbb{A})\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \int_{[R]} \varphi(ng)\overline{\theta}_{4k+2l+3}\left(u\omega_l z_l(1,ng)\right)\psi_{\mathcal{O}',p,I_l}(u)\,du\,dg\,dn. \tag{5.28}$$

Observe that $n_0 := \omega_l z_l n(\omega_l z_l)^{-1} \in R(\mathbb{A})$ for any $n \in U_{\mathcal{O}}(\mathbb{A})$, and $\psi_{\mathcal{O}',p,I_l}(n_0) = \psi_{\mathcal{O}}(n)$. Hence, we conjugate the variable n inside the function $\theta_{4k+2l+3}$ to the left and perform a change of variable by $u \mapsto un_0^{-1}$ to finally obtain that (5.28) is equal to

$$\int\limits_{U_{\mathcal{O}}(\mathbb{A})\backslash \mathrm{SO}_{2k+1}(\mathbb{A})} \left(\int\limits_{U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})} \varphi(ng)\psi_{\mathcal{O}}(n)dn \right) \int\limits_{[R]} \overline{\theta}_{4k+2l+3}\left(u\omega_l z_l(1,g)\right)\psi_{\mathcal{O}',p,I_l}(u)\,du\,dg. \tag{5.29}$$

Note that

$$F_{\psi_{\mathcal{O}},U_{\mathcal{O}}}(\varphi)(g) = \int_{U_{\mathcal{O}}(F)\backslash U_{\mathcal{O}}(\mathbb{A})} \varphi(ng)\psi_{\mathcal{O}}(n)dn \tag{5.30}$$

is a Fourier coefficient of φ associated to the unipotent orbit $\mathcal{O}$ and the generic character $\phi_{\mathcal{O}}$ on $[U_{\mathcal{O}}]$. The fact that (5.7) is non-vanishing implies that $F_{\psi_{\mathcal{O}},U_{\mathcal{O}}}(\varphi)$ is non-vanishing, which completes the proof. $\square$

6 Local theory

In this section, we establish the local counterpart to Theorem 5.1.

6.1 Local setup

Let F be a non-archimedean local field. We are still concerned with the two unipotent orbits

$$\mathcal{O} = ((2n_1+1)^{r_1}(2n_2+1)^{r_2}\cdots(2n_p+1)^{r_p})$$

and

$$\mathcal{O}' = ((2n_1+3)^{r_1}(2n_2+3)^{r_2}\cdots(2n_p+3)^{r_p}(1))$$

of the two groups SO_{2k+1} and $\mathrm{SO}_{2k+2l+2}$ respectively, where $l = r_1 + r_2 + \cdots r_p$.

Recall that we can associate the two orbits $\mathcal{O}$ and $\mathcal{O}'$ with the unipotent subgroups $U_{\mathcal{O}} = V_{1,\mathcal{O}} = V_{2,\mathcal{O}}$ and $V_{\mathcal{O}'} = V_{1,\mathcal{O}'} = V_{2,\mathcal{O}'}$ respectively. Define the generic characters $\psi_{\mathcal{O}} : U_{\mathcal{O}}(F) \to \mathbb{C}^\times$ and $\psi_{\mathcal{O}'} : V_{\mathcal{O}'}(F) \to \mathbb{C}^\times$ similarly to the global ones given by (5.3) and (5.6) respectively.

Suppose π is an irreducible genuine admissible representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. Recall from Section 4.2 that $\Theta_{4k+2l+3}$ is the local theta representation of $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$. Suppose there exists an irreducible genuine admissible representation $\Theta(\pi) = \Theta_{4k+2l+3}(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+2l+2}(F)$ such that the Hom-space

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times\widetilde{\mathrm{SO}}_{2k+2l+2}}(\Theta_{4k+2l+3}, \pi\otimes\Theta(\pi)) \qquad (6.1)$$

is non-zero. Here, we restrict $\Theta_{4k+2l+3}$ to a representation of the product subgroup $\widetilde{\mathrm{SO}}_{2k+1}(F)\times\widetilde{\mathrm{SO}}_{2k+2l+2}(F)$ which is the preimage of the product $\mathrm{SO}_{2k+1}(F)\times \mathrm{SO}_{2k+2l+2}(F)$ in $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$ via the embedding (2.3).

We extend the definition of twisted Jacquet modules in Section 3.4. If $\mathcal{O}_0$ is an odd unipotent orbit, then $U := V_{2,\mathcal{O}_0}$ is equal to the unipotent radical of some maximal parabolic subgroup of SO_{2k+1}. If the unipotent radical $U(F)$ is a Heisenberg group with its center $Z(U)(F)$ acting trivially on $\mathcal{V}$ and ψ_0 is trivial on $Z(U)(F)$, then the vector subspace $\mathcal{V}(U,\psi_0) = \mathcal{V}(U/Z(U),\psi_0)$. Denote the quotient space $\mathcal{V}/\mathcal{V}(U/Z(U),\psi_0)$ by $J_{U/Z(U),\psi}(\pi)$.

The following theorem is the local version of Proposition 4.6. For a positive integer $s < k/4$, let $R_{s,2k+1}$ be the unipotent radical of the standard maximal parabolic subgroup of SO_{2k+1} with Levi factor $\mathrm{GL}_s \times \mathrm{SO}_{2k-2s+1}$. Similarly, let $R_{2s,2k+1}$ be the unipotent radical of the maximal parabolic subgroup of SO_{2k+1} with Levi factor $\mathrm{GL}_{2s} \times \mathrm{SO}_{2k-4s+1}$. Define the character $\psi_1 : R_{s,2k+1}(F) \to \mathbb{C}^\times$ by

$$\psi_1(u) = \psi(\sum_{j=1}^{s} u_{j,j+s}), \quad u = (u_{i,j}) \in R_{s,2k+1}(F).$$

Theorem 6.1. *Consider the local theta representation Θ_{2k+1} of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. There is a surjection of $\widetilde{\mathrm{GL}}_s^{\Delta} \times \widetilde{\mathrm{SO}}_{2k-4s+1}$ -modules*

$$J_{R_{2s,2k+1}}(\Theta_{2k+1}) \twoheadrightarrow J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1}),$$

where $\mathrm{GL}_s^{\Delta} \times \mathrm{SO}_{2k-4s+1}$ is the subgroup of $\mathrm{GL}_{2s} \times \mathrm{SO}_{2k-4s+1}$ with the GL_s-factor embedded in GL_{2s} diagonally.

Proof. By Proposition 4.3, it suffices to show that the unipotent subgroup

$$R_{s,2k-2s+1}(F) \subset \mathrm{SO}_{2k-2s+1}(F) \xhookrightarrow{\iota} \mathrm{SO}_{2k+1}(F)$$

acts trivially on the Jacquet module $J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})$. The proof then proceeds in a similar fashion to the global case.

Let $R_{s,2k-2s+1}$ be the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{2k-2s+1}$ with the Levi factor $\mathrm{GL}_s \times \mathrm{SO}_{2k-4s+1}$. We identify $R_{s,2k-2s+1}$ with its embedded image in SO_{2k+1} via (2.3). Note that $R_{s,2k-2s+1}(F)$ is a Heisenberg group. We claim that its center $Z(R_{s,2k-2s+1})(F)$ acts trivially on $J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})$.

Under the action of the Levi subgroup, any non-trivial character on $Z(R_{s,2k-2s+1})(F)$ may be represented by ψ_{ξ_t} $(0 < 2t \leqslant s)$ associated to a matrix ξ_t of the form

$$\xi_t = \begin{pmatrix} 0 & z_t \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{s\times s}(F),$$

where $z_t = \mathrm{diag}(\lambda_1, \cdots, \lambda_t, \lambda_t^{-1}, \cdots, \lambda_1^{-1}) \in \mathrm{Mat}_{2t\times 2t}(F)$ with $\lambda_i \in F^\times$, $\forall i = 1, \cdots, t$. If the claim is not true, then there must be a non-trivial character ψ_{ξ_t} on $Z(R_{s,2k-2s+1})(F)$ such that

$$J_{Z(R_{s,2k-2s+1}),\psi_{\xi_t}}\left(J_{R_{s,2k+1,},\psi_1}(\Theta_{2k+1})\right) \neq 0. \tag{6.2}$$

However, the product of ψ_1 and $\psi_{\xi_t,Z}$ is a generic character attached to the unipotent orbit $\mathcal{O}_t = (4^{2t}3^{s-2t}1^{2k-2t-3s+1})$. Hence, the resulting twisted Jacquet module (6.2) is zero by Proposition 4.2. We get a contradiction.

Therefore, it remains to show that the abelian quotient $R_{s,2k-2s+1}/Z(R_{s,2k-2s+1})(F)$ also acts trivially on $J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})$. We may identify the character group of $R_{s,2k-2s+1}/Z(R_{s,2k-2s+1})(F)$ with $\mathrm{Mat}_{s\times(2k-4s+1)}(F)$. Under the action of Levi subgroup $\mathrm{GL}_s(F) \times \mathrm{SO}_{2k-4s+1}(F)$, any character on $R_{s,2k-2s+1}/Z(R_{s,2k-2s+1})(F)$ may be represented by ψ_{2,z_t}, corresponding to a matrix z_t of the form

$$z_t = \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{s\times(2k-4s+1)}(F), t = 1, \cdots, s.$$

If the action of $R_{s,2k-2s+1}/Z(R_{s,2k-2s+1})(F)$ on $J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})$ is not trivial, then there exists a character ψ_{2,z_t} such that

$$J_{R_{s,2k-2s+1}/Z(R_{s,2k-2s+1}),\psi_{2,z_t}}\left(J_{Z(R_{s,2k-2s+1})}\left(J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})\right)\right)\cong J_{R_{s^2,2k+1},\psi_1\psi_{2,z_t}}(\Theta)\neq 0, \tag{6.3}$$

where $R_{s^2,2k+1}=R_{s,2k+1}R_{s,2k-2s+1}$.

Proceed with the twisted Jacquet module $J_{R_{s^2,2k+1},\psi_1\psi_{2,z_t}}(\Theta_{2k+1})$. Apply the same argument to see that $Z(R_{s,2k-4s+1})(F)$ acts on it trivially. Hence, it suffices to check the action of the abelian quotient $R_{s,2k-4s+1}/Z(R_{s,2k-4s+1})(F)$ on $J_{R_{s^2,2k+1},\psi_1\psi_{2,z_t}}(\Theta_{2k+1})$. If the action is trivial, Proposition 4.3 implies that there is a $\widetilde{\mathrm{GL}}_{3s}(F)\times\widetilde{\mathrm{SO}}_{2k-6s+1}(F)$-module isomorphism

$$J_{R_{s,2k-4s+1}}\left(J_{R_{s^2,2k+1},\psi_1\psi_{2,z_t}}(\Theta_{2k+1})\right)\cong J_{N_{s^3},\psi_1\psi_{2,z_t}}\left(\Theta_{\mathrm{GL}_{3s}(F)}\right)\otimes\Theta_{2k-6s+1}. \tag{6.4}$$

Here $\Theta_{\mathrm{GL}_{3s}(F)}$ is the local theta representation of the double cover $\widetilde{\mathrm{GL}}_{3s}(F)$, and N_{s^3} is the unipotent radical of the parabolic subgroup of GL_{3s} with Levi subgroup GL_s^3. However, by Corollary 3.34 of [7], the twisted Jacquet module $J_{N_{s^3},\psi_1\psi_{2,t}}\left(\Theta_{\mathrm{GL}_{3s}(F)}\right)$ is zero. This is a contradiction to (6.3). Therefore, there is a non-trivial character on $R_{s,2k-4s+1}(F)$ such that the twisted Jacquet module of $J_{R_{s^2,2k+1},\psi_1\psi_{2,t}}(\Theta_{2k+1})$ with respect to $R_{s,2k-4s+1}(F)$ and this character is non-zero.

We continue by the same argument repeatedly. For each step, the corresponding unipotent radical acts trivially due to Corollary 3.34 of [7]. Eventually, we obtain a non-zero twisted Jacquet module of Θ_{2k+1} with respect to some unipotent orbit that is not comparable to $\mathcal{O}(\Theta_{2k+1})$. By Proposition 4.2, such a twisted Jacquet module must be zero, which is a contradiction. Thus, the action of $R_{s,2k-2s+1}/Z(R_{s,2k-2s+1})(F)$ on $J_{R_{s,2k+1},\psi_1}(\Theta_{2k+1})$ must be trivial, which completes the proof. □

6.2 The local main theorem

With enough tools at our disposal, we are now ready to state and prove the local result.

Theorem 6.2. *Let π be an irreducible admissible representation of $\widetilde{\mathrm{SO}}_{2k+1}(F)$. Suppose there exists an irreducible admissible representation $\Theta(\pi)$ of $\widetilde{\mathrm{SO}}_{2k+2l+2}(F)$ such that, as representations of the group $\widetilde{\mathrm{SO}}_{2k+1}(F) \times \widetilde{\mathrm{SO}}_{2k+2l+2}(F)$,*

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1} \times \widetilde{\mathrm{SO}}_{2k+2l+2}} \left(\Theta_{4k+2l+3}, \pi \otimes \Theta(\pi)\right) \neq 0. \tag{6.5}$$

Furthermore, suppose the twisted Jacquet module of $\Theta(\pi)$ with respect to the unipotent orbit $\mathcal{O}'$ and the generic character $\psi_{\mathcal{O}'}$ is non-zero, i.e.

$$J_{V_{\mathcal{O}'}, \psi_{\mathcal{O}'}}(\Theta(\pi)) \neq 0. \tag{6.6}$$

Then the twisted Jacquet module of π with respect to the unipotent orbit $\mathcal{O}$ and the generic character $\psi_{\mathcal{O}}$ is also non-zero, i.e.

$$J_{U_{\mathcal{O}}, \psi_{\mathcal{O}}}(\pi) \neq 0. \tag{6.7}$$

Throughout the proof, we identify any subgroup of SO_{2k+1} or $\mathrm{SO}_{2k+2l+2}$ with its embedded image in $\mathrm{SO}_{4k+2l+3}$ via 2.3. Similar to the proof of Theorem 5.1, we only discuss the case of $n_i - n_{i+1} = 1$ for all $i = 1, 2, \cdots, p-1$ and $n_p = 1$ for notational simplicity.

Proof. Fix a non-zero $\Psi_1 \in \mathrm{Hom}\left(\Theta_{4k+2l+3}, \pi \otimes \Theta(\pi)\right)$. It is clear that Ψ_1 must be surjective because both π and $\Theta(\pi)$ are irreducible so that $\pi \otimes \Theta(\pi)$ is irreducible as a representation of the group $\widetilde{\mathrm{SO}}_{2k+1}(F) \times \widetilde{\mathrm{SO}}_{2k+2l+2}(F)$. As a result, Ψ_1 factors through the non-zero twisted Jacquet module $J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))$. In other words, Ψ_1 induces a non-zero $\widetilde{\mathrm{SO}}_{2k+1}(F) \times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')(F)V_{\mathcal{O}'}(F)$-equivariant morphism which we still denote by Ψ_1:

$$\Psi_1 : \Theta_{4k+2l+3} \to \pi \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi)). \tag{6.8}$$

Here, $M^{\psi_{\mathcal{O}'}}(\mathcal{O}')(F)$ denotes the stabilizer of $\psi_{\mathcal{O}'}$ in $M(\mathcal{O}')(F)$.

Recall we have $V_{s_1} = V_{\mathcal{O}'} \cap R_{s_1}$ where $R_{s_1} = R_{s_1,4k+2l+3}$ is the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{4k+2l+3}$ with Levi subgroup $\mathrm{GL}_{s_1} \times \mathrm{SO}_{4k+2l-2s_1+3}$. Notice that $V_{s_1}(F)$ acts on $J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))$ by the character ψ_1, and Ψ_1 is $V_{s_1}(F)$-equivariant. As a result, Ψ_1 must factor through the non-zero twisted Jacquet module of $\Theta_{4k+2l+3}$ with respect to the unipotent subgroup $V_{s_1}(F)$ and the character $\psi_{\mathcal{O}'}$ restricted on $V_{s_1}(F)$, which we denote ψ_1. If we denote the resulting map by Ψ_1', then

$$\Psi_1' : J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3}) \to \pi \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi)) \tag{6.9}$$

is non-zero.

Consider the Heisenberg group $H_{s_1} = V_{s_1} \setminus R_{s_1}$, which may be identified with the subgroup of matrices of the form

$$H_{s_1} := \left\{ \begin{pmatrix} I_{s_1} & & x & & * \\ & I_{k+l-s_1+1} & & & \\ & & I_{2k+1} & & x^* \\ & & & I_{k+l-s_1+1} & \\ & & & & I_{s_1} \end{pmatrix} \in \mathrm{SO}_{4k+2l+3} : x \in \mathrm{Mat}_{s_1 \times (2k+1)} \right\}.$$

As $Z(H_{s_1})(F) \subset V_{s_1}(F)$ acts trivially on $J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3})$, we consider the action of the abelian quotient $H_{s_1}/Z(H_{s_1})(F)$ on the twisted Jacquet module $J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3})$. If there exists a non-zero vector in $J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3})$ on which $H_{s_1}/Z(H_{s_1})(F)$ acts by a character ψ', then Ψ_1' must factor through the non-trivial twisted Jacquet module

$$J_{H_{s_1}/Z(H_{s_1})(F),\psi'}\left(J_{Z(H_{s_1})(F)}(J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3}))\right) \cong J_{R_{s_1},\psi_1\psi'}(\Theta_{4k+2l+3}). \tag{6.10}$$

We may identify the character group of $H_{s_1}/Z(H_{s_1})(F)$ with $\mathrm{Mat}_{s_1\times(2k+1)}(F)$. Under the action of $\mathrm{GL}_{s_1}(F)\times\mathrm{SO}_{2k+1}(F)$, suppose ψ' corresponds to a matrix in $\mathrm{Mat}_{s_1\times(2k+1)}(F)$ that lies in the same conjugacy class with a matrix that contains a non-isotropic row vector. Then the product of ψ_1 and ψ' is a generic character attached to the unipotent orbit associated with the partition $(3^{s_1}1^{4k+2l+3-3s_1})$. By Proposition 4.2, the twisted Jacquet module corresponding to this character (6.10) is zero.

It remains to examine over those characters on $H_{s_1}/Z(H_{s_1})(F)$ corresponding to a matrix in $\mathrm{Mat}_{s_1\times(2k+1)}(F)$ with totally isotropic row space. Any such ψ' lies in the same conjugacy class with a character ψ_{ξ_q} corresponding to a matrix ξ_q of the form

$$\xi_q = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{s_1\times(2k+1)}(F), \quad q=0,1,\cdots,s_1. \tag{6.11}$$

For a fixed q, we check on Ψ_1' restricted to the twisted Jacquet module of the form

$$J_{H_{s_1}/Z(H_{s_1})(F),\psi_{\xi_q}}\left(J_{Z(H_{s_1})(F)}(J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3}))\right) \cong J_{R_{s_1},\psi_{1,\xi_q}}(\Theta_{4k+2l+3}),$$

where $\psi_{1,\xi_q} = \psi_1\psi_{\xi_q}$. Consider the elements $w_q, z_q \in \mathrm{SO}_{4k+2l+3}(F)$ of the form

$$z_q = \begin{pmatrix} I_{s_1} & & & & \\ & \mu_q & & & \\ & & I_{2k+1-2q} & & \\ & & & \mu_q^* & \\ & & & & I_{s_1} \end{pmatrix}, \mu_q = \begin{pmatrix} I_q & & -I_q \\ & I_{k+l+1-s_1-q} & \\ & & I_q \end{pmatrix},$$

and

$$\omega_q = \begin{pmatrix} I_{s_1} & & & & \\ & \nu_q & & & \\ & & I_{2k+1-2q} & & \\ & & & \nu_q^{-1} & \\ & & & & I_{s_1} \end{pmatrix}, \nu_q = \begin{pmatrix} & & & I_q & \\ & I_{s_1-q} & & & \\ I_q & & & & \\ & & I_{k+l+q+1-3s_1} & & \\ & & & & I_{s_1-q} \end{pmatrix}.$$

Note that z_0 and ω_0 are both the identity matrix.

Notice that the conjugation action of $w_q z_q$ on $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$ preserves $R_{s_1}(F)$. As a result, we have

$$J_{R_{s_1},\psi_{1,\xi}}(\Theta_{4k+2l+3}) \cong J_{R_{s_1},\psi_1}(\Theta_{4k+2l+3}^{w_q z_q}), \tag{6.12}$$

where $\Theta_{4k+2l+3}^{w_q z_q}$ is the representation of the group $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$ obtained by pulling back the representation $\Theta_{4r+2l+3}$ via the conjugation by $w_q z_q$ on $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$. For any $g \in \widetilde{\mathrm{SO}}_{4k+2l+3}(F)$ and any function θ in the representation $\Theta_{4k+2l+3}$, the action of g on θ is replaced by the action of $(w_q z_q)^{-1} g w_q z_q$ on θ. Moreover, by Theorem 6.1,

$$\Psi_1' : J_{R_{s_1},\psi_1}(\Theta_{4k+2l+3}^{w_q z_q}) \cong J_{R_{s_1},\psi_{1,\xi_q}}(\Theta_{4k+2l+3}) \to \pi \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi)) \tag{6.13}$$

factors through the Jacquet module $J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_qz_q})$ where we recall that $R_{s_1^2} = R_{s_1}R_{s_1,4k+2l-2s_1+3}$ and $R_{s_1,4k+2l-2s_1+3}$ is the unipotent radical of the maximal parabolic subgroup of $\mathrm{SO}_{4k+2l-2s_1+3}$ with Levi subgroup $\mathrm{GL}_{s_1} \times \mathrm{SO}_{4k+2l-4s_1+3}$. If we denote the resulting map by Ψ_1'', then

$$\Psi_1'' : J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_qz_q}) \to \pi \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi)) \tag{6.14}$$

is non-zero.

Let $V_{\mathcal{O}'}^1 = V_{\mathcal{O}'} \cap \mathrm{SO}_{2k+2l-2s_1+2}$. The intersection $R_{s_1,4k+2l-2s_1+3} \cap w_qz_qV_{\mathcal{O}'}^1(w_qz_q)^{-1}$ is non-trivial as long as $q < s_1$. It contains the one parameter subgroup $\{x_\beta(r) : r \in F\}$ associated to β given by (5.13). The root group $x_\beta(r)$ acts trivially on $J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_qz_q})$, but acts by the non-trivial character $\psi_{\mathcal{O}'}$ on $J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))$. This implies that Ψ_1'' must be zero when $q < s_1$.

It remains only the case of $q = s_1$. Note that $\psi_\xi = \psi_{\xi_{s_1}}$ is generic. Recall that the center $Z(H_{s_1}(F))$ acts trivially on $\Theta_{4k+2l+3}$. Hence, $J_{Z(H_{s_1})}(\Theta_{4k+2l+3}) \cong \Theta_{4k+2l+3}$. By Proposition 5.12(d) of [2] or Lemma A.1 of [20], there exists a short exact sequence of $Q_{s_1}(F) \times \widetilde{\mathrm{SO}}_{2k+1}(F)$-modules

$$0 \to \mathrm{ind}_{Q_{s_1}\times P_{s_1}^0}^{Q_{s_1}\times\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{H_{s_1},\psi_\xi}(\Theta_{4k+2l+3})\right) \to J_{Z(H_{s_1})}(\Theta_{4k+2l+3}) \to J_{H_{s_1}}(\Theta_{4k+2l+3}) \to 0, \tag{6.15}$$

where $Q_{s_1} = (\widetilde{\mathrm{GL}}_{s_1} \times \widetilde{\mathrm{SO}}_{2k+2l-2s_1+2})V_{s_1}$ is a maximal parabolic subgroup of $\widetilde{\mathrm{SO}}_{2k+2l+2}$. The product $Q_{s_1}(F) \times \widetilde{\mathrm{SO}}_{2k+1}(F)$ is the normalizer of $H_{s_1}(F)$ in $\widetilde{\mathrm{SO}}_{4k+2l+3}(F)$, while the stablizer of ψ_ξ in $Q_{s_1}(F) \times \widetilde{\mathrm{SO}}_{2k+1}(F)$ is

$$(\widetilde{\mathrm{GL}}_{s_1}^{\Delta} \times \widetilde{\mathrm{SO}}_{2k+2l-2s_1+2})V_{s_1} \times P_{s_1}^0 \cong Q_{s_1} \times P_{s_1}^0,$$

where $\widetilde{\mathrm{GL}}_{s_1}$ is diagonally embeded into the Levi subgroups of the two parabolic subgroups Q_{s_1} and P_{s_1}. Here, $\mathrm{ind}_{Q_{s_1}\times P^0_{s_1}}^{Q_{s_1}\times\widetilde{\mathrm{SO}}_{2k+1}}$ is the induction with compact support as in [3].

The two functors $J_{V_{s_1},\psi_1}$ and $\mathrm{ind}_{Q_{s_1}\times P^0_{s_1}}^{Q_{s_1}\times\widetilde{\mathrm{SO}}_{2k+1}}$ satisfy the following relation

$$J_{V_{s_1},\psi_1}\circ \mathrm{ind}_{Q_{s_1}\times P^0_{s_1}}^{Q_{s_1}\times\widetilde{\mathrm{SO}}_{2k+1}} \cong \mathrm{ind}_{\widetilde{\mathrm{GL}}_{s_1}\times (Q'_{s_1})^0\times P^0_{s_1}}^{\widetilde{\mathrm{GL}}_{s_1}\times (Q'_{s_1})^0\times\widetilde{\mathrm{SO}}_{2k+1}} \circ J_{V_{s_1},\psi_1}.$$

Here, $\widetilde{\mathrm{GL}}_{s_1}^{\Delta}(F)\times (Q'_{s_1})^0(F)$ is the stablizer of ψ_1 in the Levi factor $\widetilde{\mathrm{GL}}_{s_1}(F)\times \widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}(F)$, where Q'_{s_1} is the parabolic subgroup of $\widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}$ with Levi subgroup $\widetilde{\mathrm{GL}}_{s_1}\times\widetilde{\mathrm{SO}}_{2k+2l-4s_1+2}$. Hence,

$$J_{V_{s_1},\psi_1}\left(\mathrm{ind}_{Q_{s_1}\times P^0_{s_1}}^{Q_{s_1}\times\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{H_{s_1},\psi_\xi}(\Theta_{4k+2l+3})\right)\right) \tag{6.16}$$

is isomorphic to

$$\mathrm{ind}_{\widetilde{\mathrm{GL}}^{\Delta}_{s_1}\times (Q'_{s_1})^0\times P^0_{s_1}}^{\widetilde{\mathrm{GL}}_{s_1}\times (Q'_{s_1})^0\times\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1},\psi_{1,\xi}}(\Theta_{4k+2l+3})\right). \tag{6.17}$$

Moreover, $w_{s_1}z_{s_1}$ acts on $\Theta_{4k+2l+3}$ and preserves R_{s_1}. By (6.12), we further deduce that (6.17) is isomorphic to

$$\mathrm{ind}_{\widetilde{\mathrm{GL}}^{\Delta}_{s_1}\times\widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}\times P^0_{s_1}}^{\widetilde{\mathrm{GL}}_{s_1}\times\widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}\times\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1},\psi_1}(\Theta^{w_{s_1}z_{s_1}}_{4k+2l+3})\right). \tag{6.18}$$

Hereafter, we denote $\mathrm{ind}_{\widetilde{\mathrm{GL}}^{\Delta}_{s_1}\times\widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}\times P^0_{s_1}}^{\widetilde{\mathrm{GL}}_{s_1}\times\widetilde{\mathrm{SO}}_{2k+2l-2s_1+2}\times\widetilde{\mathrm{SO}}_{2k+1}}$ by $\mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}$ for simplicity.

Applying the functor $J_{V_{s_1},\psi_1}$ to (6.15), we obtain the short exact sequence

$$0\to \mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1},\psi_1}(\Theta^{w_{s_1}z_{s_1}}_{4k+2l+3})\right)\to J_{V_{s_1},\psi_1}(\Theta_{4k+2l+3})\to J_{R_{s_1},\psi_1}(\Theta_{4k+2l+3})\to 0.$$

We have just shown that, corresponding to $q=0$,

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V_{\mathcal{O}'}}\left(J_{R_{s_1},\psi_1}(\Theta_{4k+2l+3}),\pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right)=0.$$

Then, by (6.9), (6.15) and (6.18), we have

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V_{\mathcal{O}'}}\left(\mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1},\psi_1}(\Theta_{4k+2l+3}^{w_{s_1}z_{s_1}})\right), \pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right)\neq 0. \tag{6.19}$$

By Proposition 6.1, we further deduce that

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V}\left(\mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_{s_1}z_{s_1}})\right), \pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right)\neq 0. \tag{6.20}$$

Continue the same argument by replacing (6.8) by any non-zero Ψ_2 in the Hom-space (6.20). Recall the unipotent subgroup R_{s_2} defined by (5.22) and the Heisenberg quotient $H_{s_2} = V_{s_2}\backslash R_{s_2}$. Consider the twisted Jacquet modules with respect to $H_{s_2}(F)$ whose center acts trivially on $J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_{s_1}z_{s_1}})$. We note that the normalizer of H_{s_2} in $\widetilde{\mathrm{SO}}_{2k+1}$ is $\widetilde{\mathrm{SO}}_{2k-2s_1+1}\subset P^0_{s_1}$. By a similar argument, any non-zero homomorphism

$$\Psi_2 : \mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_{s_1}z_{s_1}})\right)\to \pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))$$

must factor through the non-zero twisted Jacquet module with respect to a generic character on $H_{s_2}(F)$, which is

$$\mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(\mathrm{ind}_{P^0_{s_2}}^{\widetilde{\mathrm{SO}}_{2k-2s_1+1}} J_{R_{s_2},\psi_2^\circ}\left(J_{R_{s_1^2},\psi_1}(\Theta_{4k+2l+3}^{w_{s_2}z_{s_2}})\right)\right).$$

Here, ψ_2° is $\psi_{\mathcal{O}'}$ restricted to V_{s_2}, and P_{s_2} is the standard maximal parabolic subgroup of $\widetilde{\mathrm{SO}}_{2k-2s_1+1}$ with Levi subgroup $\widetilde{\mathrm{GL}}_{s_2}\times\widetilde{\mathrm{SO}}_{2k-2(s_1+s_2)+1}$.

By the transitivity of induction

$$\mathrm{ind}_{P^0_{s_1}}^{\widetilde{\mathrm{SO}}_{2k+1}}\mathrm{ind}_{P^0_{s_2}}^{\widetilde{\mathrm{SO}}_{2k-2s_1+1}}\cong \mathrm{ind}_{P^0_{s_2}}^{\widetilde{\mathrm{SO}}_{2k+1}}.$$

By Proposition 6.1, we obtain that

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V_{\mathcal{O}'}}\left(\mathrm{ind}_{P^0_{s_2}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_2^2},\psi_2}(\Theta_{4k+2l+3}^{w_{s_2}z_{s_2}})\right), \pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right) \quad (6.21)$$

is non-zero. Continuing the same argument repeatedly, we obtain that

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V_{\mathcal{O}'}}\left(\mathrm{ind}_{\widetilde{\mathrm{SO}}_l}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_p^2},\psi_p}(\Theta_{4k+2l+3}^{w_l z_l})\right), \pi\otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right) \neq 0. \quad (6.22)$$

Here, $R_{s_p^2}$ is the unipotent radical of the standard maximal parabolic subgroup of $\mathrm{SO}_{4k+2l+3}$ with Levi part

$$\mathrm{GL}_{s_1}^{2(n_1-n_2)}\times\mathrm{GL}_{s_2}^{2(n_2-n_3)}\times\cdots\times\mathrm{GL}_{s_{p-1}}^{2(n_{p-1}-n_p)}\times\mathrm{GL}_l^{2n_p}\times\mathrm{SO}_{4l+1}\,.$$

The character $\psi_p : R_{s_p^2}(F)\to\mathbb{C}^\times$ is the product of ψ_1 and the characters corresponding to the non-zero twisted Jacquet modules in each of the repeated steps.

We now proceed to the last step. Let R_p be the unipotent radical of the maximal parabolic subgroup of SO_{4k+1} with Levi part $\mathrm{GL}_l\times\mathrm{SO}_{2l+1}$. Consider the action of the abelian quotient $H_p(F)=\omega_l z_l V_{\mathcal{O}'}^p(\omega_l z_l)^{-1}\setminus R_p$ on

$$\mathrm{ind}_{\widetilde{\mathrm{SO}}_l U}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R_{s_p^2},\psi_p}(\Theta_{4k+2l+3}^{w_l z_l})\right). \quad (6.23)$$

If there is any non-zero vector in (6.23) on which $H_p(F)$ act by a character, then any non-zero Ψ_p in (6.22) must factor through the non-zero twisted Jacquet module of (6.23) with respect to $H_p(F)$ and the corresponding character. Following the same argument as in the proof of Theorem 5.1, we only need to consider characters ψ_ξ on $H_p/Z(H_p)(F)$ such that the product of $\psi_{\mathcal{O}'}$ and ψ_ξ restricted to $R_p(F)$ is non-generic. That is, the matrix in $\mathrm{Mat}_{l\times(2l+1)}(F)$ corresponding to $\psi_{\mathcal{O}'}\psi_\xi$ has totally isotropic row space. We deduce that such a character ψ_ξ must correspond to a matrix given by

$$\xi = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_l \end{pmatrix} \in \mathrm{Mat}_{l \times l}(F)$$

satisfying the following conditions:

1. Each λ_i except $\lambda_{\frac{l+1}{2}}$ is non-zero isotropic.

2. Each pair λ_i and λ_{l+1-i} for $i = 1, 2, \cdots, \frac{l-1}{2}$ is a hyperbolic pair, i.e.

$$(\lambda_i, \lambda_{l+1-i}) = 1,\ i = 1, 2, \cdots, \frac{l-1}{2}.$$

   These hyperbolic pairs are mutually orthogonal.

3. $\lambda_{\frac{l+1}{2}}$ is non-isotropic with unit length.

The group $\mathrm{SO}_l(F)$ acts transitively on the set of these matrices. This allows us to pick the identity matrix I_l as a representative. Denote the corresponding character on $H_p/Z(H_p)(F)$ by ψ_{I_l}. Any non-zero Ψ_p in (6.22) factors through the non-zero twisted Jacquet module of (6.23) given by

$$\mathrm{ind}_{\widetilde{\mathrm{SO}}_l}^{\widetilde{\mathrm{SO}}_{2k+1}} \left(\mathrm{ind}_{\widetilde{\mathrm{SO}}_l^{\Delta}}^{\widetilde{\mathrm{SO}}_l \times \widetilde{\mathrm{SO}}_l} \left(J_{R_p, \psi_{V_{\mathcal{O}'}, I_l}} J_{R_{s_p^2}, \psi_p} (\Theta_{4k+2l+3}^{w_l z_l}) \right) \right) \cong \mathrm{ind}_{U_{\mathcal{O}}}^{\widetilde{\mathrm{SO}}_{2k+1}} \left(J_{R, \psi_{V_{\mathcal{O}'}, p, I_l}} (\Theta_{4k+2l+3}^{w_l z_l}) \right). \tag{6.24}$$

We note that the normalized induction $\mathrm{ind}_{U_{\mathcal{O}}}^{\widetilde{\mathrm{SO}}_{2k+1}}$ is $\mathrm{ind}_{L^{\Delta}(\mathcal{O}) \times U_{\mathcal{O}}}^{L(\mathcal{O}) \times \widetilde{\mathrm{SO}}_{2k+1}}$ where

$$L(\mathcal{O}) \cong \widetilde{\mathrm{GL}}_{s_1}^{n_1 - n_2} \times \widetilde{\mathrm{GL}}_{s_2}^{n_2 - n_3} \times \cdots \times \widetilde{\mathrm{GL}}_{l}^{n_p} \times \widetilde{\mathrm{SO}}_l,$$

and

$$L(\mathcal{O})^\Delta \cong \left(\widetilde{\mathrm{GL}}_{s_1}^\Delta\right)^{n_1-n_2} \times \left(\widetilde{\mathrm{GL}}_{s_2}^\Delta\right)^{n_2-n_3} \times \cdots \times \left(\widetilde{\mathrm{GL}}_{l}^\Delta\right)^{n_p} \times \widetilde{\mathrm{SO}}_l^\Delta.$$

Thus, we conclude that

$$\mathrm{Hom}_{\widetilde{\mathrm{SO}}_{2k+1}\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')V_{\mathcal{O}'}}\left(\mathrm{ind}_{U_{\mathcal{O}}}^{\widetilde{\mathrm{SO}}_{2k+1}}\left(J_{R,\psi_{V_{\mathcal{O}'},p,I_l}}(\Theta_{4k+2l+3}^{w_l z_l})\right), \pi \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right) \neq 0. \tag{6.25}$$

We now examine the action of $U_{\mathcal{O}}(F)$ on (6.25). The group $U_{\mathcal{O}}(F)$ acts on the left by the character $\psi_{\mathcal{O}}$, since $(w_l z_l)^{-1} U_{\mathcal{O}} w_l z_l \subset R$ and $\psi_{V_{\mathcal{O}'},p,I_l}$ is defined on $R(F)$. As a result, there exists a non-zero vector in π on which $U_{\mathcal{O}}(F)$ acts by the same character $\psi_{\mathcal{O}}$, and

$$\mathrm{Hom}_{M^{\psi_{\mathcal{O}}}(\mathcal{O})\times M^{\psi_{\mathcal{O}'}}(\mathcal{O}')}\left(J_{R,\psi_{V_{\mathcal{O}'},p,I_l}}(\Theta_{4k+2l+3}^{w_l z_l}), J_{U_{\mathcal{O}},\psi_{\mathcal{O}}}(\pi) \otimes J_{V_{\mathcal{O}'},\psi_{\mathcal{O}'}}(\Theta(\pi))\right) \neq 0. \tag{6.26}$$

Thus, $J_{U_{\mathcal{O}},\psi_{\mathcal{O}}}(\pi)$ is non-zero. This completes the proof. $\square$

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