



## The volumes of Miyauchi subgroups

DIDIER LESESVRE AND IAN PETROW

**Abstract.** Miyauchi described the  $L$  and  $\varepsilon$ -factors attached to generic representations of the unramified unitary group of rank three in terms of local newforms defined by a sequence of subgroups. We calculate the volumes of these Miyauchi groups.

**Mathematics Subject Classification.** Primary 11F06; Secondary 20H05, 22E50.

**Keywords.** Congruence subgroups, Newform theory, Miyauchi groups, Weyl law, Automorphic forms on unitary groups.

### 1. Introduction.

**1.1. Miyauchi's theory of local newforms.** Let  $E/F$  be a quadratic unramified extension of non-Archimedean local fields of characteristic zero. We use  $\bar{x}$  to denote the non-trivial Galois automorphism applied to  $x \in E$ . Let  $\mathfrak{o}, \mathfrak{D}$  be the rings of integers of the two fields  $F, E$  respectively. Let  $\mathfrak{p}, \mathfrak{P}$  be their maximal ideals, and  $q_F, q_E$  be the cardinalities of their residue fields. Let

$$U = \{g \in \mathrm{GL}_3(E) : {}^t\bar{g}Jg = J\}, \quad (1.1)$$

where

$$J = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \quad (1.2)$$

The group  $U$  is a rank 3 unitary group defined over  $F$ . There is a unique isomorphism class of genuine unitary groups over  $F$  [20, Section 1.9], so we are content to work with the above explicit model for this single isomorphism class of unitary groups.

Consider the family of compact open subgroups of  $U$  defined for all  $n \geq 0$  by

$$\Gamma_n = \left\{ g \in \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{P}^{-n} \\ \mathfrak{P}^n & 1 + \mathfrak{P}^n & \mathfrak{O} \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{O} \end{pmatrix} : \det(g) \in \mathfrak{O}^\times \right\} \cap U. \tag{1.3}$$

We call these the Miyauchi subgroups, as defined in [17]. These are of particular interest because of the theory of local newforms established by Miyauchi [16], relating the notions of conductors and newforms to  $L$  and  $\varepsilon$ -factors attached to representations of  $U$ , as summarized in the theorem below.

Choose a non-trivial unramified additive character  $\psi_E$  of  $E$  and a non-trivial additive character  $\psi_F$  of  $F$ . For an irreducible admissible generic representation  $\pi$  of  $U$ , let  $\mathcal{W}(\pi, \psi_E)$  be its Whittaker model with respect to  $\psi_E$ . Gelbart, Piatetski-Shapiro [7] and Baruch [1] defined the  $L$ -factor  $L(s, \pi)$  associated to an irreducible generic representation  $\pi$  of  $U$  as the greatest common divisor of a family of explicit zeta integrals  $Z(s, W, \Phi)$  where  $W \in \mathcal{W}(\pi, \psi_E)$  and  $\Phi$  is a function in  $\mathcal{C}_c^\infty(F^2)$ . The  $\varepsilon$ -factor  $\varepsilon(s, \pi, \psi_E, \psi_F)$  is then defined as the quantity appearing in the functional equation satisfied by  $L(s, \pi)$ . These fundamental quantities in the theory of automorphic forms on  $U$  can be characterized in a more explicit and computational way with the sequence of Miyauchi subgroups.

**Theorem 1.1** (Miyauchi). *Assume the residue characteristic of  $F$  is odd. Let  $\pi$  be an irreducible generic representation of  $U$ . For any  $n \geq 0$ , let  $V(n)$  be the subspace of  $\pi$  fixed by  $\Gamma_n$ . Then*

- (i) *there exists  $n \geq 0$  such that  $V(n)$  is non-zero;*
- (ii) *if  $n_\pi$  is the least such integer, then  $V(n_\pi)$  is one-dimensional and its non-zero elements are called the newforms for  $\pi$ ;*
- (iii) *if  $v$  is the newform for  $\pi$  such that the corresponding Whittaker function  $W_v$  satisfies  $W_v(I_3) = 1$ , then the corresponding Gelbart-Piatetski-Shapiro zeta integral  $Z(s, W_v, \mathbf{1}_{\mathfrak{p}^{n_\pi} \oplus \mathfrak{o}})$  matches the  $L$ -factor  $L(s, \pi)$ ;*
- (iv) *if  $\psi_F$  has conductor  $\mathfrak{o}$ , then the  $\varepsilon$ -factor of  $\pi$  satisfies*

$$\varepsilon(s, \pi, \psi_E, \psi_F) = q_E^{-n_\pi(s-1/2)}. \tag{1.4}$$

*Proof.* Points (i) and (ii) are [17, Theorem 0.3], and points (iii) and (iv) are [18, Theorem 3.6]. □

Consider the Haar measure on  $U$  normalized so that the maximal open compact subgroup  $\Gamma_0$  gets volume one. In this short note, we compute the volumes of the Miyauchi subgroups with respect to this measure.

**Theorem 1.2.** *For every  $n \geq 1$ , the volumes of the Miyauchi subgroups are given by*

$$\text{vol}(\Gamma_n) = q_F^{-3n}(1 + q_F^{-3})^{-1}. \tag{1.5}$$

Since the subgroups  $\Gamma_n$  characterize the conductor by the point (iv) in Miyauchi’s theorem, their volumes should appear as invariants in many problems involving automorphic forms on unitary groups of rank 3. Theorem 1.2 is the analogue of e.g. [15, Theorem 4.2.5] in the case of  $\text{GL}_2$ , and of [21, Lemma 3.3.3] in the case of  $\text{GSp}_4$ .

**1.2. Classical theory of local newforms.** Let us turn back to a more usual setting, the one of general linear groups over  $F$ , where the established importance of local newforms is a suitable motivation to our problem. For instance, the indices of the classical congruence subgroups in  $\mathrm{SL}_2(\mathbf{Z})$  are known to be equal to

$$[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right), \quad (1.6)$$

which is a central fact in the theory of automorphic forms, appealed to for many purposes: establishing the dimension of the space of automorphic cusp forms of given level [15], sieving out the oldforms and the spectral multiplicities in arithmetic statistic questions [9], or motivating normalization in spectral theory and trace formulas [11] for instance.

In a more representation theoretic framework, the analogous result to Theorem 1.1 was established by Casselman [3] and Deligne [4] in the case of  $\mathrm{GL}_2$  and by Jacquet, Piatetski-Shapiro, and Shalika [10] in the case of  $\mathrm{GL}_m$ . More precisely, let

$$\tilde{\Gamma}_n = \left( \begin{array}{cc} \mathrm{GL}_{m-1}(\mathfrak{o}) & M_{m-1,1}(\mathfrak{o}) \\ p^n M_{1,m-1}(\mathfrak{o}) & 1 + p^n \mathfrak{o} \end{array} \right) \quad (1.7)$$

be the classical congruence subgroups. These are of particular interest because of the following result, relating the theory of classical  $L$ -functions to the fixed vectors by these congruence subgroups, analogously to Theorem 1.1. Let  $\psi$  be a non-trivial unramified additive character of  $F$  viewed as a character of the upper-triangular Borel subgroup of  $\mathrm{GL}_m(F)$  as in [10, Eq. (2)]. For an irreducible admissible generic representation  $\pi$  of  $\mathrm{GL}_m(F)$ , let  $\mathcal{W}(\pi, \psi)$  be its Whittaker model with respect to  $\psi$ .

**Theorem 1.3** (Casselman, Jacquet, Piatetski-Shapiro, Shalika). *Let  $\pi$  be an irreducible generic representation of  $\mathrm{GL}_m$ . For  $n \geq 0$ , let  $V(n)$  be the subspace of  $\pi$  fixed by  $\tilde{\Gamma}_n$ . Then*

- (i) *there exists  $n \geq 0$  such that  $V(n)$  is non-zero;*
- (ii) *if  $n_\pi$  is the least such integer, then  $V(n_\pi)$  is one-dimensional and its elements are called the newforms for  $\pi$ ;*
- (iii) *the  $\varepsilon$ -factor  $\varepsilon(s, \pi, \psi)$  is  $q_F^{-n_\pi(s-1/2)}$  times a quantity independent of  $s$ .*

Since these congruence subgroups characterize the conductor by the point (iii) of the above theorem, their volumes govern the sizes of the families of automorphic forms on  $\mathrm{GL}_m$ . The indices (1.6) are therefore necessary to compute the main term in the automorphic Weyl law in the analytic conductor aspect for  $\mathrm{GL}_2$  and its inner forms [2, 13], where the characterization of the conductor in terms of congruence subgroups is used in a critical way. The strong similarity between Theorem 1.1 and Theorem 1.3, as well as the fundamental role played by the volumes of the subgroups  $\tilde{\Gamma}_n$  in the  $\mathrm{GL}_m$  case, suggest that the volumes of the Miyauchi subgroups  $\Gamma_n$  will play a central role in the analytic theory of  $U$ . In particular, we expect that these volumes will appear in the

main term of the automorphic Weyl law for rank 3 unitary groups, see [12, Chapter 5].

**1.3. Outline of the proof.** We follow the strategy of Roberts and Schmidt [21, Lemma 3.3.3], but encounter some new difficulties owing to the fact that  $U$  is non-split. The proof has several steps.

1. Consider the subgroups of  $\Gamma_n$  given by, for any  $n \geq 0$ ,

$$A_n = \left\{ g \in \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{P}^n & 1 + \mathfrak{P}^n & \mathfrak{O} \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{O} \end{pmatrix} : \det(g) \in \mathfrak{O}^\times \right\} \cap U,$$

which are the analogs of the Klingen subgroups introduced by Roberts and Schmidt [21, Equation (2.5)] in the  $\mathrm{GSp}(4)$  setting. Lemma 3.2 gives the decomposition of  $\Gamma_n$  into  $A_n$ -cosets in terms of the group of trace zero elements of  $\mathfrak{O}/\mathfrak{P}^n$ . The sizes of these groups can then be computed using group cohomology and the integral normal basis theorem for unramified extensions of local fields (see Lemma 2.6). In particular, Lemma 3.2 reduces the computation of the volume of  $\Gamma_n$  to the calculation of the indices  $[\Gamma_0 : A_n]$ .

2. Consider the subgroups  $B_n$  of  $\Gamma_n$  given by

$$B_n = \left\{ g \in \begin{pmatrix} \mathfrak{O} & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{P}^n & \mathfrak{O} & \mathfrak{O} \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{O} \end{pmatrix} : \det(g) \in \mathfrak{O}^\times \right\} \cap U,$$

and note that  $B_n$  contains  $A_n$ . Lemma 4.1 provides the short exact sequence

$$1 \longrightarrow A_n \longrightarrow B_n \longrightarrow E_{\mathfrak{P}^n}^1 \longrightarrow 1,$$

where  $E_{\mathfrak{P}^n}^1$  is the set of elements in  $\mathfrak{O}$  with  $x\bar{x} = 1$ , reduced mod  $\mathfrak{P}^n$ . The cardinality of  $E_{\mathfrak{P}^n}^1$  may be calculated using Hilbert’s theorem 90 (see Lemma 2.5).

3. Calculate the index of  $B_n$  in  $B_1$ . This is the content of Lemma 5.1 and uses the Iwahori factorization of reductive groups in order to state a recursive relation between the index of  $B_n$  in  $B_{n-1}$ .
4. Finally, the index of  $B_1$  in  $\Gamma_0$  is computed in Lemma 6.3 by appealing to the Bruhat decomposition of the reduction modulo  $\mathfrak{P}$  of  $B_1$ .

Altogether, these results compile into Theorem 1.2. Throughout the proofs, we may assume  $n \geq 1$  since we normalize  $\Gamma_0$  to have measure one.

**1.4. Comments on ramification.** Even though Miyauchi’s theory of newforms has only been established for unitary groups attached to unramified extensions  $E/F$ , the subgroups  $\Gamma_n$  can be defined without this assumption and it would be natural to want to extend the results of this paper to allow for ramification. We discuss here some modifications it would induce.

It seems reasonable to expect that the proof of Theorem 1.2 given in this paper could be adapted to allow for tame ramification in  $E/F$ . The generalization would force us to adapt most of the group cohomology results in Section 2

and introduce casework according to the parity of the residue characteristic. For example, since  $G$  acts trivially on  $E_{\mathfrak{p}}$  when  $E/F$  is ramified, we would have instead  $H^1(G, E_{\mathfrak{p}}^{\times}) \simeq \mathbf{Z}/2\mathbf{Z}$  in Lemma 2.4, except when the residue field has characteristic 2.

On the other hand, allowing  $E/F$  to be wildly ramified seems to be more difficult. The hypothesis that  $E/F$  is at most tamely ramified is used in Lemma 2.1 and also in the proof of Lemmas 5.1 and 6.3 to compute the cardinalities of certain unipotent subgroups. Underlying these is the existence of a normal  $\mathfrak{o}$ -basis of  $\mathfrak{D}$ , which according to Noether's integral normal basis theorem happens if and only if  $E/F$  is at most tamely ramified.

Since these generalizations would increase the length of the paper and it is not clear that the Miyauchi subgroups take the same shape for unramified extensions, we leave aside the question of ramification for the remainder of this work.

**2. Group cohomology lemmas.** The following lemmas are standard results in group cohomology that will be used often in what follows. Given a group  $G$  and a  $G$ -module  $M$ , we denote by  $H^n(G, M)$  the  $n$ th cohomology group of  $G$ . See Serre [22, Ch. VII] for more definitions and background.

**Lemma 2.1.** *Let  $A$  be a Dedekind domain,  $K$  its field of fractions,  $L$  an at most tamely ramified finite Galois extension of  $K$ ,  $G$  its Galois group, and  $B$  the integral closure of  $A$  in  $L$ . We have  $H^n(G, B) = 0$  for all  $n \geq 1$ .*

*Proof.* See e.g. [19, (6.1.10) Theorem]. □

**Corollary 2.2.** *Let  $E$  be an unramified quadratic extension of a non-Archimedean local field  $F$  with Galois group  $G$  and ring of integers  $\mathfrak{D}$ . We have that  $H^n(G, \mathfrak{D}) = 0$  for all  $n \geq 1$ .*

Let  $\varpi$  be a uniformizer of  $\mathfrak{p}$ . Since  $E/F$  is unramified,  $\varpi$  is also a uniformizer of  $\mathfrak{P}$ . Let  $E_{\mathfrak{P}^n} = \mathfrak{D}/\mathfrak{P}^n$ . Corollary 2.2 also descends to the finite additive rings  $E_{\mathfrak{P}^n}$ , as the next lemma shows.

**Lemma 2.3.** *We have that  $H^m(G, E_{\mathfrak{P}^n}) = 0$  for all  $m, n \geq 1$ .*

*Proof.* We have the short exact sequence

$$1 \longrightarrow \mathfrak{D} \xrightarrow{\times \varpi^n} \mathfrak{D} \longrightarrow E_{\mathfrak{P}^n} \longrightarrow 1,$$

and taking the long exact sequence in cohomology, we have, for all  $m, n \geq 1$ , that

$$\dots \longrightarrow H^m(G, \mathfrak{D}) \longrightarrow H^m(G, E_{\mathfrak{P}^n}) \longrightarrow H^{m+1}(G, \mathfrak{D}) \longrightarrow \dots$$

Since the outer two terms vanish by Corollary 2.2, we also have  $H^m(G, E_{\mathfrak{P}^n}) = 0$  for all  $m, n \geq 1$ . □

We also consider finite multiplicative groups. Let  $E_{\mathfrak{P}^n}^{\times} = (\mathfrak{D}/\mathfrak{P}^n)^{\times}$ .

**Lemma 2.4.** *We have  $H^1(G, E_{\mathfrak{P}^n}^{\times}) = 0$  for all  $n \geq 1$ .*

*Proof.* The case  $n = 1$  is just Hilbert’s theorem 90: since  $E/F$  is unramified,  $G$  acts on the field  $E_{\mathfrak{p}}$  by Galois automorphisms, and so  $H^1(G, E_{\mathfrak{p}}^\times) = 0$ .

Assume henceforth that  $n \geq 2$ . Hilbert’s theorem 90 no longer applies since  $E_{\mathfrak{p}^n}$  is not a field, but we shall still manage to show that  $H^1(G, E_{\mathfrak{p}^n}^\times) = 0$  as follows. By considering the sequence

$$1 \longrightarrow 1 + \varpi^n \mathfrak{D} \longrightarrow \mathfrak{D}^\times \longrightarrow E_{\mathfrak{p}^n}^\times \longrightarrow 1,$$

and taking the long exact sequence in cohomology, we get

$$\dots \longrightarrow H^1(G, \mathfrak{D}^\times) \longrightarrow H^1(G, E_{\mathfrak{p}^n}^\times) \longrightarrow H^2(G, 1 + \varpi^n \mathfrak{D}) \longrightarrow \dots. \tag{2.1}$$

We would like to show that  $H^2(G, 1 + \varpi^n \mathfrak{D}) = 0$ . For any  $m \geq 1$ , consider the exact sequence

$$1 \longrightarrow 1 + \varpi^{n+m} \mathfrak{D} \longrightarrow 1 + \varpi^n \mathfrak{D} \longrightarrow \mathfrak{p}^n / \mathfrak{p}^{n+m} \longrightarrow 1.$$

By the long exact sequence in cohomology, we see that

$$H^2(G, 1 + \varpi^{n+m} \mathfrak{D}) \simeq H^2(G, 1 + \varpi^n \mathfrak{D}) \tag{2.2}$$

since the outer two cohomology groups in the long exact sequence vanish by Lemma 2.3. For  $m$  sufficiently large, we have  $1 + \varpi^{n+m} \mathfrak{D} \simeq \mathfrak{D}$  as  $G$ -modules, so that  $H^2(G, 1 + \varpi^{n+m} \mathfrak{D})$  and thus  $H^2(G, 1 + \varpi^n \mathfrak{D})$  vanishes by Corollary 2.2.

Returning to (2.1), to prove the lemma, it now suffices to show that  $H^1(G, \mathfrak{D}^\times)$  vanishes. Consider the sequence

$$1 \longrightarrow \mathfrak{D}^\times \longrightarrow E^\times \xrightarrow{\text{val}} \mathbf{Z} \longrightarrow 1.$$

Taking the long exact sequence in cohomology and by Hilbert’s theorem 90, we have

$$1 \longrightarrow \mathfrak{o}^\times \longrightarrow F^\times \xrightarrow{\text{val}} \mathbf{Z} \longrightarrow H^1(G, \mathfrak{D}^\times) \longrightarrow 1.$$

The valuation map here is surjective since  $E/F$  is unramified. Therefore we have  $H^1(G, \mathfrak{D}^\times) = 0$ , and thus  $H^1(G, E_{\mathfrak{p}^n}^\times) = 0$  from (2.1).  $\square$

Let  $E_{\mathfrak{p}^n}^0$  be the subgroup of  $E_{\mathfrak{p}^n}$  of trace-zero elements, that is  $E_{\mathfrak{p}^n}^0 = \{x \in E_{\mathfrak{p}^n} : x + \bar{x} = 0\}$ . Likewise, write  $E_{\mathfrak{p}^n}^1$  for the subgroup of  $E_{\mathfrak{p}^n}$  consisting of norm-one elements, i.e.  $E_{\mathfrak{p}^n}^1 = \{x \in E_{\mathfrak{p}^n}^\times : x\bar{x} = 1\}$ . Our intended application of the above group cohomology lemmas is to calculate the cardinalities of the finite groups  $E_{\mathfrak{p}^n}^0$  and  $E_{\mathfrak{p}^n}^1$ . For any  $G$ -module  $M$ , we write  $Z^1(G, M)$  for the group of 1-cocycles and  $B^1(G, M)$  for the set of 1-coboundaries.

**Lemma 2.5.** *For every  $n \geq 1$ ,*

$$|E_{\mathfrak{p}^n}^1| = q_F^{n-1}(q_F + 1). \tag{2.3}$$

*Proof.* Let  $\sigma$  denote the non-identity element of  $G$ . We have

$$Z^1(G, E_{\mathfrak{p}^n}^\times) \simeq E_{\mathfrak{p}^n}^1, \tag{2.4}$$

where the map is given by  $\xi \mapsto \xi(\sigma)$ . We have an exact sequence

$$1 \longrightarrow \left( \frac{\mathfrak{o}}{\mathfrak{p}^n \cap \mathfrak{o}} \right)^\times \longrightarrow E_{\mathfrak{p}^n}^\times \xrightarrow{f} Z^1(G, E_{\mathfrak{p}^n}^\times),$$

where  $f$  is given by the composition of  $x \mapsto x/\bar{x}$  and (2.4). The image of  $f$  is by definition  $B^1(G, E_{\mathfrak{p}^n}^\times)$ . But, by Lemma 2.4, we have  $B^1(G, E_{\mathfrak{p}^n}^\times) = Z^1(G, E_{\mathfrak{p}^n}^\times)$ . Thus

$$|E_{\mathfrak{p}^n}^1| = \frac{|E_{\mathfrak{p}^n}^\times|}{|(\mathfrak{o}/\mathfrak{p}^n \cap \mathfrak{o})^\times|} = \frac{q_E^{n-1}(q_E - 1)}{q_F^{n-1}(q_F - 1)}, \quad (2.5)$$

which equals  $q_F^{n-1}(q_F + 1)$  since  $q_E = q_F^2$  because  $E/F$  is unramified.  $\square$

**Lemma 2.6.** *For every  $n \geq 1$ ,*

$$|E_{\mathfrak{p}^n}^0| = q_F^n. \quad (2.6)$$

*Proof.* We have

$$Z^1(G, E_{\mathfrak{p}^n}) \simeq E_{\mathfrak{p}^n}^0, \quad (2.7)$$

where the map is given by  $\xi \mapsto \xi(\sigma)$ . Now consider the exact sequence

$$1 \longrightarrow \frac{\mathfrak{o}}{\mathfrak{p}^n \cap \mathfrak{o}} \longrightarrow E_{\mathfrak{p}^n} \xrightarrow{f} Z^1(G, E_{\mathfrak{p}^n}),$$

where the map  $f$  is given by the composition of  $u \mapsto u - \bar{u}$  and (2.7). The image of  $f$  is  $f(E_{\mathfrak{p}^n}) = B^1(G, E_{\mathfrak{p}^n}) = Z^1(G, E_{\mathfrak{p}^n})$  by Lemma 2.3. Therefore

$$|E_{\mathfrak{p}^n}^0| = \frac{|E_{\mathfrak{p}^n}|}{|\frac{\mathfrak{o}}{\mathfrak{p}^n \cap \mathfrak{o}}|} \cdot |H^1(G, E_{\mathfrak{p}^n})|. \quad (2.8)$$

Since  $E/F$  is unramified, we have  $q_E = q_F^2$ . We have  $|E_{\mathfrak{p}^n}| = q_E^n$  and  $|\frac{\mathfrak{o}}{\mathfrak{p}^n \cap \mathfrak{o}}| = q_F^n$  (again by the unramified hypothesis). Therefore, we have shown that

$$|E_{\mathfrak{p}^n}^0| = q_E^{n/2} = q_F^n. \quad \square$$

**3. From  $A_n$  to  $\Gamma_n$ .** Recall that we have chosen a common uniformizer  $\varpi$  of  $\mathfrak{p}$  and  $\mathfrak{P}$ . Introduce, for  $n \geq 0$  and  $u \in \mathfrak{D}$ ,

$$t_n(u) = \begin{pmatrix} 1 & & u\varpi^{-n} \\ & 1 & \\ & & 1 \end{pmatrix} \quad (3.1)$$

which is in  $\Gamma_n$  if and only if  $u + \bar{u} = 0$ , and

$$\sigma_n = \begin{pmatrix} & & \varpi^{-n} \\ & 1 & \\ \varpi^n & & \end{pmatrix} \in \Gamma_n. \quad (3.2)$$

For every  $n \geq 0$  and  $k \geq 0$ , let

$$C_n^{(k)} = \left\{ g \in \begin{pmatrix} \mathfrak{D} & \mathfrak{D} & \mathfrak{P}^{k-n} \\ \mathfrak{P}^n & 1 + \mathfrak{P}^n & \mathfrak{D} \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{D} \end{pmatrix} : \det(g) \in \mathfrak{D}^\times \right\} \cap U. \quad (3.3)$$

**Lemma 3.1.** *For  $E/F$  unramified, we have*

$$\left[ C_n^{(k)} : C_n^{(k+1)} \right] = \begin{cases} q_F + 1 & \text{if } k = 0, \\ q_F & \text{if } 1 \leq k < n. \end{cases} \tag{3.4}$$

*Proof.* Introduce

$$g = \begin{pmatrix} \alpha & \beta & \gamma\varpi^{k-n} \\ \delta\varpi^n & 1 + \epsilon\varpi^n & \zeta \\ \eta\varpi^n & \theta\varpi^n & \iota \end{pmatrix}, \tag{3.5}$$

a typical element in  $C_n^{(k)}$ , where every Greek letter is a generic element in  $\mathfrak{D}$  except the uniformizer  $\varpi$ . By calculating the determinant of the matrix (3.5), we find that  $\det(g) \equiv \alpha\iota \pmod{\mathfrak{P}^k}$ . Since  $\det(g) \in \mathfrak{D}^\times$ , if  $k \geq 1$ , then we have

$$\alpha, \iota \in \mathfrak{D}^\times. \tag{3.6}$$

If  $u \in \mathfrak{D}$  satisfies  $u + \bar{u} = 0$ , then  $t_{n-k}(u) \in C_n^{(k)}$ . For such  $u$  and any  $0 \leq k < n$ , we claim that

$$t_{n-k}(u)C_n^{(k+1)} = \left\{ g \in C_n^{(k)} : \iota \in \mathfrak{D}^\times, u - \iota^{-1}\gamma \in \mathfrak{P} \right\}, \tag{3.7}$$

where  $\iota$  and  $\gamma$  on the right side of (3.7) refer to entries of  $g \in C_n^{(k)}$  as in (3.5).

We first show  $\subseteq$ . Suppose  $g \in C_n^{(k+1)}$  is written as in (3.5), but with  $k + 1$  in the upper right entry in place of  $k$ . Then  $\iota \in \mathfrak{D}^\times$  by (3.6) for any  $k \geq 0$  since  $k + 1 \geq 1$ . Calculating, we have

$$t_{n-k}(u)g = \begin{pmatrix} \alpha + u\eta\varpi^k & \beta + u\theta\varpi^k & \gamma\varpi^{k+1-n} + u\iota\varpi^{k-n} \\ * & * & * \\ * & * & \iota \end{pmatrix}.$$

To show that  $t_{n-k}(u)g$  is in the right hand side of (3.7), it then suffices to show that  $\iota \in \mathfrak{D}^\times$  and  $u - \iota^{-1}(\gamma\varpi + u\iota) \in \mathfrak{P}$ . The first of these is (3.6), and the second is clear after expanding out.

Now we show  $\supseteq$ . Suppose  $g \in C_n^{(k)}$  is written as (3.5),  $\iota \in \mathfrak{D}^\times$ , and  $u \in \mathfrak{D}$  is such that  $u + \bar{u} = 0$  and  $u - \iota^{-1}\gamma \in \mathfrak{P}$ . Then

$$t_{n-k}(-u)g = \begin{pmatrix} \alpha & \beta & (\gamma - \iota u)\varpi^{k-n} \\ * & * & * \\ * & * & * \end{pmatrix},$$

but  $\gamma - \iota u \in \mathfrak{P}$  by hypothesis, so that  $t_{n-k}(-u)g \in C_n^{(k+1)}$ . The claim (3.7) follows.

Note that if  $k \geq 1$ , then the hypothesis  $\iota \in \mathfrak{D}^\times$  can be omitted from the right hand side of (3.7) because it follows automatically from (3.6). Therefore under the hypothesis that  $k \geq 1$ , we see that

$$C_n^{(k)} = \bigsqcup_{u \in E_{\mathfrak{P}}^0} t_{n-k}(u)C_n^{(k+1)}. \tag{3.8}$$

The result now follows from Lemma 2.6.



It remains to treat the case  $k = 0$ . Under this hypothesis, we claim that

$$\sigma_n C_n^{(1)} = \left\{ g \in C_n^{(0)} : \iota \in \mathfrak{P} \right\}. \quad (3.9)$$

To see  $\subseteq$ , write  $g \in C_n^{(1)}$  in the form (3.5), and calculate

$$\sigma_n g = \begin{pmatrix} \eta & \theta & \iota \varpi^{-n} \\ \delta \varpi^n & 1 + \epsilon \varpi^n & \zeta \\ \alpha \varpi^n & \beta \varpi^n & \gamma \varpi \end{pmatrix}.$$

To see  $\supseteq$ , write  $g \in C_n^{(0)}$  in the form (3.5), and recall the hypothesis  $\iota \in \mathfrak{P}$ . Note that  $\sigma_n^2 = 1$ , and calculate

$$\sigma_n g = \begin{pmatrix} \eta & \theta & \iota \varpi^{-n} \\ \delta \varpi^n & 1 + \epsilon \varpi^n & \zeta \\ \alpha \varpi^n & \beta \varpi^n & \gamma \end{pmatrix},$$

from which we see that  $\iota \varpi^{-n} \in \mathfrak{P}^{1-n}$ , as desired.

Combining (3.7) and (3.9), we see that

$$C_n^{(0)} = \sigma_n C_n^{(1)} \sqcup \bigsqcup_{u \in E_{\mathfrak{P}}^0} t_n(u) C_n^{(1)}, \quad (3.10)$$

from which the result again follows by Lemma 2.6.  $\square$

Since  $\Gamma_n = C_n^{(0)}$  and  $A_n = C_n^{(n)}$ , the index of  $A_n$  in  $\Gamma_n$  follows by induction.

**Lemma 3.2.** *For every  $n \geq 1$ ,*

$$[\Gamma_n : A_n] = q_F^{n-1} (q_F + 1). \quad (3.11)$$

**4. From  $A_n$  to  $B_n$ .** This section aims at reducing the study of the indices of  $A_n$  in  $\Gamma_n$  to those of the more paramodularly-shaped  $B_n$ . Consider the homomorphism

$$f : B_n \rightarrow E_{\mathfrak{P}^n}^{\times}$$

given by

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \delta \varpi^n & \epsilon & \zeta \\ \eta \varpi^n & \theta \varpi^n & \iota \end{pmatrix} \mapsto \epsilon \pmod{\mathfrak{P}^n}.$$

**Lemma 4.1.** *If  $n \geq 1$ , the sequence*

$$1 \longrightarrow A_n \longrightarrow B_n \xrightarrow{f} E_{\mathfrak{P}^n}^1 \longrightarrow 1 \quad (4.1)$$

*is short exact. In particular,*

$$[\Gamma_0 : A_n] = [\Gamma_0 : B_n] \cdot |E_{\mathfrak{P}^n}^1|. \quad (4.2)$$

*Proof.* Clearly, the kernel of  $f$  is exactly  $A_n$ . It suffices to show that  $f$  has image  $E_{\mathfrak{P}^n}^1$ . Let

$$b = \begin{pmatrix} \alpha & \beta & \gamma \\ \delta\varpi^n & \epsilon & \zeta \\ \eta\varpi^n & \theta\varpi^n & \iota \end{pmatrix}$$

be an arbitrary element of  $B_n$ , where Greek letters are elements of  $\mathfrak{D}$ . Then since

$${}^t\bar{b}Jb = J,$$

we have

$$(\beta\bar{\theta} + \bar{\beta}\theta)\varpi^n + \epsilon\bar{\epsilon} = 1, \tag{4.3}$$

by expanding the center entry of  ${}^t\bar{b}Jb = J$ , and since  $\varpi$  was chosen so that  $\varpi \in \mathfrak{p} \subset F$ . Therefore we have  $\epsilon\bar{\epsilon} \equiv 1 \pmod{\mathfrak{P}^n}$ , so that  $f(B_n) \subseteq E_{\mathfrak{P}^n}^1$ . On the other hand, we have

$$\begin{pmatrix} 1 & & \\ & \epsilon & \\ & & 1 \end{pmatrix} \in B_n$$

for any  $\epsilon \in \mathfrak{D}$  such that  $\epsilon\bar{\epsilon} = 1$  by a direct calculation. Therefore  $f(B_n) \supseteq E_{\mathfrak{P}^n}^1$ , and the lemma is proved.  $\square$

**5. Index of  $B_n$ .** To compute the index of  $B_n$  in  $\Gamma_0$ , we will need the Iwahori decomposition, our reference for which is [14]. The index of  $B_n$  in  $\Gamma_0$  is given in the following lemma.

**Lemma 5.1.** *For every  $n \geq 1$ , we have*

$$[\Gamma_0 : B_n] = q_F^{3(n-1)}[\Gamma_0 : B_1]. \tag{5.1}$$

*Proof.* We use the Iwahori decomposition for the group  $U$ , which may be quickly derived from that of  $\text{GL}_3(E)$ . In [14, Proposition 2.6.4], taking  $G = \text{GL}_3(E)$  and  $S$  to be the convex hull of the following 4 points

$$\{(0, 0, 0), (n, 0, 0), (0, 0, -n), (n, n, 0)\},$$

we obtain the factorization

$$\begin{pmatrix} \mathfrak{D} & \mathfrak{D} & \mathfrak{D} \\ \mathfrak{P}^n & \mathfrak{D} & \mathfrak{D} \\ \mathfrak{P}^n & \mathfrak{P}^n & \mathfrak{D} \end{pmatrix} = \begin{pmatrix} 1 & & \\ \mathfrak{P}^n & 1 & \\ \mathfrak{P}^n & \mathfrak{P}^n & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{D}^\times & & \\ & \mathfrak{D}^\times & \\ & & \mathfrak{D}^\times \end{pmatrix} \begin{pmatrix} 1 & \mathfrak{D} & \mathfrak{D} \\ & 1 & \mathfrak{D} \\ & & 1 \end{pmatrix}, \tag{5.2}$$

where each matrix represents the indicated subgroup of  $\text{GL}_3(E)$ . We may now realize the group  $B_n$  as the fixed points of the left hand side of (5.2) by the involution  $g \mapsto J^t\bar{g}^{-1}J$ . Let  $g = n_-tn_+$  be the factorization of a typical element  $g$  as in (5.2). Since  $J^t\bar{g}^{-1}J = J^t\bar{n}_-^{-1}JJ^t\bar{t}^{-1}JJ^t\bar{n}_+^{-1}J$ , we derive for any  $n \geq 1$  that

$$B_n = \begin{pmatrix} 1 & & \\ \mathfrak{P}^n & 1 & \\ \mathfrak{P}^n & \mathfrak{P}^n & 1 \end{pmatrix} \begin{pmatrix} \mathfrak{D}^\times & & \\ & \mathfrak{D}^\times & \\ & & \mathfrak{D}^\times \end{pmatrix} \begin{pmatrix} 1 & \mathfrak{D} & \mathfrak{D} \\ & 1 & \mathfrak{D} \\ & & 1 \end{pmatrix}, \tag{5.3}$$

where each matrix appearing in (5.3) represents the intersection of the indicated subgroup with  $U$ . Moreover, the matrices of the first subgroup appearing in (5.3) are exactly those of the form

$$\begin{pmatrix} 1 & & \\ \alpha\varpi^n & 1 & \\ \beta\varpi^n & -\bar{\alpha}\varpi^n & 1 \end{pmatrix} \quad \text{where} \quad \beta + \bar{\beta} + \alpha\bar{\alpha}\varpi^n = 0. \quad (5.4)$$

For  $n \geq 1$ , this leads to the decomposition

$$\begin{pmatrix} 1 & & \\ \mathfrak{P}^{n-1} & 1 & \\ \mathfrak{P}^{n-1} & \mathfrak{P}^{n-1} & 1 \end{pmatrix} = \bigsqcup_{\substack{\alpha, \beta \in E_{\mathfrak{P}} \\ \beta + \bar{\beta} + \alpha\bar{\alpha}\varpi^n = 0}} \begin{pmatrix} 1 & & \\ \alpha\varpi^{n-1} & 1 & \\ \beta\varpi^{n-1} & -\bar{\alpha}\varpi^{n-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ \mathfrak{P}^n & 1 & \\ \mathfrak{P}^n & \mathfrak{P}^n & 1 \end{pmatrix}, \quad (5.5)$$

where the condition  $\beta + \bar{\beta} + \alpha\bar{\alpha}\varpi^n = 0$  on  $\alpha, \beta \in E_{\mathfrak{P}}$  is understood to mean that we sum over those cosets for which there exists a representative in  $\mathfrak{D}$  satisfying the indicated equation.

Given  $\alpha, \beta$  as above, it is clear by reducing the condition modulo  $\mathfrak{P}$  that  $\beta \in E_{\mathfrak{P}}^0$ . Conversely, given  $\alpha^* \in E_{\mathfrak{P}}$  and  $\beta^* \in E_{\mathfrak{P}}^0$ , there exists a representative  $\beta \in \mathfrak{D}$  for  $\beta^*$  such that  $\beta + \bar{\beta} = 0$ . Let  $\alpha$  be any choice of representative for  $\alpha^*$ . Then  $\alpha\bar{\alpha}\varpi^n \in \mathfrak{p}$ . Since  $E/F$  is at most tamely ramified, we have as a consequence of the integral normal basis theorem (see e.g. [6, §5 Theorem 2]) that  $\text{Tr } \mathfrak{P} = \mathfrak{p}$ , and so there exists  $z \in \mathfrak{P}$  such that  $\alpha\bar{\alpha}\varpi^n = z + \bar{z}$ . Then  $\beta - z$  is another representative for  $\beta^* \in E_{\mathfrak{P}}^0$ , and  $(\beta - z) + (\overline{\beta - z}) + \alpha\bar{\alpha}\varpi^n = 0$ . Thus, the decomposition (5.3) may be re-written as

$$B_n = \bigsqcup_{\substack{\alpha \in E_{\mathfrak{P}} \\ \beta \in E_{\mathfrak{P}}^0}} \begin{pmatrix} 1 & & \\ \alpha\varpi^n & 1 & \\ \beta\varpi^n & -\bar{\alpha}\varpi^n & 1 \end{pmatrix} B_{n+1}. \quad (5.6)$$

We have  $|E_{\mathfrak{P}}| = q_E = q_F^2$  and  $|E_{\mathfrak{P}}^0| = q_F$  by Lemma 2.6, so this finishes the proof by induction.  $\square$

**6. Index of  $B_1$ .** We use reduction modulo  $\mathfrak{P}$  and the Bruhat decomposition, and recall now some useful lemmas to do so.

**Lemma 6.1.** *Let  $X, G$  be two groups,  $H$  a subgroup of  $G$  and  $\phi : X \rightarrow G$  a surjective morphism. Then*

$$[G : H] = [\phi^{-1}(G) : \phi^{-1}(H)]. \quad (6.1)$$

*Proof.* We write  $G/H$  for the set of left cosets of  $H$  in  $G$ , that is we have

$$G = \bigsqcup_{g \in G/H} gH \quad \text{and so} \quad \phi^{-1}(G) = \bigsqcup_{g \in G/H} \phi^{-1}(gH).$$

Since  $\phi$  is surjective, for any representative of  $g \in G/H$ , we may choose a preimage  $g' \in \phi^{-1}(g)$ . Then  $\phi^{-1}(gH) = g'\phi^{-1}(H)$  and the lemma follows.  $\square$

**Lemma 6.2** (Hensel’s Lemma). *Let  $\mathbf{G}$  be a smooth scheme over the ring of integers  $\mathfrak{o}$  of a non-Archimedean local field. Let  $\mathfrak{p}$  the corresponding maximal ideal. Then the reduction modulo  $\mathfrak{p}$*

$$\mathbf{G}(\mathfrak{o}) \longrightarrow \mathbf{G}(F_{\mathfrak{p}}) \tag{6.2}$$

*is surjective.*

*Proof.* This is a standard version of Hensel’s lemma in algebraic geometry. For precise details, see [8, Théorème 18.5.17].  $\square$

**Lemma 6.3.** *We have*

$$[\Gamma_0 : B_1] = q_F^3 + 1. \tag{6.3}$$

*Proof.* Consider the unitary group  $\mathbf{U}$  defined over  $\mathfrak{o}$  and its standard Borel subgroup  $\mathbf{B}$ . We have  $\mathbf{U}(\mathfrak{o}) = \Gamma_0$  and  $\mathbf{B}(\mathfrak{o}) = B_1$ . Within this proof, we write  $U = \mathbf{U}(F_{\mathfrak{p}})$  and  $B = \mathbf{B}(F_{\mathfrak{p}})$ . By Hensel’s lemma (Lemma 6.2), we have surjective morphisms

$$\Gamma_0 \twoheadrightarrow U \quad \text{and} \quad B_1 \twoheadrightarrow B.$$

By Lemma 6.1, we then have that  $[\Gamma_0 : B_1] = [U : B]$ , so that the problem reduces to finite fields.

We apply the Bruhat decomposition for  $U$ , following [5, Chapter 4]. Consider the group  $\text{GL}_3$  over  $\overline{F}_{\mathfrak{p}}$  and the involution  $\tau$  defined by  $g \mapsto J^t \bar{g}^{-1} J$ , where  $\bar{g}$  denotes the Frobenius automorphism applied to  $g$ , i.e. the entries of  $g$  raised to the  $p$ th power. The fixed points of  $\text{GL}_3$  by this involution are exactly the group  $U$  (cf. [5, Example 4.3.3]). Let  $T$  be the standard maximal torus of  $\text{GL}_3$ ,  $N = N(T)$  its normalizer, and  $W = W(T)$  the associated Weyl group. Then we have a Bruhat decomposition of  $U$  of the form

$$U = \bigsqcup_{w \in W^\tau} BwU_w^\tau, \tag{6.4}$$

with  $BwU_w^\tau$  being a direct product [5, Lemma 3.2.7], where

$$U_w = \prod_{\{\alpha \in \Phi^+ : w(\alpha) < 0\}} U_\alpha$$

and  $U_\alpha$  is the root space associated to the positive root  $\alpha \in \Phi^+$  (see [5, Proposition 4.4.1(ii)]). Warning:  $U_w$  and  $U_\alpha$ , which are subgroups of  $\text{GL}_3$ , should not be confused with the unitary group  $U$ .

We proceed to calculate  $W^\tau$  and  $U_w^\tau$ . By [5, Proposition 4.4.1(i)], we have that  $W^\tau = N(T)^\tau/T^\tau$ . We can calculate explicitly that

$$T^\tau = \left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} : a\bar{c} = 1, b\bar{b} = 1 \right\}$$

and

$$N(T)^\tau = T^\tau \sqcup \left\{ \begin{pmatrix} & & c \\ & b & \\ a & & \end{pmatrix} : a\bar{c} = 1, b\bar{b} = 1 \right\}.$$

Thus  $W^\tau$  consists of two elements, which may be represented by  $I$  and  $J$  (see (1.2)).

Now we calculate  $U_w^\tau$  for  $w = I$  and  $w = J$ . We first determine the sets  $\{\alpha \in \Phi^+ : w(\alpha) < 0\}$ . Let  $U_{ij}$ ,  $1 \leq i \neq j \leq 3$ , denote the root space of  $GL_3$  that is non-zero in the  $ij$ th entry. Let  $\alpha_{ij}$  denote the corresponding root (see [5, Theorem 2.3.1(i)]). The set  $\Phi^+$  of positive roots corresponding to our choice of the standard upper triangular Borel is  $\Phi^+ = \{\alpha_{12}, \alpha_{13}, \alpha_{23}\}$ . The Weyl group  $W$  acts on  $T$  by conjugation, and thus on  $\Phi$ . Clearly,  $I$  fixes  $\Phi$ , and one may compute that  $J(\alpha_{12}) = \alpha_{32}$ ,  $J(\alpha_{13}) = \alpha_{31}$ , and  $J(\alpha_{23}) = \alpha_{21}$ . Thus,  $\{\alpha \in \Phi^+ : I(\alpha) < 0\} = \emptyset$  and  $\{\alpha \in \Phi^+ : J(\alpha) < 0\} = \Phi^+$ . Therefore we have

$$U = B \sqcup BJ \begin{pmatrix} 1 & \star & \star \\ & 1 & \star \\ & & 1 \end{pmatrix}^\tau. \tag{6.5}$$

This last group of unipotent matrices is in bijection with the set  $\cup_{\alpha \in \mathfrak{D}/\mathfrak{P}} N_\alpha$ , where  $N_\alpha = \{\beta \in \mathfrak{D}/\mathfrak{P} : \beta + \bar{\beta} + \alpha\bar{\alpha} = 0\}$  (cf. (5.4)). Recall (e.g. [6, §5 Theorem 2]) that  $\text{Tr } \mathfrak{D} = \mathfrak{o}$  since  $E/F$  is at most tamely ramified. Thus, there exists  $\theta \in \mathfrak{D}/\mathfrak{P}$  such that  $\theta + \bar{\theta} = -\alpha\bar{\alpha}$ . We then find that  $N_\alpha = \theta + E_{\mathfrak{P}}^0$ , which is of cardinality  $q_F$  by Lemma 2.6. Thus, the cardinality of the group of unipotent matrices in question is then  $q_E q_F = q_F^3$ . We conclude that  $[U : B] = q_F^3 + 1$ , hence the lemma follows.  $\square$

We now collect the previous results. By Lemmas 3.2 and 4.1, we have

$$\text{vol}(\Gamma_n) = \frac{[\Gamma_n : A_n]}{[\Gamma_0 : A_n]} \text{vol}(\Gamma_0) = \frac{q_F^{n-1}(q_F + 1)}{[\Gamma_0 : B_n] |E_{\mathfrak{P}^n}^1|}, \tag{6.6}$$

which by Lemmas 2.5 and 5.1 is  $= q_F^{3-3n} [\Gamma_0 : B_1]^{-1}$ . Finally, by Lemma 6.3, we conclude Theorem 1.2.

**Acknowledgements.** The authors are grateful to Jean Michel and Brooks Roberts for enlightening discussions, and to the Institut Henri Poincaré for providing workspace during several of the authors’ research visits to Paris. We also thank the referee for valuable comments. The first author has been partially supported by Labex CEMPI (ANR-11-LABX-0007-01) and the Deutscher Akademischer Austauschdienst (DAAD). The second author was supported by Swiss National Science Foundation grant PZ00P2\_168164.

**Open Access.** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from

the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

### References

- [1] Baruch, E.M.: On the gamma factors attached to representations of  $U(2,1)$  over a  $p$ -adic field. *Israel J. Math.* **102**, 317–345 (1997)
- [2] F. Brumley and D. Milićević. Counting cusp forms by analytic conductor. [arXiv:1805.00633](https://arxiv.org/abs/1805.00633) (2019)
- [3] Casselman, W.: On some results of Atkin and Lehner. *Math. Ann.* **201**, 301–314 (1973)
- [4] Deligne, P. Formes modulaires et représentations de  $GL(2)$ . In: *Modular Functions of one Variable, II* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 55–105. *Lecture Notes in Math.*, vol. 349. Springer, Berlin (1973)
- [5] Digne, F., Michel, J.: *Representations of Finite Groups of Lie Type*. London Mathematical Society Student Texts, vol. 95, 2nd edn. Cambridge University Press, Cambridge (2020)
- [6] Fröhlich, A.: Local fields. In: *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965), pp. 1–41. Thompson, Washington, D.C. (1967)
- [7] Gelbart, S., Piatetski-Shapiro, I.: Automorphic forms and  $L$ -functions for the unitary group. In: *Lie Group Representations, II* (College Park, Md., 1982/1983), pp. 141–184. *Lecture Notes in Math.*, vol. 1041. Springer, Berlin (1984)
- [8] Grothendieck, A.: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*. *Inst. Hautes Études Sci. Publ. Math.* **32**, 361 (1967)
- [9] Iwaniec, H., Luo, W., Sarnak, P.: Low lying zeros of families of  $L$ -functions. *Inst. Hautes Études Sci. Publ. Math.* **2000**(91), 55–131 (2001)
- [10] Jacquet, H., Piatetski-Shapiro, I.I., Shalika, J.: Conducteur des représentations du groupe linéaire. *Math. Ann.* **256**(2), 199–214 (1981)
- [11] Knightly, A., Li, C.: A relative trace formula proof of the Petersson trace formula. *Acta Arith.* **122**(3), 297–313 (2006)
- [12] Lesesvre, D.: *Arithmetic Statistics for Quaternion Algebras*. PhD thesis, Université Paris (2018)
- [13] Lesesvre, D.: Counting and equidistribution for quaternion algebras. *Math. Z.* **295**, 129–159 (2020)
- [14] Macdonald, I.G.: *Spherical Functions on a Group of  $p$ -adic Type*. Publications of the Ramanujan Institute, No. 2. University of Madras, Centre for Advanced Study in Mathematics, Ramanujan Institute, Madras (1971)
- [15] Miyake, T.: *Modular Forms*. Translated from the 1976 Japanese original by Yoshitaka Maeda. Reprint of the first 1989 English edition. Springer Monographs in Mathematics. Springer, Berlin (2006)

- [16] Miyauchi, M.: On epsilon factors attached to supercuspidal representations of unramified  $U(2, 1)$ . *Trans. Amer. Math. Soc.* **365**(6), 3355–3372 (2013)
- [17] Miyauchi, M.: On local newforms for unramified  $U(2, 1)$ . *Manuscripta Math.* **141**(1–2), 149–169 (2013)
- [18] Miyauchi, M.: On  $L$ -factors attached to generic representations of unramified  $U(2, 1)$ . *Math. Z.* **289**(3–4), 1381–1408 (2018)
- [19] Neukirch, J., Schmidt, A., Wingberg, K.: *Cohomology of Number Fields*. Second edition. *Grundlehren der mathematischen Wissenschaften*, vol. 323. Springer, Berlin (2008)
- [20] Rogawski, J.D.: *Automorphic Representations of Unitary Groups in Three Variables*. *Annals of Mathematics Studies*, vol. 123. Princeton University Press, Princeton, NJ (1990)
- [21] Roberts, B., Schmidt, R.: *Local Newforms for  $GU(2, 1)$* . *Lecture Notes in Mathematics*, vol. 1918. Springer, Berlin (2007)
- [22] Serre, J.-P.: *Local Fields*. Translated from the French by Marvin Jay Greenberg. *Graduate Texts in Mathematics*, vol. 67. Springer, New York-Berlin (1979)

DIDIER LESESVRE  
Laboratoire Paul Painlevé, CNRS UMR 8524  
Université de Lille  
59000 Lille  
France  
e-mail: [lesesvre@math.cnrs.fr](mailto:lesesvre@math.cnrs.fr)

IAN PETROW  
Department of Mathematics  
University College London  
25 Gordon Street  
London WC1H 0AY  
UK  
e-mail: [i.petrov@ucl.ac.uk](mailto:i.petrov@ucl.ac.uk)

Received: 12 May 2020

Revised: 22 October 2020

Accepted: 28 March 2022.