ON CERTAIN SMALL LIE RANK SUBGROUPS OF $E_8(2)$

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Mahah Javed
Department of Mathematics
School of Natural Sciences

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Mahah Javed
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This thesis forms a part of the much lager project whose aim is to classify the maximal subgroups of the finite simple exceptional group of Lie type $E_8(2)$. Groups H with $F^*(H)$ isomorphic to $L_2(64)$, $L_2(16)$, $L_2(8)$, $L_3(4)$, or $L_3(3)$ arise as some of the possible candidates for maximal subgroups of $E_8(2)$. We prove that if $F^*(H)$ is isomorphic to $L_2(64)$, $L_2(16)$ or $L_3(4)$ then H cannot be maximal in $E_8(2)$. Partial progress is made towards establishing whether $L_2(8)$ can be a maximal subgroup. A highlight is that we find maximal subgroups of $E_8(2)$ isomorphic to $L_3(3)$; we show that there are at most 3 conjugacy classes of them. Extensive use of the computer algebra package Magma has been made to prove our results. After the work done in this thesis not much is left to do in order to classify the maximal subgroups of $E_8(2)$.

Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

If the maximal subgroups of a finite group are known and, in turn, the maximal subgroups of those are known, and so on, then one knows everything there is to know about the subgroup structure of the group. Also, the maximal subgroup problem for all finite groups can be reduced to understanding the maximal subgroups and the 1-cohomology groups of almost simple groups, see [5]. Hence it is no surprise that following the construction of finite simple groups, the classification of their maximal subgroups, (and of the maximal subgroups of the almost simple groups), is highly sought after. We mention some pieces of literature that have contributed to the progress made towards achieving a solution to this problem.

The classification of the maximal subgroups of the alternating groups is, of course, given by the O'Nan-Scott theorem, see e.g. [31] for a proof which the authors then use in [32] for a classification.

As for the sporadic groups, we simply mention that full classifications are indeed available except in the incomplete case of the Monster. Wilson has been working on the outstanding case and has provided latest news in [51].

Turning our attention to the groups of Lie type, a classification of the maximal subgroups of the classical groups was given by Aschbacher [2] in 1984. Following this, a significantly more detailed version was provided by Kleidman and Liebeck [28]. We mention that prior to [2], classification was achieved for $L_2(q)$ by Dickson [19] in 1901 and for $L_3(q)$ by Mitchell [44] in 1911 for q odd and Hartley [22] in 1925 for q even.

There are several infinite families making up the finite simple exceptional groups of Lie type, with the group that is studied in this thesis belonging to one of them. Here things got kicked off by Suzuki, who in 1962 [48] determined the maximal subgroups of the infinite family of groups that he found and is named after him. The case of the Tits group is also a settled one, and one may look at [18], [50] or [49] for details. The eighties saw several authors providing results for the $G_2(q)$ case. Cooperstein dealt with $G_2(q)$ for q even in [17] and Migliore for q odd in [43]. Maximal subgroups of $G_2(q)$, for all q, can also be attributed to Aschbacher [3]. Note that Kleidman has also enumerated the maximal subgroups of $G_2(q)$ for q odd in [26] where he used a different approach to Migliore's and in addition described the maximal subgroups of the almost simple groups with socle $G_2(q)$ (q odd). Petrov and Tchakerian [47] listed the maximal subgroups of ${}^2G_2(q)$, $q = 3^{2m+1}$, with the same having been done by Levchuk and Nuzhin [30] earlier and independently. Note that [26], applying the methods used to analyse $G_2(q)$, q odd, also provides an answer for the case of ${}^2G_2(q)$, but extended to the almost simple groups. The list of maximals in ${}^3D_4(q)$ was determined by Kleidman [27] (almost simple groups inclusive). The list for $F_4(2)$ (and the one for $F_4(2):2$) is due to Norton and Wilson [46]. The maximal subgroups of $E_6(2)$ (and $\mathrm{Aut}(E_6(2))$) were determined by Kleidman and Wilson [29], and of ${}^2F_4(2^{2m+1})$ by Malle [40]. We mention that [4] and [39] are works on the maximal subgroups of $E_6(q)$ and $F_4(q)$, respectively, but both have some unresolved cases of candidate maximal subgroups. These are works of Aschbacher and Magaard, respectively.

Coming up to more recent years, the maximal subgroups of $E_7(2)$ were established by Ballantyne, Bates and Rowley [9] in 2015 and although the original result has been known for many years, it wasn't until 2018 that the classification of the maximal subgroups of ${}^2E_6(2)$ (and its automorphism groups) appeared in [52]. Craven [13] has completely classified the maximal subgroups of every almost simple group with socle $F_4(q)$, $E_6(q)$ or ${}^2E_6(q)$, along with correcting an error in [40] regarding the maximal subgroups of ${}^2F_4(8)$. A major contribution to classifying the maximal subgroups of $E_7(q)$ has been made by Craven [14] also.

Although maximal subgroups of all of the exceptional groups of Lie type have not yet been classified, this is a subject that has been extensively studied by various researchers, notably Liebeck and Seitz. The advances made reduce the work on finding the maximal subgroups of a finite exceptional group of Lie type to considering a finite list of almost simple groups. We state what we mean exactly by this as Theorem 1.1.

This is a result of [35] and one may look to this survey for a map of how the result came about.

In the following theorem, G denotes an adjoint simple algebraic exceptional group of Lie-type over $\overline{\mathbb{F}_q}$ and σ a standard Frobenius homomorphism of G. It will be more clear what G and σ are in the next chapter when we briefly discuss algebraic group theory.

Theorem 1.1. *Let H be a maximal subgroup of the finite exceptional group G_σ over $\mathbb{F}_q$, $q = p^a$ where p is a prime. Then one of the following holds:*

(i) *$H = M_\sigma$ where M is a maximal closed σ-stable subgroup of positive dimension in G; the possibilities are as follows;*

   (a) *Both M and H are parabolic subgroups;*
   (b) *M is a reductive group of maximal rank. The possibilities for M are determined in* [33].
   (c) *$G = E_7$, $p > 2$ and $H = (2^2 \times \Omega_8^+(q).2^2).\mathrm{Sym}(3)$ or ${}^3D_4(q).3$;*
   (d) *$G = E_8$, $p > 5$ and $H = PGL_2(q) \times \mathrm{Sym}(5)$;*
   (e) *M is as in Table 1 of* [35], *and $H = M_\sigma$ as in Table 3 of* [35].

(ii) *H is of the same type as G;*

(iii) *H is an exotic local subgroup (see* [15]*);*

(iv) *G is of type E_8, $p > 5$ and $H \sim (\mathrm{Alt}(5) \times \mathrm{Alt}(6)).2^2$;*

(v) *$F^*(H) = H_0$ is simple, and not in $\mathrm{Lie}(p)$: the possibilities for H_0 are given up to isomorphism by* [36]*;*

(vi) *$F^*(H) = H(q_0)$ is simple and in $\mathrm{Lie}(p)$; moreover $rk(H(q_0)) \leq \frac{1}{2}rk(G)$, and one of the following holds:*

   (a) *$q_0 \leq 9$;*
   (b) *$H(q_0) \cong A_2(16)$ or ${}^2A_2(16)$;*
   (c) *$q_0 \leq (2, p-1)u(G)$ and $H(q_0) \cong A_1(q_0)$, ${}^2B_2(q_0)$ or ${}^2G_2(q_0)$, where the values of $u(G)$ for each type of exceptional group are as follows:*

G	G_2	F_4	E_6	E_7	E_8
$u(G)$	12	68	124	388	1312

In cases (i)–(iv)*, H is determined up to G_σ-conjugacy.*

For the group G_σ, Theorem 1.1(v),(vi) will give a list of almost simple groups so that if H is a maximal subgroup of G_σ, not given by Theorem 1.1(i)–(iv), then H can only be isomorphic to a group in this list. Therefore achieving the classification of the maximal subgroups of G_σ, is a matter of going through the list and checking if a group in it can be maximal or not.

This is indeed how the maximal subgroups of $E_7(2)$ were determined. Of course, if one were to pick finite exceptional groups with incomplete classifications one by one, make a list of possible maximal subgroups and work their way through it, then they'd never finish. A contrasting approach to the classification problem is adopted by Craven [11]: Theorem 1.1(v) lists groups H with $F^*(H) \cong \mathrm{Alt}(n)$, $5 \leq n \leq 18$ as being possible maximal subgroups, [11] eliminates these as possibilities in almost all cases.

The only finite simple group of Lie type defined over GF(2) that we have not yet mentioned with regards to its maximal subgroups is $E_8(2)$. However, after around 8 years since the project was taken on, the classification of the maximal subgroups of $E_8(2)$ is finally near completion. This is due to efforts of Aubad, Ballantyne, Javed, McGaw, Neuhaus, Rowley and Ward and the unpublished paper [7] in the works is hoped to see the light of day before long. This thesis provides details of the latest work done on the classification problem.

For $E_8(2)$, Theorem 1.1(v),(vi) generates a list of 75 groups (after eliminations of certain alternating groups afforded by [11]) that simple maximal subgroups of $E_8(2)$ could be isomorphic to; of course automorphic extensions of these 75 groups are also candidates for maximal subgroups. Seventy of the cases have been laid to rest, [7], [42], [45], with the 5 unsettled ones being $L_2(64)$, $L_2(16)$, $L_2(8)$, $L_3(4)$ and $L_3(3)$. Theorem 1.1(v) gives rise to $L_3(3)$. In Theorem 1.1(vi)(a), $rk(H(q_0)) \leq \frac{1}{2}rk(G) = 4$ and $q_0 \leq 9$ means that $A_1(8) = L_2(8)$ and $A_2(4) = L_3(4)$ are indeed among the possibilities for $H(q_0)$. In Theorem 1.1(vi)(c), $q_0 \leq 1312$ and $H(q_0) \cong A_1(q_0)$ implies that $L_2(16)$ and $L_2(64)$ are also among the possibilities.

We now give the list of some of the maximal subgroups of $E_8(2)$. These either arise from Theorem 1.1(i)-(iv), or are the product of work done by people involved in the project, other than the author of this thesis.

$[2^{78}] : \Omega^+_{14}(2)$	$[2^{98}] : (\mathrm{Sym}(3) \times L_7(2))$
$[2^{106}] : (\mathrm{Sym}(3) \times L_3(2) \times L_5(2))$	$[2^{104}] : (\mathrm{Alt}(8) \times L_5(2))$
$[2^{97}] : (L_3(2) \times \Omega^+_{10}(2))$	$[2^{83}] : (\mathrm{Sym}(3) \times E_6(2))$
$[2^{92}] : L_8(2)$	$[2^{57}] : E_7(2)$
$\Omega^+_{16}(2)$	$\mathrm{Sym}(3) \times E_7(2)$
$L_9(2) : 2$	$3 \cdot U_9(2) : 2$
$(L_3(2) \times E_6(2)) : 2$	$3 \cdot (U_3(2) \times {}^2E_6(2)) : \mathrm{Sym}(3)$
$(L_5(2))^2.4$	$(U_5(2))^2.4$
$SU_5(4).4$	$PGU_5(4).4$
$(\Omega^+_8(2))^2.(\mathrm{Sym}(3) \times 2)$	$\Omega^+_8(4).(\mathrm{Sym}(3) \times 2)$
$({}^3D_4(2))^2.6$	${}^3D_4(4).6$
$(L_3(2))^4.GL_2(3)$	$[3^2].(U_3(2))^4.[3^2].GL_2(3)$
$(U_3(4))^2.8$	$U_3(16).8$
$3^8.(2.\Omega^+_8(2).2)$	$5^4.((4 * 2^{1+4}).\mathrm{Alt}(6).2)$
$7^4.(2.(3 \times U_4(2)))$	$11^2.(5 \times SL_2(5))$
$13^2.(12 * GL_2(3))$	$31^2.(5 \times SL_2(5))$
151.30	331.30
$L_3(5) : 2$	$PSp_4(5)$
$U_3(3) : 2 \times F_4(2)$	$L_2(31) : 2$

In Chapter 2, after briefly touching upon the topic of linear algebraic groups, leading up to the definition of a finite group of Lie type, we will focus on our particular case of $E_8(2)$. Our work heavily involves the computer algebra package Magma so we will discuss how we set $E_8(2)$ up as a group of 248×248 matrices in Magma. Information on the conjugacy classes of involutions and the semisimple elements of $E_8(2)$ plays a crucial role in calculating complete centralisers of elements and is also important in other ways; it will be provided in the chapter. Most importantly, this chapter will contain the main tool used to rule out a group as being maximal in $E_8(2)$. This is a result from [12], and a result in [37] will tell us in which situations we can immediately use it.

Chapters 3, 4 and 5 are allotted to the groups $L_2(64)$, $L_2(16)$ and $L_2(8)$, respectively. We will determine that $L_2(64)$ and $L_2(16)$ (and their extensions) cannot be maximal in $E_8(2)$. It is yet to be established whether $L_2(8)$ can be maximal in $E_8(2)$, however substantial progress has been made. Not all the progress made will make its way into Chapter 5 though. The work done in these chapters is a continuation of the work done by Neuhaus [45] on the groups $L_2(128)$ and $L_2(32)$, hence these chapters share the same basic notions. However as the size of the group decreases, the problem becomes more difficult and newer methods need to be developed. The notation will mostly remain consistent across the three chapters.

The groups $L_3(4)$ and $L_3(3)$ are collectively dealt in Chapter 6. This is because these groups will share the subgroup that is built up on to construct them inside $E_8(2)$. We will see that no group isomorphic to $L_3(4)$ or an automorphic extension of $L_3(4)$ can be maximal in $E_8(2)$. Usually it is said of groups arising from Theorem 1.1(v),(vi) that they cannot be maximal in $E_8(2)$. So it will be fascinating to see $L_3(3)$ defying the norm. Chapter 6 will see us construct maximal subgroups of $E_8(2)$ isomorphic to $L_3(3)$.

This thesis comes with two appendices. Appendix A contains programs that are used in Chapters 3, 4 and 5. Appendix B contains information from [45] on the possible embeddings in $E_8(2)$ of the groups under scrutiny in this thesis, see Chapter 2 for more.

Chapter 2

Background and Preliminaries

2.1 Linear Algebraic Groups

In this section we will briefly discuss algebraic groups G, state basic notions surrounding them, define what it means for G to be reductive or semisimple, say what the set of roots of G is, discuss the classification of semisimple algebraic groups, look at standard Frobenius homomorphisms and finally define finite groups of Lie type. The main source of the material in this section is [41] with both it and [24] being excellent books for a more detailed account.

Let k be an algebraically closed field of arbitrary characteristic. A linear algebraic group is an affine algebraic variety (so a subset of k^n, $n > 0$) such that the group operations are morphisms of varieties. We have $GL_n(k)$ as an example of an algebraic group since it can be identified with the closed (with respect to the Zariski topology) subset $\{(A, y) \in k^{n\times n} \times k : \det A \cdot y = 1\}$ with componentwise multiplication, via $A \mapsto (A, \det A^{-1})$, $A \in GL_n(k)$. Multiplication and inversion can be seen to be polynomial maps. Any closed subgroup of $GL_n(k)$ will be a linear algebraic group and in fact the converse is a well-known theorem: Any linear algebraic group can be embedded as a closed subgroup into $GL_n(k)$.

We give two more examples of linear algebraic groups that will feature later. We denote by $\mathbf{G}_a$ the additive group $(k, +)$ of k defined by the zero ideal; addition is given by a polynomial. We denote by $\mathbf{G}_m$ the multiplicative group $(k^\times, \cdot)$ of k; this can be identified with the algebraic set $\{(x, y) \in k^2 : xy = 1\}$ where multiplication is componentwise and again given by polynomials. Note that here the coordinate ring is $k[X,Y]/(XY-1) \cong k[X,X^{-1}]$ and now the inverse $((x,y)^{-1} \mapsto (y,x))$ can also be seen to be given by a polynomial.

For the rest of this section, G will always denote a linear algebraic group. We now find out what it means for G to be unipotent. It is true that for any embedding ρ of G into $GL(V)$, V an n-dimensional vector space over k, and for any $g \in G$, there exist unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, where $\rho(g_s)$ is semisimple (i.e. a diagonalisable endomorphism of V) and $\rho(g_u)$ is unipotent (i.e. some power of $\rho(g_u)-1$ is 0). The element $g \in G$ is called semisimple if $g = g_s$ and unipotent if $g = g_u$. We denote by G_u, the set of all the unipotent elements of G. If G consists entirely of unipotent elements then we say that G is a unipotent group. We remark that over $k = \overline{\mathbb{F}_p}$, unipotent elements are p-elements. This follows from the fact that over a field of positive characteristic p, u, an endomorphism of V, is unipotent if and only if it has p-power order $(0 = u^{p^i} - 1 = (u-1)^{p^i})$. It is also true that if G is unipotent then it can be embedded into the group of upper unitriangular matrices. It follows that a unipotent linear algebraic group is nilpotent, hence soluble.

The group G is called connected if it cannot be decomposed as a disjoint union of two non-empty closed subsets. We are finally in a position to see what it means for a group G to be semisimple or reductive. We denote by $R(G)$ the maximal closed connected soluble normal subgroup of G; this is called the radical of G. It is true that if a linear algebraic group is connected and soluble then the set of all its unipotent elements is a closed connected normal subgroup. Hence $R(G)_u$, the set of unipotent elements of $R(G)$, is a normal connected unipotent subgroup of $R(G)$. We observed above that a unipotent group is soluble. Therefore, any closed connected normal unipotent subgroup of G is contained in $R(G)$ and hence in $R(G)_u$. We get that $R(G)_u$ is the maximal closed connected normal unipotent subgroup of G, the so-called unipotent radical of G.

Remark 2.1.1. *In the case where $k = \overline{\mathbb{F}_p}$, the unipotent radical is the largest connected normal subgroup consisting entirely of p-elements, so the analogue of the maximal normal p-subgroup $O_p(G)$ for a finite group G.*

A linear algebraic group G is called reductive if $R(G)_u = 1$. It is called semisimple if it is connected and $R(G) = 1$. We get that a semisimple group is connected and reductive. Semisimple groups can be classified. To comment on the structure of semisimple groups, we first familiarise ourselves with some more notions.

A linear algebraic group is called a torus if it is isomorphic to a direct product $\mathbf{G}_m \times \ldots \times \mathbf{G}_m$, that is, to a group of diagonal invertible matrices. A subtorus $T \leq G$ is a maximal torus of G if it is maximal among subtori with respect to inclusion. It is true that all maximal tori of G are conjugate.

A character of G is a morphism of algebraic groups $\chi : G \to \mathbf{G}_m$. The set of characters of G is denoted by $X(G)$. Note that it can naturally be considered as a subset of $k[G]$.

The Lie algebra of G, $\mathrm{Lie}(G)$, is the space of left-invariant derivations of $k[G]$. An important use of the Lie algebra is that it defines a natural rational representation $G \to GL(\mathrm{Lie}(G))$, the so-called adjoint representation of G. We do not go into the details of defining the action of G on $\mathrm{Lie}(G)$.

It turns out that the best way to investigate reductive groups is via their adjoint action on the Lie algebra. Let $T \leq G$ be a maximal torus and write $\mathfrak{g} := \mathrm{Lie}(G)$. For $\chi \in X(T)$, consider the intersection of eigenspaces, $\mathfrak{g}_\chi = \{v \in \mathfrak{g} : t{\cdot}v = \chi(t)v \text{ for all } t \in T\}$. The set of non-zero characters with non-zero eigenspace, $\Phi(G) := \{\chi \in X(T) : \chi \neq 0, \mathfrak{g}_\chi \neq 0\}$ is called the set of roots of G with respect to T.

From now on let G be a connected reductive group. For each $\alpha \in \Phi(G)$ there exists a morphism of algebraic groups $u_\alpha : \mathbf{G}_a \to G$, which induces an isomorphism onto $u_\alpha(\mathbf{G}_a)$ such that $tu_\alpha(c)t^{-1} = u_\alpha(\alpha(t)c)$, for all $t \in T, c \in k$. $U_\alpha := \mathrm{im}(u_\alpha)$ is the unique one-dimensional connected unipotent subgroup of G normalized by T with $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$ and is known as the root subgroup of G (with respect to T) associated to the root α. It is true that $G = \langle T, U_\alpha : \alpha \in \Phi(G) \rangle$.

$W := N_G(T)/C_G(T)$ is called the Weyl group of G with respect to T. It can be shown that W stabilises $\Phi(G)$. The theory can be further developed to see that $\Phi(G)$ is an abstract root system but abstract root systems can be classified. A crucial notion is that of a base: For an abstract root system Φ in Euclidean space, E, a subset $\Delta \subseteq \Phi$ is called a base of Φ if it is a vector space basis of E and any $\beta \in \Phi$ is an integral linear combination of elements of Δ with coefficients either all negative or all positive. The roots in Δ are then called simple. For a root system, one can define its Dynkin diagram, the underlying graph of which has one node for each element of Δ. A root system is called indecomposable if its base cannot be partitioned in a certain way and it possesses this property if and only if its Dynkin diagram is connected. A semisimple algebraic group is called simple if it has no non-trivial proper closed connected normal subgroups. It can be shown that simple algebraic groups have indecomposable root systems (and conversely). Thus, as a first step in the determination of simple groups one needs a classification of indecomposable root systems: Up to isomorphism, an indecomposable root system is one of a total of nine types (see e.g. Chapter III in [23] for an excellent account on root systems and a classification).

The groups with root system of type A_n, B_n, C_n or D_n are called groups of classical type; the remaining simple groups are called groups of exceptional type. It should be noted that there exist non-isomorphic simple algebraic groups having the same root system, but not to say that the classification of semisimple algebraic groups is incomplete.

We now come to defining a finite group of Lie type. Let $k = \overline{\mathbb{F}_q}$, where $q = p^r$. The map $F_q : k \to k$, $t \mapsto t^q$, is a field automorphism of k which fixes $\mathbb{F}_q$ pointwise. Letting F_q act on the matrix entries, this induces a group homomorphism $F_q : GL_n(k) \to GL_n(k)$, $(a_{ij}) \mapsto (a_{ij}^q)$ with finite fixed point group $GL_n(k)_{F_q} := \{g \in GL_n(k) : F_q(g) = g\} = GL_n(\mathbb{F}_q)$, and F_q is called the standard Frobenius map of $GL_n(k)$ with respect to $\mathbb{F}_q$. Standard Frobenius maps can be induced by other automorphisms of k. Let $F : GL_n(k) \to GL_n(k)$, $(a_{ij}) \mapsto (a_{ij}^q)^{-\mathrm{tr}}$. Thus F is the composite of the previous F_q with the map sending a matrix to the transpose of its inverse. These two maps commute and $F^2 : GL_n(k) \to GL_n(k)$, $(a_{ij}) \mapsto (a_{ij}^{q^2})$, is the standard Frobenius map F_{q^2} with respect to $\mathbb{F}_{q^2}$. Here the fixed point group $GL_n(k)_F \leq GL_n(k)_{F_{q^2}} = GL_n(q^2)$ is the general unitary group over $\mathbb{F}_{q^2}$. The map F is an example of a Steinberg endomorphism: An endomorphism $\sigma : G \to G$ of a linear algebraic group G such that for some $m \geq 1$ the power $\sigma^m : G \to G$ is the Frobenius morphism with respect to some $\mathbb{F}_{p^a}$-structure of G is called a Steinberg endomorphism of G. We write G_σ for the group of fixed points of σ on G.

Finally, we have the following: Let G be a semisimple algebraic group, $\sigma : G \to G$ a Steinberg endomorphism, then the finite group of fixed points G_σ is called a finite group of Lie type. The G_σ we are interested in is $E_8(2)$. To see what the notions behind algebraic groups translate to in the finite case, one may refer to Part III of

[41].

2.2 Working with $E_8(2)$

2.2.1 $E_8(2)$ setup

Our work goes hand in hand with performing computations in Magma. In order to exploit a result in [12] (see Proposition 2.2.3), we would want information on how subgroups of $E_8(2)$ act on its 248-dimensional adjoint module, which henceforth will be denoted by V_{248}. Therefore, we start by constructing $E_8(2)$ as a subgroup of $GL_{248}(2)$ using its adjoint representation. First note that if Δ is a base of the root system, Φ, of type E_8 then $E_8(2)$ is generated by the root subgroups U_α, $U_{-\alpha}$ where $\alpha \in \Delta$.

As the first step in the construction, we have Magma produce $E_8(2)$ as an object in the "GrpLie" category:

```
H:=GroupOfLieType("E8",GF(2));
```

The command `Roots(H);` would then give us the ordered set of roots in Φ, with the first 8 of them being the simple roots (forming a base), the first 120 being the positive ones and the last 120 being the negative. The ordering is first by height and then by lexicographic order with respect to the labelling of the simple roots. Therefore, if α is the ith root then $-\alpha$ will be the $(120+i)$th root. Let α be the root labelled by some $i \in \{1, \ldots, 240\}$, then in order to construct the group U_α we need a generator for it. This will be given by `elt<H|<i,1>>`. We now continue with the construction.

```
//We will require the natural matrix representation,
//the adjoint representation.

f:=AdjointRepresentation(H);
Q:=Codomain(f);              //In this case Q will be GL(248,2);

//Let's get generators for H by taking elements corresponding to the
//simple roots and their negatives.

Hgens:=[];
```

```
for i:=1 to 8 do
Append(~Hgens,elt<H|<i,1>>);
end for;
for i:=1 to 8 do
Append(~Hgens,elt<H|<120+i,1>>);
end for;

//Now we map them into the matrix group.

Ggens:=[];
for h in Hgens do
Append(~Ggens,f(h));
end for;

//And now we can construct E_8(2) as a subgroup of Q.

G:=sub<Q|Ggens>;
```

Note that going forward, `Q` will always be `GL(248,2)` wherever it appears. This construction of $G \cong E_8(2)$, $G \leq GL_{248}(2)$ will be used for the majority of our computations. From [16],

$$|G| = 2^{120}.3^{13}.5^5.7^4.11^2.13^2.17^2.19.31^2.41.43.73.127.151.241.331.$$

This is an enormous size and it is desirable to work in smaller subgroups of G whose structure would also be better known. A natural example of subgroups of G are the standard parabolic subgroups. These are readily constructed using appropriate root subgroups and the structure of the Levi complements can be read off from the E_8 Dynkin diagram. Constructing certain standard parabolic subgroups marks the start of the computations involved in Chapters 3, 4 and 5 so more on parabolic subgroups can be found there.

Another example of subgroups of G that we will be working with are centralisers of its elements. Structures of centralisers of certain elements of $E_8(2)$ are known to us and given in the next subsection. A procedure, named `FindCent`, to calculate centralisers of elements of $E_8(2)$ was developed by Ballantyne and Rowley and can be found in [42]. It builds up the centraliser of an element $g \in G$ by piecing together centralisers of g found in small enough subgroups of G. The order of $C_G(g)$ needs to be checked against the actual order given in the next subsection to make sure all of it has been produced. This procedure will be making appearances in later chapters. Note that `FindCent` as written needs a subgroup $H \leq G$ conjugate to $C_G(g)$ before it can be run. Note that this isn't necessary as `FinCent` can be modified to construct $C_G(g)$ without H being available.

2.2.2 Elements of $E_8(2)$

We now give information on the conjugacy classes of involutions and semisimple elements of $E_8(2)$. The importance of this information has been hinted at in the previous subsection and will become more apparent as we progress.

Proposition 2.2.1. *With $G \cong E_8(2)$, let t be an involution in G. Also let U be the unipotent radical of $C_G(t)$ having a complement L (so $C_G(t) = UL$). Then the possibilities for t are as follows:*

(i) If $t \in 2A$, then dim$(C_{V_{248}}(t)) = 190$, $U \cong 2^{1+56}$ and $L \cong E_7(2)$,

(ii) If $t \in 2B$, then dim$(C_{V_{248}}(t)) = 156$, $U \sim [2^{78}]$ and $L \cong Sp_{12}(2)$,

(iii) If $t \in 2C$, then dim$(C_{V_{248}}(t)) = 138$, $U \sim [2^{81}]$ and $L \cong Sym(3) \times F_4(2)$,

(iv) If $t \in 2D$, then dim$(C_{V_{248}}(t)) = 128$, $U \sim [2^{84}]$ and $L \cong Sp_8(2)$.

Proof. See [6] for the shape of $C_G(t)$. The dimension of $C_{V_{248}}(t)$ can be calculated directly in Magma by asking for a random involution, t, in $G \leq GL_{248}(2)$, then using `CentraliserOfInvolution` to get a group centralising t, then using `LMGFactoredOrder` on this group to see which of the 4 possibilities it matches up with and then using `Dimension(Eigenspace(t,1))`. □

The semisimple elements of $G \cong E_8(2)$ have been investigated in [8], the main result being Theorem 2.2.2. The Lübeck number associated to a set of classes identifies it to a set in [38], where much of the data was determined.

Theorem 2.2.2. *The semisimple conjugacy classes of G, their centraliser structures, dimensions of their fixed spaces on V_{248}, together with power maps and Lübeck numbers are displayed in Table 2.1.*

Conjugacy Class	Lübeck Number	$C_G(x)$	$\|C_G(x)\|$	$\dim(C_{V_{248}}(x))$	Powers
$1A$	1	$E_8(2)$	$\|E_8(2)\|$	248	-
$3A$	294	$3 \times E_7(2)$	$2^{63} \cdot 3^{12} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 43 \cdot 73 \cdot 127$	134	-
$3B$	376	$3 \times \Omega_{14}^-(2)$	$2^{42} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 43$	92	-
$3C$	147	$3.({}^2E_6(2) \times U_3(2)).3$	$2^{39} \cdot 3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	86	-
$3D$	258	$3 \times U_9(2)$	$2^{36} \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 11 \cdot 17 \cdot 19 \cdot 43$	80	-
$5A$	480	$5 \times \Omega_{12}^-(2)$	$2^{30} \cdot 3^6 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31$	68	-
$5B$	247	$SU_5(4)$	$2^{20} \cdot 3^2 \cdot 5^5 \cdot 13 \cdot 17 \cdot 41$	48	-
$7A$	441	$7 \times E_6(2)$	$2^{36} \cdot 3^6 \cdot 5^2 \cdot 7^4 \cdot 13 \cdot 17 \cdot 31 \cdot 73$	80	-
$7B$	516	$7 \times L_3(2) \times {}^3D_4(2)$	$2^{15} \cdot 3^5 \cdot 7^4 \cdot 13$	38	-
$9A$	560	$9 \times \Omega_{10}^-(2)$	$2^{20} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	48	3C
$9B$	656	$9 \times \mathrm{Sym}(3) \times {}^3D_4(2)$	$2^{13} \cdot 3^7 \cdot 7^2 \cdot 13$	34	3C
$9C$	580	$9 \times \mathrm{Sym}(3) \times U_5(2)$	$2^{11} \cdot 3^8 \cdot 5 \cdot 11$	30	3C
$9D$	366	$9 \times \mathrm{Sym}(3) \times U_3(8)$	$2^{10} \cdot 3^7 \cdot 7 \cdot 19$	28	3C
$11A$	679	$11 \times U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11^2$	28	-
$13A$	712	$13 \times {}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13^2$	32	-
$13B$	709	$13 \times U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13^2$	20	-
$15A$	540	$15 \times \Omega_{10}^+(2)$	$2^{20} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 17 \cdot 31$	48	3B,5A
$15B$	636	$5 \times 3^2 : 2 \times \Omega_8^-(2)$	$2^{13} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 17$	34	3A,5A
$15C$	686	$15 \times U_5(2)$	$2^{10} \cdot 3^6 \cdot 5^2 \cdot 11$	28	3D,5A
$15D$	621	$5 \times GU_3(2) \times L_4(2)$	$2^9 \cdot 3^6 \cdot 5^2 \cdot 7$	26	3C,5A
$15E$	600	$15 \times L_2(4) \times U_4(2)$	$2^8 \cdot 3^6 \cdot 5^3$	24	3B,5A
$15F$	706	$15 \times U_3(4)$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 13$	20	3B,5B
$15G$	695	$15 \times L_2(16)$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 17$	16	3D,5B
$17AB$	738	$17 \times \Omega_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17^2$	32	-
$17CD$	693	$17 \times L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17^2$	16	-
$19A$	823	$19 \times 3^{\cdot}PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	-

Table 2.1: Conjugacy classes of semisimple elements of $E_8(2)$.

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_{V_{248}}(x))$	Powers
$21A$	610	$21 \times L_6(2)$	$2^{15} \cdot 3^5 \cdot 5 \cdot 7^3 \cdot 31$	38	3A,7A
$21B$	720	$21 \times {}^3D_4(2)$	$2^{12} \cdot 3^5 \cdot 7^3 \cdot 13$	32	3A,7B
$21C$	728	$21 \times \Omega_8^-(2)$	$2^{12} \cdot 3^5 \cdot 5 \cdot 7^2 \cdot 17$	32	3B,7A
$21D$	469	$7 \times 3.(3^2 : Q_8 \times L_3(4)) : 3$	$2^9 \cdot 3^6 \cdot 5 \cdot 7^2$	26	3C,7A
$21E$	594	$21 \times L_3(2) \times L_2(8)$	$2^6 \cdot 3^4 \cdot 7^3$	20	3A,7B
$21F$	697	$7 \times L_3(2) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^6 \cdot 3^5 \cdot 7^2$	20	3C,7B
$21G$	760	$21 \times 3 \times L_2(8)$	$2^3 \cdot 3^4 \cdot 7^2$	14	3B,7B
$21H$	826	$21 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 7$	14	3D,7B
$31ABC$	672	$31 \times L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31^2$	28	-
$31D$	857	31^2	31^2	8	-
$33AB$	768	$33 \times U_4(2)$	$2^6 \cdot 3^5 \cdot 5 \cdot 11$	20	3D,11A
$33CD$	748	$11 \times \mathrm{Sym}(3) \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^4 \cdot 3^5 \cdot 11$	16	3C,11A
$33E$	811	$33 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^5 \cdot 11$	14	3A,11A
$33F$	790	$33 \times 3 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^4 \cdot 11$	12	3B,11A
$35A$	778	$35 \times U_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	5A,7A
$39A$	762	$13 \times \mathrm{Sym}(3) \times L_2(8)$	$2^4 \cdot 3^4 \cdot 7 \cdot 13$	14	3A,13A
$39B$	820	$13 \times 3_+^{1+2} : 2\mathrm{Alt}(4)$	$2^3 \cdot 3^4 \cdot 13$	14	3C,13A
$39C$	872	195	$3 \cdot 5 \cdot 13$	8	3B,13B
$41AB$	864	205	$5 \cdot 41$	8	-
$43ABC$	837	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	-
$45A$	773	$45 \times L_4(2)$	$2^6 \cdot 3^4 \cdot 5^2 \cdot 7$	20	3C,9A,5A,15D
$45B$	798	$45 \times 3 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^4 \cdot 5^2$	12	3C,9A,5A,15D
$45C$	853	$45 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^4 \cdot 5$	10	3C,9C,5A,15D
$51AB$	783	$51 \times L_4(2)$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 17$	20	3B,17AB
$51CD$	764	$51 \times \mathrm{Sym}(3) \times \mathrm{Alt}(5)$	$2^3 \cdot 3^3 \cdot 5 \cdot 17$	14	3A,17AB
$51EF$	832	$17 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 17$	14	3C,17AB
$51GH$	870	255	$3 \cdot 5 \cdot 17$	8	3D,17CD

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_{V_{248}}(x))$	Powers
$55A$	877	165	$3 \cdot 5 \cdot 11$	8	5A,11A
$57AB$	823	$19 \times 3^{\cdot} PGU_3(2)$	$2^3 \cdot 3^4 \cdot 19$	14	3C,19A
$57C$	861	$3 \times 19 \times 9$	$3^3 \cdot 19$	8	3A,19A
$57DE$	863	57×3	$3^2 \cdot 19$	8	3D,19A
$63ABC$	754	$63 \times \mathrm{Sym}(3) \times L_3(2)$	$2^4 \cdot 3^4 \cdot 7^2$	16	3C,9B,7B,21F
$63D$	802	$63 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3C,9A,7A,21D
$63E$	843	$63 \times 7 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 7^2$	10	3C,9B,7A,21D
$63FGH$	849	$63 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^4 \cdot 7$	10	3C,9D,7B,21F
$65ABCD$	800	$65 \times \mathrm{Alt}(5)$	$2^2 \cdot 3 \cdot 5^2 \cdot 13$	12	5B,13B
$65EF$	858	13×5^2	$5^2 \cdot 13$	8	5A,13B
$73ABCD$	814	$73 \times L_3(2)$	$2^3 \cdot 3 \cdot 7 \cdot 73$	14	-
$85AB$	804	$85 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^2 \cdot 5 \cdot 17$	12	5A,17AB
$85CDEF$	870	255	$3 \cdot 5 \cdot 17$	8	5B,17CD
$91ABC$	817	$91 \times L_3(2)$	$2^3 \cdot 3 \cdot 7^2 \cdot 13$	14	7B,13A
$91D$	865	91×7	$7^2 \cdot 13$	8	7A,13A
$93ABC$	808	$93 \times L_3(2)$	$2^3 \cdot 3^2 \cdot 7 \cdot 31$	14	3A,31ABC
$93DEF$	788	$93 \times \mathrm{Alt}(5)$	$2^2 \cdot 3^2 \cdot 5 \cdot 31$	12	3B,31ABC
$99AB$	841	$99 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 11$	10	3C,9C,11A,33CD
$99CD$	867	99×3	$3^3 \cdot 11$	8	3C,9A,11A,33CD
$105AB$	829	$35 \times GU_3(2)$	$2^3 \cdot 3^4 \cdot 5 \cdot 7$	14	3C,5A,7A,15D,21D,35A
$105C$	794	$105 \times \mathrm{Sym}(3)^2$	$2^2 \cdot 3^3 \cdot 5 \cdot 7$	12	3B,5A,7A,15A,21C,35A
$105D$	851	$105 \times 3 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 5 \cdot 7$	10	3A,5A,7A,15B,21A,35A
$117ABC$	845	$117 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 13$	10	3C,9B,13A,39B
$119AB$	878	357	$3 \cdot 7 \cdot 17$	8	7A,17AB
$127ABCDEFGHI$	835	$127 \times \mathrm{Sym}(3)$	$2 \cdot 3 \cdot 127$	10	-

Conjugacy Class	Lübeck Number	$C_G(x)$	$\lvert C_G(x)\rvert$	$\dim(C_{V_{248}}(x))$	Powers
$129ABCDEF$	837	$129 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 43$	10	3D,43ABC
$129GHI$	859	129×3	$3^2 \cdot 43$	8	3A,43ABC
$129JKLMNO$	859	129×3	$3^2 \cdot 43$	8	3B, 43ABC
$151ABCDE$	868	151	151	8	-
$153AB$	879	153	$3^2 \cdot 17$	8	3C,9A,17AB,51EF
$155ABC$	876	465	$3 \cdot 5 \cdot 31$	8	5A,31ABC
$165AB$	877	165	$3 \cdot 5 \cdot 11$	8	3D,5A,11A,15C,33AB,55A
$171ABCDEF$	847	$171 \times \mathrm{Sym}(3)$	$2 \cdot 3^3 \cdot 19$	10	3C,9D,19A,57AB
$195ABCD$	872	195	$3 \cdot 5 \cdot 13$	8	3B,5B,13B,15F,39C,65ABCD
$205ABCDEFGH$	864	205	$5 \cdot 41$	8	5B,41AB
$217ABCDEF$	839	$217 \times \mathrm{Sym}(3)$	$2 \cdot 3 \cdot 7 \cdot 31$	10	7A,31ABC
$219ABCD$	875	219	$3 \cdot 73$	8	3A,73ABCD
$241ABCDEFGHIJ$	866	241	241	8	-
$255ABCD$	855	$255 \times \mathrm{Sym}(3)$	$2 \cdot 3^2 \cdot 5 \cdot 17$	10	3A,5A,15B,17AB,51CD,85AB
$255EF$	860	255×3	$3^2 \cdot 5 \cdot 17$	8	3B,5A,15A,17AB,51AB,85AB
$255GHIJKLMN$	870	255	$3 \cdot 5 \cdot 17$	8	3D,5B,15G,17CD,51GH,85CDEF
$273ABC$	873	273	$3 \cdot 7 \cdot 13$	8	3A,7B,13A,21B,39A,91ABC
$315AB$	871	315	$3^2 \cdot 5 \cdot 7$	8	3C,5A,7A,9A,15D,21D,35A,45A,63D,105AB
$331ABCDEFGHIJK$	869	331	331	8	-
$357ABCD$	878	357	$3 \cdot 7 \cdot 17$	8	3B,7A,17AB,21C,51AB,119AB
$381ABCDEFGHI$	874	381	$3 \cdot 127$	8	3A,127ABCDEFGHI
$465ABCDEF$	876	465	$3 \cdot 5 \cdot 31$	8	3B,5A,15A,31ABC,93DEF,155ABC
$511ABCDEFGH$	862	511	$7 \cdot 73$	8	7A,73ABCD
$651ABCDEF$	880	651	$3 \cdot 7 \cdot 31$	8	3A,7A,21A,31ABC,93ABC,217ABCDEF

2.2.3 Embeddings and Determining Maximality

Our aim is to determine whether certain groups H can be isomorphic to maximal subgroups in $G \cong E_8(2)$. If H can't be ruled out as being a subgroup of G using group some theoretic properties, which is likely if the order of H is small, then a way to proceed is by trying to construct copies of H in G and see if we are successful. Knowing what fusion patterns are possible for an embedding of H in G would give us a starting point when trying to construct H. For H isomorphic to $L_2(2^n)$, $n = 3, 4, 6$, $L_3(4)$ or $L_3(3)$ the fusion patterns possible have been calculated in [45] and are given in Appendix B. This was done by determining all the possible feasible decompositions of H on V_{248}. Note that by a feasible decomposition of H on V_{248}, we mean a GF(2)H-module V such that for every $x \in H$, there exists an element $y \in G$ of the same order as x, such that the Brauer character at y on V_{248} is equal to the Brauer character at x on V.

Therefore to find all possible feasible decompositions, one would first need the Brauer character values on V_{248} of semisimple classes of G. These have been calculated for elements of order ≤ 57, see [45]. One would also need the Brauer character values of all irreducible GF(2)H-modules of dimension ≤ 248, for H being any one of the five aforementioned groups, these can be calculated in Magma using the commands `IrreducibleModules` and `BrauerCharacter`. The Feasible Character Code in [45], written by Neuhaus, then finds all possible sums of the irreducible Brauer characters of H corresponding to feasible decompositions of H on V_{248}. Given a possible feasible decomposition, the corresponding fusion pattern can be written down.

Given a fusion pattern, one can go on to see if an embedding of H following it exists. Following the construction of H, finding $C_{V_{248}}(H)$ can possibly eliminate H as being maximal in G (see Proposition 2.2.3).

Let $\widetilde{G}$ be a simple, simply connected algebraic group of exceptional type over the algebraic closure of $\mathbb{F}_p$, p a prime, and $L(\widetilde{G})$ its Lie algebra.

Proposition 2.2.3. *If H is a finite subgroup of $\widetilde{G}$ such that H centralises a line on $L(\widetilde{G})^\circ$, then H is strongly imprimitive.*

Proof. See [12, Propostion 4.5] □

Let $\mathscr{X}$ denote the collection of maximal subgroups of positive dimension in $\widetilde{G}$. We have the following definition from [12].

Definition 2.2.4. *If σ is a Frobenius endomorphism on $\widetilde{G}$ and a subgroup $H \leq \widetilde{G}$ is contained in $G = \widetilde{G}_\sigma$, then H is called strongly imprimitive if H is contained in a σ-stable, $N_{Aut^+(\widetilde{G})}(H)$-stable member of $\mathscr{X}$.*

For us $\widetilde{G}$ is the simple algebraic group of type E_8, p is 2, and so $L(\widetilde{G})^\circ = L(\widetilde{G})$. After having constructed a subgroup H of $\widetilde{G}_\sigma \cong E_8(2)$, if we find that H fixes a non-zero vector in V_{248} then H will fix the same vector in $L(\widetilde{G})$ and thus, by Proposition 2.2.3, H will be contained in a σ-stable member, X, of $\mathscr{X}$. Let $H_0 \leq \widetilde{G}_\sigma$ be an automorphic extension of H. Then since $H_0 \leq \mathrm{Aut}^+(\widetilde{G})$ and X is $N_{\mathrm{Aut}^+(\widetilde{G})}(H)$-stable and maximal in $\widetilde{G}$, we have that $H_0 \leq X$.

Later we will attempt to construct subgroups H of $G \cong E_8(2)$, with $H \cong L_2(64)$, $L_2(16)$, $L_2(8)$, $L_3(4)$ or $L_3(3)$. Given such a H, if we find that $C_{V_{248}}(H)$ is non-zero, we will know, by Proposition 2.2.3, that H and any automorphic extension of H will not be maximal in G. Note that in order to find $\dim(C_{V_{248}}(H))$, we run `Dimension(Fix(GModule(H)))`. However, constructing H is not always necessary to see whether $\dim(C_{V_{248}}(H))$ will be non-zero.

The following result is [37, Proposition 3.6] and is based on [34, Lemma 1.2]. It tells us when we can immediately say, just by looking at a possible feasible decomposition of H on V_{248}, that if H (compatible with that feasible decomposition) existed as a subgroup of G then it'd fix a non-zero vector of V_{248}. This would then save us an attempt at constructing H.

Lemma 2.2.5. *Let S be a finite group and M a finite-dimensional kS-module, with composition factors $W_1, \ldots, W_r$, of which m are trivial. Set $n = \sum$ dim $H^1(S, W_i)$, and assume $H^1(S, k) = \{0\}$.*

(i) If $n < m$ then M contains a trivial submodule of dimension at least m-n,

(ii) If $m = n$ and M contains no nonzero trivial submodule, then $H^1(S, M) = \{0\}$,

(iii) Suppose that $m = n > 0$, and that for each i we have $H^1(S, W_i) = \{0\} \iff H^1(S, W_i^*) = \{0\}$. Then M has a nonzero trivial submodule or quotient.

Proof. See [37]. □

Keeping to the notation in the above lemma, if we're in a situation where $m = n$, we will proceed by checking if the W_i's are self-dual. If so, then M would have a nonzero trivial submodule (in which case S can't be maximal in G by the Proposition 2.2.3) or quotient. But the dual of this quotient would be a (trivial) submodule of V_{248}^*, see [1]. Since V_{248} is self-dual we again have that S cannot be maximal in G.

Chapter 3

$L_2(64)$

3.1 Methodology

We need to establish whether $L_2(2^n)$, where $n \in \{3, 4, 6\}$, can be maximal in $G \cong E_8(2)$ or not. Note that the cases $n \in \{5, 7\}$ have been dealt with in [45]. We will first need to find all copies of $L_2(2^n)$ up to conjugacy in $E_8(2)$. To do this we follow the methodology in [45] which we explain here in more detail. We also write down a few adjustments that we make and introduce a strategy that can be used to discard numerous groups of order 2^n, saving us on computations, since otherwise these groups would need to be considered to see if they can be built up to copies of $L_2(2^n)$.

The methodology explained below is at the heart of dealing with $L_2(2^n)$ for every $n \in \{3, 4, 6\}$ and we will stick to it exactly for $L_2(64)$. The implementation in code is given in A.1; the original version of this program can be found in [45]. Additional strategies that we introduce in order to deal with $L_2(16)$ and $L_2(8)$ will be discussed in the respective chapters.

We first have the following lemma on the structure of $L_2(2^n)$.

Lemma 3.1.1. *Let H be a group isomorphic to $L_2(2^n)$, and $S \in Syl_2(H)$. Then S is elementary abelian of order 2^n, there exists an element $x \in N_H(S)$ of order $2^n - 1$ and an involution t that inverts x such that $N_H(S) = \langle S, x\rangle$ and $H = \langle S, x, t\rangle$. Furthermore x acts irreducibly on S.*

Proof. By [21, Lemma 15.1.1], S is elementary abelian of order 2^n, $N_H(S) = \langle S, x\rangle$ is a Frobenius group, maximal in H, with x, an element of order $2^n - 1$, acting irreducibly on S. Then by [21, Theorem 2.7.7], $N_{N_H(S)}(\langle x\rangle) = \langle x\rangle$. By [20, Theorem 1.3], $N_H(\langle x\rangle)$ is a dihedral group of order $2(2^n - 1)$. Therefore there exists an involution $t \notin N_H(S)$ that inverts x. Since $N_H(S)$ is maximal, we have that $H = \langle S, x, t\rangle$. □

The following theorem tells us where we can find p-groups and elements that normalise them.

Theorem 3.1.2. *(Borel-Tits Theorem). Let H be a simple linear algebraic group defined over an algebraically closed field of characteristic $p \neq 0$. Let σ be a Frobenius morphism on H and H_σ the fixed point group of σ. Let U be a non-identity p-subgroup of H_σ. Then there exists a parabolic subgroup P_σ such that $N_{H_\sigma}(U) \subseteq P_\sigma$ and $U \subseteq O_p(P_\sigma)$.*

Let $J \subseteq \{1, \dots, 8\}$ and P_J the standard parabolic subgroup of G associated to the roots labelled by J. Then $P_J = Q_J L_J$, where Q_J is the unipotent radical of P_J and L_J the standard Levi complement. We make the following definition.

Definition 3.1.3. *Given $g \in L_J$, we say that $\langle g\rangle$ (or g) is L_J-*cuspidal *if $\langle g\rangle$ is not L_J-conjugate to a subgroup in any L_I, $I \subsetneq J$. Given an element $g \in G$ that is L_J-cuspidal for some $J \subseteq \{1, \dots, 8\}$, we say that $\langle g\rangle$ (or g) is a* Levi-cuspidal *subgroup (or element) of G.*

Going by the information in Lemma 3.1.1, in order to find all copies of $L_2(2^n)$ in G, we first find all copies of $S{:}\langle x\rangle$ in G, where $S \cong 2^n$ and x is an element of order $2^n - 1$. By Theorem 3.1.2, we may search for the required groups $S{:}\langle x\rangle$ in parabolic subgroups of G.

So let P be a parabolic subgroup of G, given to us by Theorem 3.1.2, so that $S{:}\langle x\rangle \leq P$ and $S \leq O_2(P)$. But we are interested in the groups $S{:}\langle x\rangle$ up to G-conjugacy. So if there is a smaller parabolic subgroup, R, of G inside P, containing a conjugate, x^p for some $p \in P$, of x then we may conjugate $S{:}\langle x\rangle$ into R whilst conjugating S into $O_2(R)$. Note that Sylow 2-subgroups of any parabolic subgroup of G are Sylow 2-subgroups of G and so $O_2(P) = \bigcap \mathrm{Syl}_2(P) \leq \bigcap \mathrm{Syl}_2(R) = O_2(R)$. Therefore, we indeed have that $S^p \leq O_2(P)^p = O_2(P) \leq O_2(R)$.

Since we are interested in $S{:}\langle x\rangle$ only up to G-conjugacy and parabolic subgroups of G are just conjugates of the standard parabolic subgroups, $P_J, J \subseteq \{1, \dots, 8\}$, we need only search for the groups $S{:}\langle x\rangle$ in the standard parabolic subgroups of G. In fact, we need only search in those standard parabolic subgroups, $P_J = Q_J L_J$, which contain elements x of order $2^n - 1$ that don't have P_J-conjugates lying in any smaller parabolic subgroups, $R \leq P_J$ of G. Such a P_J will contain an element x of order $2^n - 1$ so that $\langle x\rangle$ is not L_J-conjugate to a subgroup in any $L_I \leq P_I, I \subsetneq J$. Hence the standard parabolic subgroups we search in are those that contain Levi-cuspidal subgroups of G of order $2^n - 1$.

By Lemma 3.1.1 we know that x acts on S irreducibly. So given a standard parabolic subgroup P containing a Levi-cuspidal element, x, of G, we must search inside $O_2(P)$ for all elementary abelian subgroups, S, of order 2^n that x acts on irreducibly. Since $O_2(P) \trianglelefteq P$, x acts on it and so x also acts on $O_2(P)/\Phi(O_2(P))$, where $\Phi(O_2(P))$ is the Frattini subgroup of $O_2(P)$. Since $O_2(P)/\Phi(O_2(P))$ is elementary abelian we can use the command `GModule` to realise $O_2(P)/\Phi(O_2(P))$ as a module on which $\langle x\rangle$ acts.

Assume there exists such an S in $O_2(P)$ and let $q : O_2(P) \to O_2(P)/\Phi(O_2(P))$ be the natural map. Since S is a subgroup of $O_2(P)$ on which x acts irreducibly, $q(S)$ is an irreducible $\langle x\rangle$-submodule of $O_2(P)/\Phi(O_2(P))$. We also have that $\Phi(O_2(P)) \cap S = \{s \in S : \overline{s} = \overline{e}\} \leq S$ and is stabilised by x but since x acts on S irreducibly we have that $\Phi(O_2(P)) \cap S = \{e\}$ or S. Therefore $q(S)$ is an n- or 0-dimensional irreducible $\langle x\rangle$-submodule of $O_2(P)/\Phi(O_2(P))$. Hence we may search for the groups S in the preimages of all the n-dimensional irreducible $\langle x\rangle$-submodules of $O_2(P)/\Phi(O_2(P))$.

Assume $q(S)$ is n-dimensional. Then since x acts irreducibly on S we have that $\langle x\rangle$ acts faithfully on $q(S)$. If $\langle x\rangle$ doesn't act faithfully on $O_2(P)/\Phi(O_2(P))$ then it won't act faithfully on any submodules of $O_2(P)/\Phi(O_2(P))$ and in this case we may search for any groups S in $\Phi(O_2(P))$.

We input $\langle O_2(P), x\rangle, O_2(P)$ and $\Phi(O_2(P))$ as the arguments of `GModule`. This gives us $O_2(P)/\Phi(O_2(P))$ as a $\langle O_2(P), x\rangle$-module over GF(2). Let k be the dimension of $O_2(P)/\Phi(O_2(P))$ and $\rho : \langle O_2(P), x\rangle \to GL_k(2)$ the representation corresponding to the action of $\langle O_2(P), x\rangle$ on $O_2(P)/\Phi(O_2(P))$. The image of ρ is called the action group of $O_2(P)/\Phi(O_2(P))$ and acts faithfully on it; denote this by A.

Any element in $\langle O_2(P), x\rangle$ is of the form ox^i, $o \in O_2(P), i \in \{1, \ldots, 2^n - 1\}$. Since $O_2(P)/\Phi(O_2(P))$ is abelian any $o \in O_2(P)$ acts trivially on $O_2(P)/\Phi(O_2(P))$ and so

$\rho|_{\langle x\rangle} : \langle x\rangle \to A$ is a surjection. Hence we have that $|A|$ divides $2^n - 1$. If $|A| < 2^n - 1$ then $\rho|_{\langle x\rangle}$ is not an injection and so there exists a non-identity element in $\langle x\rangle$ that acts trivially on $O_2(P)/\Phi(O_2(P))$ and therefore the action of $\langle x\rangle$ on $O_2(P)/\Phi(O_2(P))$ is not faithful.

After using the `GModule` command, we calculate the action group of $O_2(P)/\Phi(O_2(P))$ using the `ActionGroup` command and check its order. If the order equals $2^n - 1$, we think of $O_2(P)/\Phi(O_2(P))$ as a $\langle x\rangle$-module and proceed by finding the preimages of all its irreducible n-dimensional submodules.

We run `DirectSumDecomposition` on $O_2(P)/\Phi(O_2(P))$, to get a decomposition $U \oplus V_1^1 \oplus \ldots \oplus V_{n_1}^1 \oplus \ldots \oplus V_1^m \oplus \ldots \oplus V_{n_m}^m$, where U is a direct sum of irreducible submodules of $O_2(P)/\Phi(O_2(P))$ whose dimension isn't n, and for $i \in \{1, \ldots, m\}$ and $j, k \in \{1, \ldots, n_i\}$, V_j^i is an n-dimensional irreducible submodule of $O_2(P)/\Phi(O_2(P))$ with $V_j^i \cong V_k^i$; here $m, n_i \in \mathbb{N}$. We stress here that the bigger the dimension of U is, the better this will be for us. Denote $V_1^i \oplus \ldots \oplus V_{n_i}^i$ as V^i. Let V be an n-dimensional irreducible submodule of $O_2(P)/\Phi(O_2(P))$ then V is isomorphic to one of $V_1^1, \ldots, V_1^m$. If V is isomorphic to V_1^i then V is a submodule of V^i. Hence, the preimage of V will be contained in the preimage of V^i and so we consider the preimages of $V^i, i \in \{1, \ldots, m\}$.

The preimages of $V^i, i \in \{1, \ldots, m\}$ are 2-groups smaller than $O_2(P)$ in which lie the subgroups S that we seek. If $\Phi(O_2(P))$ is trivial then each of $q^{-1}(V^i)$ is elementary abelian and we add them to a set we call `FinSub` (see the program in A.1). Otherwise we add them to a set we call `SetSub2`. We then run the process we ran on $O_2(P)$ on all the groups in `SetSub2` and we keep on repeating this until nothing more is added to `SetSub2`.

If we ever come across a 2-group b for which the order of the action group of $b/\Phi(b)$ is less than $2^n - 1$, we add it to a set we call `ActnGpDiff`.

It could also be that b is such that $b/\Phi(b)$ is a direct sum of irreducible n-dimensional modules that are all isomorphic to each other, the order of the action group of $b/\Phi(b)$ is $2^n - 1$ and $\Phi(b)$ is not trivial. In this case we would keep on adding b to `SetSub2`, resulting in an infinite loop. To stop this from happening we add b to a set we call `BadSub` instead.

Take a group b from `BadSub`, then $b/\Phi(b)$ decomposes into a direct sum of isomorphic irreducible n-dimensional modules, $V_1 \oplus \ldots \oplus V_k$ for some $k \in \mathbb{N}$. We now present a way of generating all irreducible submodules of $b/\Phi(b)$; the preimages of these will contain the required groups S. Note that b itself is too big for us to search in directly for any groups S.

Let V be an irreducible submodule of $b/\Phi(b)$, and v any non-zero vector in it, then $V = \langle v \rangle$, but moreover we have that $V = \{x^i.v : 1 \leq i \leq 2^n - 1\} \cup \{\underline{0}\}$. This is because $|\{x^i.v : 1 \leq i \leq 2^n - 1\}| = 2^n - 1$, and we know this since we check that the dimension of the space in $b/\Phi(b)$ fixed by a non-identity element of $\langle x \rangle$ is 0 (see the identifier, `bool`, in A.1). Note that this check was not needed in [45] since there, $|\{x^i.v : 1 \leq i \leq 2^n - 1\}| = 2^n - 1$ is implied by the fact that $2^n - 1$ is always prime. We aim to collect one non-zero vector from every irreducible submodule of $b/\Phi(b)$ so that we are able to generate all irreducible submodules.

Given an arbitrary vector in $b/\Phi(b)$ then either its projection to V_1 is the zero vector or it isn't. Assume first that it isn't, fix a non-zero vector $v_1 \in V_1$ and collect all vectors $v_1 + w, w \in V_2 \oplus \ldots \oplus V_k$. We know that $V_1 = \{x^i.v_1 : 1 \leq i \leq 2^n - 1\} \cup \{\underline{0}\}$ and so we don't need to consider any vector $x^i.v_1 + w, i \neq 2^n - 1$ since $x^i.v_1 + w = x^i.(v_1 + x^{-i}w)$ and $x^i.v_1 + w$ generates an irreducible submodule iff $v_1 + x^{-i}w$ generates the same irreducible submodule. We are already collecting $v_1 + x^{-i}w$, there is no need to collect $x^i.v_1 + w$ as well. Now assume that projection to V_1 is the zero vector, then the projection to V_2 is either the zero vector or it isn't; we proceed as above. What we have done is that for $i \in \{1, \ldots, k\}$ we have fixed a non-zero vector $v_i \in V_i$ and collected all vector $v_i + w, w \in V_{i+1} \oplus \ldots \oplus V_k$. Given any irreducible submodule of $b/\Phi(b)$ then this can be generated by some vector among the ones we have collected.

We pull these vectors back into b and these elements of b that we get can be used to generate the preimages of the irreducible submodules of $b/\Phi(b)$. Given one such preimage H, we now need a subgroup $A \leq \Phi(b)$ so that H/A is elementary abelian. We could then run the process we ran on $O_2(P)/\Phi(O_2(P))$ on H/A and break H up into smaller pieces in which we can search for the groups S. We could of course choose A to be $\Phi(H)$, however the way that we define A instead will allow us to rule out some of the preimages H as containing groups S.

We let C be the commutator subgroup $[b, \Phi(b)]$, we then find the Frattini subgroup of $\Phi(b)/C$ and take A to be the preimage of $\Phi(\Phi(b)/C)$ under the natural map $\Phi(b) \to \Phi(b)/C$. Since $C \leq A$, we get that $\Phi(b)/A \leq Z(b/A)$. Moreover, since $\Phi(\Phi(b)) \leq A$, we get that $\Phi(b)/A$ is elementary abelian as the homomorphic image of the elementary abelian group $\Phi(b)/\Phi(\Phi(b))$.

Now if H is the preimage of a non-zero irreducible submodule of $b/\Phi(b)$ then $H = \Phi(b) \dot{\cup} x^{-1}tx\Phi(b) \dot{\cup} x^{-2}tx^2\Phi(b) \dot{\cup} \ldots \dot{\cup} t\Phi(b)$, where t is a preimage of a non-zero vector lying in that irreducible submodule. Assume that there exists an S inside H that intersects trivially with $\Phi(b)$, then we have that $S = \{e, x^{-1}txf_1, x^{-2}tx^2f_2, \ldots, tf_{2^n-1}\}$, for some $f_1, \ldots, f_{2^n-1} \in \Phi(b)$. Since S and $\Phi(b)/A$ are elementary abelian and $\Phi(b)/A \leq Z(b/A)$, we have that, for any $i, j \in \{1, \ldots, 2^n - 1\}$, $x^{-i}t^2x^i \in A$ and the images of $x^{-i}tx^i$ and $x^{-j}tx^j$ in H/A commute.

We have seen that if there exists an S that embeds into $H/\Phi(b)$ then it must be that $t^2 \in A$ and H/A is elementary abelian. Therefore, going through the vectors we collected earlier, and calculating a preimage of each of these vectors, we may keep only those preimages, t, that square into A. These preimages are kept in a set called `SetKeep` and are used to generate subgroups H. Note that we are able to disregard a substantial number of vectors whose preimages don't square into A, making it practical to proceed and perform calculations on the groups H/A. Hence working with H/A is better than having, more naturally, considered all elementary abelian groups, $H/\Phi(H)$. We check that `SetKeep` is always non-empty (see identifiers, `bool2` and `SetKeepZero` in A.1) since otherwise we will not pick up any groups S in b if they all lie in $\Phi(b)$. `SetKeep` will indeed always be non-empty in the $L_2(64)$ case.

We will see what the possibilities for $O_2(P)$ are in Section 3.2. To summarise, the program in A.1 starts by taking $O_2(P)$ as the sole element of `SetSub2`. It then proceeds to break up $O_2(P)$ into preimages of the direct sums of isomorphic irreducible 6-dimensional $\langle x \rangle$-submodules of $O_2(P)/\Phi(O_2(P))$ and `SetSub2` is reset as empty. Each of the preimages calculated is either added to `SetSub2`, `BadSub` or `FinSub`. Computationally, we will see that an $O_2(P)$ never goes into `ActnGpDiff` and that, at this stage, all preimages get added to `SetSub2`. The process of breaking up the groups in `SetSub2` and resetting `SetSub2` is repeated until nothing more can be added to an empty `SetSub2`. Of course, along the way appropriate groups have been added to `BadSub`, `FinSub` and `ActnGpDiff`.

For every b in `BadSub`, the program then calculates the group A, as defined above, and preimages H (labelled as `Sub4aa` in A.1) of certain irreducible 6-dimensional

$\langle x\rangle$-submodules of $b/\Phi(b)$. Preimages of certain submodules of H/A for every H are then added to `SetSub2` or `FinSub`. `SetSub2` is dealt with as before, until empty, and appropriate groups get added to `BadSetNew`, `FinSub` and `ActnGpDiff` along the way. `BadSub` then gets reset as `BadSetNew` and `BadSetNew` as empty and the entire process repeated, and so on, until an empty `BadSetNew` is returned.

The program ends and returns non-empty sets `FinSub` and `ActnGpDiff` that we now turn our attention to.

For b in `ActnGpDiff`, $\langle x\rangle$ doesn't act faithfully on $b/\Phi(b)$ and so, as mentioned previously (for $O_2(P)$ if $\langle x\rangle$ doesn't act faithfully on $O_2(P)/\Phi(O_2(P))$), we need to search for any groups S in $\Phi(b)$. Therefore we add the Frattini subgroups of the groups in this `ActnGpDiff` to an empty `SetSub2` and set `BadSub`, `FinSub` and `ActnGpDiff` as empty. We break up the groups in `SetSub2` (by running the repeat loop in A.1 programmed to end when `#SetSub2 eq 0`), adding appropriate groups to our three empty sets along the way. If a non-empty `ActnGpDiff` is output then we repeat the same process but without resetting `BadSub` and `FinSub` as empty. We keep on repeating this process until an empty `ActnGpDiff` is returned.

The set `FinSub` contains elementary abelian 2-groups in which we may search for groups S, on which x acts irreducibly, such that $\langle S, x, t\rangle$ is isomorphic to $L_2(2^n)$, where $t \in G$ is an involution that inverts x. Given $F \in$ `FinSub`, we may actually be able to rule it out as containing any groups S of interest, saving us a search in F.

We denote by V_{248}, the 248-dimensional adjoint module of $G \cong E_8(2)$. Let $n \in \{3, 4, 6\}$, if $L_2(2^n)$ is a subgroup of G then the possible feasible decompositions of $L_2(2^n)$ on V_{248} are given in [45] (or see Appendix B). Call $L_2(2^n) \leq G$ as $H(= \langle S, x, t\rangle)$. Given a feasible decomposition of H on V_{248}, then this determines the composition factors of $V_{248}\!\downarrow\! H$, the restriction of V_{248} to H. We are given $V_{248}\!\downarrow\! H$ as $V_1^{n_1}/V_2^{n_2}/\ldots/V_k^{n_k}$, where V_i, an irreducible H-module, is a composition factor of $V_{248}\!\downarrow\! H$ with multiplicity n_i; here $k, n_k \in \mathbb{N}$, $i \in \{1, \ldots, k\}$.

The Steinberg module of a finite group of Lie type defined over a field of $q = p^r$ elements is irreducible, projective and of dimension equal to the order of a Sylow p-subgroup of the group (see 9.3 in [25]). This allows us to identify the Steinberg module from among the irreducible modules of $H \cong L_2(2^n)$.

The following result, as pointed out by Rowley, can be used to disregard groups in

`FinSub`.

Lemma 3.1.4. *Let $H \cong L_2(2^n)$, $S \in Syl_2(H)$, $B = N_H(S)$ and assume that H is a subgroup of $G \cong E_8(2)$. Let V_r be the composition factor of $V_{248} \downarrow H$ isomorphic to the Steinberg module, with multiplicity n_r. Then $\dim(C_{V_{248}}(B)) \geq n_r$ and if $\dim(C_{V_{248}}(B)) > n_r$, it follows that $C_{V_{248}}(H) \neq \mathbf{0}$.*

Proof. Given a non-trivial irreducible module, U, of H over GF(2) then the dimension of $C_U(B)$ is 1 if U is isomorphic to the Steinberg module and 0 otherwise. This can be checked in MAGMA. Since V_r is projective, we have that $\bigoplus_{i=1}^{n_r} V_r$ is a submodule of $V_{248}\downarrow H$. Hence, the dimension of $C_{V_{248}\downarrow H}(B)$ is at least n_r.

Assume that the dimension of $C_{V_{248}}(B)$ is $> n_r$. Then we may select $v \in C_{V_{248}}(B) \setminus \bigoplus_{i=1}^{n_r} V_r$. Now $\langle v^H \rangle$ is a quotient of the permutation module, the permutation representation being H acting on the cosets of B. To see this, let $Bx_1, \ldots, Bx_{2^n+1}$ be the right cosets of B in H and set $\alpha_i = Bx_i$. Let $W = \bigoplus_{i=1}^{2^n+1} \mathrm{GF}(2)\alpha_i$ be the permutation module and define $\varphi : W \to \langle v^H \rangle$ by $\varphi : \alpha_i \mapsto v^{x_i}$ and extend linearly. Since φ is a well-defined H-map and $\langle v^H \rangle = \langle v^{x_i} : i = 1, \ldots, 2^n+1 \rangle$, we have that $\langle v^H \rangle \cong W/\ker\varphi$. Now W has dimension $2^n + 1$ and contains the Steinberg module. But the Steinberg module is projective and of dimension 2^n and so W must be $U_1 \oplus U_2$ where $\dim(U_1) = 1$ and $U_2 \cong V_r$. Since U_1 and U_2 are irreducible and $\langle v^H \rangle \neq \mathbf{0}$, $\langle v^H \rangle \cong U_1$, U_2 or W. But if $\langle v^H \rangle \cong U_2$ then we have that $v \in \langle v^H \rangle \leq \bigoplus_{i=1}^{n_r} V_r$, a contradiction. Therefore $\mathbf{0} \neq C_{\langle v^H \rangle}(H) \leq C_{V_{248}}(H)$, and the lemma holds.

□

Given $F \in$ `FinSub` and a group $S \leq F$ of order 2^n on which x acts irreducibly, if there exists an involution $t \in G$ such that $H = \langle S, x, t \rangle \cong L_2(2^n)$, then we are interested in S only if $\dim(C_{V_{248}}(\langle S, x \rangle) = n_r$. Since otherwise H will fix a non-zero vector in V_{248} by Lemma 3.1.4 and thus, by Proposition 2.2.3, H and any automorphic extension of H will not be maximal in G. Note that n_r, as given in Lemma 3.1.4, will be known to us from Appendix B.

Going through `FinSub`, if we come across a group F such that the dimension of $C_{V_{248}}(\langle F, x \rangle) > n_r$, we discard it since any $\langle S, x \rangle \leq \langle F, x \rangle$ will also be so that $\dim(C_{V_{248}}(\langle S, x \rangle)) > n_r$. Now let F be such that $\dim(C_{V_{248}}(\langle F, x \rangle)) \leq n_r$, we must search in it for all subgroups S of order 2^n. We also want that x acts irreducibly on each of the subgroups S and so we use the `GModule` command with arguments, $\langle F, x\rangle, F, \{\mathrm{Id}_{248}\}$, to realise F as a $\langle x\rangle$-module over GF(2); call this module $\overline{F}$. The image in $\overline{F}$ of any group S that we are interested in will be an irreducible submodule of $\overline{F}$. Note that every irreducible submodule of $\overline{F}$ will have dimension n since F was realised as the preimage of a direct sum of irreducible isomorphic n-dimensional $\langle x\rangle$-modules, under a map $q : b \to b/\{\mathrm{Id}_{248}\}$, where b was an elementary abelian group once a member of the set `SetSub2`. Hence we use the command `MinimalSubmodules` to calculate all the irreducible submodules of $\overline{F}$ and then calculate their preimages in F. The set of these preimages contains all of the elementary abelian subgroups of F of order 2^n on which x acts irreducibly.

We finally have a list all subgroups S that are of interest to us. We now downsize this list by keeping only those subgroups S such that $\dim(C_{V_{248}}(\langle S, x\rangle))$ is n_r. We now calculate the extended centraliser of x in G, $C^*_G(x) = \{g \in G : x^g = x \text{ or } x^g = x^{-1}\}$. This will contain every involution that inverts x; we run through these involutions t to see if any are such that $\langle S, x, t\rangle \cong L_2(2^n)$.

Note that Lemma 3.1.4 can be used to disregard any 2-group, b (e.g. any group, or its Frattini, in `ActnGpDiff`), that we come across such that $\dim(C_{V_{248}}(\langle b, x\rangle)) > n_r$, not just the elementary abelian groups in `FinSub`. We stress here that Lemma 3.1.4 will prove to be invaluable for us when we go on to perform our computations.

3.2 Non-maximality of $L_2(64)$

Here we establish that $L_2(64)$ can't be a maximal subgroup of $E_8(2)$. For this we use the methods described in Section 3.1. First note that if $L_2(64)$ is a subgroup of $E_8(2)$, then out of the three possible feasible decompositions of $L_2(64)$ on V_{248} (see B.1), we are interested in the following only:

(iii) $12\phi_1 + 4\phi_2 + 1\phi_3 + 0\phi_4 + 1\phi_5 + 1\phi_6 + 0\phi_7 + 0\phi_8 + 0\phi_9 + 0\phi_{10} + 2\phi_{11} + 0\phi_{12} + 0\phi_{13} + 0\phi_{14}$ (3A $\to$ 3C, 5AB $\to$ 5B, 7AC $\to$ 7B, 9AC $\to$ 9B, 13A $\to$ 13B, 21AF $\to$ 21F, 63AI $\to$ 63AC, 63JR $\to$ 63AC, 65AX $\to$ 65AD)

The ϕ_i's above are all of the irreducible characters of $L_2(64)$ over GF(2), ordered in terms of increasing dimension. The first decomposition, (i), would have a trivial submodule by Lemma 2.2.5(i). Assume $H \leq E_8(2)$, $H \cong L_2(64)$ following (ii) and that $0 = V_{19} \subset V_{18} \subset V_{17} \subset \ldots \subset V_1 \subset V_0 = V_{248} \downarrow H$ is a composition series of V_0. Then we know that 8 of the V_i/V_{i+1}'s are isomorphic to ϕ_1 and 2 to ϕ_2; there are three ways these could appear in the series. Assume that for some $0 \leq i, j \leq 18$, $i < j$ and $V_i/V_{i+1}, V_j/V_{j+1} \cong \phi_2$.

- $V_k/V_{k+1} \cong \phi_1 \Rightarrow i+1 \leq k \leq j-1$ (all the ϕ_1's are trapped between the ϕ_2's): Let $m \geq i+1$ be such that for no $k < m$, V_k/V_{k+1} is isomorphic to ϕ_1. Then V_m is a H-submodule of V_0 that would contain a trivial submodule by Lemma 2.2.5(i).

- $\exists k \geq j+1, V_k/V_{k+1} \cong \phi_1$: Then V_{j+1} would contain a trivial submodule by Lemma 2.2.5(i).

- $\exists k < i, V_k/V_{k+1} \cong \phi_1$: Consider the chain $V_i \subset V_{i-1} \subset V_{i-2} \subset \ldots \subset V_3 \subset V_2 \subset V_1 \subset V_0$, then at least one factor is isomorphic to ϕ_1 and none are isomorphic to ϕ_2. Define $V_m^{\square}$, $0 \leq m \leq 19$, to be the submodule of V_0^* containing all the elements that annihilate V_m. Consider the chain $0 = V_0^{\square} \subset V_1^{\square} \subset V_2^{\square} \subset \ldots \subset V_{i-2}^{\square} \subset V_{i-1}^{\square} \subset V_i^{\square}$ of submodules of $V_0^* = V_{19}^{\square}$. Let $0 \leq m < i$, then $V_0^*/V_m^{\square} \cong V_m^*$ (consider the map $V_0^* \to V_m^*$ that sends an element to its restriction to V_m, see [10]), and so the submodule $V_{m+1}^{\square}/V_m^{\square} \leq V_0^*/V_m^{\square}$ is mapped to the submodule of V_m^* that consists of functionals (from V_m) that annihilate V_{m+1}, but this is isomorphic to $(V_m/V_{m+1})^*$, see [10]. Therefore, $V_{m+1}^{\square}/V_m^{\square} \cong V_m/V_{m+1}$ since all the irreducible modules of H over GF(2) are self-dual. Hence $V_i^{\square}$ has a trivial submodule by Lemma 2.2.5(i), but $V_i^{\square}$ is a submodule of $V_0^* \cong V_0$.

We have just proved that the decomposition (ii) would have a trivial submodule. Therefore, we now need to know which of the standard parabolic subgroups of $E_8(2)$ contain Levi-cuspidal elements of $E_8(2)$ that are in $63\mathrm{ABC}_{E_8(2)}$.

Recall that given a subset $J \subseteq \{1, \ldots, 8\}$, P_J denotes the standard parabolic subgroup of $E_8(2)$ associated to the roots labelled by J, and L_J denotes the standard Levi complement of P_J. We have the following result by P. Rowley:

Lemma 3.2.1. *Suppose that $\langle g \rangle$ is a Levi-cuspidal subgroup of $E_8(2)$ and $g \in 63\mathrm{ABC}_{E_8(2)}$. Set $\mathcal{J} = \{\{1,3,4,5,6\},\{2,4,5,6,7\},\{3,4,5,6,7\},\{4,5,6,7,8\}\}$. Then $\langle g \rangle$ is L_J-cuspidal for some $J \in \mathcal{J}$. Moreover, in each L_J, $J \in \mathcal{J}$, there is only one L_J-class of L_J-cuspidal subgroups $\langle g \rangle$ with $g \in 63\mathrm{ABC}_{E_8(2)}$.*

Proof. Will be viewable in [7], once the paper is complete and made available. □

Given $J \in \mathcal{J}$, $\mathcal{J}$ as in Lemma 3.2.1, we calculate L_J as being generated by (the image in $GL_{248}(2)$ of) all root subgroups U_α, of $E_8(2)$, where α is a root labelled by j or $120 + j$, $j \in J$. We also calculate $Q_J = O_2(P_J)$, as being generated by all root subgroups U_α, where for $i \in \{1, \ldots, 8\} \setminus J$, α is a root whose i^{th} coefficient is positive, see [6]. There is only one class of cyclic groups of order 63 in $L_J \cong L_6(2)$ and so we take x_J to be any element of order 63 in L_J.

Given $J \in \mathcal{J}$, we search for elementary abelian subgroups of order 2^6 in Q_J by running the program in A.1. The results of the runs are given in Table 3.1.

	J			
	$\{1,3,4,5,6\}$	$\{2,4,5,6,7\}$	$\{3,4,5,6,7\}$	$\{4,5,6,7,8\}$
`#FinSub`	9	12	10	0
`#BadSub`	3	1	3	1
`#ActnGpDiff`	4	2	4	0
`#FinSub`	955	14	396	78
`#BadSetNew`	0	0	0	1
`#ActnGpDiff`	4	2	4	2
`#FinSub`				81
`#BadSetNew`				0
`#ActnGpDiff`				2

Table 3.1: The outcome, at different stages, of running A.1 with Q_J and x_J. The third row shows the outcome of breaking up Q_J, fourth of breaking up the groups in `BadSub` and the fifth of breaking up the groups in `BadSetNew`.

The irreducible character of $L_2(64)$ corresponding to the Steinberg module is φ_{11} (see table in B.1), so the number of composition factors of $V_{248}\!\downarrow\! L_2(64)$ corresponding to the Steinberg module is 2.

Given any $J \in \mathcal{J}$, let b be a group in `ActnGpDiff`, then $\dim(C_{V_{248}}(\langle b, x_J\rangle))$ is either $6, 7$ or 11 and so we ignore all groups b in `ActnGpDiff`.

Let $J = \{1, 3, 4, 5, 6\}$, 953 of the elementary abelian groups, F, in `FinSub` have order 2^6 but none are such that $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ is 2. The remaining two groups,

F, both have order 2^{12}, one with $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ being 4, the other with it being 6. Therefore Q_J does not contain any desired elementary abelian groups of order 2^6.

Let $J = \{3,4,5,6,7\}$, 394 of the elementary abelian groups, F, in `FinSub` have order 2^6 but none with $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ being 2. One of the remaining two groups, F, is such that $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ is 4 and the other such that it is 6. Therefore Q_J does not contain any desired elementary abelian groups of order 2^6.

Let $J = \{2,4,5,6,7\}$, if $F \in$ `FinSub` has order 2^6 (there's 10 of these) then $\dim(C_{V_{248}}(\langle F, x_J\rangle)) \neq 2$, otherwise $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ is either 4 or 1. Let F be any one of the two groups in `FinSub` such that $|F| \neq 2^6$ and $\dim(C_{V_{248}}(\langle F, x_J\rangle)) = 1$ then $|F| = 2^{18}$, F has 4161 subgroups, S, of order 2^6 normalised by x_J, all of which are such that $\dim(C_{V_{248}}(\langle S, x_J\rangle)) \neq 2$. Therefore Q_J does not contain any desired elementary abelian groups of order 2^6.

Finally let $J = \{4,5,6,7,8\}$, if $F \in$ `FinSub` has order 2^6 (there's 11 of these) then $\dim(C_{V_{248}}(\langle F, x_J\rangle)) \neq 2$, otherwise $\dim(C_{V_{248}}(\langle F, x_J\rangle))$ is either 6, 4 or 1. Let F be any one of the 62 groups in `FinSub` such that $|F| \neq 2^6$ and $\dim(C_{V_{248}}(\langle F, x_J\rangle)) = 1$ then $|F| = 2^{12}$, F has 65 subgroups, S, of order 2^6 normalised by x_J, all of which are such that $\dim(C_{V_{248}}(\langle S, x_J\rangle)) \neq 2$. Therefore Q_J does not contain any desired elementary abelian groups of order 2^6.

If we would have proceeded to build any $L_2(64)$'s from any of the elementary abelian groups of order 2^6 that we came across above then these would've fixed non-zero vectors in V_{248}, and therefore could not have been maximal in $E_8(2)$. We have the following theorem.

Theorem 3.2.2. *If H is a subgroup of $E_8(2)$ such that $F^*(H) \cong L_2(64)$ then H is not maximal in $E_8(2)$.*

Chapter 4

$L_2(16)$

In this chapter, we establish that $L_2(16)$ and its extensions cannot be maximal in $E_8(2)$. To do this we build up on the methodology given in Section 3.1 which was used to prove that $L_2(64)$ can't be maximal in $E_8(2)$. Throughout this chapter, G will be isomorphic to $E_8(2)$ unless otherwise stated.

4.1 Methodology

From Section 3.1, we know that in order to construct copies of $L_2(2^n)$, $n \in \{3, 4, 6\}$, we first need to search for subgroups of order 2^n in the 2-cores of those standard parabolic subgroups, P, of $E_8(2)$ that contain Levi-cuspidal subgroups of order $2^n - 1$. Since we want to construct $L_2(2^n)$ up to conjugacy in $E_8(2)$, we need to consider every class of Levi-cuspidal subgroups of P of order $2^n - 1$, pick one representative, $\langle x \rangle$, from each class, consider every elementary abelian subgroup S of order 2^n in $O_2(P)$ irreducible under the action of x, and for every such S, go through all involutions, t, in $E_8(2)$ that invert x to see if $\langle S, x, t \rangle$ is isomorphic to $L_2(2^n)$ or not.

The list of parabolic subgroups, P, that we need to consider for $L_2(16)$ will be given in the next section. For almost all of these parabolic subgroups we will use a program similar to A.1, which, for every pair of $O_2(P)$ and x, will output a set `FinSub` of all elementary abelian subgroups of $O_2(P)$ that are normalised by x. The program achieves this by breaking up $O_2(P)$ into smaller and smaller subgroups b which at some point become members of the ever-changing set, `BadSub`. It is not always practical to try and break up each and every group b in `BadSub`; the following lemma tells us when we can disregard some of the groups in `BadSub`.

Lemma 4.1.1. *Let b_1 and b_2 be 2-subgroups of G normalised by x such that there exists $g \in C_G(x)$ with $b_1^g = b_2$, then groups of the form $\langle S_1, x, t_1 \rangle$ are conjugate to groups of the form $\langle S_2, x, t_2 \rangle$. Here $S_1 \leq b_1$ and $S_2 \leq b_2$ are elementary abelian groups of order 2^n irreducible under the action of x and t_1 and t_2 are involution that invert x.*

Proof. Can pick g or g^{-1} as the conjugating element. Also note that $C_G^*(x)^g = C_G^*(x)$. □

Since we are interested in constructing copies of $L_2(2^n)$ only up to conjugacy in $E_8(2)$, instead of searching for the groups S in every group in `BadSub`, we may perform the search in every group in a smaller subset of `BadSub` such that every group in `BadSub` is conjugate to some group in this subset via an element of $C_G(x)$. The sizes of the sets `BadSub` we will encounter can be a lot more than what we have seen for $L_2(64)$. Hence Lemma 4.1.1 will prove to be indispensable, not so much for $L_2(16)$ but certainly for the $L_2(8)$ case.

In order to exploit Lemma 4.1.1, we need to look for elements g in $C_G(x)$ such that there exists a group $b \in$ `BadSub` such that $b^g \in$ `BadSub`. In practice, we don't look for such elements in all of $C_G(x)$ but in the smaller group $C_P(x)$, where P is the parabolic subgroup of G containing all the groups in `BadSub`. Let g_1 be an element in $C_P(x)$ and B_1 a subset of `BadSub` such that for every $b \in$ `BadSub` $\setminus B_1$, $b^{g_1} \in B_1$ and the same does not hold true for any other subset of `BadSub` of size smaller than $|B_1|$. Pick a different element $g_2 \in C_P(x)$, we now seek a subset $B_2 \subseteq B_1$ such that for every $b \in B_1 \setminus B_2$, $b^{g_2} \in B_2$ and no other subset of B_1 of size smaller than $|B_2|$ has the same property. Given B_2, we now seek a subset of B_2 and so on, until we have exhausted all elements of $C_P(x)$. By Lemma 4.1.1, we may replace the set `BadSub` with B_r, where $r = |C_P(x)|$. It is not necessary, and indeed not always practical, to run through every element of $C_P(x)$, but just through enough random elements, say m of them, such that the set B_m can be deemed small enough to perform our computations. The code implementing the process of getting B_m is given in A.2 and we give an explanation of it next.

Let $k \in \mathbb{N}$ be the size of `BadSub`, then we write the indexed set `BadSub` as

$\{b_1, b_2, \ldots, b_k\}$. The code A.2 takes a random element h (or `h`) from $C_P(x)$ (or `cpx`) and for every $1 \leq j \leq k-1$ checks if $b_i^h = b_j$, $j+1 \leq i \leq k$. If it finds that $b_l^h = b_j$ for some $l \in \{j+1, \ldots, k\}$, it doesn't check if the same holds for any b_i, $i > l$ (see the occurrence of `break` in A.2) since this can't happen with `BadSub` being a set of distinct groups. Let $j \in \{2, \ldots, k-1\}$ and $i > j$ such that $b_i^h = b_l$ for some $l < j$, then we could improve the code by not checking if b_i^h also equals b_j since we already know that this isn't possible. Before h is picked, `orbs` is defined as the sequence $(\{i\})_{i=1}^k$. If there exist $1 \leq j, i \leq k$, $j < i$ such that $b_i^h = b_j$ then the set in `orbs` containing j and the set containing i are replaced with their union. Take a single element from every set in `orbs`, the code may define `ind` as a sequence of these elements. The set $B_1 \subseteq$ `BadSub`, described in the previous paragraph, equals $\{b_i : i \in$ `ind`$\}$. The code then takes another random element from $C_P(x)$ and repeats the same process but only with the groups in `BadSub` indexed by `ind`. We interrupt the running of A.2 once `#ind` gets small enough for our purposes or doesn't change after having selected, say 60, elements from $C_P(x)$. We now need only work with a proper subset of `BadSub` rather than all of it.

It was observed in practice that running A.2 for a certain amount of time can decrease `#ind` several times, when in the same time `#ind` stays the same as `#BadSub` if we replace $C_P(x)$ with $C_G(x)$. So it is indeed more efficient to work with $C_P(x)$ rather than $C_G(x)$.

We now move on to describe a method that enables us to deal with certain problematic groups in `BadSub`.

Let $b \in$ `BadSub` then we know that $b/\Phi(b)$ is isomorphic to a direct sum of, say k, isomorphic irreducible n-dimensional $\langle x \rangle$-modules (see Section 3.1). Then we go on to calculate the group A as the preimage of $\Phi(\Phi(b)/[b, \Phi(b)])$. Also a set we call `SetKeep` is created which contains those preimages of vectors in $b/\Phi(b)$ that square into A. This involves going through $2^{n(k-1)} + 2^{n(k-2)} + \ldots + 2^n + 1$ of the vectors in $b/\Phi(b)$. The elements in `SetKeep` are used to generate preimages, H, of irreducible submodules of $b/\Phi(b)$ and we break up H into preimages of certain submodules of H/A.

Sometimes k is so large, e.g. $k = 8$ for $n = 4$ and $k = 11$ for $n = 3$, that Magma is unable to calculate `SetKeep` entirely over the span of days. Even if we were to get our hands on a complete `SetKeep` at some point in time for large k, its size would be too big, making it impractical to continue and break up all preimages H.

To counter this problem, instead of considering $b/\Phi(b)$ we factor b out with an overgroup of $\Phi(b)$ we call F such that b/F is still elementary abelian. Write $b/\Phi(b)$ as $V_1 \oplus \ldots \oplus V_k$, for some $1 \leq r < k$, we take F to be the preimage of $V_1 \oplus \ldots \oplus V_r$. Then b/F is a direct sum of $k-r$ isomorphic irreducible n-dimensional $\langle x \rangle$-modules. We now define A as the preimage of $\Phi(F/[b,F])$, proceed to calculate `SetKeep` as normal and so on. Essentially, if we look at A.1, we now have that the group `Fb` is F instead of `FrattiniSubgroup(b)`.

Note that our preimages H will now be bigger than before. We don't want them being too big so in order to choose the best value for r we will run tests with different values and assess the situation by looking at what sizes of `SetKeep` we get and how a few of the resulting H/A behave.

To end this section, we prove a result that will help us to disregard 2-groups by giving us a bound on the dimension of the fixed spaces of involutions in $L_2(2^n) \leq G$, $G \cong E_8(2)$.

Lemma 4.1.2. *Given a group H and a H-module V_0, let $\{0\} = V_m \subset V_{m-1} \subset \ldots \subset V_1 \subset V_0$ be a composition series of V_0. Then for all $t \in H$, $dim(C_{V_0}(t)) \leq dim(C_{V_0/V_1}(t)) + dim(C_{V_1/V_2}(t)) + \ldots + dim(C_{V_{m-1}/V_m}(t))$.*

Proof. Since $V_m \subset V_{m-1} \subset \ldots \subset V_1 \subset V_0$, we have that $C_{V_m}(t) \subseteq C_{V_{m-1}}(t) \subseteq \ldots \subseteq C_{V_1}(t) \subseteq C_{V_0}(t)$. For $i \in \{0, 1, \ldots, m-1\}$, let $f_i : C_{V_i}(t) \to C_{V_i}(t)/C_{V_{i+1}}(t)$ be the quotient map, then f_i is a surjection with kernel $C_{V_{i+1}}(t)$. Therefore, by the rank-nullity theorem,

$$\begin{aligned}
\dim(C_{V_0}(t)) &= \dim(C_{V_0}(t)/C_{V_1}(t)) + \dim(C_{V_1}(t)) \\
&= \dim(C_{V_0}(t)/C_{V_1}(t)) + \dim(C_{V_1}(t)/C_{V_2}(t)) + \dim(C_{V_2}(t)) \\
&= \dim(C_{V_0}(t)/C_{V_1}(t)) + \dim(C_{V_1}(t)/C_{V_2}(t)) + \ldots \\
&\quad \ldots + \dim(C_{V_{m-1}}(t)/C_{V_m}(t)).
\end{aligned}$$

For $i \in \{0, 1, \ldots, m-1\}$, let $\rho_i : V_i/C_{V_{i+1}}(t) \to V_i/V_{i+1}$ be the map $v + C_{V_{i+1}}(t) \mapsto v + V_{i+1}$ and r_i its restriction to $C_{V_i}(t)/C_{V_{i+1}}(t)$. Then it is easy to see that $\mathrm{im}(r_i) \leq C_{V_i/V_{i+1}}(t)$ and that $\ker(r_i) = 0$. Therefore, by the rank-nullity theorem, we have that $\dim(C_{V_i}(t)/C_{V_{i+1}}(t)) = \dim(\mathrm{im}(r_i)) \leq \dim(C_{V_i/V_{i+1}}(t))$ and so the lemma is proved. □

4.2 The Cases

In this section we construct copies of $L_2(16)$ in $E_8(2)$. There are eleven possible feasible decompositions of $L_2(16)$ on V_{248} as listed in B.2. The decomposition (i) would have a trivial submodule by Lemma 2.2.5(i). All of the irreducible characters of $L_2(16)$ over GF(2), $\phi_1, \ldots, \phi_6$, are self-dual and so by Lemma 2.2.5(iii), the decomposition (iv) would also have a trivial submodule.

We see in B.2 that if $L_2(16)$ is a subgroup of $E_8(2)$ following fusion pattern (ii), (iii), (v), (vi), ..., or (xi) then the conjugacy classes of elements of order 15 of $L_2(16)$ can fuse to any class of elements of order 15 of $E_8(2)$ apart from $15\mathrm{A}_{E_8(2)}$ and $15\mathrm{B}_{E_8(2)}$. Therefore we need to know which standard parabolic subgroups of $E_8(2)$ contain Levi-cuspidal subgroups $\langle x \rangle$ such that $x \notin 15\mathrm{A}_{E_8(2)} \cup 15\mathrm{B}_{E_8(2)}$. We have the following result by P. Rowley:

Lemma 4.2.1. *Suppose that $\langle x \rangle$ is a Levi-cuspidal subgroup of $E_8(2)$ where $\langle x \rangle \cong \mathbb{Z}_{15}$ and $x \notin 15A_{E_8(2)} \cup 15B_{E_8(2)}$. Then the possibilities for x and the Levi subgroups are itemised in the following table.*

Isomorphism type of Levi subgroup L	Number of L	L-cuspidal subgroups $\langle x \rangle$	$\dim(C_{V_{248}}(x))$
$L_4(2) \times \mathrm{Sym}(3)$	20	$x \in (15\mathrm{AB}_{L_4(2)}, 3\mathrm{A}_{\mathrm{Sym}(3)})$	26
$L_4(2) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$	10	$(15\mathrm{AB}_{L_4(2)}, 3\mathrm{A}_{\mathrm{Sym}(3)}, 3\mathrm{A}_{\mathrm{Sym}(3)})$	28
		$(5\mathrm{A}_{L_4(2)}, 3\mathrm{A}_{\mathrm{Sym}(3)}, 3\mathrm{A}_{\mathrm{Sym}(3)})$	24
$L_4(2) \times L_4(2)$	2	$(15\mathrm{A}_{L_4(2)}, 15\mathrm{A}_{L_4(2)})$	16
		$(15\mathrm{A}_{L_4(2)}, 15\mathrm{B}_{L_4(2)})$	16
		$(5\mathrm{A}_{L_4(2)}, 15\mathrm{AB}_{L_4(2)})$	20
		$(15\mathrm{AB}_{L_4(2)}, 5\mathrm{A}_{L_4(2)})$	20

Proof. Will be viewable in [7], once the paper is complete and made available. □

By the above lemma, there are 48 pairs of $O_2(P)$ and x to consider; we first address the first 20 arising from the parabolic subgroups P whose Levi complements are isomorphic to $L_4(2) \times \mathrm{Sym}(3)$.

Before that, we check in Magma that there is a single class of involutions in $L_2(16)$ and the dimension of the fixed space of an involution on the modules corresponding to

$\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ and ϕ_6 is $1, 4, 4, 8, 8$ and 16 respectively. Summing up the dimensions of the fixed spaces of an involution on the composition factors in the decompositions (ii), (iii), (v), (vi), …, or (xi) gives 132 in each case. Therefore by Lemma 4.1.2, we know that if $L_2(16)$ is a subgroup of $E_8(2)$ following (ii), (iii), (v), (vi), …, or (xi) then for any involution $t \in L_2(16)$, $\dim(C_{V_{248}}(t)) \leq 132$. Hence it must be that $t \in 2\mathrm{D}_{E_8(2)}$, see Proposition 2.2.1. If we come across a subgroup of $O_2(P)$ that doesn't have any involutions t with $\dim(C_{V_{248}}(t)) = 128$, we discard it.

4.2.1 Isomorphism Type $L_4(2) \times \mathrm{Sym}(3)$

Looking at the Dynkin diagram of E_8 (one may run `DynkinDiagram("E8")` in Magma), we see that the 20 standard parabolic subgroups with Levi complements isomorphic to $L_4(2) \times \mathrm{Sym}(3)$ are the ones associated to the roots labelled by

$$\{1,3,4,6\}, \{1,3,4,7\}, \{1,3,4,8\},$$

$$\{2,3,4,6\}, \{2,3,4,7\}, \{2,3,4,8\},$$

$$\{3,4,5,7\}, \{3,4,5,8\}$$

$$\{1,2,4,5\}, \{2,4,5,7\}, \{2,4,5,8\}$$

$$\{1,4,5,6\}, \{4,5,6,8\}$$

$$\{1,5,6,7\}, \{2,5,6,7\}, \{3,5,6,7\}$$

$$\{1,6,7,8\}, \{2,6,7,8\}, \{3,6,7,8\}, \{4,6,7,8\}.$$

The above sets label all possible subdiagrams of type $A_3 \times A_1$. Consider the first row of sets above, we have chosen the nodes labelled $1, 3$ and 4 to be the three nodes forming the Dynkin diagram of type A_3. This leaves 6, 7 or 8 as the possible choices for the fourth node. In the second row we have chosen the nodes labelled by $2, 3$ and 4 to form the Dynkin diagram of type A_3, and so on.

For $J \subset \{1, \ldots, 8\}$ being one of the above sets, we construct the standard Levi complement, $L_J \cong L_4(2) \times \mathrm{Sym}(3)$, of the corresponding parabolic subgroup, P_J. We see that there is a single class of subgroups $\langle x \rangle$ of order 15 in L_J with $\dim(C_{V_{248}}(x)) = 26$. Therefore by Lemma 4.2.1, we may choose x_J to be any element of order 15 in L_J

that has a fixed space of dimension 26. We also construct $Q_J = O_2(P_J)$. The groups L_J and Q_J are generated by the appropriate root subgroups.

For an element of order 15 in $E_8(2)$, the dimension of its fixed space in V_{248} completely determines which class it's in, see Theorem 2.2.2. Elements in $15\text{D}_{E_8(2)}$ have fixed spaces of dimension 26 and so here we are interested in constructing any $L_2(16)$'s that would follow fusion pattern (iii) or (vi).

We can check in Magma that out of the two 16-dimensional irreducible modules of $L_2(16)$ over GF(2), the one that is projective has Brauer character $(16, 0, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1, -1, -1, -1)$. This matches up with ϕ_5 in B.2. Looking at decompositions (iii) and (vi), the number of composition factors corresponding to ϕ_5 is 5 in both cases. Hence we are interested in collecting only those elementary abelian subgroups, $S \leq Q_J$, of order 2^4, irreducible under the action of x_J, such that $\dim(C_{V_{248}}(\langle S, x_J \rangle))$ is 5, see 3.1.

Remark 4.2.2. *In this chapter, the process of breaking up a group $O_2(P)$ given by Lemma 4.2.1, will involve running the code A.1 or certain lines from it. However A.1 was written for $L_2(2^n)$, $n = 6$, and now n is 4. Hence before running any part of A.1, we must replace any occurrences of 6 in it with 4 and of 63 with 15 $(= 2^n - 1)$. This is what we will always be doing in this chapter even if we don't mention that we are.*

In all cases apart from $J = \{1, 2, 4, 5\}$, we run the code A.1, taking `O` to be Q_J and `x63` to be x_J. The outcome at different stages of the code runs is given in the tables that follow.

	J			
	$\{1,3,4,6\}$	$\{1,3,4,7\}$	$\{1,3,4,8\}$	$\{2,3,4,6\}$
#FinSub	7	6	5	8
#BadSub	9	8	6	9
#ActnGpDiff	6	4	3	9
#FinSub	1031	616	296	4457
#BadSetNew	61	67	64	36
#ActnGpDiff	10	8	7	9
#SetKeepZero	0	0	0	0
#FinSub	5707	5728	5707	7659
#BadSetNew	3	3	3	6
#ActnGpDiff	10	8	7	9
#SetKeepZero	0	0	0	0
#FinSub	5707	5728	5707	7659
#BadSetNew	0	0	0	0
#ActnGpDiff	10	8	7	9
#SetKeepZero	0	0	0	0

Table 4.1: Running A.1 with Q_J and x_J.

	J			
	$\{2,3,4,7\}$	$\{2,3,4,8\}$	$\{3,4,5,7\}$	$\{3,4,5,8\}$
`#FinSub`	8	7	6	6
`#BadSub`	10	10	7	9
`#ActnGpDiff`	9	8	5	6
`#FinSub`	1257	1257	208	224
`#BadSetNew`	42	45	76	70
`#ActnGpDiff`	9	8	5	6
`#SetKeepZero`	0	0	0	0
`#FinSub`	7678	7677	3696	3712
`#BadSetNew`	6	6	0	0
`#ActnGpDiff`	9	8	5	6
`#SetKeepZero`	0	0	0	0
`#FinSub`	7678	7677		
`#BadSetNew`	0	0		
`#ActnGpDiff`	9	8		
`#SetKeepZero`	0	0		

Table 4.2: Running A.1 with Q_J and x_J.

	J			
	$\{2,4,5,7\}$	$\{2,4,5,8\}$	$\{1,4,5,6\}$	$\{4,5,6,8\}$
#FinSub	9	7	7	7
#BadSub	6	6	8	4
#ActnGpDiff	4	4	5	2
#FinSub	5273	1127	1048	314
#BadSetNew	96	96	81	63
#ActnGpDiff	4	4	10	6
#SetKeepZero	0	0	0	0
#FinSub	11449	7303	7277	5241
#BadSetNew	0	0	1452	122
#ActnGpDiff	4	4	27	6
#SetKeepZero	0	0	0	0
#FinSub			7277	7913
#BadSetNew			0	0
#ActnGpDiff			27	6
#SetKeepZero			1440	0

Table 4.3: Running A.1 with Q_J and x_J.

	J		
	$\{1,5,6,7\}$	$\{2,5,6,7\}$	$\{3,5,6,7\}$
`#FinSub`	7	10	9
`#BadSub`	8	6	6
`#ActnGpDiff`	6	4	7
`#FinSub`	830	4578	845
`#BadSetNew`	297	41	302
`#ActnGpDiff`	8	5	8
`#SetKeepZero`	0	0	0
`#FinSub`	4078	7590	4093
`#BadSetNew`	0	0	0
`#ActnGpDiff`	8	5	8
`#SetKeepZero`	0	0	0

Table 4.4: Running A.1 with Q_J and x_J.

	J			
	$\{1,6,7,8\}$	$\{2,6,7,8\}$	$\{3,6,7,8\}$	$\{4,6,7,8\}$
`#FinSub`	4	7	7	9
`#BadSub`	3	5	5	6
`#ActnGpDiff`	2	3	4	4
`#FinSub`	6	4456	26	602
`#BadSetNew`	4383	4395	4395	4395
`#ActnGpDiff`	3	4	5	5
`#SetKeepZero`	0	0	0	0
`#FinSub`	12006	12361	12025	12346
`#BadSetNew`	0	0	0	0
`#ActnGpDiff`	3	4	5	5
`#SetKeepZero`	0	0	0	0

Table 4.5: Running A.1 with Q_J and x_J.

For each of the 19 cases in the tables above, if b is a group in `ActnGpDiff` or `SetKeepZero` then $\dim(C_{V_{248}}(\langle b, x_J\rangle)) > 5$. For $J = \{1,3,4,6\}$, $\{1,3,4,7\}$, $\{1,3,4,8\}$,

$\{2,3,4,6\}$, $\{2,3,4,7\}$, $\{2,3,4,8\}$, $\{3,4,5,7\}$, $\{3,4,5,8\}$, $\{1,4,5,6\}$, $\{1,5,6,7\}$ or $\{3,5,6,7\}$, if b is a group in `FinSub` of order 2^4 then the dimension of $C_{V_{248}}(\langle b, x_J\rangle)$ is not 5, otherwise $\dim(C_{V_{248}}(\langle b, x_J\rangle)) > 5$.

For $J = \{2,4,5,7\}$, $\{2,4,5,8\}$, $\{4,5,6,8\}$, $\{2,5,6,7\}$, $\{1,6,7,8\}$, $\{2,6,7,8\}$, $\{3,6,7,8\}$, $\{4,6,7,8\}$, we also have that if $b \in$ `FinSub` with $|b| = 2^4$ then $\dim(C_{V_{248}}(\langle b, x_J\rangle)) \neq 5$. However in each case, there are groups $b \in$ `FinSub` with $|b| = 2^8, 2^{12}$ or 2^{20} such that $(C_{V_{248}}(\langle b, x_J\rangle)) \leq 5$; there are 3136, 448, 2688, 2689, 2520, 2688, 2520 and 2520 such groups respectively for the 8 cases. For each of these groups we find all the subgroups S of order 2^4 normalised by the relevant x_J that it contains and keep only those with $\dim(C_{V_{248}}(\langle S, x_J\rangle)) = 5$; the number of S's we are left with is 40320, 40320, 20160, 40320, 20160, 20160, 20160 and 20160 respectively for the 8 cases. Each one of these groups is such that any involution in it has a fixed space of dimension 138; hence we discard them all.

For $J = \{1,2,4,5\}$, we break up Q_J (by having it as the sole member of `SetSub2` and running the repeat loop from A.1 programmed to end when `#SetSub2 eq 0`) to get a `BadSub` of size 4 and then our first `BadSetNew` of size 1715. Along the way, 6841 groups have been added to `FinSub`, 18 to `ActnGpDiff` and none to `SetKeepZero` but we can discard these as well as 1441 of the 1715 groups since they all are such that the dimension of the fixed space of the group generated by any one of them and x_J is greater than 5. This leaves us with 274 of the groups in `BadSetNew` to worry about; the order of a group from among these will be $2^{57}, 2^{73}$ or 2^{77} and all but one of these are such that the Frattini quotient is a direct sum of 7 isomorphic irreducible 4-dimensional $\langle x_J\rangle$-modules. With 7 summands, the size of `SetKeep` can be 2097 and it can take approximately a day to go through this and the `SetSub2` formed along the way. This is why we are unable to run A.1 on Q_J in a single Magma session. We divide the 274 groups over twenty new Magma sessions that we run in parallel. In each session, we load either 13 or 14 of the 274 groups, collected together in a set we name `BadSub`; we break up these groups as normal (by running the for loop over `[1..#BadSub]` in A.1). The results are given in the following table.

#BadSub	**13**	**13**	**13**	**13**	**13**	**13**	**14**
#FinSub	3058	3346	3058	2770	2914	3058	2946
#BadSetNew	0	0	0	0	0	0	0
#ActnGpDiff	16	16	16	16	16	16	16
#SetKeepZero	0	0	0	0	0	0	0
#BadSub	**14**	**14**	**14**	**14**	**14**	**14**	**14**
#FinSub	3234	3234	3378	2946	3090	3378	3522
#BadSetNew	0	0	0	0	0	0	0
#ActnGpDiff	16	16	16	16	16	16	16
#SetKeepZero	0	0	0	0	0	0	0
#BadSub	**14**	**14**	**14**	**14**	**14**	**14**	
#FinSub	3090	3378	3522	3090	5219	7542	
#BadSetNew	0	0	0	0	2304(0)	3751(7)	
#ActnGpDiff	16	16	16	16	16	17	
#SetKeepZero	0	0	0	0	0	0	
#FinSub						7542	
#BadSetNew						0	
#ActnGpDiff						17	
#SetKeepZero						0	

Table 4.6: Breaking up the 274 groups.

The 14 groups from the 247th till the 260th group of the 274 give a `BadSetNew` of size 2304, but no b in `BadSetNew` is such that $\dim(C_{V_{248}}(\langle b, x_J \rangle))$ is less than or equal to 5. The last 14 of the 274 groups give a `BadSetNew` of size 3751; out of all the groups b in `BadSetNew`, there's only 7 such that $\dim(C_{V_{248}}(\langle b, x_J \rangle)) \leq 5$ and these are the only ones that we proceed with (by having them make up the new `BadSub` in the same session, setting `BadSetNew` as empty and running the for loop over `[1..#BadSub]` again). For every group b in any of the twenty `FinSub`'s or twenty `ActnGpDiff`'s, $\dim(C_{V_{248}}(\langle b, x_J \rangle)) > 5$.

Let $O_2(P)$ and x be any pair from among the first 20 pairs given by Lemma 4.2.1, we have established in this subsection that if there exists an elementary abelian subgroup $S \leq O_2(P)$ of order 2^4 irreducible under the action of x and an involution

$t \in G$ inverting x such that $H := \langle S, x, t\rangle$ is isomorphic to $L_2(16)$ then H would fix a non-zero vector in V_{248}. We move on to considering the next 20 pairs.

4.2.2 Isomorphism Type $L_4(2) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$

The 10 standard parabolic subgroups with Levi complements isomorphic to $L_4(2) \times \mathrm{Sym}(3) \times \mathrm{Sym}(3)$ are the ones associated to the roots labelled by

$$\{1,3,4,6,8\},$$

$$\{2,3,4,6,8\},$$

$$\{1,2,4,5,7\},\{1,2,4,5,8\},$$

$$\{1,4,5,6,8\},$$

$$\{1,2,5,6,7\},\{2,3,5,6,7\},$$

$$\{1,2,6,7,8\},\{1,4,6,7,8\},\{2,3,6,7,8\}.$$

For $J \subset \{1,\ldots,8\}$ being one of the above sets, we construct the standard Levi complement, L_J, of the corresponding parabolic subgroup, P_J. We see that there is a single class of subgroups, $\langle x\rangle$, of order 15 in L_J with $\dim(C_{V_{248}}(x)) = 24$ and also a single class with $\dim(C_{V_{248}}(x)) = 28$. Therefore by Lemma 4.2.1 we may choose $x_{J,24}$ to be any element of order 15 in L_J with a fixed space of dimension 24, and $x_{J,28}$ to be any element of order 15 with a fixed space of dimension 28, as generators of the L_J-cuspidal subgroups we are after. We also construct $Q_J = O_2(P_J)$.

Elements in $15\mathrm{E}_{E_8(2)}$ have fixed spaces of dimension 24 and the ones in $15\mathrm{C}_{E_8(2)}$ of dimension 28, see Theorem 2.2.2. Hence when working with the pairs $Q_J, x_{J,24}$, we are interested in constructing any $L_2(16)$'s that would follow fusion pattern (ii); (vii) or (x) when working with the pairs $Q_J, x_{J,28}$. The number of composition factors corresponding to the Steinberg module is 4 in (ii), and 6 in each of (vii) and (x).

We first consider all pairs $Q_J, x_{J,24}$ and for each, run the code A.1 after replacing any occurrences of 6 in the code with 4 and of 63 with 15, as was done in 4.2.1. The outcome at different stages of the code runs is given in the tables that follow.

	J			
	$\{1,3,4,6,8\}$	$\{2,3,4,6,8\}$	$\{1,2,4,5,7\}$	$\{1,2,4,5,8\}$
`#FinSub`	6	5	7	7
`#BadSub`	8	10	8	8
`#ActnGpDiff`	5	4	4	6
`#FinSub`	40	151	719	132
`#BadSetNew`	44	592	134	422
`#ActnGpDiff`	5	4	5	7
`#SetKeepZero`	0	0	0	0
`#FinSub`	992	993	735	180
`#BadSetNew`	0	0	142	142
`#ActnGpDiff`	5	4	5	8
`#SetKeepZero`	0	0	0	0
`#FinSub`			911	308
`#BadSetNew`			0	0
`#ActnGpDiff`			5	8
`#SetKeepZero`			0	0

Table 4.7: Running A.1 with Q_J and $x_{J,24}$.

	J		
	$\{1,4,5,6,8\}$	$\{1,2,5,6,7\}$	$\{2,3,5,6,7\}$
#FinSub	7	10	9
#BadSub	8	10	10
#ActnGpDiff	5	6	5
#FinSub	73	810	6217
#BadSetNew	22	578	578
#ActnGpDiff	6	6	5
#SetKeepZero	0	0	0
#FinSub	255	6218	6217
#BadSetNew	142	0	0
#ActnGpDiff	7	6	5
#SetKeepZero	0	0	0
#FinSub	303		
#BadSetNew	0		
#ActnGpDiff	7		
#SetKeepZero	0		

Table 4.8: Running A.1 with Q_J and $x_{J,24}$.

	J		
	$\{1,2,6,7,8\}$	$\{1,4,6,7,8\}$	$\{2,3,6,7,8\}$
#FinSub	4	6	5
#BadSub	6	10	6
#ActnGpDiff	4	4	2
#FinSub	198	198	6215
#BadSetNew	578	578	578
#ActnGpDiff	4	4	2
#SetKeepZero	0	0	0
#FinSub	6214	6214	6215
#BadSetNew	0	0	0
#ActnGpDiff	4	4	2
#SetKeepZero	0	0	0

Table 4.9: Running A.1 with Q_J and $x_{J,24}$.

For each of the 10 cases in the tables above, if b is a group in `FinSub` then $\dim(C_{V_{248}}(\langle b, x_{J,24}\rangle)) > 4$; hence we are not interested in b. For $J = \{1,2,4,5,7\}$, $\{1,2,4,5,8\}$, $\{1,2,5,6,7\}$ or $\{2,3,5,6,7\}$, there are groups b in `ActnGpDiff` such that $\dim(C_{V_{248}}(\langle \Phi(b), x_{J,24}\rangle)) \leq 4$, we deal with them, for an empty output, as explained in Section 3.1, starting by adding the subgroups $\Phi(b)$ to an empty `SetSub2`.

For all J apart from $\{1,3,4,6,8\}$ and $\{2,3,4,6,8\}$, we run A.1 with Q_J and $x_{J,28}$ to get the results given in the following tables.

	J			
	$\{1,2,4,5,7\}$	$\{1,2,4,5,8\}$	$\{1,4,5,6,8\}$	$\{1,2,5,6,7\}$
#FinSub	11	9	11	11
#BadSub	7	9	11	7
#ActnGpDiff	6	6	3	6
#FinSub	6032	1935	1761	387
#BadSetNew	434	479	446	226
#ActnGpDiff	6	6	3	7
#SetKeepZero	0	0	0	0
#FinSub	51108	47011	101153	14159
#BadSetNew	209	209	269	0
#ActnGpDiff	6	6	3	7
#SetKeepZero	0	0	0	0
#FinSub	51172	47075	102113	
#BadSetNew	0	0	0	
#ActnGpDiff	6	6	3	
#SetKeepZero	0	0	0	

Table 4.10: Running A.1 with Q_J and $x_{J,28}$.

	J			
	$\{2,3,5,6,7\}$	$\{1,2,6,7,8\}$	$\{1,4,6,7,8\}$	$\{2,3,6,7,8\}$
`#FinSub`	10	5	8	5
`#BadSub`	6	6	9	5
`#ActnGpDiff`	5	4	5	3
`#FinSub`	158	279	300	279
`#BadSetNew`	214	497	813	497
`#ActnGpDiff`	6	4	5	3
`#SetKeepZero`	0	0	0	0
`#FinSub`	14158	63191	63212	63191
`#BadSetNew`	0	0	0	0
`#ActnGpDiff`	6	4	5	3
`#SetKeepZero`	0	0	0	0

Table 4.11: Running A.1 with Q_J and $x_{J,28}$.

For each of the 8 cases in the two preceding tables, there are groups b in `ActnGpDiff` such that $\dim(C_{V_{248}}(\langle\Phi(b), x_{J,28}\rangle)) \leq 6$, we deal with these as explained in Section 3.1 for an empty output. Also, if $b \in$ `FinSub` with $|b| = 2^4$ then $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$. However there are groups $b \in$ `FinSub` with $|b| = 2^8$ or 2^{12} such that $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \leq 6$; there are 52, 52, 52, 100, 100, 25, 28 and 25 such groups respectively for the 8 cases. For each of these groups we find all the subgroups S of order 2^4 normalised by the relevant $x_{J,28}$ that it contains and find that every S is such that $\dim(C_{V_{248}}(\langle S, x_{J,28}\rangle)) \neq 6$.

For $J = \{1,3,4,6,8\}$, working with the pair $Q_J, x_{J,28}$, we break up Q_J to get a `BadSub` of size 9 and then our first `BadSetNew` of size 387. Along the way, 61819 groups have been added to `FinSub`, 3 to `ActnGpDiff` and 1 to `SetKeepZero`. For $b \in$ `SetKeepZero`, $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) > 6$ and for $b \in$ `ActnGpDiff`, $\dim(C_{V_{248}}(\langle\Phi(b), x_{J,28}\rangle)) > 6$. If $b \in$ `BadSetNew` then $b/\Phi(b)$ is a direct sum of either $3, 4, 5, 6$ or 8 isomorphic irreducible 4-dimensional $\langle x_{J,28}\rangle$-modules; there are 268, 15, 6, 81 and 17 such groups, respective to the number of summands. We partition the collection of 81 groups into 8 sets; we load a partition into a separate Magma session, naming it `BadSub`. We also load the collections of groups of size 268, 15 and 6 as `BadSub` into three different Magma sessions, one collection per session. We break up the groups in these 11 sets (all named `BadSub`) as normal and get the results shown in the following table.

`#BadSub`	**268**	**15**	**6**	**10**	**10**	**10**	**10**
`#FinSub`	12306	6	340	40722	19730	2322	10514
`#BadSetNew`	65	12	497	0	0	0	0
`#ActnGpDiff`	0	0	0	0	0	0	0
`#SetKeepZero`	0	0	0	0	0	0	0
`#FinSub`	14627	23	63076				
`#BadSetNew`	0	0	0				
`#ActnGpDiff`	0	0	0				
`#SetKeepZero`	0	0	0				
`#BadSub`	**10**	**10**	**10**	**11**			
`#FinSub`	2834	10002	2834	20754			
`#BadSetNew`	0	0	0	0			
`#ActnGpDiff`	0	0	0	0			
`#SetKeepZero`	0	0	0	0			

Table 4.12: Breaking up 370 of the 387 groups; the Frattini quotient of each is a direct sum of $3, 4, 5$ or 6 isomorphic irreducible 4-dimensional modules.

We now consider the groups in the 12 `FinSub`'s that have been formed so far. All the groups in the 8 `FinSub`'s of sizes 40772, 19730, 2322, 10514, 2834, 10002, 2834 and 20754 are of order 2^4 and the dimension of the fixed space of the group generated by any one of them and $x_{J,28}$ is not 6. This leaves us with 4 `FinSub`'s of sizes 61819, 14627, 23, and 63076. After discarding any group b from each of these sets such that $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) > 6$ or $|b| = 2^4$ and $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$, we are left with sets of sizes 4, 12, 4 and 48, respectively. The union of these four sets is of size 52 and any subgroup S of size 2^4 normalised by $x_{J,28}$ of any one of the 52 groups, is such that $\dim(C_{V_{248}}(\langle S, x_{J,28}\rangle)) \neq 6$.

We are left with 17 of the 387 groups, each having a Frattini quotient that is a direct sum of 8 isomorphic irreducible 4-dimensional $\langle x_{J,28}\rangle$-modules. These are our first examples of groups, b, where the number of irreducible summands in the decomposition of $b/\Phi(b)$ is large enough so that calculating and going through `SetKeep` might be impractical. Therefore for b, one of the 17 groups, with $b/\Phi(b)$ isomorphic to $V_1 \oplus \ldots \oplus V_8$, we define `Fb` to be the preimage of $V_1 \oplus \ldots \oplus V_4$ instead of `FrattiniSubgroup(b)`, see Section 4.1. Doing this can give a `SetKeep` of size 4369 and going through approximately 1000 of these 4369 elements can take a few hours; so we divide the 17 groups over 10 Magma sessions. A non-empty `BadSetNew` will be output in each session (after running the for loop over `[1..#BadSub]` but with `Fb` defined differently) and a group in it can be broken up as usual by considering its Frattini quotient. See the following table to know what happens in the 10 sessions.

`#BadSub`	**2**	**2**	**2**	**2**	**2**
`#FinSub`	0	0	0	0	0
`#BadSetNew`	8738	8738	8738	8738	8738
`#ActnGpDiff`	0	0	0	0	0
`#SetKeepZero`	0	0	0	0	0
`#FinSub`	100626	100626	100626	100626	139538
`#BadSetNew`	0	0	0	0	0
`#ActnGpDiff`	0	0	0	0	0
`#SetKeepZero`	0	0	0	0	0
`#BadSub`	**2**	**2**	**1**	**1**	**1**
`#FinSub`	0	0	0	0	0
`#BadSetNew`	8738	8738	4369	4369	4369
`#ActnGpDiff`	0	0	0	0	0
`#SetKeepZero`	0	0	0	0	0
`#FinSub`	139538	100626	81170	81170	100626
`#BadSetNew`	0	0	0	0	0
`#ActnGpDiff`	0	0	0	0	0
`#SetKeepZero`	0	0	0	0	0

Table 4.13: Breaking up 17 of the 387 groups by factoring each out by the preimage of 4 summands in the decomposition of its Frattini quotient.

Any group in any one of the ten `FinSub`'s in the table above has order 2^4 and the dimension of the fixed space of the group generated by it and $x_{J,28}$ is not 6.

We now move on to dealing with the pair $Q_J, x_{J,28}$, for $J = \{2, 3, 4, 6, 8\}$. We break up Q_J to get a BadSub of size 9. One of these 9 groups has a Frattini quotient that is the direct sum of 7 isomorphic irreducible modules and another has a Frattini quotient that is the direct sum of 9 modules; we name these groups b_7 and b_9, respectively, and take them out of BadSub. We will consider b_7 and b_9 later. Breaking up the 7 remaining groups in the same session gives a BadSetNew of size 4383, FinSub of size 13595, ActnGpDiff of size 2 and a SetKeepZero of size 1. If $b \in$ ActnGpDiff then $\dim(C_{V_{248}}(\langle \Phi(b), x_{J,28}\rangle)) > 6$; if $b \in$ SetKeepZero then $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) > 6$. From FinSub, we take out all groups b such that $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) > 6$ or $|b| = 2^4$ and $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$. We get left with just one elementary abelian group of order 2^{12} and any subgroup, S, of this of order 2^4 normalised by $x_{J,28}$ is such that $\dim(C_{V_{248}}(\langle S, x_{J,28}\rangle)) \neq 6$.

Remark 4.2.3. *From now onwards, if mention of* ActnGpDiff *or* SetKeepZero *has been omitted from results of code runs then it's because they remain empty.*

We now consider the group b_7 in a new session. Running the for loop over [1..#BadSub] we see that the number of preimages of certain vectors in $b_7/\Phi(b_7)$ collected in SetKeep will be 61713. A FinSub of size 61714 is returned; all of these elementary abelian groups, b, are of order 2^4 with $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$.

Next, we consider b_9, with $b_9/\Phi(b_9)$ isomorphic to $V_1 \oplus \ldots \oplus V_9$. We take Fb to be the preimage of $V_1 \oplus \ldots \oplus V_5$ instead of $\Phi(b_9)$ but then proceed as normal. Just a BadSetNew of size 4369 is output. It takes approximately a day to break up 100 of these groups by considering the Frattini quotients. Hence we divide the groups over 10 Magma sessions; see the table below.

#BadSub	**440**	**440**	**440**	**440**	**440**
#FinSub	94446	43959	43978	43903	44021
#BadSetNew	0	0	0	0	0
#BadSub	**440**	**440**	**440**	**440**	**409**
#FinSub	43863	43881	43925	43767	44183
#BadSetNew	0	0	0	0	0

Table 4.14: Breaking up the 4639 subgroups of b_9.

Any group in any one of the ten FinSub's in the table above is such that its order is 2^4 and the dimension of the fixed space of the group generated by it and $x_{J,28}$ is not 6.

There is a `BadSetNew` of size 4383 left to consider. This set contains a group whose Frattini quotient is a direct sum of 10 isomorphic irreducible 4-dimensional $\langle x_{J,28}\rangle$-modules; we name this group b_{10} and consider it separately. With $b_{10}/\Phi(b_{10}) \cong V_1 \oplus \ldots \oplus V_{10}$, we take `Fb` to be the preimage of $V_1 \oplus \ldots \oplus V_5$, `A` to be the preimage of $\Phi(\texttt{Fb}/[b_{10}, \texttt{Fb}])$ and acquire `SetKeep`; this will be of size 69905. We partition `SetKeep` over 21 MAGMA sessions, loading a partition of size 3410 in each of the first 20 sessions and of size 1705 in the last one. We also load the groups `Fb` and `A` into each session; `Fb` is needed to generate the groups `Sub4aa` (preimages of submodules of $b_{10}/\texttt{Fb}$) and `A` is what we factor these groups by. Going through the 21 `SetKeep`'s (we run the for loop over `[1..#SetKeep]` immediately followed by the repeat loop programmed to end when `#SetSub2 eq 0`) outputs a `BadSetNew` of size 3410 in each of the first 20 sessions and of size 1705 in the last. In a day, around 120 groups in a `BadSetNew` can be broken down by considering the Frattini quotients. All we ever get by breaking up the groups in the `BadSetNew`'s, are elementary abelian groups b of order 2^4 such that $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$. The `BadSetNew` of size 1705 outputs a `FinSub` of size 144050. For a `BadSetNew` of size 3410, the size of `FinSub` increases to around 200000 after going through approximately 2400 of the groups and it is at this point that we interrupt the for loop over `[1..#BadSub]`, empty out `FinSub` and restart the loop, iterating over an appropriate subsequence `[k..#BadSub]`; we do this because having a large `FinSub` takes up too much memory.

It is left to consider 4382 groups and we split these over 11 MAGMA sessions; see the following table.

#BadSub	**400**	**400**	**400**	**400**	**400**
#FinSub	36114	31506	28050	30930	29202
#BadSetNew	0	0	0	0	0
#BadSub	**400**	**400**	**400**	**400**	**400**
#FinSub	30354	29778	30482	30226	118290
#BadSetNew	0	0	0	0	0
#BadSub	**382**				
#FinSub	165164				
#BadSetNew	21862				
#ActnGpDiff	1				
#SetKeepZero	1				

Table 4.15: Breaking up all the groups in the `BadSetNew` of size 4383 apart from b_{10}.

We now consider the non-empty groups in the above table. Any group in any of the first 10 `FinSub`'s is such that its order is 2^4 and the dimension of the fixed space of the group generated by it and $x_{J,28}$ is not 6. From the `FinSub` of size 165164, we take out any group b such that $\dim(C_{v_{248}}(\langle b, x_{J,28}\rangle)) > 6$ or $|b| = 2^4$ and $\dim(C_{v_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$, and are left with 4 groups of order 2^{12}. Any subgroup S of order 2^4 normalised by $x_{J,28}$ of any one of these 4 groups is such that $\dim(C_{v_{248}}(\langle S, x_{J,28}\rangle)) \neq 6$. If $b \in$ `ActnGpDiff` then $\dim(C_{v_{248}}(\langle \Phi(b), x_{J,28}\rangle)) > 6$; if $b \in$ `SetKeepZero` then $\dim(C_{v_{248}}(\langle b, x_{J,28}\rangle)) > 6$.

Testing the code on a few of the groups in the `BadSetNew` of size 21862, it seems like it'd take around 15 minutes on average to break up one of 21862 groups. It also looks like a lot of (too many to collect them all if breaking up several groups in `BadSetNew` together in a single session) elementary abelian groups, b, of order 2^4 will be output but none of them such that $\dim(C_{v_{248}}(\langle b, x_{J,28}\rangle)) = 6$. We partition the 21862 into six sets of sizes 3644, 3644, 3644, 3644, 3643 and 3643. We open six parallel MAGMA sessions and load the six sets into them, one each. Each set has been named `BadSub` as usual. We try to see if any of the groups in a `BadSub` are conjugate to each other via elements that centralise $x_{J,28}$ (see Lemma 4.1.1); in order to do this we first calculate a subgroup of $C_{P_J}(x_{J,28})$.

The order of $P_J = \langle Q_J, L_J\rangle$ is $2^{120}.3^4.5.7$. This is too big a subgroup of $GL_{248}(2)$

for us to comfortably perform computations in and we are unable to get a permutation representation of it. We can however factor out P_J by its soluble radical using the command `LMGRadicalQuotient` and get $\overline{P_J}$ as a permutation group. Asking for centralisers of elements in a permutation setting works much better (indeed several group theoretic operations work much faster in permutation or pc-group settings) and so we ask for $C_{\overline{P_J}}(\overline{x_{J,28}})$ using the command `Centraliser`. This gives us a group whose preimage K is a proper subgroup of P_J containing $C_{P_J}(x_{J,28})$. The order of K is $2^{114}.3^3.5$; this is less than $|P_J|$ but we are still unable to directly ask for $C_K(x_{J,28})$ by using `Centraliser`. We don't need to calculate all of $C_K(x_{J,28}) = C_{P_J}(x_{J,28})$ anyway; Lemma 4.2.4 will help us calculate a subgroup of $C_K(x_{J,28})$ which will prove to be enough for our purpose of finding conjugating elements.

Lemma 4.2.4. *Given a group G, let R and H be subgroups of G, V a G-module and W the fixed space of H in V. Then $N_R(H)$ is contained in $Stab_R(W)$.*

Proof. For any $g \in N_R(H)$ and $v \in W$, we need to show that $g.v \in W$. For any $h \in H$, $h.g.v = g.h'.v$ (for some $h' \in H$) $= g.v$ and the lemma is proved. □

Remark 4.2.5. *For us G, H and V from Lemma 4.2.4 will be $E_8(2)$, a 2-group and V_{248}, respectively. We will then be able to use the command* `UnipotentStabiliser` *to find $Stab_H(W)$, which will be a smaller subgroup of H containing $N_H(R)$; possibly small enough to calculate all of $N_H(R)$ in. Essentially, pairing Lemma 4.2.4 with the command* `UnipotentStabiliser` *enables one to find normalisers of groups in large unipotent subgroups of $E_8(2)$. This method will make a reappearance in a later chapter.*

The soluble radical of P_J is a subgroup of K of order $2^{114}.3^2$ and so contains a Sylow 2-subgroup of K. Since the size of the soluble radical is less than $|K|$, we prefer to run the command `LMGSylow` on the soluble radical rather than on K. We obtain a Sylow 2-subgroup R of K. The index of R in K is 135; this is small enough to allow a smooth run of the command `Transversal`. We obtain Γ as a right transversal of R in K. The set of all Sylow 2-subgroups of K, $\{R^{r\gamma} : r \in R, \gamma \in \Gamma\}$ can of course be calculated as $\{R^\gamma : \gamma \in \Gamma\}$. We find that K has 9 Sylow 2-subgroups, $R_1, \ldots, R_9$. Let $W = C_{V_{248}}(x_{J,28})$, we compute the group $U = \langle \mathrm{Stab}_{R_i}(W) : i \in \{1, \ldots, 9\}\rangle$. We find that $U \leq C_G(x_{J,28})$ and that $|U| = 2^{10}$. We take `cpx` to be the group $\langle U, x_{J,28}\rangle \leq C_{P_J}(x_{J,28})$ in each of our 6 sessions and run A.2.

We run A.2 until we get `#ind` (= `#orbs`, see A.2) as 30, 166, 214, 218, 260 and 223 respectively in our 6 sessions; these will be the sizes of our new `BadSub`'s, since each `BadSub` is replaced by a subset of itself containing the groups indexed by `ind`. Every group in the original `BadSub` will be conjugate to a group in the replacement via an element that centralises $x_{J,28}$ (see A.2 and Section 4.1). Before breaking up groups b in any of our six `BadSub`'s we make changes to the for loop over `[1..#SetKeep]`: If `A` (the preimage of $\Phi(\Phi(b)/[b, \Phi(b)])$) is ever trivial then subgroups, `IncGrp`, of `Sub4aa` (see A.1) are added to `FinSub` only if $|\texttt{IncGrp}| = 2^4$ and $\dim(C_{V_{248}}(\langle \texttt{IncGrp}, x_{J,28}\rangle)) = 6$ or $|\texttt{IncGrp}| \neq 2^4$ and $\dim(C_{V_{248}}(\langle \texttt{IncGrp}, x_{J,28}\rangle)) \leq 6$. In each session apart from the first, a `BadSetNew` of size 12 is output (the five `BadSetNew`'s across the sessions are not all the same), we take this to be the new `BadSub`, set `BadSetNew` as empty, and run the usual for loop over `[1..#BadSub]`. In each of the five sessions the final size of `FinSub` is 39; we take out all the groups b from `FinSub` such that $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) > 6$ or $|b| = 2^4$ and $\dim(C_{V_{248}}(\langle b, x_{J,28}\rangle)) \neq 6$. None of the 4 groups we are left with contain any subgroups S of order 2^4 normalised by $x_{J,28}$ such that $\dim(C_{V_{248}}(\langle S, x_{J,28}\rangle)) = 6$.

Let Q_J, x be any one of the pairs $Q_J, x_{J,24}$ or $Q_J, x_{J,28}$, we have established in this subsection that if there exists an elementary abelian subgroup $S \leq Q_J$ of order 2^4 irreducible under the action of x and an involution $t \in G$ inverting x such that $H := \langle S, x, t\rangle$ is isomorphic to $L_2(16)$ then H would fix a non-zero vector in V_{248}. We move on to considering the last 8 pairs given by Lemma 4.2.1.

4.2.3 Isomorphism Type $L_4(2) \times L_4(2)$

The 2 standard parabolic subgroups with Levi complements isomorphic to $L_4(2) \times L_4(2)$ are the ones associated to the roots labelled by

$$\{1, 3, 4, 6, 7, 8\}, \{2, 3, 4, 6, 7, 8\}.$$

Let J be either one of the above sets, we construct the standard Levi complement, L_J, of P_J. We see that L_J has 14 subgroups of order 15 up to conjugacy and given such a subgroup, any two elements g, h of order 15 in it are such that $\dim(C_{V_{248}}(g))$ $= \dim(C_{V_{248}}(h))$. Only two of the 14 subgroups are such that every element, x, of order 15 in them has a fixed space of dimension 16 and only two more are such that $\dim(C_{V_{248}}(x)) = 20$. By Lemma 4.2.1, these 4 subgroups must be the L_J-cuspidal subgroups we are after. We call the generators of the four subgroups as $x^1_{J,16}$, $x^2_{J,16}$, $x^1_{J,20}$ and $x^2_{J,20}$, respectively. We also compute $Q_J = O_2(P_J)$.

Since $\dim(C_{V_{248}}(x^i_{J,16})) = 16$ and $\dim(C_{V_{248}}(x^i_{J,20})) = 20$, by Theorem 2.2.2, $x^i_{J,16}$ is in $15\mathrm{G}_{E_8(2)}$ and $x^i_{J,20}$ is in $15\mathrm{F}_{E_8(2)}$. Hence when working with the pairs $Q_J, x^i_{J,16}$, we are interested in constructing any $L_2(16)$'s that would follow fusion pattern (viii) or (xi) (see B.2); (v) or (ix) when working with the pairs $Q_J, x^i_{J,20}$. The number of composition factors corresponding to the Steinberg module is 0 in each of (viii) and (xi), and 2 in each of (v) and (ix).

Remark 4.2.6. *Note that if we find any overgroup $H \cong L_2(16)$ of $\langle S, x^i_{J,16}\rangle$, S an elementary abelian group of order 2^4 irreducible under the action of $x^i_{J,16}$, then H will not fix any non-zero vectors in V_{248} since we will have chosen S so that $\langle S, x^i_{J,16}\rangle$ doesn't. This means that we won't be able to immediately discard H or any of its extensions as being non-maximal in $E_8(2)$ using Proposition 2.2.3.*

We run A.1 with the pairs $Q_J, x^i_{J,16}$ and $Q_J, x^1_{J,20}$ for $J = \{2, 3, 4, 6, 7, 8\}$ and also with $Q_J, x^i_{J,20}$ for $J = \{1, 3, 4, 6, 7, 8\}$; see table below.

	$\{2,3,4,6,7,8\}$			$\{1,3,4,6,7,8\}$	
	$x^1_{J,16}$	$x^2_{J,16}$	$x^1_{J,20}$	$x^1_{J,20}$	$x^2_{J,20}$
#FinSub	6	7	7	6	7
#BadSub	6	10	6	6	8
#ActnGpDiff	4	4	4	3	3
#FinSub	552	4136	117	41	1048
#BadSetNew	4402	275	21	24	312
#ActnGpDiff	4	6	4	3	4
#SetKeepZero	0	0	0	0	0
#FinSub	841	4393	374	1399	10793
#BadSetNew	31	0	48	48	257
#ActnGpDiff	4	6	4	3	4
#SetKeepZero	0	0	0	0	1
#FinSub	871		1158	1399	10921
#BadSetNew	0		0	0	0
#ActnGpDiff	4		4	3	4
#SetKeepZero	0		0	0	193

Table 4.16: Running A.1 with five of the eight pairs under consideration.

We first look at the non-empty sets output in the cases $x^i_{J,16}$, $i \in \{1,2\}$, $J = \{2,3,4,6,7,8\}$. If $b \in$ `ActnGpDiff` for the $x^1_{J,16}$ case then $\langle \Phi(b), x^1_{J,16}\rangle$ fixes at least one non-zero vector. We stumble upon groups b in `ActnGpDiff` for the $x^2_{J,16}$ case such that $\dim(C_{V_{248}}(\langle \Phi(b), x^2_{J,16}\rangle)) = 0$. We deal with `ActnGpDiff`, for an empty output, as explained in Section 3.1, starting by adding the Frattini subgroups of the groups in this `ActnGpDiff` to an empty `SetSub2`.

We now turn our attention to the `FinSub` of size 871 for $J = \{2,3,4,6,7,8\}$. There are 240 groups b in `FinSub` such that $|b| = 2^4$ and $\langle b, x^1_{J,16}\rangle$ does not fix any non-zero vectors in V_{248}; we collect these elementary abelian groups in a set we name $E^1_{J,16}$. There are also 287 groups, b, in `FinSub` such that $|b| = 2^8$ and $\langle b, x^1_{J,16}\rangle$ does not fix any non-zero vectors. We find all subgroups S of b of order 2^4 normalised by $x^1_{J,16}$ and add those such that $\dim(C_{V_{248}}(\langle S, x^1_{J,16}\rangle)) = 0$ to $E^1_{J,16}$. We get that $|E^1_{J,16}| = 3600$. Working in the same way with the `FinSub` of size 4393, we create a set $E^2_{J,16}$ of size

6960. Every involution in every group in $E^1_{J,16}$ or $E^2_{J,16}$ is in $2\mathrm{D}_{E_8(2)}$. We come back to these two sets later.

We now deal with the non-empty sets output in the remaining 3 cases in Table 4.16. Let b be a 2-group and x the relevant element of order 15 acting on it. If b is in any one of the three `ActnGpDiff`'s then $\dim(C_{V_{248}}(\langle \Phi(b), x\rangle)) > 2$. If b is in the `SetKeepZero` of size 193 then $\dim(C_{V_{248}}(\langle b, x\rangle)) > 2$. There is a group b in the `FinSub` of size 10921 with $|b| = 2^{12}$ and $\dim(C_{V_{248}}(\langle b, x^2_{J,20}\rangle)) \leq 2$, $J = \{1,3,4,6,7,8\}$, such that $\langle x^2_{J,20}\rangle$, doesn't act faithfully on it, and so we discard b. This is the first time we have come across such an elementary abelian group. Working through the three `FinSub`'s in a similar way to the two before, we create sets $E^1_{J,20}$, $E^2_{J,20}$ (for $J = \{1,3,4,6,7,8\}$) and $E^1_{J,20}$ (for $J = \{2,3,4,6,7,8\}$) of sizes 480, 5760 and 480, respectively. Each of these sets contains elementary abelian subgroups b of order 2^4 such that the dimension of the fixed space of the subgroup generated by b and the relevant element of order 15 is exactly 2 and any involution in b is in $2\mathrm{D}_{E_8(2)}$.

For $J = \{1,3,4,6,7,8\}$, let $x = x^1_{J,16}$. Working with the pair Q_J, x, we break up Q_J to get a `BadSub` of size 5 but only 4 of these groups b are such that $C_{V_{248}}(\langle b, x\rangle)$ is zero; we care about these 4 groups only. We also get a `FinSub` of size 8 and an `ActnGpDiff` of size 3 (if $b \in$ `ActnGpDiff` then $C_{V_{248}}(\langle \Phi(b), x\rangle)$ is non-zero). There is a group in `BadSub` with a Frattini quotient isomorphic to a direct sum of 5 irreducible submodules. We calculate the subgroups `Fb` (the Frattini) and `A` of this group as normal and acquire a `SetKeep` of size 8465. We split `SetKeep` over 8 Magma sessions and also load `Fb` and `A` into each session; see the following table to know what happens when we run appropriate parts of A.1.

#SetKeep	**1100**	**1100**	**1100**	**1100**
#FinSub	38	38	39	41
#BadSetNew	1101	1101	1101	1101
#ActnGpDiff	2	2	2	3
#SetKeepZero	0	0	0	0
#FinSub	582	310	311	313
#BadSetNew	7	0	1	114
#ActnGpDiff	2	2	2	3
#SetKeepZero	1021	1027	1025	962
#FinSub	589		312	350
#BadSetNew	0		0	0
#ActnGpDiff	2		2	3
#SetKeepZero	1021		1025	1007
#SetKeep	**1065**	**1000**	**1000**	**1000**
#FinSub	7	7	7	9
#BadSetNew	1066	1001	1001	1001
#ActnGpDiff	3	3	3	3
#SetKeepZero	0	0	0	0
#FinSub	284	284	284	285
#BadSetNew	2066	1926	1941	1895
#ActnGpDiff	3	3	3	3
#SetKeepZero	0	0	0	15
#FinSub	357	355	356	358
#BadSetNew	0	0	0	0
#ActnGpDiff	3	3	3	3
#SetKeepZero	945	870	885	870

Table 4.17: Dealing with the elements in the `SetKeep` of size 8465.

If b is a group in any one of the eight `ActnGpDiff`'s in Table 4.17 then dimension of $C_{V_{248}}\langle b, x\rangle$ is non-zero. If b is a group in any one of the eight `SetKeepZero`'s then dimension of $C_{V_{248}}(\langle \Phi(b), x\rangle)$ is non-zero.

There are still 3 groups in the `BadSub` obtained from breaking up Q_J left to consider.

Running the for loop over `[1..#BadSub]` on these 3 groups returns a `BadSetNew` of size 45, running the loop again with `BadSetNew` as the new `BadSub` returns a `FinSub` of size 53 as the only non-empty set.

A total of ten `FinSub`'s, of sizes 8, 589, 310, 312, 350, 357, 355, 356, 358 and 53, have been formed during our computations. The union of these sets has size 886. Working through the groups in this union in the usual way we form a set $E^1_{J,16}$ of size 3600 containing groups b of order 2^4 such that $\langle b, x\rangle$ doesn't fix any non-zero vectors and every involution in b is in $2\mathrm{D}_{E_8(2)}$.

Now for $J = \{1, 3, 4, 6, 7, 8\}$, let $x = x^2_{J,16}$. Working with the pair Q_J, x, we break up Q_J to get a `BadSub` of size 7. There are two groups, b_5 and b_7, in `BadSub` whose Frattini quotients are isomorphic to a direct sum of 5 and 7 irreducible modules, respectively. We take b_5 and b_7 out of `BadSub`. We then proceed to break up the remaining groups in `BadSub` to get 17 groups in `BadSetNew` which in turn break up to return a `FinSub` of size 4122 and an `ActnGpDiff` of size 5. There are groups, b, in `ActnGpDiff` such that $\langle \Phi(b), x\rangle$ doesn't fix non-zero vectors. We treat this `ActnGpDiff` in the same way as the `ActnGpDiff` in the $x^2_{J,16}$, $J = \{2, 3, 4, 6, 7, 8\}$ case was treated, and get an empty output.

We can run the for loop over `[1..#BadSub]` with `BadSub:={@` b_7 `@}` to get a `FinSub` of size 483 as the only non-empty set. It does take a little while to calculate `SetKeep` but it turns out to be a small set of size 481.

Working with b_5, we calculate the groups `Fb` $(= \Phi(b_5))$ and `A` and acquire a `SetKeep` of size 4385 which we split over four MAGMA sessions; see table below.

#SetKeep	**1100**	**1100**	**1100**	**1085**
#FinSub	4	4	4	6
#BadSetNew	1102	1102	1102	1103
#ActnGpDiff	2	2	2	3
#SetKeepZero	0	0	0	0
#FinSub	1171	945	958	1192
#BadSetNew	0	0	0	4353
#ActnGpDiff	2	2	2	4
#SetKeepZero	0	0	0	0
#FinSub				1208
#BadSetNew				0
#ActnGpDiff				4
#SetKeepZero				4080

Table 4.18: Dealing with the elements in the `SetKeep` of size 4385.

Let b be a group in any `ActnGpDiff` or `SetKeepZero` in the above table, then $\langle \Phi(b), x \rangle$ will fix at least one non-zero vector.

A total of six `FinSub`'s, of sizes 4122, 483, 1171, 945, 958 and 1208, have been formed during our computations. The union of these sets has size 8010. Working through the groups in this union in the usual way we form the set $E^2_{J,16}$ containing groups b of order 2^4 such that $\langle b, x \rangle$ doesn't fix any non-zero vectors and $\langle x \rangle$ acts irreducibly on b; we find that every involution in b is in $2\mathrm{D}_{E_8(2)}$ and the size of $E^2_{J,16}$ is 57360.

We finally address the last pair Q_J, x of Lemma 4.2.1, where $J = \{2, 3, 4, 6, 7, 8\}$ and $x = x^2_{J,20}$. We break up Q_J to get a `BadSub` of size 10 and then our first `BadSetNew` of size 548. Only 275 of the groups in `BadSetNew` are such that the dimension of the group generated by any one of them and x is less than or equal to 2. Along the way 10840 group have been added to `FinSub`, 2 to `ActnGpDiff` and 1 to `SetKeepZero`. If $b \in$ `ActnGpDiff` then $\dim(C_{V_{248}}(\langle \Phi(b), x \rangle)) > 2$. If $b \in$ `SetKeepZero` then $\dim(C_{V_{248}}(\langle b, x \rangle)) > 2$. Mostly, the Frattini quotients of the 275 groups are isomorphic to direct sums of 7 irreducible modules and it seems like MAGMA can work through 5 in around a day. We split the 275 groups over ten MAGMA sessions and run the for loop over `[1..#BadSub]` in each session; see the table below.

`#BadSub`	**30**	**30**	**30**	**30**	**30**
`#FinSub`	29610	29610	29834	30058	30170
`#BadSetNew`	0	0	0	0	0
`#BadSub`	**30**	**30**	**25**	**20**	**20**
`#FinSub`	30002	30002	25142	20058	11745
`#BadSetNew`	0	0	0	0	0

Table 4.19: Breaking up 275 of the groups in `BadSetNew`.

We have 11 `FinSub`'s of sizes 10840, 29610, 29610, 29834, 30058, 30170, 30002, 30002, 25142, 20058 and 11745 to consider. Each of these sets has a group of order bigger than 2^4 such that the dimension of the fixed space of the group generated by it and x is less than or equal to 2, but $\langle x\rangle$ doesn't act faithfully on it and so we discard it. We create sets ${}^1E^2_{J,20}$, ${}^2E^2_{J,20}$, ${}^3E^2_{J,20}$, ${}^4E^2_{J,20}$, ${}^5E^2_{J,20}$, ${}^6E^2_{J,20}$, ${}^7E^2_{J,20}$, ${}^8E^2_{J,20}$, ${}^9E^2_{J,20}$, ${}^{10}E^2_{J,20}$ and ${}^{11}E^2_{J,20}$, each containing those subgroups b of the groups in the respective `FinSub` such that $|b| = 2^4$, $\dim(C_{V_{248}}(\langle b, x\rangle)) = 2$ and x acts irreducibly on b. We find that every involution in b will be in $2\mathrm{D}_{E_8(2)}$. The sizes of the 11 sets created turn out to be 5760, 22230, 22230, 22590, 22950, 23130, 22860, 22860, 19185, 15150 and 6495, respectively.

In this subsection we have constructed 18 sets of elementary abelian groups of order 2^4 and we must now see if we can build up any of the groups to a copy of $L_2(16)$.

4.2.4 Constructing Copies of $L_2(16)$

From the previous subsection, we carry over elements x of order 15 and the corresponding sets E of elementary abelian groups of order 2^4. For a given pair of x and E, x acts irreducibly on each group, S, in E and $\dim(C_{V_{248}}(\langle S, x\rangle))$ is either 0 or 2. We also know that any involution in S is in $2\mathrm{D}_{E_8(2)}$. By Lemma 3.1.1, we must now go through all involutions t in $E_8(2)$ inverting x and check whether $\langle S, x, t\rangle$ is isomorphic to $L_2(16)$. Recall that it follows from Lemma 4.1.2 that t must be in $2\mathrm{D}_{E_8(2)}$. The pairs x and E are listed in Table 4.20.

	x	E	$\|E\|$
$J = \{1,3,4,6,7,8\}$	$x^1_{J,16}$	$E^1_{J,16}$	3600
	$x^2_{J,16}$	$E^2_{J,16}$	57360
	$x^1_{J,20}$	$E^1_{J,20}$	480
	$x^2_{J,20}$	$E^2_{J,20}$	5760
$J = \{2,3,4,6,7,8\}$	$x^1_{J,16}$	$E^1_{J,16}$	3600
	$x^2_{J,16}$	$E^2_{J,16}$	6960
	$x^1_{J,20}$	$E^1_{J,20}$	480
	$x^2_{J,20}$	${}^1E^2_{J,20}$	5760
		${}^2E^2_{J,20}$	22230
		${}^3E^2_{J,20}$	22230
		${}^4E^2_{J,20}$	22590
		${}^5E^2_{J,20}$	22950
		${}^6E^2_{J,20}$	23130
		${}^7E^2_{J,20}$	22860
		${}^8E^2_{J,20}$	22860
		${}^9E^2_{J,20}$	19185
		${}^{10}E^2_{J,20}$	15150
		${}^{11}E^2_{J,20}$	6495

Table 4.20: The 18 pairs of x and E.

We will now get our hands on all the involutions inverting x, where x is one of the 8 elements of order 15 listed in Table 4.20. For $G = E_8(2)$, consider the extended centraliser $C^*_G(x) = \{g \in G : x^g = x \text{ or } x^g = x^{-1}\}$ of x. Let $t \in G$ be any involution such that $x^t = x^{-1}$, then $C^*_G(x) = \langle C_G(x), t\rangle$: Let $g \in C^*_G(x)$ so that $x^g = x^{-1}$ then $g = (gt)t$, where $gt \in C_G(x)$. Hence, given that we have $C_G(x)$, if we can find a single involution inverting x, we can find them all.

Our x is in $15\mathrm{F}_{E_8(2)}$ or $15\mathrm{G}_{E_8(2)}$ and so by Theorem 2.2.2, we know that $x^3 \in 5\mathrm{B}_{E_8(2)}$. Also, $C^*_G(x) \leq C^*_G(x^3)$ and so we attempt to construct $C^*_G(x^3)$ since centralisers of elements in $5\mathrm{B}_{E_8(2)}$ are readily available to us. The centraliser of an element in $5\mathrm{B}_{E_8(2)}$ is given to us by Neuhaus, and we take this group as the fourth argument of `FindCent` (see [42]); G, x^3 and 4 are taken as the first three arguments, where 4 is the dimension of the non-trivial irreducible $\langle x^3 \rangle$-module over GF(2). Running `FindCent` then gives us $C_G(x^3)$.

Let L_J be the standard Levi complement containing x^3 then running `LMGClasses` shows us that L_J has three classes of elements of order 5, one containing x^3 ($5\mathrm{C}_{L_J}$) and two containing elements in $5\mathrm{A}_{E_8(2)}$ ($5\mathrm{AB}_{L_J}$). We can randomly search in L_J for elements $f_1 \in 5\mathrm{A}_{L_J}$ and $f_2 \in 5\mathrm{B}_{L_J}$ so that both centralise x^3. Taking the copy of the centraliser of an element in $5\mathrm{A}_{E_8(2)}$ calculated by Neuhaus, we use `FindCent` to compute $C_G(f_1)$ and $C_G(f_2)$. These centralisers contain x^3 and are a good selection of subgroups of G in which we may search for an involution inverting x^3.

We factor out $C_G(f_1)$ by its soluble radical and calculate the preimage N_1, of the normaliser of $\langle \overline{x^3} \rangle$ in $\overline{C_G(f_1)}$. The order of N_1 is $2^6.3^2.5^4$ and it doesn't contain any elements inverting x^3. Hence we add to N_1, the preimage of the normaliser of $\langle \overline{x^3} \rangle$ in $\overline{C_G(f_2)}$ to get an overgroup N_2 of order $2^8.3^2.5^4$. Searching in N_2 we do indeed find an involution r inverting x^3. We have the group $C_G^*(x^3) = \langle C_G(x^3), r \rangle$ of order $2^{21}.3^2.5^5.13.17.41$.

We ask for the centraliser in the radical quotient, $\overline{C_G^*(x^3)}$, of $\overline{x}$ and then for the normaliser of this centraliser. The preimage of this normaliser will contain $C_G^*(x)$ and so in this preimage we ask for the centraliser of x and also search for an involution t, inverting x. We have the wanted group $\langle C_G(x), t \rangle$ of order either $2^5.3^2.5^2.17$ or $2^7.3^2.5^3.13$.

We first consider all pairs x, E from Table 4.20 with $\dim(C_{V_{248}}(x)) = 20$; there are 14 such pairs. The order of $C_G^*(x)$ will be $2^7.3^2.5^3.13$. Going through all elements of $C_G^*(x)$, we collect all involutions that invert x in a set we name `I1`; there are 15600 such involutions and they are all in $2\mathrm{D}_{E_8(2)}$. We now introduce a way of cutting down the number, 15600.

Let $t \in$ `I1` then $\langle x, t \rangle \cong \mathrm{Dih}(30)$ is a subgroup of $C_G^*(x)$ containing 14 other involutions of `I1`, each of which along with an $S \in E$ and x would generate the same group, $\langle S, x, t \rangle$. Therefore going through every involution $t \in$ `I1` to see if $\langle S, x, t \rangle$, $S \in E$, could be isomorphic to $L_2(16)$ is redundant. By running the following code we collect all involutions, in a set called `I2`, such that each along with $x =$ `x15` would generate a distinct subgroup of $C_G^*(x)$ isomorphic to $\mathrm{Dih}(30)$; we get the size of `I2` as 1040.

```
I2:={Random(I1)};
for t in I1 do
if forall{g : g in I2 | t notin sub<Q|x15,g>} then
Include(~I2,t);
end if;
end for;
```

For practicality, we should first check that the order of $\langle S, x, t\rangle$, $S \in E$, $t \in$ `I2` equals $|L_2(16)|$ before checking for isomorphism. But even checking the order of all possible groups $\langle S, x, t\rangle$ is not the best if a lot of them will be large. Note that if $\langle S, x, t\rangle$ turns out to be a large subgroup of $E_8(2)$ then there are a lot of possibilities for the orders of its elements just because the same is true for elements of $E_8(2)$. The set of possible orders of elements in $L_2(16)$ is $\{1, 2, 3, 5, 15, 17\}$. If the orders of certain chosen elements of $\langle S, x, t\rangle$ are not all in $\{1, 2, 3, 5, 15, 17\}$ then we disregard $\langle S, x, t\rangle$; this is employed in code as follows.

```
subE:=[]; subI2:=[];
for S in E do
for t in I2 do
if {Order(s*t) : s in S} subset {1,2,3,5,15,17} then
Append(~subE,S); Append(~subI2,t);
end if;
end for;
end for;
```

Note that if $\langle S, x, t\rangle \cong L_2(16)$ then $\langle S, x, t\rangle = \langle S, t\rangle$. In the above code `E`:= E and our choice of elements whose orders we check and the way of collecting groups S and involutions t that pass this check is very similar to [45]. The above code returns a sequence, `subE`, of groups in E and a sequence, `subI2`, of involutions in `I2` and we must now check if the ith term of `subE` along with the ith term of `subI2` generates a group of order equal to $|L_2(16)|$. Note that checking a group S in E against 1040 involutions indeed proves to be a lot more practical than having $|E| \times 15600$ iterations. Just to give an idea to the reader of the code run times, we mention that if $|E|$ is approximately 22000 then it can take around 2 weeks for the above code to finish running.

Working with the pairs x, E in Table 4.20 with $\dim(C_{V_{248}}(x)) = 20$, we get that in all 14 cases the sequences `subE` and `subI2` are returned as empty. Working with a pair x, E with $\dim(C_{V_{248}}(x)) = 16$, we again have that every involution in $C^*_G(x)$ that inverts x is in $2\mathrm{D}_{E_8(2)}$. The size of `I2` will be 256. This time we do get a non-empty `subE`. Let $m = |\texttt{subE}| = |\texttt{subI2}|$ and then for $i \in \{1, \ldots, m\}$, let S_i be the ith term of `subE` and t_i the ith term of `subI2`, we construct the set $L = \{\langle S_i, t_i\rangle : i \in \{1, \ldots, m\}\}$. We get that $|L| = m$ and that every group in L is isomorphic to $L_2(16)$. See the following table.

	x	E	$\lvert E\rvert$	m	$\lvert L\rvert$
$J = \{1, 3, 4, 6, 7, 8\}$	$x^1_{J,16}$	$E^1_{J,16}$	3600	3600	3600
	$x^2_{J,16}$	$E^2_{J,16}$	57360	57360	57360
$J = \{2, 3, 4, 6, 7, 8\}$	$x^1_{J,16}$	$E^1_{J,16}$	3600	3600	3600
	$x^2_{J,16}$	$E^2_{J,16}$	6960	6960	6960

Table 4.21: The number of copies of $L_2(16)$ that we have constructed in $E_8(2)$.

Let H be any copy of $L_2(16)$ in $E_8(2)$ that we have constructed above. We know that H will not fix any non-zero vectors in V_{248} since it contains a subgroup that doesn't either (see Remark 4.2.6). The order of $\mathrm{Aut}(H)$ is $2^6.3.5.17$; to show that H or any of its automorphic extensions can't be maximal in $E_8(2)$, we construct an overgroup of H bigger that this, but smaller than $E_8(2)$. To this end we have the following lemmas.

Lemma 4.2.7. *Given a group G, let H be a subgroup of G, g an element of $N_G(H)$ and V a G-module. If W is an irreducible H-submodule of $V{\downarrow}H$ then so is W^g.*

Proof. It is easy to see that W^g is a subspace of V, actually isomorphic to the underlying vector space of W. Let $h \in H$, then $W^{gh} = W^{h'g}$ (for some $h' \in H$) $= W^g$. Hence W^g is a H-module.

Let $U \neq 0$ be a proper H-submodule of W^g. Then we have just proved that $U^{g^{-1}}$ is a proper non-zero H-submodule of W, a contradiction. Hence W^g is irreducible. □

Lemma 4.2.8. *Given a group G, let H be a subgroup of G, g an element of $N_G(H)$, and V a G-module. If W is the socle of $V{\downarrow}H$, then $W^g = W$.*

Proof. The socle of $V \downarrow H$ is the sum of all the irreducible H-submodules of $V \downarrow H$, say $W = W_1 + W_2 + \ldots + W_k$. Then by Lemma 4.2.7, $W^g = W_1^g + W_2^g + \ldots + W_k^g$ is also a sum of some irreducible H-submodules. For $i \in \{1, \ldots, k\}$, we have that $W_i^{g^{-1}}$, being an irreducible H-submodule, appears as a summand in W and so W_i appears as a summand in W^g. Therefore $W^g = W$. □

Going back to H being a copy of $L_2(16)$ in $E_8(2)$ that we have constructed, let W be the socle of $V_{248} \downarrow H$, then we have learned that any extension of H would stabilise W. In all our cases, W will have a non-zero dimension less than 248, hence its stabiliser can't be all of $E_8(2)$ (V_{248} is irreducible). Therefore if H, or an extension, is a maximal subgroup of $E_8(2)$, then it will be equal to the stabiliser of W in $E_8(2)$. In order to show non-maximality, we construct a partial stabiliser of W in $E_8(2)$ of order bigger than $2^6.3.5.17$.

Let $0 < k < 248$ be the dimension of W. Asking for W in MAGMA using the command `Socle` returns W with $\mathbb{F}_2^k$ as its underlying vector space; our 248-dimensional matrices (elements of $E_8(2)$) can't act on vectors of dimension k. The code we present below incorporates a solution, as suggested by Ballantyne, to this problem.

```
kspace:=VectorSpace(GF(2),248);
ProbL216s:={@@};
ustabos:={};
for i in [1..#L216s] do
gp:=L216s[i];
gpM:=GModule(gp); //The restriction of the 248-dimensional
//G-module to gp.
W:=Socle(gpM);
Wphi:=Morphism(W,gpM);
genW:=[kspace!Wphi(v): v in Generators(W)];
gpN:=sub<kspace|genW>; //The socle as a subspace of the
//248-dimensional vector space
//over GF(2).
ustab:=UnipotentStabiliser(O,gpN);
ustabo:=Order(ustab);
```

```
Include(~ustabos,Factorisation(ustabo));
if ustabo le 2^6 then Include(~ProbL216s,gp); end if;
if i mod 1000 eq 0 then i; end if;
end for;
#ProbL216s;
#ustabos;
ustabos;
```

In the above code, `L216s` is one of our (indexed) sets L from Table 4.21 containing groups H isomorphic to $L_2(16)$ and `O` is the unipotent radical Q_J, $J = \{1, 3, 4, 6, 7, 8\}$ or $\{2, 3, 4, 6, 7, 8\}$. The code calculates the stabiliser `ustab` in `O` of the socle of $V_{248}{\downarrow}H$. If the order of `ustab` is less than or equal to 2^6 then H is added to a set called `ProbL216s`. Running the code with all four sets L in parallel, we see that the order of `ustab` is either $2^{20}, 2^{24}, 2^{38}, 2^{40}, 2^{42}, 2^{44}, 2^{54}$ or 2^{56} and so `ProbL216s` always remains empty. We have the following theorem.

Theorem 4.2.9. *If H is a subgroup of $E_8(2)$ such that $F^*(H) \cong L_2(16)$ then H is not maximal in $E_8(2)$.*

Chapter 5

$L_2(8)$

In this chapter, we make partial progress towards establishing whether $L_2(8)$ can be maximal in $E_8(2)$. To do this we first build up on the methodology given in Sections 3.1 and 4.1.

5.1 Methodology

As usual we will need a list of pairs of $O_2(P)$ and x, where P is a standard parabolic subgroup of $E_8(2)$ containing a Levi-cuspidal subgroup $\langle x \rangle$ of order 7. This list will be given in the next section. The group $O_2(P)$ needs to be broken down and during the process of doing so, we will encounter elementary abelian subgroups and also groups b such that $b/\Phi(b)$ is a direct sum of isomorphic 3-dimensional irreducible $\langle x \rangle$-modules, say $V_1 \oplus \ldots \oplus V_k$. The elementary abelian groups will be added to sets called `FinSub` and the groups b to sets called `BadSub` (or `BadSetNew`). It may be gleaned from the previous chapter that things that can get in the way of having a smooth run of the program A.1 are (a) the size of `BadSub` becomes too big, or (b) the number of summands, k, is too big. In the $L_2(8)$ case we will very frequently encounter these problems and so in this section we introduce more ways of countering them. But before that, we recall some notation.

Let b with $b/\Phi(b)$ being $V_1 \oplus \ldots \oplus V_k$ be a group in `BadSub`, then `Fb` is the Frattini subgroup of b or the preimage of the sum of the first r of the k summands. The group `A` is the preimage of $\Phi(\mathtt{Fb}/[b, \mathtt{Fb}])$. The set `SetKeep` contains those preimages in b of certain vectors in $b/\mathtt{Fb}$ that square into `A`. Let $t \in$ `SetKeep`, $\mathtt{Sub4aa} = \langle \mathtt{Fb}, t^{x^i} : i \in$

$\{1, \ldots, 7\}\rangle$. Whenever we encounter a Frattini quotient or a quotient `Sub4aa/A` that is not a direct sum of isomorphic 3-dimensional irreducible modules, we consider its submodule generated by isomorphic 3-dimensional summands and add its preimage to a set called `SetSub2`; there can be more than one such submodule. See A.1 and Sections 3.1 and 4.1 for more information.

Note that in the $L_2(8)$ case, given a `BadSub`, before we attempt to break up a group $b \in$ `BadSub`, we will often calculate its order first. If the order is large then we will proceed to calculate the number k and the size of `SetKeep`. Although, we don't attempt to calculate `SetKeep` if $k \geq 11$. This information associated to b will help us establish the best way to break up b into smaller subgroups. Information on groups in `BadSub` will be given more often in the $L_2(8)$ case than was given in the $L_2(16)$ case. No such information was calculated in the $L_2(64)$ case since for every parabolic subgroup of $E_8(2)$ arising from Lemma 3.2.1, it was possible to run A.1 and finish within realistic time.

We now present solutions to the problem of sizes of sets in which we collect subgroups of $O_2(P)$ becoming too large.

- We will see in the next section that we are interested in a subgroup of $O_2(P)$ only if it, along with x, generates a group whose fixed space has dimension less than or equal to 5. Previously we have checked groups collected in `FinSub` against a similar condition and discarded them if the condition was not met; at times we did the same to groups in `BadSub`. In the $L_2(8)$ case, if we break up a group $b \in$ `BadSub` by running the for loop over `[1..#SetKeep]` followed by the repeat loop programmed to end when `#SetKeep eq 0`, we don't add a subgroup of b to `SetSub2`, `BadSetNew` or `FinSub` at all if the dimension of the fixed space of the group generated by the subgroup and x is greater than 5. Note that the code was modified once in a similar way before towards the end of 4.2.2.

  Given a group $b \in$ `BadSub`, the for loop over `[1..#SetKeep]` breaks up b by computing its subgroups `Sub4aa`. The number of subgroups computed equals `#SetKeep` of course. The loop breaks up a `Sub4aa` by computing preimages of certain submodules of `Sub4aa/A`. Each preimage is a subgroup of `Sub4aa` and is added to `SetSub2` or `FinSub`. As stated above, now we don't add a subgroup of `Sub4aa` to `SetSub2` or `FinSub` if the dimension of the fixed space of the group generated by it and x is greater than 5.

While performing computations for the $L_2(8)$ case, groups b were encountered such that running the new for loop over `[1..#SetKeep]` on any one b yielded an empty `SetSub2` and `FinSub`. After investigating, it was found that every group `Sub4aa` $\leq b$ was such that $\dim(C_{V_{248}}(\langle \texttt{Sub4aa}, x \rangle))$ was greater than 5. Hence, the same would hold true for any subgroup of $\langle \texttt{Sub4aa}, x \rangle$, and this is why `SetSub2` and `FinSub` would be returned as empty.

But if $\dim(C_{V_{248}}(\langle \texttt{Sub4aa}, x \rangle)) > 5$, then computing any subgroups of it is redundant. Hence we adjust the code to always ignore such `Sub4aa`'s. Note that if a group b is small, say of order 2^{31}, then it is likely that many of its subgroups `Sub4aa` are such that $\dim(C_{V_{248}}(\langle \texttt{Sub4aa}, x \rangle)) > 5$, and we will very often come across large `BadSub`'s containing small groups. Making the mentioned adjustment to the code may allow us to deal with these `BadSub`'s a lot faster than before.

This adjustment also means that any small group b with large k, giving rise to a very large `SetKeep`, no longer needs to be factored out by an `Fb` that is the preimage of the sum of the first $0 < r < k$ summands. Test running code with different values, in order to choose the best one for r, may be avoided in favour of running a code that may work even better than any non-zero value for r we could choose.

- We simply turn the sets into sequences at the end of which new items will be appended rather than MAGMA first checking if an item is already in a collection. The code will then output `BadSetNew` and `FinSub` as sequences of, possibly, non-distinct groups which we may then turn into sets if we wish.

  Given an indexed set `BadSub`, the for loop over `[1..#BadSub]` considers the first group b in `BadSub`, and for every `Sub4aa` $\leq b$ adds appropriate subgroups of `Sub4aa` to `SetSub2` or `FinSub`. The loop then defines `SetSub` as `SetSub2`, `SetSub2` as empty and breaks up every group in `SetSub` into smaller subgroups, with each subgroup being added to `SetSub2`, `FinSub` or `BadSetNew`. The process of breaking up the groups in (the new) `SetSub2` is repeated, and so on, until an empty `SetSub2` is returned. The loop then moves on to considering the next group in BadSub, and so on. The sizes of the sets SetSub2, FinSub and BadSetNew will affect the speed of the loop.

Say that we run the for loop over [1..#BadSub] on a given BadSub but with SetSub2, FinSub and BadSetNew as sequences. If after the code run we decide to turn BadSetNew into a set (to obtain a list of distinct groups), then this would be equivalent to not having changed it to a sequence in the first place. But it could've been that more than one large SetSub2's were created while running the for loop, and so it's important in this case that the SetSub2's remain as sequences even if we decide to not have BadSetNew as a sequence. Another reason to keep SetSub2's as sequences is that groups added to them may be bigger than the ones added to BadSetNew and forming a set of large objects is slower than forming a set, of the same size, of smaller objects.

Note that if after having gotten the sequences BadSetNew and FinSub of non-distinct groups, we find that these groups are small, it can be much faster to perform subsequent calculations on a given group more than once rather than converting the sequences into sets first.

We now give an example demonstrating that switching from sets to sequences can largely decrease the time taken to run the code. A particular pair of $O_2(P)$ and x from among the ones listed in the next section will be such that $O_2(P)$ will contain 24 groups of order 2^{55}, each having a Frattini quotient that is a direct sum of 7 isomorphic 3-dimensional irreducible $\langle x \rangle$-modules. Any one of the 24 groups will give rise to a SetKeep of size 3017. We choose one particular group of order 2^{55} and call it b_7. Running the for loop over [1..#BadSub] on b_7 and collecting subgroups of it in sets takes around a day and a half to give a BadSetNew of size 3017 and a FinSub of size 1. Running the loop again but now collecting groups in sequences takes less than 12 hours to give a BadSetNew of size 3017 (so we know all these groups will actually be distinct) and a FinSub also of size 3017; we know that all groups in FinSub will be the same so it'll be better to keep it as a set in this case. Just like in this example, we will quite often have that the groups produced to be added to BadSetNew will all be distinct and so in these cases having it and SetSub2 as sequences will work exceptionally well for us.

- Let's say we have `BadSub` as a collection of groups that are not too big and on which the for loop over `[1..#BadSub]` seems to be running smoothly, but a lot of groups are being added to the set or sequence `BadSetNew` (this will happen quite often). The loop would run smoothly on the even smaller groups in `BadSetNew` as well and this could result in an even bigger `BadSetNew` being formed subsequently. Instead of collecting large `BadSetNew`'s we fix our original `BadSub` as `OrigBadSub`, and run the for loop on just the first group in it. We then keep running the loop on any `BadSetNew`'s that arise, breaking this first group all the way down to its elementary abelian subgroups. After this we move on to the second group in `OrigBadSub` and do the same. Rather than dealing with the `BadSetNew`'s one by one, this is a slightly different way of automating the process, by keeping on running the for loop on all `BadSetNew`'s that arise until an empty one is output, than the one in A.1. This method means that we don't have to worry about using too much memory forming large `BadSetNew`'s and could even, at times, have them as sets of distinct groups rather than sequences.

  On occasions, it will be better to break down all elementary abelian subgroups collected in `FinSub` into subgroups of order 2^3 and reset `FinSub` as empty before moving on to collect elementary abelian subgroups of the next group in `OrigBadSub`. If we don't do this then as more and more groups are added to `FinSub` and its size increases, either the code will slow down too quickly (if we have `FinSub` as a set) or it'll keep running at the same speed but too much memory will get used up (if we have `FinSub` as a sequence).

  As subgroups of $O_2(P)$ get smaller the sizes of `BadSetNew`'s get bigger and we have just discussed a way of bypassing constructions of large `BadSetNew`'s. Note that A.2 is a way of downsizing `BadSub` but it is mainly a tool against large subgroups of $O_2(P)$ and won't work well with a `BadSub` of size over, say, 4000; the size of `BadSetNew` can well exceed this if smaller groups of size approximately 2^{30} are being added to it. For example consider the set `BadSetNew` of size 3017 containing subgroups of b_7; the possible orders for these groups are $2^{21}, 2^{24}, 2^{27}$ and 2^{28}. Picking a second group of order 2^{55} from among the 24 and running the for loop over `[1..#BadSub]` on it will add 3016 more groups to our existing `BadSetNew` of size 3017. Running the for loop on all the 24 groups together seems to have the potential of returning a `BadSetNew` containing approximately 3017×24 distinct groups.

Let b be such that $b/\Phi(b)$ is $V_1 \oplus \ldots \oplus V_k$, as before. The bigger k is, the bigger `#SetKeep` will be. We now discuss solutions to the problem of coming across a large k, or k is not large but there are a lot of groups b to go through. In the latter case, even if sizes of the `SetKeep`'s associated to the groups b are not large, running code on all the groups together would take too long unless sizes of the `SetKeep`'s are decreased.

- We will very often come across large `BadSub`'s containing groups of order approximately 2^{30}. Let b have order $\leq 2^{30}$, very often we will see that $b/Z(b)$ is elementary abelian. If so then $\Phi(b) \leq Z(b)$ and $b/Z(b)$ will be a direct sum of $\leq k$ isomorphic modules. We would then take `Fb` to be $Z(b)$ and if this is bigger than the Frattini then the `SetKeep` produced will be of a smaller size and so the for loop over `[1..#BadSub]` on b will run faster than if we were to keep `Fb` as $\Phi(b)$.

  Continuing to look at the example of the `BadSetNew` of size 3017, we have that 2297 of these have order $\leq 2^{24}$ among which are those whose quotient by the centre is elementary abelian. Running the for loop over `[1..#BadSub]` on the 2297 groups but with `Fb` as the centre whenever the quotient is elementary abelian, as the Frattini otherwise, takes around 6 hours (the next `BadSetNew` is output as empty but `FinSub` will be of size 1153). In the same time, the loop runs through only 133 of the 2297 groups if we take `Fb` to always be the Frattini. Note that in addition to taking `Fb` as the centre whenever possible, if we adjust the code to ignore any `Sub4aa`'s such that $\dim(C_{V_{248}}(\langle \texttt{Sub4aa}, \texttt{x} \rangle)) > 5$ as suggested in the first bullet point, we see that the loop takes just an hour and a half to finish running.

Remark 5.1.1. *The code incorporating all of the methods explained in the above bullet points is given in A.3, where, also,* `SetKeepZero` *will now contain* `Fb`*'s instead of* `b`*'s.*

- We've said in the first bullet point that if k, and so `SetKeep`, is large then as long as the order of the group b is small we can get away with factoring out with

$\Phi(b)$ if we ignore any `Sub4aas`'s such that the dimension of the fixed space of $\langle$`Sub4aa`$, x\rangle$ is > 5; there can be many such `Sub4aa`'s since smaller groups tend to fix a bigger subspace of V_{248}. However if $|b|$ is large then we have no choice but to take `Fb` as the preimage of the sum of the first r summands in $V_1 \oplus \ldots \oplus V_k$ (see Section 4.1 for more details). We've had to do this for some of the groups we came across in 4.2.2, but here we present a different way of going about it.

Very often we will choose r so that $k-r$ is 4 or 5. This means that the number of vectors in $b/$`Fb` whose preimages are considered for inclusion in `SetKeep` is 585 or 4681, respectively. Frequently, it was seen that `#SetKeep` turned out to be exactly 585 or 4681 and moreover, every `Sub4aa` was such that `Sub4aa/A` was a direct sum of isomorphic modules (and so `SetSub2` was just the set of all `Sub4aa`'s), as was the Frattini quotient of `Sub4aa` (and so `BadSetNew` equalled `SetSub2`).

If `BadSetNew` is going to be output as the set of all `Sub4aa`'s then instead wasting hours on calculating a quotient of every `Sub4aa` and mapping the preimage of the entire quotient back into $GL_{248}(2)$, two times, collecting the preimage in `SetSub2` the first time and `BadSetNew` the second (the process is especially slow if `SetSub2` and `BadSetNew` are sets instead of sequences), as soon as `SetKeep` has been calculated we should simply calculate and collect all groups $\langle$`Fb`$, t^{\langle x\rangle}\rangle$, $t \in$ `SetKeep`, to immediately obtain all `Sub4aa`'s in a sequence. We collect the `Sub4aa`'s in a sequence since quite often we will see that all, or many of them, are distinct. We call this sequence `OrigBadSub`.

In short, whenever `#SetKeep` equals $2^{3(k-r-1)} + 2^{3(k-r-2)} + \ldots + 2^3 + 1$, we take this as an indication that `BadSetNew` is likely to be output as the collection of all `Sub4aa`'s and instead of going through the process of calculating `BadSetNew`, we simply put all `Sub4aa`'s in a sequence called `OrigBadSub`.

Given our `OrigBadSub`, it is sometimes possible that a group b in it is such that $b/\Phi(b)$ has an irreducible module not isomorphic to all of the others after all. To account for this, instead of dealing with `OrigBadSub` as explained in the third bullet point above, we straight away add this b to `SetSub2`, skipping the for loop over `[1..#SetKeep]`.

The code incorporating the method in this bullet point is given in A.4.

The last method we discuss will not make many appearances but is immensely helpful in situations it can be applied to.

- Let b be a group in `BadSub` such that $Z(b)$ is elementary abelian but $b/Z(b)$ isn't. Let S be an elementary abelian subgroup of b of order 2^3 on which x acts irreducibly. Then $S = \{e, t^{x^i} : i \in \{1, \ldots, 7\}\}$ for any involution $t \in S$. Consider the image of S in $b/Z(b)$ then its preimage is $\tilde{S} = \langle Z(b), t^{\langle x \rangle}\rangle$ $= Z(b) \cup Z(b)t^x \cup \ldots \cup Z(b)t^{x^7}$. So assuming the index of $Z(b)$ in b is small enough for the command `Transversal` to work, we must search for desired elementary abelian subgroups of order 2^3 in the groups $\langle Z(b), \gamma_1^{\langle x \rangle}\rangle, \ldots, \langle Z(b), \gamma_m^{\langle x \rangle}\rangle$, where $\Gamma := \{\gamma_1, \ldots, \gamma_m\}$ is a transversal for $Z(b)$ in b. However, for $1 \leq i \leq m$, $z \in Z(b)$, $(z\gamma_i)^2 = z\gamma_i z\gamma_i = z^2\gamma_i^2 = \gamma_i^2$, since $Z(b)$ is elementary abelian. So the only cosets with involutions in them are $Z(b)\gamma_i$ where γ_i is the identity or an involution. Hence we are interested in constructing subgroups $\langle Z(b), \gamma_i^{\langle x \rangle}\rangle$ with $o(\gamma_i) = 1$ or 2, only, which we then add to `SetSub2`.

  Looking at the `BadSetNew` of size 3017 from above, we have that 384 of these groups have order 2^{28}. None of the 384 groups have an elementary abelian quotient by the centre (so we can't take `Fb` to be the centre) but all have elementary abelian centres. Using the above method on these 384 groups b (adding the groups $\langle Z(b), \gamma_i^{\langle x \rangle}\rangle$ to `SetSub2` and then running the repeat loop in A.3 programmed to end when `#SetKeep eq 0`, doing the same to the next b and so on) enables us to break them down to their elementary abelian subgroups in approximately 9 hours. In comparison, taking `Fb` to always be the Frattini (running the for loop over `[1..#BadSub]` in A.3) doesn't even get us through 31 of the 384 groups in the same time.

  The codes incorporating the method in this bullet point are given in A.5 and A.6. It will become clearer why these codes are written as they are when we use them later on.

We now move on to listing all the possible pairs of $O_2(P)$ and x for the $L_2(8)$ case, and dealing with some of them by utilising the methods described in this section.

5.2 The Cases

We embark on our journey of trying to construct copies of $L_2(8)$ in $E_8(2)$. In B.3, all the possible fusion patterns for an embedding of $L_2(8)$ in $E_8(2)$ are listed. By Lemma 2.2.5(i) and Proposition 2.2.3, we are not interested in decompositions (i) and (xi)-(xxi). Also (ii)-(iv) and (viii)-(x) are not realisable since the class of elements of order 3 of an $L_2(8)$ following any one of them will fuse to $3B_{E_8(2)}$ or $3D_{E_8(2)}$ and so these are the possible classes that powers of elements of order 9 of the $L_2(8)$ can lie in. This contradicts the fact that the 3rd power of any class of elements of order 9 of $E_8(2)$ is $3C_{E_8(2)}$, see Theorem 2.2.2.

We have that the conjugacy classes of elements of order 7 of an $L_2(8)$ embedded in $E_8(2)$ according to (v),(vi) or (vii) will fuse to $7B_{E_8(2)}$. The following result by Rowley tells us where we can find Levi cuspidal subgroups of $E_8(2)$ generated by elements in $7B_{E_8(2)}$.

Lemma 5.2.1. *Suppose that $\langle x \rangle$ is a Levi-cuspidal subgroup of $E_8(2)$ with $x \in 7B_{E_8(2)}$. Then $\langle x \rangle$ is L-cuspidal for $L \cong L_3(2) \times L_3(2)$ with $\langle x \rangle$ being one of two diagonal $\mathbb{Z}_7$-subgroups in L.*

Proof. Will be viewable in [7], once the paper is complete and made available. □

The standard parabolic subgroups with Levi complements isomorphic to $L_3(2) \times L_3(2)$ are the ones associated to the roots labelled by

$$\{1,3,5,6\}, \{1,3,6,7\}, \{1,3,7,8\},$$

$$\{3,4,6,7\}, \{3,4,7,8\},$$

$$\{2,4,6,7\}, \{2,4,7,8\},$$

$$\{4,5,7,8\}.$$

Out of all the elements of order 7 in $E_8(2)$ only the ones in $7B_{E_8(2)}$ fix spaces of dimension 38, see Theorem 2.2.2. For J being one of the above sets we calculate L_J and then its subgroups of order 7. There's 4 of these with only 2 among them containing elements of order 7 that fix spaces of dimension 38. These two must be the L_J-cuspidal subgroups given to us by Lemma 5.2.1. We take $x_{J,a}$ to be the generator of one of them and $x_{J,b}$ of the other. We also calculate the groups Q_J.

We have our pairs, 16 of them, $Q_J, x_{J,a}$ and $Q_J, x_{J,b}$. The number of composition factors isomorphic to the Steinberg module in decompositions (v)-(vii) is 2, 4 or 5. So we care about a subgroup of Q_J only if the dimension of the fixed space of the group generated by it and $x_{J,a}$, or $x_{J,b}$, is ≤ 5; from now onwards, if a subgroup of Q_J is like so then we say that it *satisfies the Steinberg bound*.

We've not managed to finish running computations on all 16 pairs yet. Note that A.2 and the methods outlined in Section 5.1 were developed while running computations on the different pairs for $L_2(8)$ and so may not always be used when we describe our work with some of the pairs in the subsections that follow. We fix notation for the other/less effective programs used sometimes, below; referring to them will now be convenient. Denote by:

(†) : the for loop over `[1..#BadSub]` in A.3 except that `Fb` will always be the Frattini.

(††) : the for loop over `[1..#BadSub]` in A.3 except that for every $b \in$ `BadSub`, where $b/\Phi(b) \cong V_1 \oplus \ldots \oplus V_k$, we take `Fb` to be the preimage of the sum of the first r summands. The identifier `STH` will denote the number r. Note that (††) is what we did for groups with large k in 4.2.2 and is different from A.4 as detailed in Section 5.1. Also note that running (††) with `STH` $= 0$ is the same as running (†).

($*$) : A.3 except that `Fb` is always the Frattini.

For every code that we run, `FinSub`, `SetSub2` and `BadSetNew` will always be sets rather than sequences (i.e. we will be using the command `Include` rather then `Append`), unless stated otherwise. Also `ActnGpDiff` will always remain empty and hence we make no further mention of it. If a `BadSetNew`, `FinSub` or `SetKeepZero` is empty after a code run then we may omit mentioning this. For $b \in$ `BadSub` or `BadSetNew`, we will always use k to denote the number of summands in a direct sum decomposition of $b/\Phi(b)$ into irreducible modules. Frequently, before trying to break up b, we will be using appropriate lines from A.4 to examine how the sizes of `SetKeep` change as `STH` is varied. Finally we remark that any code will run slower on the larger groups.

5.2.1 $Q_J, x_{J,a}$ for $J = \{2, 4, 7, 8\}$

Firstly, we mention that before we were able to bring the Steinberg bound down to 5 we were working with a bound of 6 and so a group among the ones collected below could be so that the dimension of the fixed space generated by it and $x_{J,a}$ is 6. Also, since ignoring any `Sub4aa`'s not satisfying the Steinberg bound wasn't incorporated into A.3 until later, none of the codes used below do so; if they did then any code run times mentioned below could possibly have been shorter.

We now break up Q_J by having it as the sole member of `SetSub2` and running the repeat loop in A.3 programmed to end when `#SetSub2 eq 0`. This just outputs a single group of order 2^{113} with $k = 4$. Running (†) on this group returns a `BadSetNew` of size 95; 3 are of order 2^{82} with $k = 4$, 73 with possible orders 2^{82} and 2^{76} and $k = 5$, 4 with possible orders 2^{30} and 2^{40} and $k = 6$, 10 with possible orders $2^{27}, 2^{33}$ and 2^{38} and $k = 7$, 3 with possible orders $2^{50}, 2^{54}$ and 2^{79} and $k = 8$, and finally 2 with possible orders 2^{39} and 2^{70} and $k = 9$.

We first look at the groups with $k = 7$ or 9 and two of the groups with $k = 8$:

- 6 of the 10 groups with $k = 7$ have order 2^{38}. Running (††) with `STH` $= 3$ on them returns a `BadSetNew` of size 3510 on which we successfully run ($*$) but with `FinSub` a sequence.

  We run (†) on the remaining 4 out of 10 and get a `FinSub` containing 2772 groups of order 2^9, none of which contain subgroups of order 2^3 normalised by $x_{J,a}$, satisfying the Steinberg bound.

- Consider the groups with $k = 9$. We run (††) with `STH` $= 5$ on the group of order 2^{39} to get a `BadSetNew` of size 585 on which we run (†) to get a `FinSub` containing 10752 groups of order 2^6; nothing in `FinSub` contains subgroups of interest to us.

  Running (†) on the group of order 2^{70} returns a `BadSetNew` containing 7632 groups with possible orders $2^{18}, 2^{21}$ and 2^{24}; we run ($*$) but with `FinSub` a sequence, on the `BadSetNew`.

- Consider the group of order 2^{50} with $k = 8$. Running (†) on it returns a `BadSetNew` containing 3264 groups of order 2^{18}. We run ($*$) but with `FinSub` a sequence, on the BadSetNew.

We run (††) with STH $= 1$ on the group of order 2^{54}. Doing so can get #SetKeep down to 2377 and returns a BadSetNew containing 2352 groups with possible orders 2^{18} and 2^{21}. We run ($*$) but with FinSub a sequence, on the BadSetNew.

We now run (†) on the group of order 2^{79} with $k = 8$ and get a BadSetNew containing 5386 groups, each of order $\leq 2^{44}$. We divide these groups according to the associated values for k.

- 2881 of the 5386 have $k = 6$. One of the 2881 has order 2^{30}. Factoring it out with its Frattini subgroup would give a SetKeep of size 4689 and so instead we run (††) with STH $= 2$ to get a BadSetNew containing 576 groups with possible orders 2^{18} and 2^{21}. We run ($*$) but with FinSub a sequence, on the BadSetNew.

  528 of the 2881 have order 2^{36} with a SetKeep of size 657 each, with STH $= 0$. Instead of running ($*$) on the lot straight away, we test how (†) runs on them; we see that the 1st group adds 72 groups to BadSetNew and every subsequent group seems to be adding approximately 8. It seems like if we were to run (†), the size of BadSetNew won't increase detrimentally, whereas running ($*$) would give us 528 separate BadSetNew's intersecting in a lot of groups and we will be wasting time repeating calculations. Therefore we decide to run (†) on the 528 groups to get a BadSetNew containing 5568 groups of order 2^{15}. We run A.3 on these but with the line `loop:=0;` changed to `loop:=1;` so that for each of the 5568 groups it's checked whether the quotient by the centre is elementary abelian.

  1008 of the 2881 have order 2^{41} with a SetKeep of size 649 each, with STH $= 0$. Again it seems like running (†) won't increase the size of BadSetNew by a lot, therefore we do so but after dividing the groups over 2 Magma sessions. Each session will contain 504 of the groups and return a BadSetNew of size 704. These two BadSetNew's aren't the same but each is a subset of the BadSetNew of size 5568 encountered in the previous paragraph.

  1344 of the 2881 have order 2^{43} with a SetKeep of size 1161 each, with STH $= 0$. After investigating how well a particular 2^{43} breaks up by choosing different values for STH, it seems like sticking with 0 will work best. Hence we divide the 1344 groups over 6 Magma sessions with 224 groups in each and run A.3.

- 2128 of the 5386 have $k = 7$. The quickest we are able to break up a chosen group from the 2128 is by taking `STH` as 0; this takes 3h. We divide the groups over 14 sessions with 152 groups in each and run A.3.

- 367 of the 5368 have $k = 8$. Out of the 367, 85 have possible orders 2^{36} and 2^{41}. Further, 14 of these have `#SetKeep` as 11337 and the rest as ≤ 7753, with `STH` $= 0$. Running (†) on a test group with `#SetKeep` $= 11337$ takes approximately 4h; we run (†) on the 85 groups together.

  24 of the 367 have order 2^{43} and `#SetKeep` 19017. Running (†) on one of the 24 takes $\approx$ 9h; we run (†) on the lot.

  48 of the 367 have order 2^{43} and `#SetKeep` 11337. Running (†) on one of the 48 takes $\approx$ 5h; we run (†) on the lot.

  210 of the 367 have order 2^{44} and `#SetKeep` ≤ 7753. Running (†) on one of the 210 takes $\approx$ 4h. We divide the 210 over 3 sessions with 70 groups in each and run (†).

  In this bullet point, every time (†) has run to give an empty output. Also, in every run, despite `#SetKeep` being quite big, the size of `SetSub2` seems to remain either insignificant or not detrimentally large, and so it makes sense to keep `SetSub2` a set.

- Finally 10 of the 5368 have $k = 9$. The possible orders are 2^{36} and 2^{41}, the possible values for `#SetKeep` with `STH` $= 0$ are $66121, 291401$ and 348745. We can get `#SetKeep` down to 2121 for two of the groups with `STH` $= 3$. Hence we run (††) on them (with `STH` $= 3$) to get a `BadSetNew` containing 2552 groups of order 2^{18}; we run $(*)$ on `BadSetNew` but with `FinSub` a sequence.

  We run (††) with `STH` $= 5$ and everything (`FinSub`, `SetSub2` and `BadSetNew`) a sequence, on the remaining 8 groups to get a `BadSetNew` containing 4680 groups with possible orders $2^{21}, 2^{24}, 2^{27}$ and 2^{32}. We run $(*)$ but with `FinSub` a sequence, on the `BadSetNew`.

It is left to consider the groups with $k = 4, 5$ or 6 arising from the initial split of Q_J. We run (†) on them to get a `BadSetNew` of size 4839 and a `FinSub` containing 4874 groups of order $2^{12}, 2^{15}$ or 2^{18}. Nothing in `FinSub` contains subgroups of interest to us. We divide the groups in `BadSetNew` according to the associated values for k.

- 1314 have $k = 3$ and 1393 have $k = 4$. Together these groups have order 2^{15} or 2^{18} and we run (∗) on them but with `FinSub` a sequence.

- 1003 have $k = 5$ and order $\leq 2^{55}$. We separate out the 3 groups of order 2^{43} because our test runs show that breaking these up will result in groups with $k = 7$ being formed. We run (†) on these 3 and get a `BadSetNew` containing 432 groups of order $2^{21}, 2^{24}$ or 2^{28}. A `FinSub` containing 3 groups of order 2^{21} is also given but; none of the groups contain subgroups of interest. 216 of the 432 have $k = 5$; we run (∗), but with `FinSub` a sequence, on them. The other 216 have $k = 7$ and we run (††) on them, with `STH` $= 4$ and everything a sequence. We obtain a `FinSub` containing 512 groups of order 2^{15}, none of which contain any subgroups of interest. We also obtain a `BadSetNew` of size 15207, on which we run (∗) but with `FinSub` a sequence.

  We divide the remaining 1000 groups over 3 sessions and run (∗) but with `FinSub` a sequence.

- 828 of the 4839 have $k = 6$ and order $\leq 2^{57}$. We divide them up according to what `#SetKeep` with `STH` $= 0$ can be. We run (∗), with everything a sequence, on the following collections: 64 groups with $393, 649, 1161, 1225$ and 1617 as the possible values for `#SetKeep`, 42 group with `#SetKeep` $= 593$, and 48 groups with `SetKeep` $= 713$.

  There are 81 groups with `#SetKeep` $= 1609$. We run (∗) on them with everything a sequence apart from `FinSub`. We also comment out the lines in the code, towards the end, that deal with groups in the `FinSub` produced from breaking up a group in `OrigBadSub`, before moving on to the next group. A `FinSub` containing 584 distinct groups of order 2^{15} is returned; none of these groups contain any subgroups of interest. Note that if we hadn't adjusted the code as stated before running it, several intersecting `FinSub`'s would've been produced, and calculating subgroups of the groups in them would've slowed the code down and increased memory usage. Indeed, an order of 2^{15} for an elementary abelian group is a tad too big for our liking.

There is one group of order 2^{24} with `#SetKeep` $= 2057$. We run (††) with everything a sequence and `STH` $= 2$. This returns a `BadSetNew` of size 585 on which we run ($*$).

There is another group of order 2^{37} with `#SetKeep` $= 4689$. Taking `STH` to be 1 we can get `#SetKeep` down to 593 and so we run (††) with `STH` $= 1$ and everything a sequence. This returns a `FinSub` containing 585 groups of order 2^{18}, none of which contain any subgroups of interest.

The remaining groups are divided over 5 sessions. In each of two sessions, we load a collection of 147 groups all having `#SetKeep` $= 585$. In each of the other three sessions, we load 99 groups having `#SetKeep` $= 1097$. We run A.3 in each session but with everything a sequence. In each session, a `SetKeepZero` containing a single elementary abelian group of order 2^{15} is returned; this group doesn't contain any subgroups of interest.

- 234 of the 4839 have $k = 7$ and order $\leq 2^{51}$. Most of these have `#SetKeep` as 8777 or 9801 with `STH` $= 0$. Taking `STH` to be 1 won't help much with bringing the values for `#SetKeep` down; we work with `STH` $= 2$ or 3. We have the following list of 2-tuples: $\langle 49, 4\rangle$, $\langle 90, 1\rangle$, $\langle 153, 5\rangle$, $\langle 257, 1\rangle$, $\langle 265, 1\rangle$, $\langle 585, 108\rangle$, $\langle 649, 9\rangle$, $\langle 713, 13\rangle$, $\langle 841, 1\rangle$, $\langle 1097, 30\rangle$, $\langle 1609, 2\rangle$, $\langle 4681, 59\rangle$. The first entry in each tuple is a possible value for `#SetKeep` with `STH` $= 2$ and the second entry is the number of groups out of 234 having that value associated to them.

  We run (††) with everything a sequence and `STH` $= 2$ on the 67 groups not having `#SetKeep` as 585 or 4681. A `FinSub` containing 73 groups of order 2^{18} is returned; none of which contain any subgroups of interest. A `BadSetNew` of size 52878 is also returned, running ($*$) on which, but with `FinSub` a sequence, takes a month, with a third of the time being spent breaking up the 512 groups of order 2^{23} among the 52878.

  With `STH` $= 3$, `#SetKeep` can go down to under 585 for 148 of the remaining groups. We divide the 148 over two sessions and in each run (††) with everything a sequence and STH $= 3$. A non-empty BadSetNew will be returned in each session and we run $(*)$, but with everything a sequence, on it.

We run $(\dagger\dagger)$ on the remaining 19 of the 234 groups, but with everything a sequence and STH $= 3$ (#SetKeep won't get bigger than 585). A BadSetNew of size 10154 is returned on which we run $(*)$ but with everything a sequence.

- 36 of the 4839 have $k = 8$ and possible orders $2^{36}, 2^{42}$ and 2^{47}. We run $(\dagger\dagger)$ on them with STH $= 4$ and everything a sequence, to get a BadSetNew, of size 20073, on which we run $(*)$ but with everything a sequence.

- 1 of the 4839 has $k = 9$ and order 2^{39}. We run $(\dagger\dagger)$ on it with STH $= 5$ and everything a sequence, to get a BadSetNew, of size 585, on which we run $(*)$ but with FinSub a sequence.

We have established that Q_J does not contain any elementary abelian subgroups of order 2^3, irreducible under the action of $x_{J,a}$, satisfying the Steinberg bound.

5.2.2 $Q_J, x_{J,a}$ for $J = \{2, 4, 6, 7\}$

Again, some of the groups collected below might be satisfying a bigger Steinberg bound of 6 and any Sub4aa's not satisfying the bound were hardly ever ignored.

We break up Q_J as in the previous subsection. We have the following information on the 11 groups b contained in the BadSetNew output: $\langle |b|, k,$#SetKeep (STH $= 0$)$\rangle = \langle 2^{98}, 4, 21\rangle$, $\langle 2^{85}, 6, 273\rangle$, $\langle 2^{37}, 9, 2527817\rangle$, $\langle 2^{69}, 5, 91\rangle$, $\langle 2^{40}, 7, 5001\rangle$, $\langle 2^{58}, 7, 5369\rangle$, $\langle 2^{52}, 8, 2993\rangle$, $\langle 2^{55}, 5, 83\rangle$, $\langle 2^{21}, 6, 8777\rangle$, $\langle 2^{24}, 7, 70217\rangle$, $\langle 2^{33}, 7, 6217\rangle$. A FinSub containing a single group of order 2^{21} is also output but this group doesn't contain any subgroups of interest.

We first look at the last 9 groups in BadSetNew:

- We run $(\dagger\dagger)$ with STH $= 5$ and everything a sequence on the group of order 2^{37}. A BadSetNew containing 585 groups of order 2^{28} is returned, running $(\dagger)$ on which gives an empty output. All of this takes $\approx$ 10d; a test group of order 2^{28} had $k = 6$ and #SetKeep$= 4937$.

- We run $(\dagger)$ on the group of order 2^{40} to get a FinSub containing 4968 groups of order 2^{12}, none of which contain any subgroups of interest.

- We run (†) on the 2^{52} to get a `FinSub` containing 616 groups of order 2^{15}, none of which contain any subgroups of interest. A `BadSetNew` containing 1792 groups of order 2^{18} is also returned and we run $(*)$ on it but with `FinSub` a sequence.

- We run (††) with $\texttt{STH} = 2$ and everything a sequence on the group of order 2^{21} to get a `BadSetNew` containing 585 groups of order 2^{12}; we run $(*)$, but with `FinSub` a sequence, on the `BadSetNew`.

- We run (††) with $\texttt{STH} = 3$ and everything a sequence on the groups of order 2^{24} and 2^{33}, together. We get a `FinSub` containing 72 groups of order 2^{12}, none of which contain any subgroups of interest. A `BadSetNew` containing 200 groups with possible orders 2^{18} and 2^{21} is also returned; we run $(*)$, but with `FinSub` a sequence, on it.

- Running (†) on the 2^{55} gives a `FinSub` containing 10 groups of order 2^{18} or 2^{24}, none of which contain any subgroups of interest. A `BadSetNew` containing 74 groups with possible orders $2^{27}, 2^{30}$ and 2^{31} is also returned; 72 of these have $k = 6$ and we run $(*)$, but with `FinSub` a sequence, on them. The remaining two groups have $k = 8$ and we run (††) with $\texttt{STH} = 4$ on them; a `BadSetNew` of size 1170 is returned and we run $(*)$, but with `FinSub` a sequence, on it.

- We run (†) on the 2^{58} and get a `FinSub` containing a single group of order 2^{18} which doesn't contain any subgroups of interest. We also get a `BadSetNew` containing 5369 groups of order $\leq 2^{31}$; 4992 of these have $k = 5$ or 6 and we run $(*)$, but with `FinSub` a sequence, on them.

  329 of the 5369 have order $\leq 2^{30}$ and $k = 7$ and all are such that the quotient by the centre is elementary abelian. Hence we run A.3 (with everything a sequence) on them but with the line `loop:=0;` changed to `loop:=1;`.

  The remaining 48 of the 5369 all have order 2^{31} and $k = 7$. None have an elementary abelian quotient by the centre. A group we test from among the 48 had a `SetKeep` of size 41545, going through which would take 29h, it seemed. On the other hand, running A.4 on the group, with $\texttt{STH} = 3$, would be a lot faster; we run this code on all of the 48 groups together.

- Running (†) on the 2^{69} gives a `FinSub` containing 58 groups none of which contain any subgroups of interest. A `BadSetNew` of size 92 is also returned. Three of the 92 have order 2^{46} and $k = 9$; we run A.4 with `STH` = `5` on them. One of the 92 has order 2^{27} and $k = 6$; we run (∗), but with `FinSub` a sequence, on it.

  The remaining 88 groups have $k = 7$ and possible orders $2^{40}, 2^{43}$ and 2^{46}. With `STH` = 0, 1 of these has `#SetKeep` as 5001, 7 have 6793 and 80 have 7241. Running (†) but with `SetSub2` and `BadSetNew` as sequences on a test group of order 2^{46} with `#SetKeep` = 7241 takes ≈ 12h to give a `BadSetNew` containing 2560 groups, all distinct. A `FinSub` containing 4673 groups of order 2^{12} or 2^{15} is also returned; its size would've been a lot more if we would've appended groups to it instead. It takes ≈ 1.5h to break down the groups in `FinSub`. Breaking up the 2560 groups by allowing `Fb` to be the centre is 4 times faster than if it's fixed as the Frattini. We divide the 88 groups over 4 sessions, with 22 groups in each, and run A.3 but with `SetSub2` and `BadSetNew` as sequences.

We now run (†) on the group of order 2^{98} and get a `FinSub` containing a single group of order 2^{24}; this elementary abelian group doesn't contain any subgroups of interest. A `BadSetNew` of size 35 is also returned and we deal with this as below:

- 6 of the 35 have $k = 6$ and order $\leq 2^{34}$; we run (∗), but with `FinSub` a sequence, on them.

- 1 of the 35 has $k = 9$ and order 2^{39}; we run (††), with `STH` = 5 and everything a sequence, on it to get a `BadSetNew` of size 585. We run (∗), but with everything a sequence, on the 585 groups.

- 5 of the 35 have $k = 8$. One of the 5 has order 2^{67}, `#SetKeep` for it can go down to 713 with `STH` = 3. We run (††) on it with `STH` = 3 and get a `BadSetNew` of size 713. 505 of the 713 have $k = 4$ and we run A.3 on them. 134 of the remaining have $k = 5$, 65 have $k = 6$ (with possible orders $2^{40}, 2^{41}, 2^{43}$ and 2^{48}) and 9 have $k = 7$ (with possible orders 2^{33} and 2^{44}). We pick 6 groups, one of each order and run (†) on them, separately. In each case the `SetSub2`'s that are formed are small except one of size 4020 for the group of order 2^{44}. This isn't bad considering there's only 9 groups with $k = 7$. It seems like we would get a successful run of ($*$), with SetSub2 and BadSetNew as sets (FinSub a sequence), on all the 208 groups together within a reasonable amount of time; this indeed holds true.

We calculate #SetKeep with STH $= 0$ for the remaining 4 of the 5 groups. The group of order 2^{66} has #SetKeep as 14793 and the three of order 2^{33} have 365129 each. We run (††) on the 4 groups with STH $= 4$ and get a BadSetNew of size 2342. 2324 of these have $k = 5$, 74 of which have order 2^{18}, 1682 have order 2^{22} and 568 have order 2^{57}. With STH $= 0$, a test group of order 2^{18} had #SetKeep as 1097, a 2^{22} had it as 713, one 2^{57} had 89 and another had 153. Running (†) on the four test cases took 42s, 78s, 10m and 35m, respectively. In each case the size of FinSub, SetSub2 and BadSetNew remained small. We run A.3 on the 1756 groups of order 2^{18} or 2^{22}. We divide the 568 groups of order 2^{57} evenly over two sessions and run A.3 in each.

9 of the 2342 have $k = 6$, with all but one of these having order 2^{54} and #SetKeep as 4697. It seems like running (†) on a test 2^{54} will give the first SetSub2 as a set of size 4698. With STH $= 1$, however, the #SetKeep can go down to ≤ 713 for six of the groups and 1097 for two of them. Hence we run (††) with STH $= 1$ on the 9 groups and get a FinSub of size 1 (this elementary abelian group doesn't contain any subgroups of interest) and a BadSetNew of size 5794. The 5794 groups have order $\leq 2^{35}$ and we run A.3 on them but with `loop:=0;` changed to `loop:=1;` and the code lines towards the end dealing with a FinSub commented out. A FinSub containing 213 groups of order $\leq 2^{15}$ is output, none of which contain any subgroups of interest.

Lastly, 9 of the 2342 have $k = 7$ and order 2^{47}. With STH $= 0$, #SetKeep is ≤ 521 for 8 of them and 1737 for one of them. We run A.3 on the 9 groups.

- 11 of the 35 have $k = 5$. We run (†) on them and get a BadSetNew of size 2262 and also a FinSub containing a single group of order 2^{27} (this doesn't contain any subgroups of interest). We divide the 2262 groups according to the associated values for k.

  5 of the 2262 have $k = 9$ with possible orders $2^{34}, 2^{37}$ and 2^{39}. These are dealt with below:

– We run (††) with `STH` $= 5$ and everything a sequence on the three groups of order 2^{34}. This returns a `BadSetNew` containing 1755 groups of order 2^{25} on which we run ($*$) but with `FinSub` a sequence.

– We run (††) with `STH` $= 5$ and everything a sequence on the group of order 2^{37}. This returns a `BadSetNew` of size 585 on which we run ($*$) but with `FinSub` a sequence.

– We run (††) with `STH` $= 5$ and everything a sequence on the group of order 2^{39}. This returns a `BadSetNew` containing 585 groups each of which has an elementary abelian quotient by the centre; we run A.3 on the `BadSetNew` but with `loop:=0;` changed to `loop:=1;`.

857 of the 2262 have $k = 6$. These are dealt with below:

– 2 of the 857 are of order $\leq 2^{30}$, both having a quotient by the centre that is elementary abelian, and so we run A.3 on them but with `loop:=0;` changed to `loop:=1;`.

– 27 of the 857 have order 2^{34} and 108 have 2^{37}. We pick a test group of each order and run (†) on them. This takes 11m on the 2^{34} and 30m on the 2^{37} to give `FinSub`'s of sizes 590 and 421, respectively; these sizes aren't big. We run A.3 on the 135 groups together.

– 720 of the 857 have order 2^{40}. A test 2^{40} had `#SetKeep`$=$ 1673 (with `STH` $= 0$). Running (†) but with everything a sequence gives a `FinSub` containing 504 distinct groups and a `BadSetNew` containing 1088 distinct groups; this takes $\approx$ 1h. We divide the 720 groups evenly across four sessions and in each run A.3 but with everything a sequence.

1400 of the 2262 have $k = 5$. We deal with them as below:

– 1176 of the 1400 have order 2^{40}. Test running (†) on these groups adds 48 groups to `FinSub` and 128 groups to `BadSetNew` after the first iteration. Each additional iteration seems to be adding 64 more groups to `BadSetNew` while the size of `FinSub` remains the same. We divide the 1176 groups evenly across 2 sessions and in each run A.3 but the code lines dealing with `FinSub` commented out. In each session, a `FinSub` containing 48 groups of order 2^{15} and a `SetKeepZero` containing a single elementary abelian group of order 2^{15}, are returned; none of these elementary abelian groups contain any subgroups of interest.

– 42 of the 1400 have order 2^{38}. We run (†) on them to get a FinSub of size 48. 168 of the 1400 have order 2^{37}. We run (†) on them to get a FinSub of size 48. The two FinSub are not the same; they contain groups of order 2^{15}, none of which contain any subgroups of interest.

– 14 of the 1400 have order 2^{34} and we run ($*$) on them but with everything a sequence.

- 12 of the 35 have $k = 7$. 3 of these have order $2^{24}, 2^{27}$ and 2^{40} each. We run (††) on them with STH = 3 to get a BadSetNew of size 1170 on which we run ($*$) but with FinSub a sequence.

  3 of the 12 have 2^{63} and #SetKeep for them can go down to ≤ 777 with STH $= 1$. Hence we run (††) on them (with STH $= 1$) and get a BadSetNew of size 794. The 794 groups are dealt with as follows:

  – 641 of the 794 have $k = 6$ and order $\leq 2^{30}$; we run ($*$), but with FinSub a sequence, on them.

  – 3 of the 794 have $k = 8$ and order 2^{30}. Each has a centre bigger than its Frattini and an elementary abelian quotient by the centre. With Fb as the Frattini, #SetKeep was seen to be > 60000 for one of the three groups whereas with Fb as the centre, it was 1225. We run A.3 on the 3 groups but after changing `loop:=0;` to `loop:=1;`.

  – 150 of the 794 have $k = 7$ and possible orders $2^{27}, 2^{30}, 2^{33}$ and 2^{34}. We take three test groups, one of each possible order $\leq 2^{33}$ and run (†) on them. Each run takes ≤ 35m and the first SetSub2 itself is output as empty. There are 102 groups with order $\leq 2^{33}$ and we run ($*$) on them. Running (†), but with everything sequence, on a test group of order 2^{34} took ≈ 1.5h and collections of distinct groups were created along the way; we run ($*$), but with everything a sequence on the remaining 48 groups of order 2^{34}.

  The last 6 of the 12 also have order 2^{63} and #SetKeep for them can go down to ≤ 713 with STH $= 2$. Hence we run (††) on them (with STH $= 2$) and get a `FinSub` of size 114 (none of these contain any subgroups of interest) and also `BadSetNew` of size 2814. The 2814 groups are dealt with as follows:

– 1289 of the 2814 have $k = 5$ and we run A.3 on them.

– 17 of the 2814 have $k = 8$, we divide them across two sessions and in each run A.4 with `STH` $= 4$.

– 1 of the 2814 has $k = 9$ and order 2^{37}, we run (††) on it (with `STH` $= 5$) and get a `FinSub` containing 73 groups of order 2^{18} (none of which contain any subgroups of interest) and also a `BadSetNew` containing 512 groups of order 2^{28}. We run ($*$), but with `FinSub` a sequence, on the 512 groups.

– 77 of the 2814 have $k = 7$. 71 of the 77 have possible orders 2^{33} and 2^{43}. With `STH` $= 0$, all 2^{43}'s have `#SetKeep`$= 13385$. Running (†) on a test 2^{43} takes < 6h to give the first `SetSub2` as a set of size 1; we run ($*$) on the 71 groups. The remaining 6 of the 77 have order 2^{37} and `#SetKeep`$= 5705$ with `STH` $= 0$. There's only 6 groups and we run ($*$) on them but with `FinSub` a sequence.

– The last 1430 of the 2814 have $k = 6$. We have the following information on them: $\langle 1609, 210, \{2^{40}, 2^{43}\}\rangle$, $\langle 1673, 1154, \{2^{40}, 2^{43}\}\rangle$, $\langle 5193, 1, \{2^{24}\}\rangle$, $\langle 657, 24, \{2^{36}\}\rangle$, $\langle 713, 14, \{2^{37}\}\rangle$, $\langle 777, 15, \{2^{41}\}\rangle$, $\langle 1161, 12, \{2^{40}\}\rangle$. The first entry in each tuple is a possible value for `#SetKeep` with `STH` $= 0$, the second is the number of groups out of the 1430 having that value associated to them and the third is the set of possible orders of these groups. There's only 54 groups with `#SetKeep` ≤ 777 or 5193; we A.3 on them. Now, we take a test group with `#SetKeep` $= 1161$, one with 1609 and a group of order 2^{40} with `#SetKeep` $= 1673$; running (†) on them, but with everything a sequence, returns a `FinSub` and `BadSetNew` containing distinct groups, and takes 45m, 1.5h and 1h, respectively. However trying to break a group of order 2^{40} or 2^{43} down to its elementary abelian subgroups can lead to at least 1673 (all same) groups of order 2^{15} being appended to `FinSub`. The code A.3 will break these elementary abelian groups down before moving on to the next group in `OrigBadSub`; this will increase memory usage (and by a lot if more groups in `OrigBadSub` behave the same), so having `FinSub` as a sequence is not a good idea (an undesirable increase in memory usage was indeed witnessed). We divide the remaining 1376 of the 1430 groups over 6 sessions, 2 containing 230 groups each and the rest 229. One session turns out to have all 229 groups as groups of order 2^{40}, we run A.3 with everything a sequence in this one, in the rest we run the same code but leave FinSub a set.

All that is left to consider now is the group of order 2^{85} with $k = 6$. We run (†) on it and get a FinSub containing a single group of order 2^{24} (this doesn't contain any subgroups of interest) and also a BadSetNew of size 299. We deal with the 299 groups as below:

- 24 of the 299 have $k = 5$ and 2 have 6. These 26 groups are of order $\leq 2^{37}$ and we run (∗), but with FinSub a sequence, on them.

- 8 of the 299 groups have $k = 8$. One of these is a group of order 2^{52} with #SetKeep = 2993 (with STH = 0). We run (†) on it to get a FinSub containing 616 groups of order 2^{15} (none of these contain any subgroups of interest), and also a BadSetNew containing 1792 groups of order 2^{18}. We run (∗), but with FinSub a sequence, on the groups in BadSetNew.

  The remaining 7 have order 2^{59} and we run (††) on them with STH = 4 and everything a sequence. This gives a BadSetNew containing 4095 groups of order 2^{50}. Running (†) on a few test cases shows that elementary abelian groups of order 2^{21} are being created. Breaking up a group of order 2^{21} can take a while so we want to avoid repeating computations on the same group of order 2^{21}. We split the 4095 groups over 4 sessions with 1024 groups in 3 of the sessions and 1023 in one. In each session, we run A.3 but with the code lines dealing with FinSub commented out. A FinSub containing 610 groups of order $\leq 2^{21}$ is output in each session, none of which contain any subgroups of interest.

- 248 of the 299 have $k = 7$. 56 of these are of order 2^{59} with #SetKeep = 2505 (with STH = 0), 168 are also of order 2^{59} but with #SetKeep = 2521 and the remaining 24 are of order 2^{55} with #SetKeep = 3017. Note that the groups of order 2^{55} have been talked about in Section 5.1. We take a test group from each of the three types and run (†) but with everything a sequence; in each case the code will take ≈ 12h to finish running. The three BadSetNew's output will have sizes 2496, 2521 and 3017, respectively (three non-empty `FinSub`'s are also output); each `BadSetNew` will contain distinct groups. Among the 2496 are groups of order 2^{25} and some of the groups in each of the other two `BadSetNew`'s are of order 2^{28}. None of these groups of order 2^{25} or 2^{28} has an elementary abelian quotient by the centre so we can't take `Fb` to be the centre for any of them. Taking `Fb` to be the Frattini turns out to be impractical. However all of the groups do have elementary abelian centres and so we can use the method described in the last bullet point in Section 5.1 to break them up and be done in realistic time. The code A.5 will do this for us. But first we run A.2 on the 248 groups and manage to find 12 groups such that each group from among the 248 is conjugate to some group from among the 12 via an element that centralises $x_{J,a}$ (see A.2 and Section 4.1). We run A.5 on these 12 groups.

- 17 of the 299 have $k = 11$. Taking `STH` to be 0, even after two days MAGMA is unable to calculate `SetKeep` entirely; the size of the partial `SetKeep` at which point was 114816. We run A.2 on the 17 groups and manage to get `#ind` (see A.2 and Section 4.1) down to 5. We load each of the 5 groups indexed by `ind` into a separate MAGMA session and run A.4 with `STH` = 6. However elementary abelian groups of order 2^{21} will be created and so we also comment out the code lines dealing with `FinSub` before running A.4. The code run in each of 4 of the sessions take ≈ 1 month to finish; the `FinSub`'s output all have size 1 and share the same group of order 2^{21} (this doesn't contain any subgroups of interest).

  As for the last session, after having calculated an `OrigBadSub` containing 4681 groups, the code took 2 months to break down 2165 of them (into the same elementary abelian group of order 2^{21} output in the other 4 sessions). This is because `#SetKeep` for these groups increases to ≈ 670 while being ≈ 140 for the groups in the `OrigBadSub`'s in the other 4 sessions. We have interrupted the code and will deal with the remaining 2516 groups by splitting them across 4 sessions. This still needs to be done.

As just mentioned, establishing whether Q_J contains any elementary abelian subgroups of order 2^3 that'd be of interest to us is still pending.

5.2.3 $Q_J, x_{J,a}$ for $J = \{3, 4, 7, 8\}$

Here every group collected will be satisfying a Steinberg bound of 5 and any `Sub4aa`'s not satisfying the bound were almost always ignored.

We break up Q_J as usual. We have the following information on the 5 groups b contained in the `BadSetNew` output: $\langle |b|, k,$`#SetKeep` (`STH` $= 0$)$\rangle = \langle 2^{94}, 6, 385\rangle$, $\langle 2^{109}, 5, 111\rangle$, $\langle 2^{76}, 7, 5769\rangle$, $\langle 2^{27}, 6, 377\rangle$, $\langle 2^{42}, 8, -\rangle$. We first look at all the groups apart from the one of order 2^{109}:

- We run A.4 with `STH` $= 4$ on the group of order 2^{42}.

- We run (†) on the group of order 2^{27} for an empty output.

- The group of order 2^{76} is a big group with a big `#SetKeep` and so running (†) on it would take a while. To speed things up a bit we do run (†) but with everything a sequence. This returns a `BadSetNew` containing 5769 distinct groups of order $\leq 2^{40}$. We run A.2 on these groups until `#ind` $= 195$. To finish, we run A.3, but with `SetSub2` and `BadSetNew` as sequences, on the 195 groups indexed by `#ind`.

- We run (†) on the 2^{94} and get a `BadSetNew` of size 387. One of the 387 has order 2^{52} and $k = 9$, also `#SetKeep` $= 327241$ with `STH` $= 0$. With `STH` $= 5$ though, `#SetKeep` can go down to 201, and so we run (††) (with `STH` $= 5$) on the group. We get a `BadSetNew` containing 201 groups of order 2^{33} or 2^{40} and we run A.3 on it but with `SetSub2` and `BadSetNew` as sequences.

  184 of the 387 have $k = 6$ and order $\leq 2^{60}$. With `STH` $= 0$, `#SetKeep` for these groups is ≤ 161 (which is nice and small) except when it's 377 for one group or 1161 for eight (the only ones of order 2^{60}) others. Essentially, these 184 groups don't look like they'd give us much grief so we collect them together with the 56 of the 387 with $k = 5$ and the 56 with $k = 4$. We run A.2 on the 296 groups until `#ind` $= 21$. We run (†) on the groups indexed by `ind` and get a `BadSetNew` containing 1097 groups of order 2^{31}. To finish, we run A.3, but with `SetSub2` and `BadSetNew` as sequences, on the 1097 groups.

  One of the 387 has $k = 8$, order 2^{42} and `#SetKeep` $= 7561$ with `STH` $= 0$. But only 2816 of the 7561 `Sub4aa`'s will satisfy the Steinberg bound. The remaining 89 of the 387 have $k = 7$ and `#SetKeep` ≤ 2185, being 1609 forty-two times, with

STH $= 0$; for each possible value for #SetKeep we selected a group and tested running code on it. It seemed like if we were to run A.3 (with SetSub2 and BadSetNew as sequences) on the 89 groups, it'd take $\approx$ 14d to finish. Hence we do run this code but together on the one group with $k = 8$ and the 14 groups indexed by ind (gotten after running A.2 on the 89).

We now run (†) on the group of order 2^{109} and get a BadSetNew of size 120. We deal with the 120 groups as below:

- 6 of the 120 have $k = 6$ and order $\leq 2^{27}$; we run A.3 on them. We saw that for each group, #SetKeep was either 8777 or 377 but none of the Sub4aa's satisfied the Steinberg bound, and so the code was very quick to run.

- 8 of the 120 have $k = 5$ and order 2^{76}. We run A.2 on them and get 4 groups (the ones indexed by ind). Running (†) on the 4 groups gives us a FinSub containing a single group of order 2^{18} (this doesn't contain any subgroups of interest) and a BadSetNew containing 457 groups of order $\leq 2^{44}$. We run A.2 on the 457 groups followed by A.3 (with SetSub2 and BadSetNew as sequences) on the 63 groups indexed by ind that we obtain.

- 9 of the 120 have $k = 8$ and order 2^{67}. 97 of the 120 have $k = 7$; 3 of these are of order 2^{33} (#SetKeep $= 1449$ with STH $= 0$), 2^{41} (#SetKeep $= 2017$) and 2^{76} each, 94 of 2^{73}. Running A.2 on the 9 of order 2^{67} gives #ind $= 5$. With STH $= 0$, #SetKeep for the 5 groups indexed by ind is 3401 two times and 1977 the rest. Running (†) (with everything a sequence) on a test group with #SetKeep $= 3401$ and on one with 1977, gives BadSetNew's of sizes 3312 and 1952, respectively, both containing distinct groups; empty FinSub's are also output. We run A.3 (with SetSub2 and BadSetNew as sequences) together on the 5 groups and the two groups of order 2^{33} and 2^{41} each.

  The group of order 2^{76} with $k = 7$ has #SetKeep $= 310$ (with STH $= 0$). We run (†) on it and get a BadSetNew containing 302 groups of order $\leq 2^{41}$. We then run A.2 on the 302 groups followed by A.3 (with SetSub2 and BadSetNew as sequences) on the 25 groups indexed by ind that we obtain.

  It is left to deal with the 94 groups of order 2^{73}. With STH $= 0$, #SetKeep can either be 6217 or 2633 for any one of these groups. Running (†) (with everything a sequence) on a test group with `#SetKeep` $= 6217$ returns an empty `FinSub` and a `BadSetNew` containing 6205 groups of order $\leq 2^{34}$, at least 1500 of which will be distinct; this takes $\approx$ 2d. For a test group with `#SetKeep` $= 2633$, running the same code takes $\approx$ 14.5h to give an empty `FinSub` and a `BadSetNew` containing 2633 groups (at least 900 of which will be distinct) of order $\leq 2^{34}$. All 2633 groups have elementary abelian centres; among these are 441 groups b such that $|b| = 2^{32}$, $|\Phi(b)| = 2^{20}$ and $|Z(b)| = 2^{14}$ (of course, $b/Z(b)$ is not elementary abelian). Take a particular b then running (†) on it takes $\approx$ 1.75m (to return non-empty but small `FinSub` and `BadSetNew`). In comparison, taking `SetSub2` to be the set $\{\langle Z(b), \gamma^{\langle x_{J,a} \rangle}\rangle : \gamma \in \Gamma, o(\gamma) \leq 2\}$ takes approximately a minute less; here Γ is a transversal for $Z(b)$ in b. Indeed, running the for loop over `[1..#BadSub]` in A.3 (with `SetSub2` and `BadSetNew` as sequences and `loop` $= 2$) on all the 2633 groups takes $\approx$ 20.5h whereas running the corresponding loop in A.6 takes $\approx$ 14h.

We run A.2 on the 94 groups until `#ind` $= 15$. We split the 15 groups indexed by `ind` over 2 Magma sessions and in each session, run A.6.

Note that in A.6, before asking for a transversal of the centre of a group it is checked that the index is $\leq 2^{18}$. This is because the larger the index gets the longer `Transversal` will take to execute.

We have established that Q_J does not contain any elementary abelian subgroups of order 2^3, that'd be of interest to us.

Remark 5.2.2. *Note that* 5 *of the pairs* $Q_J, x_{J,a}$ *and all* 8 $Q_J, x_{J,b}$ *have not been discussed in this thesis. However, computations on five of the remaining cases are indeed finished (giving rise to no subgroups of interest) and partial progress has been made with the rest. Finishing the* $L_2(8)$ *problem is simply a matter of setting off code and, given enough computer time, will be achievable before long.*

Chapter 6

$L_3(4)$ and $L_3(3)$

In this chapter, we will establish that $L_3(4)$ cannot be maximal in $E_8(2)$. We will also see that the same cannot be said for all copies of $L_3(3)$ inside $E_8(2)$. Throughout this chapter, $G \leq GL_{248}(2)$, will be $E_8(2)$. We mention that the work done up until Lemma 6.1.4 is by other people involved in classifying the maximal subgroups of $E_8(2)$, with no involvement from the author of this thesis.

6.1 Commonalities

As usual we want to construct copies of $L_3(4)$ and $L_3(3)$ inside $E_8(2)$. Therefore, we need information on the structure of these groups; this is given in the following two lemmas.

Lemma 6.1.1. *Given $H \cong L_3(4)$ and $E \in Syl_3(H)$ then:*

(i) $E \cong 3^2$;

(ii) $N_H(E) = E : Q$, *with* $Q \cong Q_8$;

(iii) The central involution, t, of Q inverts E;

(iv) $H = \langle N_H(E), C_H(Q) \rangle$, *with* $C_H(Q) \cong 2^2$.

Lemma 6.1.2. *Given $H \cong L_3(3)$ and $S \in Syl_3(H)$ then:*

(i) S is contained in a subgroup, $P_1 = E : \langle t \rangle^{\cdot} K$, of H, with $E \cong 3^2$ and $\langle t \rangle^{\cdot} K \cong 2^{\cdot} Sym(4)$;

(ii) *The central involution, t, of $\langle t\rangle^{\cdot}K$ inverts E;*

(iii) *There is exactly one subgroup of H, other than P_1, also containing S and having shape $3^2 : 2^{\cdot}Sym(4)$, call it P_2;*

(iv) $H = \langle P_1, P_2\rangle$;

(v) *$N_{P_1}(S)$ has exactly two normal subgroups of order 3^2, one is E, call the other F. We have that $N_{P_1}(S)$ has 9 involutions, s, inverting F, and there are involutions, x, in $C_H(s)$ ($F : C_H(s) = P_2$) such that $H = \langle P_1, x\rangle$.*

Looking at the above lemmas we see that in order to construct copies of $L_3(4)$ and $L_3(3)$ in G, we first need to construct subgroups of G having shape $3^2 : Q_8$ and also those having shape $3^2 : 2^{\cdot}\mathrm{Sym}(4)$. To do this we will need a subgroup E of G, isomorphic to 3^2, and an involution $t \in G$, inverting E. We would then proceed by searching in $C_G(t)$ for groups isomorphic to Q_8 and those isomorphic to $2^{\cdot}\mathrm{Sym}(4)$. We start by narrowing down our choices for E and t.

From B.4, we have the following as all the possible fusion patterns for an embedding of $L_3(4)$ in G:

(i) $2\phi_1 + 3\phi_2 + 3\phi_3 + 4\phi_4 + 2\phi_5$ (3A→3D, 5AB→5B, 7A→7B, 7B∗∗ →7B)

(ii) $4\phi_1 + 2\phi_2 + 2\phi_3 + 1\phi_4 + 3\phi_5$ (3A→3C, 5AB→5B, 7A→7B, 7B∗∗ →7B)

For the purpose of ruling out pattern (i), we have the following lemma from [7].

Lemma 6.1.3. *Suppose that $A \leq G$ with A elementary abelian of order 9. If $A^\# \subseteq 3D_{E_8(2)}$, then up to G-conjugacy there are at most six classes. Further, $\dim(C_{V_{248}}(A)) = 26$ (three times), 32 (once) and 44 (two times).*

Proof. Selecting $g \in A^\#$, we calculate $C_G(g) \cong 3 \times U_9(2)$ (this can be done using `FindCent`). Employing `LMGRadicalQuotient` gives us $C_G(g)/\langle g\rangle$ as a permutation group in which we may determine the conjugacy classes of elements of order 3. Taking inverse images in $C_G(g)$, we then check which elementary abelian subgroups of order 9 have all their non-trivial elements in $3\mathrm{D}_{E_8(2)}$. This results in six $C_G(g)$-classes of such subgroups for which we may then calculate $\dim(C_{V_{248}}(A))$. □

If our subgroup $E \leq G$ can be built up to a copy of $L_3(4)$ that embeds in G as described in pattern (i) then it must be that $E^{\#} \subseteq 3\mathrm{D}_{E_8(2)}$ since $3\mathrm{A}_{L_3(4)} \to 3\mathrm{D}_{E_8(2)}$. Hence by Lemma 6.1.3, $\dim(C_{V_{248}}(E)) = 26, 32$ or 44. But the dimension of the fixed space of any Sylow 3-subgroup of $L_3(4)$ on the modules corresponding to ϕ_2, ϕ_3, ϕ_4 and ϕ_5 is $1, 1, 0$ and 8, respectively. Hence, it must be that $\dim(C_{V_{248}}(E)) = 24$, a contradiction.

Now that we know pattern (i) isn't possible, we are interested in trying to build up a subgroup $E \leq G$ to an embedding of $L_3(4)$ in G only if $E^{\#} \subset 3\mathrm{C}_{E_8(2)}$ and $\dim(C_{V_{248}}(E)) = 32$.

It can be checked that any involution of $L_3(4)$ fixes a space of dimension $5, 5, 8$ and 32 on the modules corresponding to ϕ_2, ϕ_3, ϕ_4 and ϕ_5, respectively. Hence by Lemma 4.1.2, we are interested in an involution, t, inverting E (the E we are trying to build up to an $L_3(4)$) only if $\dim(C_{V_{248}}(t)) \leq 128$, i.e., $t \in 2\mathrm{D}_{E_8(2)}$ (see Proposition 2.2.1).

We can say the same things about E and t if we are trying to construct an overgroup of $\langle E, t\rangle$ isomorphic to $L_3(3)$ instead of $L_3(4)$. The possible feasible decompositions of $L_3(3)$ on V_{248} are given in B.5. The decompositions (iii)-(v) are ignored due to Lemma 2.2.5(i) and Proposition 2.2.3.

Pattern (ii) is eliminated in [45]. $L_3(3)$ has subgroups of the form $13 : 3$. If $H \cong L_3(3)$ is a subgroup of $E_8(2)$ following (ii) then all elements of order 3 of H are in $3\mathrm{C}_{E_8(2)}$ and all its elements of order 13 are in $13\mathrm{B}_{E_8(2)}$. It is shown in [45], by working with the normaliser of a Sylow 13-subgroup of G, that no element in $3\mathrm{C}_{E_8(2)}$ can act on an element in $13\mathrm{B}_{E_8(2)}$. Hence pattern (ii) is not achievable after all.

We have that if $H \cong L_3(3)$ is a subgroup of G then H must follow (i). Assume $E : \langle t\rangle^{\cdot}\mathrm{Sym}(4) \leq H$ but any elementary abelian subgroup of order 9 of H whose normaliser in H has shape $3^2 : 2^{\cdot}\mathrm{Sym}(4)$ has all its non-identity elements in $3\mathrm{A}_H$ and so again we have that $E^{\#} \subset 3\mathrm{C}_{E_8(2)}$. Also, it is true that $\dim(C_{\phi_2}(E)) = 4$ and $\dim(C_{\phi_3}(E)) = 2$ (ϕ_2 and ϕ_3 as in B.5), therefore $\dim(C_{V_{248}}(E)) = 32$.

Note that $E : \langle t\rangle^{\cdot}\mathrm{Sym}(4) \leq H$ contains a Sylow 3-subgroup of H, call it S. Then since $3\mathrm{A}_H \to 3\mathrm{C}_{E_8(2)}$ and $3\mathrm{B}_H \to 3\mathrm{D}_{E_8(2)}$, it must be that 14 of the non-identity elements of S are in $3\mathrm{C}_{E_8(2)}$ and 12 of them are in $3\mathrm{D}_{E_8(2)}$. This fact will be used later to discard constructed groups of shape $3^2 : 2^{\cdot}\mathrm{Sym}(4)$ whose Sylow 3-subgroups behave any differently.

In [45] while working out how $V_{248} \downarrow H$ (H following (ii)) decomposes, it was established that any involution t in H must be in $2\mathrm{D}_{E_8(2)}$. Hence the choices for E and t have been narrowed down to wanting only those such that $E^\# \subset 3\mathrm{C}_{E_8(2)}$, $\dim(C_{V_{248}}(E)) = 32$ and $t \in 2\mathrm{D}_{E_8(2)}$. Of course, we are interested in $\langle E, t\rangle$ only up to G-conjugacy. The next result by Rowley et al. gives us all the possibilities for E and t, but first we make mention of certain subgroups that will be featured in it.

For $x \in 3\mathrm{C}_{E_8(2)}$, by Theorem 2.2.2,

$$C_G(x) \sim 3^{\cdot}({}^2E_6(2) \times U_3(2)).3.$$

Let N_x denote the subgroup of $C_G(x)$ of index 3 with $N_x \sim 3^{\cdot}({}^2E_6(2) \times U_3(2))$, L_x the full inverse image of ${}^2E_6(2)$ in N_x and M_x the full inverse image of $U_3(2)$ in N_x. So $L_x \sim 3^{\cdot\,2}E_6(2)$. Note that if we write $G_1 \sim G_2$, then we mean that groups G_1 and G_2 have the same shape.

We want only the groups E containing x, since those containing $x' \in 3\mathrm{C}_{E_8(2)}$, $x' \neq x$, will be contained in $C_G(x')$ and be conjugate to the ones (containing x) in $C_G(x)$.

Lemma 6.1.4. *Suppose that $E \leq G$ where E is elementary abelian of order 3^2 and t is an involution of G which inverts E. Further assume that*

(i) $E^\# \subseteq 3C_{E_8(2)}$;

(ii) $\dim(C_{V_{248}}(E)) = 32$; *and*

(iii) $t \in 2D_{E_8(2)}$.

Then $\langle E, t\rangle$ is G-conjugate to one of $\langle E_i, t_{ij}\rangle$ where $i = 1, j = 1; i = 2, j = 1, 2, 3, 4, 5; i = 3, j = 1, 2$. The E_i are elementary abelian of order 3^2 and t_{ij} are involutions where $\langle E_i, t_{ij}\rangle \leq N_G(\langle x\rangle)$, some $x \in E^\#$. Further $E_1 \leq L_x, E_2 \leq N_x$, but $E_2 \nleq L_x$ and $E_2 \nleq M_x$ and $E_3 \nleq N_x$.

Proof. We start with $L \sim 3^8.2.\Omega_8^+(2).2$ (this is a subgroup of G constructed in [7] since it contains a Sylow 3-subgroup of G) for which we readily find a faithful permutation representation. In this setting we carry out the following calculations. Selecting $F \in \mathrm{Syl}_3(L)$, we use `ElementaryAbelianSubgroups` to find, up to F-conjugacy, 13416 elementary abelian subgroups of F of order 3^2. Of these only 5078 satisfy condition (ii).

Now we sieve again for those satisfying (i), using $\dim(C_{V_{248}}(y)) = 86$ for $y \in 3C_{E_8(2)}$. Only 1192 subgroups survive this sieve. We now take $x \in Z(F)$ of order 3 (in fact $\langle x \rangle = Z(F)$). Note that $x \in 3C_{E_8(2)}$. Now we focus on those 88 subgroups which contain x. Then running `IsConjugate` we find there are 13 L-classes of 3^2-subgroups which satisfy conditions (i) and (ii). Let $F_1, \ldots, F_{13}$ be representatives of these classes.

Employing `FindCent` gives us N_x (with $[C_G(x) : N_x] = 3$) and hence $C_G(x) = \langle F, N_x \rangle$. We have $F_i \leq N_x$ for $i \in \{1, \ldots, 12\}$ and $F_{13} \not\leq N_x$. Looking in $N_L(\langle x \rangle)$ we find an involution $s \in 2D_{E_8(2)}$ which inverts x. Thus $C_G^\star(x) = \langle C_G(x), s \rangle$. Using elements of order 19 in $C_G(x)$, we generate $L_x \sim 3^{\cdot 2}E_6(2)$. We find f_i such that $\langle x, f_i \rangle = F_i$ ($i \in \{1, \ldots, 13\}$). Then using `LMGIsIn` to test whether $f_i \in L_x$ we discover, up to labelling, that $F_i \leq L_x$ for $i \in \{1, \ldots, 5\}$ and $F_i \not\leq L_x, F_i \not\leq M_x$ for $i \in \{6, \ldots, 12\}$.

We now show that the F_i for $i \in \{1, \ldots, 5\}$ are all L_x-conjugate. Deploying `FindCent` in L_x to produce a partial centralizer for each f_i ($i \in \{1, \ldots, 5\}$) we find in each case $X \leq C_{L_x}(f_i)$ with $|X| = 2^9 3^8$ and X being 3-closed. Set $\overline{L_x} = L_x/\langle x \rangle (\cong {}^2E_6(2))$. From [52], $\overline{L_x}$ has three classes of elements of order 3 with $C_{\overline{L_x}}(3\mathrm{A}_{\overline{L_x}}) \cong 3 \times U_6(2)$, $C_{\overline{L_x}}(3\mathrm{B}_{\overline{L_x}}) \cong 3 \times \Omega_8^+(2) : 3$, and $C_{\overline{L_x}}(3\mathrm{C}_{\overline{L_x}}) \sim 3^{1+6} : 2^{3+6}.(3 \times 3)$. Taking a Sylow 3-subgroup of either $3 \times U_6(2)$ or $3 \times \Omega_8^+(2) : 3$ (which has order 3^7) we calculate its normaliser in the respective groups getting in each case a group of order $2^3 3^7$. Now $\overline{X}$ will be 3-closed with $|\overline{X}| = 2^9 3^7$, but this is bigger than the order of the normaliser of a Sylow 3-subgroup possible in $C_{\overline{L_x}}(3\mathrm{A}_{\overline{L_x}})$ or $C_{\overline{L_x}}(3\mathrm{B}_{\overline{L_x}})$, and so we deduce that $\overline{f_i} \in 3C_{\overline{L_x}}$ for $i \in \{1, \ldots, 5\}$. As a consequence the F_i, for $i \in \{1, \ldots, 5\}$ are all L_x-conjugate. Set $E_1 = F_1$.

For $i \in \{6, \ldots, 12\}$, similar arguments show that these F_i are all L_x-conjugate. There we calculate partial centralizers of f_i in N_x getting $3 \times \Omega_8^+(2)$ in each case. Set $E_2 = F_6$, and $E_3 = F_{13}$.

We now hunt for the possible inverting involutions for $E_i, i = 1, 2, 3$, looking in $H_i = \langle C_G(E_i), s \rangle$. In H_1 there is only one H_1-conjugacy class of inverting involutions, namely the one containing s. For $i = 2$, there are six H_2-classes of inverting involutions, with one of them not in $2D_{E_8(2)}$. While H_3 (where $C_G(E_3) \sim 3 \times 3 \times {}^3D_4(2) : 3$) may be turned into a 1638 degree permutation group and yields three H_3-classes of inverting involutions, two of which are in $2D_{E_8(2)}$.

This completes the proof of Lemma 6.1.4.

□

Remember, for a pair of E and t given by Lemma 6.1.4, we need to construct groups $E : Q$, where $Q \cong Q_8$ and $Z(Q) = \langle t \rangle$, and also groups $E : \langle t \rangle^{\cdot} K$ where $K \cong \mathrm{Sym}(4)$. Therefore we may search for the groups Q and $\langle t \rangle^{\cdot} K$ inside $N_{C_G(t)}(E)$. By Lemma 4.2.4, we know that $N_{C_G(t)}(E) \leq \mathrm{Stab}_{C_G(t)}(C_{V_{248}}(E))$. Running `CentraliserOfInvolution` gives us $C_G(t)$ if a group of order $2^{100}.3^5.5^2.7.17$ has been returned, see Proposition 2.2.1. We need elements of order 4 to generate Q but these elements square to t and so map to the identity or involutions in $\overline{C_G(t)}$ (since $\bar{t}$ is the identity), the radical quotient of $C_G(t)$. Therefore, if we can get our hands on a set of Sylow 2-subgroups of $\overline{C_G(t)}$ forming an involution cover then the preimages of these Sylow 2-subgroups will contain the generating elements we seek. We would then need to calculate the stabiliser of $C_{V_{248}}(E)$ inside these preimages only rather than the whole of $C_G(t)$. Call the preimages as $S_1, \ldots, S_r$, $r \in \mathbb{N}$, we wish to calculate the group,

$$J = \langle \mathrm{Stab}_{S_i}(C_{V_{248}}(E)) : i \in \{1, \ldots, r\} \rangle.$$

Note that $\langle t \rangle^{\cdot} K \cong Q_8 : \mathrm{Sym}(3)$ where $Z(Q_8 : \mathrm{Sym}(3)) = Z(Q_8)$. Let's say we have found all the wanted Q's in J, then the $\langle t \rangle^{\cdot} K$'s we seek can only be realised as overgroups of the Q's. The involutions needed to generate the Sym(3)'s will all lie in J since $S_1, \ldots, S_r$, being preimages of groups forming an involution cover of $\overline{C_G(t)}$, form an involution cover of $C_G(t)$.

We have just discussed that the groups we seek all lie in J. We now provide further details on how J is calculated. The space $C_{V_{248}}(E)$ is calculated as the intersection of fixed spaces of the generators of E and we call it `CVE`. The following code for calculating J has been adapted from [42]:

```
rq1,rq2,rq3:=LMGRadicalQuotient(CGt);
//CGt is the centraliser of the inverting involution.
//rq1 is the radical quotient of the centraliser.
//rq2 is a map from the centraliser to the quotient.
//rq3 is the soluble radical of the centraliser.

Crq1:=Classes(rq1);
```

```
//With rq1 a permutation group, the command Classes is executable.

Irq1:={};
for i in [1..#Crq1] do
if Crq1[i][1] eq 2 then
Irq1:=Irq1 join Class(rq1,Crq1[i][3]);
end if;
end for;
//Irq1 is the set of all involution of the quotient.

Srq1:=Sylow(rq1,2);
ISrq1:={};
for i in Srq1 do
if Order(i) eq 2 then Include(~ISrq1,i); end if;
end for;
//ISrq1 is the set of all involutions of a Sylow 2-subgroup of rq1.

ICov:={};
Itest:={};
repeat
old:=#Itest;
r:=Random(rq1);
Itest:=Itest join {k^r : k in ISrq1};
//Itest is created as the set of involutions of conjugates of Srq1.
new:=#Itest;
if new gt old then Include(~ICov,r); end if;
until Itest eq Irq1;
//The set ICov contains elements r of rq1 so that the groups Srq1^r
//form an involution cover of rq1.

SCG:=sub<Q|rq3,{i@@rq2 : i in Generators(Srq1)}>;
//SCG, the preimage of Srq1, is a Sylow 2-subgroup of CGt.
```

```
Gamma:=[r@@rq2 : r in ICov];
//Elements g in Gamma are so that the groups SCG^g form an involution
//cover of CGt.

SubGamma:={Random(Gamma) : i in [1..50]}; //A subset of Gamma.
J:=sub<Q|Id(Q)>;
for g in SubGamma do
SCGg:=SCG^g;
JJ:=UnipotentStabiliser(SCGg,CVE);
J:=sub<Q|J,JJ>;
end for;
LMGFactoredOrder(J);
//In most cases J will be the entire group that we are after rather
//than being a proper subgroup of it.
//The size of Gamma will be >3500 so if we can make all of J just
//using up to 50 Sylow 2-subgroups then this is a lot better than
//the alternative of trying to use all >3500 since that'd just add
//more and more generators to J (without increasing its size), making
//it impractical to work with.

//We do still need to check if we have all of J:
for g in Gamma do
if g notin SubGamma then
SCGg:=SCG^g;
JJ:=UnipotentStabiliser(SCGg,CVE);
if LMGIsSubgroup(J,JJ) eq false then
J:=sub<Q|J,JJ>;
end if;
end if;
end for;
LMGFactoredOrder(J);
```

The result of running the above code on all pairs E, t, is given in the table below.

E	t	$\|J\|$
E_1	t_{11}	$2^8.3^5$
E_2	t_{21}	$2^{19}.3^5$
	t_{22}	$2^{17}.3^3.5$
	t_{23}	$2^{17}.3^3.5$
	t_{24}	$2^{17}.3^3.5$
	t_{25}	$2^{17}.3^3$
E_3	t_{31}	$2^{15}.3^2.7$
	t_{32}	$2^{13}.3$

Table 6.1: Orders of J.

We are, of course, not interested in all of J but the normaliser of E in it.

Lemma 6.1.5. *For $i = 1, 3$ (both cases) $J = N_J(E_i)$ and for $i = 2$ (all five cases) $[J : N_J(E_i)] = 8$.*

Proof. We check whether all the generators of J normalise E and find that they do if $E = E_1$ or E_3 (both cases). In the other cases, forming the subgroup of J generated by those generators of J which do normalise E, gives a subgroup, J_s, of index 8 (4 times) or 24 (once). Note that an element in $g \in J \setminus J_s$ normalises E iff everything in $J_s g$ normalises E. Hence we ask for a transversal of J_s in J and find that when index is 8 only one element of the transversal normalises E and so $N_J(E) = J$. When index is 24 we find two elements, other than the one in J_s, that normalise E; the subgroups generated by them and J_s is our $N_J(E)$. □

Having found $N_J(E)$, we can now search in it for the groups Q to end this section.

Lemma 6.1.6. *Up to conjugacy in $N_J(E)$, the following holds.*

(i) $N_J(E_1)$ with $t = t_{11}$ has a unique Q_8 subgroup.

(ii) $N_J(E_2)$ for $t = t_{21}$, respectively $t = t_{25}$, has six Q_8 subgroups, respectively, four Q_8 subgroups. For $t = t_{22}, t_{23}, t_{24}, N_J(E_2)$ has no Q_8 subgroups.

(iii) $N_J(E_3)$ for $t = t_{31}$, respectively, $t = t_{32}$, has fourteen Q_8 subgroups, respectively, two Q_8 subgroups.

Proof. In each case, we are able to use `PermutationRepresentation` on $N_J(E)$ and will now perform calculations in the permutation setting. We ask for a Sylow 2-subgroup, S, of $N_J(E)$ and then for all its subgroups of order 8 up to conjugacy. In the t_{31} case, S, is a group of a very high degree of 781956 (`DegreeReduction` doesn't do us any good) and will further need to be converted into a pc-group before we ask for its subgroups. Let $\mathcal{L}_1$ be the set of these groups of order 8. We want only those groups in $\mathcal{L}_1$ that have t in them; we collect these in a set we name $\mathcal{L}_2$. In $\mathcal{L}_3$, we collect together all the groups in $\mathcal{L}_2$ that are isomorphic to Q_8. The groups in $\mathcal{L}_3$ are unique up to conjugacy in S; we need to check if any are conjugate in $N_J(E)$. To do this in the t_{31} case we need to map the groups in $\mathcal{L}_3$ back to the permutation group $N_J(E)$ of degree 781956. Even though the degree is quite large, `IsConjugate` will work quickly enough for our purposes. The code written below turns `orbs` into a sequence of sets where each set is a collection of indices that correspond to the positions of groups in $\mathcal{L}_3$ that are conjugate in $N_J(E)$; indices in different sets will label non-conjugate groups. Below `List3` is $\mathcal{L}_3$, `P` is the permutation group $N_J(E)$, `p` is the isomorphism from the matrix group to `P`.

```
orbs:=[{@i@} : i in [1..#List3]];
for i in [1..(#List3-1)] do
for k in orbs do
if i in k then size:=#k; end if;
end for;
if size eq 1 then
for j in [(i+1)..#List3] do
for k in orbs do
if j in k then size2:=#k; end if;
end for;
if size2 eq 1 then
if IsConjugate(P,List3[i],List3[j]) then
for o in orbs do
if i in o then oi:=o; end if;
if j in o then oj:=o; end if;
end for;
```

```
Exclude(~orbs,oi);
Exclude(~orbs,oj);
Include(~orbs,oi join oj);
end if;
end if;
end for;
end if;
end for;
List4:=[(List3[i[1]])@@p : i in orbs];
```

After the first iteration of the for loop over `[1..(#List3-1)]`, the indices labelling all the groups conjugate to `List3[1]` are collected together in a set, say O. Going forward, we only want to consider `i` > 1, `j` $>$ `i` such that `i`, `j` $\notin O$ and so we make sure that the identifiers `size` and `size2` have a value of 1. Let $\mathcal{L}_4$ be `List4`, the collection of all groups in $N_J(E)$ isomorphic to Q_8, with $\langle t \rangle$ as the centre, up to $N_J(E)$-conjugacy.

Our findings are displayed in Table 6.2 below.

E	t	$\vert\mathcal{L}_1\vert$	$\vert\mathcal{L}_2\vert$	$\mathcal{L}_3$	$\vert\mathcal{L}_4\vert$
E_1	t_{11}	162	37	1	1
E_2	t_{21}	17589	1891	428	6
	t_{22}	19941	893	0	0
	t_{23}	19941	893	0	0
	t_{24}	19941	893	0	0
	t_{25}	14041	583	14	4
E_3	t_{31}	19405	3039	840	14
	t_{32}	10801	502	6	2

Table 6.2: Finding the groups Q.

□

6.2 $L_3(4)$

Continuing directly from the construction in Lemma 6.1.6 of the groups Q isomorphic to Q_8, we will now construct copies of $L_3(4)$ in G. To do this, we need to run through all ($2\mathrm{D}_{E_8(2)}$) involutions x in G that centralise Q and see if $\langle E, Q, x\rangle \cong L_3(4)$ (see Lemma 6.1.1(iv)); here, with t being the central involution of Q, E and t are given in Lemma 6.1.4.

We label the groups Q in Lemma 6.1.6(i) as Q_{11}, from (ii) as Q_{21i}, $1 \le i \le 6$ and Q_{25i}, $1 \le i \le 4$, and from (iii) as Q_{31i}, $1 \le i \le 14$ and Q_{32i}, $1 \le i \le 2$. Of course, $C_G(Q) \le C_G(t)$. We first look at the cases Q_{11}, Q_{253}, Q_{254} and Q_{322}.

Let $\overline{C}$ be the radical quotient of $C_G(t_{11})$ then $|\overline{Q_{11}}| = 2^2$, $|C_{\overline{C}}(\overline{Q_{11}})| = 2^{10}.3$. The preimage of $C_{\overline{C}}(\overline{Q_{11}})$ is a soluble group of order $2^{94}.3$ containing $C_G(Q_{11})$; we turn it into a pc-group using `LMGSolubleRadical` and ask in it for the centraliser of Q_{11}. We get that $|C_G(Q_{11})| = 2^{28}.3$. This group is too big to go through all of its elements and pick out the ones in $2\mathrm{D}_{E_8(2)}$, and so staying in the pc-group setting, we run `Classes` on $C_G(Q_{11})$. Picking out the representatives which (when mapped back into $GL_{248}(2)$) would fix a space of dimension 128, we take the union of their classes. This gives us a set of 315392 involutions x. One at a time, we map x back into the matrix setting, if for every $y \in E_1^{\#}$, $o(xy) = 5$, we compute $L = \langle E_1, Q_{11}, x\rangle$. We then check if $|L| = |L_3(4)|$, if so, we check if $L \cong L_3(4)$. Any L's that survive the checks are kept in a set called $\mathcal{L}$; there will be only 3 of them. Note that starting by trying to collect 315392 248×248 matrices instead would not have been a good idea. We repeat the same process for Q_{253} and Q_{254} but starting with $\overline{C}$ as the radical quotient of $C_G(t_{25})$, and then for Q_{322}. Our findings are displayed in Table 6.3 below.

E_i	t_{ij}	Q	$\lvert C_{\overline{C}}(\overline{Q})\rvert$	$\lvert C_G(Q)\rvert$	$\lvert C_G(Q) \cap 2\mathrm{D}_{E_8(2)}\rvert$	$\lvert\mathcal{L}\rvert$
E_1	t_{11}	Q_{11}	$2^{10}.3$	$2^{28}.3$	315392	3
E_2	t_{25}	Q_{253}	$2^{10}.3$	$2^{28}.3$	315392	0
		Q_{254}	$2^{10}.3$	$2^{28}.3$	315392	0
E_3	t_{32}	Q_{322}	$2^{10}.3$	$2^{28}.3$	315392	0

Table 6.3: Finding possible $L_3(4)$'s arising from Q_{11}, Q_{253}, Q_{254} and Q_{322}.

With $Q = Q_{251}, Q_{252}$ or Q_{321}, we define $\overline{C}$ to be the radical quotient of $C = C_G(t_{25})$ or $C_G(t_{32})$. We get $C_{\overline{C}}(\overline{Q})$ as a group of order $2^{12}.3.5$ and we won't be able to turn its preimage in C into a pc-group. We're interested only in the preimages of its Sylow 2-subgroups anyway, call these groups of order 2^{96} as $S_1, \ldots, S_k$, $k \in \mathbb{N}$. We then calculate the group $U = \langle \mathrm{Stab}_{S_i}(C_{V_{248}}(Q)) : 1 \le i \le k\rangle$, by Lemma 4.2.4, U will contain all the involutions in G that centralise Q. In the $Q = Q_{252}$ case, U can't be turned into a pc-group and so we come back to it later. Proceeding with the other two cases, we convert U into a pc-group, calculate $C_U(Q)$, get $C_U(Q) \cap 2\mathrm{D}_{E_8(2)}$ and proceed as above, collecting any $L_3(4)$'s that arise in $\mathcal{L}$.

Going back to the $Q = Q_{252}$ case, we have that $|U| = 2^{34}.3.5$. We are still able to find $C_U(Q)$: Take the subgroup of U generated by all generators of U that centralise Q and call it U_s, this has a small index in U; we then take $C_U(Q)$ to be the group generated by U_s and all the elements in a transversal of U_s that centralise Q. We get that $|C_U(Q)| = 2^{32}.3.5$. We take a Sylow 2-subgroup, S, of $C_U(Q)$, turn it into a pc-group, find all the $2\mathrm{D}_{E_8(2)}$ involutions in it (there will be 3325952 of them) and proceed as usual, collecting any $L_3(4)$'s that arise. It turns out we don't have to repeat this with the rest of the Sylow 2-subgroups of $C_U(Q)$: All of the 3325952 involutions are contained in $O_2(C_U(Q))$ (membership checked in pc-group setting after taking the image of $O_2(C_U(Q))$ in the pc-group S). Our findings are displayed in Table 6.4 below.

E_i	t_{ij}	Q	$\lvert C_{\overline{C}}(\overline{Q})\rvert$	$\lvert U\rvert$	$\lvert C_U(Q)\rvert$	$\lvert C_U(Q) \cap 2\mathrm{D}_{E_8(2)}\rvert$	$\lvert\mathcal{L}\rvert$
E_2	t_{25}	Q_{251}	$2^{12}.3.5$	2^{34}	2^{32}	704512	0
		Q_{252}	$2^{12}.3.5$	$2^{34}.3.5$	$2^{32}.3.5$	3325952	16
E_3	t_{32}	Q_{321}	$2^{12}.3.5$	2^{32}	2^{30}	573440	4

Table 6.4: Finding possible $L_3(4)$'s arising from Q_{251}, Q_{252} and Q_{321}.

We now deal with the $t = t_{21}$ and $t = t_{31}$ cases. Let Q be one of the 20 groups isomorphic to Q_8 then Q lies in the soluble radical of $C_G(t)$ and so $C_{\overline{C}}(\overline{Q})$ will be all of $\overline{C}$. Recall that an involution cover of $C_G(t)$ was calculated in Section 6.1, call this $\mathcal{C}$. We use the code in Section 6.1 (for calculating `J`) to calculate the group $U = \langle \mathrm{Stab}_S(C_{V_{248}}(Q)) : S \in \mathcal{C}\rangle$. We turn U into a pc-group, compute $C_U(Q)$ and proceed as normal. Our findings are displayed in Table 6.5 below.

E_i	t_{ij}	Q	$\lvert U\rvert$	$\lvert C_U(Q)\rvert$	$\lvert C_U(Q) \cap 2\mathrm{D}_{E_8(2)}\rvert$	$\lvert\mathcal{L}\rvert$
E_2	t_{21}	Q_{211}	$2^{41}.3^4$	$2^{36}.3^3$	5066752	0
		Q_{212}	2^{41}	2^{34}	5545984	0
		Q_{213}	$2^{41}.3$	2^{34}	14897152	0
		Q_{214}	$2^{41}.3^2$	$2^{34}.3$	6094848	0
		Q_{215}	2^{41}	2^{34}	6967296	0
		Q_{216}	2^{41}	2^{34}	13832192	0
E_3	t_{31}	Q_{311}	2^{40}	2^{34}	4059136	0
		Q_{312}	2^{40}	2^{34}	4059136	0
		Q_{313}	2^{40}	2^{34}	3268608	0
		Q_{314}	2^{40}	2^{34}	4059136	0
		Q_{315}	2^{40}	2^{34}	4059136	0
		Q_{316}	2^{40}	2^{34}	4059136	0
		Q_{317}	2^{40}	2^{34}	3268608	0
		Q_{318}	2^{40}	2^{34}	4059136	0
		Q_{319}	2^{40}	2^{34}	4059136	0
		$Q_{31(10)}$	2^{40}	2^{34}	4059136	0
		$Q_{31(11)}$	2^{40}	2^{34}	3268608	0
		$Q_{31(12)}$	2^{40}	2^{34}	3272704	0
		$Q_{31(13)}$	2^{40}	2^{34}	5578752	0
		$Q_{31(14)}$	2^{40}	2^{34}	4059136	0

Table 6.5: Finding possible $L_3(4)$'s arising in the $t = t_{21}$ and $t = t_{31}$ cases.

Each of the twenty-three subgroups of G isomorphic to $L_3(4)$ that we have found fixes a subspace of V_{248} of dimension 2, and so we have proved the following result.

Theorem 6.2.1. *If H is a subgroup of $E_8(2)$ such that $F^*(H) \cong L_3(4)$ then H is not maximal in $E_8(2)$.*

6.3 $L_3(3)$

In order to construct subgroups of G isomorphic to $L_3(3)$, by Lemma 6.1.2, we first need to construct subgroups $E : \langle t\rangle^{\cdot} K \cong 3^2 : 2^{\cdot}\mathrm{Sym}(4)$. The possible E and t are given in Lemma 6.1.4. We know from Section 6.1 that the groups $\langle t\rangle^{\cdot}K$ will lie in the groups $N_J(E)$. But $\langle t\rangle^{\cdot}K = Q : S$, where $Q \cong Q_8$, $S \cong \mathrm{Sym}(3)$, and $Z(Q : S) = \langle t\rangle = Z(Q)$. So if we had all the subgroups, Q, of $N_J(E)$ isomorphic to Q_8 and containing t, we could go through involutions in $N_J(E)$, normalising Q, and check if a pair of them along with Q generated a group isomorphic to $2^{\cdot}\mathrm{Sym}(4)$. But we do have all the subgroups Q; they are given to us by Lemma 6.1.6.

Lemma 6.3.1. *Up to conjugacy in $N_J(E)$, the following holds.*

(i) $N_J(E_1)$ with $t = t_{11}$ has two $2^{\cdot}Sym(4)$ subgroups.

(ii) $N_J(E_2)$ for $t = t_{21}$ has ten $2^{\cdot}Sym(4)$ subgroups and $N_J(E_2)$ for $t = t_{25}$ has twelve $2^{\cdot}Sym(4)$ subgroups.

(iii) $N_J(E_3)$ for $t = t_{31}$ and $t = t_{32}$ has no $2^{\cdot}Sym(4)$ subgroups.

Proof. (i) Let Q_1 be the group isomorphic to Q_8 in Lemma 6.1.6(i). $N_J(E_1)$ is a small group of order $2^8.3^5$ and we can simply use `Normaliser` to calculate $N_{N_J(E_1)}(Q_1)$. This will turn out to be a group of order $2^5.2^3$ containing 43 involutions of $2\mathrm{D}_{E_8(2)}$, name the involutions as $r_1, \ldots, r_{43}$. One by one, we compute the groups $\langle Q_1, r_i, r_j : 1 \le i \le 42, i+1 \le j \le 43\rangle$. If one such group has order 48, we check if it's isomorphic to $2^{\cdot}\mathrm{Sym}(4)$. Only 9 of the groups survive the checks. Using a permutation representation of $N_J(E_1)$ and the code in the proof of Lemma 6.1.6, we see that only 2 of the 9 are unique up to $N_J(E_1)$-conjugacy.

(iii) To tackle the t_{31} case, we use the same method as above except that the normalisers of the 14 groups isomorphic to Q_8 from Lemma 6.1.6(iii) have to be calculated in the permutation group $N_J(E)$. Thirteen of the normalisers have order 2^{10} and so can't possibly contain any Sym(3)'s. The only normaliser of order $2^{10}.3$ has 55 $2\mathrm{D}_{E_8(2)}$ involutions but no groups isomorphic to $2^{\cdot}\mathrm{Sym}(4)$ arise. As for the t_{32} case, the normalisers of both the Q_8's in $N_J(E)$ have order 2^6.

(ii) In the t_{25} case, given the 4 Q_8's, the order of the normaliser is $2^9.3$ two times and $2^9.3^2$, also two times of course. If the order is $2^9.3$ then the number of $2\mathrm{D}_{E_8(2)}$ involutions in the normaliser is 103, it is 295 otherwise. Across all the four Q_8's and normalising involutions, 416 groups isomorphic to $2^{\cdot}\mathrm{Sym}(4)$ are formed, with 12 of them being unique up to $N_J(E)$-conjugacy.

In the t_{21} case, we convert $N_J(E)$ into a pc-group and work in this setting. Given the six Q_8's from Lemma 6.1.6(ii), the orders of the normalisers are 2^{11} (three times), $2^{11}.3$, $2^{11}.3^2$ and $2^{11}.3^4$. Ignoring the three 2-groups, the number of $2\mathrm{D}_{E_8(2)}$ involutions contained in the normalisers is 247, 631 and 1783, respectively. The number of involutions can be too big to go through all pairs so instead let N be one of the three normalisers in question, say of the group Q, in $N_J(E)$, and label with $C_1, \ldots, C_n$, $n \in \mathbb{N}$, the conjugacy classes of $2\mathrm{D}_{E_8(2)}$ involutions in N in descending order of length. For $1 \leq i \leq n-1$, we fix an involution $r \in C_i$ and go through all involutions $r' \in C_{i+1} \cup \ldots \cup C_n$ to see if $\langle Q, r, r' \rangle$ is isomorphic to $2^{\cdot}\mathrm{Sym}(4)$. Note that if s is any involution in C_i other than r then we don't need to consider any group $\langle Q, s, r' \rangle$, $r' \in C_{i+1} \cup \ldots \cup C_n$, since $s \sim_g r$, some $g \in N$, and $\langle Q^g, s^g, r'^g \rangle = \langle Q, r, r'^g \rangle$, a conjugate of $\langle Q, s, r' \rangle$ in $N_J(E)$, is already being considered. Now, for $1 \leq i \leq n$, let $r_1, \ldots, r_{|C_i|}$ be all the involutions in C_i, we check if any of $\langle Q, r_1, r_j \rangle$, $2 \leq j \leq |C_i|$, are isomorphic to $2^{\cdot}\mathrm{Sym}(4)$. Across the three Q_8's we obtain 30 groups isomorphic to $2^{\cdot}\mathrm{Sym}(4)$, with 10 of them being unique up to $N_J(E)$-conjugacy. □

We have all the wanted subgroups $\langle t \rangle^{\cdot} K \cong 2^{\cdot}\mathrm{Sym}(4)$ as given by Lemma 6.3.1. Recall that if $E : \langle t \rangle^{\cdot} K$ can be built up to an $L_3(3)$ that we are after, then a Sylow 3-subgroup, S, of it should have 14 elements in $3\mathrm{C}_{E_8(2)}$ and 12 in $3\mathrm{D}_{E_8(2)}$. One of the two $E : \langle t \rangle^{\cdot} K$, with $E = E_1$ and $t = t_{11}$, has a Sylow 3-subgroup, S, containing 6 $3\mathrm{B}_{E_8(2)}$ elements and 20 $3\mathrm{C}_{E_8(2)}$ ones, and so is eliminated. The other has the right G-fusion in S. None of the ten $E : \langle t \rangle^{\cdot} K$, $E = E_2$, $t = t_{21}$ possess the right G-fusion in S and can be eliminated; with $t = t_{25}$ all but two of the $E : \langle t \rangle^{\cdot} K$ can also be ruled out in this way.

Now that we have all the groups, $P_1 = E : \langle t \rangle^{\cdot} K$, from Lemma 6.1.2(i), three of them, we may proceed to construct possible overgroups isomorphic to $L_3(3)$ as described in Lemma 6.1.2(v). Let S be a Sylow 3-subgroup of P_1, we calculate $N_{P_1}(S)$ and then its normal subgroups of order 3^2, we denote the one not equal to E by F. Out of the 9 involutions in $N_{P_1}(S)$ inverting F, we choose just one, say s (it doesn't matter which one we choose). Working in $C_G(s)$ we repeat the process for s, F as for t, E to get a corresponding J, which we denote by J_F (i.e. we use the code for calculating J in Section 6.1). Then, again as for J, we find $N_{J_F}(F)$; this group will contain all involutions in G that centralise s and normalise F.

For $E = E_1$ we get $|J_F| = 2^{17}.3^3, |N_{J_F}(F)| = 2^{14}.3^3$ and $|N_{J_F}(F) \cap 2\mathrm{D}_{E_8(2)}| = 5839$ and for $E = E_2$ with $t = t_{25}$ in both cases we get $|J_F| = 2^8.3^5 = |N_{J_F}(F)|$ and $|N_{J_F}(F) \cap 2\mathrm{D}_{E_8(2)}| = 1201$. Running through the $x \in N_{J_F}(F) \cap 2\mathrm{D}_{E_8(2)}$ we check whether $\langle P_1, x\rangle \cong L_3(3)$. As usual we first sieve the x's according to element orders – for $y \in E^{\#}$ we must have that xy has order 6 or 8. For those x's that survive, we must check that $|\langle P_1, x\rangle| = |L_3(3)|$ before employing `IsIsomorphic`. The outcome is that in the case $E = E_1$, we obtain exactly one $L_3(3)$ subgroup of G and for $E = E_2$, one of the possibilities yields no $L_3(3)$ subgroups whereas the other gives two $L_3(3)$ subgroups.

Name the three $L_3(3)$ subgroups of G as L_1, L_2 and L_3. We find that each L_i doesn't fix any non-zero vectors in V_{248}.

It is true that $\mathrm{Aut}(L_3(3)) \cong L_3(3) : 2$. We have the following result.

Proposition 6.3.2. *For $i \in \{1, 2, 3\}$, there are no subgroups isomorphic to $L_i : 2$ in G.*

Proof. Take a Sylow 2-subgroup of L_i, then this has a centre of order 2. Call the involution in the centre z. Let g be an involution in G that normalises L_i. Then g is L_i-conjugate to an involution $x \in G$ that fixes z. Also, since x normalises L_i, by Lemma 4.2.8, it'll stabilise the socle of $V_{248}\downarrow L_i$.

Therefore, if $\mathcal{C}$ is a set of Sylow 2-subgroups of $C_G(z)$ forming an involution cover of $C_G(z)$ and W is the socle of $V_{248}\downarrow L_i$, then we may search for x in $J = \langle \mathrm{Stab}_S(W) : S \in \mathcal{C}\rangle$. We use the code in Section 6.1 to calculate J and find that in all three cases $|J| = 2^4.3$ and $J \leq L_i$.

Therefore $x \in L_i$ and so $g \in L_i$. We have proved that there is no involution in $G \setminus L_i$ normalising L_i. □

One begins to wonder if each L_i could be a maximal subgroup of G and the following result, whose proof is by Rowley, tells us that this is indeed the case. First note that if $A.B$, where $A \trianglelefteq A.B$, $B \cong (A.B)/A$, is a group containing L_i then either $L_i \leq A$ or $L_i \cap A$ is trivial: For any $g \in L_i$, $(L_i \cap A)^g = L_i^g \cap A^g = L_i \cap A$, but L_i is simple. Further, if $L_i \cap A$ is trivial then L_i embeds into B.

Theorem 6.3.3. *There are at most three conjugacy classes of maximal subgroups of $E_8(2)$ isomorphic to $L_3(3)$.*

Proof. Let L be in $\{L_1, L_2, L_3\}$, the set containing the three groups in question. If L is not a maximal subgroup of G, then $L < M$ where M is one of the maximal subgroups given in the list in Chapter 1.

Recall that $C_{V_{248}}(L) = 0$. If M is a parabolic subgroup of G, then by constructing $O_2(M)$ in G, we find that either $1 \leq \dim(C_{V_{248}}(O_2(M))) \leq 8$ or $\dim(C_{V_{248}}(O_2(M)) = 14$. Note that $C_{V_{248}}(O_2(M))$, being generated by all the 1-dimensional $O_2(M)$-submodules of $V_{248}{\downarrow}O_2(M)$, is, by Lemma 4.2.7, stabilised by all of M. If $L < M$ then from the table in B.5, we see that the Brauer character of $C_{V_{248}}(O_2(M)){\downarrow}L$ is $n\phi_1$, $1 \leq n \leq 8$, $14\phi_1$ or $2\phi_1 + \phi_2$; in any case by Lemma 2.2.5(i) we have that $C_{V_{248}}(L) \neq 0$, a contradiction.

Because no involution can centralise L (see proof of Proposition 6.3.2), we see that L cannot be a subgroup of M if M has shape $(L_3(2) \times E_6(2)) : 2$, $3.(U_3(2) \times {}^2E_6(2)) : \mathrm{Sym}(3)$ or $\mathrm{Sym}(3) \times E_7(2)$.

The non-abelian groups whose order is divisible by 13 and involved in one of the remaining possibilities for M to contain L are: $F_4(2)$, $PSp_4(5)$, $U_3(4)$, ${}^3D_4(2)$, ${}^3D_4(4)$, $\Omega_8^+(4)$, $SU_5(4)$, $PGU_5(4)$, $\Omega_{16}^+(2)$.

Now $|PSp_4(5)|$, $|U_3(4)|$, $|SU_5(4)|$ and $|PGU_5(4)|$ are not divisible by 3^3, and so cannot contain an $L_3(3)$ subgroup.

In B.6, all the possible feasible decomposition (i)-(iv) of $F_4(2)$ on V_{248} would have a trivial submodule by Lemma 2.2.5(i). In B.7, $\phi_5, \ldots, \phi_8, \phi_{12}, \phi_{13}, \phi_{14}$ are the only irreducible characters involved in the decompositions (i)-(iv), but these are all self-dual and so by Lemma 2.2.5(iii) any copy of $\Omega_8^+(4)$ in G would fix a non-zero vector of V_{248}. Therefore we get that if $L < M$ with $M \sim U_3(3) : 2 \times F_4(2)$ or $M \sim \Omega_8^+(4).(\mathrm{Sym}(3) \times 2)$, then $C_{V_{248}}(L) \neq 0$.

If ${}^3D_4(2)$ has an $L_3(3)$ subgroup then this subgroup would contain a Sylow 13-subgroup of ${}^3D_4(2)$. The normaliser of this Sylow 13-subgroup in ${}^3D_4(2)$ would have shape $13 : 4$, whereas in the $L_3(3)$ subgroup it'd have shape $13 : 3$, a contradiction. Therefore $M \sim ({}^3D_4(2))^2 : 6$ is ruled out as a possible maximal subgroup containing L.

The maximal subgroups of ${}^3D_4(4)$ are given in [27]; they are $[4^9]SL_2(2^6) \circ \mathbb{Z}_3$, $[4^{11}]\mathbb{Z}_{63} \circ SL_2(4)$, $G_2(4)$, ${}^3D_4(2)$, $L_2(2^6) \times L_2(4)$, $(SL_2(2^6) \circ SL_2(2))2$, $(\mathbb{Z}_{21} \circ SL_3(4)).3.2$, $(\mathbb{Z}_{13} \circ SU_3(4)).2$, $(\mathbb{Z}_{21})^2 SL_2(3)$ and $\mathbb{Z}_{241}.4$. A group in this list can't contain $L_3(3)$ subgroups for one of the following reasons: 13 doesn't divide its order, $L_2(2^n)$ has abelian Sylow 2-subgroups whereas $L_3(3)$ doesn't or the normaliser of a Sylow 13-subgroup of ${}^3D_4(2)$ has shape $13:4$. Therefore ${}^3D_4(4)$ can contain no $L_3(3)$ subgroups, and so if $L \leq M$ then $M \not\cong {}^3D_4(4).6$.

Finally, $\Omega_{16}^+(2)$, being a subgroup of maximal rank can be constructed in G as being generated by the root subgroups associated to the roots of the extended Dynkin diagram apart from the one labelled by 1. We then find that the socle of $\Omega_{16}^+(2)$ on V_{248} is $1 \oplus 128$, meaning that $M \cong \Omega_{16}^+(2)$ cannot contain L.

Therefore L is a maximal subgroup of $G = E_8(2)$. □

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Appendix A

Programs

A.1 Code for $L_2(64)$

Given $J \in \mathcal{J}$, see Lemma 3.2.1, below we denote by `O`, the group Q_J, and by `x63`, the element x_J.

```
function Code(O,x63);
GROUPS:=[**];
RESULTS:=[**];

FinSub:={@@};
BadSub:={@@};
SetSub2:={@O@};
ActnGpDiff:={@@};
count:=0;
repeat
countt:=0;
SetSub:=SetSub2; count+:=1;
SetSub2:={@@};
for x in SetSub do countt+:=1;
Sub63:=sub<Q|x,x63>;
FX63:=FrattiniSubgroup(x);
MNt5aa,phit5aa:=GModule(Sub63,x,FX63);
```

```
if Order(ActionGroup(MNt5aa)) ne 63 then Include(~ActnGpDiff,x);

else

Com:=DirectSumDecomposition(MNt5aa);
Dim:=[Dimension(Com[i]): i in [1..#Com]];

CheckSet:= {@ 1 @};
ModSet:= {@ Com[1] @};
for i in [2..#Com] do
check:=0;
for j in CheckSet do
if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
end for;
if check eq 0 then
Include(~CheckSet,i); Include(~ModSet,Com[i]);
end if;
end for;

if Order(FX63) eq 1 then

for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
Include(~FinSub,IncGrp);
end if;
end for;
```

```
else

if #ModSet eq 1 then
if Dimension(Com[1]) eq 6 then Include(~BadSub,x); end if;
else
for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
Include(~SetSub2,IncGrp);
end if;
end for;
end if;

end if;

count,#SetSub,countt,Dim,#ModSet,#ActnGpDiff,"FinSub",#FinSub,\
"BadSub",#BadSub,#SetSub2,#RESULTS;

end if;

end for;
until #SetSub2 eq 0;

Append(~RESULTS, [*#FinSub,#BadSub,#ActnGpDiff*]);
Append(~GROUPS,BadSub);
```

```
BadSetNew:=BadSub;
loopn:=0;
bool:={@@};
bool2:={@@};
SetKeepZero:={@@};

repeat

BadSub:=BadSetNew; BadSetNew:={@@};

for k in [1..#BadSub] do

b:=BadSub[k];
Fb:=FrattiniSubgroup(b);
Pb,pmap:=PCGroup(b);
PFb:=pmap(Fb);
C:=CommutatorSubgroup(Pb,PFb);
QPFb,qPFb:=quo<PFb|C>;
FQPFb:=FrattiniSubgroup(QPFb);
A:=(FQPFb@@qPFb)@@pmap;

MNt,phit:= GModule(sub<Q|x63,b>,b,Fb);

actMNtstar:={@@};
for g in ActionGroup(MNt) do
if Order(g) ne 1 then Include(~actMNtstar,g); end if;
end for;
Include(~bool, forall{g : g in actMNtstar | \
Dimension(Eigenspace(g,1)) eq 0});

Com:= DirectSumDecomposition(MNt);
```

```
IsLarge:=[Dimension(Com[i]): i in [1..#Com]];


SetKeep:= {@@};
for i in [1..#Com-1] do
repeat xm:= Random(Com[i]);
until xm ne Zero(Com[i]);
x:= xm@@phit;
setym:={@@};
for j in [i+1..#Com] do
Include(~setym,Com[j]);
end for;
YM:= sub<MNt|setym>;
countym:=0;
for ym in YM do
countym:= countym+1;
y:= ym@@phit;
t:= x*y;
if t*t in A then Include(~SetKeep,t);
end if;
end for;
end for;
repeat x:= Random(Com[#Com]);
until x ne Zero(Com[#Com]);
t:=x@@phit;
if t*t in A then Include(~SetKeep,t);
end if;


Include(~bool2, #SetKeep ne 0);
if #SetKeep eq 0 then Include(~SetKeepZero,b); end if;


for r in [1..#SetKeep] do
x:=SetKeep[r];
```

```
set63:={@@};
for i in [1..63] do
Include(~set63,x^(x63^i));
end for;
Sub63:=sub<Q|Fb,x,x63>;
Sub4aa:=sub<Q|Fb,set63>;
MNt4aa,phit4aa:=GModule(Sub63,Sub4aa,A);

Com:=DirectSumDecomposition(MNt4aa);
Dim:=[Dimension(Com[i]): i in [1..#Com]];

CheckSet:= {@ 1 @};
ModSet:= {@ Com[1] @};
for i in [2..#Com] do
check:=0;
for j in CheckSet do
if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
end for;
if check eq 0 then
Include(~CheckSet,i); Include(~ModSet,Com[i]);
end if;
end for;

if Order(A) eq 1 then

for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt4aa|GenSet>;
```

```
IncGrp:= IncMod@@phit4aa;
Include(~FinSub,IncGrp);
end if;
end for;

else

if #ModSet eq 1 then
if Dimension(Com[1]) eq 6 then Include(~SetSub2,Sub4aa); end if;
else
for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt4aa|GenSet>;
IncGrp:= IncMod@@phit4aa;
Include(~SetSub2,IncGrp);
end if;
end for;
end if;

end if;

IsLarge,"loopn",loopn,#BadSub,"k",k,bool,bool2,#SetKeep,r,Dim,#ModSet,\
"FinSub",#FinSub,"BadSetNew",#BadSetNew,#SetSub2;

end for;

count:=0;
repeat
```

```
countt:=0;
SetSub:=SetSub2; count+:=1;
SetSub2:={@@};
for x in SetSub do countt+:=1;
Sub63:=sub<Q|x,x63>;
FX63:=FrattiniSubgroup(x);
MNt5aa,phit5aa:=GModule(Sub63,x,FX63);

if Order(ActionGroup(MNt5aa)) ne 63 then Include(~ActnGpDiff,x);

else

Com:=DirectSumDecomposition(MNt5aa);
Dim:=[Dimension(Com[i]): i in [1..#Com]];

CheckSet:= {@ 1 @};
ModSet:= {@ Com[1] @};
for i in [2..#Com] do
check:=0;
for j in CheckSet do
if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
end for;
if check eq 0 then
Include(~CheckSet,i); Include(~ModSet,Com[i]);
end if;
end for;

if Order(FX63) eq 1 then

for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
```

```
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
Include(~FinSub,IncGrp);
end if;
end for;

else

if #ModSet eq 1 then
if Dimension(Com[1]) eq 6 then Include(~BadSetNew,x); end if;
else
for m in ModSet do
if Dimension(m) eq 6 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
Include(~SetSub2,IncGrp);
end if;
end for;
end if;

end if;

"loopn",loopn,count,#SetSub,countt,Dim,#ModSet,#ActnGpDiff,\
"FinSub",#FinSub,"BadSetNew",#BadSetNew,#SetSub2;
```

```
end if;

end for;
until #SetSub2 eq 0;

end for;

loopn+:=1; Append(~RESULTS, <loopn,#BadSetNew,#FinSub,#ActnGpDiff,\
bool eq {@true@},bool2 eq {@true@},#SetKeepZero>);
Append(~GROUPS,BadSetNew);

until #BadSetNew eq 0;

BadSub:={@@};

Append(~GROUPS,FinSub);
Append(~GROUPS,ActnGpDiff);
Append(~GROUPS,SetKeepZero);

return RESULTS,GROUPS;
end function;

RESULTS,GROUPS:=Code(O,x63);
```

A.2 Code for Conjugating Groups in a BadSub

Below cpx is $C_P(x)$, where P is a standard parabolic subgroup (see Section 4.1).

```
ind:=[1..#BadSub];
orbs:=[{@i@} : i in ind];
count:=0;
repeat h:=Random(cpx); count+:=1; old:=#orbs;
for j in [1..(#ind-1)] do
```

```
for i in [(j+1)..#ind] do
if BadSub[ind[j]] eq BadSub[ind[i]]^h then
for o in orbs do
if ind[j] in o then oj:=o; end if;
if ind[i] in o then oi:=o; end if;
end for;
Exclude(~orbs,oj); Exclude(~orbs,oi); Include(~orbs,oj join oi);
break;
end if;
end for;
end for;
new:=#orbs;
if new lt old then ind:=[k[1] : k in orbs]; end if;
count,#elts,#orbs;
until 1 eq 2;

BadSub:=[BadSub[orbs[i][1]] : i in [1..#orbs]];
```

A.3 $L_2(8)$ Code 1

Below occurrences of `Include(~FinSub`, `Include(~SetSub2` or `Include(~BadSetNew` can be replaced by `Append` if seen fit according to the situation.

```
Prob23:={@@};
dimnotmetbool:={@@};

FinSub:=[];
SetSub2:=[];
ActnGpDiff:={@@};

bool:={@@};
bool2:={@@};
bool3:={@@};
```

```
SetKeepZero:=[];

for obs in [1..#OrigBadSub] do

BadSetNew:={@OrigBadSub[obs]@};

loop:=0;

repeat

BadSub:=BadSetNew; loop+:=1;
BadSetNew:=[];

for k in [1..#BadSub] do

b:=BadSub[k];
Pb,pmap:=PCGroup(b);
if loop eq 1 then Fb:=FrattiniSubgroup(b);
else if IsElementaryAbelian(Pb/Centre(Pb)) then Fb:=Centre(b);
else Fb:=FrattiniSubgroup(b);
end if;
end if;
PFb:=pmap(Fb);
C:=CommutatorSubgroup(Pb,PFb);
QPFb,qPFb:=quo<PFb|C>;
FQPFb:=FrattiniSubgroup(QPFb);
A:=(FQPFb@@qPFb)@@pmap;

MNt,phit:= GModule(sub<Q|x7,b>,b,Fb);

Include(~bool3, Order(ActionGroup(MNt)) eq 7);
```

```
actMNtstar:={@@};
for g in ActionGroup(MNt) do
if Order(g) ne 1 then Include(~actMNtstar,g); end if;
end for;
Include(~bool, forall{g : g in actMNtstar | \
Dimension(Eigenspace(g,1)) eq 0});

Com:= DirectSumDecomposition(MNt);

IsLarge:=[Dimension(Com[i]): i in [1..#Com]];

SetKeep:= {@@};
for i in [1..#Com-1] do
repeat xm:= Random(Com[i]);
until xm ne Zero(Com[i]);
x:= xm@@phit;
setym:={@@};
for j in [i+1..#Com] do
Include(~setym,Com[j]);
end for;
YM:= sub<MNt|setym>;
countym:=0;
for ym in YM do
countym:= countym+1;
y:= ym@@phit;
t:= x*y;
if t*t in A then Include(~SetKeep,t);
end if;
end for;
end for;
repeat x:= Random(Com[#Com]);
until x ne Zero(Com[#Com]);
```

```
t:=x@@phit;
if t*t in A then Include(~SetKeep,t);
end if;

Include(~bool2, #SetKeep ne 0);
if (#SetKeep eq 0 and Dimension(Fix(GModule(sub<Q|Fb,x7>))) le 5) \
then Include(~SetKeepZero,Fb); end if;

beta:=0;

for r in [1..#SetKeep] do
x:=SetKeep[r];
set7:={@@};
for i in [1..7] do
Include(~set7,x^(x7^i));
end for;
Sub7:=sub<Q|Fb,x,x7>;
Sub4aa:=sub<Q|Fb,set7>;

if Dimension(Fix(GModule(sub<Q|Sub4aa,x7>))) le 5 then beta+:=1;

MNt4aa,phit4aa:=GModule(Sub7,Sub4aa,A);

Com:=DirectSumDecomposition(MNt4aa);
Dim:=[Dimension(Com[i]): i in [1..#Com]];

CheckSet:= {@ 1 @};
ModSet:= {@ Com[1] @};
for i in [2..#Com] do
check:=0;
for j in CheckSet do
if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
```

```
end for;
if check eq 0 then
Include(~CheckSet,i); Include(~ModSet,Com[i]);
end if;
end for;

if Order(A) eq 1 then

for m in ModSet do
if Dimension(m) eq 3 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt4aa|GenSet>;
IncGrp:= IncMod@@phit4aa;
if Dimension(Fix(GModule(sub<Q|IncGrp,x7>))) le 5 then
Include(~FinSub,IncGrp);
end if;
end if;
end for;

else

if #ModSet eq 1 then
if Dimension(Com[1]) eq 3 then
if Dimension(Fix(GModule(sub<Q|Sub4aa,x7>))) le 5 then
Include(~SetSub2,Sub4aa);
end if;
end if;
else
for m in ModSet do
```

```
if Dimension(m) eq 3 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt4aa|GenSet>;
IncGrp:= IncMod@@phit4aa;
if Dimension(Fix(GModule(sub<Q|IncGrp,x7>))) le 5 then
Include(~SetSub2,IncGrp);
end if;
end if;
end for;
end if;

end if;

IsLarge,#OrigBadSub,"obs",obs,#BadSub,"k",k,bool,bool2,bool3,\
#SetKeep,r,Dim,#ModSet,#ActnGpDiff,"SetKeepZero",#SetKeepZero,\
"FinSub",#FinSub,"BadSetNew",#BadSetNew,"Prob23",#Prob23,\
"dimnotmetbool",dimnotmetbool,#SetSub2;

end if;

end for;

"obs",obs,"k",k,"#SetKeep",#SetKeep,"beta",beta;

count:=0;
repeat
countt:=0;
SetSub:=SetSub2; count+:=1;
SetSub2:=[];
```

```
for x in SetSub do countt+:=1;
Sub7:=sub<Q|x,x7>;
FX7:=FrattiniSubgroup(x);
MNt5aa,phit5aa:=GModule(Sub7,x,FX7);

if Order(ActionGroup(MNt5aa)) ne 7 then Include(~ActnGpDiff,x);

else

Com:=DirectSumDecomposition(MNt5aa);
Dim:=[Dimension(Com[i]): i in [1..#Com]];

CheckSet:= {@ 1 @};
ModSet:= {@ Com[1] @};
for i in [2..#Com] do
check:=0;
for j in CheckSet do
if IsIsomorphic(Com[i],Com[j]) then check:=1; end if;
end for;
if check eq 0 then
Include(~CheckSet,i); Include(~ModSet,Com[i]);
end if;
end for;

if Order(FX7) eq 1 then

for m in ModSet do
if Dimension(m) eq 3 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
```

```
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
if Dimension(Fix(GModule(sub<Q|IncGrp,x7>))) le 5 then
Include(~FinSub,IncGrp);
end if;
end if;
end for;

else

if #ModSet eq 1 then
if Dimension(Com[1]) eq 3 then Include(~BadSetNew,x); end if;
else
for m in ModSet do
if Dimension(m) eq 3 then
GenSet:={@@};
for n in Com do
if IsIsomorphic(n,m) then Include(~GenSet,n); end if;
end for;
IncMod:= sub<MNt5aa|GenSet>;
IncGrp:= IncMod@@phit5aa;
if Dimension(Fix(GModule(sub<Q|IncGrp,x7>))) le 5 then
Include(~SetSub2,IncGrp);
end if;
end if;
end for;
end if;

end if;

count,#SetSub,countt,Dim,#ModSet,#ActnGpDiff,"FinSub",#FinSub,\
"BadSetNew",#BadSetNew,#SetSub2;
```

```
end if;

end for;
until #SetSub2 eq 0;

end for;

until #BadSetNew eq 0;

BadSub:={@@};

dimnotmet:=FinSub;
for i in [1..#dimnotmet] do
Mdnmi,phidnmi:=GModule(sub<Q|dimnotmet[i],x7>,dimnotmet[i]);
Sdnmi:=MinimalSubmodules(Mdnmi);
Include(~dimnotmetbool,Order(ActionGroup(Mdnmi)) eq 7);
for s in Sdnmi do ps:=s@@phidnmi;
if Dimension(Fix(GModule(sub<Q|ps,x7>))) le 5 then
Include(~Prob23,ps);
end if;
end for;
end for;
dimnotmet:=[];
FinSub:=[];

end for;

#BadSetNew;
#FinSub;
#ActnGpDiff;
bool eq {@true@};
```

```
bool2 eq {@true@};
bool3 eq {@true@};
#SetKeepZero;

#Prob23;
dimnotmetbool;
```

A.4 $L_2(8)$ Code 2

Let b be such that $b/\Phi(b)$ is $V_1 \oplus \ldots \oplus V_k$. If we want to factor b out by the preimage of the sum of the first r summands then before running the following we must replace STH with the number r.

Below occurrences of .... mean that the code here is the same as in the relevant parts of A.3.

```
OrigBadSub:=[];

bool:={@@};
bool2:={@@};
bool3:={@@};
SetKeepZero:={@@};

for k in [1..#BadSub] do

b:=BadSub[k];

ML,phiL:=GModule(sub<Q|b,x7>,b,FrattiniSubgroup(b));
ComL:=DirectSumDecomposition(ML);
BSet:={@@};
for i in [1..STH] do Include(~BSet,ComL[i]); end for;
IncMod:=sub<ML|BSet>;
Fb:=IncMod@@phiL;
```

```
Pb,pmap:=PCGroup(b);
PFb:=pmap(Fb);
C:=CommutatorSubgroup(Pb,PFb);
QPFb,qPFb:=quo<PFb|C>;
FQPFb:=FrattiniSubgroup(QPFb);
A:=(FQPFb@@qPFb)@@pmap;

MNt,phit:= GModule(sub<Q|x7,b>,b,Fb);

Include(~bool3, Order(ActionGroup(MNt)) eq 7);

actMNtstar:={@@};
for g in ActionGroup(MNt) do
if Order(g) ne 1 then Include(~actMNtstar,g); end if;
end for;
Include(~bool, forall{g : g in actMNtstar | \
Dimension(Eigenspace(g,1)) eq 0});

Com:= DirectSumDecomposition(MNt);

SetKeep:= {@@};
//Insert usual method of adding elements to SetKeep here.

Include(~bool2, #SetKeep ne 0);
if #SetKeep eq 0 then Include(~SetKeepZero,Fb); end if;

for r in [1..#SetKeep] do
x:=SetKeep[r];
set7:={@@};
for i in [1..7] do
Include(~set7,x^(x7^i));
end for;
```

```
Sub4aa:=sub<Q|Fb,set7>;
Append(~OrigBadSub,Sub4aa);
end for;

end for;

#OrigBadSub;
bool eq {@true@};
bool2 eq {@true@};
bool3 eq {@true@};
#SetKeepZero;

Prob23:={@@};

.........

for obs in [1..#OrigBadSub] do

BadSetNew:={@OrigBadSub[obs]@};

loop:=0;

repeat

BadSub:=BadSetNew; loop+:=1;
BadSetNew:=[];

for k in [1..#BadSub] do

.........

MNt,phit:= GModule(sub<Q|x7,b>,b,Fb);
```

```
Include(~bool3, Order(ActionGroup(MNt)) eq 7);

Com:= DirectSumDecomposition(MNt);

IsLarge:=[Dimension(Com[i]): i in [1..#Com]];

SetKeep:= {@@};

if forall{z : z in Com | (Dimension(z) eq 3) and \
IsIsomorphic(z,Com[1])} eq false then Include(~SetSub2,b);

else

actMNtstar:={@@};
for g in ActionGroup(MNt) do
if Order(g) ne 1 then Include(~actMNtstar,g); end if;
end for;
Include(~bool, forall{g : g in actMNtstar | \
Dimension(Eigenspace(g,1)) eq 0});

for i in [1..#Com-1] do
repeat xm:= Random(Com[i]);
until xm ne Zero(Com[i]);
x:= xm@@phit;
setym:={@@};
for j in [i+1..#Com] do
Include(~setym,Com[j]);
end for;
YM:= sub<MNt|setym>;
countym:=0;
for ym in YM do
```

```
countym:= countym+1;
y:= ym@@phit;
t:= x*y;
if t*t in A then Include(~SetKeep,t);
end if;
end for;
end for;
repeat x:= Random(Com[#Com]);
until x ne Zero(Com[#Com]);
t:=x@@phit;
if t*t in A then Include(~SetKeep,t);
end if;

Include(~bool2, #SetKeep ne 0);
if (#SetKeep eq 0 and Dimension(Fix(GModule(sub<Q|Fb,x7>))) le 5) then
Include(~SetKeepZero,Fb);
end if;

end if;

beta:=0;

for r in [1..#SetKeep] do

.........

end for;

"obs",obs,"k",k,"#SetKeep",#SetKeep,"beta",beta;

count:=0;
repeat
```

```
countt:=0;
SetSub:=SetSub2; count+:=1;
SetSub2:=[];

.........

until #SetSub2 eq 0;

end for;

until #BadSetNew eq 0;

.........

end for;

.........

#Prob23;
dimnotmetbool;
```

A.5 $L_2(8)$ Code 3

Below occurrences of .... mean that the code here is the same as in the relevant parts of A.3 except that occurrences of `Include(~SetSub2` and `Include(~BadSetNew` have been changed to `Append`.

```
Prob23:={@@};

.........

for obs in [1..#OrigBadSub] do
```

```
BadSetNew:={@OrigBadSub[obs]@};

loop:=0;

repeat

BadSub:=BadSetNew; loop+:=1;
BadSetNew:=[];

for k in [1..#BadSub] do

.........

SetKeep:= {@@};

if ((Order(Pb) in {2^(25),2^(28)}) and \
(IsElementaryAbelian(Pb/Centre(Pb)) eq false) and \
(IsElementaryAbelian(Centre(Pb)))) then
zb:=Centre(b);
trp:=Transversal(Pb,Centre(Pb));
for t in trp do
if Order(t) le 2 then
gp:=sub<Q| zb, {(t@@pmap)^(x7^i) : i in [1..7]}>;
if Dimension(Fix(GModule(sub<Q|gp,x7>))) le 5 then
Include(~SetSub2,gp);
end if;
end if;
end for;

else

for i in [1..#Com-1] do
```

```
repeat xm:= Random(Com[i]);
until xm ne Zero(Com[i]);
x:= xm@@phit;
setym:={@@};
for j in [i+1..#Com] do
Include(~setym,Com[j]);
end for;
YM:= sub<MNt|setym>;
countym:=0;
for ym in YM do
countym:= countym+1;
y:= ym@@phit;
t:= x*y;
if t*t in A then Include(~SetKeep,t);
end if;
end for;
end for;
repeat x:= Random(Com[#Com]);
until x ne Zero(Com[#Com]);
t:=x@@phit;
if t*t in A then Include(~SetKeep,t);
end if;

Include(~bool2, #SetKeep ne 0);
if #SetKeep eq 0 then Include(~SetKeepZero,Fb); end if;

end if;

beta:=0;

for r in [1..#SetKeep] do
```

. . . .

```
#Prob23;
dimnotmetbool;
```

A.6 $L_2(8)$ Code 4

Below occurrences of . . . . mean that the code here is the same as in the relevant parts of A.5.

```
Prob23:={@@};
```

. . . .

```
SetKeep:= {@@};

if (((Order(Pb)/Order(Centre(Pb))) le 2^(18)) and \
(IsElementaryAbelian(Pb/Centre(Pb)) eq false) and \
(IsElementaryAbelian(Centre(Pb)))) then
zb:=Centre(b);
trp:=Transversal(Pb,Centre(Pb));
for t in trp do
if Order(t) le 2 then
gp:=sub<Q| zb, {(t@@pmap)^(x7^i) : i in [1..7]}>;
if Dimension(Fix(GModule(sub<Q|gp,x7>))) le 5 then
Include(~SetSub2,gp);
end if;
end if;
end for;

else

for i in [1..#Com-1] do
```

```
.........

#Prob23;
dimnotmetbool;
```

Appendix B

Brauer Character Tables

The following information is taken from [45] which is where it was calculated.

B.1 $L_2(64)$

Brauer Character Table

$\boldsymbol{L_2(64)}$	1A	3A	5AB	7AC	9AC	13AF	21AF
ϕ_1	1	1	1	1	1	1	1
ϕ_2	12	-6	-3	-2	0	-1	1
ϕ_3	12	3	-3	5	6	-1	-4
ϕ_4	16	-2	-4	2	-2	3	5
ϕ_5	24	6	9	-4	-6	-2	-1
ϕ_6	24	6	-6	-4	-6	-2	-1
ϕ_7	48	-6	3	-8	-6	9	1
ϕ_8	48	-6	3	6	-6	-4	-6
ϕ_9	48	-6	3	6	12	-4	-6
ϕ_{10}	48	3	3	6	9	-4	3
ϕ_{11}	64	1	-1	1	1	-1	1
ϕ_{12}	96	6	6	-2	0	5	-8
ϕ_{13}	96	6	-9	-2	0	5	13
ϕ_{14}	192	-6	-3	-4	6	-3	8

Feasible Decompositions

(i) $12\phi_1+1\phi_2+4\phi_3+2\phi_4+0\phi_5+0\phi_6+0\phi_7+2\phi_8+0\phi_9+1\phi_{10}+0\phi_{11}+0\phi_{12}+0\phi_{13}+0\phi_{14}$
(3A→ 3C, 5AB→ 5B, 7AC→ 7A, 9AC→ 9A, 13A→ 13B, 21AF→ 21D, 63AI→ 63D, 63JR→ 63E, 65AX→ 65AD)

(ii) $8\phi_1+2\phi_2+0\phi_3+2\phi_4+3\phi_5+0\phi_6+1\phi_7+2\phi_8+0\phi_9+0\phi_{10}+1\phi_{11}+0\phi_{12}+0\phi_{13}+0\phi_{14}$
(3A→ 3C, 5AB→ 5A, 7AC→ 7B, 9AC→ 9D, 13A→ 13B, 21AF→ 21F, 63AI→ 63FH, 63JR→ 63FH, 65AX→ 65EF)

(iii) $12\phi_1+4\phi_2+1\phi_3+0\phi_4+1\phi_5+1\phi_6+0\phi_7+2\phi_8+0\phi_9+0\phi_{10}+2\phi_{11}+0\phi_{12}+0\phi_{13}+0\phi_{14}$
(3A→ 3C, 5AB→ 5B, 7AC→ 7B, 9AC→ 9B, 13A→ 13B, 21AF→ 21F, 63AI→ 63AC, 63JR→ 63AC, 65AX→ 65AD)

Cohomological Dimensions

$\phi_2 = 6$, $\phi_3 = 0$, $\phi_4 = 0$, $\phi_5 = 0$, $\phi_6 = 0$, $\phi_7 = 0$, $\phi_8 = 0$, $\phi_9 = 0$, $\phi_{10} = 0$, $\phi_{11} = 0$, $\phi_{12} = 0$, $\phi_{13} = 0$, $\phi_{14} = 0$.

B.2 $L_2(16)$

Brauer Character Table

$\boldsymbol{L_2(16)}$	1A	3A	5AB	15AD	17AD	17EH
ϕ_1	1	1	1	1	1	1
ϕ_2	8	-4	-2	1	b17	$*$
ϕ_3	8	2	3	-3	$*$	b17
ϕ_4	16	4	-4	-1	-1	-1
ϕ_5	16	1	1	1	-1	-1
ϕ_6	32	-4	2	-4	$*$	2b17-1

Feasible Decompositions

(i) $32\phi_1+1\phi_2+8\phi_3+8\phi_4+1\phi_5+0\phi_6$ (3A→3A, 5AB→5A, 15AD→15B, 17AD→17AB, 17EH→17AB)

(ii) $16\phi_1+8\phi_2+9\phi_3+2\phi_4+4\phi_5+0\phi_6$ (3A→3B, 5AB→5A, 15AD→15E, 17AD→17CD, 17EH→17CD)

(iii) $16\phi_1+9\phi_2+8\phi_3+1\phi_4+5\phi_5+0\phi_6$ (3A→3C, 5AB→5A, 15AD→15D, 17AD→17CD, 17EH→17CD)

(iv) $32\phi_1+8\phi_2+1\phi_3+1\phi_4+8\phi_5+0\phi_6$ (3A→3B, 5AB→5A, 15AD→15A, 17AD→17AB, 17EH→17AB)

(v) $16\phi_1+7\phi_2+4\phi_3+5\phi_4+2\phi_5+1\phi_6$ (3A→3B, 5AB→5B, 15AD→15F, 17AD→17CD, 17EH→17CD)

(vi) $16\phi_1+7\phi_2+6\phi_3+1\phi_4+5\phi_5+1\phi_6$ (3A→3C, 5AB→5A, 15AD→15D, 17AD→17CD, 17EH→17CD)

(vii) $16\phi_1+8\phi_2+5\phi_3+0\phi_4+6\phi_5+1\phi_6$ (3A→3D, 5AB→5A, 15AD→15C, 17AD→17CD, 17EH→17CD)

(viii) $16\phi_1+9\phi_2+4\phi_3+4\phi_4+0\phi_5+2\phi_6$ (3A→3D, 5AB→5B, 15AD→15G, 17AD→17CD, 17EH→17CD)

(ix) $16\phi_1+5\phi_2+2\phi_3+5\phi_4+2\phi_5+2\phi_6$ (3A→3B, 5AB→5B, 15AD→15F, 17AD→17CD, 17EH→17CD)

(x) $16\phi_1+6\phi_2+3\phi_3+0\phi_4+6\phi_5+2\phi_6$ (3A→3D, 5AB→5A, 15AD→15C, 17AD→17CD, 17EH→17CD)

(xi) $16\phi_1+7\phi_2+2\phi_3+4\phi_4+0\phi_5+3\phi_6$ (3A→3D, 5AB→5B, 15AD→15G, 17AD→17CD, 17EH→17CD)

Cohomological Dimensions

$\phi_2 = 4,\ \phi_3 = 0,\ \phi_4 = 0,\ \phi_5 = 0,\ \phi_6 = 0.$

B.3 $L_2(8)$

Brauer Character Table

$\boldsymbol{L_2(8)}$	1A	3A	7AC	9AC
ϕ_1	1	1	1	1
ϕ_2	6	-3	-1	0
ϕ_3	8	-1	1	-1
ϕ_4	12	3	-2	-3

Feasible Decompositions

(i) $64\phi_1 + 2\phi_2 + 8\phi_3 + 9\phi_4$ (3A→3A, 7AC→7A, 9AC→9A)

(ii) $30\phi_1 + 13\phi_2 + 4\phi_3 + 9\phi_4$ (3A→3B, 7AC→7B, 9AC→9D)

(iii) $32\phi_1 + 14\phi_2 + 3\phi_3 + 9\phi_4$ (3A→3B, 7AC→7B, 9AC→9C)

(iv) $36\phi_1 + 16\phi_2 + 1\phi_3 + 9\phi_4$ (3A→3B, 7AC→7B, 9AC→9B)

(v) $28\phi_1 + 14\phi_2 + 5\phi_3 + 8\phi_4$ (3A→3C, 7AC→7B, 9AC→9D)

(vi) $30\phi_1 + 15\phi_2 + 4\phi_3 + 8\phi_4$ (3A→3C, 7AC→7B, 9AC→9C)

(vii) $34\phi_1 + 17\phi_2 + 2\phi_3 + 8\phi_4$ (3A→3C, 7AC→7B, 9AC→9B)

(viii) $26\phi_1 + 15\phi_2 + 6\phi_3 + 7\phi_4$ (3A→3D, 7AC→7B, 9AC→9D)

(ix) $28\phi_1 + 16\phi_2 + 5\phi_3 + 7\phi_4$ (3A→3D, 7AC→7B, 9AC→9C)

(x) $32\phi_1 + 18\phi_2 + 3\phi_3 + 7\phi_4$ (3A→3D, 7AC→7B, 9AC→9B)

(xi) $32\phi_1 + 0\phi_2 + 24\phi_3 + 2\phi_4$ (3A→3B, 7AC→7A, 9AC→9C)

(xii) $36\phi_1 + 2\phi_2 + 22\phi_3 + 2\phi_4$ (3A→3B, 7AC→7A, 9AC→9B)

(xiii) $50\phi_1 + 9\phi_2 + 15\phi_3 + 2\phi_4$ (3A→3B, 7AC→7A, 9AC→9A)

(xiv) $28\phi_1 + 0\phi_2 + 26\phi_3 + 1\phi_4$ (3A→3C, 7AC→7A, 9AC→9D)

(xv) $30\phi_1 + 1\phi_2 + 25\phi_3 + 1\phi_4$ (3A→3C, 7AC→7A, 9AC→9C)

(xvi) $34\phi_1 + 3\phi_2 + 23\phi_3 + 1\phi_4$ (3A→3C, 7AC→7A, 9AC→9B)

(xvii) $48\phi_1 + 10\phi_2 + 16\phi_3 + 1\phi_4$ (3A→3C, 7AC→7A, 9AC→9A)

(xviii) $26\phi_1 + 1\phi_2 + 27\phi_3 + 0\phi_4$ (3A→3D, 7AC→7A, 9AC→9D)

(xix) $28\phi_1 + 2\phi_2 + 26\phi_3 + 0\phi_4$ (3A→3D, 7AC→7A, 9AC→9C)

(xx) $32\phi_1 + 4\phi_2 + 24\phi_3 + 0\phi_4$ (3A→3D, 7AC→7A, 9AC→9B)

(xxi) $46\phi_1 + 11\phi_2 + 17\phi_3 + 0\phi_4$ (3A→3D, 7AC→7A, 9AC→9A)

Cohomological Dimensions

$\phi_2 = 3$, $\phi_3 = 0$, $\phi_4 = 0$.

B.4 $L_3(4)$

Brauer Character Table

$L_3(4)$	1A	3A	5AB	7A	7B**
ϕ_1	1	1	1	1	1
ϕ_2	9	0	-1	b7-1	**
ϕ_3	9	0	-1	**	b7-1
ϕ_4	16	-2	1	2	2
ϕ_5	64	1	-1	1	1

Feasible Decompositions

(i) $2\phi_1 + 3\phi_2 + 3\phi_3 + 4\phi_4 + 2\phi_5$ (3A→3D, 5AB→5B, 7A→7B, 7B** →7B)

(ii) $4\phi_1 + 2\phi_2 + 2\phi_3 + 1\phi_4 + 3\phi_5$ (3A→3C, 5AB→5B, 7A→7B, 7B** →7B)

Cohomological Dimensions

$\phi_2 = 2$, $\phi_3 = 2$, $\phi_4 = 0$, $\phi_5 = 0$.

B.5 $L_3(3)$

Brauer Character Table

$\boldsymbol{L_3(3)}$	1A	3A	3B	13AD
ϕ_1	1	1	1	1
ϕ_2	12	3	0	-1
ϕ_3	26	-1	-1	0
ϕ_4	64	-8	4	-1

Feasible Decompositions

(i) $4\phi_1 + 3\phi_2 + 8\phi_3 + 0\phi_4$ (3A→3C, 3B→3D, 13AD→13B)

(ii) $6\phi_1 + 4\phi_2 + 5\phi_3 + 1\phi_4$ (3A→3C, 3B→3C, 13AD→13B)

(iii) $14\phi_1 + 0\phi_2 + 9\phi_3 + 0\phi_4$ (3A→3C, 3B→3C, 13AD→13A)

(iv) $8\phi_1 + 5\phi_2 + 2\phi_3 + 2\phi_4$ (3A→3C, 3B→3B, 13AD→13B)

(v) $16\phi_1 + 1\phi_2 + 6\phi_3 + 1\phi_4$ (3A→3C, 3B→3B, 13AD→13A)

Cohomological Dimensions

$\phi_2 = 1$, $\phi_3 = 1$, $\phi_4 = 0$.

B.6 $F_4(2)$

Brauer Character Table

$\boldsymbol{F_4(2)}$	1A	3A	3B	3C	5A	7A	7B	9A	9B	13A	15A	15B	17A	17B	21A	21B
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	26	8	-1	-1	1	-2	5	2	-1	0	4	-2	*	b17+1	1	-1
ϕ_3	26	-1	8	-1	1	5	-2	-1	2	0	-2	4	b17+1	*	-1	1
ϕ_4	246	12	-6	3	-4	1	1	-3	0	-1	-1	2	b17	*	-2	1
ϕ_5	246	-6	12	3	-4	1	1	-3	0	-1	2	-1	*	b17	1	-2

Feasible Decompositions

(i) $2\phi_1 + 0\phi_2 + 0\phi_3 + 0\phi_4 + 1\phi_5$ (3A→3D, 3B→3B, 3C→3C, 5A→5B, 7A→7B, 7B→7B, 9A→9D, 9B→9C, 13A→13B, 15A→15F, 15B→15G, 17A→17CD, 17B→17CD, 21A→21H, 21B→21E)

(ii) $2\phi_1 + 0\phi_2 + 0\phi_3 + 1\phi_4 + 0\phi_5$ (3A→3D, 3B→3B, 3C→3C, 5A→5B, 7A→7B, 7B→7B, 9A→9D, 9B→9C, 13A→13B, 15A→15G, 15B→15F, 17A→17CD, 17B→17CD, 21A→21E, 21B→21H)

(iii) $14\phi_1 + 1\phi_2 + 8\phi_3 + 0\phi_4 + 0\phi_5$ (3A→3B, 3B→3A, 3C→3C, 5A→5A, 7A→7A, 7B→7B, 9A→9B, 9B→9A, 13A→13A, 15A→15B, 15B→15A, 17A→17AB, 17B→17AB, 21A→21C, 21B→21B)

(iv) $14\phi_1 + 8\phi_2 + 1\phi_3 + 0\phi_4 + 0\phi_5$ (3A→3A, 3B→3B, 3C→3C, 5A→5A, 7A→7B, 7B→7A, 9A→9A, 9B→9B, 13A→13A, 15A→15A, 15B→15B, 17A→17AB, 17B→17AB, 21A→21B, 21B→21C)

Cohomological Dimensions

$\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 1$, $\phi_5 = 1$.

B.7 $\Omega_8^+(4)$

Brauer Character Table

$\mathbf{\Omega_8^+(4)}$	1A	3A	3B	3C	3D	3E	5AB	5CD	5EF	5GH	5IJ	5KL	5MN	5O	5P	5Q	5RS	7A
ϕ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ϕ_2	16	10	-8	-8	4	-2	11	-4	-4	1	-4	-4	6	6	-4	-4	1	2
ϕ_3	16	-8	10	-8	4	-2	-4	11	-4	-4	1	-4	6	-4	6	-4	1	2
ϕ_4	16	-8	-8	10	4	-2	-4	-4	11	-4	-4	1	6	-4	-4	6	1	2
ϕ_5	52	16	16	16	-2	-2	22	22	22	7	7	7	7	2	2	2	-3	-4
ϕ_6	64	25	16	16	4	1	29	-16	-16	-11	-1	-1	4	9	4	4	-1	1
ϕ_7	64	16	25	16	4	1	-16	29	-16	-1	-11	-1	4	4	9	4	-1	1
ϕ_8	64	16	16	25	4	1	-16	-16	29	-1	-1	-11	4	4	4	9	-1	1
ϕ_9	96	12	-24	-24	0	6	26	-24	-24	16	9	9	16	-4	-4	-4	1	-2
ϕ_{10}	96	-24	12	-24	0	6	-24	26	-24	9	16	9	16	-4	-4	-4	1	-2
ϕ_{11}	96	-24	-24	12	0	6	-24	-24	26	9	9	16	16	-4	-4	-4	1	-2
ϕ_{12}	128	32	-40	-40	8	2	-32	-12	-12	-2	13	13	8	8	-12	-12	-2	2
ϕ_{13}	128	-40	32	-40	8	2	-12	-32	-12	13	-2	13	8	-12	8	-12	-2	2
ϕ_{14}	128	-40	-40	32	8	2	-12	-12	-32	13	13	-2	8	-12	-12	8	-2	2

Feasible Decompositions

(i) $4\phi_1+0\phi_2+0\phi_3+0\phi_4+1\phi_5+1\phi_6+1\phi_7+1\phi_8+0\phi_9+0\phi_{10}+0\phi_{11}+0\phi_{12}+0\phi_{13}+0\phi_{14}$ (3A→ 3A, 3B→ 3A, 3C→ 3A, 3D→ 3B, 3E→ 3C, 5AB→ 5A, 5CD→ 5A, 5EF→ 5A, 5GH→ 5B, 5IJ→ 5B, 5KL→ 5B, 5MN→ 5A, 5O→ 5A, 5P→ 5A, 5Q→ 5A, 5RS→ 5B, 7A→ 7B)

(ii) $4\phi_1+0\phi_2+0\phi_3+0\phi_4+1\phi_5+1\phi_6+0\phi_7+0\phi_8+0\phi_9+0\phi_{10}+0\phi_{11}+1\phi_{12}+0\phi_{13}+0\phi_{14}$ (3A→ 3A, 3B→ 3D, 3C→ 3D, 3D→ 3B, 3E→ 3C, 5AB→ 5A, 5CD→ 5B, 5EF→ 5B, 5GH→ 5B, 5IJ→ 5A, 5KL→ 5A, 5MN→ 5A, 5O→ 5A, 5P→ 5B, 5Q→ 5B, 5RS→ 5B, 7A→ 7B)

(iii) $4\phi_1+0\phi_2+0\phi_3+0\phi_4+1\phi_5+0\phi_6+1\phi_7+0\phi_8+0\phi_9+0\phi_{10}+0\phi_{11}+0\phi_{12}+1\phi_{13}+0\phi_{14}$ (3A→ 3D, 3B→ 3A, 3C→ 3D, 3D→ 3B, 3E→ 3C, 5AB→ 5B, 5CD→ 5A, 5EF→ 5B, 5GH→ 5A, 5IJ→ 5B, 5KL→ 5A, 5MN→ 5A, 5O→ 5A, 5P→ 5B, 5Q→ 5A, 5RS→ 5B, 7A→ 7B)

(iv) $4\phi_1+0\phi_2+0\phi_3+0\phi_4+1\phi_5+0\phi_6+0\phi_7+1\phi_8+0\phi_9+0\phi_{10}+0\phi_{11}+0\phi_{12}+0\phi_{13}+1\phi_{14}$ (3A→ 3D, 3B→ 3D, 3C→ 3A, 3D→ 3B, 3E→ 3C, 5AB→ 5B, 5CD→ 5B, 5EF→ 5A, 5GH→ 5A, 5IJ→ 5A, 5KL→ 5B, 5MN→ 5A, 5O→ 5B, 5P→ 5B, 5Q→ 5A, 5RS→ 5B, 7A→ 7B)

Cohomological Dimensions

$\phi_2 = 0$, $\phi_3 = 0$, $\phi_4 = 0$, $\phi_5 = 4$, $\phi_6 = 0$, $\phi_7 = 0$, $\phi_8 = 0$, $\phi_9 = 0$, $\phi_{10} = 0$, $\phi_{11} = 0$, $\phi_{12} = 0$, $\phi_{13} = 0$, $\phi_{14} = 0$.