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EXPLICIT TIME STEPPING FOR THE WAVE EQUATION USING CUTFEM WITH DISCRETE EXTENSION*

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Abstract. In this paper we develop a fully explicit cut finite element method for the wave equation. The method is based on using a standard leap frog scheme combined with an extension operator that defines the nodal values outside of the domain in terms of the nodal values inside the domain. We show that the mass matrix associated with the extended finite element space can be lumped leading to a fully explicit scheme. We derive stability estimates for the method and provide optimal order a priori error estimates. Finally, we present some illustrating numerical examples.

Key words. wave equation, explicit time stepping, CutFEM, discrete extension operator, a priori error estimates

AMS subject classifications. 65M12, 65M15, 65M60

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1. Introduction.

New contributions. Let $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, be an open connected domain with smooth boundary $\partial\Omega$. We consider the wave equation: find $u : [0,T) \to H^2(\Omega)$ such that

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{in } (0,T)\times\Omega, \qquad u = 0 \quad \text{on } (0,T)\times\partial\Omega \tag{1.1}$$

with initial data $u = u_0$ and $\partial u/\partial t = u_1$ at $t = 0$, and right-hand side $f : [0,T) \to L^2(\Omega)$. The objective of the present note is to design an explicit cut finite element method (CutFEM) for the approximation of solutions to (1.1). The method uses a leapfrog scheme for the time discretization combined with an extension operator which provides values in nodes outside of the domain in terms of the interior nodal values. The extension is based on a composition of an extension operator from interior elements into the space of discontinuous piecewise polynomials and an average operator that projects into the continuous finite element space. The framework is quite general, allows for several natural implementations, is convenient for analysis, and may be viewed as a generalization of previous constructions; see [1]. We prove stability and interpolation results for the extended finite element space. To construct a purely explicit scheme, we show that the mass matrix associated with the extended finite element space can indeed be lumped while preserving optimal order for piecewise linear elements. Key to this result is that with a properly constructed extension

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operator we can show that the lumped mass matrix is positive definite. This is in contrast to popular stabilization procedures of the mass matrix such as stabilization of the jump in derivatives across faces where lumping is, in general, not possible. Combining cut finite elements, the extension operator, and mass lumping we obtain a very simple fast explicit method which can handle complex geometric situations thanks to the flexibility provided by the CutFEM.

We note that the discrete extension operator provides an alternative to weak stabilization of the cut elements through the bilinear form which controls jumps in derivatives across faces. The extension operator is therefore of interest in its own right and may find other applications, for instance, for the computation of physical fluxes in the shifted boundary method. Furthermore, our construction and theory of the extension operator may be extended to higher order nodal finite element spaces; see [9] for a general framework. In this paper we restrict our attention to explicit lumped methods based on piecewise linears for the wave equation.

Previous work. Cut finite elements allow the boundary of the domain to cut through an underlying fixed mesh in an arbitrary manner. This procedure manufactures so called cut elements in the vicinity of the boundary that may lead to stability problems and bad conditioning of the resulting algebraic equations. The remedy is to add some form of stabilization, for instance, a weak least squares control on the jump in the normal gradient across element faces, so called ghost penalty; see [4, 8, 18, 23] for various applications of this concept. Another approach to handle cut elements is to eliminate them using agglomeration where small elements are connected to larger elements in order to form an element with a sufficiently large intersection with the domain; see [20, 7] for discontinuous Galerkin methods, and [1] for an extension operator where degrees of freedom associated with external nodes are eliminated using a local average of internal node values. For a general introduction to CutFEMs, we refer to the overview article [5].

Error analysis of finite element methods for the wave equation was originally developed in early papers including [13, 2, 3], space time methods were proposed and analysed in [19, 21]. Recent works on wave equations focus on explicit schemes [11, 12] and discontinuous Galerkin methods [16, 17]. CutFEMs for the wave equation were developed in [27, 28], in particular the authors consider higher order elements with face stabilization combined with an explicit Runge–Kutta time stepping scheme which involves inversion of the mass matrix.

Outline. In section 2 we first introduce the discrete extension operator and derive stability estimates and interpolation error bounds for the extended finite element space. Then we formulate the finite element method. In section 3 we prove a stability estimate for the method and then prove optimal order a priori error estimates taking also lumping of the mass matrix into account. Finally, in section 4 we present illustrating numerical examples.

2. The finite element method.

2.1. Standard notation. We shall use the following standard notation: $H^s(\omega)$ denotes the Sobolov spaces of order s over the set ω with norm $\|\cdot\|_{H^s(\omega)}$. For $s=0$ we write $L^2(\omega) = H^0(\omega)$ and $\|\cdot\|_{L^2(\omega)} = \|\cdot\|_\omega$. In the case $\omega = \Omega$, we further simplify and write $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$. The $L^2(\omega)$ inner product is denoted by $(v,w)_\omega = \int_\omega vw$, and for $\omega = \Omega$ we write $(v,w)_\Omega = (v,w)$.

2.2. Mesh and finite element spaces. We introduce the following notation.

- We let Ω_0 be a polygonal domain with $\Omega \subset \Omega_0$ and assume that $\mathcal{T}_{h,0}$ is a quasi-uniform triangulation of Ω_0 with mesh parameter $h \in (0, h_0]$ for some $h_0 > 0$. We let $\mathcal{T}_h$ denote the active mesh $\mathcal{T}_h = \{T \in \mathcal{T}_{h,0} : T \cap \Omega \neq \emptyset\}$. We let $\mathcal{F}_h$ denote the set of interior faces in $\mathcal{T}_h$.

- We let $\mathcal{X}_h$ be the set of vertices in $\mathcal{T}_h$ and denote its cardinality by N_h.
- We define the space of piecewise linear discontinuous functions W_h on $\mathcal{T}_h$ and the subspace of continuous piecewise linear functions $V_h := W_h \cap C^0(\Omega_h)$, where $\Omega_h = \cup_{T \in \mathcal{T}_h} T$.
- We shall often use scalar products and norms defined on a set of mesh entities. For instance, let $\widetilde{\mathcal{T}}_h \subset \mathcal{T}_h$ be a subset of elements; then

$$(v, w)_{\widetilde{\mathcal{T}}_h} = \sum_{T \in \widetilde{\mathcal{T}}_h} (v, w)_T, \qquad \|v\|^2_{\widetilde{\mathcal{T}}_h} = \sum_{T \in \widetilde{\mathcal{T}}_h} \|v\|^2_T. \tag{2.1}$$

2.3. Discrete extension. It is well known [26, Theorem 5, page 181] that for domains with sufficiently smooth boundary, there exists a universal stable extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^d)$, $s \in \mathbb{N}_+$,

$$\|Eu\|_{H^s(\mathbb{R}^d)} \lesssim \|u\|_{H^s(\Omega)}, \tag{2.2}$$

Here and below $x \lesssim y$ means $x \leq Cy$ for some positive constant C. We will now construct a stable discrete extension operator. The construction is based on polynomial extension into the discontinuous finite element space W_h and then application of an average operator to obtain a continuous piecewise linear function in V_h. We first recall such an average operator A_h.

Average operator. Let the nodal averaging operator $A_h : W_h \to V_h$ be defined by

$$A_h : W_h \ni w \mapsto \sum_{x \in \mathcal{X}_h} \langle w \rangle_x \varphi_x \in V_h, \tag{2.3}$$

where the average of the discontinuous function $w \in W_h$ at a node $x \in \mathcal{X}_h$ is defined by

$$\langle w \rangle_x = \sum_{T \in \mathcal{T}_h(x)} \kappa_{T,x} w|_T(x), \tag{2.4}$$

where the weights $\kappa_{T,x}$ satisfy

$$\kappa_{T,x} \geq 0, \qquad \sum_{T \in \mathcal{T}_h(x)} \kappa_{T,x} = 1, \tag{2.5}$$

and $\mathcal{T}_h(x) = \{T \in \mathcal{T}_h : x \in T\}$ with cardinality $|\mathcal{T}_h(x)|$. The operator A_h was introduced in [24] and is often called the Oswald interpolation operator. We have the following estimate (see, for instance, [6]):

$$\|w - A_h w\|_{\mathcal{T}_h} \lesssim h^{1/2} \|[w]\|_{\mathcal{F}_h}, \tag{2.6}$$

where for $x \in F$, $[w](x) = w_+(x) - w_-(x)$, with $w_\pm(x) = \lim_{\epsilon \to 0_+} w(x \pm \epsilon n_F)$ and n_F a fixed unit normal associated with the face F, denotes the jump in the discontinuous function w across the face F. For completeness, we include a brief derivation.

Proof of (2.6). Letting $w_T = w|_T$ and using an inverse estimate to pass from the elements to the nodes, we obtain

$$\|w - A_h w\|^2_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h} \|w_T - A_h w\|^2_T \lesssim \sum_{T\in\mathcal{T}_h} h^d \|w_T - A_h w\|^2_{\mathcal{X}_h(T)} \tag{2.7}$$

$$\lesssim \sum_{T\in\mathcal{T}_h} \sum_{x\in\mathcal{X}_h(T)} h^d |w_T(x) - \langle w\rangle_x|^2 \lesssim \sum_{T\in\mathcal{T}_h} \sum_{x\in\mathcal{X}_h(T)} \sum_{F\in\mathcal{F}_h(x)} h\|[w]\|^2_F \lesssim h\|[w]\|^2_{\mathcal{F}_h}, \tag{2.8}$$

where $\mathcal{X}_h(T)$ is the set of nodes associated with T, $\|w\|^2_{\mathcal{X}_h(T)} = \sum x \in \mathcal{X}_h(T)|w(x)|^2$, $\mathcal{F}_h(x)$ is the set of faces belonging to node $x \in \mathcal{X}_h$, and we finally used the following inverse estimate:

$$|w_T(x) - \langle w\rangle_x|^2 = \sum_{S\in\mathcal{T}_h(x)} \kappa^2_{S,x} |w_T(x) - w_S(x)|^2 \lesssim \sum_{F\in\mathcal{F}_h(x)} h^{1-d}\|[w]\|^2_F. \tag{2.9}$$

Here we used the fact that the weights in the average sum to one; then to estimate $|w_T(x) - w_S(x)|^2$ for two arbitrary elements $S, T \in \mathcal{T}_h(x)$ we note that there is a sequence of face neighboring elements $\{T_j\}_{j=1}^n$, with n uniformly bounded thanks to quasi-uniformity, such that $T_1 = S$ and $T_n = T$ and using the triangle inequality

$$h^d|w_T(x) - w_S(x)|^2 = h^d|w_n(x) - w_1(x)|^2 \le h^d\left(\sum_{j=2}^n |w_j(x) - w_{j-1}(x)|\right)^2 \tag{2.10}$$

$$\lesssim \sum_{j=2}^n h^d|w_j(x) - w_{j-1}(x)|^2 \lesssim \sum_{F\in\mathcal{F}_h(x)} h^d|[w(x)]|^2 \lesssim \sum_{F\in\mathcal{F}_h(x)} h\|[w]\|^2_F, \tag{2.11}$$

where at last we us used the inverse estimate $|v(x)|^2 \lesssim h^{1-d}\|v\|^2_F$ with $v = [w]$. ∎

Extension operator. To define the extension operator, we split $\mathcal{T}_h$ as follows:

$$\mathcal{T}_h = \mathcal{T}_{h,B} \cup \mathcal{T}_{h,I}, \tag{2.12}$$

where $\mathcal{T}_{h,I}$ is the set of elements in the interior of Ω (or with sufficiently large intersection with Ω; see Remark 2.1) and $\mathcal{T}_{h,B}$ are the elements that intersect the boundary,

$$\mathcal{T}_{h,I} = \{T \in \mathcal{T}_h : T \subset \Omega\}, \qquad \mathcal{T}_{h,B} = \mathcal{T}_h \setminus \mathcal{T}_{h,I}. \tag{2.13}$$

Let $W_{h,I} = W_h|_{\mathcal{T}_{h,I}}$ and $V_{h,I} = V_h|_{\mathcal{T}_{h,I}}$. We construct an extension operator $F_h : W_{h,I} \to F_h W_{h,I} \subset W_h$ by using canonical polynomial extensions from a nearest neighboring element $T \in \mathcal{T}_{h,I}$. Restricting F_h to $V_{h,I}$ and composing with the average operator A_h, we obtain a discrete extension operator $E_h : V_{h,I} \to E_h V_{h,I} \subset V_h$. The space $E_h V_{h,I}$ will be our approximation space and we will use the notation

$$V_h^E = E_h V_{h,I}. \tag{2.14}$$

Observe that V_h^E is a proper subspace of V_h; however as we shall see under mild assumptions on the mesh geometry, it has similar approximation properties.

To make things precise, let $S_h : \mathcal{T}_{h,B} \to \mathcal{T}_{h,I}$ be a mapping that associates an element $T \in \mathcal{T}_{h,I}$ with each element $T \in \mathcal{T}_{h,B}$ and assume that there is a constant such that for all $h \in (0, h_0]$ and $T \in \mathcal{T}_{h,B}$,

$$\operatorname{diam}(T \cup S_h(T)) \lesssim h. \tag{2.15}$$

For h_0 small enough there is such a mapping S_h; see Lemma 2.4 below. We extend S_h from $\mathcal{T}_{h,B}$ to $\mathcal{T}_h$ by letting $S_h(T) = T$ for $T \in \mathcal{T}_{h,I}$.

For $v \in \mathbb{P}_1(T)$ we let $v^e \in \mathbb{P}_1(\mathbb{R}^d)$ denote the canonical extension such that $v^e|_T = v$. We can then define the discrete extension operator $F_h : W_{h,I} \to W_h$ as follows:

$$(F_h v)|_T = (v|_{S_h(T)})^e|_T \tag{2.16}$$

and then define the discrete extension operator $E_h : V_{h,I} \to V_h$,

$$E_h = A_h \circ F_h. \tag{2.17}$$

Remark 2.1. In practice, we can generalize the definition of the set of elements that have a large intersection with the domain as follows:

$$\mathcal{T}_{h,I,\tau} = \{T \in \mathcal{T}_h : |T \cap \Omega| \geq \tau h^d\} \tag{2.18}$$

for some positive parameter τ. Then for small enough τ we have $\mathcal{T}_{h,I} \subset \mathcal{T}_{h,I,\tau}$ and we extend to the small elements $\mathcal{T}_{h,B,\tau} = \mathcal{T}_h \setminus \mathcal{T}_{h,I,\tau}$. This approach has the advantage that fewer elements are mapped resulting in a simpler map F_h. We will employ this construction in section 3.3 to show that the lumped mass matrix is positive definite.

Remark 2.2. The construction of the extension operator and the forthcoming theory may be extended to higher order polynomials and more generally to nodal finite element spaces; see [9] for details.

We will now prove that the extension is stable and that the associated interpolation operator has optimal approximation properties.

LEMMA 2.3. *Global piecewise linear polynomials,* $\mathbb{P}_1(\mathbb{R}^d)$*, are invariant under the extension operator,*

$$E_h(v|_{\mathcal{T}_{h,I}}) = v|_{\mathcal{T}_h}, \qquad v \in \mathbb{P}_1(\mathbb{R}^d). \tag{2.19}$$

Proof. Using (2.16) we note that for $T \in \mathcal{T}_{h,B}$,

$$(F_h v)|_T = (v|_{S_h(T)})^e|_T = v_T \tag{2.20}$$

since $v \in \mathbb{P}_1(\mathbb{R}^d)$. Therefore,

$$F_h v|_{\mathcal{T}_{h,I}} = v|_{\mathcal{T}_h}, \qquad v \in \mathbb{P}_1(\mathbb{R}^d). \tag{2.21}$$

Furthermore, for $v \in V_h$ we have $A_h v = v$, since v is continuous, and therefore. in particular. $A_h v = v$ for $v \in \mathbb{P}_1(\mathbb{R}^d) \subset \mathbb{R}^d$. Since $E_h = A_h \circ F_h$ and both operators preserve $v \in \mathbb{P}_1(\mathbb{R}^d)$. the proof is complete. $\square$

LEMMA 2.4. *For h_0 small enough there is a mapping $S_h : \mathcal{T}_h \to \mathcal{T}_{h,I}$ that satisfies* (2.15).

Proof. Let $\rho_{\partial\Omega}$ be the signed distance function associated with the boundary $\partial\Omega$, and let $U_\delta(\partial\Omega) = \{y \in \mathbb{R}^d : |\rho_{\partial\Omega}(x)| < \delta\}$ be the tubular neighborhood of thickness $2\delta > 0$ associated with $\partial\Omega$. Then there is $\delta_0 > 0$ such that the closest point mapping $p : U_\delta(\partial\Omega) \to \partial\Omega$ is well defined for $\delta \in (0, \delta_0]$. For $T \in \mathcal{T}_{h,B}$ take $x \in T \cap \partial\Omega$, and let $T_x(\partial\Omega)$ be the tangent plane to $\partial\Omega$ at x with exterior unit normal n_x. Let $\rho_{T_x(\partial\Omega)}$ be a signed distance function associated with $T_x(\partial\Omega)$ such that $\nabla\rho_{T_x(\partial\Omega)} = -n_x$. Let

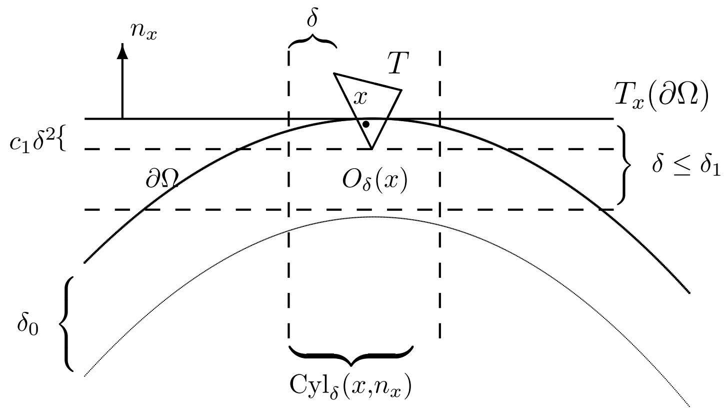

FIG. 1. *Illustration of the construction in the proof of Lemma* 2.4.

$U_\delta(T_x(\partial\Omega)) = \{y \in \mathbb{R}^d : |\rho_{T_x(\partial\Omega)}| < \delta\}$ be the tubular neighborhood of $T_x(\partial\Omega)$, and let $U_\delta^+(T_x(\partial\Omega)) = \{y \in \mathbb{R}^d : 0 < \rho_{T_x(\partial\Omega)} < \delta\}$ be the one sided tubular neighborhood. Define the cylinder $\mathrm{Cyl}_\delta(x, n_x)$ with radius δ and center axis aligned with the normal n_x at $x \in \partial\Omega$. We then note that since $\partial\Omega$ is smooth and $T_x(\partial\Omega)$ is a tangent plane to $\partial\Omega$ at x we may apply Taylor's formula to conclude that there are constants $c_1 > 0$ and $\delta_1 > 0$, independent of $x \in \partial\Omega$, such that

$$\partial\Omega \cap \mathrm{Cyl}_\delta(x, n_x) \subset U_{c_1\delta^2}(T_x(\partial\Omega)) \cap \mathrm{Cyl}_\delta(x, n_x) \tag{2.22}$$

for all $\delta \in (0, \delta_1]$. Therefore, it follows that

$$O_\delta(x) = (U_\delta^+(T_x(\partial\Omega)) \setminus U^+_{c_1\delta^2}(T_x(\partial\Omega))) \cap \mathrm{Cyl}_\delta(x, n_x) \subset U_{\delta_0}(\partial\Omega) \cap \Omega \tag{2.23}$$

for $\delta \in (0, \delta_1]$; see Figure 1. Then we may take $\delta_2 \in (0, \delta_1]$ small enough to guarantee that $\delta - c_1\delta^2 \geq \delta/2$ for $\delta \in (0, \delta_2]$. Taking $\delta = c_2 h$ for $h \in (0, h_0]$ with $h_0 = h_0(c_2)$ small enough to guarantee that $\delta \in (0, \delta_2]$, we note that $O_\delta(x)$ is a cylinder of radius $c_2 h$ and height $c_2 h/2$. Taking the constant c_2 large enough, we conclude from quasi-uniformity that there is an element $T \in \mathcal{T}_{h,I}$ such that $T \subset O_\delta(x)$ for $h \in (0, h_1]$. Finally, we note that we can take a ball centered at x of radius proportional to h that contains both T and $O_\delta(x)$ which completes the proof. ∎

LEMMA 2.5. *There are constants such that for all* $v \in V_{h,I}$,

$$\|F_h v\|_{\mathcal{T}_h} \lesssim \|v\|_{\mathcal{T}_{h,I}}, \tag{2.24}$$

$$\|\nabla F_h v\|_{\mathcal{T}_h} + h^{-1/2}\|[F_h v]\|_{\mathcal{F}_h} \lesssim \|\nabla v\|_{\mathcal{T}_{h,I}} \tag{2.25}$$

Proof. To prove (2.24) we note that for each $T \in \mathcal{T}_{h,B}$ we have the inverse inequality

$$\|v^e\|_T \leq \|v^e\|_{B_\delta} \lesssim \|v\|_{S_h(T)}, \tag{2.26}$$

where B_δ is a ball with diameter $\delta \sim h$ such that $T \cup S_h(T) \subset B_\delta$. Summing over $T \in \mathcal{T}_{h,B}$ and noting that thanks to (2.15) the number of elements in $\mathcal{T}_{h,B}$ that S_h

maps to T is uniformly bounded over all $T \in \mathrm{Im}(S_h)$,

$$\sum_{T\in\mathcal{T}_{h,B}} \|v^e\|^2_T \lesssim \sum_{T\in\mathcal{T}_{h,B}} \|v\|^2_{S_h(T)} \lesssim \sum_{T\in\mathrm{Im}(S_h)} \|v\|^2_T \lesssim \|v\|^2_{\mathcal{T}_{h,I}}, \tag{2.27}$$

where for the last inequality we used the inclusion $\mathrm{Im}(S_h) \subset \mathcal{T}_{h,I}$. For (2.25), we obtain

$$\|\nabla F_h v\|_{\mathcal{T}_h} \lesssim \|\nabla v\|_{\mathcal{T}_{h,I}} \tag{2.28}$$

using the same argument. To estimate the remaining term

$$h^{-1}\|[F_h v]\|^2_{\mathcal{F}_h} = \sum_{F\in\mathcal{F}_h} h^{-1}\|[F_h v]\|^2_F, \tag{2.29}$$

we have for each $F \in \mathcal{F}_h$, $[v] = [v - w_F]$ for an arbitrary constant w_F. Using the triangle inequality followed by an inverse inequality to pass from the face F to the elements $\mathcal{T}_h(F)$ sharing F,

$$h^{-1}\|[F_h v]\|^2_F \lesssim h^{-2}\|F_h v - w_F\|^2_{\mathcal{T}_h(F)} \lesssim h^{-2}\|v - w_F\|^2_{S_h(\mathcal{T}_h(F))}. \tag{2.30}$$

Next, there is an open ball B_δ with diameter $\delta \sim h$ such that

$$S_h(\mathcal{T}_h(F)) \subset B_\delta, \tag{2.31}$$

and then we have

$$h^{-2}\inf_{w_F\in\mathbb{R}}\|v - w_F\|^2_{S_h(\mathcal{T}_h(F))} \tag{2.32}$$

$$\leq h^{-2}\inf_{w_F\in\mathbb{R}}\|v - w_F\|^2_{\mathcal{T}_{h,I}(B_\delta)} \lesssim \delta^2 h^{-2}\|\nabla v\|^2_{\mathcal{T}_{h,I}(B_\delta)} \lesssim \|\nabla v\|^2_{\mathcal{T}_{h,I}(B_\delta)}, \tag{2.33}$$

which concludes the proof. □

A key property of CutFEM stabilized using ghost penalty is that the weakly consistent penalty term allows for control of the finite element solution on the whole mesh domain, by the combination of the stability from coercivity on the physical domain and the penalty terms. We will now show that such a stability property holds by construction for the extended space, thereby eliminating the need for additional stabilization.

LEMMA 2.6 (stability of the extension). *There are constants such that for all* $v \in V_{h,I}$,

$$\|\nabla^m E_h v_h\|_{\mathcal{T}_h} \lesssim \|\nabla^m v_h\|_{\mathcal{T}_{h,I}}, \qquad m = 0, 1. \tag{2.34}$$

Proof. For $m = 0$, we add and subtract $F_h v$ and use (2.24) and (2.6) to conclude that

$$\|E_h v\|_{\mathcal{T}_h} = \|A_h F_h v\|_{\mathcal{T}_h} \tag{2.35}$$

$$\leq \|F_h v\|_{\mathcal{T}_h} + \|(I - A_h)F_h v\|_{\mathcal{T}_h} \tag{2.36}$$

$$\lesssim \|F_h v\|_{\mathcal{T}_h} + h^{1/2}\|[F_h v]\|_{\mathcal{F}_h} \tag{2.37}$$

$$\lesssim \|F_h v\|_{\mathcal{T}_h} + \|F_h v\|_{\mathcal{T}_h} \tag{2.38}$$

$$\lesssim \|v\|_{\mathcal{T}_{h,I}}. \tag{2.39}$$

For $m = 1$, we proceed in the same way but we instead employ the stronger stability (2.25) of the operator F_h,

$$\|\nabla E_h v\|_{\mathcal{T}_h} = \|\nabla A_h F_h v\|_{\mathcal{T}_h} \tag{2.40}$$
$$\leq \|\nabla F_h v\|_{\mathcal{T}_h} + \|\nabla (I - A_h) F_h v)\|_{\mathcal{T}_h} \tag{2.41}$$
$$\leq \|\nabla F_h v\|_{\mathcal{T}_h} + h^{-1}\|(I - A_h) F_h v\|_{\mathcal{T}_h} \tag{2.42}$$
$$\lesssim \|\nabla F_h v\|_{\mathcal{T}_h} + h^{-1/2}\|[F_h v]\|_{\mathcal{F}_h} \tag{2.43}$$
$$\lesssim \|\nabla v\|_{\mathcal{T}_{h,I}}, \tag{2.44}$$

and thus the proof is complete. ∎

2.4. Interpolation. We begin by defining some interpolation operators that will be needed in the analysis.

- Let $\pi_h : H^1(\Omega_h) \to V_h$ be an interpolation operator of average type (see [10] or [25]), which satisfies the standard elementwise estimate

  $$\|v - \pi_h v\|_{H^m(T)} \lesssim h^{2-m}\|v\|_{H^2(\mathcal{T}_h(T))}, \qquad m = 0, 1, \tag{2.45}$$

  with $\mathcal{T}_h(T) \subset \mathcal{T}_h$ the neighboring elements of T. Composing π_h with the continuous extension operator E we obtain an interpolation operator $\pi_h \circ E : H^1(\Omega) \to V_h$, and using the stability (2.2) of the continuous extension operator we have

  $$\|Ev - \pi_h Ev\|_{H^m(\mathcal{T}_h)} \lesssim h^{2-m}\|Ev\|_{H^2(\Omega_h)} \lesssim h^{2-m}\|v\|_{H^2(\Omega)}, \qquad m = 0, 1. \tag{2.46}$$

  For simplicity, we will use the notation $Ev = v$ and $\pi_h v = \pi_h Ev$ when appropriate.
- We shall also need an interpolation operator $P_h : L^2(\Omega) \to F_h W_{h,I}$, which we define by noting that the sets $S_h^{-1}(T)$ for $T \in \mathcal{T}_{h,I}$ provides a partition of $\mathcal{T}_h$. Then there is $\delta \sim h$ and a ball $B_{\delta,T}$ such that

  $$S_h^{-1}(T) \subset B_{\delta,T}. \tag{2.47}$$

  On each ball $B_{\delta,T}$ there is $P_{h,T} v \in \mathbb{P}_1(B_{\delta,T})$ such that

  $$\|\nabla^m (v - P_{h,T} v)\|_{B_{\delta,T}} \lesssim h^{2-m}\|v\|_{H^2(B_{\delta,T})}, \qquad m = 0, 1. \tag{2.48}$$

  Defining $P_h : L^2(\mathcal{T}_h) \to W_h$ by

  $$(P_h v)|_{S_h^{-1}(T)} = (P_{h,T} Ev)|_{S_h^{-1}(T)}, \tag{2.49}$$

  we obtain the global error estimate

  $$\|\nabla^m (v - P_h v)\|_{\mathcal{T}_h} \lesssim h^{2-m}\|v\|_{H^2(\Omega)}, \qquad m = 0, 1. \tag{2.50}$$

  Observe also that P_h satisfies $P_h v = F_h(P_h v)_I$, where we introduced the shorthand notation $(v)_I := v|_{\mathcal{T}_{h,I}}$.
- We define the interpolation operator $I_h : H^1(\Omega) \to V_h^E$ by $I_h u := E_h(\pi_h E u)_I$.

LEMMA 2.7. *There is a constant such that for all $v \in H^2(\Omega)$,*

$$\|Ev - I_h v\|_{\mathcal{T}_h} + h\|\nabla(Ev - I_h v)\|_{\mathcal{T}_h} \lesssim h^2\|v\|_{H^2(\Omega)}. \tag{2.51}$$

Proof. Adding and subtracting $\pi_h Ev$ and $F_h(\pi_h Ev)_I$ and using the triangle inequality,

$$\|Ev - I_h v\|_{H^m(\mathcal{T}_h)} = \|Ev - E_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.52}$$
$$\leq \|Ev - \pi_h Ev\|_{H^m(\mathcal{T}_h)} + \|\pi_h Ev - E_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.53}$$
$$\leq \|(I - \pi_h)Ev\|_{H^m(\mathcal{T}_h)} + \|\pi_h Ev - F_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.54}$$
$$+ \|(I - A_h)F_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.55}$$
$$= I + II + III. \tag{2.56}$$

Term I. Using (2.46), we directly have

$$\|(I - \pi_h)Ev\|_{H^m(\mathcal{T}_h)} \lesssim h^{2-m}\|v\|_{H^2(\Omega)}. \tag{2.57}$$

Term II. Adding and subtracting $P_h v$, recalling the identity $P_h v = F_h(P_h v)_I$, and using the triangle inequality, we obtain

$$\|\pi_h Ev - F_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.58}$$
$$\leq \|\pi_h Ev - P_h v\|_{H^m(\mathcal{T}_h)} + \|P_h v - F_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.59}$$
$$\leq \|\pi_h Ev - P_h v\|_{H^m(\mathcal{T}_h)} + \|F_h(P_h v - \pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \tag{2.60}$$
$$\lesssim \|\pi_h Ev - P_h v\|_{H^m(\mathcal{T}_h)} \tag{2.61}$$
$$\lesssim \|\pi_h Ev - v\|_{H^m(\mathcal{T}_h)} + \|v - P_h v\|_{H^m(\mathcal{T}_h)} \tag{2.62}$$
$$\lesssim h^{2-m}\|v\|_{H^2(\Omega)}, \tag{2.63}$$

where we used the stability estimates (2.24) for $m = 0$ and (2.25) for $m = 1$ for F_h, added and subtracted v and used the triangle inequality, and used the interpolation error estimates (2.46) and (2.50).

Term III. Using the approximation result (2.6) for the average operator A_h, inserting the continuous function $\pi_h Ev$ into the jump, and using an inverse estimate to pass from faces to elements, we obtain

$$\|(I - A_h)F_h(\pi_h Ev)_I\|_{H^m(\mathcal{T}_h)} \lesssim h^{-m}\|(I - A_h)F_h(\pi_h Ev)_I\|_{\mathcal{T}_h} \tag{2.64}$$
$$\lesssim h^{1/2-m}\|[F_h(\pi_h Ev)_I]\|_{\mathcal{F}_h} \tag{2.65}$$
$$\lesssim h^{1/2-m}\|[F_h(\pi_h Ev)_I - \pi_h Ev]\|_{\mathcal{F}_h} \tag{2.66}$$
$$\lesssim h^{-m}\|F_h(\pi_h Ev)_I - \pi_h Ev\|_{\mathcal{T}_h} \tag{2.67}$$
$$\lesssim h^{2-m}\|v\|_{H^2(\Omega)}, \tag{2.68}$$

where we used the estimate for Term II with $m = 0$ in the last step. ◻

2.5. Finite element method. In order to formulate the finite element method we use the following notation.

- Partition $[0, T]$ into N intervals of length $k = T/N$, and let $t_n = nk$ for $n = 0, 1, \ldots, N$. We let $u^n = u(t_n)$ and $v^n : \Omega \to \mathbb{R}$ denote a function at time t_n. Define the discrete first (forward) and second (central) time differences

$$\partial_t v^n = \frac{v^{n+1} - v^n}{k}, \tag{2.69}$$

$$\partial_t^2 v^n = \frac{v^{n+1} - 2v^n + v^{n-1}}{k^2} = \frac{1}{k}(\partial_t v^n - \partial_t v^{n-1}). \tag{2.70}$$

Note that ∂_t^2 should not be interpreted as $\partial_t \circ \partial_t$ but as an operator defined by (2.70).

- Define the central difference

$$\delta_t v^n = \frac{1}{2}(\partial_t v^n + \partial_t v^{n-1}), \tag{2.71}$$

and note for use below that we have the summation by parts formula

$$\sum_{n=1}^{N-1} 2k(v^n, \delta_t w^n) \tag{2.72}$$
$$= (v^{N-1}, w^N) + (v^{N-1}, w^{N-1}) - (v^1, w^1) - (v^1, w^0) \tag{2.73}$$
$$- \sum_{n=2}^{N-1} 2k(\delta_t v^n, w^n). \tag{2.74}$$

- For the spatial discretization we employ Nitsche's method and define the bilinear form

$$a_h(u, v) = (\nabla u, \nabla v) - (\nabla_n u, v)_{\partial\Omega} - (u, \nabla_n v)_{\partial\Omega} + \gamma h^{-1}(u, v)_{\partial\Omega}, \tag{2.75}$$

where $\nabla_n = n \cdot \nabla$ with n the exterior unit normal and $\gamma > 0$ a parameter.

Method. The CutFEM takes the following form: for $n = 1, \ldots, N-1$, find $u_h^{n+1} \in V_h^E$, such that

$$(\partial_t^2 u_h^n, v) + a_h(u_h^n, v) = (f^n, v) \qquad \forall v \in V_h^E \tag{2.76}$$

with initial data $u_h^0, u_h^1 \in V_h$ specified below. The resulting updating formula takes the form

$$(u_h^{n+1}, v) = (2u_h^n, v) - (u_h^{n-1}, v) + k^2 a_h(u_h^n, v) + k^2(f^n, v). \tag{2.77}$$

2.6. Matrix formulation and mass lumping. We formulate the method on matrix form and replace the mass matrix with a diagonal matrix obtained by lumping the mass matrix in order to obtain an explicit method.

- Let $\{\psi_i\}_{i\in\mathcal{I}_h}$ be the nodal basis in V_h enumerated by the index set $\mathcal{I}_h$, let $\{\varphi_i\}_{i\in\mathcal{I}_{I,h}}$ be the nodal basis in $V_{h,I}$ enumerated by the index set $\mathcal{I}_{h,I}$, and let $\{\varphi_i^E\}_{i\in\mathcal{I}_{h,I}}$, with $\varphi_i^E = E_h\varphi_i$, be the corresponding basis in V_h^E. Denote the dimension of V_h by N_h, and the common dimension of $V_{h,I}$ and V_h^E by $N_{h,I}$.
- To keep track of the different expansions, we employ the notation

$$V_h \ni v = \sum_{i\in\mathcal{I}_h} \widetilde{v}_i\psi_i, \quad V_{h,I} \ni v = \sum_{i\in\mathcal{I}_{h,I}} \hat{v}_i\varphi_i, \quad V_h^E \ni v = \sum_{i\in\mathcal{I}_{h,I}} \widehat{v}_i\varphi_i^E. \tag{2.78}$$

- Define the mass matrix, stiffness matrix, and load vector associated with the full finite element space V_h by

$$(\widetilde{M}_h\widetilde{v}, \widetilde{w})_{\mathcal{I}_h} = (v, w), \quad (\widetilde{A}_h\widetilde{v}, \widetilde{w})_{\mathcal{I}_h} = a_h(v, w), \quad (\widetilde{b}_h, \widetilde{w})_{\mathcal{I}_h} = (f, w) \tag{2.79}$$

for all $v, w \in V_h$.

- Define the mass matrix, stiffness matrix, and load vector associated with the extended finite element space by

$$(\widehat{M}_{h,I}\widehat{v},\widehat{w})_{\mathcal{I}_{h,I}} = (v,w), \quad (\widehat{A}_{h,I}\widehat{v},\widehat{w})_{\mathcal{I}_{h,I}} = a_h(v,w), \quad (\widehat{b}_{h,I},\widehat{w})_{\mathcal{I}_{h,I}} = (f,w) \tag{2.80}$$

  for all $v, w \in V_h^E$.
- For implementation purposes it is convenient to work with the standard nodal bases in V_h and $V_{h,I}$ and to express the matrices associated with V_h^E in terms of the standard matrices associated with V_h using a matrix representation of the extension operator. Recalling first that the extended basis is defined by $\varphi_i^E = E_h\varphi_i$, $i \in \mathcal{I}_{h,I}$, we have $\widehat{E_h v} = \hat{v}$ for all $v \in V_{h,I}$, and therefore the matrix representation of $E_h : V_{h,I} \to V_h^E$ is the $N_{h,I} \times N_{h,I}$ identity matrix and $V_{h,I} \cong V_h^E$. Next, define the matrix representation of $E_h : V_{h,I} \to V_h$ by

$$(\widetilde{\hat{E}}_h\hat{v},\widetilde{w})_{\mathcal{I}_h} = (\widetilde{E_h v},\widetilde{w})_{\mathcal{I}_h} \tag{2.81}$$

  for all $v \in V_{h,I}$, $w \in V_h$. We note that $\widetilde{\hat{E}}_h$ is an $N_h \times N_{h,I}$, matrix and that it follows from (2.81) that $\widetilde{\hat{E}}_h\hat{v} = \widetilde{E_h v}$. We then have for $v, w \in V_{h,I}$,

$$(\hat{v},\widehat{M}_{h,I}\hat{w})_{\mathcal{I}_{h,I}} = (\widehat{E_h v},\widehat{M}_{h,I}\widehat{E_h w})_{\mathcal{I}_{h,I}} = (E_h v, E_h w) = (\widetilde{E_h v},\widetilde{M}_h\widetilde{E_h w})_{\mathcal{I}_h} \tag{2.82}$$

$$= (\widetilde{\hat{E}}_h\hat{v},\widetilde{M}_h(\widetilde{\hat{E}}_h\hat{w}))_{\mathcal{I}_h} = (\hat{v},(\widetilde{\hat{E}}_h^T\widetilde{M}_h\widetilde{\hat{E}}_h)\hat{w})_{\mathcal{I}_{h,I}}. \tag{2.83}$$

  Therefore, the mass matrix on the extended finite element space can be expressed in terms of the mass matrix on the full finite element space as follows:

$$\widehat{M}_{h,I} = \widetilde{\hat{E}}_h^T\widetilde{M}_h\widetilde{\hat{E}}_h, \tag{2.84}$$

  and in the same way

$$\widehat{A}_{h,I} = \widetilde{\hat{E}}_h^T\widetilde{A}_h\widetilde{\hat{E}}_h, \qquad \widehat{b}_{h,I} = \widetilde{\hat{E}}_h^T\hat{b}_h. \tag{2.85}$$

- Define the lumped mass matrix $\widehat{M}_L$ as the diagonal matrix with diagonal elements equal to the row sums of the mass matrix $\widehat{M}_{h,I}$,

$$\widehat{M}_{L,ij} = \begin{cases} 0, & i \neq j, \\ \sum_{l\in\mathcal{I}_{h,I}(i)} \widehat{m}_{il}, & i = j, \end{cases} \tag{2.86}$$

  where for each $i \in \mathcal{I}_{h,I}$,

$$\mathcal{I}_{h,I}(i) = \{j \in \mathcal{I}_{h,I} : \widehat{m}_{ij} \neq 0\} \tag{2.87}$$

  is the set of indices for which there is a nonzero entry in the ith row (and column due to symmetry) of $\widehat{M}_{h,I}$. We also define the induced lumped mass inner product

$$(v,w)_L = (\widehat{M}_L\widehat{v},\widehat{w})_{\mathcal{I}_{h,I}}, \qquad v, w \in V_h^E. \tag{2.88}$$

Explicit method. We define the lumped mass method: for $n = 1, \ldots, N-1$, find $u^{n+1} \in V_h^E$, such that

$$(\partial_t^2 u_h^n, v)_L + a_h(u_h^n, v) = (f_h^n, v)_L \qquad \forall v \in V_h^E \tag{2.89}$$

with initial data $u_h^0, u_h^1 \in V_h$ and $f_h^n \in V_h^E$ a suitable approximation of $f(t^n)$. Using the fact that $\widehat{M}_L$ is diagonal, we obtain the explicit updating formula for $n = 2, \ldots, N-1$,

$$\widehat{u}_h^{n+1} = 2\widehat{u}_h^n - \widehat{u}_h^{n-1} - k^2 \widehat{M}_L^{-1} \widehat{A}_{h,I} \widehat{u}_h^n + k^2 \widehat{b}_L^n, \tag{2.90}$$

where $\widehat{b}_L^n$ is the load vector associated with the lumped mass inner product

$$(\widehat{b}_L^n, \widehat{v})_{\mathcal{I}_{h,I}} = (f_h^n, v)_L, \qquad v \in E_h V_{h,I}. \tag{2.91}$$

It follows that $\widehat{b}_L^n = \widehat{M}_L \widehat{f}_h^n$, where $\widehat{f}_h^n$ is the internal nodal values of f_h^n.

3. Analysis of the method. The forthcoming error analysis essentially relies on discrete stability of the method and approximation properties of the extended space. In particular, we employ the Ritz projection onto the extended finite element space and derive the corresponding basic approximation results. Related error estimates for standard finite element spaces have been derived in [2, 3, 13], and in the context of elastodynamics in [31, 32]. We also prove an estimate for the lumping error which demand a more complicated analysis compared with the standard estimates for piecewise linear elements; see Chapter 15 in [29], for instance. Furthermore, we show using some natural restrictions on the construction of the extension operator that the lumped mass matrix is indeed positive definite.

3.1. Ritz projection. In this section we will discuss the Ritz projection on the extended finite element space V_h^E. This will provide us with an interpolant with properties suitable for the error analysis of the wave equation. It also provides an analysis of Poisson's equation discretized using the extended space V_h^E in a CutFEM framework.

Let

$$|||v|||_h^2 = \|\nabla v\|^2 + h\|\nabla_n v\|_{\partial\Omega}^2 + h^{-1}\|v\|_{\partial\Omega}^2. \tag{3.1}$$

LEMMA 3.1. *The form a_h defined in* (2.75) *is continuous,*

$$a_h(v, w) \lesssim |||v|||_h |||w|||_h, \qquad v, w \in H^{3/2+\epsilon}(\Omega) + V_h, \tag{3.2}$$

and for γ large enough coercive,

$$|||v|||_h^2 \lesssim a_h(v, v), \qquad v \in V_h^E. \tag{3.3}$$

Proof. The continuity follows directly from Cauchy–Schwarz, and to establish the coercivity we start from

$$a_h(v, v) = \|\nabla v\|^2 - 2(\nabla_n v, v)_{\partial\Omega} + \gamma h^{-1}\|v\|_{\partial\Omega}^2. \tag{3.4}$$

We have the estimate

$$2(\nabla_n v, v)_{\partial\Omega} \leq 2\|\nabla_n v\|_{\partial\Omega}\|v\|_{\partial\Omega} \tag{3.5}$$

$$\leq C\|\nabla v\|_{\mathcal{T}_h(\partial\Omega)} h^{-1/2}\|v\|_{\partial\Omega} \tag{3.6}$$

$$\leq C^2\delta\|\nabla v\|^2_{\mathcal{T}_h(\partial\Omega)} + \delta^{-1}h^{-1}\|v\|^2_{\partial\Omega} \tag{3.7}$$

$$\leq C^2\delta\|\nabla v\|^2_{\mathcal{T}_{h,I}} + \delta^{-1}h^{-1}\|v\|^2_{\partial\Omega} \tag{3.8}$$

$$\leq C^2\delta\|\nabla v\|^2 + \delta^{-1}h^{-1}\|v\|^2_{\partial\Omega}, \tag{3.9}$$

where we used the inverse estimate $h^{1/2}\|\nabla v\|_{\partial\Omega\cap T} \leq C\|\nabla v\|_T$, the stability (2.34) of the discrete extension operator E_h, and finally the fact that $\mathcal{T}_{h,I} \subset \Omega$. Combining the estimates we find that

$$a_h(v,v) \geq (1 - C^2\delta)\|\nabla v\|^2 + (\gamma - \delta^{-1})h^{-1}\|v\|^2_{\partial\Omega} \gtrsim \|\nabla v\|^2 + h^{-1}\|v\|^2_{\partial\Omega}, \tag{3.10}$$

where we chose δ small enough and γ large enough. Finally, (3.5)–(3.9) give the estimate $h\|\nabla_n v\|^2_{\partial\Omega} \lesssim \|\nabla v\|^2$ the coercivity (3.3) follows. ∎

In view of Lemma 3.1, we note that we can define the norm

$$\|v\|^2_{a_h} = a_h(v,v), \qquad v \in V_h^E, \tag{3.11}$$

directly associated with the Nitsche form, which is equivalent to $|||\cdot|||_h$ on V_h^E,

$$\|v\|_{a_h} \sim |||v|||_h, \qquad v \in V_h^E, \tag{3.12}$$

It will later be convenient to work with $\|\cdot\|_{a_h}$ instead of $|||\cdot|||_h$.

The Ritz projection $R_h : H^s(\Omega) \to E_h(V_{h,I})$, for $s > 3/2$, is defined by

$$a_h(R_h v, w) = a_h(v,w) \qquad \forall w \in V_h^E, \tag{3.13}$$

and we have the following error estimates.

Lemma 3.2. *There are constants such that*

$$|||v - R_h v|||_h \lesssim h\|v\|_{H^2(\Omega)}, \qquad \|v - R_h v\| \lesssim h^2\|v\|_{H^2(\Omega)}. \tag{3.14}$$

Proof. Here we add and subtract an interpolant

$$|||v - R_h v|||_h \leq |||v - I_h v|||_h + |||I_h v - R_h v|||_h \tag{3.15}$$

$$\lesssim h^{2-m}\|v\|_{H^2(\Omega)} + |||I_h v - R_h v|||_h, \tag{3.16}$$

where for the first term on the right-hand side we used the interpolation error estimate (2.51) together with the cut trace inequality (see [30])

$$\|w\|^2_{\partial\Omega\cap T} \lesssim h^{-1}\|w\|^2_T + h\|\nabla w\|^2_T, \qquad w \in H^1(T), \tag{3.17}$$

to estimate the boundary contributions.

For the second term, coercivity (3.3), orthogonality (3.13), and continuity (3.2), give

$$|||I_h v - R_h v|||^2_h \lesssim a_h(I_h v - R_h v, I_h v - R_h v) \tag{3.18}$$

$$= a_h(I_h v - v, I_h v - R_h v) \tag{3.19}$$

$$\lesssim |||I_h v - v|||_h |||I_h v - R_h v|||_h, \tag{3.20}$$

and therefore, using once more the interpolation estimate (2.51) for I_h,

$$|||I_h v - R_h v|||_h \lesssim |||I_h v - v|||_h \lesssim h\|v\|_{H^2(\Omega)}. \tag{3.21}$$

The L^2 estimate is established using duality in the usual way. ∎

Remark 3.3. Note that $u_h = R_h u$ is the finite element solution to

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{3.22}$$

and thus (3.14) provides error estimates for a CutFEM based on the extension operator E_h for the Poisson equation.

3.2. Estimate of the lumping error. We begin by showing a stability estimate for the lumped inner product; then we prove an estimate of the consistency error resulting from lumping the mass matrix. Finally, we use the representation of the lumping error to show that the lumped mass matrix is indeed positive definite.

Let $\|v\|_L^2 = (v, v)_L$ be the norm associated with the lumped scalar product. We then have the stability

$$\|v\|_{\mathcal{T}_h} \lesssim \|v\|_L, \qquad v \in V_h^E. \tag{3.23}$$

This estimate follows from the L^2 stability (2.34) of the extension operator followed by equivalence of the lumped product and the full L^2 product on the set of interior triangles

$$\|E_h v\|_{\mathcal{T}_h} \lesssim \|v\|_{\mathcal{T}_{h,I}} \sim h^{d/2}\|\widehat{v}\|_{\mathcal{X}_{h,I}} \sim \|v\|_L, \tag{3.24}$$

where $\mathcal{X}_{h,I}$ denotes the set of nodes in $\mathcal{T}_{h,I}$ and for a discrete set $\mathcal{X}$ the norm $\|v\|_{\mathcal{X}}^2 = \|v\|_{l^2(\mathcal{X})}^2 = \sum_{x\in\mathcal{X}} v^2(x)$. Note that the last relation above holds since all elements of M_L must be $O(h^d)$, since only interior nodes are considered.

LEMMA 3.4. *There is a constant such that*

$$|(v, w) - (v, w)_L| \lesssim h^2 \|\nabla v\|_\Omega \|\nabla w\|_\Omega, \quad v, w \in V_h^E. \tag{3.25}$$

Proof. Using the definitions (2.80) and (2.86) of the mass matrix $\widehat{M}_{h,I}$ and the lumped mass matrix $\widehat{M}_L$ we have

$$(v, w)_L - (v, w) = (\widehat{v}, \widehat{M}_L \widehat{w})_{\mathcal{I}_{h,I}} - (\widehat{v}, \widehat{M}_{h,I}\widehat{w})_{\mathcal{I}_{h,I}} \tag{3.26}$$

$$= (\widehat{v}, (\widehat{M}_L - \widehat{M}_{h,I})\widehat{w})_{\mathcal{I}_{h,I}} = (\widehat{v}, \widehat{B}\widehat{w})_{\mathcal{I}_{h,I}} \tag{3.27}$$

with $\widehat{B} = \widehat{M}_L - \widehat{M}_{h,I}$. We note that the elements $\widehat{b}_{ij}$ of $\widehat{B}$ are

$$\widehat{b}_{ij} = \begin{cases} \sum_{l\in\mathcal{I}_{h,I}(i)\setminus\{i\}} \widehat{m}_{il}, & i = j, \\ -\widehat{m}_{ij}, & i \neq j \end{cases} \tag{3.28}$$

with $\widehat{m}_{ij} \geq 0$. Since $\widehat{b}_{ii} = -\sum_{l\in\mathcal{I}_{h,I}(i)\setminus\{i\}} \widehat{b}_{il}$ it follows that $\widehat{B}$ is a graph Laplacian on the undirected weighted graph with vertices $\mathcal{X}_{h,I}$, enumerated by $\mathcal{I}_{h,I}$, and edges

$$\mathcal{E}_{h,I} = \{(i, j) \in \mathcal{I}_{h,I} \times \mathcal{I}_{h,I} : \widehat{m}_{ij} \neq 0\} \tag{3.29}$$

with weights $\widehat{m}_{ij} \geq 0$. We can then write the form $(\widehat{w}, \widehat{B}\widehat{w})_{\mathcal{I}_h}$ as a sum of positive semidefinite forms associated with the edges. To that end we associate with each graph edge $(i,j) \in \mathcal{E}_{h,I}$ the positive semidefinite $N_{h,I} \times N_{h,I}$ matrix

$$\widehat{B}_{(i,j)} = \widehat{m}_{ij}(e_i \otimes e_i - e_i \otimes e_j - e_j \otimes e_i + e_j \otimes e_j), \tag{3.30}$$

where $\{e_i\}_{\mathcal{I}_{h,I}}$ is the canonical basis in $\mathbb{R}^{N_{h,I}}$. Note that $\widehat{B}_{(i,j)}$ maps the two dimensional space $\mathrm{span}\{e_i, e_j\}$ into itself, and the corresponding matrix takes the form

$$\widehat{B}_{(i,j)}|_{\mathrm{span}\{e_i,e_j\}} = \widehat{m}_{ij} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{3.31}$$

One can then verify that

$$\widehat{B} = \sum_{(i,j)\in\mathcal{E}_{h,I}} \widehat{B}_{(i,j)}, \tag{3.32}$$

which gives the identity

$$(\widehat{v}, \widehat{B}\widehat{w})_{\mathcal{I}_{h,I}} = \sum_{(i,j)\in\mathcal{E}_{h,I}} \widehat{m}_{ij}(\widehat{v}_i - \widehat{v}_j)(\widehat{w}_i - \widehat{w}_j); \tag{3.33}$$

see [15, Chapter 13]. Using the Cauchy–Schwarz inequality together with the fact that $\widehat{m}_{ij} \sim h^d$ we obtain

$$(\widehat{v}, \widehat{B}\widehat{w})_{\mathcal{I}_{h,I}} \lesssim h^d \Big(\sum_{(i,j)\in\mathcal{E}_{h,I}} (v_i - v_j)^2 \Big)^{1/2} \Big(\sum_{(i,j)\in\mathcal{E}_{h,I}} (w_i - w_j)^2 \Big)^{1/2}. \tag{3.34}$$

We next note that

$$\sum_{(i,j)\in\mathcal{E}_{h,I}} (v_i - v_j)^2 \leq \sum_{i\in\mathcal{I}_{h,I}} \sum_{j\in\mathcal{I}_{h,I}(i)} (v_i - v_j)^2 = \bigstar, \tag{3.35}$$

where $\mathcal{I}_{h,I}(i) \subset \mathcal{I}_{h,I}$ is the set of indices connected to the node i by an edge $E \in \mathcal{E}_{h,I}$. Next, let $\mathcal{T}_h(i)$ be the set of elements with at least one node in $\mathcal{I}_{h,I}(i)$ and note that it follows from the construction of the extension operator and shape regularity that there is a uniform bound, independent of $h \in (0, h_0]$ and $i \in \mathcal{I}_{h,I}$, on the number of elements in $\mathcal{T}_h(i)$ and that $\mathrm{diam}(\mathcal{T}_h(i)) \lesssim h$. We then have

$$\sum_{j\in\mathcal{I}_{h,I}(i)} h^d \, (v_i - v_j)^2 \lesssim \|v_i - v\|^2_{\mathcal{T}_h(i)} \lesssim h^2 \|\nabla v\|^2_{\mathcal{T}_h(i)} \tag{3.36}$$

since $v \in V_h$. It follows that

$$\bigstar = \sum_{i\in\mathcal{I}_{h,I}} \sum_{j\in\mathcal{I}_{h,I}(i)} (v_i - v_j)^2 \lesssim \sum_{i\in\mathcal{I}_{h,I}} h^2 \|\nabla v\|^2_{\mathcal{T}_h(i)} \lesssim h^2 \|\nabla v\|^2_{\mathcal{T}_h}. \tag{3.37}$$

Combining (3.34) and (3.37) and applying Lemma 2.6 we arrive at the desired estimate. ∎

3.3. Positive definiteness of the lumped mass matrix. The mass matrix associated with the extended basis $\{\varphi_i^E\}_{i\in\mathcal{I}_{h,I}}$ is obviously positive definite but the extended basis functions may in fact be negative on elements at the boundary. To guarantee that the lumped mass matrix is positive definite we will therefore restrict the construction of the extension operator in the following ways:

- Let $\mathcal{T}_{h,I,\tau} = \{T \in \mathcal{T}_h : |T\cap\Omega| \geq \tau|T|\}$, with $\tau \in (0,1]$ a parameter, and let $\mathcal{T}_{h,B,\tau} = \mathcal{T}_h \setminus \mathcal{T}_{h,I,\tau}$. Then we have $\mathcal{T}_{h,I} \subseteq \mathcal{T}_{h,I,\tau}$ and $\mathcal{T}_{h,B,\tau} \subseteq \mathcal{T}_{h,B}$, with equality for $\tau = 1$. We extend the definition of $V_{h,I}$ to $V_{h,I} = V_h|_{\mathcal{T}_{h,I,\tau}}$ and let

$$E_h : V_{h,I} \to V_h^E \subset V_h \tag{3.38}$$

  be defined as in (2.16) and (2.17) with the mapping $S_h : \mathcal{T}_{h,B,\tau} \to \mathcal{T}_{h,I}$, which means that we extend from elements $T \in \mathcal{T}_{h,I}$ in the interior of Ω not from elements $T \in \mathcal{T}_{h,I,\tau} \setminus \mathcal{T}_h$ that intersect the boundary.
- We will use an average operator A_h with unit weight on all elements $T \in \mathcal{T}_{h,I,\tau}$ which implies

$$(v^E)|_{\mathcal{T}_{h,I,\tau}} = v|_{\mathcal{T}_{h,I,\tau}}. \tag{3.39}$$

These assumptions enable us to confine potential negative values of the extended basis functions to elements $T \in \mathcal{T}_{h,B,\tau}$ with a measure that can be bounded by τh^d. This restriction of the construction of the extension operator is, in particular, important if the local mesh size varies, which may be the case even if we are using quasi-uniform meshes. The parameter τ will depend on the constants in the quasi-uniformity assumptions, which we can see in the forthcoming proof of Lemma 3.5. Note, however, that in CutFEM regular meshes are, in general, used and then the quasi-uniformity constants are moderate.

LEMMA 3.5. *Assume that the extension operator satisfies the above restrictions; then for τ small enough the lumped mass matrix is positive definite.*

Proof. First, we observe that the lumped mass elements takes the form

$$\widehat{m}_{L,ii} = \sum_{l\in\mathcal{I}_{h,I}} \widehat{m}_{il} = \sum_{l\in\mathcal{I}_{h,I}} \int_\Omega \varphi_i^E \varphi_l^E = \int_\Omega \varphi_i^E \Bigg(\sum_{l\in\mathcal{I}_{h,I}} \varphi_l^E \Bigg) = \int_\Omega \varphi_i^E. \tag{3.40}$$

Here we used the fact that $\sum_{l\in\mathcal{I}_{h,I}} \varphi_l^E = 1$, which holds since the sum of all basis functions associated with nodes belonging to elements in $\mathcal{T}_{h,I,\tau}$ is one on $\mathcal{T}_{h,I,\tau}$ and the extension operator is linear so that

$$\sum_{i\in\mathcal{I}_{h,I},\tau} \varphi_i^E = \sum_{i\in\mathcal{I}_{h,I},\tau} E_h\varphi_i = E_h\Bigg(\sum_{i\in\mathcal{I}_{h,I},\tau} \varphi_i \Bigg) = E_h 1 = 1, \tag{3.41}$$

where in the last step we used Lemma 2.3. To show that $\int_\Omega \varphi_i^E > 0$ is positive we first note that if $\mathrm{supp}(\varphi_i^E) \cap S_h(\mathcal{T}_{h,B,\tau}) = \emptyset$, then $\varphi_i^E = \varphi_i$ and $|\mathrm{supp}(\varphi_i)| \geq \tau h^d$ which together with the fact that φ_i is positive on its support guarantees that $\int_\Omega \varphi_i > 0$. Next, if there is $T \in \mathrm{supp}(\varphi_i^E) \cap S_h(\mathcal{T}_{h,B,\tau})$ we have $T \in \mathcal{T}_{h,I}$ since S_h maps into $\mathcal{T}_{h,I}$.

We then have the estimates

$$\int_\Omega \varphi_i^E = \int_{\Omega\cap\mathcal{T}_{h,I,\tau}} \varphi_i^E + \int_{\Omega\cap\mathcal{T}_{h,B,\tau}} \varphi_i^E \tag{3.42}$$

$$\geq \int_{\Omega\cap\mathcal{T}_{h,I,\tau}} \varphi_i - |\Omega\cap\mathcal{T}_{h,B,\tau}|\,\|\varphi_i^E\|_{L^\infty(\Omega\cap\mathcal{T}_{h,B,\tau})} \tag{3.43}$$

$$\geq C_1 h^d - C_2\tau h^d \tag{3.44}$$

for τ small enough. Here we used the fact that $\varphi_i^E = \varphi_i$ on $\mathcal{T}_{h,I,\tau}$, due to the property (3.39) of the average operator, and the fact that there is an element $T \in \mathcal{T}_{h,I}$ such that $T \subset \mathrm{supp}(\varphi_i^E)$ and therefore

$$\int_{\Omega\cap\mathcal{T}_{h,I,\tau}} \varphi_i \geq \int_T \varphi_i \geq C_1 h^d, \tag{3.45}$$

where in the last step we use nodal quadrature to compute the integral of the linear function $\varphi_i|_T$ which is equal to one in at least one node. Furthermore, we used the bound $\|\varphi_i^E\|_{L^\infty(\Omega\cap\mathcal{T}_{h,B,\tau})} \lesssim 1$, which holds since

$$\|A_h F_h \varphi_i\|_{L^\infty(T)} \lesssim \|F_h\varphi_i\|_{L^\infty(\mathcal{N}(T))} \lesssim 1, \tag{3.46}$$

where $\mathcal{N}(T)$ is the set of elements in $\mathcal{T}_h$ sharing a node with T and finally we used the fact that $F_h\varphi_i|_T = (\varphi_i|_{S_h(T)})^e$ and the gradient of φ_i is bounded by Ch^{-1} and according to Lemma 2.4 the distance between T and $S_h(T)$ is bounded by Ch. ∎

3.4. Discrete stability. To prepare the terrain for the error analysis we will prove stability for a slightly more general version of (2.89). Indeed, we introduce a right-hand side that consists of two parts, expressed as functionals on V_h, $r_1 = \{r_1^n\}_{n=1}^N$ and $r_2 = \{r_2^n\}_{n=1}^N$, $r_i^n : V_h \mapsto \mathbb{R}$. They will later be identified with two different sources of approximation error driving the perturbation equation. The reason for this split is that optimal estimates require r_1 and r_2 to be continuous with respect to different (discrete) topologies, r_1 with respect to a discrete H^1-norm, and r_2 with respect to a discrete L^2-norm. This is a consequence of fact that the test function in the derivation of the stability estimate is a discrete first order time derivative and that the lumped mass approximation estimate (3.25) requires control of the gradient of the test function. To avoid the appearance of mixed derivatives, which cannot be controlled, we apply summation by parts in the r_1 part and move the discrete time derivative from the test function to the functional. To provide bounds in term of these functionals, we recall the standard definition of norms for linear functionals $l : V_h \to \mathbb{R}$, using the appropriate norms,

$$\|l\|_{a_h,\star} = \sup_{v\in V_h^E\setminus\{0\}} \frac{|l(v)|}{\|v\|_{a_h}}, \qquad \|l\|_{L,\star} = \sup_{v\in V_h^E\setminus\{0\}} \frac{|l(v)|}{\|v\|_L}. \tag{3.47}$$

The abstract scheme that we consider takes the following form: for $n = 1,\dots,N-1$, find $v^{n+1} \in V_h^E$, such that

$$(\partial_t^2 v^n, w)_L + a_h(v^n, w) = r^n(w) \qquad \forall w \in V_h^E \tag{3.48}$$

given $v^0, v^1 \in V_h$. Here $r^n : V_h \to \mathbb{R}$ are the linear functionals of the form

$$r^n(v) = r_1^n(v) + r_2^n(v). \tag{3.49}$$

Let us first derive the bounds necessary for the two contributions r_1 and r_2, when their argument is a central difference of the form $k\delta_t v^n$. For $r_1(k\delta_t v^n)$, we sum over the contributions r_1^n and apply the summation by parts formula (2.72) to move the central difference from the test function of the form $k\delta_t v^n$ to the functional,

$$\left|\sum_{n=1}^{N-1} 2kr^n(\delta_t v^n)\right| = \left|\sum_{n=1}^{N-1} 2k(r_1^n(\delta_t v^n) + r_2^n(\delta_t v^n))\right| \tag{3.50}$$

$$\leq \left|r_1^{N-1}(v^N) + r_1^{N-1}(v^{N-1}) - r_1^1(v^1) - r_1^1(v^0)\right| \tag{3.51}$$

$$+ \left|\sum_{n=2}^{N-1} 2k((\delta_t r_1^n)(v^n) + \sum_{n=1}^{N-1} r_2^n(\delta_t v^n))\right|. \tag{3.52}$$

Next, turning our attention to r_2, we have in view of (2.71),

$$r_2^n(\delta_t v^n) = \frac{1}{2} r_2^n(\partial_t v^n + \partial_t v^{n-1}), \tag{3.53}$$

and therefore

$$\sum_{n=1}^{N-1} r_2^n(\delta_t v^n) = r_2^{N-1}(\partial_t v^{N-1}) + \sum_{n=2}^{N-1} \frac{1}{2}(r_2^n + r_2^{n-1})(\partial_t v^{n-1}) + r_2^1(\partial_t v^0). \tag{3.54}$$

Applying this in the right-hand side of (3.52), it follows that, using $\overline{r}_2^n = \frac{1}{2}(r_2^n + r_2^{n-1})$,

$$\left|\sum_{n=1}^{N-1} 2kr^n(\delta_t v^n)\right| \leq |r_1^{N-1}(v^N)| + |r_1^N(v^{N-1})| + |r_1^1(v^0)| + |r_1^0(v^1)| \tag{3.55}$$

$$+ |r_2^{N-1}(\partial_t v^{N-1})| + |r_2^1(\partial_t v^0)| \tag{3.56}$$

$$+ \sum_{n=2}^{N-1} 2k|((\delta_t r_1^n)(v^n) + (\overline{r}_2^n)(\partial_t v^{n-1}))|. \tag{3.57}$$

For each term of the sum in the right-hand side we have,

$$(\delta_t r_1^n)(v^n) + (\overline{r}_2^n)(\partial_t v^{n-1}) \leq (\|\delta_t r_1^n\|_{a_h,\bigstar}^2 + \|\overline{r}_2^n\|_{L,\bigstar}^2)^{\frac{1}{2}} (\|v^n\|_{a_h}^2 + \|\partial_t v^{n-1}\|_L^2)^{\frac{1}{2}}. \tag{3.58}$$

Therefore,

$$\sum_{n=2}^{N-1} 2k|((\delta_t r_1^n)(v^n) + (\overline{r}_2^n)(\partial_t v^{n-1}))| \leq \sum_{n=2}^{N-1} 4k\,(\|\partial_t r_1^n\|_{a_h,\bigstar} + \|\overline{r}_2^n\|_{L,\bigstar}) \tag{3.59}$$

$$\times \max_{2\leq n\leq N-1} (\|v^n\|_{a_h}^2 + \|\partial_t v^{n-1}\|_L^2)^{\frac{1}{2}}. \tag{3.60}$$

Note that we used the identity (2.71) and a triangle inequality to pass from δ_t to ∂_t, and in a similar fashion we have passed from $\overline{r}_2^n$ to r_2^n. Now introducing the relevant norms of the functionals $\{r_1^n\}_{n=1}^{N-1}$ and $\{r_2^n\}_{n=1}^{N-1}$,

$$|||r_1|||_{a_h,\bigstar} = \|r_1^{N-1}\|_{a_h,\bigstar} + \|r_1^1\|_{a_h,\bigstar} + \sum_{n=1}^{N-1} k\|\partial_t r_1^n\|_{a_h,\bigstar}, \tag{3.61}$$

$$|||r_2|||_{L,\bigstar} = \|r_2^{N-1}\|_{L,\bigstar} + \|r_2^1\|_{L,\bigstar} + \sum_{n=1}^{N-1} 2k\|r_2^n\|_{L,\bigstar}. \tag{3.62}$$

Note that we used the identity (2.71) to pass from δ_t to ∂_t. Combining (3.52) and (3.60), we get for all $\varepsilon > 0$,

$$\left|\sum_{n=1}^{N-1} 4kr^n(\delta_t v^n)\right| \lesssim \varepsilon^{-1}C(|||r_1|||^2_{a_h,\star} + |||r_2|||^2_{L,\star}) \tag{3.63}$$

$$+\,\varepsilon\Big(\|\partial_t v^{N-1}\|^2_L + \|v^N\|^2_{a_h} + \|v^{N-1}\|^2_{a_h} + \|\partial_t v^0\|^2_L + \|v^1\|^2_{a_h} + \|v^0\|^2_{a_h} \tag{3.64}$$

$$+ \max_{2\le n\le N-1}(\|v^n\|^2_{a_h} + \|\partial_t v^{n-1}\|^2_L)\Big). \tag{3.65}$$

Lemma 3.6. *Let v^{n+1}, $n = 1,\dots,N-1$, be defined by* (3.48), *and assume that* (3.63) *is satisfied. If $k/h \le c$ with c sufficiently small, then the following stability estimate holds:*

$$\max_{2\le n\le N}\Big(\|\partial_t v^{n-1}\|^2_L + \|v^n\|^2_{a_h} + \|v^{n-1}\|^2_{a_h}\Big) \tag{3.66}$$

$$\lesssim \|\partial_t v^0\|^2_L + \|v^1\|^2_{a_h} + \|v^0\|^2_{a_h} + |||r_1|||^2_{a_h,\star} + |||r_2|||^2_{L,\star}. \tag{3.67}$$

Proof. To prove stability, we test (3.48) with $w = 4k\delta_t v^n = 2k(\partial_t v^n + \partial_t v^{n-1})$ for $n = 1,\dots,N-1$, and sum over the time levels,

$$\sum_{n=1}^{N-1} 2k(\partial_t^2 v^n, \partial_t v^n + \partial_t v^{n-1})_L + \sum_{n=1}^{N-1} 2ka_h(v^n, \partial_t v^n + \partial_t v^{n-1}) \tag{3.68}$$

$$= \sum_{n=1}^{N-1} 2kr^n(\partial_t v^n + \partial_t v^{n-1}). \tag{3.69}$$

Here the first term on the left-hand side satisfies

$$\sum_{n=1}^{N-1} 2k(\partial_t^2 v^n, \partial_t v^n + \partial_t v^{n-1})_L = 2\|\partial_t v^{N-1}\|^2_L - 2\|\partial_t v^0\|^2_L \tag{3.70}$$

since

$$k(\partial_t^2 v^n, \partial_t v^n + \partial_t v^{n-1})_L = (\partial_t v^n - \partial_t v^{n-1}, \partial_t v^n + \partial_t v^{n-1})_L \tag{3.71}$$

$$= \|\partial_t v^n\|^2_L - \|\partial_t v^{n-1}\|^2_L. \tag{3.72}$$

Next, for the second term we have

$$\sum_{n=1}^{N-1} 2ka_h(v^n, \partial_t v^n + \partial_t v^{n-1}) = \sum_{n=1}^{N-1} 2a_h(v^n, v^{n+1} - v^{n-1}) \tag{3.73}$$

$$= 2a_h(v^{N-1}, v^N) - 2a_h(v^1, v^0). \tag{3.74}$$

Inserting (3.70) and (3.74) into (3.68), we obtain

$$2\|\partial_t v^{N-1}\|^2_L + 2a_h(v^{N-1}, v^N) = 2\|\partial_t v^0\|^2_L + 2a_h(v^0, v^1) \tag{3.75}$$

$$+ \sum_{n=1}^{N-1} r^n(2k(\partial_t v^n + \partial_t v^{n-1})). \tag{3.76}$$

Using the identities

$$k^2\|\partial_t v^{N-1}\|_{a_h}^2 + 2a_h(v^{N-1}, v^N) = \|v^N\|_{a_h}^2 + \|v^{N-1}\|_{a_h}^2, \tag{3.77}$$

$$k^2\|\partial_t v^0\|_{a_h}^2 + 2a_h(v^0, v^1) = \|v^1\|_{a_h}^2 + \|v^0\|_{a_h}^2, \tag{3.78}$$

we may write (3.75) in the form

$$2\|\partial_t v^{N-1}\|_L^2 - k^2\|\partial_t v^{N-1}\|_{a_h}^2 + \|v^N\|_{a_h}^2 + \|v^{N-1}\|_{a_h}^2 \tag{3.79}$$

$$= 2\|\partial_t v^0\|_L^2 - k^2\|\partial_t v^0\|_{a_h}^2 + \|v^1\|_{a_h}^2 + \|v^0\|_{a_h}^2 \tag{3.80}$$

$$+ \sum_{n=1}^{N-1} 2kr^n(\partial_t v^n + \partial_t v^{n-1}). \tag{3.81}$$

Using an inverse inequality followed by the stability (3.23), we get

$$\|w\|_{a_h}^2 \lesssim h^{-2}\|w\|_{\mathcal{T}_h}^2 \lesssim h^{-2}\|w\|_L^2, \tag{3.82}$$

which, with $w = \partial_t v^{N-1}$, gives

$$k^2\|\partial_t v^{N-1}\|_{a_h}^2 \lesssim h^{-2}k^2\|\partial_t v^{N-1}\|_{\mathcal{T}_h}^2 \lesssim h^{-2}k^2\|\partial_t v^{N-1}\|_L \tag{3.83}$$

Using the Courant–Friedrichs–Lewy (CFL) condition $Ch^{-2}k^2 \leq Cc^2 \leq 1$, where C is the hidden constant in (3.83), and $k/h \leq c$ with c small enough due to assumption in the lemma, we arrive at

$$\|\partial_t v^{N-1}\|_L^2 + \|v^N\|_{a_h}^2 + \|v^{N-1}\|_{a_h}^2 \tag{3.84}$$

$$\leq 2\|\partial_t v^0\|_L^2 + \|v^1\|_{a_h}^2 + \|v^0\|_{a_h}^2 + 2\left|\sum_{n=1}^{N-1} 2kr^n(\delta_t v^n)\right|. \tag{3.85}$$

Applying now the bound (3.63) in the last term of the right-hand side of the last expression we obtain, for all $\varepsilon > 0$,

$$(1-\varepsilon)(\|\partial_t v^{N-1}\|_L^2 + \|v^N\|_{a_h}^2 + \|v^{N-1}\|_{a_h}^2) \tag{3.86}$$

$$\leq 2(1+\varepsilon)(\|\partial_t v^0\|_L^2 + \|v^1\|_{a_h}^2 + \|v^0\|_{a_h}^2) \tag{3.87}$$

$$+ \varepsilon^{-1}C(|||r_1|||_{a_h,\bigstar}^2 + |||r_2|||_{L,\bigstar}^2) \tag{3.88}$$

$$+ \varepsilon\Big(\max_{2\leq n\leq N-1}(\|v^n\|_{a_h}^2 + \|\partial_t v^{n-1}\|_L^2)\Big). \tag{3.89}$$

Next, keeping N fixed on the right-hand side, we note that (3.86) holds if N in the left-hand side is replaced by an arbitrary $n = 2, \dots, N-1$. Taking the maximum over n on the left-hand side we get

$$(1-\varepsilon)\max_{2\leq n\leq N-1}(\|\partial_t v^{n-1}\|_L^2 + \|v^n\|_{a_h}^2 + \|v^{n-1}\|_{a_h}^2) \tag{3.90}$$

$$\leq 2(1+\varepsilon)(\|\partial_t v^0\|_L^2 + \|v^1\|_{a_h}^2 + \|v^0\|_{a_h}^2) \tag{3.91}$$

$$+ \varepsilon^{-1}C(|||r_1|||_{a_h,\bigstar}^2 + |||r_2|||_{L,\bigstar}^2) \tag{3.92}$$

$$+ \varepsilon\Big(\max_{2\leq n\leq N-1}(\|v^n\|_{a_h}^2 + \|\partial_t v^{n-1}\|_L^2)\Big). \tag{3.93}$$

Finally, using a kick back argument and taking ε small enough, we obtain

$$\max_{2\le n\le N-1}\left(\|\partial_t v^{n-1}\|_L^2+\|v^n\|_{a_h}^2+\|v^{n-1}\|_{a_h}^2\right) \tag{3.94}$$

$$\lesssim \|\partial_t v^0\|_L^2+\|v^1\|_{a_h}^2+\|v^0\|_{a_h}^2 \tag{3.95}$$

$$+|||r_1|||_{a_h,\star}^2+|||r_2|||_{L,\star}^2, \tag{3.96}$$

which completes the proof. ∎

Remark 3.7. Clearly the bound also holds for $\|v^n\|_{a_h}^2$ and $\|\partial_t v^{n-1}\|_L^2$ separately. For instance,

$$\max_{2\le n\le N-1}\|v^n\|_{a_h}^2\le\max_{2\le n\le N-1}\left(\|\partial_t v^{n-1}\|_L^2+\|v^n\|_{a_h}^2+\|v^{n-1}\|_{a_h}^2\right), \tag{3.97}$$

If we assume that the max in the left-hand side is taken for n^*, we immediately see that

$$\|v^{n^*}\|_{a_h}^2\le\|\partial_t v^{n^*-1}\|_L^2+\|v^{n^*}\|_{a_h}^2+\|v^{n^*-1}\|_{a_h}^2 \tag{3.98}$$

$$\le\max_{2\le n\le N-1}\left(\|\partial_t v^{n-1}\|_L^2+\|v^n\|_{a_h}^2+\|v^{n-1}\|_{a_h}^2\right). \tag{3.99}$$

3.5. Error estimates. We will now combine the approximation properties and stability estimates proved in the previous section to derive error estimates for the CutFEM approximation. To simplify the notation we denote a continuous function at a certain time level t^n, $v^n:=v(t^n)$ and its partial derivatives

$$d_t^m v(t):=\frac{\partial^m v}{\partial t^m}(t)\text{ and }(d_t^m v)^n:=\frac{\partial^m v}{\partial t^m}(t^n),\quad m\in\mathbb{N}_+. \tag{3.100}$$

For $m=1$ we will drop the superscript.

Before we derive the error estimates we recall the following elementary results for the finite difference discretization in time.

LEMMA 3.8. *For functions $v\in L^\infty(0,T;L^2(\Omega))$ there exists positive constants such that*

$$\|\partial_t^m v^n\|\lesssim\|d_t^m v\|_{L^\infty(0,T;L^2(\Omega))},\quad m\in\{1,2\},\quad n\ge 0, \tag{3.101}$$

$$\|\partial_t^2 v^n-\partial_t^2 v^{n-1}\|\lesssim k\|d_t^3 v\|_{L^\infty(0,T;L^2(\Omega))},\quad n\ge 2, \tag{3.102}$$

$$\|\partial_t^2 v^n-d_t^2 v^n\|\lesssim k^2\|d_t^4 v\|_{L^\infty(0,T;L^2(\Omega))},\quad n\ge 2, \tag{3.103}$$

Proof. We only prove the first inequality (3.101). For $m=2$, the case $m=1$ is similar. Taking the L^2-norm over Ω and using partial integration in time and

(3.104)

$$\|\partial_t^2 v^n\|=\left\|\frac{1}{k^2}(v^{n+1}-2v^n+v^{n-1})\right\|$$

$$=\left\|\frac{1}{k^2}\left(\int_{t^n}^{t^{n-1}}(t^{n-1}-t)\frac{\partial^2 v}{\partial t^2}(t)\ \mathrm{d}t+\int_{t^n}^{t^{n+1}}(t^{n+1}-t)\frac{\partial^2 v}{\partial t^2}(t)\ \mathrm{d}t\right)\right\| \tag{3.105}$$

$$\lesssim\frac{1}{k^2}\int_{t^n}^{t^{n-1}}(t^{n-1}-t)\left\|\frac{\partial^2 v}{\partial t^2}(t)\right\|\ \mathrm{d}t+\frac{1}{k^2}\int_{t^n}^{t^{n+1}}(t^{n+1}-t)\left\|\frac{\partial^2 v}{\partial t^2}(t)\right\|\ \mathrm{d}t \tag{3.106}$$

$$\lesssim\|d_t^2 v\|_{L^\infty(t^{n-1},t^{n+1};L^2(\Omega))} \tag{3.107}$$

and the proof of the first estimate (3.101) is complete. The second estimate (3.102) follows using the same technique. Finally, for (3.103), once again using partial integration it follows that

$$\partial_t^2 v^n - d_t^2 v^n = \frac{1}{k^2}\left(\int_{t^n}^{t^{n-1}} \frac{(t^{n-1}-t)^3}{6}\frac{\partial^4 v}{\partial t^4}(t)\ \mathrm{d}t + \int_{t^n}^{t^{n+1}} \frac{(t^{n+1}-t)^3}{6}\frac{\partial^4 v}{\partial t^4}(t)\ \mathrm{d}t\right). \tag{3.108}$$

Taking the norm and supremum over $d_t^4 u$ in time in the right-hand side, we have

$$\left\|\frac{1}{k^2}\int_{t^n}^{t^{n+1}} \frac{(t^{n+1}-t)^3}{6}\frac{\partial^4 v}{\partial t^4}(t)\ \mathrm{d}t\right\| \lesssim k^2\|d_t^4 v\|_{L^\infty(t^{n-1},t^{n+1};L^2(\Omega))}. \tag{3.109}$$

□

THEOREM 3.9. *Let u_h^{n+1}, for $n = 1,\dots,N-1$, be defined by* (2.89) *with initial data $u_h^0 = R_h u^0$ and $u_h^1 = R_h u^1$. Then if u is a sufficiently smooth solution to* (1.1), *the following error estimates hold:*

$$\|(d_t u)^{N-1} - \partial_t(u_h^{N-1})\| + \|u^N - u_h^N\| \lesssim h^2 + k^2, \tag{3.110}$$

$$\|\nabla(u^N - u_h^N)\| \lesssim h + k^2. \tag{3.111}$$

Proof. We first note that the exact solution satisfies

$$((d_t^2 u)^n, v) + a_h(u^n, v) = (f^n, v^n) \qquad \forall v \in V_h, t \in (0,T). \tag{3.112}$$

and for $n = 1,\dots,N-1$, the numerical scheme satisfies

$$(\partial_t^2 u_h^n, v)_L + a_h(u_h^n, v) = (f_h^n, v^n)_L \qquad \forall v \in V_h^E. \tag{3.113}$$

Subtracting the two equations we obtain the error equation

$$((d_t^2 u)^n, v) - (\partial_t^2 u_h^n, v)_L + a_h(u^n - u_h^n, v) = (f^n, v) - (f_h^n, v)_L \qquad \forall v \in V_h^E. \tag{3.114}$$

In order to estimate the error, we split it into two contributions using the Ritz projection,

$$u^n - u_h^n = u^n - R_h u^n + R_h u^n - u_h^n = \rho^n + \theta^n. \tag{3.115}$$

In the standard manner, we then split the norms in the left-hand side of (3.110) and (3.111) using the triangle inequality in the contributions from ρ^n and θ^n, $\|u^n - u_h^n\| \le \|\rho^n\| + \|\theta^n\|$. In the following paragraphs we estimate the two contributions to the error emanating from the interpolation error ρ and the discrete part of the error θ. The ρ contribution can be directly estimated using the error estimates (3.14) for the Ritz projection. For the θ contribution we derive an error equation with a right-hand side that accounts for the lumping error and the error in the difference approximation of the second order time derivative. The bound for θ is then obtained by applying the stability estimate (3.67) followed by a priori bounds for the right-hand side.

The ρ contribution. Applying the error estimate (3.14) for the Ritz projection we have the estimates

$$\|\nabla^m \rho^n\| \lesssim h^{2-m}\|u^n\|_{H^2(\Omega)}, \qquad m = 0,1, \tag{3.116}$$

$$\|(d_t^m \rho)^n\| \lesssim h^2\|(d_t^m u)^n\|_{H^2(\Omega)}, \quad m = 1,2,3,4, \tag{3.117}$$

where we used the commutation $(d_t R_h v)^n = R_h(d_t v)^n$.

The θ contribution. We note that we have the identity

$$((d_t^2 u)^n, v) - (\partial_t^2 u_h^n, v)_L \tag{3.118}$$
$$= ((d_t^2 u)^n, v) - (\partial_t^2 (R_h u^n), v)_L + (\partial_t^2 (R_h u - u_h)^n, v)_L \tag{3.119}$$
$$= ((d_t^2 u)^n, v) - (\partial_t^2 (R_h u^n), v)_L + (\partial_t^2 \theta^n, v)_L, \tag{3.120}$$

and using the orthogonality of R_h,

$$a_h(u^n - u_h^n, v) = a_h^n(\rho^n, v) + a_h(\theta^n, v) = a_h(\theta^n, v). \tag{3.121}$$

Combining (3.114), (3.118), (3.120), and (3.121), we get the following error equation for the discrete part θ of the error:

$$(\partial_t^2 \theta^n, v)_L + a_h(\theta^n, v) = \underbrace{(f^n, v) - (f_h^n, v)_L + (\partial_t^2 (R_h u^n), v)_L - ((d_t^2 u)^n, v)}_{r^n(v)}, \tag{3.122}$$

where we introduced the functional $r^n : V_h \to \mathbb{R}$. We now split r^n, by adding and subtracting suitable terms, in order to apply a stability bound of the form (3.63),

$$r^n(v) = (f^n, v) - (f_h^n, v)_L + (\partial_t^2 R_h u^n, v)_L - ((d_t^2 u)^n, v) \tag{3.123}$$
$$= \underbrace{(f^n, v) - (f_h^n, v)_L + (\partial_t^2 R_h u^n, v)_L - (\partial_t^2 R_h u^n, v)}_{r_1^n(v)} \tag{3.124}$$
$$+ \underbrace{(\partial_t^2 R_h u^n, v) - ((d_t^2 R_h u)^n, v) + ((d_t^2 R_h u)^n, v) - ((d_t^2 u)^n, v)}_{r_2^n(v)} \tag{3.125}$$
$$= r_1^n(v) + r_2^n(v), \tag{3.126}$$

where we have collected the terms associated with the lumping error in r_1 and the remaining terms in r_2. Below we will prove the following bounds on the residuals r_1 and r_2:

$$|||r_1|||_{a_h,\star} \lesssim h^2 (\|u\|_{W^{3,\infty}(0,T;H^1(\Omega))} + \|f\|_{W^{1,\infty}(0,T;H^2(\Omega))}), \tag{3.127}$$

$$|||r_2|||_{L,\star} \lesssim k^2 \|d_t^4 u\|_{L^\infty(0,T;L^2(\Omega))} + h^2 \|u\|_{W^{2,\infty}(0,T;H^2(\Omega))}. \tag{3.128}$$

Here we have omitted higher order terms. Anticipating the approximation error estimates (3.127) and (3.128), we may use the stability estimate (3.67), where $\theta^0 = \theta^1 = 0$ since $u_h^0 = R_h u^0$ and $u_h^1 = R_h u^1$, to obtain

$$\max_{2 \le n \le N} \left(\|\partial_t \theta^{n-1}\|_L^2 + \|\theta^n\|_{a_h}^2 + \|\theta^{n-1}\|_{a_h}^2 \right) \lesssim |||r_1|||_{a_h,\star}^2 + |||r_2|||_{L,\star}^2 \tag{3.129}$$
$$\lesssim h^4 \left(\|u\|_{W^{3,\infty}(0,T;H^1(\Omega))} + \|f\|_{W^{1,\infty}(0,T;H^2(\Omega))} \right)^2 \tag{3.130}$$
$$+ (k^2 \|d_t^4 u\|_{L^2(0,T;L^2(\Omega))} + h^2 \|u\|_{W^{2,\infty}(0,T;H^2(\Omega))})^2 \tag{3.131}$$
$$\lesssim (h^2 + k^2)^2. \tag{3.132}$$

Verification of (3.127). Starting from the definition (3.61),

$$|||r_1|||_{a_h,\star} = \|r_1^{N-1}\|_{a_h,\star} + \|r_1^1\|_{a_h,\star} + \sum_{n=1}^{N-1} k \|\partial_t r_1^n\|_{a_h,\star} \tag{3.133}$$

with

$$r_1^n(v) = \underbrace{(f^n, v) - (f_h^n, v)_L}_{\text{I}} + \underbrace{(\partial_t^2 R_h u^n, v)_L - (\partial_t^2 R_h u^n, v)}_{\text{II}}. \tag{3.134}$$

We start with estimates of the first four terms in the right-hand side of (3.61), by considering an arbitrary n. By adding and subtracting (f_h^n, v) we have

$$I = (f^n, v) - (f_h^n, v)_L \leq |(f^n, v) - (f_h^n, v)| + |(f_h^n, v) - (f_h^n, v)_L|. \tag{3.135}$$

Assuming that f_h^n has optimal approximation properties, we see that

$$|(f^n, v) - (f_h^n, v)| \lesssim h^2 \|f^n\|_{H^2(\Omega)} \|v\|_{a_h}, \tag{3.136}$$

where we used the Poincaré inequality $\|v\| \lesssim \|v\|_{a_h}$. For the second term and term II, we apply Lemma 3.4 to obtain

$$|(f_h^n, v) - (f_h^n, v)_L| \lesssim h^2 \|\nabla f_h^n\| \|v\|_{a_h} \lesssim h^2 (\|\nabla f^n\| + h\|f^n\|_{H^2(\Omega)}) \|v\|_{a_h} \tag{3.137}$$

and

$$(\partial_t^2 R_h u^n, v) - (\partial_t^2 R_h u^n, v)_L \lesssim h^2 \|\nabla \partial_t^2 R_h u^n\| \|v\|_{a_h}. \tag{3.138}$$

Applying the first inequality of Lemma 3.8, adding and subtracting $\nabla d_t^2 u^n$ and applying approximation shows that

$$\|\nabla \partial_t^2 R_h u^n\| \leq \|\nabla d_t^2 \rho^n\| + \|\nabla d_t^2 u^n\| \lesssim \|\nabla d_t^2 u^n\| + h\|d_t^2 u^n\|_{H^2(\Omega)}. \tag{3.139}$$

To sum up we have (neglecting higher order terms)

$$\|r_1^{N-1}\|_{a_h,\star} + \|r_1^1\|_{a_h,\star} \lesssim h^2 (\|u\|_{W^{2,\infty}(0,T;H^1(\Omega))} + \|f\|_{L^\infty(0,T;H^2(\Omega))}). \tag{3.140}$$

To control the last term in the right-hand side of (3.133), we simply apply the above arguments to $\partial_t f^n$, $\partial_t f_h^n$, and $\partial_t(\partial_t^2 R_h u^n)$. For the bound of $\partial_t(\partial_t^2 R_h u^n)$ we use (3.102). This results in similar bounds, but with an additional time derivative,

$$\sum_{n=1}^{N-1} k \|\partial_t r_1^n\|_{a_h,\star} \tag{3.141}$$

$$\lesssim k \sum_{n=0}^{N-1} h^2 (\|u\|_{W^{3,\infty}(t^n,t^{n+1};H^1(\Omega))} + \|f\|_{W^{1,\infty}(t^n,t^{n+1};H^2(\Omega))}) \tag{3.142}$$

$$\lesssim h^2 (\|u\|_{W^{3,\infty}(0,T;H^1(\Omega))} + \|f\|_{W^{1,\infty}(0,T;H^2(\Omega))}). \tag{3.143}$$

Verification of (3.128). We recall the definition (3.62),

$$|||r_2|||_{L,\star} = \|r_2^{N-1}\|_{L,\star} + \|r_2^1\|_{L,\star} + \sum_{n=1}^{N-1} 2k \|r_2^n\|_{L,\star}. \tag{3.144}$$

It follows that

$$|||r_2|||_{L,\star} \leq 4 \max_{1 \leq n \leq N-1} \|r_2^n\|_{L,\star}. \tag{3.145}$$

The $\|r_2^n\|_{L,\star}$ contribution in the right-hand side can be bounded as follows. Using (3.23) we see that for all $w \in L^2(\Omega)$,

$$(w, v_h) \leq \|w\|\|v_h\| \lesssim \|w\|\|v_h\|_L. \tag{3.146}$$

In particular,

$$(\partial_t^2 R_h u^n - (d_t^2 R_h u)^n, v) + ((d_t^2 R_h u)^n - (d_t^2 u)^n, v) \tag{3.147}$$

$$\lesssim (\|\partial_t^2 R_h u^n - (d_t^2 R_h u)^n\| + \|((d_t^2 R_h u)^n - (d_t^2 u)^n\|)\|v_h\|_L. \tag{3.148}$$

By the definition of r_2^n we then have

$$\|r_2^n\|_{L,\star} = \underbrace{\|\partial_t^2 R_h u^n - (d_t^2 R_h u)^n\|}_{\mathrm{I}} + \underbrace{\|(d_t^2 \rho)^n\|}_{\mathrm{II}}. \tag{3.149}$$

Term I is first bounded using inequality (3.103) in Lemma 3.8 and then, since we have not proved L^2-stability of R_h, we add and subtract $d_t^4 u$, use the triangle inequality and the inequality (3.117),

$$I \lesssim k^2 \|d_t^4 R_h u\|_{L^\infty(0,T;L^2(\Omega))} \tag{3.150}$$

$$\lesssim k^2(\|d_t^4 u\|_{L^\infty(0,T;L^2(\Omega))} + \|d_t^4(u - R_h u)\|_{L^\infty(0,T;L^2(\Omega))}) \tag{3.151}$$

$$\lesssim k^2(\|d_t^4 u\|_{L^\infty(0,T;L^2(\Omega))} + h^2\|d_t^4 u\|_{L^\infty(0,T;H^2(\Omega))}). \tag{3.152}$$

For II we apply (3.117) to obtain

$$\|(d_t^2 \rho)^n\| \lesssim h^2 \|u\|_{W^{2,\infty}(0,T;H^2(\Omega))}. \tag{3.153}$$

We conclude that, omitting high order terms, we have as claimed,

$$|||r_2|||_{L,\star} \lesssim k^2 \|d_t^4 u\|_{L^\infty(0,T;L^2(\Omega))} + h^2 \|u\|_{W^{2,\infty}(0,T;H^2(\Omega))}, \tag{3.154}$$

which completes the proof. ∎

4. Numerical examples. In the numerical examples below we use the following implementation of the extension operator. The mapping S_h is constructed by associating with each element $T \in \mathcal{T}_h \setminus \mathcal{T}_{h,I}$ the element S in $\mathcal{T}_{h,I}$ which minimizes the distance between the element centroids. For each $x \in \mathcal{X}_h \setminus \mathcal{X}_{h,I}$ the weights in the nodal average $\langle \cdot \rangle_x$ (see (2.4)) is taken to be 1 on precisely one element $T_x \in \mathcal{T}_h(x)$ and zero on all elements in $\mathcal{T}_h(x) \setminus T_x$, where we recall that $\mathcal{T}_h(x)$ is the set of elements which has x as a vertex. Note that this choice of weights corresponds to simply defining the nodal value in $x \in \mathcal{X}_h \setminus \mathcal{X}_{h,I}$ by $((F_h v)|_{T_x})|_x$, where F_h is defined in (2.16). The Nitsche parameter was set to $\gamma = 10$ in all computations and the initial data is the extension of nodal interpolant in interior nodes. For definiteness, we define $h := \sqrt{2|T|}$, where $|T|$ is the area of the element (or extended element).

4.1. Space-time convergence.

4.1.1. A Dirichlet problem. On the disc $\Omega = \{(x, y) \in \mathbb{R}^2 : \; r < 0.5\}$, with $r = \sqrt{x^2 + y^2}$, we consider a problem with manufactured solution

$$u = (1 - 4r^2)\cos(\omega t) \tag{4.1}$$

corresponding to the right-hand side

$$f = (4\omega^2 r^2 - \omega^2 + 16)\cos(\omega t) \tag{4.2}$$

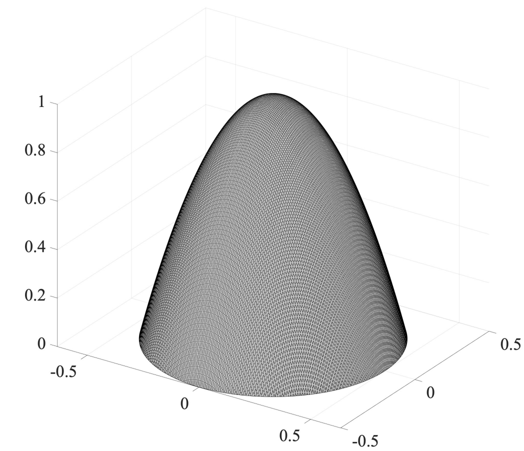

FIG. 2. *Elevation of the computed solution on a particular mesh.*

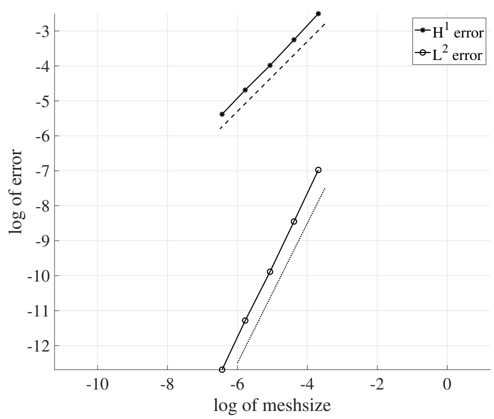

FIG. 3. *Convergence at time* $T = 1$*, Dirichlet case. Dashed line has inclination* 1:1*, dotted line has inclination* 2:1.

with $\omega = 2\pi$. We solve this problem over one period, i.e., with $T = 1$. The timestep k is coupled to the meshsize h by $k = Ch$ with C fixed. On our initial mesh, $h = 0.025$ and $k = 1/500$.

In Figure 2 we show the solution (on the third mesh in a sequence of halving the meshsize) after one period, and in Figure 3 we show the convergence at time T in $L^2(\Omega)$ and in $H^1(\Omega)$. The expected convergence of $O(h^2)$ is attained in L^2 and $O(h)$ in H^1.

4.1.2. A Neumann problem. In this example we use the same domain and the same meshes, timesteps, and final time T as in section 4.1.1. We use the fabricated solution

$$u = 8(r^2/2 - r^4)\cos(\omega t) \tag{4.3}$$

corresponding to the right-hand side

$$f = 4(\omega^2(2x^4 + 2y^4 + 4x^2y^2 - x^2 - y^2) + 32x^2 + 32y^2 - 4)\cos(\omega t) \tag{4.4}$$

with $\omega = 2\pi$.

In Figure 4 we show the convergence at time T in $L^2(\Omega)$ and in $H^1(\Omega)$. Again, the expected convergence of $O(h^2)$ is attained in L^2 and $O(h)$ in H^1.

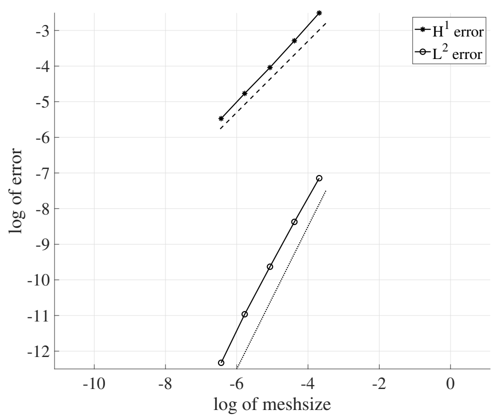

FIG. 4. *Convergence at time* $T = 1$*, Neumann case. Dashed line has inclination* 1:1*, dotted line has inclination* 2:1*.*

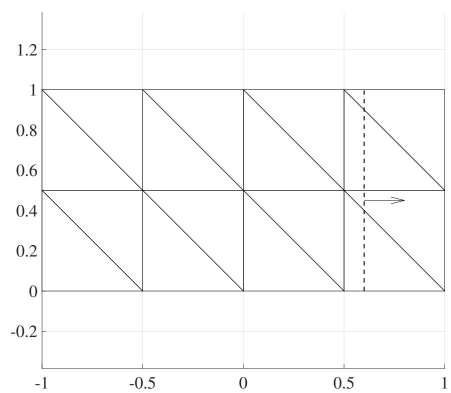

FIG. 5. *Small mesh used for computation of the CFL condition. Cut moves across the last row of elements.*

4.2. CFL condition. As is well known, e.g., [22, 14], the leapfrog scheme has a CFL condition on the timestep that can be written

$$k \leq \alpha h, \quad \text{where} \quad \alpha := \frac{2}{h\sqrt{\lambda_{\max}\left(\widehat{M}_L^{-1}\widehat{A}_{h,I}\right)}} \tag{4.5}$$

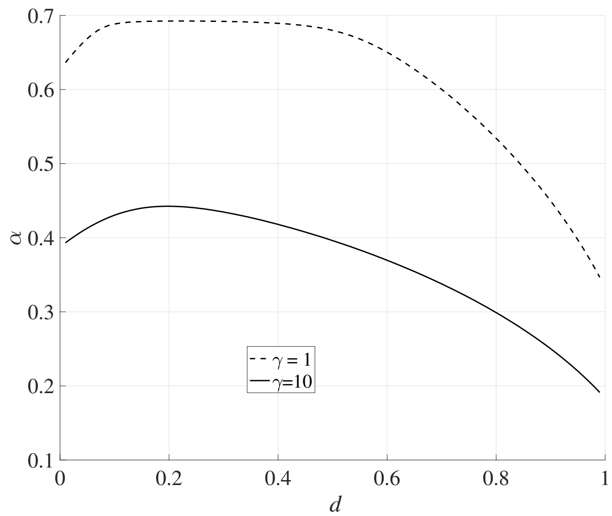

FIG. 6. *CFL condition depending on placement of cut.*

with $\lambda_{\max}(S)$ the maximum eigenvalue of the matrix S. To investigate how the cut elements affect the CFL condition, we consider a small problem with Neumann boundary conditions on all uncut boundaries, and a Dirichlet condition on the cut boundary. The mesh with cut elements indicated is shown in Figure 5. We move the cutting line from close to the uncut element to far from the uncut element, as indicated in Figure 5. We denote by d the distance from the uncut cell divided by h, $0 < d < 1$. The corresponding α is given in Figure 6 for two choices of γ. The choice of the Nitsche parameter γ affects the conditioning of $\widehat{A}_{h,I}$, so in Figures 8 and 9 we show graphs of $\lambda_{\min}(\widehat{A}_{h,I})$ and $\lambda_{\max}(\widehat{A}_{h,I})$, respectively. We note that $\gamma = 1$ is too small to give a coercive problem when d is small, but that $\lambda_{\min}$ is not affected by increasing γ beyond the point where $\widehat{A}_{h,I}$ is positive definite. It is the maximum eigenvalue that is adversely affected by γ, and we conclude that for large distances from the uncut mesh it would be beneficial for the CFL condition to either lower γ, change the definition of h in the Nitsche term, or modify the method according to Remark 2.1. This question is left for future work.

4.3. Increasing frequency. Here we show the effect of a pulse with decreasing support approaching the boundary. Our domain is $(-0.81, 0.79) \times (-0.8, 0.8)$ and has Neumann boundary conditions on the uncut boundaries $y = \pm 0.8$. On the uncut boundary $x = -0.81$ we impose Dirichlet conditions strongly, and on the cut boundary at $x = 0.79$ we impose zero Dirichlet boundary conditions weakly. In Figure 7 we show how the mesh is cut in a closeup. We set $h = 8.9 \times 10^{-3}$ and $k = 3.93 \times 10^{-4}$. The initial solution is given by

$$u(x,y) = (1 + \cos(\pi|x + 0.01|/d_0) \ \text{ if } |x + 0.01| < d_0, u(x,y) = 0 \ \text{ elsewhere} \tag{4.6}$$

and $\partial_t u = 0$, with different d_0. This pulse splits into two, one going left and hitting the uncut boundary and one going right and hitting the cut boundary. We show snapshots of the solutions different times and for different d_0 in Figures 10–18. Note the dispersion error becomes more pronounced as d_0 decreases. The difference in quality of the solution at the uncut and cut boundaries boundary is small and does not become more pronounced as the support of the pulse decreases. We note that as the frequency increases and the meshsize must (eventually) be decreased to avoid dispersion errors, which means the weak Dirichlet data will also be better resolved.

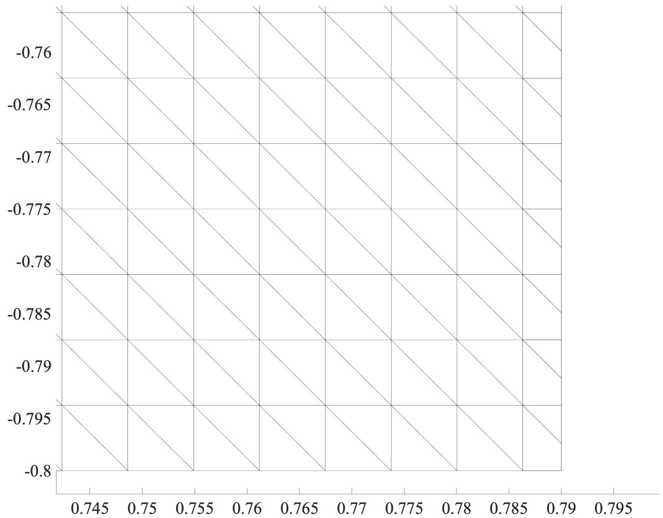

FIG. 7. *Closeup of the mesh at the lower right corner.*

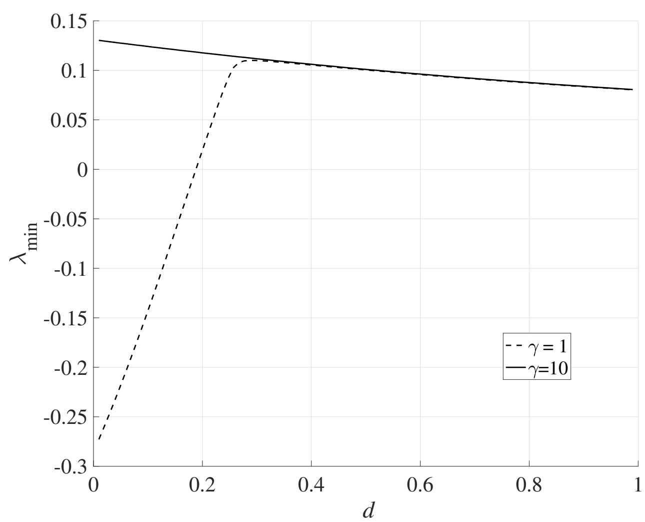

FIG. 8. *Minimum eigenvalue of $\widehat{A}_{h,I}$ depending on placement of cut.*

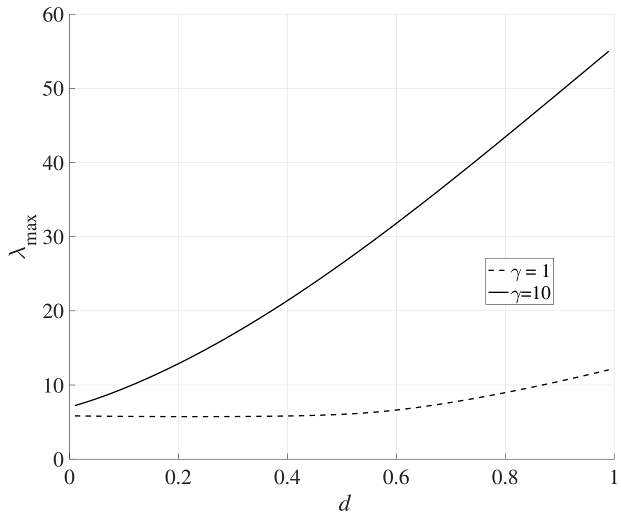

Fig. 9. *Maximum eigenvalue of $\widehat{A}_{h,I}$ depending on placement of cut.*

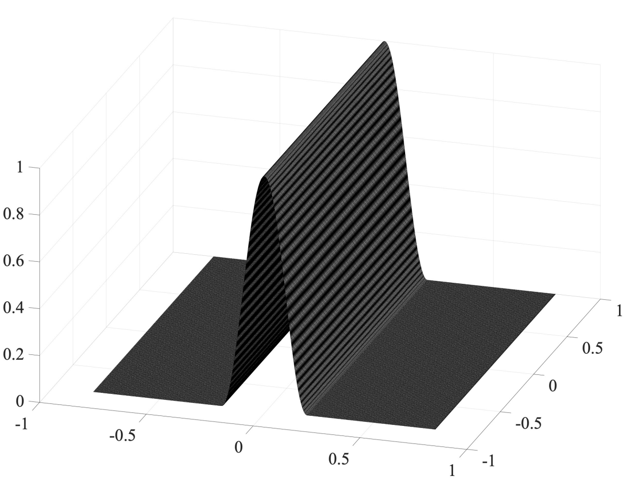

Fig. 10. *Pulse at $t = 0$ for $d_0 = 0.2$.*

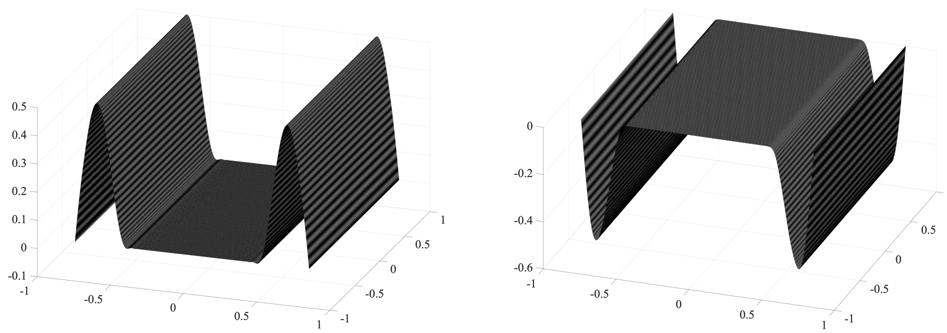

FIG. 11. *Pulse at* $t = 0.65$ *(left) and* $t = 0.9$ *(right) for* $d_0 = 0.2$.

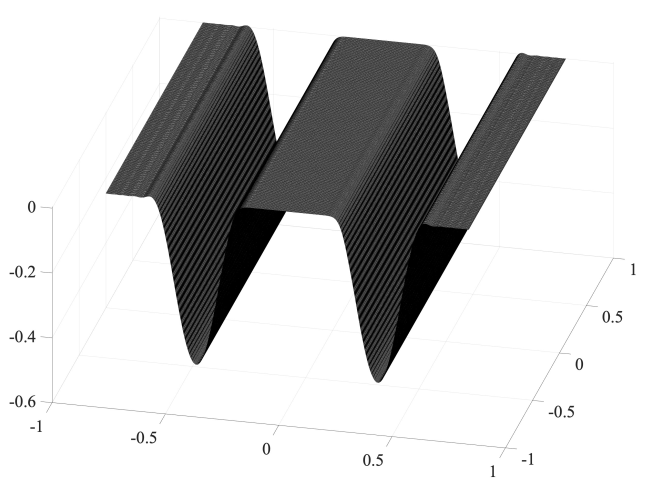

FIG. 12. *Pulse at* $t = 1.2$ *for* $d_0 = 0.2$.

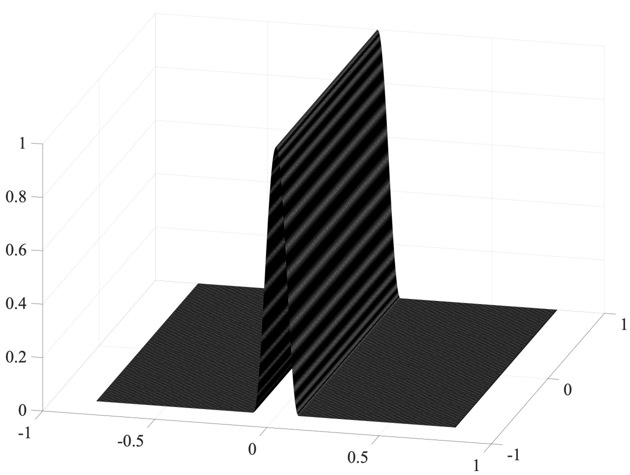

Fig. 13. *Pulse at $t = 0$ for $d_0 = 0.1$.*

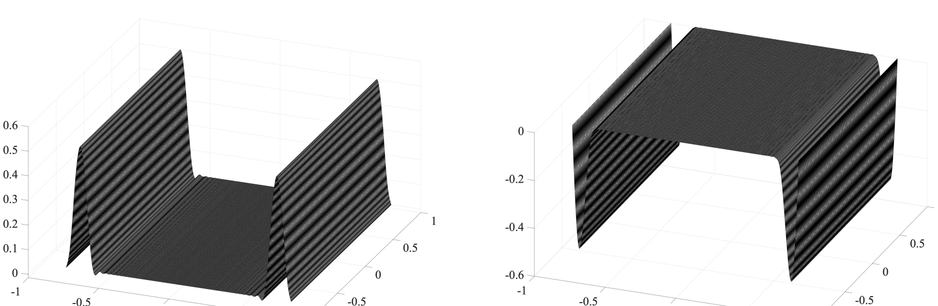

Fig. 14. *Pulse at $t = 0.7$ (left) and $t = 0.85$ (right) for $d_0 = 0.1$.*

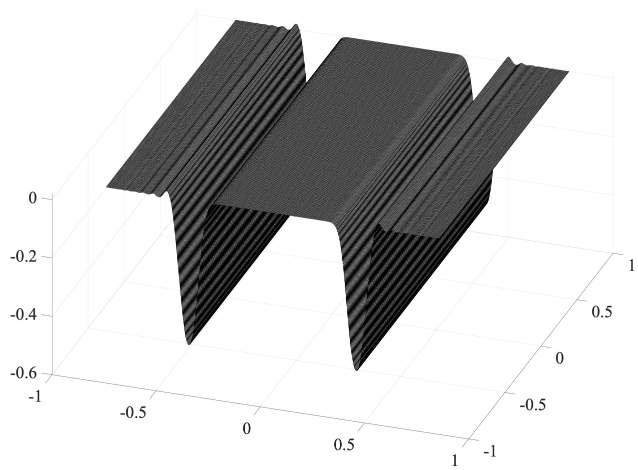

FIG. 15. *Pulse at $t = 1.2$ for $d_0 = 0.1$.*

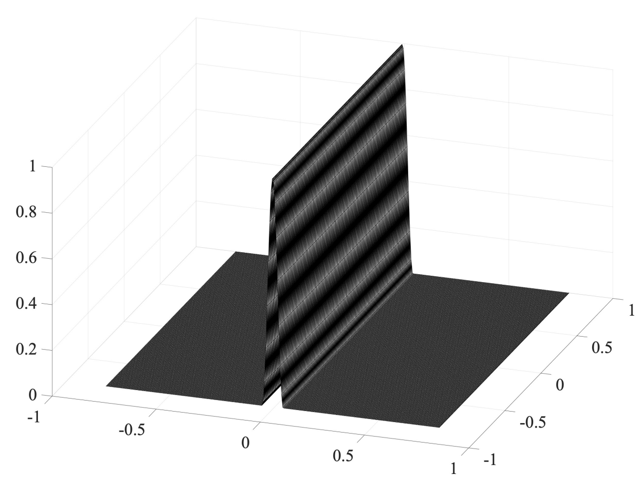

FIG. 16. *Pulse at $t = 0$ for $d_0 = 0.05$.*

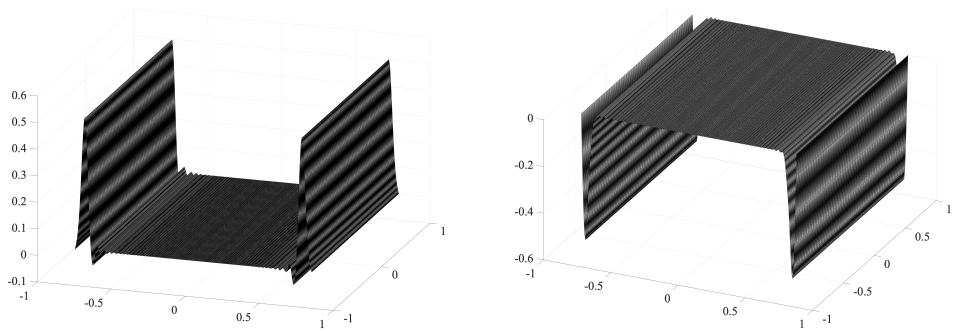

Fig. 17. *Pulse at $t = 0.74$ (left) and $t = 0.84$ (right) for $d_0 = 0.05$.*

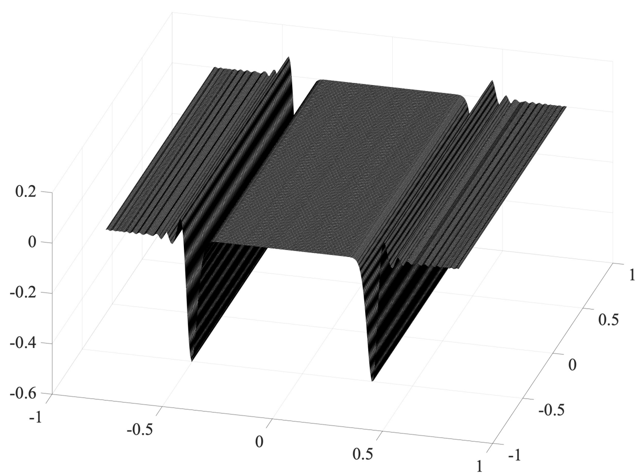

Fig. 18. *Pulse at $t = 1.2$ for $d_0 = 0.05$.*

REFERENCES

[1] S. Badia, F. Verdugo, and A. F. Martín, *The aggregated unfitted finite element method for elliptic problems*, Comput. Methods Appl. Mech. Engrg., 336 (2018), pp. 533–553, https://doi.org/10.1016/j.cma.2018.03.022.

[2] G. A. Baker, *Error estimates for finite element methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 13 (1976), pp. 564–576, https://doi.org/10.1137/0713048.

[3] G. A. Baker and V. A. Dougalis, *The effect of quadrature errors on finite element approximations for second order hyperbolic equations*, SIAM J. Numer. Anal., 13 (1976), pp. 577–598, https://doi.org/10.1137/0713049.

[4] E. Burman, *Ghost penalty*, C. R. Math. Acad. Sci. Paris, 348 (2010), pp. 1217–1220, https://doi.org/10.1016/j.crma.2010.10.006.

[5] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing, *CutFEM: Discretizing geometry and partial differential equations*, Internat. J. Numer. Methods Engrg., 104 (2015), pp. 472–501, https://doi.org/10.1002/nme.4823.

[6] E. Burman and A. Ern, *Continuous interior penalty hp-finite element methods for advection and advection-diffusion equations*, Math. Comp., 76 (2007), pp. 1119–1140, https://doi.org/10.1090/S0025-5718-07-01951-5.

[7] E. Burman and A. Ern, *An unfitted hybrid high-order method for elliptic interface problems*, SIAM J. Numer. Anal., 56 (2018), pp. 1525–1546, https://doi.org/10.1137/17M1154266.

[8] E. Burman and P. Hansbo, *Fictitious domain finite element methods using cut elements:* II. *A stabilized Nitsche method*, Appl. Numer. Math., 62 (2012), pp. 328–341, https://doi.org/10.1016/j.apnum.2011.01.008.

[9] E. Burman, P. Hansbo, and M. G. Larson, *CutFEM Based on Extended Finite Element Spaces*, preprint, https://arxiv.org/abs/2101.10052, 2021.

[10] P. Clément, *Approximation by finite element functions using local regularization*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér., 9 (1975), pp. 77–84.

[11] J. Diaz and M. J. Grote, *Energy conserving explicit local time stepping for second-order wave equations*, SIAM J. Sci. Comput., 31 (2009), pp. 1985–2014, https://doi.org/10.1137/070709414.

[12] M. Drolia, M. S. Mohamed, O. Laghrouche, M. Seaid, and A. El Kacimi, *Explicit time integration with lumped mass matrix for enriched finite elements solution of time domain wave problems*, Appl. Math. Model., 77 (2020), pp. 1273–1293, https://doi.org/10.1016/j.apm.2019.07.054.

[13] T. Dupont, *L^2-estimates for Galerkin methods for second order hyperbolic equations*, SIAM J. Numer. Anal., 10 (1973), pp. 880–889, https://doi.org/10.1137/0710073.

[14] J. C. Gilbert and P. Joly, *Higher Order Time Stepping for Second Order Hyperbolic Problems and Optimal CFL Conditions*, in Partial Differential Equations, Comput. Methods Appl. Sci. 16, Springer, Dordrecht, 2008, pp. 67–93, https://doi.org/10.1007/978-1-4020-8758-5_4.

[15] C. Godsil and G. Royle, *Algebraic Graph Theory*, Grad. Texts in Math. 207, Springer-Verlag, New York, 2001, https://doi.org/10.1007/978-1-4613-0163-9.

[16] M. J. Grote, A. Schneebeli, and D. Schötzau, *Discontinuous Galerkin finite element method for the wave equation*, SIAM J. Numer. Anal., 44 (2006), pp. 2408–2431, https://doi.org/10.1137/05063194X.

[17] M. J. Grote and D. Schötzau, *Optimal error estimates for the fully discrete interior penalty DG method for the wave equation*, J. Sci. Comput., 40 (2009), pp. 257–272, https://doi.org/10.1007/s10915-008-9247-z.

[18] P. Hansbo, M. G. Larson, and K. Larsson, *Cut finite element methods for linear elasticity problems*, in Geometrically Unfitted Finite Element Methods and Applications, Lect. Notes Comput. Sci. Eng. 121, Springer, Cham, 2017, pp. 25–63, https://doi.org/10.1007/978-3-319-71431-8_2.

[19] G. M. Hulbert and T. J. R. Hughes, *Space-time finite element methods for second-order hyperbolic equations*, Comput. Methods Appl. Mech. Engrg., 84 (1990), pp. 327–348, https://doi.org/10.1016/0045-7825(90)90082-W.

[20] A. Johansson and M. G. Larson, *A high order discontinuous Galerkin Nitsche method for elliptic problems with fictitious boundary*, Numer. Math., 123 (2013), pp. 607–628, https://doi.org/10.1007/s00211-012-0497-1.

[21] C. Johnson, *Discontinuous Galerkin finite element methods for second order hyperbolic problems*, Comput. Methods Appl. Mech. Engrg., 107 (1993), pp. 117–129, https://doi.org/10.1016/0045-7825(93)90170-3.

[22] P. Joly, *Variational methods for time-dependent wave propagation problems*, in Topics in

Computational Wave Propagation, Lect. Notes Comput. Sci. Eng. 31, Springer, Berlin, 2003, pp. 201–264, https://doi.org/10.1007/978-3-642-55483-4_6.

[23] A. Massing, M. G. Larson, A. Logg, and M. E. Rognes, *A stabilized Nitsche fictitious domain method for the Stokes problem*, J. Sci. Comput., 61 (2014), pp. 604–628, https://doi.org/10.1007/s10915-014-9838-9.

[24] P. Oswald, *On a BPX-preconditioner for* P1 *elements*, Computing, 51 (1993), pp. 125–133, https://doi.org/10.1007/BF02243847.

[25] L. R. Scott and S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493, https://doi.org/10.2307/2008497.

[26] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, Volume 30, Princeton University Press, Princeton, NJ, 1970.

[27] S. Sticko and G. Kreiss, *Higher order cut finite elements for the wave equation*, J. Sci. Comput., 80 (2019), pp. 1867–1887, https://doi.org/10.1007/s10915-019-01004-2.

[28] S. Sticko, G. Ludvigsson, and G. Kreiss, *High-order cut finite elements for the elastic wave equation*, Adv. Comput. Math., 46 (2020), 45, https://doi.org/10.1007/s10444-020-09785-z.

[29] V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, 2nd ed., Springer Ser. Comput. Math. 25, Springer-Verlag, Berlin, 2006.

[30] H. Wu and Y. Xiao, *An unfitted hp-interface penalty finite element method for elliptic interface problems*, J. Comput. Math., 37 (2019), pp. 316–339, https://doi.org/10.4208/jcm.1802-m2017-0219.

[31] S. R. Wu, *A priori error estimates for explicit finite element for linear elasto-dynamics by Galerkin method and central difference method*, Comput. Methods Appl. Mech. Engrg., 192 (2003), pp. 5329–5353, https://doi.org/10.1016/j.cma.2003.08.002.

[32] S. R. Wu, *Lumped mass matrix in explicit finite element method for transient dynamics of elasticity*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 5983–5994, https://doi.org/10.1016/j.cma.2005.10.008.