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Research Paper

Stochastic finite element analysis of the free vibration of non-uniform beams with uncertain material

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This paper deal with the stochastic finite element method for investigating the eigenvalues of free vibration of non-uniform beams due to a random field of elastic modulus. The formulation of stochastic analysis of the non-uniform beam is established using perturbation method in conjunction with finite element method. Monte Carlo simulation (MCS) used for validation with stochastic finite element approach. The spectral representation was used to generate a random field to employ the Monte Carlo simulation. The performance of results of the uncertain eigenvalue problem of non-uniform beams with random field of elastic modulus by comparing the first-order perturbation technique with the same moments evaluated from the Monte Carlo simulation. The numerical results show that the response of coefficient of variation of eigenvalue increases when the ratio of correlation distance of random field increases.

1 Introduction

Non-uniform beams may provide a better or more suitable distribution of mass and strength than uniform beams and therefore can meet special functional requirements in architecture, robotics, aeronautics, and other innovative engineering applications and they have been studied by many authors. The mass density of materials is heterogeneous due to the manufacturing process, so properties of the material are non-uniform distribution In most engineering applications, computational structures ignore the random heterogeneity of materials (e.g. soil, concrete, composites, etc.) and loading (e.g. vehicle, wind, wave, etc.) and favor to use deterministic models with average or extreme valued parameters. So that it leads to a coarse representation of physical behavior and a false sense of precision.

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In the past, many studies have been performed on the behavior of the beams. The analytical and numerical solutions for the static and dynamic response of a beam have been obtained. P.V.Phe and N.X Huy [1] studied the behavior of GFRP-flexural strengthened steel beams using finite element methods. N.T Anh [2] improved the multi-fiber finite element beam model for torsional behaviour of reinforced concrete members. The dynamic responses of a non-uniform Timoshenko beam are developed by generalizing the exact solutions of non-uniform Timoshenko beam vibration given by Lee and Gutierrez [3, 4]. Faruk Fırat Calım [5] applied the Laplace transform to determine the dynamic responses of non-uniform composite beams. The free vibration of non-uniform beams is available only for some special types of non-uniform beams on elastic foundations and elastic end restraints. Hence, the problems were mainly treated by different beams such as the Bernoulli-Euler beams [6], Timoshenko beams [7], high-order beams [8]. However, deterministic analysis cannot provide complete information on structural responses. A probabilistic approach is a mathematical tool, capable of handling uncertainties in the material properties to compute the statistic in terms of mean and variance of structural response. At the micromechanics level, the state-of-the-art computational multi-scale homogenization method couple with the stochastic finite element method proposed by Zhou et al. [9] to calculate the effect of the randomness of material density on the overall elastic properties. Hien [10] developed the stochastic finite element method for non-uniform columns with the random field of elastic modulus. Khurshudyan and Arakelyan [11] solved the dynamic problem of Euler–Bernoulli sandwich beam using Green's function approach with random loads. Chang et al. [12] assumed the random field of elastic modulus to be uniform within each element to calculate the dynamic response of a non-uniform beam. Xu Yalan et al. [13] computed random natural frequencies of the functionally graded beam, so authors limited research with random variables of material properties. Hien and Cuong [14] proposed stochastic isogeometric analysis for computing random displacement of plate consider the random field of elastic modulus. Although certain efforts have been made in the past to predict the dynamic behaviour of structures with randomness, but the research works concerning the non-uniform beams with uncertain material are still limited. Assumption of Chang et al. [12] on discretization of the random field of elastic modulus will have a limited precision when the length distance of random fields is small, so it needs to improve technique of discretization of random field in the element. In this paper, a simple and effective method, which increase number of point to discretise within each elements, is presented to study the stochastic analysis of the eigenvalue problem of free vibration of non-uniform beam with random field of elastic modulus by stochastic finite element method using perturbation technique.

2 Governing equation and characteristic equation

2.1 Formulation beam finite element for non-uniform beams

Consider a non-uniform beam with the coordinate system *(x, z)* is shown in Fig. 1. The parameters of the model non-uniform beam are as follows: L is the length of the cross-section beam, h is the thickness of the cross-section beam.

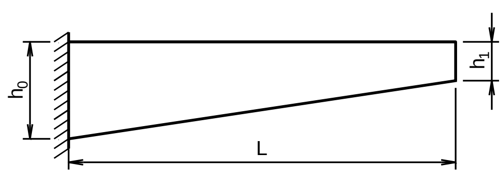


Fig. 1 – Model of non-uniform beams

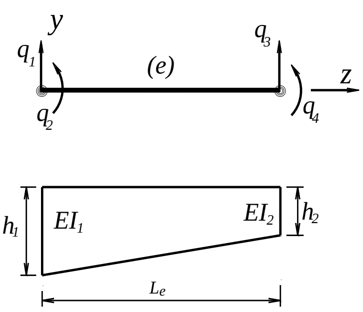


Fig. 2 – Cross-section beam element with a linear profile

The beam element is assumed to be two degrees of freedom, one rotation, and one translation at each end. The location and positive directions of these displacements in a typical linearly tapered beam element are shown in Fig. 2, where L_e is the length of the element, *E* and *I* are the area and inertia moment of the cross-section beam. The depth of the cross-sections at the smaller and of the beam is denoted as and h_2 and the larger as end h_1, respectively. The longitudinal axis of the element lies along the *x*-axis.

We obtain the lateral displacement field is

$$w_e = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = [N]\{q\} \tag{1}$$

where $N = \langle N_1 \quad N_2 \quad N_3 \quad N_4 \rangle$, and N_i is the shape function of *i-th* degree of freedom, and the Hermite polynomial functions [15] as follows:

$$\begin{aligned} N_1 &= 1 - 3\frac{z^2}{L_e^2} + 2\frac{z^3}{L_e^3}; \quad N_2 = z\left(1 - 2\frac{z}{L_e} + \frac{z^2}{L_e^2}\right) \\ N_3 &= 3\frac{z^2}{L_e^2} - 2\frac{z^3}{L_e^3}; \qquad N_4 = z\left(-\frac{z}{L_e} + \frac{z^2}{L_e^2}\right) \end{aligned} \tag{2}$$

In this paper, the equation for the natural frequency of the beam is derived using Hamilton's principle. The strain energy expression U_e for bending is given as follows:

$$U_e = \frac{1}{2}\int_0^L EI(z)\left(\frac{d^2 w}{dz^2}\right)^2 dz \tag{3}$$

The kinetic energy of the beam element is

$$T_e = \frac{1}{2}\int_0^{L_e} bh(z)\rho \dot{w}^2 dz \tag{4}$$

Linear approximation cross-section moment of inertia in element:

$$EI_e(z) = EI_{1e}\left(1 - \frac{z}{L_e}\right) + EI_{2e}\frac{z}{L_e} \tag{5}$$

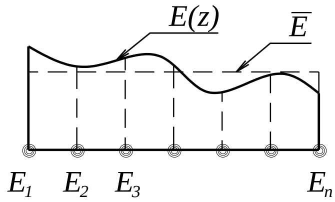


Fig. 3 – Model for approximation random field of elastic modulus

In context elastic modulus is a random field, it need discretize to random variables for deriving finite element formulation. The random field of elastic modulus is assumed as follows:

$$E(z) = E_0\left[1 + r\ (z)\right] \tag{6}$$

where *r(z)* is a one-dimensional Gaussian random field with a mean equal to zero.

The form of the autocorrelation function of random field r(x) is:

$$R(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r(z)r(z+\tau)\, f_z(z, z+\tau, \tau)\, dr(z)\, dr(z+\tau) \tag{7}$$

By averaging random variables within the element, the random field of elastic modulus in the element is calculated as

$$\overline{E} = E_0\left[1 + \frac{r_1 + r_2 + ... + r_n}{n}\right] \tag{8}$$

The governing differential equations of motion and the related governing equation can be derived using Hamilton's principle;

$$\delta\int_{t_1}^{t_2}\left(\sum_{1}^{N}U_e-\sum_{1}^{N}T_e\right)dt=0 \tag{9}$$

where N denotes the number of finite elements.

Taking the variation with respect to q, we obtain the resulting equation [16] for the eigenvalues and vectors:

$$\left(\left[K\right]-\lambda\left[M\right]\right)\left\{q\right\}=0 \tag{10}$$

In the foregoing equations, $\left[K\right]$ and $\left[M\right]$ denote the assembled global stiffness, respectively, of the cross-section beam, and λ_i denotes the square of the circular frequency ω_i.

where the stiffness matrix

$$\left[K\right]_e=E_0\left[1+\frac{r_1+r_2+...+r_n}{n}\right]\begin{bmatrix}\frac{6\left(I_{1e}+I_{2e}\right)}{L_e^3} & \frac{4I_{1e}+2I_{2e}}{L_e^2} & -\frac{6\left(I_{1e}+I_{2e}\right)}{L_e^3} & \frac{2I_{1e}+4I_{2e}}{L_e^2}\\ & \frac{3I_{1e}+I_{2e}}{L_e} & -\frac{4I_{1e}+2I_{2e}}{L_e^2} & \frac{I_{1e}+I_{2e}}{L_e}\\ & & \frac{6\left(I_{1e}+I_{2e}\right)}{L_e^3} & -\frac{2I_{1e}+4I_{2e}}{L_e^2}\\ Sym. & & & \frac{I_{1e}+3I_{2e}}{L_e}\end{bmatrix} \tag{11}$$

Mass matrix is defined as:

$$\left[M\right]_e=\int_0^{L_e}\rho b\left\{h_1\left(1-\frac{z}{L_e}\right)+h_2\frac{z}{L_e}\right\}\left[N\left(z\right)\right]^T\left[N\left(z\right)\right]dx \tag{12}$$

$$\left[M\right]_e=\rho bL_e\begin{bmatrix}\frac{3h_2+10h_1}{35} & \frac{\left(7h_2+15h_1\right)L_e}{420} & \frac{9\left(h_2+h_1\right)}{140} & \frac{-\left(h_2+7h_1\right)L_e}{420}\\ & \frac{\left(3h_2+5h_1\right)L_e^2}{840} & \frac{\left(7h_2+6h_1\right)L_e}{420} & \frac{-\left(h_2+h_1\right)L_e^2}{280}\\ & & \frac{10h_2+3h_1}{35} & \frac{-\left(15h_2+7h_1\right)L_e}{420}\\ Sym. & & & \frac{\left(5h_2+3h_1\right)L_e^2}{840}\end{bmatrix} \tag{13}$$

2.2 Formulation of stochastic finite element method using perturbation technique

The free vibration equation (10) contains random variables, it is difficult to find the eigenvalue and eigenvector. The free vibration equation can be perturbed with respect to the mean of the random variables as follows:

$$\left(\left[K\right]_0+\sum_{i=1}^{Nr}\frac{\partial\left[K\right]}{\partial r_i}r_i-\left(\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)\left[M\right]\right)\left\{q_0+\sum_{i=1}^{Nr}\frac{\partial q}{\partial r_i}r_i\right\}=0 \tag{14}$$

Solve the stochastic equation (14), we obtain the formulation for:

zeroth-order:

$$\left(\left[K\right]_0-\lambda_0\left[M\right]\right)\left\{q_0\right\}=0 \tag{15}$$

and first-order:

$$\left([K]_0-\lambda_0[M]\right)\left\{\frac{\partial q}{\partial r_i}\right\}=-\left(\frac{\partial[K]}{\partial r_i}-\frac{\partial\lambda}{\partial r_i}[M]\right)\{q_0\} \tag{16}$$

Premultiplication of the equation (14) by $\left\{q_0+\sum_{i=1}^{Nr}\frac{\partial q}{\partial r_i}r_i\right\}^T$ gives:

$$\left\{q_0+\sum_{i=1}^{Nr}\frac{\partial q}{\partial r_i}r_i\right\}^T\left([K]_0+\sum_{i=1}^{Nr}\frac{\partial[K]}{\partial r_i}r_i-\left(\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)[M]\right)\left\{q_0+\sum_{i=1}^{Nr}\frac{\partial q}{\partial r_i}r_i\right\}=0 \tag{17}$$

Using the orthonormal property and collecting by a random variable, we obtain:

$$\frac{\partial\lambda}{\partial r_i}\approx\frac{\{q_0\}_i^T\dfrac{\partial[K]}{\partial r_i}\{q_0\}}{\{q_0\}_i^T[M]\{q_0\}} \tag{18}$$

The first-order perturbation solutions to calculate the mean and variance of the eigenvalue:

$$\begin{aligned}\boldsymbol{\mu}_\lambda&=\int_{-\infty}^{\infty}\left\{\left(\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)-\lambda_0\right\}p(r_i)\,dr_i=\lambda_0\\ \boldsymbol{Var}_\lambda&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(\lambda_0+\sum_{i=1}^{Nr}\frac{\partial\lambda}{\partial r_i}r_i\right)-\lambda_0\right\}\left\{\left(\lambda_0+\sum_{j=1}^{Nr}\frac{\partial\lambda}{\partial r_j}r_j\right)-\lambda_0\right\}f_z\left(r_i,r_j,r_j-r_i\right)dr_idr_j\\ &=\sum_{i=1}^{Nr}\sum_{j=1}^{Nr}\frac{\partial\lambda}{\partial r_i}\frac{\partial\lambda}{\partial r_i}R(\tau)\end{aligned} \tag{19}$$

where, $R(\tau)$ denotes the autocorrelation function of the random field, and the relative distance vector is defined as $\tau=r_j-r_i$. We assumed the autocorrelation function in the form:

$$R(\tau)=\sigma^2\,exp\left(-\frac{\tau^2}{4d^2}\right) \tag{20}$$

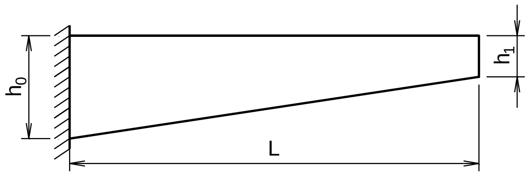
with correlation distance *d* of the random field.

3 Numerical example

Use of the stiffness and mass matrices derived above is illustrated in the vibration analysis of linearly tapered cantilever and simply-supported beams of concrete rectangular cross-section as shown in Fig. 5. The modulus of elasticity and mass density of the beam is assumed to be 32×10^3 MPa and 2.500 kg/m^3, respectively. The geometric dimensions of the example cross-section beams are: h_0 = 0.8 m, h_1 = 0.4 m, L= 8 m, and b=0.3 m

In order to evaluate the response variability in the natural frequency of the cross-section beam, we employed the scheme of Monte Carlo simulation (MCS). The MCS is equivalent to the deterministic analysis on a set of heterogeneous models of the given structure in that the material properties have different values depending on the position in the domain of the structure for individual samples. The numerically generated random field using spectral representation method [17]. There are various Monte Carlo methods, but it tends four steps in practice:

- Define a possible inputs random field.
- Generate inputs randomly from a probability distribution of random field.
- Perform a repeated deterministic computation on the inputs.
- Aggregate the results and calculate statistic of output.



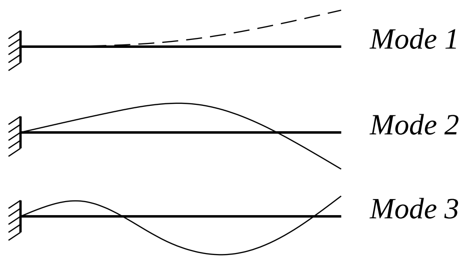


Fig. 4 – Dimension of non-uniform beam

Fig. 5 – First three mode shapes of non-uniform beam

It can be found from Fig. 6 that the effect of the correlation distance d of the random field on the variability of the natural frequency, where the results of the proposed formulation were compared with those of the Monte Carlo simulation, for the same cases of stochastic finite element method. The results designated by the dotted line denote the corresponding results of the MCS for a standard deviation of the stochastic field 0.1, 0.15, and 0.2. As shown in Fig. 6, the response variability of natural frequency is converging to certain values, as the correlation distance tends to move to infinity in both analysis schemes. In this study, the converging value was obtained to be approximately 100% in the perturbation method and slightly over 100% in the Monte Carlo simulation of the assumed standard deviation of the stochastic field. We observe that the increasing rate of the correlation distances is accelerated with an increase in the coefficient of variation of the stochastic field. Also, the first-order perturbation values of eigenvalues' coefficient of variation obtained by the proposed scheme are close to those determined by the Monte Carlo simulation.

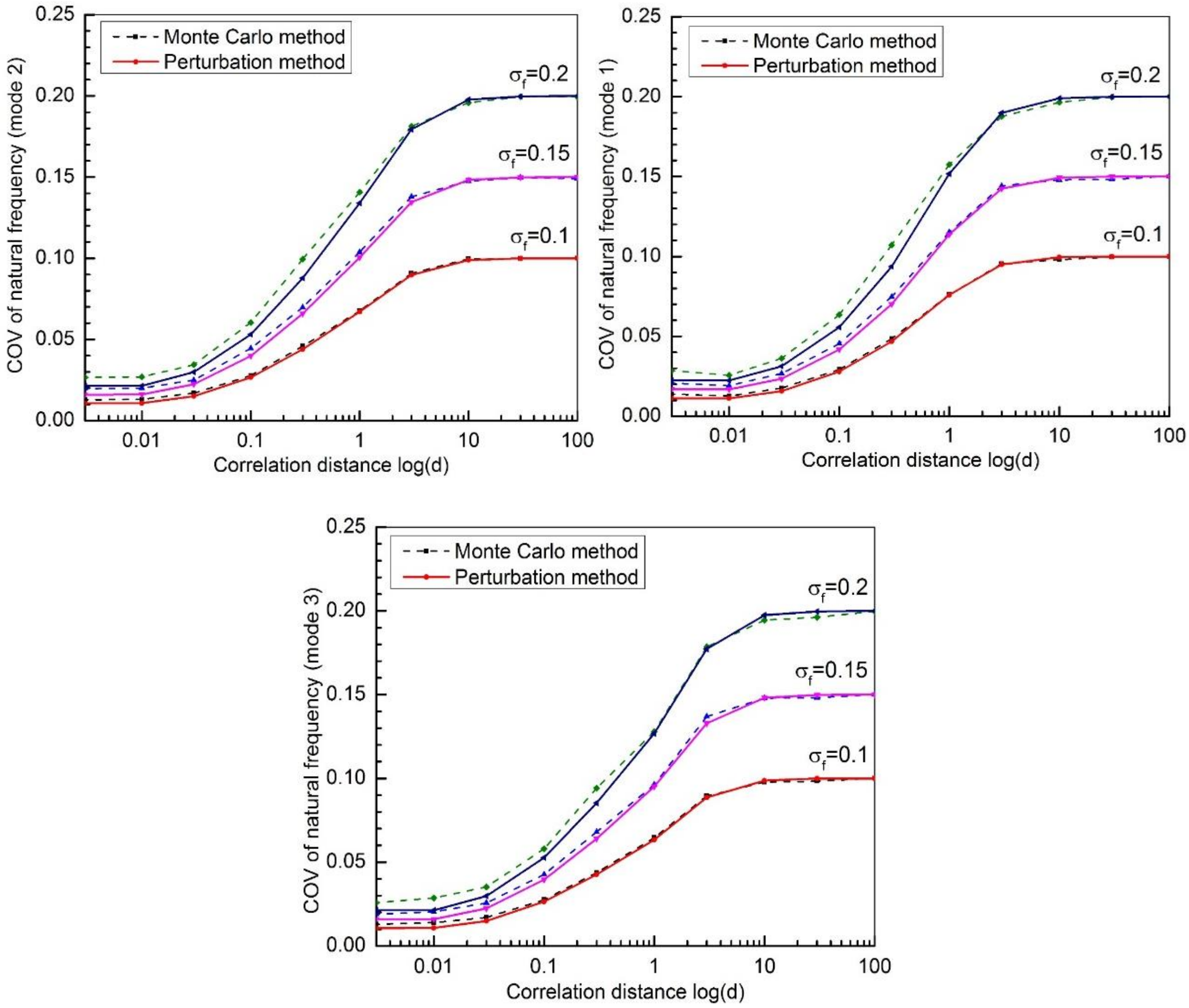


Fig. 6 – Effects of correlation distance d on the different standard deviation σ of the natural frequency

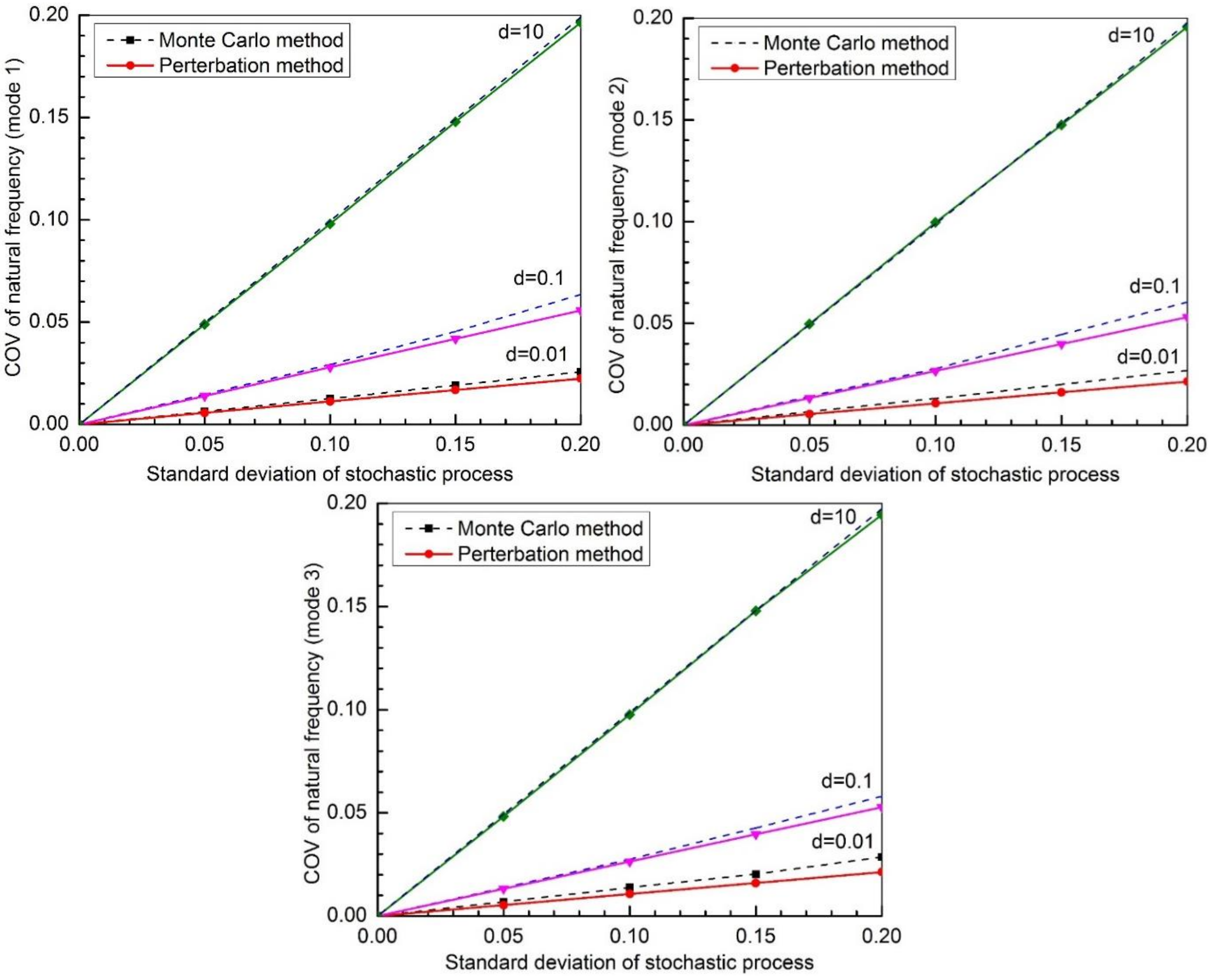


Fig. 7 – COV of natural frequency as a function of the standard deviation of stochastic process

Figure 7 displays the additional response variability of natural frequency by the proposed scheme for various values of the three cases of the correlation distances (0.01, 0.1 and 10). The results of the Monte Carlo and perturbation method are discussed in four cases of the standard deviation of the random field 0.05, 0.1, 0.15 and 0.2. As shown in Fig. 7, the linear extrapolation of the response variability sets original and 0.01 up the dotted lines. As noted by the comparison between analysis results and linear extrapolations, the response variability tends to be nonlinear in MCS and slightly nonlinear in the perturbation method results, as the small correlation distance value.

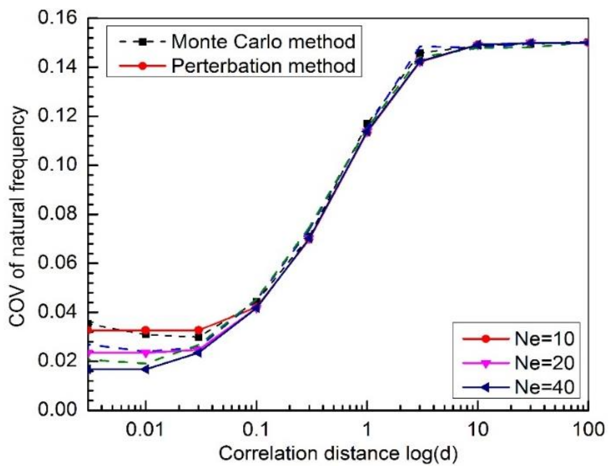


Fig. 8 – Effect of mesh refinement on the COV of natural frequency in case σ=0.15

Figure 8 present the effect of mesh refinement on the COV of the natural frequency of the cross-section beam for different the number of finite element N_e= 10, 20, 40. As it can be seen from Fig. 8, it is observed that the response COV of eigenvalue is not influences by the number of finite elements of the cross-section beam. This result, of course, is not to say that we can use highly coarse mesh in the analysis but to say that if we are using a reasonable number of finite elements in the mesh then we can obtain reasonable results of response variability.

4 Conclusion

In this paper, a perturbation technique in conjugation with finite element analysis is successfully developed for the stochastic natural frequency problem of non-uniform beams having a random field of elastic modulus. In order to check the validity of the proposed first-order stochastic field function, a Monte Carlo simulation is performed employing 10,000 random samples to simulate the results of the desired response. The efficacy of the first-order perturbation method has been verified using a homogeneous Gaussian random field by stochastic finite element method is in perfect agreement with the Monte Carlo simulation where the correlation distance was as high as 10. Furthermore, in both methods of analysis, the results showed that the response variability of natural frequency is not affected by the number of elements finite of the non-uniform beam. In the foregoing, a superior method of stochastic finite element analysis is developed, which does not suffer from the deficiencies of existing procedures.

Acknowledgments

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