# SPECIAL SUBMANIFOLDS IN NEARLY KÄHLER 6-MANIFOLDS

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A dissertation submitted in partial fulfillment of the requirements for the degree of

**Doctor of Philosophy**

of

**University College London**.

March 2022

I, Benjamin Aslan confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

# Abstract

This thesis is on $J$-holomorphic curves and special Lagrangians of nearly Kähler manifolds, with a focus on nearly Kähler $\mathbb{CP}^3$. We consider the following four problems.

Firstly, we relate classical geometric properties of surfaces in four-manifolds to properties of their twistor lifts, based on the work of Eells, Salamon and Friedrich. This leads to the construction of deformation invariant quantities for $J$-holomorphic curves in certain twistor spaces, such as $\mathbb{CP}^3$ or the manifold of complete flags in $\mathbb{C}^3$. We give an example of how the twistor lift of the discriminant locus of a family of quadrics in $\mathbb{CP}^3$ performs a desingularisation.

Secondly, we introduce the class of transverse $J$-holomorphic curves in $\mathbb{CP}^3$, for which we define angle functions. It turns out that the angle functions essentially encode the geometry of the curve, which results in classification results for $J$-holomorphic curves with special geometric properties. We derive a system of PDEs for the angle functions which enables us to establish a Bonnet-type theorem for transverse $J$-holomorphic curves. By constructing toric moment-type maps we relate them to the theory of $\mathrm{U}(1)$ invariant minimal surfaces in $S^4$.

Thirdly, we consider the deformation problem for $J$-holomorphic curves in general nearly Kähler manifolds. We turn to infinitesimal deformations and show that they are eigensections of a twisted Dirac operator on the normal bundle of the curve. By solving this equation explicitly we show that homogeneous tori in $\mathbb{CP}^3$ and $S^6$ are rigid and compute the spectrum of the Dirac operator in these cases.

Lastly, we derive the structure equations for special Lagrangians in $\mathbb{CP}^3$. This yields a classification of totally geodesic special Lagrangians. By introducing moment maps we also classify all $\mathrm{SU}(2)$ invariant special Lagrangians in $\mathbb{CP}^3$ and provide new homogeneous examples.

# Impact Statement

This thesis is about six-dimensional spaces equipped with a special geometric structure called nearly Kähler structure. Whilst nearly Kähler six-manifolds have been studied since the 1970s, they have become a hot topic in the last two decades, after fundamental theorems about the structure of these manifolds were proven. However, there are still many open questions in the field and it is a typical approach in geometry to study a higher-dimensional space by the lower-dimensional objects, called submanifolds, it contains.

However, the submanifolds of nearly Kähler manifolds have been relatively unexplored, except for a few special cases. The key objective of this thesis is to address this issue by performing new studies of submanifolds in nearly Kähler six-manifolds, leading to new examples, new invariants and classification results using techniques from geometry, topology and analysis.

The submanifold theory of nearly Kähler manifolds is connected to various topics in geometry, such as the surface theory of Riemannian four-manifolds or calibrated geometry and exceptional holonomy.

Since the formulation of general relativity it has been clear that insights in differential geometry are important for advances in theoretical physics. Indeed, the connection to exceptional holonomy leads to exciting applications of nearly Kähler geometry in theoretical physics. For example, the two fields medallists Michael Atiyah and Edward Witten identify the nearly Kähler $\mathbb{CP}^3$, which is the focus of this thesis, as a model for spacetime in $M$-theory [AW02].

# Acknowledgements

First and foremost, I would like to thank my advisors Prof. Jason Lotay and Prof. Simon Salamon. Their constant advice, effort, patience and guidance have made the last four years a very pleasurable experience for me. The feedback of my examiners Dr. Lorenzo Foscolo and Prof. Luc Vrancken has helped me to improve the thesis. I am grateful for the many helpful discussions with Daniel Platt.

I would also like to thank my family, especially Andrea, Sara and Karla, my friends, in particular Simon, Makan, Prell, Yannic, Fiona, Anand, David, Mino, Sophie, Calvin and my girlfriend Simonetta for their great company and moral support.

# Contents

# Chapter 1

# Introduction

The six-dimensional sphere $S^6$ admits an almost complex structure $J$ arising from octonionic multiplication. This almost complex structure is orthogonal for the round metric on $S^6$. However, $J$ is not integrable and it is a famous open problem as to whether $S^6$ admits an integrable complex structure. In particular, $(S^6, J, g)$ is not a Kähler manifold, but since both $J$ and $g$ have a simple definition one expects another underlying geometric structure. Historically, this question led Gray to the definition of a *nearly Kähler* manifold [Gra70].

A nearly Kähler manifold has the weaker integrability condition that $\nabla J$ is anti-symmetric, we want to exclude Kähler manifolds from the definition so we also require $\nabla J \neq 0$. Nearly Kähler manifolds in dimension six share some important properties with $S^6$, they are Einstein with positive Einstein constant and their Riemannian cone is a torsion-free $G_2$-manifold.

Only six examples of compact simply-connected nearly Kähler manifolds are known: homogeneous structures on $S^6, S^3 \times S^3, \mathbb{CP}^3$ and on $\mathbb{F}_{1,2}(\mathbb{C}^3)$, the manifold of complete flags in $\mathbb{C}^3$. More recently, two inhomogeneous structures have been constructed on $S^3 \times S^3$ and $S^6$ by Foscolo and Haskins [FH17].

It is a general strategy in geometry to define invariants from submanifolds. In symplectic geometry, this approach has led to the construction of Gromov-Witten invariants. It has been suggested to construct invariants for special holonomy manifolds by appropriately counting their calibrated submanifolds, for example for special Lagrangians in Calabi-Yau manifolds [Joy02] or for associative submanifolds in $G_2$-geometry [Joy18]. In nearly Kähler geometry there is less need for invariants because only few compact six-dimensional examples are known.

The motivation comes from the idea that the particular examples of nearly Kähler manifolds can be understood better from their submanifolds. The most important submanifolds of nearly Kähler manifolds are $J$-holomorphic curves and special Lagrangians. Besides revealing the structure of the ambient nearly Kähler geometry, they are important from the Riemannian and special holonomy perspective, since they are minimal and have associative and coassociative cones respectively.

Nearly Kähler manifolds are neither complex nor symplectic, so many general statements about $J$-holomorphic curves or special Lagrangians do not apply. Most of the work on submanifolds of nearly Kähler manifolds has consequently been concerned with homogeneous ambient spaces, with a focus on the construction and classification of examples. The homogeneous structure on $S^6$ has received the most attention [Bry82b; BVW94; Vra03; Lot11b] but recently important results for the ambient space $S^3 \times S^3$, such as in [Bol+15] and also for $\mathbb{F}_{1,2}(\mathbb{C}^3)$ have been obtained [Sto20b; CV21].

The focus of this thesis is on $J$-holomorphic curves and special Lagrangians in nearly Kähler $\mathbb{CP}^3$. Previous work on this topic is Bryant's Weierstraß parametrisation of superminimal $J$-holomorphic curves in $\mathbb{CP}^3$ [Bry82a]. They are $J$-holomorphic with the additional property that they are also holomorphic for the usual complex structure on $\mathbb{CP}^3$. Later, Xu identified another special class of $J$-holomorphic curves in $\mathbb{CP}^3$ and showed that they are in correspondence with superminimal curves [Xu10]. Recently, Storm and Konstantinov showed that any superminimal surface parametrises the fibres of a unique special Lagrangian in $\mathbb{CP}^3$ [Sto20a; Kon17]. All of these classes of submanifolds are relatively well understood since they are amenable to techniques from complex geometry. Our interest lies in submanifolds that are transverse to these distinguished classes.

Some of our results also apply to other ambient spaces, for example to $\mathbb{F}_{1,2}(\mathbb{C}^3)$, which is as $\mathbb{CP}^3$ a twistor space over a four-manifold. Furthermore, we apply and motivate more general statements for the in some way simpler ambient space $S^6$.

Although $J$-holomorphic curves and special Lagrangians in a nearly Kähler manifold are generally not calibrated, we can still use methods that are typical in calibrated geometry. There is no hope of describing all special submanifolds of a given nearly Kähler manifold. The challenge is to impose additional conditions on the submanifold that are restrictive enough to allow for a classification, but not too restrictive to still admit a large class of examples.

We use a moving-frame setup to define angle functions for special submanifolds. Imposing conditions on the angle functions defines subclasses of special submanifolds and they usually translate into geometric conditions, such as being totally geodesic or flat, which are then classified.

Another method we use is to impose symmetry on the submanifolds. This reduces the analytic complexity of the problem, for example from a PDE to an ODE. In the classification of such submanifolds, moment-type maps will play an important role in the thesis. By this, we mean suitable contractions of nearly Kähler differential forms with Killing vector fields coming from the symmetry. The idea is to identify group invariant submanifolds as (being contained in) particular fibres of the moment map.

## Summary of Results

### Chapter 3

The idea of twistor theory is to relate the Riemannian geometry of a base manifold to the (almost) complex geometry of the twistor space, which is a fibre bundle over the base space. The relation between Riemannian and almost complex geometry also applies to submanifolds: via the Eells-Salamon correspondence [ES85], $J$-holomorphic curves in the twistor spaces $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}(\mathbb{C}^3)$ are in correspondence with minimal surfaces in $S^4$ and $\mathbb{CP}^2$. This is the background of chapter 3.

Just as in the symplectic case, the volume remains constant along deformations of $J$-holomorphic curves in a nearly Kähler manifold $M$ [Ver13]. In theorem 3.2.6, we refine this result for the case when $M$ is the twistor space of a four-manifold $N$ by introducing quantities representing horizontal and vertical volume, whose sum gives the volume of the curve, and show that each of them stays constant too. This is achieved by generalising Friedrich's Euler number formula [Fri84] to arbitrary immersions. This result has consequences for the moduli space of $J$-holomorphic curves in $M$. For example, families of tori that are not superminimal, i.e. with non-vanishing vertical volume, cannot degenerate to spheres.

### Chapter 4

By the Frenet-Serret formula, a curve $\gamma\colon (0,1) \to \mathbb{R}^3$ is essentially classified by two functions $\kappa, \tau\colon I \to \mathbb{R}$, which describe the torsion and curvature of the curve respectively. The complex version of this problem is to describe $J$-holomorphic curves in an almost complex manifold of dimension six by a reduced set of functions. By Cartan-Kähler theory, the curves are locally described by four functions of one variable. Two functions $\alpha_-, \alpha_+$ of two variables that satisfy a Laplace-type equation are parametrised by four functions of one variable too.

By using an appropriate adaption of frames, we distil two such $\mathbb{R}$-valued functions $\alpha_-$, and $\alpha_+$ for $J$-holomorphic curves in $\mathbb{CP}^3$ geometrically. These functions essentially classify the $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$ if $X$ is simply-connected. They satisfy the affine 2D Toda lattice equations for the Lie algebra $\mathfrak{sp}(2)$, see theorem 4.3.8. Furthermore, the first and second fundamental form can be expressed in terms of $\alpha_-$ and $\alpha_+$ directly. As an application, we classify flat $J$-holomorphic tori and $J$-holomorphic curves with vanishing or holomorphic second fundamental form.

Given a suitable action of a torus $\mathbb{T}^n$ on a symplectic manifold $M^{2n}$, the image of the moment map is always a convex polytope, which essentially classifies the symplectic manifold together with its action. For a nearly Kähler manifold $M^6$, no such theory exists. The nearly Kähler moment map studied by Russo and Swann [RS19] for the torus action is $\mathbb{R}$-valued.

For the example $M = \mathbb{CP}^3$, we construct a map $P\colon M \to \mathbb{R}^2$, using the functions $\alpha_-$ and $\alpha_+$, which resembles the symplectic moment map: Its image is a rectangle $\mathcal{R}$ which encodes information about the toric nearly Kähler geometry of $M$. The fibres of $P$ contain U(1) invariant $J$-holomorphic curves and degenerate over the boundary of $\mathcal{R}$ where they consist of distinguished U(1) invariant $J$-holomorphic curves in $\mathbb{CP}^3$. This is the content of theorem 4.6.17 and gives a twistor perspective on the study of U(1) invariant minimal surfaces in $S^4$ [Fer+92].

**Chapter 5**

One way to obtain new examples of $J$-holomorphic curves is to start from a given $J$-holomorphic curve and try to deform it. To investigate such deformations we study the infinitesimal deformation problem in chapter 5. We show that infinitesimal deformations of $J$-holomorphic curves in nearly Kähler manifolds are described by eigensections of a twisted Dirac operator $\bar{D}$ defined for sections of the normal bundle of the curve. This statement fits into the results obtained by Kawai for other submanifolds in special geometries [Kaw17]. As an application, we show in theorem 5.2.10 that homogeneous $J$-holomorphic tori in $S^6$ and $\mathbb{CP}^3$ are rigid up to automorphisms.

**Chapter 6**

For special Lagrangian submanifolds $L \subset \mathbb{CP}^3$ we introduce a function $L \to [0, \pi/4]$ which is a measure of the angle between $TL$ and the twistor fibres. If $\theta \equiv \pi/4$ then $L$ projects to a superminimal surface in $S^4$. By using a moving frame set-up similar to [Bry06b] we show that all totally geodesic special Lagrangians and those with $\theta = 0$ are homogeneous.

The group Sp(2) is the identity component of the group of automorphisms of $\mathbb{CP}^3$. We classify all special Lagrangian submanifolds on which an SU(2)-subgroup of Sp(2) acts in theorem 6.3.11. There are two examples for each of $\theta = 0$ and $\theta = \pi/4$ and one example that arises from the irreducible action of SU(2) on $S^3(\mathbb{C}^2)$ and satisfies $\theta = \frac{1}{2}\arccos(\frac{7\sqrt{2}}{5\sqrt{5}}) \approx 0.24$. In the end, we outline further directions, which includes two approaches to construct non-homogeneous compact special Lagrangians and results on U(2) moment maps on nearly Kähler manifolds.

The material of the author's paper [Asl21] is contained in chapter 2, chapter 3 and above all chapter 4.

# Chapter 2

# Background

This chapter aims to review results that are important throughout the thesis. More specific background material is explained at the beginning of each chapter. We outline important results in nearly Kähler geometry and discuss the nearly Kähler structure on $\mathbb{CP}^3$ in more detail. For context, we give an overview of different notions of special submanifolds in nearly Kähler geometry. At the end of this chapter, we give an account of the previous work on two-torus multi-moment maps.

## 2.1 Nearly Kähler Six-Manifolds

Consider an almost Hermitian manifold $(M, g, J)$ with the two-form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. If $\omega$ is covariant constant with respect to the Levi-Civita connection $\nabla$ then $(M, J, g)$ is a Kähler manifold. The tensor $\nabla\omega$ has generally four irreducible components [GH80]. By requiring only some of them to vanish one obtains different notions of an almost Hermitian manifold which satisfy weaker integrability conditions than a Kähler manifold. One of them is the class of nearly Kähler manifolds.

**Definition 2.1.1.** An almost Hermitian manifold is a nearly Kähler manifold if

$$(\nabla_\xi J)\xi = 0$$

for every vector field $\xi$ on $M$.

A nearly Kähler manifold which is not Kähler is called a strictly nearly Kähler manifold. The case $\dim(M) = 6$ is the one most studied in the literature. Firstly, it turns out that every nearly Kähler manifold in dimension two or four is automatically Kähler. Secondly, six-dimensional nearly Kähler manifolds are one of the building blocks of higher dimensional nearly Kähler manifolds. More precisely, the structure theorem, due to Nagy [Nag02], says that every strict and complete nearly Kähler manifold is locally a Riemannian product of homogenous nearly Kähler spaces, twistor spaces over quaternionic Kähler manifolds, and six-dimensional nearly Kähler manifolds.

Strictly nearly Kähler manifolds in dimension six are Einstein and their first Chern class vanishes [Gra76]. Furthermore, strictly nearly Kähler six-manifolds are intimately related to $G_2$-geometry. The Riemannian cone $C(M) = M \times \mathbb{R}^{>0}$ of a nearly Kähler manifold can be equipped with a three form $\varphi \in \Lambda^3(C(M))$ which turns $(C(M), \varphi)$ into a torsion-free $G_2$-manifold and induces the cone metric $\mathrm{dr}^2 + r^2 g$ [Bär93, Theorem 2]. Similarly, the sine cone of $M$ carries a nearly parallel $G_2$ structure [BG08].

There is a convenient characterisation of nearly Kähler manifolds using differential forms in dimension six. Let $(M, g, J, \omega)$ be a six-dimensional almost Hermitian manifold. Then $M$ is nearly Kähler if and only if there is a three-form $\psi = \operatorname{Re}\psi + i \operatorname{Im}\psi \in \Lambda^{3,0}(M)$ defining an SU(3)-structure and a constant $\mu \in \mathbb{R}$ satisfying

$$\begin{aligned} \mathrm{d}\omega &= 3\mu \operatorname{Re}\psi \\ \mathrm{d}\operatorname{Im}\psi &= -2\mu\omega \wedge \omega. \end{aligned}$$

In this thesis we are exclusively concerned with the case $\mu \neq 0$ which we make part of the definition of a nearly Kähler manifold. By rescaling, we assume $\mu = 1$.

Every nearly Kähler manifold carries a unique connection $\overline{\nabla}$ with skew-symmetric torsion and holonomy contained in SU(3), i.e. $\overline{\nabla} g = \overline{\nabla} J = \overline{\nabla} \psi = 0$ [Gra70]. Furthermore, a connected, simply-connected Riemannian six-manifold $(M^6, g)$ admits a real Killing spinor if and only if there is an almost complex structure $J$ on $M$ which turns $(M, g, J)$ into a nearly Kähler manifold [Gru90].

Examples of (compact) nearly Kähler manifolds are very scarce. In fact, there are only six known examples of compact simply-connected nearly Kähler manifolds.

**Proposition 2.1.2.** [But10, Theorem 1] If $M = G/H$ is a homogeneous strictly nearly Kähler manifold of dimension six, then $M$ is an element of the following list:

- $G = G_2$ and $H = \mathrm{SU}(3)$ such that $M = S^6$
- $G = S^3 \times S^3 \times S^3$ and $H = \{(g, g, g) \mid g \in S^3\}$ such that $M = S^3 \times S^3$
- $G = \mathrm{Sp}(2)$ and $H = S^1 \times S^3$ such that $M = \mathbb{CP}^3$
- $G = \mathrm{SU}(3)$ and $H = \mathbb{T}^2$ such that $M = \mathbb{F}_{1,2}(\mathbb{C}^3)$ is the manifold of complete complex flags of $\mathbb{C}^3$

In each case, the identity component of the group of nearly Kähler automorphisms is equal to $G$. There are infinitely many freely-acting finite subgroups of the automorphism group of the homogeneous nearly Kähler $S^3 \times S^3$ [CV15].

In addition, there are two known examples of compact, simply-connected nearly Kähler manifolds which are not homogeneous. They were constructed by Foscolo and Haskins via cohomogeneity one actions on $S^3 \times S^3$ and $S^6$ [FH17].

## 2.2 The Nearly Kähler Structure on $\mathbb{CP}^3$

The nearly Kähler structure on $\mathbb{CP}^3$ can be defined from the twistor fibration. For explicit computations it is convenient to define the nearly Kähler structure from the homogeneous space structure $\mathbb{CP}^3 = \mathrm{Sp}(2)/S^1 \times S^3$. Identify $\mathbb{H}^2$ with $\mathbb{C}^4$ via $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. This identification gives an action of Sp(2) on $\mathbb{C}^4$ which descends to $\mathbb{CP}^3$ and acts transitively on that space. The stabiliser of the element $(1, 0, 0, 0)$ is

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & q \end{pmatrix} \mid z \in S^1 \subset \mathbb{C}, \quad q \in S^3 \subset \mathbb{H} \right\}$$

which shows $\mathbb{CP}^3 = \mathrm{Sp}(2)/S^1 \times S^3$. Following [Xu10], consider the Maurer-Cartan form on Sp(2) which can be written in components as

$$\Omega_{MC} = \begin{pmatrix} i\rho_1 + j\overline{\omega_3} & -\frac{\overline{\omega_1}}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} \\ \frac{\omega_1}{\sqrt{2}} + j\frac{\omega_2}{\sqrt{2}} & i\rho_2 + j\tau \end{pmatrix}. \tag{2.2.1}$$

Since $\Omega_{MC}$ has values in $\mathfrak{sp}(2)$, the one-forms $\omega_1, \omega_2, \omega_3$ and $\tau$ are complex-valued and $\rho_1, \rho_2$ are real-valued. The equation

$$\mathrm{d}\Omega_{MC} + [\Omega_{MC}, \Omega_{MC}] = 0$$

implies the following differential identities for the components of $\Omega_{MC}$

$$\mathrm{d}\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = -\underbrace{\begin{pmatrix} i(\rho_2 - \rho_1) & -\overline{\tau} & 0 \\ \tau & -i(\rho_1 + \rho_2) & 0 \\ 0 & 0 & 2i\rho_1 \end{pmatrix}}_{A_\omega :=} \wedge \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} + \begin{pmatrix} \overline{\omega_2 \wedge \omega_3} \\ \overline{\omega_3 \wedge \omega_1} \\ \overline{\omega_1 \wedge \omega_2} \end{pmatrix}. \tag{2.2.2}$$

Let furthermore

$$\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} = \begin{pmatrix} i(\rho_2 - \rho_1) & -\overline{\tau} \\ \tau & -i(\rho_1 + \rho_2) \end{pmatrix},$$

so one obtains

$$\mathrm{d}\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} = -\begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} \wedge \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} + \begin{pmatrix} \omega_1 \wedge \overline{\omega_1} - \omega_3 \wedge \overline{\omega_3} & \omega_1 \wedge \overline{\omega_2} \\ \omega_2 \wedge \overline{\omega_1} & \omega_2 \wedge \overline{\omega_2} - \omega_3 \wedge \overline{\omega_3} \end{pmatrix}.$$

Let finally $\kappa_{33} = 2i\rho_1$ which satisfies $\mathrm{d}\kappa_{33} = -\omega_1 \wedge \overline{\omega_1} - \omega_2 \wedge \overline{\omega_2} + 2\omega_3 \wedge \overline{\omega_3}$. The nearly Kähler structure on $\mathbb{CP}^3$ is defined by declaring the forms $s^*(\omega_1), s^*(\omega_2)$ and $s^*(\omega_3)$ to be unitary $(1,0)$ forms for any local section $s$ of the bundle $\mathrm{Sp}(2) \to \mathbb{CP}^3$. The resulting almost complex structure and metric do not depend on the choice of $s$. The nearly Kähler forms $\omega, \psi$ are pullbacks of

$$\frac{i}{2}\sum_{i=1}^{3} \omega_i \wedge \overline{\omega}_i, \qquad -i\omega_1 \wedge \omega_2 \wedge \omega_3$$

respectively. The matrix $A_\omega$ is in fact the connection matrix of the nearly Kähler connection $\overline{\nabla}$ in the frame dual to the local co-frame $(\omega_1, \omega_2, \omega_3)$.

Note that the forms $(\omega_1, \omega_2, \omega_3, \tau)$ are complex-valued invariant forms on $\mathrm{Sp}(2)$ and as such they can be seen as elements in $\mathfrak{sp}(2)^\vee \otimes \mathbb{C}$. They span different root spaces with respect to the maximal torus $S^1 \times S^1$, see fig. 4.1.

In general, we will treat the nearly Kähler forms as basic forms on $\mathrm{Sp}(2)$. However, Killing vector fields typically have a simple expression in local coordinates. To contract the nearly Kähler forms on $\mathbb{CP}^3$ with Killing vector fields we pull back the local unitary $(1,0)$ forms $\omega_1, \omega_2, \omega_3$ on the chart $\mathbb{A}_0 = \{Z_0 \neq 0\}$ with the local section

$$s\colon \mathbb{A}_0 \to \mathrm{Sp}(2), \quad (1, Z_1, Z_2, Z_3) \mapsto \begin{pmatrix} h_1|Z|^{-1} & -\overline{h_1}^{-1}\overline{h_2}a \\ h_2|Z|^{-1} & a \end{pmatrix}.$$

Here,

$$|Z|^2 = 1 + |Z_1|^2 + |Z_2|^2 + |Z_3|^2, \quad h_1 = 1 + jZ_1, \quad h_2 = Z_2 + jZ_3, \quad a = (1 + \frac{|h_2|^2}{|h_1|^2})^{-1/2}.$$

This gives the following expressions for the pull-back

$$\begin{aligned} s^*\omega_1 &= \sqrt{2}|Z|^{-2}((\overline{Z_3} - \overline{Z_1}Z_2)\mathrm{d}Z_1 + (1 + |Z_1|^2)\mathrm{d}Z_2) \\ s^*\omega_2 &= \sqrt{2}|Z|^{-2}((-\overline{Z_2} - \overline{Z_1}Z_3)\mathrm{d}Z_1 + (1 + |Z_1|^2)\mathrm{d}Z_3) \\ s^*\omega_3 &= |Z|^{-2}(\mathrm{d}\overline{Z_1} - \overline{Z_3}\mathrm{d}\overline{Z_2} + \overline{Z_2}\mathrm{d}\overline{Z_3}). \end{aligned} \tag{2.2.3}$$

To show these formulae, note that the pullback of the Maurer-Cartan form via $s$ is

$$s^*(\Omega_{MC}) = \begin{pmatrix} \overline{h}_1|Z|^{-1} & \overline{h_2}|Z|^{-1} \\ -h_2h_1^{-1}a & a \end{pmatrix} \begin{pmatrix} \mathrm{d}(|Z|^{-1}h_1) & d(-\overline{h_1}^{-1}\overline{h_2}a) \\ \mathrm{d}(|Z|^{-1}h_2) & \mathrm{d}a \end{pmatrix}.$$

Combining this with eq. (2.2.1) yields

$$(is^*\rho_1 + js^*\overline{\omega_3}) = |Z|^{-2}(\overline{h_1}\mathrm{d}h_1 + \overline{h_2}\mathrm{d}h_2) + R$$
$$\frac{1}{\sqrt{2}a|Z|}(s^*\omega_1 + js^*\omega_2) = -h_2h_1^{-1}\mathrm{d}h_1 + \mathrm{d}h_2,$$

where $R$ is a real term. Equations 2.2.3 follow by splitting the quaternionic-valued differential forms on the right-hand side into their $\mathbb{C}$ and $j\mathbb{C}$ part.

## 2.3 Special Submanifolds

The motivation to study special submanifolds comes from the idea that nearly Kähler manifolds can be understood better from their submanifolds. Most of the work on submanifolds of nearly Kähler manifolds has been developed for specific ambient spaces. We outline notions of special submanifolds in different codimensions. This discussion is not exhaustive, we discuss submanifolds related to the SU(3)-structure and not those which are only defined in terms of the Riemannian metric, e.g. minimal or totally geodesic submanifolds.

Any **codimension one** submanifold $N$ carries an almost contact metric structure, given by $T \in \mathrm{End}(TN)$. If one equips $N$ with an orientation the shape operator $A$ is also an endomorphism of $TN$. For classical space-forms there is a rich class of hypersurfaces on which $A$ and $T$ commute. However, recently it has been shown that this ansatz does not produce interesting hypersurfaces for the homogeneous nearly Kähler manifolds. More precisely, the standard embeddings $S^2 \times S^3 \to S^3 \times S^3$ and $S^5 \to S^6$ are the only hypersurfaces in nearly Kähler manifolds where $T$ and $A$ commute [DL19; HYZ18].

There is another negative result in **codimension two**. From complex geometry one might expect numerous four-dimensional $J$ invariant submanifolds, but in fact there are none in a nearly Kähler manifold. This has been shown for compact nearly Kähler manifolds in [PS10] and for arbitrary nearly Kähler manifolds in [LVW20], which also includes an analogous statement for higher-dimensional nearly Kähler manifolds. We give another proof in dimension six which holds for a slightly more general class of SU(3)-structures.

**Proposition 2.3.1.** *A nearly Kähler six-manifold does not admit any four-dimensional $J$ invariant submanifolds.*

*Proof.* Assume that $N$ is such a manifold. Then $\psi|_N$ is a $(3,0)$-form on $N$, so it must vanish. Since $\mathrm{d}\,\mathrm{Im}\,\psi = -2\omega \wedge \omega$ this implies that $\omega \wedge \omega$ vanishes on $N$. This is a contradiction because four-dimensional $J$-inariant subspaces are calibrated by $\frac{1}{2}\omega \wedge \omega$. □

There is a weaker condition for submanifolds of $M$ to be compatible with the almost complex structure $J$. A submanifold $N \subset M$ is called *CR manifold* if it carries a $J$ invariant distribution $U$ such that $J(U^\perp)$ is orthogonal to $TN$, where $U^\perp$ denotes the orthogonal complement of $U$ in $TN$. There are many examples of CR-manifolds in dimension three and four and some subclasses of such submanifold have been classified, for example in [Ant18; Ant+19] and references therein.

Motivated by symplectic geometry one calls a submanifold $L \subset M$ Lagrangian if $\omega|_L = 0$. Because of the identity $\mathrm{d}\omega = 3 \operatorname{Re} \psi$ we have $\operatorname{Re} \psi \,|_L$. So $L$ is a special Lagrangian in the sense that it is calibrated by $\operatorname{Im} P$, even though this form is not closed. The cone of a special Lagrangian in $M$ is coassociative in the $G_2$ cone $C(M)$.

The vanishing condition of $\omega$ also yields a distinguished class of manifolds in **codimension four**. Such two-manifolds have not received much attention in their own right but are relevant for special Lagrangian submanifolds since any real-analytic two-manifold on which $\omega$ vanishes can locally be thickened to a special Lagrangian submanifold.

Apart from this, $J$-holomorphic curves are of much interest in nearly Kähler geometry. Let $X$ be a Riemann surface with complex structure $I$. A $J$-holomorphic curve is a map $\varphi\colon X \to M$ such that $\mathrm{d}\varphi \circ I = J \circ \mathrm{d}\varphi$. We call a two-dimensional submanifold which is $J$ invariant an embedded $J$-holomorphic curve.

Every real-analytic curve can locally be thickened to a $J$-holomorphic curve in a nearly Kähler manifold. To the author's knowledge, real curves, i.e. **codimension five** manifolds have not been studied on their own right in nearly Kähler geometry.

The focus of this thesis is the study of $J$-holomorphic curves and special Lagrangian submanifolds. In addition to revealing the structure of the ambient space they are interesting from a Riemannian perspective, as they are minimal submanifolds of an Einstein manifold. The other source of motivation comes from special holonomy. Nearly Kähler manifolds are related to $G_2$-geometry via the cone and sine-cone construction. Taking the cones of $J$-holomorphic curves and Lagrangian submanifolds then gives associative and coassociative submanifolds. One can also produce associative submanifolds of the nearly parallel $G_2$ sine cone from each of them [Kaw17].

## 2.4 Nearly Kähler Multi-Moment Maps

The interaction of symmetries and special submanifolds is an important theme of this thesis. Previous work has been concerned with actions of tori on nearly Kähler manifolds and we review the multi-moment map of Russo and Swann for two-torus symmetries [RS19].

Let $M^6$ be a compact strictly nearly Kähler manifold which admits an effective two-torus-action $\mathbb{T}^2 \curvearrowright M$ of automorphisms. Let $\xi_0$ and $\xi_1$ be a basis of $\mathfrak{t}^2$ and denote the corresponding infinitesimal symmetries by $K^{\xi_0}$ and $K^{\xi_1}$. Since the $K^{\xi_i}$ preserve $J$ we have

$$[K^{\xi_0}, K^{\xi_1}] = [JK^{\xi_0}, K^{\xi_1}] = [K^{\xi_0}, JK^{\xi_1}] = 0.$$

Consider the multi-moment map $\nu\colon M \to \mathbb{R}$, introduced in [RS19], given by $\nu = \omega(K^{\xi_0}, K^{\xi_1})$. Then

$$d\nu = \operatorname{Re}\psi(K^{\xi_0}, K^{\xi_1}, \cdot)$$

and let

$$M^* = \{x \in M \mid \mathrm{d}_x\nu \neq 0\} = \{x \in M \mid K^{\xi_0}(x) \text{ and } K^{\xi_1}(x) \text{ are lin. indep. over } \mathbb{C}\}.$$

The action of $\mathbb{T}^2$ on $M^*$ is free, which is proven in the same way as [RS19, Proposition 3.3].

For a general nearly Kähler manifold, there is no abstract existence result for compact $J$-holomorphic curves. However, in the presence of a two-torus symmetry, extrema of $\nu$ consist of $J$-holomorphic tori. The following is also proven in [Rus20, Theorem 2.3].

**Lemma 2.4.1.** *The set $M \setminus M^*$ is a union of fixed points of the $\mathbb{T}^2$ action and embedded $J$-holomorphic curves on which $\mathbb{T}^2$ acts. The curves are tori if $\nu \neq 0$ and totally geodesic spheres or tori if $\nu = 0$. Every nearly Kähler manifold with a two-torus symmetry has at least two $J$-holomorphic curves which are orbits of the torus action.*

*Proof.* Note that $\mathrm{d}_x\nu = 0$ if and only if $K^{\xi_0}_x$ and $K^{\xi_1}_x$ are linearly dependent over $\mathbb{C}$. If $\nu(x) \neq 0$ then this means that $\operatorname{span}_{\mathbb{R}}(K^{\xi_0}, K^{\xi_1})$ is two-dimensional and invariant under $J$. This condition is invariant under the torus action, so the orbit will be a $J$-holomorphic curve since the tangent space of the orbit is spanned by $K^{\xi_0}$ and $K^{\xi_1}$. If $\nu(x) = 0$ then $K^{\xi_0}$ and $K^{\xi_1}$ are linearly dependent over $\mathbb{R}$, say $aK^{\xi_0}(x) + bK^{\xi_1}(x) = 0$. The vector field $aK^{\xi_0} + bK^{\xi_1}$ also preserves the nearly Kähler structure and so its zeros are unions of totally geodesic, $J$-holomorphic curves. They must be homeomorphic to spheres or tori since they admit a cohomogeneity one action. It has been shown in [RS19] that the image of $\nu$ is a closed interval $[\nu_{\min}, \nu_{\max}]$ which contains 0. Hence, the orbit of any point in $\nu^{-1}(\nu_{\min})$ or $\nu^{-1}(\nu_{\max})$ is a $J$-holomorphic torus. □

In the general setting, $\nu$ has the distinguished values 0 as well as its minimum $\nu_{\min}$ and maximum $\nu_{\max}$. The set $S$ has been computed by Russo for all homogeneous nearly Kähler manifolds. Furthermore, it has been shown that $S$ always projects to a trivalent graph under the map $M^6 \to M^6/\mathbb{T}^2$. The vertices in this graph are fixed points of the torus action, the edges consist of points with one-dimensional stabilisers [Rus20].

The focus of this thesis is on $\mathbb{CP}^3$, and the torus action on that space will be studied in detail in section 4.6.2. In particular, we will see that $\nu^{-1}(\nu_{\max})$ and $\nu^{-1}(\nu_{\min})$ both consist of Clifford tori.

# Chapter 3

# Properties of Twistor Lifts

One important construction of nearly Kähler manifold comes from the twistor theory of quaternionic Kähler manifolds. They are one of the building blocks of Nagy's structure theorem of higher-dimensional nearly Kähler manifolds. In four dimensions, the twistor space $Z$ is a $\mathbb{CP}^1$ fibre bundle over the base manifold. In fact, $Z$ carries two canonical almost complex structures and a family of Riemannian metrics. The homogeneous nearly Kähler manifold structures on $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}(\mathbb{C}^3)$ arise as twistor spaces of $S^4$ and $\mathbb{CP}^2$.

In the first section, we follow mainly [ES85] to review the twistor theory of four-manifolds and the Eells-Salamon correspondence of $J$-holomorphic curves in the twistor space and minimal surfaces in the base manifold.

In section 3.2 we establish how classical geometric quantities of surfaces in four-manifolds are related to components of the twistor lift of that surface. These computations yield a generalisation of Friedrich's formula in [Fri84] for the Euler number of a surface in a self-dual Einstein manifold. It has been shown in [Ver13] that the volume of $J$-holomorphic curves is constant along smooth deformations. In theorem 3.2.6 we refine this result for nearly Kähler twistor spaces by defining a horizontal and vertical volume for $J$-holomorphic curves and show that they are also invariant.

We interpret the transformation of $J$-holomorphic curves in $\mathbb{CP}^3$ found in [Xu10] in terms of the twistor fibration in section 4.2.

The focus of section 3.3 is on the conventional complex structure $J_1$. The study of algebraic varieties in $\mathbb{CP}^3$ dates back more than a century. Recently, understanding algebraic varieties with respect to the twistor fibration $\mathbb{CP}^3 \to S^4$ has attracted a lot of interest, see for example [SV09; GSS14; AB19; AB20]. We study a specific one-parameter family of quadrics $\mathcal{Q}_\lambda$ with discriminant locus diffeomorphic to a real circular cone. In theorem 3.3.5 we show that its twistor lift desingularises the cone to a two-torus lying in $\mathcal{Q}_\lambda$.

## 3.1 Twistor Spaces

To each even-dimensional Riemannian manifold $N$ one can associate a twistor space $Z_\pm(N)$, which is a fibre bundle over $N$. The fibre of this bundle over $x$ is given by

$$\begin{aligned} Z_\pm(N) = \{J_x \colon T_xN \to T_xN \mid J_x^2 = -1, \quad g(J_xv, J_xw) = g(v,w) \\ J_x \text{ preserves/reverses orientation}\}. \end{aligned} \tag{3.1.1}$$

Here we refer to orientation preserving or orientation reversing as the sign of the Pfaffian of $J_x$ in a local frame as $\det(J)$ is always positive for an almost complex structure. We shall restrict to the case when $N$ is a four manifold.

The twistor bundle can be constructed as an associated bundle of the SO(4)-frame bundle $P_{\mathrm{SO}(4)}$ of $N$. Let $\mathrm{U}(2)_-$ and $\mathrm{U}(2)_+$ be the stabiliser of the almost complex structure

$$J_\pm = e_1 \mapsto e_2, \quad e_2 \mapsto -e_1, \quad e_3 \mapsto \pm e_4, \quad e_4 \mapsto \mp e_3$$

in SO(4). Then $J_+$ preserves and $J_-$ reverses the orientation on $\mathbb{R}^4$ in the sense above and we have

$$Z_\pm = P_{\mathrm{SO}(4)} \times_{\mathrm{SO}(4)} \mathrm{SO}(4)/\mathrm{U}(2)_\pm.$$

For now fix the negative orientation for the twistor space and simply write $Z = Z_-(N^4)$. In dimension four, the twistor space can be described explicitly as a subbundle of $\Lambda^2_-(T^\vee N)$. Let $J$ be an almost complex structure on $\mathbb{R}^4$ and let $\{e_1, e_2, e_3, e_4\}$ be an orthonormal basis in which $J(e_1) = e_2$ and $J(e_3) = e_4$. The corresponding two-form is $\frac{1}{2}(e^1 \wedge e^2 - e^3 \wedge e^4)$, which is a self-dual two-form and of norm one (in the appropriate convention). Since any self-dual two form of norm one is of the form $\frac{1}{2}(e^1 \wedge e^2 \pm e^3 \wedge e^4)$ for some basis and this assignment is SO(4) invariant, we obtain that $Z$ is a sphere bundle inside $\Lambda^2_-(T^\vee N)$.

Let $E = \Lambda^2_-(T^\vee N)$ and $\pi_E$ be the bundle projection $E \to N$. For $y \in E$ with $\pi_E(y) = x \in N$ the vertical subspace $\mathcal{V}_E$ at $y$ is the kernel of $\mathrm{d}_y\pi \colon T_yE \to T_xN$. The map

$$E_x \to T_yE, \quad v \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\,|_{t=0}\,(y + vt)$$

is an isomorphism. Furthermore, $E$ is equipped with the Levi-Civita connection which gives rise to a splitting $T_yE = \mathcal{H}_y \oplus E_x$. For $x \in N$ and $y \in \pi_E^{-1}(x)$ the horizontal space $\mathcal{H}_y$ is expressed in terms of the covariant derivative by

$$\mathcal{H}_y = \{\mathrm{d}s(X) - \nabla_X s \mid X \in T_xN\} \tag{3.1.2}$$

for any local section $s \colon U \to E$. In particular, $\nabla_X s$ is equal to the projection of $\mathrm{d}s(X)$ onto the vertical subspace. Since $\nabla$ is a metric connection $\mathcal{H}_y \subset TZ_y$ and because $Z_y$

is a sphere bundle in $E$

$$TZ_y = \mathcal{H}_y \oplus \{v \in E_x \mid g_E(v, y) = 0\} \tag{3.1.3}$$

where $g_E$ denotes the induces metric on $E$. We declare the second summand to be the vertical subbundle of $Z \to N$, so we have

$$TZ = \mathcal{H} \oplus \mathcal{V}. \tag{3.1.4}$$

The almost complex structures will be defined with respect to this decomposition. Since $\mathcal{H}_y \cong T_xN$ and $y$ is an almost complex structure on the latter space, we obtain an almost complex structure on $\mathcal{H}$. We now study the vertical space on the bundle $Z$. Let $y \in Z$, which is an almost complex structure on $T_xN$ and hence gives rise to a splitting

$$\Lambda^2(T_xN) = \underbrace{[\![(\Lambda^{2,0} \oplus \Lambda^{0,2}]\!] \oplus y\mathbb{R}}_{\Lambda^2_-} \oplus \underbrace{\Lambda_0^{1,1}}_{\Lambda^2_+}. \tag{3.1.5}$$

Note that this differs by sign from [ES85, Proposition 3.4] since $y$ takes values in the negative twistor space here. Since the splitting eq. (3.1.5) is orthogonal with respect to $g_E$, eq. (3.1.3) gives the identification

$$\mathcal{V}_y = [\![(\Lambda^{2,0} \oplus \Lambda^{0,2}]\!]. \tag{3.1.6}$$

Here $[\![\cdot,\cdot]\!]$ denotes the underlying real vector space, as in [Sal89]. In other words, $\mathcal{V}_y \otimes \mathbb{C} = \Lambda^{2,0} \oplus \Lambda^{0,2}$ and we get two different almost complex structures on $\mathcal{V}_y$ by either declaring $\Lambda^{2,0}$ or $\Lambda^{0,2}$ to be the $+i$ eigenspace. The first choice is called $J_1$ or the Atiyah-Hitchin-Singer almost complex structure and the second choice $J_2$ or the Eells-Salamon almost complex structure.

There is a neat relationship between the twistor and spinor bundles on a four-manifold, which we outline briefly following [Sal85, section 8]. Recall that $\mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Let $\mathbb{H}_-$ denote the representation where the first and $\mathbb{H}_+$ where second factor of $\mathrm{Sp}(1)$ acts from the right on $\mathbb{H}$. Regarding $\mathbb{H}$ as a complex two-dimensional vector space where scalars act my left multiplication gives

$$\mathbb{H}_- \otimes_{\mathbb{C}} \mathbb{H}_+ \cong \mathbb{C}^4.$$

Here $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts via the complexified action of $\mathrm{SO}(4)$ on $\mathbb{R}^4$ and the real structure on the left-hand side is $a \otimes b \mapsto ja \otimes jb$. Consequently, there is a bundle isomorphism

$$TN \otimes \mathbb{C} \cong \Delta_- \otimes \Delta_+$$

where $\Delta_-$ and $\Delta_+$ denote the spinor bundles associated to $\mathbb{H}_-$ and $\mathbb{H}_+$. It turns out that a choice of a vector in $\Delta_-$ defines a maximally isotropic subspace in $TN$ which then gives that $\mathbb{P}(\Delta_\pm) = Z_\pm(N)$, i.e. the twistor bundle is the projectivised spinor bundle on a four-manifold. As such, there is a tautological complex line bundle $\zeta$ on $Z_\pm$. The almost complex structures can then be formulated in terms of the splitting eq. (3.1.4) and the $(1,0)$ subbundle as

$$T^{1,0}Z_\pm = \begin{cases} \bar{\zeta}\Delta_\mp \oplus \bar{\zeta}^2 \text{ for } J_1 \\ \bar{\zeta}\Delta_\mp \oplus \zeta^2 \text{ for } J_2 \end{cases}.$$

From this characterisation, it is shown that the first Chern class of $(Z, J_2)$ always vanishes [Sal85, Proposition 8.1.]. In contrast, the first Chern class of the almost complex structure $J_1$ does not vanish in general. The two almost complex structures are not only fundamentally different from a topological point of view but also differ with respect to their integrability.

**Proposition 3.1.1.** [Sal85, Theorem 3.3, Proposition 3.4] and [AHS78, Theorem 4.1] Let $N$ be a Riemannian four-manifold, then

- $(Z(N), J_1)$ is a complex manifold if and only if $N$ is self-dual. The almost complex structure $J_1$ only depends on the conformal class of the metric on $N$.
- $(Z(N), J_2)$ is never integrable.

This result fits into the philosophy of twistor geometry, to relate the Riemannian geometry of $N$ to the (almost) complex geometry of $Z$.

Similarly, the twistor space $Z$ carries a canonical family of Riemannian metrics

$$g_\lambda = g_{\mathcal{H}} \oplus \lambda g_{\mathcal{V}} \tag{3.1.7}$$

for $\lambda > 0$, i.e. the splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ is orthogonal. Since $N$ carries a metric one obtains $g_{\mathcal{H}}$ on $\mathcal{H}$ via $\mathcal{H} \cong \pi^*(TN)$. The metric $g_{\mathcal{V}}$ comes from the metric on $\Lambda^2_-(N)$ and the identification eq. (3.1.6). The normalisation is chosen such that the two-form $\frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4)$ has norm one.

The focus of this thesis is on the case when there is a $\lambda > 0$ such that $(J_2, g_\lambda)$ equips $Z$ with a nearly Kähler metric. For compact four-manifolds this only happens for $N = \mathbb{CP}^2$ and $N = S^4$ with their standard metric. The point is that one needs the base to be quaternionic-Kähler, which translates in dimension four to Einstein and self-dual [Hit81]. In each of these two cases, there are unique values $\lambda_1, \lambda_2$ such that $(g_{\lambda_1}, J_1)$ is the Kähler and $(g_{\lambda_2}, J_2)$ the nearly Kähler structure. The normalisations are chosen in such a way that for $N = S^4$ we have $\lambda_2 = 1$. The splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ is parallel with respect to $\overline{\nabla}$ when $N = S^4$ or $\mathbb{CP}^2$.

To give more details on the twistor fibration $\mathbb{CP}^3 \to S^4$ consider the quaternionic projective line $\mathbb{HP}^1$. Note that since $\mathbb{H}$ is non-commutative, there are two possible conventions to define it, we stick to the one where two non-zero vectors $v, v' \in \mathbb{H}^2$ are identified when $v' = vq$ for a quaternion $q$. Let

$$\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1, \quad [Z_0, Z_1, Z_2, Z_3] \mapsto [Z_0 + jZ_1, Z_2 + jZ_3].$$

In fact, one can identify $\mathbb{HP}^1 = \mathbb{H} \cup \{\infty\} = S^4$ via the stereographic projection or via the double cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. This makes it possible to consider $\pi$ as the twistor fibration of by the means of

$$\begin{aligned} Z_-(S^4) &\cong \mathrm{SO}(5) \times_{\mathrm{SO}(4)/\mathrm{U}(2)_-} \cong \mathrm{Sp}(2)/(\mathrm{U}(1) \times \mathrm{Sp}(1)) \cong \mathbb{CP}^3 \\ Z_+(S^4) &\cong \mathrm{SO}(5) \times_{\mathrm{SO}(4)/\mathrm{U}(2)_+} \cong \mathrm{Sp}(2)/(\mathrm{Sp}(1) \times \mathrm{U}(1)) \cong \mathbb{CP}^3. \end{aligned} \tag{3.1.8}$$

Since the subgroups $\mathrm{Sp}(1) \times \mathrm{U}(1)$ and $\mathrm{U}(1) \times \mathrm{Sp}(1)$ are conjugate to each other the spaces $Z_-(S^4)$ and $Z_+(S^4)$ are not only diffeomorphic (to $\mathbb{CP}^3$) but also isomorphic as bundles over $S^4$. This is fact is not true for general four manifolds and we will see that it is responsible for symmetries in equations describing $J$-holomorphic curves in $\mathbb{CP}^3$.

Regarding $\mathbb{CP}^3$ as a quotient $\mathbb{H}^2 \setminus \{0\}/\mathbb{C}^*$, right multiplication by $j$ is a map $\mathbb{CP}^3 \to \mathbb{CP}^3$ whose square equals $-1$. We will denote this map by $j$ as well. In homogeneous coordinates it is given by $[Z_0, Z_1, Z_2, Z_3] \mapsto [-\overline{Z_1}, \overline{Z_0}, -\overline{Z_3}, \overline{Z_2}]$. This map encodes the twistor structure of $\mathbb{CP}^3$. Given $x \in \mathbb{CP}^3$, the twistor line $\pi^{-1}(\pi(x))$ is the unique projective line through $x$ and $jx$.

For any immersion $f\colon X \to N^4$ the differential $\mathrm{d}f$ defines a lift, called the Gauss lift, $\hat{\varphi}$ from $X$ into the oriented Grassmannian bundle $\widetilde{\mathrm{Gr}}_2(TN)$. This bundle in turn projects to $Z$ such that by composition with $\hat{\varphi}$ one obtains a map $\varphi\colon X \to Z(N)$ which is called the twistor lift of $f$

$$\begin{tikzcd} & & \widetilde{\mathrm{Gr}}_2(TN) \arrow[dll] \arrow[dd] \\ Z(N) \arrow[drr] & & \\ X \arrow[u, "\varphi"] \arrow[uurr, "\hat{\varphi}"] \arrow[rr, "f"] & & N \end{tikzcd} .$$

Proposition 3.1.1 establishes a relationship between the Riemannian geometry of $N$ and (almost) complex geometry of $Z(N)$. The Eells-Salamon correspondence deepens this connection by relating minimal surfaces in $N$ with $J_2$-holomorphic curves in $Z(N)$.

**Proposition 3.1.2** (Eells-Salamon)**.** [ES85, Corollary 5.4] Let $X$ be a Riemann surface and $N$ be a four-dimensional Riemannian manifold. Then $f\colon X \to N$ is a minimal branched immersion if and only if $\varphi\colon X \to Z(N)$ is a $J_2$-holomorphic non-vertical curve.

Note that if $f\colon X \to N$ is a branched minimal immersion, i.e. a minimal immersion off a discrete set of points, then there is a rank two subbundle of $f^*(TN)$ which contains $\mathrm{d}f(TX)$ so the Gauß lift is still well-defined in this case. Since the domain is two-dimensional, a branched minimal immersion is the same thing as a conformal harmonic map. The following lemma is contained in the proof of proposition 3.1.2, we summarise the argument of the proof.

**Lemma 3.1.3.** *If $\varphi$ is either $J_1$ or $J_2$ holomorphic and nowhere vertical then the twistor lift of $\pi \circ \varphi$ is $\varphi$.*

*Proof.* Observe that a two-dimensional subspace $V \subset T_xN$ is a complex subspace with respect to the almost complex structure $y \in \pi^{-1}(x)$ if and only if $y$ is the twistor lift of $V$. If $W$ is a two-dimensional subspace $T_yZ$ invariant under $J_1$ or $J_2$ then if $W$ is not vertical, the horizontal projection $W_{\mathcal{H}} \subset \mathcal{H}_y$ is also complex linear and hence $V = \mathrm{d}\pi(W_{\mathcal{H}})$ is complex linear with respect to $y$. This means that the twistor lift of $V$ is in fact $y$. $\square$

Via this correspondence, branched minimal surfaces in $S^4$ are in one-to-one correspondence with $J_2$-holomorphic curves in the nearly Kähler twistor space $\mathbb{CP}^3$. The splitting $TZ = \mathcal{H} \oplus \mathcal{V}$ also distinguished a special class of minimal immersions in $N$.

**Definition.** A branched minimal immersion in $N$ is called *superminimal* if the twistor lift $\varphi$ is always tangent to $\mathcal{H}$.

For $N = S^4$, Bryant found a Weierstraß parametrisation for them: each of them is a projective line or is parametrised by

$$\Theta(f,g) = [1, f - \frac{1}{2}g\Big(\frac{\mathrm{df}}{\mathrm{dg}}\Big), g, \frac{1}{2}\Big(\frac{\mathrm{d}f}{\mathrm{d}g}\Big)] \tag{3.1.9}$$

for $f, g$ meromorphic functions on $X$ with $g$ being non-constant [Bry82a, Theorem F].

## 3.2 Encoding the Second Fundamental Form

A smooth embedded minimal surface $X \subset S^n$ satisfies

$$1 - K = \frac{1}{2}\|\mathrm{I\!I}_{S^n}\|^2 \tag{3.2.1}$$

by the Gauß equation. Smoothly deforming a minimal surface *along minimal surfaces* leaves the volume invariant, since minimal surfaces are stationary for the volume functional. Integrating eq. (3.2.1) gives that $\int_X \|\mathrm{II}_{S^n}\|^2$ stays invariant under deformations of minimal surfaces in $S^n$.

**Example 3.2.1.** *The space $\mathcal{M}_0$ of totally geodesic $J$-holomorphic curves in $S^6$ is diffeomorphic to $G_2/\mathrm{SO}(4)$. This space is a symmetric space known not to admit an almost complex structure. Any smooth deformation of elements in $\mathcal{M}_0$ as $J$-*holomorphic curves *stays in $\mathcal{M}_0$. Without giving a formal definition of a moduli space, $\mathcal{M}_0$ can be viewed as one connected component of the moduli space of $J$-holomorphic curves.*

If $M$ is a nearly Kähler manifold which is not the six-sphere the Gauß equation involves a more complicated curvature term, so one cannot define a deformation invariant from the second fundamental form in the same way. In this section, we construct such invariants for the case when $M$ is a nearly Kähler twistor space, i.e. $\mathbb{CP}^3$ or $\mathbb{F}_{1,2}(\mathbb{C}^3)$. $J$-holomorphic curves in these spaces are in one-to-one correspondence with minimal surfaces in $S^4$ and $\mathbb{CP}^2$. We start with a more general setup of general immersions of surfaces in self-dual Einstein spaces.

The almost complex and Riemannian structure on $Z_\pm(N)$ and the twistor lift have been defined invariantly in the previous section. For computational purposes, we use the ad-hoc setup of [Fri84] in this section. However, our focus is on $J_2$ instead of $J_1$. Since the second fundamental form will be considered in different ambient spaces, we use the notation $\mathrm{II}_Y$ where $Y$ is the ambient space or an ambient bundle.

Let $(N, g)$ be a Riemannian four-manifold and $f\colon X^2 \to N$ be an isometric immersion. The second fundamental form $\mathrm{II}_N$ encodes local geometric information about $f$. The aim of this section is to relate the norm of $\mathrm{II}_N$ to quantities defined for the twistor lift of $f$.

Let $\nu$ be the normal bundle of $TX$ in $f^*(TN)$. Locally, fix an oriented orthonormal frame $\{e_1, e_2, e_3, e_4\}$ such that $\{e_1, e_2\}$ is an oriented basis of $f^*(TX)$. Let $I$ be the (integrable) almost complex structure on $X$, in the frame given by $e_1 \mapsto e_2$. Denote by $\sigma_{ij}(v) = g(\nabla_v e_i, e_j)$ the locally defined connection one-forms of the Levi-Civita connection $\nabla$ on $N$. Furthermore, let $\sigma_{ijk} = \sigma_{ij}(e_k)$. Then $\sigma_{ijk} = -\sigma_{jik}$ and if $i, k \in \{1, 2\}$ and $j \in \{3, 4\}$ then $\sigma_{ijk} = \sigma_{kji}$ holds by torsion-freeness. The Riemann curvature tensor is expressed as $R_{ij} = \mathrm{d}\sigma_{ij} + \sum_k \sigma_{jk} \wedge \sigma_{ki}$ and we let $R_{ijkl} = R_{ij}(e_k, e_l)$.

As in the previous subsection, the term twistor space will refer to the negative twistor space, unless specified otherwise. The vertical component of the twistor lift is given as follows [Fri84]

$$(\mathrm{d}\varphi_-)^{\mathcal{V}} = \frac{\sigma_{13} + \sigma_{24}}{2} y_5 + \frac{\sigma_{14} - \sigma_{23}}{2} y_6. \tag{3.2.2}$$

Here $\{y_5, y_6\}$ is a local orthonormal basis for the metric $g_\mathcal{V}$ on the vertical space of $Z$. This agrees with the invariant definition of $g_\mathcal{V}$ given in the previous section. One can regard $(\mathrm{d}\varphi)^\mathcal{V}$ as a one-form with values in $\mathrm{Hom}_\mathbb{C}(TX, \nu)$. Under this identification,

$$y_5(e_1, e_2, e_3, e_4) = (e_3, e_4, -e_1, -e_2), \quad y_6(e_1, e_2, e_3, e_4) = (e_4, e_3, -e_2, -e_1).$$

The almost complex structures $J_1$ and $J_2$ act by $J_1(y_5, y_6) = (-y_6, y_5)$ and $J_2(y_5, y_6) = (y_6, -y_5)$ on the vertical bundle. Let $f\colon X \to N$ be an immersion. Changing between $\varphi_-$ and $\varphi_+$ amounts to changing the orientation of $N$, i.e. swapping the indices $3 \leftrightarrow 4$. This gives the analogous formula

$$(\mathrm{d}\varphi_+)^\mathcal{V} = \frac{\sigma_{14} + \sigma_{23}}{2} y_5 + \frac{\sigma_{13} - \sigma_{24}}{2} y_6. \tag{3.2.3}$$

From the definition of the twistor lift, the horizontal component of $\varphi$ is $J_a$-holomorphic for $a = 1, 2$ in the sense that

$$J_a((\mathrm{d}\varphi)^\mathcal{H}) = (\mathrm{d}\varphi)^\mathcal{H} \circ I$$

which means that $\varphi$ is $J_a$ holomorphic if and only if

$$J_a((\mathrm{d}\varphi)^\mathcal{V}) = (\mathrm{d}\varphi)^\mathcal{V} \circ I.$$

It follows from eq. (3.2.2) and eq. (3.2.3) that $(\mathrm{d}\varphi)^\mathcal{V}$ is $J_2$ holomorphic if and only if

$$\begin{aligned} \sigma_{131} + \sigma_{232} &= 0 \\ \sigma_{141} + \sigma_{242} &= 0. \end{aligned}$$

Changing the indices $(3, 4) \leftrightarrow (4, 3)$ amounts to changing the orientation of $N$ and hence changing between $Z_+ \leftrightarrow Z_-$. We recover proposition 3.1.2, namely

$$\varphi_+ \text{ is } J_2\text{-holomorphic} \quad \Leftrightarrow \quad \varphi_- \text{ is } J_2\text{-holomorphic} \quad \Leftrightarrow \quad \varphi \text{ is minimal.}$$

The norms of the two twistor lifts $\|\mathrm{d}\varphi_-^\mathcal{V}\|$ and $\|\mathrm{d}\varphi_+^\mathcal{V}\|$ contain information about the ambient geometry of $X$ in $N$. As a consequence of eq. (3.2.2) and eq. (3.2.3) they are related to the second fundamental form by

$$\begin{aligned} 4\|\mathrm{d}\varphi_-^\mathcal{V}\|^2 &= (\sigma_{131} + \sigma_{241})^2 + (\sigma_{132} + \sigma_{242})^2 + (\sigma_{141} - \sigma_{231})^2 + (\sigma_{142} - \sigma_{232})^2 \\ 4\|\mathrm{d}\varphi_+^\mathcal{V}\|^2 &= (\sigma_{141} + \sigma_{231})^2 + (\sigma_{142} + \sigma_{232})^2 + (\sigma_{131} - \sigma_{241})^2 + (\sigma_{132} - \sigma_{242})^2. \end{aligned} \tag{3.2.4}$$

These formulas imply

$$\|\mathrm{d}\varphi_-^{\mathcal{V}}\|^2 + \|\mathrm{d}\varphi_+^{\mathcal{V}}\|^2 = \frac{1}{2}\|\mathrm{I\!I}_N\|^2. \tag{3.2.5}$$

The sum $\|\mathrm{d}\varphi_-^{\mathcal{V}}\|^2 + \|\mathrm{d}\varphi_+^{\mathcal{V}}\|^2$ is also equal to the norm of the vertical component of the lift of $f$ into the Grassmannian bundle $\mathrm{Gr}_2(TN)$. In particular, $f$ is totally geodesic in a point if and only if both twistor lifts are horizontal.

Let $H$ and $K$ be the mean curvature vector and Gauß curvature and let

$$G = \sigma_{131}\sigma_{232} - \sigma_{132}^2 + \sigma_{141}\sigma_{242} - \sigma_{142}^2$$

which is independent of the chosen frame. Denote by $K^N$ the induced curvature on the normal bundle and note that the component $R_{1234}$ of the Riemann curvature tensor does not depend on the choice of $\{e_1, \dots, e_4\}$. These quantities satisfy the relationship

$$\pm K^N = 2(-|H|^2 + \|\varphi^{\mathcal{V}_\mp}\|^2) + G \mp R_{1234}. \tag{3.2.6}$$

This equation can be checked explicitly by expressing everything in a local frame

$$\begin{aligned} K^N &= \mathrm{d}\sigma_{43}(e_1, e_2) = -(\sigma_{31} \wedge \sigma_{14} + \sigma_{32} \wedge \sigma_{24})(e_1 \wedge e_2) - R_{1234} \\ &= \sigma_{142}(\sigma_{131} - \sigma_{232}) + \sigma_{132}(-\sigma_{141} + \sigma_{242}) - R_{1234} \\ |H|^2 &= \frac{1}{4}((\sigma_{131} + \sigma_{232})^2 + (\sigma_{141} + \sigma_{242})^2). \end{aligned} \tag{3.2.7}$$

Sometimes, $\mathrm{I\!I\!I}(v, w) = g(\mathrm{I\!I}(v), \mathrm{I\!I}(w))$ is called the third fundamental form of a surface. Note that $\mathrm{Tr}(\mathrm{I\!I\!I}) = \|\mathrm{I\!I}\|^2$ and if $N = \mathbb{R}^4$ we have $R_{1234} = 0$ and $G = 1$ by the Gauß equation. So eq. (3.2.6) simplifies to

$$0 = \mathrm{Tr}(-2H\mathrm{I\!I}_N + \mathrm{I\!I\!I}_N + K\mathrm{I}_N).$$

In fact, there is a well known relation of the three fundamental forms for surfaces in $\mathbb{R}^3$ [ASG17, p. 402]

$$\mathrm{I\!I\!I} = 2\mathrm{I\!I}H - \mathrm{I}K.$$

One consequence of eq. (3.2.6) is that

$$K^N + R_{1234} = \|\mathrm{d}\varphi_-^{\mathcal{V}}\|^2 - \|\mathrm{d}\varphi_+^{\mathcal{V}}\|^2. \tag{3.2.8}$$

The Eells-Salamon correspondence relates minimal surfaces in $N$ to $J$-holomorphic curves in $Z$. For the Riemannian metric $g_1$ and almost complex structure $J_2$ on $Z$ consider the Hermitian two-form $\omega$. A distinguished class of two-manifolds in $Z$ are those on which $\omega$ vanishes. The following lemma relates this condition to surfaces in $N$, the statements for $J_1$ and other choices of $\lambda$ can be derived similarly.

**Lemma 3.2.2.** *We have* $\varphi_\pm^*\omega = (1 - |H|^2 + |\varphi^{\mathcal{V}_\pm}|^2)\mathrm{vol}_{\mathcal{H}}$.

*Proof.* We restrict ourselves to showing the identity for $\varphi_-$. The other case follows by reversing the orientation on $N$. Since $\mathrm{d}\varphi_-^{\mathcal{H}}$ is always $J_a$ holomorphic and $\mathrm{d}\pi\colon \mathcal{H} \to TN$ is an isometry we have that $\varphi_-^*\omega_{\mathcal{H}} = \mathrm{vol}_{\mathcal{H}}$. For the vertical component we have by eq. (3.2.2)

$$\varphi_-^*\omega_{\mathcal{V}} = \frac{1}{4}(\langle u_1, v_2\rangle - \langle v_2, u_1\rangle)\mathrm{vol}_{\mathcal{H}}$$

where $u_i$ and $v_i$ are elements in $\mathbb{R}^2$ given by

$$u_1 = (\sigma_{131} + \sigma_{241}, \sigma_{141} - \sigma_{231}), \quad u_2 = (\sigma_{132} + \sigma_{242}, \sigma_{142} - \sigma_{232}),$$
$$v_1 = (\sigma_{141} - \sigma_{231}, -\sigma_{131} - \sigma_{241}), \quad v_2 = (\sigma_{142} - \sigma_{232}, -\sigma_{132} - \sigma_{242}).$$

The statement follows from an explicit simplification of the quadratic expressions. □

Note that in contrast to proposition 3.1.2 the previous lemma does not give a one-to-one correspondence, since lemma 3.1.3 does not hold for surfaces with $\omega = 0$. The set $\{\mathrm{II}(v,v) \mid \|v\| = 1, \quad v \in T_xX\} \subset \nu_x$ is an ellipse in the normal space and is called the ellipse of curvature at $x \in X$. The centre of the ellipse is the mean curvature vector $H$. The degeneracy condition of the ellipse of curvature can be formulated in terms of the twistor lifts.

**Lemma 3.2.3.** *If $f\colon X \to N$ is minimal then $\|\mathrm{d}\varphi_-^{\mathcal{V}}\| = \|\mathrm{d}\varphi_+^{\mathcal{V}}\|$ at $x \in X$ if and only if the ellipse of curvature is degenerate. In particular, if $f$ takes values in a totally geodesic submanifold then $\|\mathrm{d}\varphi_-^{\mathcal{V}}\| = \|\mathrm{d}\varphi_+^{\mathcal{V}}\|$.*

*Proof.* By eq. (3.2.4) $\|\mathrm{d}\varphi_-^{\mathcal{V}}\| = \|\mathrm{d}\varphi_+^{\mathcal{V}}\|$ is equivalent to

$$\sigma_{142}(\sigma_{131} - \sigma_{232}) + \sigma_{132}(\sigma_{242} - \sigma_{141}) = 0.$$

This condition is satisfied if and only if $\mathrm{II}_3$ and $\mathrm{II}_4$ commute. Here,

$$\mathrm{II}_j = (\sigma_{ijk})_{i,k=1,2}, \quad j \in \{3, 4\}.$$

This in turn is equivalent to $\mathrm{II}_3$ and $\mathrm{II}_4$ being simultaneously diagonalisable, which is equivalent to the ellipse of curvature being degenerate [GR83]. □

If the ellipse is not degenerate then there is a positive orthonormal frame $\{e_1, e_2\}$ of $T_xX$ such that $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and $\sigma_{12}$ are orthogonal and equal to the semi-axes of the curvature ellipse [GR83]. If the ellipse of curvature is a circle, this statement is true for any orthonormal frame. Otherwise, such a frame is unique up to a rotation of $\pi/4$. This is because a rotation of the frame $\{e_1, e_2\}$ about an angle $\theta$ results in a rotation of $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and $\sigma_{12}$ around $2\theta$. If $f$ is minimal then the second fundamental form is entirely determined by the vertical components of the twistor lifts.

**Proposition 3.2.4.** *If the ellipse of curvature is not degenerate at $x \in X$ there is an oriented orthonormal frame of $f^*(T_xN)$ such that*

$$\sigma_{132} = 0, \quad \sigma_{141} = \sigma_{142}, \quad \frac{1}{2}|\sigma_{131} - \sigma_{232}| \geq |\sigma_{142}|, \quad \sigma_{131} \geq \sigma_{232}.$$

*If the ellipse of curvature is degenerate then there is an oriented frame such that $\sigma_{142} = 0, \sigma_{141} = \sigma_{242}$ and $\frac{1}{2}(\sigma_{131} - \sigma_{232}) = \sigma_{132}$. If $f$ is a minimal immersion then the second fundamental form is determined by $\|\mathrm{d}\varphi_-^{\mathcal{V}}\|$ and $\|\mathrm{d}\varphi_+^{\mathcal{V}}\|$ in this frame in the non-degenerate case by*

$$\begin{aligned} 2\|\mathrm{d}\varphi_-^{\mathcal{V}}\|^2 &= \sigma_{141}^2 + (\sigma_{131} + \sigma_{141})^2 \\ 2\|\mathrm{d}\varphi_+^{\mathcal{V}}\|^2 &= \sigma_{141}^2 + (\sigma_{131} - \sigma_{141})^2 \end{aligned} \tag{3.2.9}$$

*and in the degenerate case by*

$$2\|\mathrm{d}\varphi_-^{\mathcal{V}}\|^2 = 2\|\mathrm{d}\varphi_+^{\mathcal{V}}\|^2 = \sigma_{131}^2.$$

*Proof.* If the ellipse of curvature is non-degenerate then we choose $e_1$ and $e_2$ as above such that $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and $\sigma_{12}$ are orthogonal. Using the freedom of rotation we can achieve that $|\frac{1}{2}(\sigma_{11} - \sigma_{22})| \leq |\sigma_{12}|$. Choose $e_3$ and $e_4$ parallel to $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and $\sigma_{12}$, respectively. This implies that $\sigma_{132} = 0, \quad \sigma_{141} = \sigma_{142}$ and $\frac{1}{2}|\sigma_{131} - \sigma_{232}| \geq |\sigma_{142}|$. By changing the sign of $\{e_3, e_4\}$ if necessary we can assume that $\sigma_{131} \geq \sigma_{232}$. If the ellipse of curvature is degenerate then $\frac{1}{2}(\sigma_{11} - \sigma_{22})$ and $\sigma_{12}$ are parallel for any choice of frame $\{e_1, e_2\}$. Hence, there is choice of $\{e_1, e_2\}$, unique up to rotation of $\pi/2$, such that $\sigma_{12} = \frac{1}{2}(\sigma_{11} - \sigma_{22})$. □

The metrics $g_{\mathcal{H}}$ and $g_{\mathcal{V}}$ pull back to metrics on $X$ via $\varphi_-$. Using $\varphi^* g_{\mathcal{H}}$ one gets a volume form vol $= \mathrm{vol}_{\mathcal{H}}$ on $X$. Let us also consider the two-forms $\mathrm{vol}_\pm = \|\mathrm{d}\varphi_\pm^{\mathcal{V}}\|^2 \mathrm{vol}_{\mathcal{H}}$. If $f\colon X \to N$ is minimal, then $\varphi_\pm$ is $J_2$-holomorphic and $\mathrm{vol}_\pm$ is the volume form of $\varphi^* g_{\mathcal{V}}$ by the proof of lemma 3.2.2 (if $\varphi_\pm$ is horizontal then both forms extend to zero). Integrating these forms over $X$ gives real numbers Vol, $\mathrm{Vol}_\pm$. From the pointwise curvature equations one obtains formulae for the Euler number $e$ of the normal bundle of $TX \subset f^*(TN)$.

**Proposition 3.2.5.** *Let $f\colon X \to N$ be an immersion and $\chi$ be the Euler characteristic of $X$. Then*

1. *If $N$ is self-dual and Einstein with scalar curvature $\tau$ then*

$$e = \chi - \frac{\tau \mathrm{Vol}}{24\pi} + \frac{1}{\pi}\mathrm{Vol}_- - \frac{1}{\pi}\int_X |H|^2.$$

2. *If $N$ is anti-self-dual and Einstein with scalar curvature $\tau$ then*

$$-e = +\chi - \frac{\tau \mathrm{Vol}}{24\pi} + \frac{1}{\pi}\mathrm{Vol}_+ - \frac{1}{\pi}\int_X |H|^2.$$

*Proof.* Statement 2 follows from statement 1 by reversing the orientation. By the generalised Gauß-Bonnet theorem, $e = \frac{1}{2\pi}\int_X K^N$ and by the Gauß equation $K = R_{1221} + G$. Furthermore, if $N$ is Einstein and self-dual then $R_{1234} + R_{1221} = \frac{\tau}{12}$. Combining these identities with eq. (3.2.6) and integrating over $X$ proves statement 1. □

In the case when $\varphi_-$ is $J_1$-holomorphic $2|(\mathrm{d}\varphi)^{\mathcal{V}}|^2 = |H|^2$ and we recover Friedrich's formula [Fri84, Theorem 1].

The bundle $f^*(TN)$ carries two natural almost complex structures $J_+$ and $J_-$. They preserve the splitting $f^*(TN) = TX \oplus \nu$ and in the local frame $J_\pm$ is given by

$$e_1 \mapsto e_2, \quad e_2 \mapsto -e_1, \quad e_3 \mapsto \pm e_4, \quad e_4 \mapsto \mp e_3.$$

The Euler number is then expressed as the first Chern-class by $e = c_1(\nu, J_+)$. The normal bundle and almost complex structures are also well-defined when $f$ is not necessarily an immersion but conformal and harmonic. In that case, $\mathrm{d}f$ has only isolated zeros and there is a unique complex subbundle $L_N$ of $f^*(TN)$ which contains $\mathrm{d}f(TX)$. Let $r_N = \deg(T^\vee X \otimes L_N)$ be the ramification number of $\mathrm{d}f$.

**Proposition.** *Let $N$ be self-dual Einstein $f\colon X \to N$ be conformal and harmonic. Then the Euler number of the normal bundle of $L_N$ in $f^*(TN)$ is given by*

$$e = \chi + r_N - \frac{\tau \mathrm{Vol}}{24\pi} + \frac{1}{\pi}\mathrm{Vol}_-.$$

The computation in the proof of this proposition is the same as before. The bundle $L_N$ inherits a metric from $f^*(TN)$ and thus carries a curvature form. The integral over this curvature is now due to Chern-Weil given by $c_1(L_N)$ which is equal to $\chi + r_N$. For a conformal harmonic map $f\colon X \to N$ the twistor degrees $\mathrm{d}_\pm$ are defined as $\frac{1}{2}\varphi_\pm^* c_1(\mathcal{V}, J_1) = -\frac{1}{2}\varphi_\pm^* c_1(\mathcal{V}, J_2)$. The formula for $e$ can be combined with the formula $e = \pm(2d_\pm - \chi - r)$ for a branched minimal immersion [ES85, Proposition

8.3] to yield a formula for the twistor degree into a self-dual Einstein manifold

$$d_- = \frac{\tau \mathrm{Vol}}{48\pi} - \frac{1}{2\pi}\mathrm{Vol}_-$$

and for anti-self dual Einstein manifolds

$$d_+ = \frac{\tau \mathrm{Vol}}{48\pi} - \frac{1}{2\pi}\mathrm{Vol}_+.$$

This should be seen as a generalisation of [Bry82a, Proposition 2.4] which says that the volume of a superminimal surface in $S^4$ is $4\pi$ times the algebraic degree of $\varphi_+$. [Bry82a, Proposition 2.4]. When $\varphi_\pm$ is not holomorphic, there is no notion of an algebraic degree but the twistor degree is the suitable generalisation as it is equal to the topological degree for $N = S^4$. The quantity $\mathrm{Vol}_\pm$ can be regarded as a measure of how far a $J_2$ holomorphic curve is from being $J_1$ holomorphic, i.e. horizontal.

### 3.2.1 Invariance under Homotopy

The Euler number $e$ is the crucial topological invariant to determine if two immersions are regularly homotopic, i.e. homotopic as immersions. If either $X = S^2$ or $N = \mathbb{R}^4$ then this is in fact the only obstruction [Hir59; Sma59]. We are interested when two minimal immersions are homotopic via *minimal* immersions.

**Theorem 3.2.6.** *Let $N$ be self-dual Einstein and $f_t \colon I \times X \to N$ be a family of minimal immersions in $N$. Then* $\mathrm{vol}_-$ *and* $\mathrm{vol}_{\mathcal{H}}$ *both stay constant along $f_t$.*

*Proof.* The quantity $\mathrm{Vol}(f_t)$ stays constant by minimality and so does $e$. As a consequence of proposition 3.2.5 $\mathrm{Vol}_-$ is also constant. □

This has a consequence for superminimal curves and twistor lines, since they are characterised by the vanishing of either $\mathrm{Vol}_-$ or Vol.

**Corollary 3.2.7.** *Smooth deformations of twistor lines and superminimal curves stay twistor lines and superminimal curves respectively.*

Note that this result does not hold for example for $J_1$ on $\mathbb{CP}^3$. As holomorphic curves, a twistor line can be deformed to any other projective line in $\mathbb{CP}^3$. As in example 3.2.1 we only use the term moduli space informally here.

**Example 3.2.8.** *For the nearly Kähler twistor spaces $\mathbb{CP}^3$ and $\mathbb{F}_{1,2}(\mathbb{C}^3)$, the set of twistor lines, is a connected component of the moduli space of $J$-holomorphic curves. The set of twistor lines is parametrised by the base space of the twistor fibration. Hence, it admits a complex structure in the case of $\mathbb{F}_{1,2}(\mathbb{C}^3)$ and no almost complex structure for $\mathbb{CP}^3$.*

So Vol and $\mathrm{Vol}_-$ should be regarded as obstructions for two minimal immersions to be homotopic via minimal immersions. The sum $\mathrm{vol}_{\mathcal{H}} + \mathrm{vol}_-$ is the volume of the twistor lift of $f$ with respect to the metric $g_1$. When $(Z, J_2)$ is $(1, 2)$ symplectic, such as in the case of $N = S^4$ or $\mathbb{CP}^2$, then theorem 3.2.6 is consistent with the following.

**Proposition 3.2.9.** *Let $(M, g, \omega)$ be an almost complex Hermitian manifold satisfying $\mathrm{d}\omega \in \Omega^{3,0}(X) \oplus \Omega^{0,3}(X)$. Then the volume function $\mathrm{Vol}_M$ is constant along smooth variations of $J$-holomorphic curves.*

*Proof.* Let $\gamma\colon I \times X \to M$ be a path of $J$-holomorphic curves.

$$
\begin{aligned}
\mathrm{Vol}_M(\gamma_1) - \mathrm{Vol}_M(\gamma_0) &= \int_X \gamma_1^*\omega - \int_X \gamma_0^*\omega = \int_{\partial(X\times I)} \gamma^*\omega = \int_{X\times I} \mathrm{d}(\gamma^*\omega) \\
&= \int_{X\times I} (\gamma^*\mathrm{d}\omega) = \int_I \int_X \iota_{\frac{\partial}{\partial t}}(\gamma_t^*(\mathrm{d}\omega)).
\end{aligned}
$$

The differential form $\iota_{\frac{\partial}{\partial t}}(\gamma_t^*(\mathrm{d}\omega))$ vanishes on $X$. Let $\tau \in \Lambda^2(TX) \subset T^{1,1}X$ then

$$
\iota_{\frac{\partial}{\partial t}}(\gamma_t^*(\mathrm{d}\omega))(\tau) = (\mathrm{d}\omega)((\gamma_t)_*(\frac{\partial}{\partial t}), (\gamma_t)_*(\tau)) = 0.
$$

Since $\gamma_t$ is a $J$-holomorphic curve, so $(\gamma_t)_*(\tau)$ is in $T^{1,1}X$ but $\mathrm{d}\omega \in \Lambda^{3,0}(X) \oplus \Lambda^{0,3}(X)$. □

This result and a similar proof have first been obtained by Verbitsky [Ver13] where $J$-holomorphic curves are modelled as subsets with discrete singularities instead of maps, equipped with the Hausdorff distance. In this case the proof is more involved since working with singular subsets is a more delicate matter. For example, consider the family of algebraic curves in $\mathbb{CP}^2$ given by $Z_0^2 + c^2 Z_1 Z_2 = 0$ for $c \in [0, 1]$. The volume of each curve is equal to its degree up to a multiple. The degree jumps from two to one for $c \to 0$. Note that on the level of maps this is poses no problem, for example $\mathbb{CP}^1 \times [0, 1] \to \mathbb{CP}^2$, $\quad [s, t] \mapsto [tsc, s^2, t^2]$ is not an embedding of $\mathbb{CP}^1$ at $c = 0$ but instead a branched double cover.

### 3.2.2 Example: Holomorphic Curves in $\mathbb{CP}^2$

Equip $\mathbb{CP}^2 = S^5/S^1$ with the Fubini-Study metric coming from the invariant round metric on $S^5$. It is known that $N = \mathbb{CP}^2$ is self-dual Einstein with $\tau = 24$ [Huy06, p. 224]. Let $C$ be an algebraic curve in $\mathbb{CP}^2$ and $f\colon C \to \mathbb{CP}^2$ the tautological embedding. Then $f$ is holomorphic and $\sigma_{13} = \sigma_{24}$ and $\sigma_{23} = -\sigma_{14}$ and hence the twistor lift $\varphi^+$ is horizontal, however the $-$ twistor lift need not be horizontal and $\mathbb{CP}^2$ is not anti self-dual. Furthermore, $C$ is known to be minimal because $\mathbb{CP}^2$ is also

Kähler, such that proposition 3.2.5 simplifies to

$$e = \chi - \frac{\tau \mathrm{vol}}{24\pi} + \frac{1}{\pi}\mathrm{Vol}_-.$$

It is a consequence of Wirtinger's Theorem that the volume of $C$ is given by its degree times a constant $\kappa$. Computing the volume of a projective line reveals that in this normalisation $\kappa = \pi$. So we obtain

$$e = -d^2 + 2d + \frac{1}{\pi}\mathrm{vol}_{\mathcal{V}}$$

by the degree-genus formula. On the other hand, it is known that $e$ is the self-intersection number of $C$. There is an algebraic way to compute this number for a curve, it is the intersection number of $D$ and $D'$ where $D$ is the divisor representing $C$ and $D'$ is a generic linear equivalent divisor to $D$. In particular, $\deg(D) = \deg(D') = d$, so by Bézout's theorem $e = d^2$ which implies

$$\mathrm{Vol}_- = 2\pi(d^2 - d).$$

As a corollary, we obtain the expected result that a curve of degree $d$ can only be totally geodesic when $d = 1$.

## 3.3 Twistor Lifts and Quadrics in $\mathbb{CP}^3$

We now turn our attention to the integrable complex structure on $\mathbb{CP}^3$ and study quadrics with respect to the twistor fibration. It is a classical result, which is proved by using characteristic classes, that $S^4$ does not admit an almost complex structure [Bor53]. Hence, $\mathbb{CP}^3 \to S^4$ does not have a global section. However, each section $U \to \mathbb{CP}^3$ from an open subset $U \subset S^4$ will give rise to an almost complex structure on $U$. This structure is integrable if and only if the graph of $J$ is a complex submanifold in $\mathbb{CP}^3$ [ES85]. As a warm-up we consider projective planes with respect to the twistor fibration.

**Lemma 3.3.1.** *Every projective plane in $\mathbb{CP}^3$ contains exactly one twistor line.*

*Proof.* The plane must intersect any twistor fibre, which is always a projective line in $\mathbb{CP}^3$. If it intersected every twistor fibre transversally the plane would define an almost complex structure on $S^4$. As a result, the plane must contain at least one twistor line. It can not contain more than one twistor line since twistor lines are always skew. □

The involution

$$j([Z_0, Z_1, Z_2, Z_3]) = [-\bar{Z}_1, \bar{Z}_0, -\bar{Z}_3, \bar{Z}_2] \tag{3.3.1}$$

is important for the twistor fibration. Twistor lines in $\mathbb{CP}^3$ are exactly $j$ invariant projective lines.

To see the correspondence $\mathbb{CP}^3 \cong Z(S^4)$ in more explicit terms, consider the following local coordinates for the fibration $\pi\colon \mathbb{CP}^3 \to \mathbb{HP}^1$ as in [SV09]. Firstly, we restrict to $\mathbb{H} = \{[1, q] \mid q \in \mathbb{H}\} \subset \mathbb{HP}^1$. Let $q = z_1 + jz_2$ and $[Z_0, Z_1, Z_2, Z_3] \in \mathbb{CP}^3$, this point lies in the fibre $\pi^{-1}(q)$ if and only if

$$q = (Z_2 + jZ_3)(Z_0 + jZ_1)^{-1}.$$

Let $a \in \mathbb{C}$ and let

$$Z_0 = 1, \quad Z_1 = a, \quad Z_2 = W_1, \quad Z_3 = W_2.$$

Then the equation above becomes

$$(z_1 + jz_2)(1 + ja) = W_1 + jW_2.$$

This results in

$$\begin{aligned} W_1 &= z_1 - a\bar{z}_2 \\ W_2 &= z_2 + a\bar{z}_1. \end{aligned} \tag{3.3.2}$$

In fact we can allow $a = \infty \in \mathbb{CP}^1$ and then get $W_1 = -\bar{z}_2$ as well as $W_2 = \bar{z}_1$. The advantage of this notation is that $(z_1, z_2)$ can be regarded as coordinates on $\mathbb{H} = \mathbb{R}^4$ where $a$ holomorphically parametrises the fibre of $\pi^{-1}(z_1, z_2)$ such that $W_1$ and $W_2$ become coordinates on $\mathbb{CP}^3$.

We have written down a local trivialisation of the bundle $\pi\colon \mathbb{CP}^3|_{\mathbb{H}} \to \mathbb{H}$. In fact, an almost complex structure $J$ on an open subset $U \subset S^4$ is given by $a\colon U \to \mathbb{CP}^1$ and $J$ is integrable if and only if $a$ is holomorphic. To see explicitly how $\mathbb{CP}^3|_{\mathbb{H}}$ parametrises orthogonal almost complex structures over $\mathbb{R}^4$, let $\sigma_1 = \mathrm{d}W_1 = \mathrm{d}z_1 - a\mathrm{d}\overline{z_2}$ and $\sigma_2 = \mathrm{d}W_2 = \mathrm{d}z_2 + a\mathrm{d}\overline{z_1}$. We can associate this to an almost complex structure on $\mathbb{R}^4$ which is either orientation preserving or reversing. The difference between these two cases depends on the orientation on $\mathbb{R}^4$. We study the negative twistor space and choose coordinates $z_1 = x_1 + ix_2$ and $z_2 = x_3 - ix_4$ with the orientation that $e_i = \mathrm{d}x_i$ is a positively oriented frame.

The forms $\sigma_1$ and $\sigma_2$ are a maximally isotropic subspace of one forms on $\mathbb{C}^4$. So we can declare them to be $(1, 0)$ forms for an orthogonal almost complex structure on

$\mathbb{R}^4$. This defines an orientation reversing almost complex structure since the resulting two-form

$$i(\sigma_1 \wedge \overline{\sigma_1} + \sigma_2 \wedge \overline{\sigma_2}) = (1 - |a|^2)b_1 + 2\mathrm{Re}(a)b_2 + 2\mathrm{Im}(a)b_3 \tag{3.3.3}$$

for $b_1 = \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4), b_2 = -\frac{1}{2}(e_1 \wedge e_3 + e_2 \wedge e_4)$ and $b_3 = -\frac{1}{2}(e_1 \wedge e_4 - e_2 \wedge e_3)$ is anti-self dual. We summarise this discussion in the following proposition.

**Proposition 3.3.2.** *With the identifications of the restricted bundles*

$$\mathbb{R}^4 \times (\mathbb{C} \cup \{\infty\}) \cong \mathbb{CP}^3|_{\mathbb{R}^4}, \quad Z_-(S^4)|_{\mathbb{R}^4} \cong \mathbb{R}^4 \times S^2$$

*the isomorphism between $\mathbb{CP}^3$ and $Z_-(S^4)$ is the stereographic projection $\mathbb{C} \cup \{\infty\} \to S^2$.*

Note that a change of orientation on the surface corresponds to the antipodal map on the fibre $S^2$. In $\mathbb{CP}^1$, this is equal to the inversion $a \mapsto -1/\bar{a}$. Taking the twistor coordinates in eq. (3.3.2) into account, one sees that this induces the real structure $j$ on $\mathbb{CP}^3$. In other words, changing the orientation of the surface will result in applying $j$ to the twistor lift.

All non-degenerate quadrics on $\mathbb{CP}^3$ are projectively equivalent. However, this equivalence will not necessarily respect the twistor fibration. In fact, with respect to the twistor fibration, there are different equivalence classes of quadrics on $\mathbb{CP}^3$. The linear transformations of $\mathbb{CP}^3$ which respect the fibration are those given by the action of $\mathrm{SL}(2, \mathbb{H})$ on $\mathbb{CP}^3$. Note that the space of quadrics on $\mathbb{CP}^3$ can be identified with $\mathbb{P}(S^2(\mathbb{C}^4))$. The real dimension of this space is $2(10-1) = 18$. The real dimension of $\mathrm{SL}(2, \mathbb{H})$ is $16 - 1 = 15$. Hence, one expects a three-dimensional space of inequivalent quadrics. Indeed, [SV09, Theorem 1.9] gives an explicit three-parameter family of inequivalent quadrics representing the $\mathrm{SL}(2, \mathbb{H})$ orbits of quadrics in $\mathbb{CP}^3$.

One invariant for a quadric $\mathcal{Q}$ is the nature of its discriminant locus. Each fibre of $\pi$ is a projective line in $\mathbb{CP}^3$, so it hits $\mathcal{Q}$ generically in exactly two points. The discriminant locus is the subset of elements where this generic condition does not hold. A natural question to ask is which subsets can arise as discriminant loci.

**Theorem 3.3.3.** *[SV09, Theorem 1.11] Let $\mathcal{Q}$ be a non-singular quadric in $\mathbb{CP}^3$, then there are four possibilities for the discriminant locus $D$. Either*

- *$\mathcal{Q}$ is real, then $D$ is a circle in $S^4$ and $\mathcal{Q}$ contains all twistor lines over the circle*
- *the discriminant locus of $\mathcal{Q}$ is a smooth torus and $\mathcal{Q}$ does not contain any twistor lines*
- *$D$ is a pinched torus and $\mathcal{Q}$ contains exactly one twistor line*

- *$D$ is a double pinched torus and $\mathcal{Q}$ contains exactly two twistor lines.*

In the last two cases the discriminant locus is not smooth everywhere. A quadric is called real if it is invariant under the involution $j$. Inspecting the proof of theorem 3.3.3 gives a characterisation of the smooth points of $D$.

**Lemma 3.3.4.** *If $\mathcal{Q}$ is not real then $D$ is smooth at $x \in D$ if and only if $\mathcal{Q}$ does not contain the twistor line $\pi^{-1}(\{x\})$. In this case, $\pi^{-1}(\{x\})$ is tangent to $\mathcal{Q}$.*

Denote the set of smooth points of $D$ by $D^*$. As an example, we investigate the family of quadrics

$$\mathcal{Q}_\lambda\colon \quad \lambda Z_0 Z_3 = Z_1 Z_2 \quad \text{equivalently} \quad \lambda W_2 = a W_1$$

for a parameter $\lambda \in \mathbb{R}$. In the coordinates eq. (3.3.2), the quadric $\mathcal{Q}_\lambda$ is described by the equation

$$a^2 \overline{z_2} + a(\lambda \overline{z_1} - z_1) + \lambda z_2 = 0.$$

The discriminant of this quadratic expression in $a$ vanishes if the following equations in $\mathbb{R}^4$ are satisfied

$$\begin{aligned} 4\lambda(x_2^2 + y_2^2) + (\lambda+1)^2 y_1^2 - x_1^2(\lambda-1)^2 &= 0 \\ x_1 y_1 (\lambda+1)(\lambda-1) &= 0. \end{aligned}$$

We see that the resulting discriminant locus $\subset \mathbb{R}^4$ is a line for $\lambda = \pm 1$ and a cone given by

$$D = \{\mu^2(x_2^2 + y_2^2) - x_1^2 = 0 \quad y_1 = 0\} \cup \{\infty\}$$

for $\mu^2 = \frac{4\lambda}{(\lambda-1)^2}$ otherwise. On the other hand, the fibre over $\infty$ lies in the quadric for all values of $\lambda$. As a result, the discriminant locus in $S^4 = \mathbb{R}^4 \cup \infty$ is the one-point compactification of the discriminant locus in $\mathbb{R}^4$, i.e. a circle for $\lambda = \pm 1$ and a double pinched torus otherwise. This is consistent with theorem 3.3.3 since $\mathcal{Q}_\lambda$ always contains the twistor lines over 0 and $\infty$ and is real if and only $\lambda = \pm 1$.

We fix $\lambda > 1$ and compute the twistor lift of the set of smooth points of $D$, i.e. $D^* = D \setminus \{0, \infty\}$. It can be parametrised by $f_\pm\colon (s,t) \mapsto (\pm \mu r, 0, s, t)$ with $r = \sqrt{s^2 + t^2}$. Note that this computation only takes place in the negative twistor space. The sign in the index of $f$ only stands for the sign of $r$ in the parametrisation. We first investigate $f_+$

$$\frac{\partial f_+}{\partial s} \wedge \frac{\partial f_+}{\partial t} = e_3 \wedge e_4 + \frac{\mu s}{r} e_1 \wedge e_4 + \frac{\mu t}{r} e_3 \wedge e_1.$$

The projection of this onto the sphere bundle $Z \subset \Lambda^2_-(\mathbb{R}^4)$ is

$$\frac{1}{\sqrt{1+\mu^2}}\begin{pmatrix} -1 \\ \frac{\mu t}{r} \\ -\frac{\mu s}{r} \end{pmatrix}$$

in the basis $\{b_1, b_2, b_3\}$. According to equation 3.3.3, the inverse stereographic projection has to be computed, which yields

$$a = \frac{\sqrt{\lambda} z}{|z|}$$

where $z = -s + it$. Write $z = r\exp(i\theta)$, so the image of the lift $\varphi_+$ is parametrised by

$$\begin{aligned} X_+ &= \{[1, \exp(i\theta)\sqrt{\lambda}, kr\sqrt{\lambda}, r\exp(i\theta)k] \mid (r, \exp(i\theta)) \in \mathbb{R}^{>0} \times S^1]\} \\ &= \{[1, \exp(i\theta)\sqrt{\lambda}, r\sqrt{\lambda}, r\exp(i\theta)] \mid (r, \exp(i\theta)) \in \mathbb{R}^{>0} \times S^1\} \end{aligned}$$

where $k = \frac{\lambda+1}{\lambda-1}$. Analogously, the image of the lift of $\varphi_-$ is

$$\{[1, \exp(i\theta)\sqrt{\lambda}, r\sqrt{\lambda}, r\exp(i\theta)] \mid (r, \exp(i\theta)) \in \mathbb{R}^{<0} \times S^1\}.$$

Clearly, $X_- \cup X_+$ extends smoothly over the two singular points in $D$. The twistor lift can be considered a desingularisation of $D$.

**Theorem 3.3.5.** *The twistor lift of the discriminant locus of the quadric $\mathcal{Q}_\lambda$ for $\lambda > 1$ is the intersection of $\mathcal{Q}_\lambda$ and the non-holomorphic quadric $Z_1\overline{Z_3} = \overline{Z_0}Z_2$ which is diffeomorphic to a two-torus.*

One part of this statement can be generalised to any quadric in $\mathbb{CP}^3$.

**Proposition 3.3.6.** *Let $\mathcal{Q}$ be a smooth quadric in $\mathbb{CP}^3$, which is not real. Denote by $D$ the discriminant locus of $\mathcal{Q}$ and by $D^*$ the set of smooth points of $D$. Then the twistor lift of $D^*$ lies in $\mathcal{Q}$.*

*Proof.* Let $q \in \mathcal{Q}$ such that $x = \pi(q) \in D^*$. By lemma 3.3.4 the smoothness condition implies that $\pi^{-1}(x)$ is not a subset of $\mathcal{Q}$ and $T_q(\pi^{-1}(x)) \subset T_q Q$. Since $T_q(\pi^{-1}(x)) = \ker(\mathrm{d}\pi_q \colon T_q\mathbb{CP}^3 \to T_x S^4)$ the real rank of $\mathrm{d}\pi_q \colon T_q\mathcal{Q} \to T_x S^4$ is two. Furthermore, $T_x D = \mathrm{d}\pi_q(T_q\mathcal{Q})$ and both spaces are two-dimensional, so they are equal.

By the definition of the complex structure $J_1$ on the twistor space $\mathbb{CP}^3$, the map $\mathrm{d}\pi_q \colon T_q\mathbb{CP}^3 \to T_x S^4$ is complex-linear when the complex structure on the vector space $T_x S^4$ is given by $q$ itself. Since $\mathcal{Q}$ is a complex submanifold of $\mathbb{CP}^3$, this means that $\mathrm{Im}(\mathrm{d}\pi_q \colon T_q Q) = T_x D$ is a complex line with respect to $q$. This implies that the Gauß lift of $T_x D$ is $q$. $\square$

For complex hypersurfaces in $\mathbb{CP}^3$ of higher degree $k > 2$ one can define the discriminant locus as the set of points in $S^4$ which intersect the hypersurface in less than $k$ points or contains a twistor line. To generalise proposition 3.3.6 to this situation one needs to understand the smoothness condition as in lemma 3.3.4 for $k > 2$. Another question is whether discriminant loci or their lifts are in general submanifolds with special geometric properties.

# Chapter 4

# Transverse $J$-holomorphic Curves in $\mathbb{CP}^3$

The nearly Kähler manifold for which $J$-holomorphic curves have been studied the most is $M = S^6$. This was initiated by the work of Bryant defining a Frenet-frame and constructing null-torsion curves via integrals of a holomorphic distribution on an auxiliary space, a quadric in $\mathbb{CP}^6$. Bolton, Vrancken and Woodward categorised $J$-holomorphic curves into four classes and gave a characterisation of $J$-holomorphic curves amongst minimal surfaces in $S^6$ [BVW94]. More recently, L. Fernández showed that the space of $J$-holomorphic spheres in $S^6$ is a complex manifold and computed the dimension of each component [Fer15].

$J$-holomorphic curves in $S^3 \times S^3$ have attracted interest since the work of Bolton, Dillen, Dioos and Vrancken [Bol+15]: The homogeneous nearly Kähler structure on $S^3 \times S^3$ admits an almost product structure $P$. If $z$ denotes a local coordinate on such a $J$-holomorphic curve there is a holomorphic differential $\Lambda \mathrm{d}z^2 = g(P\partial_z, \partial_z)\mathrm{d}z^2$. Curves on which $\Lambda$ vanishes are locally in 1:1 correspondence with constant mean curvature surfaces in $\mathbb{R}^3$ [Bol+15, Theorem 3.10]. If $\Lambda$ is non-vanishing then by a local change of the complex coordinate one can assume $\Lambda = 1$. Such curves are locally characterised by a complex valued function $\mu$ on $U \subset X$ and the conformal factor $\omega$ satisfying the system of equations

$$\omega_{z\bar{z}} \sinh(\omega) - \frac{e^{-\omega}}{2}|\omega_z|^2 + \frac{4}{3}\sinh^2\omega(1 - |\mu|^2) = 0$$
$$\mu_{\bar{z}} + \frac{\omega_{\bar{z}} e^{\omega}\mu - \omega_z \bar{\mu}}{2\sinh\omega} = 0.$$

Furthermore, the second fundamental form can be expressed in terms of the functions $\mu$ and $\omega$.

One of the aims of this chapter is to provide a similar analysis for $J$-holomorphic curves in $\mathbb{CP}^3$. Instead of an almost product structure, the parallel splitting $T\mathbb{CP}^3 =$

$\mathcal{H} \oplus \mathcal{V}$ plays a key role in this case. By an appropriate frame adaption we will describe transverse $J$-holomorphic curves by two functions $\alpha_-, \alpha_+ \colon X \to \mathbb{R}$ which carry local geometric information such as the curvature and second fundamental form of the curve.

The cone of a nearly Kähler manifold carries a torsion-free $G_2$-structure and the cone of a $J$-holomorphic curve is an associative submanifold of this space. There are further relationships between $J$-holomorphic curves and other geometries which are specific to the ambient space $\mathbb{CP}^3$.

By 3.1.2 non-vertical $J$-holomorphic curves in nearly Kähler $\mathbb{CP}^3$ are in one-to-one correspondence with minimal surfaces in $S^4$. By viewing $\mathbb{CP}^3$ as a sphere bundle in $\Lambda^2_-(S^4)$ a $J$-holomorphic curve can be thickened to an associative submanifold for the Bryant-Salamon metric. Remarkably, this associative submanifold is complete and is constructed as the total space of a line bundle where each fibre passes through the zero section of $\Lambda^2_-(S^4)$ and the $J$-holomorphic curve [KM05].

The Hopf fibration $S^7 \to S^4$ has $S^3$-fibres and can be realised as a quotient of Lie groups $\mathrm{Sp}(2)/\{e\} \times \mathrm{Sp}(1) \to \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. This fibration carries a natural connection and by squashing the metric on the fibres one obtains a nearly parallel $G_2$-structure on $S^7$. This metric is different from the round metric, it is called the squashed seven-sphere $S^7_{\mathrm{sq}}$. The squashing factor is different from the nearly Kähler squashing factor $\mathbb{CP}^3 \to S^4$. But $J$-holomorphic curves do not depend on the metric and they in fact give rise to associative submanifolds in $S^7_{\mathrm{sq}}$ [Kaw15].

In proposition 4.3.5 we will show that $J$-holomorphic curves in $\mathbb{CP}^3$ also give rise to $\tau$-primitive maps in the flag manifold $\mathrm{Sp}(2)/\mathbb{T}^2$ as in [BW94]. The following diagram summarises the relationship between $J$-holomorphic curves in $\mathbb{CP}^3$ and other geometries

$$\begin{tikzcd}
\text{associative in } S^7_{\text{squashed}} & & & & \text{associative in } \Lambda^2_-(S^4) \\
\tau - \text{primitive map in } \mathrm{Sp}(2)/\mathbb{T}^2 & & & & \text{associative in the cone } \mathbb{CP}^3 \times \mathbb{R}^{>0} \\
& & J\text{-holomorphic curve in } \mathbb{CP}^3 \arrow[uull] \arrow[ull] \arrow[uurr] \arrow[urr] \arrow[d] & & \\
& & \text{minimal surface in } S^4 & &
\end{tikzcd}.$$

In section 4.1 we give some background on $J$-holomorphic curves in general almost complex manifolds. We review the local behaviour of such curves with a Cartan-Kähler set-up. We show that the differential of a $J$-holomorphic curve can be seen as a holomorphic section in an appropriate bundle.

Xu defines a special class of $J$-holomorphic curves, which he calls null-torsion curves, and shows that they are in correspondence with superminimal curves [Xu10].

Section 4.2 gives a twistor interpretation of this correspondence.

In section 4.3, we establish properties of the angle functions $\alpha_\pm$. They always satisfy the 2D periodic Toda lattice equation for $\mathfrak{sp}(2)$ which are equivalent to the system

$$\Delta_0 \log(\alpha_-^2) = -4(3\alpha_-^2 + \alpha_+^2 - 2)\gamma^2$$
$$\Delta_0 \log(\alpha_+^2) = -4(3\alpha_+^2 + \alpha_-^2 - 2)\gamma^2$$

where $\gamma^2 = (\alpha_-\alpha_+)^{-1/2}$ is the conformal factor of the induced metric on $X$ and $\Delta_0$ the Laplacian for the corresponding flat metric.

Viewing the surface in the ambient space $\mathbb{CP}^3$ means we can characterise additional data equipped to the curve, such as the first and second fundamental form in $\mathbb{CP}^3$. It turns out that both of them can be expressed through the functions $\alpha_-$ and $\alpha_+$. As an application, we show in proposition 4.5.5 that any flat $J$-holomorphic torus is a Clifford torus.

The second fundamental form $\mathrm{II}_{\mathbb{CP}^3}$ is a complex linear tensor and its differential $\bar{\partial}\mathrm{II}_{\mathbb{CP}^3}$ can be computed neatly in terms of $\alpha_-, \alpha_+$ and vanishes exactly in points where the curve is non-transverse. The normal bundle $\nu$ of a $J$-holomorphic curve carries a natural holomorphic structure. We show that the normal bundles of all transverse $J$-holomorphic tori are isomorphic to each other, in particular all of them admit a holomorphic section.

Section 4.5 features a proof of a Bonnet-type theorem for $J$-holomorphic curves stating that the first and second fundamental form on $X$ are determined by $\alpha_-$ and $\alpha_+$. The essence of theorem 4.5.2 is that if $X$ is simply-connected then a $\mathbb{C}^*$-family of transverse $J$-holomorphic curves can be recovered from a solution to the 2D Toda lattice equation. Roughly speaking, this can be seen as a complex analogue to the statement that curves in $\mathbb{R}^3$ are essentially classified by their curvature and torsion.

To get hold of examples of transverse $J$-holomorphic curves we impose an arbitrary $\mathrm{U}(1)$ symmetry on them in section 4.6, given by a certain element $\xi \in \mathfrak{sp}(2)$. Such an action commutes with a $\mathbb{T}^2$-action of automorphisms on $\mathbb{CP}^3$. Given a $\mathbb{T}^3$-action on a torsion-free $G_2$ manifold $M^7$ the multi-moment maps give rise to a local homeomorphism $M^7/\mathbb{T}^3 \to \mathbb{R}^4$ [MS12a, Theorem 4.5]. The only known example of a nearly Kähler manifold admitting a $\mathbb{T}^3$-action is $S^3 \times S^3$ whose geometry has been described with multi-moment maps by Dixon [Dix19].

Assuming $\mathbb{T}^2$ symmetry on a nearly Kähler manifold $M^6$ is less restrictive but the corresponding multi-moment map $\nu \colon M^6 \to \mathbb{R}$ is only real-valued. The question arises whether there is a geometric construction of a map into $\mathbb{R}^4$ which descends to a local homeomorphism to the $\mathbb{T}^2$ quotient of $M$, at least away from a singular set. We construct a map $p \colon {}_{\mathbb{T}^2}\backslash^{(\mathbb{CP}^3 \backslash \mathcal{S})} \to \mathbb{R}^4$ for a certain singular set $\mathcal{S}$ and in theorem 4.6.13

it is shown that $p$ descends to a branched double cover from ${}_{\mathbb{T}^2}\backslash^{\mathbb{CP}^3\backslash\mathcal{S}}$ onto its image $D \subset \mathbb{R}^4$. The map $p$ converts U(1) invariant $J$-holomorphic curves in $\mathbb{CP}^3$ to solutions of the 1D Toda lattice equation for $\mathfrak{sp}(2)$ in $D$. Derived from the Lax representation of this equation one derives two preserved quantities, giving rise to a map $u\colon D \to \mathbb{R}^2$.

If one equips $\mathbb{CP}^3$ with its Kähler structure then a $\mathbb{T}^2$-action gives rise to a symplectic moment map whose image is a quadrilateral. Composing $u$ with $p$ gives a $\mathbb{T}^2$ invariant map $P\colon \mathbb{CP}^3 \to \mathbb{R}^2$ whose fibres contain U(1) invariant $J$-holomorphic curves and whose image is a rectangle $\bar{\mathcal{R}} \subset \mathbb{R}^2$. Just as in the symplectic case, the fibre of $P$ degenerate over the boundary $\partial\bar{\mathcal{R}}$ and are geometrically distinguished sets. In fact, theorem 4.6.17 relates $P^{-1}(\partial\bar{\mathcal{R}})$ to the nearly Kähler multi-moment map $\nu$, Clifford tori and families of minimal tori in $S^4$ discovered by Lawson [Law70].

Most of the material of this chapter is found in the author's paper [Asl21].

## 4.1 $J$-holomorphic Curves

For now, assume that $M^n$ is a general almost complex manifold. It is a classical result by Nijenhuis and Woolf [NW63] that given any $J$ invariant two-plane $E_2 \subset T_xM$ there is a local $J$-holomorphic curve passing through $E_2$. We review how this result can be shown using Cartan-Kähler theory and how many functions $J$-holomorphic curves locally depend on.

We give a very brief account of some important notions from exterior differential systems, based on [Bry99]. An exterior differential system (EDS) is a pair of a smooth manifold $M$ and a graded ideal $\mathcal{I}$ in the algebra of smooth differential forms $\Omega^*(M)$ on $M$ that is closed under the exterior differential d. In other words, $\mathcal{I}$ is a linear subspace of $\Omega^*(M)$ such that if $\alpha$ is in $\mathcal{I}$ then so are $\mathrm{d}\alpha$ and $\beta \wedge \alpha$ for any $\beta$. Given an EDS $(M, \mathcal{I})$, the key task is to find a submanifold $f\colon N \to M$, such that $f^*\alpha = 0$ for all $\alpha \in \mathcal{I}$. Such a submanifold is called integral submanifold. An $n$-dimensional subspace $E \subset T_xM$ is called an integral element of $\mathcal{I}$ at $x$ if $\alpha|_E = 0$ for all $\alpha \in \mathcal{I}^n := \Omega^n(M) \cap \mathcal{I}$. The set of all integral elements is denoted by $V_n(\mathcal{I}) \subset \mathrm{Gr}_n(TM)$. For an integral element $E \in V_n(\mathcal{I})$ at $x \in M$ let

$$H(E) = \{v \in T_xM \mid \alpha|_{\mathrm{span}(E,v)} = 0, \text{ for all } \alpha \in \mathcal{I}^{n+1}\} \subset T_xM$$

be the polar space of $E$ and let $c(E) = \dim(T_xM) - \dim(H(E))$.

A flag of subspaces of $T_xM$

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

with $\dim(E_k) = k$ is called an integral flag if $E_k \in V_k(\mathcal{I})$ for $1 \leq k \leq n = \dim(E)$.

The following follows from a combination the Cartan-Kähler theorem and Cartan's test, as found in [Bry99].

**Theorem 4.1.1.** *Let $(M, \mathcal{I})$ be a real analytic EDS and let*

$$\{0\} = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

*be an integral flag of $\mathcal{I}$ at $x \in M$. If $V_n(\mathcal{I})$ is a smooth submanifold of $\mathrm{Gr}_n(TM)$ of codimension $c(E_0) + c(E_1) + \cdots + c(E_{n-1})$ near $E$ then there is a real-analytic $n$-dimensional integral submanifold $P \subset M$ of $\mathcal{I}$ passing through $x$ with $T_xP = E$.*

In the situation above let furthermore $s_k = c(E_k) - c(E_{k-1})$. Then the integral submanifolds from theorem 4.1.1 locally depend on $s_0$ constants, $s_1$ functions of one variable, ... , $s_n$ functions of $n$ variables.

We now describe $J$-holomorphic curves in an almost complex manifold $M$ by an EDS. Let $\mathcal{I}$ be the ideal in $\Omega^*(M)$ which is generated by $\Omega^{2,0}(M) \oplus \Omega^{0,2}(M)$ i.e.

$$\mathcal{I} = \Omega^{2,0}(M) \oplus \Omega^{0,2}(M) \oplus \bigoplus_{p+q \geq 3} \Omega^{p,q}(M).$$

Then integral submanifolds of $\mathcal{I}$ are exactly embedded $J$-holomorphic curves. Note that

$$V_k(\mathcal{I}) = \begin{cases} \mathrm{Gr}_1(TM) & \text{if } k = 1 \\ \{J \text{ invariant planes in } TM\} & \text{if } k = 2 \\ \emptyset & \text{if } k \geq 3. \end{cases}$$

Furthermore, any integral element is regular. Fix an arbitrary $J$ invariant two-plane $E_2$ in $T_xM$ and any one-dimensional subspace $E_1 \subset E_2$ then

$$E_0 = (0) \subset E_1 \subset E_2 \subset TM$$

is a regular flag. Furthermore, the polar spaces are

$$H(E_0) = T_xM, \quad H(E_1) = E_2, \quad H(E_2) = E_2$$

which means that

$$c_0 = 0, \quad c_1 = n - 2, \quad c_2 = n - 2.$$

Consequently, an embedded $J$-holomorphic curve in an almost complex manifold of dimension $n$ is locally described by $n - 2$ functions in one variable. This count refers to $J$-holomorphic curves viewed as submanifolds. If a $J$-holomorphic curve is viewed as a map $\mathbb{C} \to M$ one adds two functions of one variable which parametrise local holomorphic maps $\mathbb{C} \to \mathbb{C}$.

Another point of view is to write the $J$-holomorphic curve equation $\varphi\colon X \to M$ in local coordinates. Locally, they are described by solutions $\varphi\colon B \to \mathbb{R}^n$ of the non-linear elliptic equation

$$\partial_x \varphi + J(\varphi)\partial_y \varphi = 0$$

where $B$ is an open disk in $\mathbb{C}$, $x + iy$ is a coordinate on $B$ and $J$ is thought of as a function $\mathbb{R}^n \to \mathrm{GL}(2n, R)$ with appropriate smoothness assumptions [MS12b, Eq. 2.3.1]. Squaring this equation gives a second order equation where the Laplace operator on $\mathbb{C}$ is the highest order term [MS12b, Eq. 2.3.2]. Performing a Cartan-Kähler analysis for this type of equation gives the same result.

**Lemma 4.1.2.** *On a domain in $\mathbb{R}^2$ with coordinates $x, y$ consider the equation*

$$\Delta u = F(u, u_x, u_y, x, y), \tag{4.1.1}$$

*then solutions are locally parametrised by $n$ harmonic functions, i.e. $F$ does not change "how many solutions" this equation has locally.*

*Proof.* Let $p = u_x, q = u_y$. On $M = \mathbb{R}^2 \times (\mathbb{R}^3)^n$, define the differential ideal $\mathcal{I} = \langle \Upsilon_i, \theta_i \rangle$ where $\Upsilon_i = \mathrm{d}p_i \wedge \mathrm{d}y - \mathrm{d}q_i \wedge \mathrm{d}x + F_i \mathrm{d}x \wedge \mathrm{d}y$ and $\theta_i = \mathrm{d}u_i - p_i \mathrm{d}x - q_i \mathrm{d}y$ for $i \in \{1, \ldots, n\}$. A two-dimensional integral submanifold $N \subset M$ on which $\mathrm{d}x$ and $\mathrm{d}y$ are linearly independent is locally a graph of a solution of eq. (4.1.1), see [Bry99]. Note that $\mathcal{I}$ is algebraically generated by $\Upsilon_i, \theta_i$ and $-d\theta_i = \mathrm{d}p_i \wedge \mathrm{d}x + \mathrm{d}q_i \wedge \mathrm{d}y$. The forms $\theta_i$ are linearly independent and

$$V_1(\mathcal{I}) = \mathbb{P}(\cap_{i=1}^n \ker(\theta_i))$$

is a co-dimension $n$ subbundle of $\mathrm{Gr}_1(M)$. Let $E_2$ be an element in $V_2(\mathcal{I})$ such that $\mathrm{d}x \wedge \mathrm{d}y \neq 0$, i.e. the components of $\mathrm{d}u, \mathrm{d}p, \mathrm{d}q$ are linear combinations of $\mathrm{d}x$ and $\mathrm{d}y$. Since $E_2$ vanishes on $\mathcal{I}$ one has that

$$\mathrm{d}u_i = p_i \mathrm{d}x + q_i \mathrm{d}y, \quad \mathrm{d}p_i = a_i \mathrm{d}x + b_i \mathrm{d}y, \quad \mathrm{d}q_i = b_i \mathrm{d}x - (a_i + F_i)\mathrm{d}y$$

for some $a_i, b_i \in \mathbb{R}$. This shows that $V_2(\mathcal{I})$ is a smooth manifold of codimension $\dim(Gr_2(TM)) - 2n = 4n$ in $\mathrm{Gr}_2(TM)$ in a neighbourhood of $E_2$. Let $E_1 \subset E_2$ and $E_1 \in V_1(\mathcal{I})$. This means that $E_1$ is spanned by a vector with the components satisfying the equations

$$v^{u_i} = p_i v^x + q_i v^y, \quad v^{p_i} = a_i v^x + b_i v^y, \quad v^q = b_i v^x - (a_i + F_i)$$

with $(v^x, v^y) \neq (0, 0)$. Then $H(E_1) = E_2$ and $c_0 = n, c_1 = 3n, c_2 = 3n$ and the flag $\{0\} = E_0 \subset E_1 \subset E_2$ is regular by Cartan's test, which implies the statement. $\square$

We now establish that the differential of a $J$-holomorphic curve in an almost complex manifold is a holomorphic section when $TM$ is equipped with an appropriate connection. In the nearly Kähler case, the torsion of $\overline{\nabla}$ is totally skew-symmetric, so it can be viewed as an element $\Omega^3(M)$. It is in fact a component of $\psi$ and in particular of type $(3,0)+(0,3)$ [Nag02]. For a general almost complex manifold the torsion of an arbitrary connection is an element in $\Omega^2(M,TM)$.

**Lemma 4.1.3.** *Any almost complex manifold $(M,J)$ has a connection $\bar{\nabla}$ with $\bar{\nabla}J=0$ and $(2,0)+(0,2)$ torsion.*

*Proof.* Let $\nabla$ be a torsion-free connection on $TM$ and define

$$\bar{\nabla}_X(Y)=\nabla_X(Y)-Q(X,Y)$$

with

$$4Q(X,Y)=(\nabla_{JY}J)X+J(\nabla_Y J)X+2J(\nabla_X J)Y.$$

Then $\bar{\nabla}$ preserves $J$ and has torsion

$$4\alpha(X,Y):=4(Q(X,Y)-Q(Y,X))=(\nabla_{JY})X-J(\nabla_Y J)X-(\nabla_{JX}J)Y+J(\nabla_X J)Y$$

which equals minus the Nijenhuis tensor of $M$. This is part of [MS12b, Lemma C.7.2] which is stated for the case when $J$ is tamed by a symplectic structure but it can be checked that this part of the Lemma does not require the symplectic condition. We have $\alpha(JX,JY)=-\alpha(X,Y)$ and hence $\alpha$ is of type $(2,0)+(0,2)$. □

Let $\bar{\nabla}$ be a connection on $M$ such that $\overline{\nabla}J=0$. Denote by $\theta$ the tautological one-form in $\Omega^1(M,TM)$. The torsion $T$ of $\overline{\nabla}$ is equal to $\overline{\nabla}_{\mathrm{d}}(\theta)$ which denotes the induced covariant derivative on differential forms, i.e. $\overline{\nabla}_{\mathrm{d}}(\theta)\in\Omega^2(M,TM)$. Let $\varphi\colon X\to M$ be a $J$-holomorphic curve. Then $\varphi^*\theta=\mathrm{d}\varphi\in\Omega^1(X,\varphi^*(TM))$, which would hold for any smooth map. Furthermore, $\overline{\nabla}J=0$ ensures that $\overline{\nabla}$ induces a connection on $T^{1,0}M$ which defines a covariant derivative on $\Omega^{1,0}(M,T^{1,0}M)$ and also on $\Omega^{1,0}(X,\varphi^*(T^{1,0}M)))$ such that

$$\overline{\nabla}_{\mathrm{d}}(\mathrm{d}\varphi)=\overline{\nabla}_{\mathrm{d}}(\varphi^*\theta)=\varphi^*(\overline{\nabla}_{\mathrm{d}}(\theta))=\varphi^*(T)\in\Omega^{1,1}(X,\varphi^*(T^{1,0}M)).$$

Note that we have $\overline{\nabla}^{0,1}=\overline{\nabla}_{\mathrm{d}}$ as maps from $\Omega^{1,0}(X,E)$ to $\Omega^{1,1}(X,E)$ because $X$ is a Riemann surface. We have shown the following proposition, which relates the holomorphicity of $\mathrm{d}\varphi$ to the torsion form of $\overline{\nabla}$.

**Proposition 4.1.4.** *Let $\varphi\colon X\to M$ be a $J$-holomorphic curve and let $\overline{\nabla}$ be a connection on $TM$ with $\overline{\nabla}J=0$ and torsion form $T\in\Omega^2(M,TM)$. Then $\mathrm{d}\varphi\in$*

$\Omega^{1,0}(X, \varphi^*(TM))$ *is holomorphic for the holomorphic structure on* $\varphi^*(TM)$ *induced from* $\overline{\nabla}$ *if and only if* $\varphi^*(T)$ *vanishes on* $X$.

Combining this statement with lemma 4.1.3 yields the following.

**Corollary 4.1.5.** *Let* $\varphi\colon X \to M$ *be a* $J$*-holomorphic curve then there exists a connection of* $\varphi^*(TM)$ *such that* $\mathrm{d}\varphi$ *is a holomorphic section of* $\Omega^{1,0}(X, \varphi^*(T^{1,0}M))$.

In particular, the set $\{x \in X \mid \mathrm{d}_x\varphi = 0\}$ is discrete. This is a well known fact and even holds for $C^1$ almost complex structures and is usually proven using PDE techniques, see [MS12b, Lemma 2.4].

**Corollary 4.1.6.** *Let* $M$ *be a nearly Kähler six-manifold and let* $\nabla$ *be the nearly Kähler connection. Let* $\varphi\colon X \to M$ *be a* $J$*-holomorphic curve. Then* $\mathrm{d}\varphi \in \Omega^{1,0}(X, \varphi^*TM)$ *is holomorphic.*

One consequence of this corollary is that if $\varphi\colon X \to M$ is a $J$-holomorphic curve in a nearly Kähler manifold then there is a unique holomorphic line bundle $L \subset \varphi^*(TM)$ that contains $\mathrm{d}\varphi(TX)$. This line bundle has a second fundamental form $\mathrm{I\!I}_L$ in $\varphi^*(TM)$. For [Bry82b] it is crucial that this second fundamental form, which he calls torsion, can also be viewed as a holomorphic section. The curves with zero torsion are then classified by identifying them as integrals of a holomorphic differential system on the reduced twistor space $\tilde{\mathrm{Gr}}(2, \mathbb{R}^7)$ of $S^6$.

The following proposition makes clear that this is special to $M = S^6$, as this space has a simple curvature tensor.

**Proposition 4.1.7.** *Let* $X$ *a Riemann surface and let* $E$ *be a Hermitian vector bundle with a compatible connection* $\nabla$. *The connection defines a holomorphic structure on* $E \to X$. *Let* $F \subset E$ *be a holomorphic subbundle with normal bundle* $\nu = E/F$. *Then the second fundamental form* $\mathrm{I\!I}_F$ *lies in* $\Omega^{1,0}(X, \mathrm{Hom}(F, \nu))$ *and is holomorphic for the induced holomorphic structure on* $\mathrm{Hom}(F, \nu)$ *if and only if the curvature tensor* $R_\nabla \in \Omega^2(X, \mathrm{End}(E))$ *preserves the bundle* $F$ *for any two-vector in* $X$.

*Proof.* We follow an approach similar to [Don17, p. 9]. Pick a Hermitian metric that is compatible with $\nabla$ which splits

$$0 \to F \to E \to \nu \to 0$$

as $C^\infty$ bundles. Consequently,

$$\nabla = \begin{pmatrix} \nabla_F & -\bar{\mathrm{I\!I}}_F \\ \mathrm{I\!I}_F & \nabla_\nu \end{pmatrix},$$

with $\mathrm{I\!I}_F \in \Omega^1(X, \mathrm{Hom}(F,\nu))$. Since $F$ is a holomorphic sub-bundle of $E$ a holomorphic section $s$ of $F$ is also holomorphic for $E$ which implies

$$0 = \nabla^{0,1}(s) = \nabla_F^{0,1}(s) + \mathrm{I\!I}_F^{0,1}(s) = \mathrm{I\!I}_F^{0,1}(s),$$

i.e. $\mathrm{I\!I}_F$ is a $(1,0)$ form. Furthermore, $R$ is of type $(1,1)$ and can be computed as

$$R^{1,1} = [\nabla^{1,0}, \nabla^{0,1}] = [\begin{pmatrix} \nabla_F^{1,0} & 0 \\ \mathrm{I\!I}_F & \nabla_\nu^{1,0} \end{pmatrix}, \begin{pmatrix} \nabla_F^{0,1} & -\bar{\mathrm{I\!I}}_F \\ 0 & \nabla_\nu^{0,1} \end{pmatrix}].$$

The $(1,2)$-entry of this matrix is

$$-\mathrm{I\!I}_F \circ \nabla_F^{0,1} + \nabla_\nu^{0,1} \circ \mathrm{I\!I}_F = \nabla_{\mathrm{Hom}(F,\nu)}(\mathrm{I\!I}_F)$$

which vanishes if and only if $R$ preserves the bundle $F$. □

Clearly, the condition of proposition 4.1.7 does not hold for arbitrary $J$-holomorphic curves in $\mathbb{CP}^3$. One can however consider the second fundamental form in the rank two bundle $\mathcal{H}$ as done in [Xu10], which turns out to be holomorphic.

Let $(M, g, J)$ be an almost Hermitian manifold. If $M$ is compact then the group of diffeomorphisms $A(M)$ preserving the almost complex structure $J$ has a Lie group structure [Wol69, Sec. 4]. Let $H(M)$ be the intersection of $A(M)$ with the isometry group of $M$. If $A(M)$ is larger than $H(M)$ then acting with elements in $A(M)\setminus H(M)$ produces examples of potentially non-isometric $J$-holomorphic curves from a known one. However, for homogeneous nearly Kähler manifolds different from $S^3 \times S^3$ one can apply [Wol69, Theorem 4.1] and conclude that the identity components of $A(M)$ and $H(M)$ agree.

Recall that a nearly Kähler manifold does not admit any four-dimensional almost complex submanifolds by proposition 2.3.1. This implies that there is no $J$-holomorphic submersion $M \to \mathbb{C}$, not even locally. A stronger statement is also true.

**Proposition 4.1.8.** *Let $(N^4, I)$ be an almost complex submanifold and $(M, J)$ be a nearly Kähler manifold. Then there is no submersion $f \colon M \to N$ with $\mathrm{d}f \circ J = I \circ \mathrm{d}f$.*

*Proof.* Assume there is an $x \in M$ such that $\mathrm{d}_x f$ has full rank. Then there is a local special unitary co-frame $\omega_1, \omega_2, \omega_3$ near $x$ on $M$ such that $\omega_1$ and $\omega_2$ are pullbacks of $(1,0)$ forms $\alpha_1, \alpha_2$, on $N$, i.e. $f^*\alpha_i = \omega_i$ for $i = 1, 2$. In particular, this implies that the $(0,2)$-part of $\mathrm{d}\omega_1$ is a multiple of $\bar{\omega_1} \wedge \bar{\omega_2}$. But the Nijenhuis tensor on a nearly Kähler manifold is a multiple of $\mathrm{Im}\,\psi$, so the $(0,2)$-part of $\mathrm{d}\omega_1$ is a non-zero multiple of $\bar{\omega_2} \wedge \bar{\omega_3}$, see [Bry06a]. □

There are however fibrations from $M$ to a four-manifold, where the fibres are $J$-holomorphic curves in $M$ but the fibration is not $J$-holomorphic. Examples of that are not only the twistor fibrations but also the fibration

$$S^3 \times S^3 \to S^2 \times S^2$$

considered in [MNS05]. The fibres are $J$-holomorphic totally geodesic tori invariant under an $S^1$ action.

## 4.2 Twistor Interpretation of Xu's correspondence

As seen in eq. (3.1.1), there is a notion of a negative and a positive twistor space for a Riemannian four-manifold $N^4$. In general, they can be different fibre bundles over $N$. However, for $N = S^4$ both twistor space are isomorphic as fibre bundles. We exploit this to give a twistor interpretation of Xu's correspondence between superminimal curves and curves with vanishing torsion [Xu10].

Recall the structure equations of $\mathbb{CP}^3$ from section 2.2 and note that the horizontal bundle $\mathcal{H}$ is locally the kernel of the one-form $\omega_3$. For a general $J$-holomorphic curve $\varphi$ there is a unique holomorphic line bundle $L \subset \varphi^*\mathcal{H}$ which contains the projection of $\mathrm{d}\varphi(TX)$ to $\varphi^*\mathcal{H}$. Let $N$ be the quotient bundle $\varphi^*\mathcal{H}/L$, which naturally carries a holomorphic structure. Define

$$P = \{p \in \varphi^*\mathrm{Sp}(2) \mid \omega_1|_p = 0\}$$

which is an $S^1 \times S^1$ sub-bundle of $\varphi^*\mathrm{Sp}(2)$. The bundle $P$ can be equipped with a connection such that the forms $\tau$ and $\omega_2$ restrict to basic forms on $P$. By multiplying $\bar{\tau}$ by an appropriate section in $L^\vee \otimes N$ one obtains the form $\mathrm{II}_{\mathcal{H}} \in \Omega^{1,0}(X, L^\vee \otimes N)$.

**Proposition 4.2.1.** *[Xu10, Theorem 1.3] In $\mathbb{CP}^3$, there is a one-to-one correspondence between horizontal holomorphic curves and $J_2$-holomorphic curves on which $\mathrm{II}_{\mathcal{H}}$ vanishes.*

Let $I_{\mathcal{H}} \in \Omega^{1,0}(X, L)$ and $I_{\mathcal{V}} \in \Omega^{1,0}(X, \mathcal{V})$ be $\mathrm{d}\varphi$ composed with orthogonal projections onto $\mathcal{H}$ and $\mathcal{V}$. Then $I_{\mathcal{H}}$, $I_{\mathcal{V}}$ and $\mathrm{II}_{\mathcal{H}}$ are all holomorphic sections and if neither of them vanishes everywhere and $X$ is compact with genus $g$ and $\varphi$ is simple (in the sense of [MS12b]), then

$$8(g-1) = 2r_{\mathcal{H}} + r_{\mathcal{V}} + r_{\mathrm{II}} \tag{4.2.1}$$

where $r_{\mathcal{H}}, r_{\mathcal{V}}, r_{\mathrm{II}}$ denote the number of zeros of $I_{\mathcal{H}}$, $I_{\mathcal{V}}$ and $\mathrm{II}_{\mathcal{H}}$, counted with multiplicities [Xu10, Remark 4.11.]. These integers are related to invariants of the corresponding minimal surface in $S^4$, $r_{\mathcal{H}} = r_N, r_{\mathcal{V}} = d_- - \chi, r_{\mathrm{II}} = d_+ - \chi$. The correspondence of proposition 4.2.1 can be entirely described in terms of twistor lifts.

The twistor theory of $S^4$ has the particularity that both the negative and the positive twistor space can be identified with $\mathbb{CP}^3$. To distinguish the two spaces as bundles over $S^4$ we denote them by $\mathbb{CP}^3_\pm$. Both spaces are quotients of Sp(2) by different but conjugate subgroups

$$\begin{tikzcd}[column sep=small] & Sp(2)/S^1 \times S^1 \arrow[dl] \arrow[dr] & \\ \mathbb{CP}^3_- = \mathrm{Sp}(2)/S^1 \times S^3 & & \mathbb{CP}^3_+ = \mathrm{Sp}(2)/S^3 \times S^1 \end{tikzcd}.$$

For a $J_2$-holomorphic curve $\varphi_-\colon X \to \mathbb{CP}^3_-$, Xu constructs a lift $\tilde{\varphi}_-\colon X \to Sp(2)/S^1 \times S^1$ and then considers the projection onto $\mathbb{CP}^3_+$ which yields a map $X \to \mathbb{CP}^3_+$. He constructs a similar lift $\tilde{\varphi}_+$ when starting with a curve in $\mathbb{CP}^3_+$ and shows that both constructions are inverse to each other.

Observe that $Sp(2)/S^1 \times S^1$ is nothing but $\widetilde{\mathrm{Gr}}_2(S^4)$. Let $\pi_\pm$ be the projection of $\mathbb{CP}^3_\pm$ onto $S^4$. It turns out that the lifts $\tilde{\varphi}_\pm$ equal the Gauß lift of $\pi_\pm \circ \varphi$ into $\mathrm{Sp}(2)/S^1 \times S^1$. This immediately shows that the two constructions are inverse to each other since the transformation leaves the underlying map into $S^4$ unchanged. In fact, this procedure gives a way to pass between $J_2$-holomorphic curves in $\mathbb{CP}^3_+$ and $\mathbb{CP}^3_-$ due to proposition 3.1.2. However, when starting with a horizontal $J_2$-holomorphic curve in $\mathbb{CP}^3_-$ the resulting curve in $\mathbb{CP}^3_+$ need not be horizontal. In fact the tangent bundle of $F$ admits a natural splitting

$$T(Sp(2)/S^1 \times S^1) = \mathbf{H} \oplus \mathbf{V}_+ \oplus \mathbf{V}_- \tag{4.2.2}$$

which is a connection of the fibration $Sp(2)/S^1 \times S^1 \to S^4$. Let $p_\pm$ be the fibration maps $\mathrm{Sp}(2)/S^1 \times S^1 \to \mathbb{CP}^3_\pm$ and let $T\mathbb{CP}^3_\pm = \mathcal{H}_\pm \oplus \mathcal{V}_\pm$ be the splitting from the twistor fibration, then

$$\mathbf{V}_\pm = \ker(\mathrm{d}p_\mp), \qquad p_\pm^*(\mathcal{H}_\pm) = \mathbf{H}, \qquad p_\pm^*(\mathcal{V}_\pm) = \mathbf{V}_\pm.$$

If $\{f_1, f_2, f_3, f_4\}$ denotes a frame dual to $\{\omega_1, \omega_2, \omega_3, \tau\}$ then $\mathbf{H}$ is locally spanned by $f_1$ and $f_2$, $V_-$ by $f_3$ and $V_+$ by $f_4$. Consider a $J_2$-holomorphic curve $\varphi_-\colon X \to \mathbb{CP}^3_-$ and the Gauß lift $\hat{\varphi}_-\colon X \to \mathrm{Sp}(2)/S^1 \times S^1$. The curve is horizontal if the lift does not have a component in $V_-$. Eq. 4.2.2 describes the splitting of $\mathfrak{sp}(2)/\mathfrak{t}^2$ into root spaces. In fact, eq. (2.2.1) reveals that the component in $\mathbf{V}_+$ vanishes if and only if

$$\varphi^*\tau = 0 \quad \Leftrightarrow \quad \mathrm{II}_{\mathcal{H}} = 0$$

on $X$ which proves proposition 4.2.1. Finally, observe that eq. (2.2.2) implies that $\mathrm{II}_{\mathcal{H}}$ is equal to the second fundamental form of $L$ in $\varphi^*\mathcal{H}$ which is an element in

$\Omega^{1,0}(X, \mathrm{Hom}(L, N))$ because $L$ is a holomorphic sub bundle of $\varphi^*\mathcal{H}$. The holomorphicity of $\mathrm{II}_\mathcal{H}$ can then be related to properties of the curvature tensor of $\varphi^*\mathcal{H}$.

## 4.3 Adapting Frames on $J$-holomorphic Curves

The aim of this section is to describe $J$-holomorphic curves in $\mathbb{CP}^3$ by solutions of an equation of type eq. (4.1.1). Instead of local coordinates, we will derive two geometric functions $\alpha_\pm$ coming from the twistor fibration $\mathbb{CP}^3 \to S^4$ to achieve that.

On simply-connected domains, we will be able to describe $J$-holomorphic curves by flat $\mathrm{Sp}(2)$ connections satisfying appropriate conditions and modulo $\mathrm{U}(1)\times\mathrm{Sp}(1)$–valued gauge transformations.

Recall from section 4.2 that there are three distinguished classes of $J$-holomorphic curves in $\mathbb{CP}^3$. The first are curves which are always tangent to the vertical bundle $\mathcal{V}$. They are twistor lines and are parametrised by elements in $S^4$. Since the horizontal bundle $\mathcal{H}$ is of complex rank two it is less restrictive to require a curve being tangent to $\mathcal{H}$. These curves are called superminimal and classified in eq. (3.1.9). There is a third class of curves, namely those on which $\mathrm{II}_\mathcal{H}$ vanishes identically. Such curves are in one to one correspondence with superminimal curves by proposition 4.2.1. Since all of these classes are relatively well understood we are interested in studying $J$-holomorphic curves which do not belong to any of these three classes and are defined as follows.

**Definition 4.3.1.** A $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$ is called transverse if one of the equivalent conditions is satisfied

- $\mathrm{II}_\mathcal{H} \neq 0$ everywhere and $\varphi$ is nowhere tangent to the horizontal bundle $\mathcal{H}$ or the vertical bundle $\mathcal{V}$
- Both twistor lifts of $\pi_- \circ \varphi\colon X \to S^4$ are nowhere horizontal or vertical
- The Gauß lift of $\pi_- \circ \varphi$ into $\mathrm{Sp}(2)/S^1 \times S^1$ is nowhere tangent to either bundle $\mathbf{H}$,$\mathbf{V}_-$ or $\mathbf{V}_+$.

If $X$ is homeomorphic to a two-sphere then the curve is superminimal or satisfies $\mathrm{II}_\mathcal{H} \equiv 0$ [Xu06]. The first characterisation in definition 4.3.1 says that a $J$-holomorphic curve that is not superminimal, vertical or null-torsion everywhere is characterised by $r_\mathcal{H} = r_\mathcal{V} = r_{\mathrm{II}} = 0$, so eq. (4.2.1) implies the following two statements. Since all of these numbers are necessarily positive, we have that if $X$ is a torus then the curve is automatically transverse if it is not superminimal or satisfies $\mathrm{II}_\mathcal{H} \equiv 0$. If $X$ is compact with genus $g \geq 2$ there are non-transverse points. This also follows from eq. (4.2.1). For this reason, we are mainly interested in the case when $X$ has genus one. However, the set of non-transverse points is discrete and in the end of this section we describe the behaviour of the curve near these points.

We start by investigating the action of $H = \mathrm{U}(1) \times \mathrm{Sp}(1)$ on $\mathfrak{sp}(2)$. This is crucial for adapting frames along a transverse $J$-holomorphic curve.

Consider the embedding

$$i\colon \mathrm{U}(2) \to \mathrm{SU}(3), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det(A^{-1}) \end{pmatrix}. \tag{4.3.1}$$

Let $(v_1, v_2, v_3)^T \in \mathbb{C}^3$ with $|v_1|^2 + |v_2|^2 \neq 0$ and $|v_3|^2 \neq 0$. Then there is $A \in \mathrm{U}(2)$ such that if $(w_1, w_2, w_3)^T = A(v_1, v_2, v_3)^T$ then $w_2 = 0$ and $w_3/w_1 \in \mathbb{R}^{>0}$. The choice of such an $A$ is unique up to multiplication by an element in the subgroup $K' = \{\mathrm{diag}(e^{i\vartheta}, e^{-2i\vartheta}, e^{i\vartheta})\} \subset \mathrm{SU}(3)$.

Define the double cover $u\colon H = \mathrm{U}(1) \times \mathrm{Sp}(1) \to \mathrm{U}(2)$ where $u(\lambda, q)$ acts on $\mathbb{C}^2 = \mathbb{C} \oplus j\mathbb{C} = \mathbb{H}$ by $h \mapsto qh\lambda^{-1}$. Let $\rho$ be the action of $\mathrm{U}(1) \times \mathrm{Sp}(1)$ on $V_1 = \mathbb{C}^3$ coming from $i \circ u$. Consider the adjoint action of $H \subset \mathrm{Sp}(2)$ on $\mathfrak{sp}(2)$. It splits as

$$\mathfrak{sp}(2) = \mathfrak{h} \oplus V_1.$$

The action of $H$ on $\mathfrak{h}$ is the adjoint action while $H$ acts on $V_1$ by $\rho$. Here $V_1$ embeds into $\mathfrak{sp}(2)$ as follows

$$(z_1, z_2, z_3) \mapsto \begin{pmatrix} j\bar{z}_3 & -\overline{z_1} + j\overline{z_2} \\ z_1 + jz_2 & 0 \end{pmatrix}.$$

Since for any element $v$ in $\mathbb{C}^3$ there is an element $A$ in $\mathrm{U}(2)$, unique up to a multiple in $\mathbb{T}^2$, such that the second component of $Av$ is zero, we have shown the following lemma.

**Lemma 4.3.2.** *For any $\zeta = \eta + (v_1, v_2, v_3) \in \mathfrak{h} \oplus \mathbb{C}^3$ there is a $h \in H$ such that the second component of $h\zeta h^{-1}$ vanishes . Such an $h$ is unique up to multiplication by an element in $S^1 \times S^1$.*

This statement of this lemma can be improved for a generic element in $\mathbb{C}^3$. Note that $K = \rho^{-1}(K') = \{\mathrm{diag}(e^{i\theta}, e^{i3\theta})\}$ and define $W = \{(v_1, 0, v_3) \in V_1 \mid v_3/v_1 > 0\}$. The observations after eq. (4.3.1) imply the following.

**Lemma 4.3.3.** *For any $\zeta = \eta + (v_1, v_2, v_3) \in \mathfrak{h} \oplus \mathbb{C}^3$ with $(v_1, v_2) \neq (0, 0)$ and $v_3 \neq 0$ there is a $h \in H$ such that $h\zeta h^{-1}$ lies in $W$. Such an $h$ is unique up to multiplication by an element in $K$.*

The adjoint action of $K$ on $\mathfrak{h}$ splits into one-dimensional subspaces. Let

$$V_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & jw \end{pmatrix} \mid w \in \mathbb{C}\}, \quad V_3 = \{\begin{pmatrix} ix_1 & 0 \\ 0 & ix_2 \end{pmatrix} \mid x_1, x_2 \in \mathbb{R}\}.$$

The action of $K$ on $V_3$ is trivial while it acts on $V_2$ by multiplication of $e^{-6i\theta}$. On the other hand, $K$ acts on $(z_1, 0, z_3) \in W$ by multiplication of $e^{2i\theta}$ in each component. Define

$$\mathfrak{r} = \{\begin{pmatrix} ix_1 + j\bar{z}_3 & -\bar{z}_1 \\ z_1 & ix_2 + jw \end{pmatrix} \mid z_1 \neq 0, \quad z_3/z_1 \in \mathbb{R}^{>0}, \quad w/z_1 \in \mathbb{R}^{>0}\}. \tag{4.3.2}$$

We conclude

**Lemma 4.3.4.** *Let $v = (z_1, z_2, z_3, w, x_1, x_2) \in \mathfrak{sp}(2)$ with $(z_1, z_2) \neq (0, 0), w \neq 0$ and $z_3 \neq 0$. Then there is an element $A \in K$ such that $Av \in \mathfrak{r}$. The choice of such an element $A$ is unique up to multiplication by an element of the subgroup $K_F := \{\mathrm{diag}(e^{i\theta}, e^{i3\theta}) \mid e^{8i\theta} = 1\} \cong \mathbb{Z}_8$.*

This lemma will enable us to reduce the structure group of a transverse $J$-holomorphic curve $\varphi \colon X \to \mathbb{CP}^3$ from $H$ to the discrete group $K_F$.

Denote by $g$ the nearly Kähler metric on $\mathbb{CP}^3$ which splits as $g_{\mathcal{H}} + g_{\mathcal{V}}$ since the splitting $T\mathbb{CP}^3 = \mathcal{H} \oplus \mathcal{V}$ is orthogonal. We will consider the pull-back metrics of $g, g_{\mathcal{H}}, g_{\mathcal{V}}$ to $X$ via $\varphi$ and denote them with the same symbols. The metric $2g_{\mathcal{H}}$ is equal to the induced metric from $\pi \circ \varphi \colon X \to S^4$.

Note that Sp(2) pulls back to an $S^1 \times S^3$ bundle over $X$. The structure equations are formulated with respect to differential forms on Sp(2). To simplify matters, one reduces the $S^1 \times S^3$ bundle $\varphi^*(\mathrm{Sp}(2))$ which ensures that additional relations of the differential forms are satisfied.

To begin with, by lemma 4.3.2, one can reduce the bundle $\varphi^*(\mathrm{Sp}(2))$ to an $S^1 \times S^1$ bundle $P$ by imposing the equation $\omega_2 = 0$. This uses a different convention from [Xu10], where the reduction is defined by $\omega_1 = 0$, but both reductions are related by the right action of a constant element in Sp(2). This gives a lift of $\varphi$ into $\mathrm{Sp}(2)/S^1 \times S^1$. On this reduction, $\tau$ becomes a basic form of type $(1, 0)$. Since $\varphi$ is $J$-holomorphic $\mathrm{d}\varphi(T^{1,0})$ takes values in the subbundle corresponding to the root spaces $\{(2, 0), (0, -2), (-1, 1)\}$ which are associated to $(\omega_3, \tau, \omega_1)$ under the isomorphism $\Omega^1(\mathrm{Sp}(2), \mathbb{C})^{\mathrm{Sp}(2)} \cong \mathfrak{sp}(2)^\vee \otimes \mathbb{C}$, see fig. 4.2.

Given a semi-simple Lie group with maximal torus $T$ the flag manifold $G/T$ carries a natural $m$-symmetric structure $\tau$ where $m$ is the height of the Lie algebra of $G$. The $m$-symmetric structure $\tau$ induces a splitting $T_{\mathbb{C}}(G/T) = \oplus_{k=0}^m [\mathcal{M}_k]$. From the decomposition into root spaces the bundle $[\mathcal{M}_1]$ splits into line bundles.

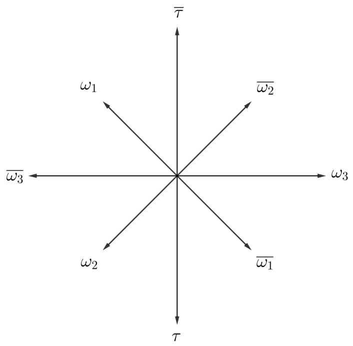


Figure 4.1: Root spaces associated to different components of $\Omega_{MC}$ under the identification $\Omega^1(\mathrm{Sp}(2),\mathbb{C})^{\mathrm{Sp}(2)} \cong \mathfrak{sp}(2)^\vee \otimes \mathbb{C}$

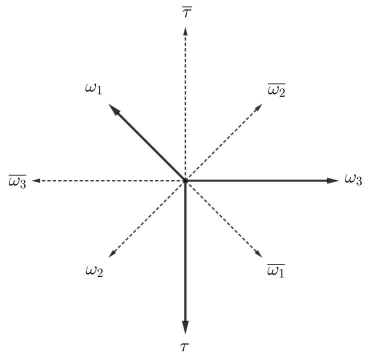


Figure 4.2: The thickened arrows represent a basis of $[\mathcal{M}_1]$ highlighting that transverse $J$-holomorphic curves are in one-to-one correspondence with $\tau$-primitive maps in $\mathrm{Sp}(2)/S^1 \times S^1$.

A map $\psi\colon X \to G/T$ is called *$\tau$-primitive* if $\mathrm{d}\psi(T^{1,0}X)$ lies in $[\mathcal{M}_1]$ with non-zero components in all of the line bundles, see [BW94]. In the case of $G = \mathrm{Sp}(2)$, $\{(2,0),(0,-2),(-1,1)\}$ is a basis of $[\mathcal{M}_1]$.

**Proposition 4.3.5.** *Any $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$ admits a lift into $\mathrm{Sp}(2)/S^1 \times S^1$ which is $\tau$-primitive in the sense of [BW94]. Conversely, composing any $\tau$-primitive map $X \to \mathrm{Sp}(2)/S^1 \times S^1$ with the projection $\mathrm{Sp}(2)/S^1 \times S^1 \to \mathbb{CP}^3$ gives a $J$-holomorphic curve $X \to \mathbb{CP}^3$.*

Note that under this identification, the formula eq. (4.2.1) becomes an application of the more general Plücker formula for $\tau$-primitive maps [BW11].

We will now explain how $\varphi^*(\mathrm{Sp}(2))$ can be reduced to a discrete bundle for a transverse $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$. We define the function $\alpha_-\colon X \to \mathbb{R}^{>0}$ by

$$\alpha_-(x) = \frac{\|\xi\|_\mathcal{V}}{\|\xi\|_\mathcal{H}},$$

where $\xi$ is any non-zero vector in $T_xX$. Here $\|\cdot\|_\mathcal{H}$ and $\|\cdot\|_\mathcal{V}$ denote the norms with respect to the metric $g_\mathcal{H}$ and $g_\mathcal{V}$ on $X$. The value of $\alpha_-$ does not depend on this choice because $\varphi$ is $J$-holomorphic. The function $\alpha_-$ is a measure of the angle in which $TX$ lies between $\varphi^*\mathcal{H}$ and $\varphi^*\mathcal{V}$ inside $\varphi^*(T\mathbb{CP}^3)$.

In accordance with lemma 4.3.3 we now adapt frames in the following way. Over $X$, define the principal bundle $Q$ by the relations

$$\begin{aligned} \omega_2 &= 0 \\ \omega_3 &= \alpha_-\omega_1. \end{aligned}$$

The bundle $Q$ has structure group $K \cong S^1$ by lemma 4.3.3. From now on, we will consider the restrictions of all differential forms to $Q$ without changing the notation. Since the structure group of $Q$ is $K$, the forms $\tau, 3\rho_1 - \rho_2$ and $\omega_1 \wedge \overline{\omega_1}$ become basic, i.e. they descend to forms on $\mathbb{CP}^3$. The structure equations then yield

$$\begin{aligned} \mathrm{d}\omega_1 &= -i(\rho_2 - \rho_1) \wedge \omega_1 \\ \mathrm{d}\omega_2 &= \tau \wedge \omega_1 = 0 \\ \mathrm{d}\omega_3 &= -2i\rho_1 \wedge \omega_3. \end{aligned} \tag{4.3.3}$$

Combining all of the equations gives that $\tau$ and $-\mathrm{dlog}(\alpha_-) + i(-3\rho_1 + \rho_2)$ are $(1,0)$-forms since their wedge product with $\omega_1$ vanishes. Since $\rho_1, \rho_2$ and $-\mathrm{dlog}(\alpha_-)$ are real-valued we get that

$$\mathrm{dlog}(\alpha_-)(J\xi) = (-3\rho_1 + \rho_2)(\xi)$$

for any tangent vector $\xi$. In other words,

$$\mathrm{d}^C\mathrm{log}(\alpha_-) = -3\rho_1 + \rho_2. \tag{4.3.4}$$

Note that a-priori this is an equation on $Q$ but it also holds on $X$ since $-3\rho_1 + \rho_2$ is a basic form. Eq. 4.3.4 implies that

$$-\Delta\mathrm{log}(\alpha_-) = \mathrm{d}(\rho_2 - 3\rho_1). \tag{4.3.5}$$

Here, $\Delta$ denotes the positive definite Laplace operator $\Omega^0(X) \to \Omega^2(X)$.

To obtain another differential equation, we make a further frame adaption to reduce the structure group from $K$ to $K_F$, using lemma 4.3.4. Since $\varphi$ is assumed to be transverse, $\tau$ is nowhere vanishing. On $Q$, both $\tau$ and $\omega_1$ reduce to forms on $X$ with values in the same line bundle. So we can define

$$\alpha_+(x) = \frac{|\omega_1(\xi)|}{|\tau(\xi)|}$$

for any $\xi \in T_xX$ which does not depend on $\xi \in T_xX$ since $\tau$ is a $(1,0)$-form. In fact, by proposition 4.3.5 transverse $J$-holomorphic curves correspond to $\tau$-primitive maps $\hat{\varphi}\colon X \to \mathrm{Sp}(2)/S^1 \times S^1$. As seen in eq. (4.2.2)

$$T(\mathrm{Sp}(2)/S^1 \times S^1) = H \oplus V_- \oplus V_+.$$

Note that $\alpha_-$ is a measure of the angle of $TX$ between $\hat{\varphi}^*(H)$ and $\hat{\varphi}^*(V_-)$ while $\alpha_+$ is a measure of the angle of $TX$ between $\hat{\varphi}^*(H)$ and $\hat{\varphi}^*(V_+)$. This is why we will refer to $\alpha_-$ and $\alpha_+$ as *angle functions.*

By lemma 4.3.3, we can adapt frames further. The bundle $Q$ restricts to a $K_F$ bundle $R$ which is characterised by the equation $\tau = \alpha_+\omega_1$. From the structure equations we now get

$$\mathrm{d}\tau = 2i\rho_2 \wedge \tau \tag{4.3.6}$$
$$-2i\mathrm{d}\rho_1 = (1 - 2\alpha_-^2)\omega_1 \wedge \overline{\omega_1} \tag{4.3.7}$$
$$2i\mathrm{d}\rho_2 = 2\overline{\tau} \wedge \tau + \omega_1 \wedge \overline{\omega_1} = (-2\alpha_+^2 + 1)\omega_1 \wedge \overline{\omega_1}. \tag{4.3.8}$$

Combining eq. (4.3.7) and eq. (4.3.6) one infers that $-\mathrm{dlog}(\alpha_+) + i(-\rho_1 + 3\rho_2)$ is a $(1,0)$-form. As before, we get

$$\mathrm{d}^C \log(\alpha_+) = -\rho_1 + 3\rho_2.$$

This implies

$$\Delta\log(\alpha_+) = \mathrm{d}(\rho_1 - 3\rho_2). \tag{4.3.9}$$

Let us summarise the results so far.

**Lemma 4.3.6.** *Let $\varphi\colon X \to \mathbb{CP}^3$ be a transverse $J$-holomorphic curve. Then the bundle $\varphi^*(\mathrm{Sp}(2))$ restricts to a $K_F$ bundle $R$ on which the following equations hold*

$$\begin{aligned} \omega_3 &= \alpha_-\omega_1, \qquad \omega_2 = 0, \qquad \tau = \alpha_+\omega_1 \\ \rho_1 &= \frac{1}{8}(-3\mathrm{d}^C \log(\alpha_-) + \mathrm{d}^C \log(\alpha_+)) \\ \rho_2 &= \frac{1}{8}(-\mathrm{d}^C \log(\alpha_-) + 3\mathrm{d}^C \log(\alpha_+)). \end{aligned} \tag{4.3.10}$$

The reduction of the bundle Sp(2) over a transverse $J$-holomorphic curve is summarised in the following table.

| Bundle | Structure Group | Reduction characterised by | Restriction implies |
|---|---|---|---|
| $\varphi^*\mathrm{Sp}(2)$ | $H = S^1 \times S^3$ | | |
| $P$ | $S^1 \times S^1$ | $\omega_2 = 0$, lemma 4.3.2 | $\tau$ is of type $(1,0)$ |
| $Q$ | $K \cong S^1$ | $\omega_3 = \alpha_-\omega_1, \omega_2 = 0$, lemma 4.3.3 | $\mathrm{d}^C\mathrm{log}(\alpha_-) = -3\rho_1 + \rho_2$ |
| $R$ | $K_F \cong \mathbb{Z}_8$ | $\tau = \alpha_+\omega_1$, lemma 4.3.4 | $\mathrm{d}^C\mathrm{log}(\alpha_+) = -\rho_1 + 3\rho_2$ |

Table 4.1: Stepwise reductions of the bundle $\varphi^*\mathrm{Sp}(2)$. In each line, all equations of the rows above also hold. Note that $P$ is defined for any $J$-holomorphic curve while $Q$ and $R$ need the assumption of transversality.

By the uniformisation theorem, any metric on a Riemann surface is (globally) conformally equivalent to a constant curvature metric. For a transverse $J$-holomorphic curve, this factor is given explicitly in terms of the angle functions in the following proposition.

**Proposition 4.3.7.** *The metrics $g_{\mathcal{H}}$ and $g$ are globally conformally flat for any transverse $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$. The conformal factor for $g_{\mathcal{H}}$ is $\gamma^2 = (\alpha_-\alpha_+)^{-1/2}$ and for $g$ it is $\gamma^2(1+\alpha_-^2)$.*

*Proof.* Since the metrics $g_{\mathcal{H}}$ and $g$ only differ by the conformal factor $(1+\alpha_-^2)$ it suffices to prove this statement for $g_{\mathcal{H}}$. First assume that $X$ is simply-connected, i.e. $X \cong \mathbb{D}$ or $\mathbb{C}$. In this case, the bundle $R$ admits a global section $s$. Then $s^*\omega_1$ is a unitary $(1,0)$-form on $X$, satisfying the equation

$$\mathrm{d}(s^*\omega_1) = s^*(i(\rho_1-\rho_2))\wedge\omega_1 = d^C(-i/4\log(\alpha_-\alpha_+))\wedge s^*\omega_1.$$

Hence

$$\mathrm{d}((\gamma^{-1}s^*(\omega_1)) = (\mathrm{d}(\gamma^{-1}) - i\gamma^{-1}d^C\log(\gamma^{-1}))\wedge s^*\omega_1 = (\mathrm{d}(\gamma^{-1}) - id^C(\gamma^{-1}))\wedge s^*\omega_1 = 0$$

since $\mathrm{d}(\gamma^{-1}) - id^C(\gamma^{-1})$ is a $(1,0)$-form. This means that the metric $\gamma^{-2}g_{\mathcal{H}}$ has a closed, unitary $(1,0)$-form $\gamma^{-1}s^*\omega_1$, hence it is flat. If $X$ is not simply-connected we can show the statement by passing to the universal cover. $\square$

Putting eq. (4.3.5), eq. (4.3.9), eq. (4.3.7) and eq. (4.3.8) together gives

$$\begin{aligned} i\Delta\log(\alpha_-) &= (3\alpha_-^2+\alpha_+^2-2)\omega_1\wedge\overline{\omega_1}\\ i\Delta\log(\alpha_+) &= (3\alpha_+^2+\alpha_-^2-2)\omega_1\wedge\overline{\omega_1}. \end{aligned}$$

If we equip $X$ with the metric $g_{\mathcal{H}}$ then $-\frac{1}{2i}\omega_1\wedge\overline{\omega_1}$ becomes the volume form $\mathrm{vol}_{\mathcal{H}}$ on $X$ and we may rewrite the equations as

$$\begin{aligned} \Delta\log(\alpha_-^2) &= -4(3\alpha_-^2+\alpha_+^2-2)\mathrm{vol}_{\mathcal{H}}\\ \Delta\log(\alpha_+^2) &= -4(3\alpha_+^2+\alpha_-^2-2)\mathrm{vol}_{\mathcal{H}}. \end{aligned} \tag{4.3.11}$$

The curvature form on $X$ is then given by

$$\begin{aligned} \mathrm{d}\kappa_{11} &= \overline{\tau}\wedge\tau + \omega_1\wedge\overline{\omega_1} - \omega_3\wedge\overline{\omega_3}\\ &= (1-\alpha_-^2-\alpha_+^2)\omega_1\wedge\overline{\omega_1} = -2i(1-\alpha_-^2-\alpha_+^2)\mathrm{vol}_{\mathcal{H}}. \end{aligned} \tag{4.3.12}$$

Let $\gamma = (\alpha_-\alpha_+)^{-1/4}$ as in proposition 4.3.7, i.e. $\gamma^{-2}g_{\mathcal{H}}$ is flat. Denote by $\Delta_0$ the Laplace operator on functions for the metric $\gamma^{-2}g_{\mathcal{H}}$.

**Theorem 4.3.8.** *Let $\varphi\colon X \to \mathbb{CP}^3$ be a transverse $J$-holomorphic curve. Then the* *functions $\alpha_-, \alpha_+$ satisfy*

$$\begin{aligned}\Delta_0 \log(\alpha_-^2) &= -4(3\alpha_-^2 + \alpha_+^2 - 2)\gamma^2 \\ \Delta_0 \log(\alpha_+^2) &= -4(3\alpha_+^2 + \alpha_-^2 - 2)\gamma^2.\end{aligned} \tag{4.3.13}$$

*These equations are equivalent to the affine 2D Toda lattice equations for $\mathfrak{sp}(2)$. The induced Gauß curvature on $X$ is $2(1 - \alpha_-^2 - \alpha_+^2)$.*

*Proof.* Eq. 4.3.13 is a direct consequence of the discussion preceding the theorem. The formula for the Gauß curvature follows from eq. (4.3.12) since $-i\kappa_{11}$ is the Levi-Civita connection form of $X$. If we define $\hat{\alpha}_\pm = \gamma\alpha_\pm$ such that $\gamma = (\hat{\alpha}_-\hat{\alpha}_+)^{1/2}$ and eq. (4.3.13) becomes

$$\begin{aligned}\Delta_0 \log(\hat{\alpha}_-^2) &= -4(2\hat{\alpha}_-^2 - \hat{\alpha}_-^{-1}\hat{\alpha}_+^{-1}) \\ \Delta_0 \log(\hat{\alpha}_+^2) &= -4(2\hat{\alpha}_+^2 - \hat{\alpha}_-^{-1}\hat{\alpha}_+^{-1}).\end{aligned} \tag{4.3.14}$$

So if we let $\hat{\alpha}_-^2 = \frac{1}{\sqrt{2}}\exp(\Omega_1)$ and $\hat{\alpha}_+^2 = \frac{1}{\sqrt{2}}\exp(-\Omega_2)$ these equations are equivalent to the 2D affine Toda equations for $\mathfrak{sp}(2)$. □

Note the metric induced from $\pi \circ \varphi \colon X \to S^4$ equals $2g_{\mathcal{H}}$ and has hence Gauß curvature equal to $1 - \alpha_-^2 - \alpha_+^2$.

*Remark* 4.3.9. The result could also be deduced from proposition 4.3.5 since a $\tau$-primitive map into $G/T$ is described by the Toda lattice equations for $\mathfrak{g}$ [BW94]. Furthermore, the relationship between Toda lattice equations and minimal surfaces in $S^4$ has already been observed in [Fer+92].

From section 4.2 recall that $I_{\mathcal{H}}, I_{\mathcal{V}}, \mathrm{II}_{\mathcal{H}}$ are all holomorphic sections in different line bundles over $X$. If one of these sections vanishes at a point, then the curve is not transverse at this point and $\alpha_-$ or $\alpha_+$ becomes singular. In proposition 4.3.11 we will show how the defining equations for $\alpha_\pm$ eq. (4.3.13) can locally be extended to the singular set. To this aim, we will make use of the following observation.

**Lemma 4.3.10.** *Let $L$ be a line bundle equipped with a hermitian metric over a Riemann surface $X$ with holomorphic section $s$. Then for each point $x \in X$ there is a neighbourhood $U$ such that $|s| = |z|^{k_x}u$ with $u$ positive and smooth, $z \colon U \to \mathbb{C}$ biholomorphic onto its image, $z(0) = x$ and $k_x$ a non-negative integer.*

*Proof.* For $x \in X$ choose a local non-vanishing section $s' \colon U \to L$ and write $s = fs'$ for a holomorphic function $f \colon U \to \mathbb{C}$. Then $k_x$ is the order of the pole of $f$ at $x$, possibly 0 and the statement follows. □

This lemma can now be applied to study the singularities of $\alpha_\pm$.

**Proposition 4.3.11.** *Let $\varphi\colon X \to \mathbb{CP}^3$ be a $J$-holomorphic curve with neither $I_{\mathcal{H}}, I_{\mathcal{V}}, \mathrm{II}_{\mathcal{H}}$ vanishing everywhere. Then there is a discrete set $S$ such that $\varphi|_{X\setminus S}$ is transverse with angle functions $\alpha_\pm\colon X\setminus S \to (0,\infty)$ satisfying eq. (4.3.13). For each point $x \in S$ there are integers $k_\pm$, not both zero, a neighbourhood $U \subset \mathbb{CP}^3$ of $x$ and a chart $z\colon U \to \mathbb{C}$ with $z(x) = 0$ such that $\alpha_\pm = \hat{\alpha}_\pm |z|^{k_\pm}$ for $\hat{\alpha}_\pm$ smooth and positive on $U$. The metric $g\hat{\gamma}^{-2}$ for $\hat{\gamma} = (\hat{\alpha}_-\hat{\alpha}_+)^{-1/4}$ is flat on $U$. The functions $\hat{\alpha}_\pm$ satisfy the equations*

$$
\begin{aligned}
\hat{\Delta}_0 \log(\hat{\alpha}_-^2) &= -4|z|^{-\frac{1}{2}(k_-+k_+)}(3\hat{\alpha}_-^2|z|^{2k_-} + \hat{\alpha}_+^2|z|^{2k_+} - 2)\hat{\gamma}^2 \\
\hat{\Delta}_0 \log(\hat{\alpha}_+^2) &= -4|z|^{-\frac{1}{2}(k_-+k_+)}(3\hat{\alpha}_+^2|z|^{2k_+} + \hat{\alpha}_-^2|z|^{2k_-} - 2)\hat{\gamma}^2
\end{aligned}
$$

*which are defined on all of $U$. Here $\hat{\Delta}_0$ is the Laplace operator for $g\hat{\gamma}^{-2}$.*

*Proof.* Observe that

$$\alpha_- = \frac{\|I_{\mathcal{V}}\|}{\|I_{\mathcal{H}}\|}, \quad \alpha_+ = \frac{\|\mathrm{II}_{\mathcal{H}}\|}{\|I_{\mathcal{H}}\|}$$

with the norms on the different line bundles induced from the metric on $\varphi^* T\mathbb{CP}^3$. Hence, $S$ is the discrete set where $I_{\mathcal{H}}, I_{\mathcal{V}}$ or $\mathrm{II}_{\mathcal{H}}$ vanish. By lemma 4.3.10 we can locally write $\alpha = \hat{\alpha}|z|^{k_\pm}$ for $k \in \mathbb{Z}$. The rest of the statement follows from the fact that $\log|z|$ is harmonic since $z$ is holomorphic. □

## 4.4 The Second Fundamental Form

As a nearly Kähler manifold, $\mathbb{CP}^3$ comes equipped with two natural connections. The Levi-Civita connection $\nabla$ and the nearly Kähler connection $\overline{\nabla}$. The second fundamental form of a $J$-holomorphic curve is the same for $\nabla$ and $\overline{\nabla}$. Despite $\overline{\nabla}$ having torsion it is the connection that is easier to work with since it preserves the almost complex structure $J$.

For a fixed $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$ consider the map $\Theta\colon TX \to \nu$ which is defined as $\Theta = -\alpha_-^2 \mathrm{Id}_{\mathcal{H}} + \mathrm{Id}_{\mathcal{V}}$. Observe that $\Theta$ is injective and let $N_1 = \Theta(TX)$. Denote by $N_2$ the orthogonal complement of $N_1$ in $\nu$. It turns out that $N_2$ is in fact equal to the kernel of the orthogonal projection $\nu \to \mathcal{V}_X$. In other words, $\varphi^*(T\mathbb{CP}^3)$ splits into an orthogonal sum of complex line bundles

$$\varphi^*(T\mathbb{CP}^3) = TX \oplus N_1 \oplus N_2 \tag{4.4.1}$$

where $N_1 \cong TX$ and $N_2 \cong (TX)^{-2}$. This splitting is related to the reduction of

$\varphi^*(\mathrm{Sp}(2))$ to $Q$. If

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\alpha_-^2+1}} & 0 & \frac{\alpha_-}{\sqrt{\alpha_-^2+1}} \\ \frac{-\alpha_-}{\sqrt{\alpha_-^2+1}} & 0 & \frac{1}{\sqrt{\alpha_-^2+1}} \\ 0 & 1 & 0 \end{pmatrix}}_{T^{-1}:=} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \tag{4.4.2}$$

and $s$ is a (local) section $s\colon X \to Q$ then $s^*(u_1, u_2, u_3)$ is a unitary frame as well. Let $(f_1, f_2, f_3)$ be the dual frame of $s^*(u_1, u_2, u_3)$. Then $f_1$ always takes values in $TX$, $f_2$ in $N_1$ and $f_3$ in $N_2$. A frame with this property will be called a $Q$-adapted frame from now on. The connection matrix $A_u$ for the frame $\{f_1, f_2, f_3\}$ is then computed via applying the base change eq. (4.4.2) to the connection matrix $A_\omega$ in eq. (2.2.2) and using eq. (4.3.4)

$$A_u = T^{-1}A_\omega T + T^{-1}\mathrm{d}T = \begin{pmatrix} \frac{i\left(\left(2\alpha_-^2-1\right)\rho_1+\rho_2\right)}{\alpha_-^2+1} & \frac{2}{\alpha_-^2+1}\mathrm{d}^{0,1}\alpha_- & -\frac{\bar{\tau}}{\sqrt{\alpha_-^2+1}} \\ -\frac{2}{\alpha_-^2+1}\mathrm{d}^{1,0}\alpha_- & -\frac{i\left(\left(\alpha_-^2-2\right)\rho_1-\alpha_-^2\rho_2\right)}{\alpha_-^2+1} & -\frac{\alpha_-\bar{\tau}}{\sqrt{\alpha_-^2+1}} \\ \frac{\tau}{\sqrt{\alpha_-^2+1}} & \frac{\alpha_-\tau}{\sqrt{\alpha_-^2+1}} & -i(\rho_1+\rho_2) \end{pmatrix}. \tag{4.4.3}$$

**Lemma 4.4.1.** *For a $Q$-adapted frame $f_1, f_2, f_3$ and co-frame $u_1, u_2, u_3$, the second fundamental form of $X$ is equal to*

$$\mathrm{I\!I}_{\mathbb{CP}^3} = \mathrm{I\!I}_1 \otimes f_2 \otimes u_1 + \mathrm{I\!I}_2 \otimes f_3 \otimes u_1 \tag{4.4.4}$$

*for $\mathrm{I\!I}_1 = -\frac{2}{\alpha_-^2+1}\mathrm{d}^{1,0}\alpha_-$ and $\mathrm{I\!I}_2 = \frac{\tau}{\sqrt{\alpha_-^2+1}} = \frac{\alpha_+\omega_1}{\sqrt{\alpha_-^2+1}}$. We see that transverse points of a $J$-holomorphic curve are never totally geodesic. Conversely, the frame $\{f_1, f_2, f_3\}$ and the first and second fundamental form of a transverse curve determine $\alpha_-$ and $\alpha_+$.*

*Proof.* The expression for $\mathrm{I\!I}_{\mathbb{CP}^3}$ can be read off from eq. (4.4.3). For the last statement, note that $\mathrm{I\!I}_1 = -2\mathrm{d}^{1,0}\arctan(\alpha_-)$. Assume that $\alpha_-, \alpha_+$ and $\alpha'_-, \alpha'_+$ are two pairs of functions inducing the same first and second fundamental form. Then $\arctan(\alpha_-)$ and $\arctan(\alpha'_-)$ differ by a real constant. Equivalently,

$$\alpha'_- = \frac{\alpha_- + C}{1 - \alpha_- C}$$

for a constant $C \in \mathbb{R}$. Furthermore, the equation for $\mathrm{I\!I}_2$ implies that

$$\alpha'_+ = \alpha_+\sqrt{\frac{{\alpha'_-}^2 + 1}{\alpha_-^2 + 1}}.$$

Putting all together gives that

$$\frac{\alpha_-'\alpha_+'}{\alpha_-\alpha_+} = -\frac{(\alpha_- + C)\sqrt{\frac{C^2+1}{(\alpha_-C-1)^2}}}{\alpha_-(\alpha_-C-1)}$$

which is a constant by proposition 4.3.7. However, this is only possible if $C = 0$ or if $\alpha_-$ is constant. But solutions with $\alpha_-$ constant force $\alpha_-^2 = \alpha_+^2 = 1/2$ and hence $C = 0$. $\square$

Different $Q$-adapted frames are related by an action of $K \cong S^1$. To work out tensors which are invariant under this action, let $\lambda = (e^{i\theta}, e^{3i\theta})$ be an element in $K$ and denote $-2\theta = \vartheta$. The action of $\lambda$ introduces a gauge transformation leading to a transformed set of tensors $(\omega_1', \omega_2', \omega_3', \tau')$ on $Q$. Note that

$$(\omega_1', \omega_2', \omega_3') = (e^{i\vartheta}\omega_1, e^{-2i\vartheta}\omega_2, e^{i\vartheta}\omega_3)$$

which leads to

$$(u_1', u_2', u_3', \tau') = (e^{i\vartheta}u_1, e^{i\vartheta}u_2, e^{-2i\vartheta}u_3, e^{-3i\vartheta}\tau)$$

and hence

$$(f_1', f_2', f_3') = (e^{-i\vartheta}f_1, e^{-i\vartheta}f_2, e^{2i\vartheta}f_3).$$

Consequently, the tensors $\tau \otimes u_i \otimes f_3, u_i \otimes f_j, u_3 \otimes f_3$ for $i, j = 1, 2$ are all invariant under $K$ and hence correspond to tensors on $X$. Regard the second fundamental form $\mathrm{I\!I}_{\mathbb{CP}^3}$ as a section in $\Omega^{1,0}(X, \mathrm{Hom}(TX, \nu))$. The connection $\overline{\nabla}$ induces connections $\overline{\nabla}^T$ and $\overline{\nabla}^\perp$ on $TX$ and $\nu$ so that

$$\bar{\partial}_{\overline{\nabla}}\mathrm{I\!I}_{\mathbb{CP}^3} = \mathrm{d}_{\overline{\nabla}}\mathrm{I\!I}_{\mathbb{CP}^3} \in \Omega^{1,1}(X, \mathrm{Hom}(TX, \nu)).$$

We have seen that a transverse $J$-holomorphic curve has no points which are totally geodesic. Having holomorphic second fundamental form is a natural generalisation of being totally geodesic.

**Theorem 4.4.2.** *The differential of the second fundamental form is equal to*

$$\mathrm{d}_{\overline{\nabla}}(\mathrm{I\!I}_{\mathbb{CP}^3}) = \frac{2i\alpha_-}{\alpha_-^2 + 1}(-2 + 3\alpha_-^2)\mathrm{vol}_{\mathcal{H}} \otimes u_1 \otimes f_2.$$

*In particular, there is no transverse $J$-holomorphic curve with holomorphic second fundamental form.*

*Proof.* Using eq. (4.4.4), the differential of $\mathrm{I\!I}_{\mathbb{CP}^3}$ is computed by

$$\begin{aligned}\mathrm{d}_{\overline{\nabla}}\mathrm{I\!I}_{\mathbb{CP}^3} =& \mathrm{d}\mathrm{I\!I}_1 u_1 \otimes f_2 - \mathrm{I\!I}_1 \wedge \overline{\nabla}^T u_1 \otimes f_2 - u_1 \otimes \mathrm{I\!I}_1 \wedge \overline{\nabla}^\perp f_2 \\ &+ \mathrm{d}\mathrm{I\!I}_2 u_1 \otimes f_3 - \mathrm{I\!I}_2 \wedge \overline{\nabla}^T u_1 \otimes f_3 - u_1 \otimes \mathrm{I\!I}_2 \wedge \overline{\nabla}^\perp f_3.\end{aligned}$$

This expression takes values in $\Omega^{1,1}(X, TX^\vee \otimes \nu)$ and we compute the components in $u_1 \otimes f_2$ and $u_1 \otimes f_3$ separately. Note that

$$\begin{aligned}\overline{\nabla}^T(u_1) &= -\frac{i\left(\left(2\alpha_-^2 - 1\right)\rho_1 + \rho_2\right)}{\alpha_-^2 + 1} \otimes u_1 \\ \overline{\nabla}^\perp(f_2) &= -\frac{i\left(\left(\alpha_-^2 - 2\right)\rho_1 - \alpha_-^2\rho_2\right)}{\alpha_-^2 + 1} \otimes f_2 + \frac{\alpha_-\tau}{\sqrt{\alpha_-^2 + 1}} \otimes f_3 \\ \overline{\nabla}^\perp(f_3) &= -\frac{\alpha_-\bar{\tau}}{\sqrt{\alpha_-^2 + 1}} \otimes f_2 - i(\rho_1 + \rho_2) \otimes f_3\end{aligned}$$

and

$$\begin{aligned}\mathrm{d}\mathrm{I\!I}_1 &= i\frac{1 - \alpha_-^2}{(1 + \alpha_-^2)^2}\mathrm{d}\alpha_- \wedge \mathrm{d}^C\log(\alpha_-) - \frac{i\alpha_-}{1 + \alpha_-^2}\Delta\log(\alpha_-) \\ &= 2i\frac{1 - \alpha_-^2}{(1 + \alpha_-^2)^2}\mathrm{d}^{1,0}\alpha_- \wedge \mathrm{d}^C\log(\alpha_-) + \frac{\alpha_-}{1 + \alpha_-^2}(-3\alpha_-^2 - \alpha_+^2 + 2)\omega_1 \wedge \overline{\omega_1}.\end{aligned}$$

Furthermore, the $u_1 \otimes f_2$ component of $-\mathrm{I\!I}_1 \wedge \overline{\nabla}^T u_1 \otimes f_2 - u_1 \otimes \mathrm{I\!I}_1 \wedge \overline{\nabla}^\perp f_2$ is equal to

$$\begin{aligned}&\frac{\mathrm{I\!I}_1 i}{\alpha_-^2 + 1} \wedge ((2\alpha_-^2 - 1)\rho_1 + \rho_2 + (\alpha_-^2 - 2)\rho_1 - \alpha_-^2\rho_2) \\ =& \frac{i(\alpha_-^2 - 1)}{\alpha_-^2 + 1}\mathrm{I\!I}_1 \wedge (3\rho_1 - \rho_2) = i\frac{1 - \alpha_-^2}{1 + \alpha_-^2}\mathrm{I\!I}_1 \wedge \mathrm{d}^C\log(\alpha_-).\end{aligned}$$

Finally, the $u_1 \otimes f_2$ component of $-u_1 \otimes \mathrm{I\!I}_2 \wedge \overline{\nabla}^\perp f_3$ is equal to

$$\mathrm{I\!I}_2 \wedge \frac{\alpha_-\bar{\tau}}{\sqrt{\alpha_-^2 + 1}} = \frac{\alpha_-\alpha_+^2}{\alpha_-^2 + 1}\omega_1 \wedge \overline{\omega_1}. \tag{4.4.5}$$

Observe that various terms cancel and that the $u_1 \otimes f_2$ component of $\mathrm{d}_{\overline{\nabla}}\mathrm{I\!I}_{\mathbb{CP}^3}$ is

$$\frac{\alpha_-}{\alpha_-^2 + 1}(-3\alpha_-^2 + 2)\omega_1 \wedge \overline{\omega_1} = \frac{2i\alpha_-}{\alpha_-^2 + 1}(3\alpha_-^2 - 2)\mathrm{vol}_{\mathcal{H}}.$$

The $u_1 \otimes f_3$ component is computed in an analogous way and equal to

$$\frac{\alpha_-}{\alpha_-^2 + 1}\mathrm{I\!I}_2 \wedge (\mathrm{d}\alpha_- - i\mathrm{d}^C\alpha_-) = 0 \tag{4.4.6}$$

since both $\mathrm{I\!I}_2$ and $\mathrm{d}\alpha_- - i\mathrm{d}^C\alpha_-$ are $(1,0)$ forms. This proves the formula

$$\mathrm{d}_{\overline{\nabla}}(\mathrm{I\!I}_{\mathbb{CP}^3}) = \frac{2i\alpha_-}{\alpha_-^2+1}(-2+3\alpha_-^2)\mathrm{vol}_{\mathcal{H}} \otimes u_1 \otimes f_2.$$

For a transverse $J$-holomorphic curve, $\alpha_-$ is always positive. So the second fundamental form is holomorphic if and only if $\alpha_-$ is constant to $\sqrt{2/3}$. However, no such solution exists for eq. (4.3.13). □

*Remark* 4.4.3. Let $f\colon X \to S^4$ be a minimal immersion with twistor lift $\varphi\colon X \to \mathbb{CP}^3$ and associated angle functions $\alpha_\pm$. Then $f$ induces the metric $2g_{\mathcal{H}}$ on $X$ from $S^4$ and $\alpha_\pm = 2\|\mathrm{d}\varphi_\pm^{\mathcal{V}}\|$. In this setting, we have defined the second fundamental forms $\mathrm{I\!I}_{S^4}, \mathrm{I\!I}_{\mathcal{H}}$ and $\mathrm{I\!I}_{\mathbb{CP}^3}$, depending on the ambient space or bundle. Curves with $\mathrm{I\!I}_{\mathcal{H}} \equiv 0$ are in one-to-one correspondence with superminimal curves while

$$\mathrm{I\!I}_{S^4} \equiv 0 \Leftrightarrow \mathrm{I\!I}_{\mathbb{CP}^3} \equiv 0 \Leftrightarrow \varphi \text{ parametrises a superminimal projective line.}$$

In fact, by proposition 3.2.4 and lemma 4.4.1 each of $\alpha_\pm$, $\mathrm{I\!I}_{\mathbb{CP}^3}$ and $\mathrm{I\!I}_{S^4}$ determines the other two.

The nearly Kähler connection $\overline{\nabla}$ preserves $J$ and $\nu$ is a complex subbundle of $T\mathbb{CP}^3$. Hence, $\overline{\nabla}^\perp$ defines a holomorphic structure on $\nu$. In the rest of this section we determine this holomorphic structure for a transverse torus $\varphi\colon X \to \mathbb{CP}^3$. The degree of $\nu$ is zero since the first Chern class of any nearly Kähler manifold vanishes, i.e.

$$c_1(\nu) = c_1(T\mathbb{CP}^3) - c_1(TX) = 0.$$

Let $\mathrm{Bun}(r,d)$ be the space of indecomposable holomorphic bundles of rank $r$ and degree $d$ over $X$. By Atiyah's classification of holomorphic vector bundles over elliptic curves [Ati57], $\mathrm{Bun}(2,0)$ is isomorphic to a two-torus. For any element $E \in \mathrm{Bun}(2,0)$ the line bundle $\Lambda^2(E)$ is trivial. This is consistent with the fact that $\Lambda^2(\nu) = TX$. The space $\mathrm{Bun}(2,0)$ has a distinguished element $E_0$, the unique non-trivial extension of the sequence

$$0 \to \mathbb{C} \to E \to \mathbb{C} \to 0.$$

Based on this, there are a-priori three possibilities for $\nu$. Either $\nu$ is decomposable, isomorphic to $E_0$ or another element in $\mathrm{Bun}(2,0)$. In the following we will see that for a transverse $J$-holomorphic torus in $\mathbb{CP}^3$, $\nu$ is always isomorphic to $E_0$.

By eq. (4.2.1), not only $TX \cong L$ but also $N$ is trivial for a transverse torus. Furthermore, $\mathrm{I\!I}_{\mathcal{H}}$ is a non-vanishing holomorphic section of $\Omega^{1,0}(X, L^\vee \otimes N)$. Since $H^{1,0}(X, L^\vee \otimes N) \cong \mathbb{C}$, all non-trivial extensions of the sequence

$$0 \to L \to E \to N \to 0$$

are isomorphic to each other, and $\varphi^*\mathcal{H}$ is such an extension and in particular $\varphi^*\mathcal{H} \cong E_0$. On the other hand, the composition

$$\varphi^*\mathcal{H} \to \varphi^*(T\mathbb{CP}^3) \to \nu$$

is an isomorphism since $\varphi$ is transverse, we have proven.

**Proposition 4.4.4.** *As a holomorphic bundle, the normal bundle $\nu$ of a transverse torus is the unique non-trivial extension of the sequence*

$$0 \to \mathbb{C} \to \nu \to \mathbb{C} \to 0.$$

In particular, $\nu$ always has a holomorphic section, which can be written down explicitly in an adapted frame. Let $\sigma_{ij}$ be the components of the connection matrix $A_u$. Assume that the frame $\{f_1, f_2, f_3\}$ is $R$-adapted, such that eq. (4.3.10) hold. Then $s = s_2 f_2 + s_3 f_3$ describes a general section in the normal bundle for $s_2, s_3 \in \Omega(X, \mathbb{C})$. By the Leibniz rule for $\bar{\partial}$, holomorphic sections are solutions of the equation

$$s_2\sigma_{32}(\frac{\partial}{\partial \bar{z}}) + s_3\sigma_{33}(\frac{\partial}{\partial \bar{z}}) + \frac{\partial}{\partial \bar{z}}(s_3) = 0 \tag{4.4.7}$$

$$s_2\sigma_{22}(\frac{\partial}{\partial \bar{z}}) + s_3\sigma_{23}(\frac{\partial}{\partial \bar{z}}) + \frac{\partial}{\partial \bar{z}}(s_2) = 0. \tag{4.4.8}$$

Since $\sigma_{32}$ is of type $(1,0)$ it annihilates $\frac{\partial}{\partial \bar{z}}$. Note that on $X$, $u_1 = \sqrt{1+\alpha_-^2}\omega_1$ and hence by proposition 4.3.7 we can find a local coordinate $z$ on $X$ such that

$$\mathrm{d}z = \frac{(\alpha_-\alpha_+)^{1/4}}{\sqrt{1+\alpha_-^2}} u_1.$$

Then eq. (4.4.7) reduces to

$$s_3\sigma_{33}(\frac{\partial}{\partial \bar{z}}) + \frac{\partial}{\partial \bar{z}}(s_3) = 0$$

and

$$\sigma_{33}(\frac{\partial}{\partial \bar{z}}) = \frac{\partial}{\partial \bar{z}} \log(\alpha_+^{-1/2}\alpha_-^{1/2}).$$

Hence, all solutions of eq. (4.4.7) are given by

$$s_3 = c\alpha_-^{-1/2}\alpha_+^{1/2}$$

for a constant $c \in \mathbb{C}$. Furthermore, we have

$$\sigma_{23}(\frac{\partial}{\partial \bar{z}}) = \frac{-(\alpha_-\alpha_+)^{3/4}}{\sqrt{1+\alpha_-^2}}$$
$$\sigma_{22}(\frac{\partial}{\partial \bar{z}}) = \frac{\partial}{\partial \bar{z}}(\log(\alpha_+^{1/4}\alpha_-^{-3/4}\sqrt{1+\alpha_-^2})).$$

So, define $s_2' = s_2\alpha_+^{1/4}\alpha_-^{-3/4}\sqrt{1+\alpha_-^2}$ which makes eq. (4.4.8) equivalent to

$$s_3\frac{-(\alpha_-\alpha_+)^{3/4}}{\sqrt{1+\alpha_-^2}} + {\alpha_+}^{-1/4}{\alpha_-}^{+3/4}(1+\alpha_-^2)^{-1/2}\frac{\partial}{\partial \bar{z}}s_2' = 0$$

and this in turn results in

$$\frac{\partial}{\partial \bar{z}}s_2' = c{\alpha_+}^{3/2}\alpha_-^{-1/2}.$$

But the function $\alpha_-^{-1/2}\alpha_+^{1/2}$ is nowhere vanishing which forces $c = 0$ and $s_2'$ to be a constant. This computation implies the following two lemmas.

**Lemma 4.4.5.** *Let $X$ be a transverse torus, then $h^0(X, \nu) = 1$.*

**Lemma 4.4.6.** *The holomorphic section $s_2'f_2$ induces an exact sequence of holomorphic bundles*

$$0 \to \mathbb{C} \to \nu \to \mathbb{C} \to 0.$$

*Proof.* Since $s_2$ is non-vanishing the section gives an injection $\mathbb{C} \to \nu$. The quotient bundle admits a non-vanishing holomorphic section because $\sigma_{33}^{0,1}$ is $\bar{\partial}$-exact. Hence, the quotient is trivial. □

## 4.5 A Bonnet-type Theorem

The uniqueness part of the classical Bonnet theorem says that two surfaces $\Sigma, \Sigma' \subset \mathbb{R}^3$ with the same first and second fundamental form I and II necessarily differ by an isometry of $\mathbb{R}^3$. The existence part states that given a simply-connected surface $\Sigma$ with the tensors $I, \mathrm{II}$ defined on $\Sigma$ satisfying the Gauß and Codazzi equation there is an immersion $\Sigma \to \mathbb{R}^3$ with induced first and second fundamental form equal to I and II.

One way to prove this statement is via the theorem of Maurer-Cartan: Let $G$ be a Lie Group with Maurer-Cartan form $\Omega_{MC}$, let $N$ be connected and simply-connected, equipped with $\eta \in \Omega^1(N, \mathfrak{g})$ satisfying $\mathrm{d}\eta + \frac{1}{2}[\eta, \eta] = 0$. Then there exists a smooth map $f\colon N \to G$, unique up to left translation in $G$, such that $f^*\Omega_{MC} = \eta$. In this section we will show, also using the theorem of Maurer-Cartan, an analogue of Bonnet's theorem for $J$-holomorphic curves in $\mathbb{CP}^3$.

We have seen that given a transverse $J$-holomorphic curve $\varphi\colon X \to \mathbb{CP}^3$ there are functions $\alpha_-, \alpha_+, \gamma$ that satisfy eq. (4.3.13), where $\gamma$ is determined by $\alpha_-$ and $\alpha_+$. Up to a constant factor, the functions $(\alpha_-, \alpha_+, \gamma)$ determine the first and second fundamental form of $X$. Composing $\varphi$ with an element in an automorphism in $\mathrm{Sp}(2)$ leaves the quantities $(\alpha_-, \alpha_+)$ invariant since they are defined via components of $\Omega_{MC}$. Besides, the data $(\alpha_-, \alpha_+)$ determines the first and second fundamental form up to a constant.

If a Bonnet-theorem holds for $J$-holomorphic curves in $\mathbb{CP}^3$ then the first and second fundamental form determine the curve up to isometries. Furthermore it would say that such a curve exists if the Gauß and Codazzi equations are satisfied. In our setting, the system eq. (4.3.11) plays the role of these equations. This raises the following question. *Are the solutions of eq.* (4.3.11) *in one to one correspondence to transverse $J$-holomorphic curves up to isometries?* Later, we will see that the answer to this question is no because there are periodic solutions of eq. (4.3.11) which do not descend to a two-torus. However, there is a positive result for when $X$ is simply-connected.

**Lemma 4.5.1.** *Let $X$ be a simply-connected Riemann surface equipped with a metric $k$. Let furthermore $\alpha_-, \alpha_+\colon X \to \mathbb{R}^{>0}$ such that eq.* (4.3.13) *are satisfied for $\gamma = (\alpha_-\alpha_+)^{-1/4}$ and that $\gamma^{-2}k$ is flat. Then there is a $J$-holomorphic immersion $\varphi\colon X \to \mathbb{CP}^3$ such that the induced metric satisfies $g_{\mathcal{H}} = k$. The angle functions of $\varphi$ are $\alpha_-$ and $\alpha_+$. The immersion is unique up to isometries of $\mathbb{CP}^3$ and an element in $S^1/\mathbb{Z}_4 \cong S^1$ which parametrises a choice of a unitary $(1,0)$-form $\omega_0$ on $X$ such that* $\mathrm{d}(\gamma^{-1}\omega_0) = 0$.

*Proof.* Since $X$ is simply-connected and globally conformally flat, it is isomorphic to $\mathbb{C}$ as a complex manifold by the uniformisation theorem. In particular, $\Omega^{1,0}(X)$ is trivial as a bundle, let $\omega_0$ be a unitary $(1,0)$-form on $X$. Arguing similarly as in the proof of proposition 4.3.7 we see that, since $\gamma^{-2}k$ is flat,

$$\mathrm{d}\omega_0 = i\mathrm{d}^C \log(\gamma) \wedge \omega_0. \tag{4.5.1}$$

Now, define a $\mathfrak{sp}(2)$-valued one-form on $X$ by

$$\eta_{\omega_0} = \begin{pmatrix} \frac{i}{8}(-3\mathrm{d}^C\log(\alpha_-) + \mathrm{d}^C\log(\alpha_+)) + j\alpha_-\overline{\omega_0} & -\frac{\omega_0}{\sqrt{2}} \\ \frac{\omega_0}{\sqrt{2}} & \frac{i}{8}(-\mathrm{d}^C\log(\alpha_-) + 3\mathrm{d}^C\log(\alpha_+)) + j\alpha_+\omega_0. \end{pmatrix}.$$

And observe that

$$\mathrm{d}\eta_{\omega_0} + \frac{1}{2}[\eta_{\omega_0}, \eta_{\omega_0}] = 0$$

is equivalent to eq. (4.3.13) and eq. (4.5.1). Hence, by Cartan's theorem, there is an immersion $\Phi\colon X \to \mathrm{Sp}(2)$ such that $\Phi^*(\Omega_{MC}) = \eta$ which is unique up to left multiplication in Sp(2). Note that $\Phi^*(\Omega_{MC}) = \eta$ is equivalent to the equations

$$\begin{aligned} \omega_0 &= \Phi^*(\omega_1) & \mathrm{d}^C(\log(\alpha_-)) &= -\Phi^*(3\rho_1 - \rho_2) & \omega_0 &= \alpha_-\Phi^*(\omega_3) \\ 0 &= \Phi^*(\omega_2) & \mathrm{d}^C(\log(\alpha_+)) &= -\Phi^*(\rho_1 - 3\rho_2) & \omega_0 &= \alpha_+\Phi^*\tau. \end{aligned} \tag{4.5.2}$$

Consider the map $\varphi = \pi \circ \Phi \colon X \to \mathbb{CP}^3$

$$\begin{tikzcd} & \mathrm{Sp}(2) \arrow[d, "\pi"] \\ X \arrow[ur, "\Phi"] \arrow[r, "\varphi"] & \mathbb{CP}^3 \end{tikzcd} .$$

Then $\varphi$ is also an immersion because $v \in \ker(\mathrm{d}\varphi) \subset T^{1,0}(X)$ is equivalent to $\Phi^*\omega_i(v) = \omega_i(\mathrm{d}\Phi(v)) = 0$ for $i = 1, 2, 3$. By eq. (4.5.2), this implies that $\omega_0(v) = 0$ and hence $v = 0$. Furthermore, $\varphi$ is $J$-holomorphic since $\Phi$ pulls back the forms $\omega_i$ to multiples of $\omega_0$. Since $\omega_1$ is unitary for $g_{\mathcal{H}}$ and $\omega_0$ for $k$ and $\Phi^*(\omega_1) = \omega_0$ the metric $g_{\mathcal{H}}$ induced by $\varphi$ is equal to $k$. Since left-multiplication on Sp(2) acts on $\mathbb{CP}^3$ by isometries it remains to prove that choosing $e^{i\vartheta}\omega_0$ as unitary $(1,0)$-form on $X$ yields, up to isometries, the same immersion $\varphi \colon X \to \mathbb{CP}^3$ as $\omega_0$ if $e^{4i\vartheta} = 1$. Let $R_\lambda$ be the right multiplication of an element $\lambda = (e^{i\vartheta}, e^{i\vartheta})$ on Sp(2). Then $R_\lambda^*(\Omega_{MC}) = \mathrm{Ad}_{\lambda^{-1}}(\Omega_{MC})$. From our knowledge of the adjoint action of $K$ on $\mathfrak{sp}(2)$ we infer that

$$R_\lambda^*(\omega_1, \omega_2, \omega_3, \tau, \rho_1, \rho_2) = (e^{-2i\vartheta}\omega_1, e^{4i\vartheta}\omega_2, e^{-2i\vartheta}\omega_3, e^{6i\vartheta}\tau, \rho_1, \rho_2).$$

Hence, if $e^{4i\vartheta} = 1$ then $\omega_1$ and $\tau$ transform in the same way. This means that in this case if $\varphi$ satisfies $\varphi^*\Omega_{MC} = \eta_{\omega_0}$ then $(\varphi \circ R_\lambda)^*(\Omega_{MC}) = \eta_{e^{i\vartheta}\omega_0}$. But right multiplication on Sp(2) does not affect the immersion $\varphi \colon X \to \mathbb{CP}^3$. $\square$

Note that lemma 4.5.1 as it is stated requires $X$ to be equipped with a fixed metric. However, it is more natural to let the metric be one of the quantities to be determined, such as $\alpha_-$ and $\alpha_+$. Think of $X$ as equipped with a flat metric and consider a solution of eq. (4.3.13) with $\gamma = (\alpha_-\alpha_+)^{-1/4}$. Then $(\alpha_-, \alpha_+)$ is a solution of eq. (4.3.11) with the metric $g = \gamma^2 g_0$. However, we could also have chosen the metric $\lambda^2 g_0$ for a constant $\lambda > 0$. In general, lemma 4.5.1 will produce a different immersion $\varphi_\lambda \colon X \to \mathbb{CP}^3$. The following theorem is then a reformulation of lemma 4.5.1.

**Theorem 4.5.2.** *Let $X$ be a simply-connected Riemann surface and let $(\alpha_-, \alpha_+)$ be solutions of eq.* (4.3.13) *for some flat metric on $X$. Then there is a $J$-holomorphic immersion $\varphi \colon X \to \mathbb{CP}^3$ such that the angle functions $\varphi$ are exactly $\alpha_-$ and $\alpha_+$. The immersion is unique up to isometries of $\mathbb{CP}^3$, a choice $\lambda > 0$ and an element in $S^1/\mathbb{Z}_4 \cong S^1$ which parametrises a choice of a unitary $(1,0)$-form $\omega_0$ on $X$ such that* $\mathrm{d}(\gamma^{-1}\omega_0) = 0$.

In other words, a solution $(\alpha_-, \alpha_+)$ of eq. (4.3.13) specifies the immersion up to isometries and a choice of a constant $(\lambda, \omega_0) \in \mathbb{C}^*$. In the case when the solutions $(\alpha_-, \alpha_+)$ and hence $g$ have symmetries, the different embeddings $\varphi_{\lambda,\gamma}$ might come from reparametrisations of $X$. Consider for example, $X = \mathbb{C}$ and let $L_\lambda \colon \mathbb{C} \to \mathbb{C}$ be the multiplication by $\lambda$. Define furthermore $\alpha_{-\lambda} = \alpha_- \circ L_\lambda, \alpha_{+\lambda} = \alpha_+ \circ L_\lambda, g_\lambda = g \circ L_\lambda$. If $(\alpha_-, \alpha_+)$ solves eq. (4.3.11) for a metric $g$ on $\mathbb{C}$ then $(\alpha_{-\lambda}, \alpha_{+\lambda})$ is a solution of eq. (4.3.11) for the metric $\lambda^2 g_\lambda$. This solution comes from the $J$-holomorphic curve $\varphi \circ L_\lambda$. In particular, if $g_\lambda = g$ then the different immersions $\varphi_\lambda$ come from reparametrisations of $X$ by $L_\lambda$. Similarly, if the induced $g$ has radial symmetry, different choices of $\omega_0$ correspond to reparametrisations by rotations.

### 4.5.1 Special Solutions of the Toda Equations

Note that eq. (4.3.13) are symmetric in $\alpha_-$ and $\alpha_+$, meaning a distinguished set of solutions is of the form $\alpha = \alpha_- = \alpha_+$. This reduces eq. (4.3.13) to

$$\Delta_0 \alpha = -8\sqrt{2}\sinh(\alpha) \tag{4.5.3}$$

which becomes the Sinh-Gordon equation after rescaling the metric with a constant factor. This is somewhat similar to the situation in $S^3 \times S^3$ where $J$-holomorphic curves with $\Lambda = 0$ are locally described by the same equation.

**Proposition 4.5.3.** *Transverse $J$-holomorphic curves in $\mathbb{CP}^3$ with $\alpha_- = \alpha_+$ are locally described by the same equation as constant mean curvature tori in $\mathbb{R}^3$.*

Geometrically, the condition $\alpha_- = \alpha_+$ is satisfied for a $J$-holomorphic curve if the corresponding minimal surface $X \to S^4$ lies in a totally geodesic $S^3$, see lemma 3.2.3.

The Clifford torus plays a special role because it is a $\mathbb{T}^2$ group orbit. Any $J$-holomorphic curve that is isometric to its twistor lift will be referred to as Clifford torus. Since isometries leave $\alpha_-$ and $\alpha_+$ invariant this means that $\alpha_-$ and $\alpha_+$ are constant on Clifford tori. There is in fact only one solution for $\alpha_-$ and $\alpha_+$ constant.

**Lemma 4.5.4.** *If either $\alpha_-$ or $\alpha_+$ is constant then both must be constant and equal to $1/\sqrt{2}$.*

The following result is known for minimal tori in $S^4$.

**Proposition 4.5.5.** *Let $\varphi \colon X \to \mathbb{CP}^3$ be a transverse $J$-holomorphic curve such that the induced metric $g_{\mathcal{H}}$ is flat and $X$ is equal to $\mathbb{T}^2$ or $\mathbb{C}$ as a complex manifold. Then $\varphi$ parametrises a Clifford torus for $X = \mathbb{T}^2$ and its universal cover if $X = \mathbb{C}$.*

*Proof.* By passing to the universal cover it suffices to assume $X = \mathbb{C}$. Furthermore, by eq. (4.3.12), flatness of $g_{\mathcal{H}}$ is equivalent to $1 = \alpha_-^2 + \alpha_+^2$ and so eq. (4.3.13) implies that $\log(\alpha_-^2(1-\alpha_-^2))$ is harmonic. It is also bounded because $\alpha_-^2, \alpha_+^2 > 0$ and hence constant. Since $\alpha_-$ and $\alpha_+$ must be constant on Clifford tori it follows from lemma 4.5.4 that $\alpha_-^2 = \alpha_+^2 = 1/2$. Now, theorem 4.5.2 can be applied to prove uniqueness. Note that since $\mathbb{C}$ carries the flat metric, a different choice of $(\omega_0, \lambda)$ amounts to applying an isometry of $X$. $\square$

Consider $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ equipped with the metric $g_{m,k} = \mathrm{d}x^2 + f_{m,k}(x)\mathrm{d}y^2$ for $f_{m,k}(x) = m^2\cos(x)^2 + k^2\sin(x)^2$ and $k, m \in \mathbb{R}\setminus\{0\}$. Let $\hat{f}_{m,k} = \frac{\sqrt{8}}{km} f_{m,k}$ and $\alpha_- = \alpha_+ = 2\hat{f}_{m,k}^{-1}$, which implies $\gamma = \alpha_-^{-1/2}$. Furthermore, $\gamma^{-2}g = g_0$ is a flat metric with Laplace operator

$$\Delta_0 = (\sqrt{\hat{f}}\frac{\partial}{\partial x})^2.$$

Observe that these $\alpha_\pm$ and $g$ are a solution of eq. (4.5.3) and hence give rise to transverse $J$-holomorphic curves in $\mathbb{CP}^3$. In fact, they correspond to the minimal surfaces $\Psi_{m,k}$ described in the following section.

The case when $X$ is superminimal can be regarded as the limit $\alpha_- \to 0$. A similar frame adaption gives

$$\Delta\log(\alpha_+^2) = -4(3\alpha_+^2 - 2)\mathrm{vol}_{\mathcal{H}} \tag{4.5.4}$$

and Gauß curvature $2(1-\alpha_+^2)$ for $g_{\mathcal{H}}$. The solution $\alpha_+^2 = 2/3$ results in constant curvature $1/3$. The induced metric from the immersion in $S^4$ is $2g_{\mathcal{H}}$, i.e. the corresponding minimal surface has constant curvature $1/3$, volume $12\pi$ and corresponds to the Veronese surface.

If $\varphi$ is superminimal it is also a holomorphic curve for the standard Kähler structure on $\mathbb{CP}^3$. Similarly to eq. (4.2.1) one derives for a simple superminimal curve

$$r_{\mathrm{II}} = 6(g-1) + 2\deg(\varphi) - 2r_{\mathcal{H}} \tag{4.5.5}$$

where $\deg(\varphi)$ is the algebraic degree of $\varphi$. Since $\varphi$ is tangent to $\mathcal{H}$, $r_{\mathcal{H}}$ is the number of branching points of $\varphi$, and by eq. (3.2.5), $r_{\mathrm{II}}$ is the number of totally geodesic points of the corresponding minimal branched immersion in $S^4$.

To obtain an explicit solution of eq. (4.5.4), consider the superminimal curve $\Theta = \Theta(f, g)$ parametrised via eq. (3.1.9), for $f(z) = z^k$ and $g(z) = z$ regarded as maps $\mathbb{CP}^1 \to \mathbb{CP}^1$. This example has been considered by Friedrich [Fri84], who computed the degree of $\Theta$, which equals $k$ for $k \geq 3$. The induced metric, curvature and $\alpha_+^2$ on $X$ can then be computed[1] by pulling back the Fubini-Study Kähler potential on $\mathbb{CP}^3$

[1]see https://github.com/deepfloe/superminimal-curves for an implementation in Mathematica

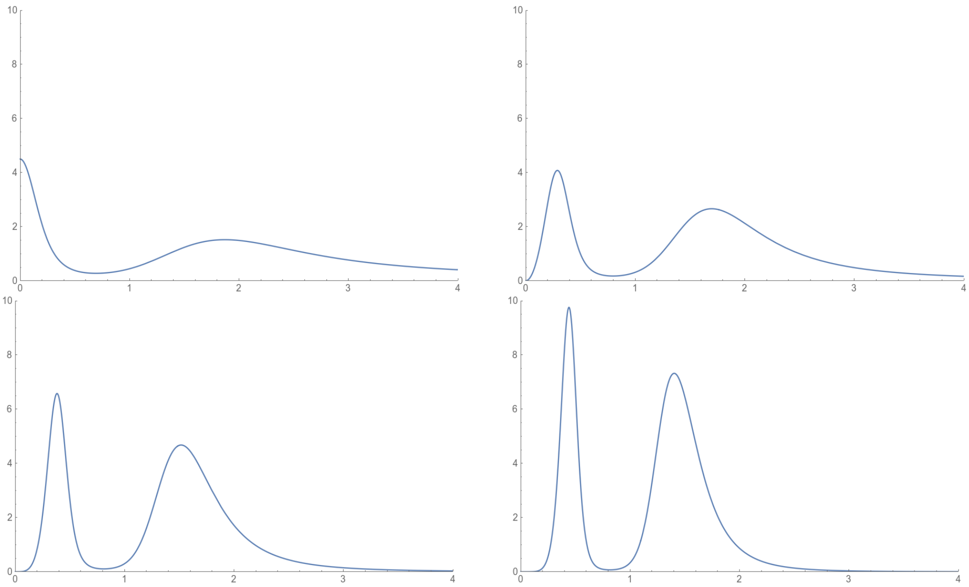

Figure 4.3: The radial dependence of the function $\alpha_+^2$ for different superminimal spheres of degree $k = 3, 4, 5, 6$. For $k \geq 3$, $\alpha_+$ has a zero of order $k-3$ at 0 and $\infty$, corresponding to totally geodesic points of the immersion $S^2 \to S^4$. The number of totally geodesic points for a general superminimal immersion is computed in eq. (4.5.5).

resulting in

$$\alpha_+^2(z) = \frac{2(k-2)^2(k-1)^2k^2r^{2k-2}\left(\left(k^2+(k-2)^2r^2\right)r^{2k}+4\left(r^4+r^2\right)\right)^4}{\left((k-2)^2k^2r^{4k}+4\left(\left(k^2-3k+2\right)^2r^4+2(k-2)^2k^2r^2+(k-1)^2k^2\right)r^{2k}+16r^4\right)^3}$$

for $r = |z|$. This function is plotted for $k = 3, \ldots, 6$ in fig. 4.3. We see that $\alpha_+^2$ vanishes everywhere if $k = 0, 1, 2$, so assume from now on $k \geq 3$. Then $\alpha_+$ has a zero of order $k-3$ at both $z = 0$ and $z = \infty$, yielding $r_{\mathrm{I\!I}} = 2k-6$. This confirms $\deg(\Theta) = k$ since $\Theta$ is an immersion, which implies $r_{\mathcal{H}} = 0$.

## 4.6 U(1) invariant $J$-holomorphic Curves

U(1) actions on the nearly Kähler $\mathbb{CP}^3$ have been studied on the resulting $G_2$-cone by Atiyah and Witten in the context of dimensional reductions of M-theory [AW02]. Closed expressions for the induced metric, curvature and the symplectic structure on $\mathbb{R}^6$ have recently been found by Acharya, Bryant and Salamon [ABS20] for one particular U(1) action.

While superminimal curves can be parametrised very explicitly our description of transverse curves has been, with the exception of Clifford tori, rather indirect so far. Imposing U(1) symmetry on the curves reduces a system of PDE's to a system of ODE's. In terms of $\alpha_-$ and $\alpha_+$, the 2D Toda lattice equation will reduce to the 1D

Toda lattice equation.

Killing vector fields on $\mathbb{CP}^3$ are in one to correspondence with Killing vector fields on $S^4$. We will provide a twistor perspective on the work of U(1) invariant minimal surfaces [FP90]. The geodesic equation on $S^3$ for the Hsiang-Lawson metric is replaced by the computationally less involved 1D Toda equation. The twistor perspective establishes a relationship between U(1) invariant curves and the toric nearly Kähler geometry of $\mathbb{CP}^3$. We start by giving an account of some important results for U(1) invariant minimal surfaces in $S^3$ and $S^4$.

In [Law70], Lawson developed a rich theory of minimal surfaces in $S^3$. For $m, k \in \mathbb{R} \setminus \{0\}$ there is a minimal immersion

$$\Psi_{m,k}(x, y) = (\cos(mx)\cos(y), \sin(mx)\cos(y), \cos(kx)\sin(y), \sin(kx)\sin(y))$$

with induced metric $g_{m,k}$ from section 4.5.1. Assume that $k, m$ are coprime integers, such that $T_{m,k} = \mathrm{Im}(\Psi_{km})$ represent Klein bottles $2|(mk)$ and tori otherwise. The surface $T_{1,1}$ is the Clifford torus $T$. They are geodesically ruled and are invariant under the U(1) action $\rho$ given by

$$e^{i\vartheta}(z, w) = (e^{ki\vartheta}z, e^{mi\vartheta}w).$$

It is furthermore shown that closed minimal surfaces invariant under this action are in one to one correspondence with closed geodesics in the orbit space $S^3/\mathrm{U}(1)$ equipped with an ovaloid metric. By studying this metric explicitly, a rationality condition on the initial values for the geodesic to be closed is obtained. This gives a countable family of tori $(T_{k,m,a})_{a \in A_{k,m}}$ where $A_{k,m}$ is a certain countable dense subset of $(0, \pi/2)$. This family is bounded by $T_{k,m}$ corresponding to the boundary case $a = 0$ and the Clifford torus $T$ for $a = \pi/2$. Similarly, for either $m = 0, k = 0$ or $m = \pm k$ there is a countable family $C_a$ of minimal tori bounded by the Clifford torus and a totally geodesic two-sphere [HL71, Theorem 8]. Any cohomogeneity-one minimal surface in $S^3$ belongs to the families $C_a$ or $T_{k,m,a}$ [HL71, Theorem 9].

For U(1) invariant minimal surfaces in spheres the work of [Uhl82] is important. Ferus and Pedit studied minimal tori in $S^4$, invariant under the action $\rho$ on $\mathbb{C}^2 \oplus \mathbb{R}$. Their approach is closer to the one of Hsiang and Lawson. Indeed, they establish that U(1) invariant tori are in one to one correspondence with closed geodesics in the orbit space $S^4/\mathrm{U}(1) \cong S^3$ equipped with the Hsiang-Lawson metric. The geodesic equation is translated into a Hamiltonian flow equation on the total space of $TS^3$. Using the fact that the Hopf differential is holomorphic and hence constant on tori, two preserved quantities $H_1, H_2$ are constructed from the second fundamental form.

Furthermore, the speed of the corresponding geodesic in $S^3$ is another preserved quantity $H_0$, which is also the Hamiltonian of the flow equation on $TS^3$. It turns out that $\{H_0, H_1, H_2\}$ are in involution.

The generic fibre of the map $(H_0, H_1, H_2)$ is a three-torus. It suffices to consider the map $(H_1, H_2)$ since $H_0$ can be normalised to be 1. By explicitly working out the action angle variables for the geodesic flow, the condition of the geodesic to be closed is translated into a rationality condition which is computed from $H_0, H_1, H_2$.

The final result is that for any $k \geq m \geq 1$ with coprime $k, m$ there is a countable set of values for $H_1, H_2$ for which the corresponding level set $\mathbb{T}^2 \subset TS^3$ consists of closed geodesics. They give rise to a two-parameter family of minimal tori in $S^4$ where one parameter is coming from an isometry action while the other is producing non-congruent deformations.

The aim of this section is to view these results from the twistor perspective, i.e. to obtain a Hamiltonian description of U(1) invariant $J$-holomorphic curves in $\mathbb{CP}^3$. Since the nearly Kähler $\mathbb{CP}^3$ has isometry group Sp(2) any element $\xi \in \mathfrak{sp}(2)$ defines the Killing vector field

$$K^\xi(x) = \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi)x.$$

Assume that there is $t > 0$ such that $\exp(t\xi)$ equals the identity $e \in \mathrm{Sp}(2)$, i.e. $\xi$ corresponds to an action $\rho$ of U(1) on $\mathbb{CP}^3$. Acting via $\rho$ on integral curves of $JK^\xi$ gives rise to U(1) invariant $J$-holomorphic curves in $\mathbb{CP}^3$.

Another way of stating this is that $[K^\xi, JK^\xi] = 0$ and that the span of $K^\xi$ and $JK^\xi$ defines an integrable distribution $V_\xi$ on $M = \mathbb{CP}^3 \setminus (K^\xi)^{-1}(0)$. The integral submanifolds are exactly the $\rho$ invariant $J$-holomorphic curves in $\mathbb{CP}^3$ which locally foliate $M$. Due to the isomorphism between $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, the element $\xi$ also defines a Killing vector field $K^\xi_{S^4}$. This results in a uniqueness statement for minimal surfaces in $S^4$.

**Lemma 4.6.1.** *Let $x \in S^4$ such that $V \subset T_xS^4$ is a two-dimensional subspace containing $K^\xi_{S^4}(x)$ and let $p \in \pi^{-1}(x) \subset \mathbb{CP}^3$ be the twistor lift of $V$. If $K^\xi_p$ is non-vertical there is a locally unique minimal surface $\Sigma$ with $x \in \Sigma$ and $V = T_x\Sigma$.*

Let $X$ be an integral submanifold of the distribution $V_\xi$. Since it is a $J$-holomorphic curve the bundle $\mathrm{Sp}(2)|_X$ reduces to the $\mathbb{Z}_8$ bundle $R_X$, see lemma 4.3.6. There is a $\mathbb{Z}_8$ bundle $R$ over $M$ which restricts to $R_X$ on each integral submanifold. Note that the construction of $R$ is linked to the subspace $\mathfrak{r} \subset \mathfrak{sp}(2)$ from eq. (4.3.2), which becomes apparent in the following lemma.

**Lemma 4.6.2.** *If a section $s\colon \mathbb{CP}^3 \supset U \to \mathrm{Sp}(2)$ lies in $R$ then $s^*\Omega_{MC}(K^\xi) = s^{-1}\xi s \in \mathfrak{r}$.*

*Proof.* Let $s$ be a section of $R$. Consider the section $s' = L_g \circ s \circ L_{g^{-1}}$ where $g = \exp(t\xi)$ for some $t \in \mathbb{R}$. Then $s'^*\Omega_{MC}(K^\xi_x) = s^*\Omega_{MC}(K^\xi_{gx}) \in \mathfrak{r}$. In other words, $s'$ satisfies eq. (4.3.10) and hence $s'$ also has values in $R$. Since the sections $s$ and $s'$ are joined by a continuous path and $R$ has discrete structure group it follows that $s' = s$, i.e.

$$s(gx) = gs(x).$$

This implies

$$s^*(\Omega_{MC})(K^\xi) = s^{-1}\mathrm{d}s(K^\xi) = s^{-1}\xi s,$$

as required. □

The point of the previous lemma is that it gives a more explicit description of the bundle $R$.

**Corollary 4.6.3.** *The bundle $R$ is equal to $\{g \in \mathrm{Sp}(2) \mid g^{-1}\xi g \in \mathfrak{r}\}$.*

The presence of the vector field $K^\xi$ means we can define the functions $h = \|K^\xi\|^2_{\mathcal{H}}$, $v_- = \|K^\xi\|^2_{\mathcal{V}}$. Furthermore, if we restrict $\tau$ to $R$ the quantity $v_+ = |\tau(K^\xi)|^2$ is well-defined. This gives

$$\alpha_- = \sqrt{v_-/h}, \qquad \alpha_+ = \sqrt{v_+/h}. \tag{4.6.1}$$

Let $X$ be a transverse integral submanifold. Since $K^\xi$ and $JK^\xi$ commute, these vector fields yield coordinates $(u,t)$ such that $\frac{\partial}{\partial u} = K^\xi$ and $\frac{\partial}{\partial t} = JK^\xi$. The induced metric $g_{\mathcal{H}}$ on $X$ is equal to $h(\mathrm{d}u^2 + \mathrm{d}t^2)$. In particular, due to proposition 4.3.7 and since $h$ is the conformal factor of the metric, the quantity $C = h\sqrt{\alpha_-\alpha_+} = h^{1/2}(v_-v_+)^{1/4}$ is constant along $X$. Hence, eq. (4.3.13) reduces to

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log(v_-) &= 4\Big(\frac{C^2}{\sqrt{v_-v_+}} - 2v_-\Big)\\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\log(v_+) &= 4\Big(\frac{C^2}{\sqrt{v_-v_+}} - 2v_+\Big).\end{aligned} \tag{4.6.2}$$

It is clear that the equations of lemma 4.3.6 hold when the forms are restricted to $R$. Moreover, the fact that $K^\xi$ is a Killing vector field guarantees that the following additional equations are satisfied

$$\rho_1(JK^\xi) = \rho_2(JK^\xi) = 0 \tag{4.6.3}$$
$$\omega_1(K^\xi) = \lambda\sqrt{h} \tag{4.6.4}$$
$$\omega_3(K^\xi) = \lambda\sqrt{v_-} \tag{4.6.5}$$
$$\tau(K^\xi) = \lambda\sqrt{v_+} \tag{4.6.6}$$

for a constant $\lambda \in S^1$. Note that eq. (4.6.3) are satisfied because isometries preserve the quantities $\alpha_-$ and $\alpha_+$, eq. (4.6.4) holds because $(u,t)$ are isothermal coordinates for $g_{\mathcal{H}}$ on the integral submanifold. Lastly, eq. (4.6.5) and eq. (4.6.6) follow from eq. (4.6.4) and the definition of $v_-, v_+$, i.e. eq. (4.6.1) as well as the equations defining $R$, i.e. eq. (4.3.10). Since $\frac{\mathrm{d}}{\mathrm{d}t}C^2 = 0$ we have

$$\begin{aligned}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}(K^\xi) &= \frac{1}{8}\begin{pmatrix}-3 & 1\\ -1 & 3\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\log(\alpha_-)\\ \log(\alpha_+)\end{pmatrix}\\ &= \frac{1}{8}\begin{pmatrix}-3 & 1\\ -1 & 3\end{pmatrix}\frac{1}{4}\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\log(v_-)\\ \log(v_+)\end{pmatrix} = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}-\log(v_-)\\ \log(v_+)\end{pmatrix}.\end{aligned}$$

Furthermore, $\lambda = 0$ is preserved along $K^\xi$ since $R$ has a discrete structure group.

We have abused notation slightly here. By how $K_F$ acts on $\mathfrak{r}$ the forms $\omega_1, \dots, \omega_3$ are basic on $R$ but take values in a possibly non-trivial line bundle on $X$ whose structure group reduces to $K_F$. In other words, $\lambda$ is only well-defined up to multiplication of a fourth root of unity. However, this poses no problem since the results which follow will only depend on $\mu = \lambda^4$.

### 4.6.1 Lax Representation and Toda Lattices

Since a general transverse curve satisfies the 2D periodic Toda lattice equation, U(1) invariant curves will satisfy the 1D periodic Toda lattice equations for the same Lie algebra, i.e. eq. (4.6.2). These equations admit a Lax representation [Bog76]. In the following, we will work out this Lax representation directly from the formalism of adapted frames. Consider the restriction of $\Omega_{MC}$ to the reduced bundle $R$, which will still be denoted by $\Omega = \Omega_{MC}$. We have

$$\mathrm{d}\Omega = -[\Omega, \Omega].$$

Note that eq. (4.6.3)-eq. (4.6.6) imply

$$\Omega(JK^\xi) = \begin{pmatrix}-ji\bar{\lambda}\sqrt{v_-} & i\frac{\bar{\lambda}\sqrt{h}}{\sqrt{2}}\\ i\frac{\lambda\sqrt{h}}{\sqrt{2}} & ji\sqrt{v_+}\lambda\end{pmatrix} \tag{4.6.7}$$

$$\Omega(K^\xi) = \begin{pmatrix}\frac{-i}{4}(\frac{\mathrm{d}}{\mathrm{d}t}\log(v_-)) + j\bar{\lambda}\sqrt{v_-} & -\frac{\bar{\lambda}\sqrt{h}}{\sqrt{2}}\\ \frac{\lambda\sqrt{h}}{\sqrt{2}} & \frac{i}{4}(\frac{\mathrm{d}}{\mathrm{d}t}\log(v_+)) + j\lambda\sqrt{v_+}\end{pmatrix}. \tag{4.6.8}$$

Consequently, $K^\xi(\Omega(JK^\xi)) = 0$ because $\lambda, h, v_-, v_+$ are constant along $K^\xi$. With this in mind, let us evaluate both two-forms at $K^\xi \wedge JK^\xi$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\Omega(K^\xi)) &= (JK^\xi)(\Omega(K^\xi)) = -\mathrm{d}\Omega(K^\xi, JK^\xi)\\ &= [\Omega, \Omega](K^\xi, JK^\xi) = [\Omega(K^\xi), \Omega(JK^\xi)].\end{aligned} \tag{4.6.9}$$

Evaluating the right-hand side of eq. (4.6.9) gives

$$[\Omega(K^\xi), \Omega(JK^\xi)] = \begin{pmatrix} i(-h+2v_-) + j\bar{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{v_-} & \frac{\bar{\lambda}\sqrt{h}}{4\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}t}\log(v_-v_+) \\ -\frac{\lambda\sqrt{h}}{4\sqrt{2}}\frac{\mathrm{d}}{\mathrm{d}t}\log(v_-v_+) & i(h-2v_+) + j\lambda\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{v_+} \end{pmatrix}.$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}t}\lambda = 0$ and eq. (4.6.9) is equivalent to eq. (4.6.2). In other words, we have found a Lax representation of the ODE system. This proves the following lemma.

**Lemma 4.6.4.** *The eigenvalues of $\Omega(K^\xi) \in \mathfrak{sp}(2) \subset \mathfrak{su}(4)$ are constant along $JK^\xi$.*

Introduce the variables

$$q_- = \frac{1}{2}\log(v_-), \quad r_- = \dot{q}_-, \quad q_+ = \frac{1}{2}\log(v_+), \quad r_+ = \dot{q}_+$$

where the dot denotes the derivative with respect to $\frac{\mathrm{d}}{\mathrm{d}t}$, i.e. along $JK^\xi$. Then eq. (4.6.2) is equivalent to

$$\begin{aligned} \dot{q}_- &= r_-, & \dot{r}_- &= 2(C^2\exp(-(q_-+q_+)) - 2\exp(2q_-)) \\ \dot{q}_+ &= r_+, & \dot{r}_+ &= 2(C^2\exp(-(q_-+q_+)) - 2\exp(2q_+)). \end{aligned} \tag{4.6.10}$$

This system is Hamiltonian with

$$H = 2(C^2\exp(-(q_-+q_+)) + \exp(2q_-) + \exp(2q_+)) + \frac{1}{2}r_-^2 + \frac{1}{2}r_+^2.$$

The Hamiltonian is in the form of [Bog76, Theorem 1]. In other words, eq. (4.6.10) are the equations for a generalised, periodic Toda lattice for the Lie algebra $\mathfrak{sp}(2)$. Bogoyavlensky's Lax representation for such a system coincides with eq. (4.6.9). Thus, we have proven.

**Proposition 4.6.5.** *Simply connected, embedded, transverse $J$-holomorphic curves with a one-dimensional symmetry define solutions to the 1D periodic Toda lattice equations for the Lie algebra $\mathfrak{sp}(2)$ with Lax representation*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Omega(K^\xi)) = [\Omega(K^\xi), \Omega(JK^\xi)].$$

We have seen that a choice of a Killing vector field on $\mathbb{CP}^3$ gives rise to the above ODE system. The converse statement is also true.

**Proposition 4.6.6.** *Let $X \cong \mathbb{C}$ be a Riemann surface equipped with coordinates $(u,t)$, a metric $k = C\gamma^2(\mathrm{d}u^2 + \mathrm{d}t^2)$ and $\alpha_-, \alpha_+ \colon X \to \mathbb{R}^{>0}$ that satisfy the ODE system eq. (4.6.2) for some $C > 0$. Then there is an element $\xi \in \mathfrak{sp}(2)$ such that $X$ is an integral manifold of the distribution $V_\xi$ and such that $\alpha_-, \alpha_+$ are the angle functions and the tautological embedding $X \to \mathbb{CP}^3$ is $J$-holomorphic and isometric.*

*Proof.* Let $(\alpha_-, \alpha_+)$ be a solution of eq. (4.6.2) with $C > 0$. We will now give a $\xi \in \mathfrak{sp}(2)$ such that the integral submanifold through $[e] = [1, 0, 0, 0] \in \mathbb{CP}^3$ has angle functions $\alpha_-, \alpha_+$. We have already seen that such integral submanifolds satisfy eq. (4.6.2). The system eq. (4.6.2) itself does not depend on the choice of $\xi$. However, $\xi$ will determine its initial condition. Hence, it suffices to show that a $\xi$ with the desired properties can be found for any $C > 0$ and fixed initial conditions for $\alpha_-(0), \alpha_+(0), \dot{\alpha}_-(0), \dot{\alpha}_+(0)$. To this end, let

$$h(0) = C\alpha_-(0)^{1/2}\alpha_+(0)^{1/2}, \quad v_-(0) = C\alpha_-(0)^{3/2}\alpha_+(0)^{-1/2}, \quad v_+(0) = C\alpha_+(0)^{3/2}\alpha_-(0)^{-1/2}$$

and

$$\xi = \begin{pmatrix} \frac{i}{8}(-3\frac{\dot{\alpha}_-(0)}{\alpha_-(0)} + \frac{\dot{\alpha}_+(0)}{\alpha_+(0)}) + j\sqrt{v_-(0)} & -\frac{h(0)}{\sqrt{2}} \\ \frac{h(0)}{\sqrt{2}} & \frac{i}{8}(-\frac{\dot{\alpha}_-(0)}{\alpha_-(0)} + 3\frac{\dot{\alpha}_+(0)}{\alpha_+(0)}) + j\sqrt{v_+(0)} \end{pmatrix}.$$

Let $X$ be the integral manifold of $V_\xi$ passing through the point $[e] = [1, 0, 0, 0] \in \mathbb{CP}^3$. Since $\xi \in \mathfrak{r}$ the identity element $e \in \mathrm{Sp}(2)$ lies in $R$ by corollary 4.6.3. Let $s$ be a local section of $R$ with $s([e]) = e$, then by lemma 4.6.2 $s^*\Omega_{MC}(K^\xi) = \xi$ at $[e]$. Apply lemma 4.3.6 and evaluate $\rho_1(K^\xi)$ and $\rho_2(K^\xi)$ at $([e])$ to see that the angle functions $\alpha_-, \alpha_+$ of $X$ satisfy the given initial conditions at $[e]$. ☐

Define

$$\begin{aligned} H_1 &= 2H = 4h + 4v_+ + 4v_- + r_+^2 + r_-^2 \\ H_2 &= (r_-^2 - r_+^2 + 4v_- - 4v_+)^2 + 8h((r_+ - r_-)^2 + 4(v_+ + v_-)). \end{aligned} \tag{4.6.11}$$

A calculation shows that the eigenvalues of $\Omega(K^\xi)$ are equal to

$$\pm\frac{i}{\sqrt{2}2}\sqrt{H_1 \pm \sqrt{H_2 + 64C^2\mathrm{Re}(\lambda^4)}}. \tag{4.6.12}$$

In particular, the quantities $H_1, H_2, C^2$ are all preserved along $JK^\xi$. To study their relationship to each other regard $U = (H_1, H_2, C^2)$ as a function from $\mathbb{R}^3_{>0} \times \mathbb{R}^2$, parametrised by $(h, v_-, v_+, r_-, r_+)$, to $\mathbb{R}^3_{>0}$. The following proposition establishes important properties of $U$.

**Proposition 4.6.7.** *The image of $U$ is the subset of $\mathbb{R}^3_{>0}$ satisfying the inequalities*

$$H_1^2 \geq H_2 + 64C^2, \quad H_2 \geq 64C^2.$$

*Equality in the first inequality occurs if and only if*

$$2h = -r_+r_- + 4\sqrt{v_+v_-}, \quad r_+\sqrt{v_-} + r_-\sqrt{v_+} = 0 \tag{4.6.13}$$

*and in the second inequality if and only if*

$$v_- = v_+, \quad r_- = r_+. \tag{4.6.14}$$

*For a fixed $C^2$, and any $(q_-, q_+, r_-, r_+)$ not satisfying either eq.* (4.6.13) *or eq.* (4.6.14) *there are local coordinates*

$$I_1, I_2, \vartheta_1, \vartheta_2, \quad \vartheta_i \in \mathbb{R}/\mathbb{Z}$$

*for $\mathbb{R}^2_{>0} \times \mathbb{R}^2$, parametrised by $(q_-, q_+, r_-, r_+)$, such that the angles $\vartheta_1, \vartheta_2$ are coordinates on the fibre of $(H_1, H_2)$ and $I_1, I_2$ only depend on $H_1, H_2$ and $C^2$. In particular, the fibres are diffeomorphic to two-tori in this case. In these coordinates eq.* (4.6.2) *is transformed to*

$$\dot{I}_i = 0, \quad \dot{\vartheta} = \omega_i(I_1, I_2, C^2) \quad i = 1, 2.$$

*Proof.* First observe that the fibres of $U$ are bounded by the definition of $H_1$. Since $C^2 > 0$ every fibre has a positive distance from the boundary of $\mathbb{R}^3_{>0} \times \mathbb{R}^2$ and is hence closed as a subset of $\mathbb{R}^5$. This implies that the fibres of $U$ are compact. One argues similarly to show that $U$ is in fact a proper map.

Let $a_\pm = r_\pm/\sqrt{2h}$ and $b_\pm = \sqrt{2v_\pm}/\sqrt{h}$ and $z_\pm = a_\pm + ib_\pm$. Then

$$\begin{aligned} H_1 &= 4h(1 + \frac{|z_+|^2}{2} + \frac{|z_-|^2}{2}) \\ H_1^2 - (H_2 + 64C^2) &= 16h^2|z_+z_- + 1|^2 \\ H_2 - 64C^2 &= 16h^2(|z_+ - z_-|^2 + (\frac{|z_+|^2}{2} - \frac{|z_-|^2}{2})^2). \end{aligned} \tag{4.6.15}$$

From the second equation it follows that $H_1^2 \geq (H_2 + 64C^2)$ with equality if and only if $z_+z_- = -1$. The third equation implies that $H_2 = 64C^2$ if and only if $z_+ = z_-$. From these equations one also checks that the image of $U$ is the set

$$H_1, H_2, C^2 > 0, \quad H_2 \geq 64C^2, \quad H_1^2 \geq H_2 + 64C^2$$

and that $U$ is a submersion on the interior of this set. Since $U$ is proper it is a fibration $U_0$ over the interior of its image, by Ehresmann's fibration theorem. The interior of the image is furthermore simply-connected. The set $\mathbb{R}^3_{>0} \times \mathbb{R}^2 \setminus (\{z+ = z_-\} \cup \{z_+z_- = -1\})$ is connected since the sets $\{z_+ = z_-\}$ and $\{z_+z_- = -1\}$ are of co-dimension two. Hence, the fibres of $U_0$ are connected.

Viewing $C^2$ as parameter for the ODE system and $H_1, H_2$ as a map from $\mathbb{R}^2_{>0} \times \mathbb{R}^2$ parametrised by $(v_-, v_+, r_-, r_+)$, we introduce the symplectic form

$$\mathrm{d}q_- \wedge \mathrm{d}r_- + \mathrm{d}q_+ \wedge \mathrm{d}r_+ = \frac{1}{2}(\frac{1}{v_-}\mathrm{d}v_- \wedge \mathrm{d}r_- + \frac{1}{v_+}\mathrm{d}v_+ \wedge \mathrm{d}r_+)$$

and check that $H_1$ and $H_2$ are in involution for this symplectic form. The statement now follows from the Liouville-Arnold theorem. $\square$

Using

$$\frac{\mathrm{d}}{\mathrm{d}t}v_\pm = 2r_\pm v_\pm, \quad \frac{\mathrm{d}}{\mathrm{d}t}r_\pm = 2(h-2v_\pm), \quad \frac{\mathrm{d}}{\mathrm{d}t}h = -h(r_- + r_+)$$

we can check that taking the derivative of either of the equations in 4.6.13 is equivalent to the other.

### 4.6.2 The $\mathbb{T}^2$-Action

In this subsection, we will investigate the geometry of a general U(1) action on $\mathbb{CP}^3$ with respect to the vector field $JK^\xi$. We make use of the fact that such an action commutes with a subgroup of Sp(2), which is generically a two-torus. To that end, we fix

$$\xi = \begin{pmatrix} ik & 0 \\ 0 & im \end{pmatrix} \in \mathfrak{sp}(2)$$

which integrates to the U(1)-action

$$\rho(e^{i\vartheta}[Z_0, Z_1, Z_2, Z_3]) = [e^{ki\vartheta}Z_0, e^{-ki\vartheta}Z_1, e^{mi\vartheta}Z_2, e^{-mi\vartheta}Z_3] \tag{4.6.16}$$

on $\mathbb{CP}^3$. We fix an isomorphism $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$ which maps $\xi$ to the element $\mathrm{diag}(k+m, -k-m, k-m, m-k, 0) \in \mathfrak{so}(5)$ and thus corresponds to the U(1) action with weights $\tilde{k} = m-k, \tilde{m} = m+k$ on $S^4 \subset \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{R}$. The Lawson torus $T_{\tilde{k},\tilde{m}}$ admits a $J$-holomorphic twistor lift which is invariant under $\rho$ and will be denoted by $\tau_{k,m}$. Observe that

$$\mathrm{Stab}(\xi) = \{g \in \mathrm{Sp}(2) \mid g^{-1}\xi g = \xi\} \cong \begin{cases} \mathrm{U}(1) \times \mathrm{Sp}(1) & \text{for } k = 0 \text{ or } m = 0 \\ \mathrm{U}(2) & \text{for } k = \pm m \\ \mathbb{T}^2 & \text{otherwise} \end{cases}.$$

We will refer to the first two cases as degenerate and to the last case as non-degenerate. Due to the presence of a larger symmetry group, the degenerate cases are simpler and are treated in the end of this chapter.

For now, we restrict ourselves to the non-degenerate case in which $\mathrm{Stab}(\xi)$ is equal to the standard two-torus in Sp(2), i.e. $\{\mathrm{diag}(e^{i\theta}, e^{i\phi})\}$, to which we will simply refer as $\mathbb{T}^2$ unless stated otherwise. Let $\xi_1, \xi_2$ be the elements in $\mathfrak{sp}(2)$ corresponding to the action of $e^{i\theta}$ and $e^{i\phi}$ respectively.

The torus action yields the multi-moment map $\nu = \omega(K^{\xi_1}, K^{\xi_2})$, introduced in [RS19]. With the help of eq. (2.2.3), it can be computed explicitly on $\mathbb{CP}^3$.

**Proposition 4.6.8.** *The nearly Kähler multi-moment map on $\mathbb{CP}^3$ is given in homogeneous coordinates by*

$$\nu = \frac{12}{|Z|^4}\mathrm{Im}(Z_0 Z_1 \overline{Z_2 Z_3}),$$

*where $|Z|^2 = \sum_{i=0}^3 |Z_i|^2$.*

Given a nearly Kähler manifold with a $\mathbb{T}^2$ action, the possibly singular space ${}_{\mathbb{T}^2}\backslash^{\nu^{-1}(0)}$ captures important geometric information. In the case of $\mathbb{CP}^3$ this space is an orbifold.

**Corollary 4.6.9.** *The set ${}_{\mathbb{T}^2}\backslash^{\nu^{-1}(0)}$ is homeomorphic to $\mathbb{RP}^3/\{\pm 1\}$ where the action of $-1$ is given by $[X_0, X_1, X_2, X_3] \mapsto [-X_0, -X_1, X_2, X_3]$.*

There are two Clifford tori which arise as orbits of $\mathbb{T}^2$ and are equal to the preimages of the extremal values of $\nu$ by lemma 2.4.1. They are $\mathbb{T}^2$ orbits of the points $[1, 1, 1, i]$ and $[1, 1, 1, -i]$.

When investigating $\rho$ invariant $J$-holomorphic curves, it proves worthwhile to consider a lower-dimensional subset $Y$ of $\mathbb{CP}^3$ which is both invariant under $\rho$ and the flow of $JK^\xi$. In other words, the distribution $V_\xi$ is then tangent to $Y$ and the problem of finding $\rho$ invariant $J$-holomorphic curves can be done separately on each such $Y$.

Since $\mathrm{d}\nu = \mathrm{Re}\,\psi(K^{\xi_1}, K^{\xi_2}, \cdot)$ the distribution $V_\xi$ is tangent to each preimage $\nu^{-1}(c)$. The value 0 as well as extrema of $\nu$ have a distinguished geometrical importance. All in all, there are the following sets which arise in a natural geometric way and to which $V_\xi$ is tangent

- $\mathcal{C} = \nu^{-1}(\{\nu_{\min}, \nu_{\max}\})$
- $\mathcal{B} = \nu^{-1}(0)$
- $\mathcal{Q}$, the quadric associated to $\xi$ under the identification $\mathfrak{sp}(2)$ with the real part of $S^2(\mathbb{C}^4)$
- $\mathcal{S}$, the set where $\mathbb{T}^2$ does not act freely
- $\mathcal{T}$, a distinguished $S^1$ bundle over $S^3_0 \subset \mathbb{R}^4 \oplus \{0\}$ (for $(k, m)$ non-degenerate).

In the following we will define and outline the properties of each of the subsets.

The four points $\{[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]\}$ are the fixed points of the $\mathbb{T}^2$ action. It turns out that $\mathbb{T}^2$ acts on the projective line going through any two of them with cohomogeneity one. More specifically, let

$$\begin{aligned} &L_1 = \{Z_0 = Z_1 = 0\}, \quad L_2 = \{Z_0 = Z_2 = 0\}, \quad L_3 = \{Z_0 = Z_3 = 0\}, \\ &L_4 = \{Z_1 = Z_3 = 0\}, \quad L_5 = \{Z_1 = Z_2 = 0\}, \quad L_6 = \{Z_2 = Z_3 = 0\} \end{aligned} \tag{4.6.17}$$

and observe that

$$L_2 \cup L_4 = \left(K^{\xi_1}\right)^{-1}(0) \quad L_3 \cup L_5 = \left(K^{\xi_2}\right)^{-1}(0),$$
$$L_1 = (K^{\xi_1} + K^{\xi_2})^{-1}(0), \quad L_6 = (K^{\xi_1} - K^{\xi_2})^{-1}(0).$$

The action of $\mathbb{T}^2$ on $\mathbb{CP}^3$ is free away from the projective lines $L_1, \ldots, L_6$, i.e. $\mathcal{S} = L_1 \cup \cdots \cup L_6$. Note that $L_1$ and $L_6$ are twistor lines, i.e. $h = 0$, while $L_2, \ldots, L_5$ are superminimal, i.e. $v_- = 0$. Since $L_2, \ldots, L_5$ project to totally geodesic two-spheres they furthermore satisfy $v_+ = 0$, see eq. (3.2.5). Via the isomorphism $\mathfrak{sp}(2) \otimes \mathbb{C} \cong S^2(\mathbb{C}^4)$ the element $\xi$ is identified with the polynomial

$$i(kZ_0Z_1 + mZ_2Z_3). \tag{4.6.18}$$

Furthermore, since $\mathfrak{sp}(2) \cong \mathfrak{so}(5)$, $\xi$ defines a vector field $K^{\xi}_{S^4}$ on $S^4$. Because it is a Killing vector field, $\nabla(K^{\xi}_{S^4})$ can be identified with a two form on $S^4$. Its anti-self-dual part satisfies the twistor equation and thus defines a quadric on $\mathbb{CP}^3 = Z_-(S^4)$, holomorphic with respect to $J_1$, see [Bes07, ch. 13]. This quadric is

$$\mathcal{Q} = \{kZ_0Z_1 + mZ_2Z_3 = 0\} \tag{4.6.19}$$

and coincides with the vanishing set of the quadratic expression eq. (4.6.18).

For $x \in \mathbb{CP}^3$, let $X_x$ be the unique $\rho$ invariant embedded $J$-holomorphic curve containing $x$ and let $\Sigma_x$ be the corresponding minimal surface in $S^4$. Define

$$\mathcal{T} = \{x \in \mathbb{CP}^3 \mid \Sigma_x \text{ is contained in a totally geodesic } S^3\}.$$

Note that $\mathcal{T}$ is invariant under $\rho$. If $y \in X_x$ then $X_y = X_x$ and in particular if $x \in \mathcal{T}$ then $X_x$ is contained $\mathcal{T}$. Since $JK^{\xi}$ is tangent to $X_x$ for any $x \in \mathbb{CP}^3$ it is also tangent $\mathcal{T}$.

Let $S^3_0$ be the totally geodesic three-sphere $\mathbb{R}^4 \oplus \{0\} \cap S^4$. For non-degenerate $k, m$ we derive an explicit expression for $\mathcal{T}$ in section 4.6.5. For one of the degenerate actions, every U(1) invariant minimal surface lies in a totally geodesic $S^3$.

**Lemma 4.6.10.** *If $k = m$ then $\mathcal{T} = \mathbb{CP}^3$.*

*Proof.* For $k = m = 1$ the action on $S^4$ only rotates the first two components of $\mathbb{R}^2 \oplus \mathbb{R}^3$. This action commutes with $\mathrm{SO}(2) \times \mathrm{SO}(3)$. Let $\Sigma$ be a minimal surface containing the orbit $\mathcal{O}_x$ for some $x$. Let $\nu$ be the normal bundle of $\mathcal{O}_x$ in $\Sigma$ and $v \in \nu_x$. By an action of $\mathrm{SO}(2) \times \mathrm{SO}(3)$ we can assume that $x$ and $v$ lie in $\mathbb{R}^4 \oplus \{0\}$. In particular, $x$ is contained in $S^3_0 \subset \mathbb{R}^4 \oplus \{0\}$ and $v$ is tangent to it. Because $\nu$ and $S^3$ are $\rho$ invariant we have that $\mathcal{O}_x \subset S^3$ and $\nu_x \subset TS^3$. By [Uhl82], there is an embedded, $\rho$ invariant minimal surface $\Sigma'$ such that $\mathcal{O}_x \subset \Sigma' \subset S^3$. Now, lemma 4.6.1 implies that $\Sigma = \Sigma'$, i.e. $\Sigma \subset S^3_0$. $\square$

We need the following auxiliary lemma.

**Lemma 4.6.11.** *Let $X \subset Y \subset Z$ all be compact Riemannian manifolds with $Y$ totally geodesic. Assume that the second fundametal form $\mathrm{II}_Y$ of $X$ in $Y$ is surjective as a bundle map $S^2(TX) \to \nu_Y$, where $\nu_Y$ is the normal bundle of $X$ in $Y$. If a Lie group $G$ acts by isometries on $X$ and $Z$ then $Y$ is also $G$ invariant.*

*Proof.* We say $G$ acts on a bundle $E$ over $X$ if $G$ acts on the total space of $E$ covering the action of $G$ on $X$. Then $G$ clearly acts on $S^2(TX)$. Since $G$ acts by isometries on $Z$ it also acts on the normal bundle $\nu_Z$ of $X$ in $Z$ and the second fundametal form $\mathrm{II}_Z$ of $X$ in $Z$ is an equivariant map $S^2(TX) \to \nu_Z$. As $Y$ is totally geodesic, the second fundametal form $\mathrm{II}_Z$ takes values in $\nu_Y$ and equals $\mathrm{II}_Y$. By assumption, the image of $\mathrm{II}_Y$ is the normal bundle $\nu_Y$ and since $\mathrm{II}_Z = \mathrm{II}_Y$ is equivariant $G$ acts on $\nu_Y$.

Because $X, Y$ are compact, $Y$ is the image of $\exp\colon \nu_Y \to Y$. The map exp is $G$ equivariant since $G$ acts by isometries, which implies the statement. $\square$

**Lemma 4.6.12.** *If $k \neq m$ and $\Sigma$ is a $\rho$ invariant minimal surface, which is not superminimal and which is contained in a totally geodesic $N \cong S^3$ then $N = S^3_0$.*

*Proof.* One can check that $S^3_0$ is the only totally geodesic $S^3$ on which $\rho$ acts. If $\Sigma$ has a totally geodesic point then it lies inside the quadric $\mathcal{Q}$ and is superminimal everywhere. Since $\Sigma$ is of codimension one in $N$, this implies that the condition on the second fundamental form in lemma 4.6.11 is satisfied and the statement follows. $\square$

From now on, assume $k \neq m$. Over $S^3_0$ we introduce the reduced Grassmannian bundle

$$\mathrm{Gr}(K^\xi_{S^4}, S^3_0) = \{V \in \widetilde{\mathrm{Gr}}_2(S^4) \mid K^\xi \in V \subset TS^3_0\}$$

and notice that $\mathrm{Gr}(K^\xi, S^3_0)$ is in fact an $S^1$-bundle over $S^3_0$ which is mapped diffeomorphically to $\mathcal{T} \subset \mathbb{CP}^3 = Z_-(S^4)$ by the projection $\widetilde{\mathrm{Gr}}_2(S^4) \to Z_-(S^4)$.

### 4.6.3 Separating $\mathbb{T}^2$-Orbits

As highlighted in the introduction, it is desirable to have a geometric construction of a map into $\mathbb{R}^4$ which descends to a local homeomorphism to the $\mathbb{T}^2$ quotient of $\mathbb{CP}^3$, at least away from a singular set. In general, one candidate for such a map is $(\nu, \|K^{\xi_1}\|, \|K^{\xi_2}\|, g(K^{\xi_1}, K^{\xi_2}))$. For $S^3 \times S^3$ such a map cannot have a four-dimensional image however, due to the presence of a unit Killing vector field on $S^3 \times S^3$. This is a special case, because $S^3 \times S^3$ is the only nearly Kähler manifold admitting a unit Killing vector field [MNS05]. Nevertheless, it seems difficult to compute the differentials of $g(K^{\xi_i}, K^{\xi_j})$ in a general setting.

For the case $M = \mathbb{CP}^3$, we use the map

$$p = (v_+, v_-, r_+, r_-) \colon M_F \to \mathbb{R}^4$$

where $M_F$ denotes the smooth quotient ${}_{\mathbb{T}^2}\backslash^{\mathbb{CP}^3 \setminus \{L_1 \cup \cdots \cup L_6\}}$. We will see in corollary 4.6.20 that, up to a sign, the multi-moment map $\nu$ can be expressed as a function of $p$. Note that the functions $(v_+, v_-, r_+, r_-)$ are all $\mathbb{T}^2$ invariant.

**Theorem 4.6.13.** *The image of $p$ is a bounded set $D \subset \mathbb{R}^4$ over which $p$ is a branched double cover. The two different points in the fibres of $p$ are complex conjugates of each other and the branch locus is equal to $\mathcal{B} = \nu^{-1}(0)$.*

*Proof.* Let

$$\mathfrak{sp}(2)_F = \left\{ \begin{pmatrix} q_1 & -\overline{q_3} \\ q_3 & q_2 \end{pmatrix} \in \mathfrak{sp}(2) \mid q_3 \in \mathbb{H} \setminus \{0\}, \quad q_1 \in \mathbb{H} \setminus \mathbb{C} \right\}$$

and $\mathfrak{r}_F = \mathfrak{r} \cap \mathfrak{sp}(2)_F$, where $\mathfrak{r}$ is defined in eq. (4.3.2). The aim is to identify $p$ as a composition of

$$p \colon M_F \xrightarrow{c_F} \mathfrak{sp}(2)_F / S^1 \times S^3 \xrightarrow{\Pi} \mathfrak{r}_F / K_F \xrightarrow{\bar{\zeta}} \bar{D} \subset \mathbb{R}^6 \xrightarrow{\mathrm{pr}} D \subset \mathbb{R}^4. \tag{4.6.20}$$

We will now define each of the maps individually and establish its properties.

$c_F$

Consider the map $c\colon \mathrm{Sp}(2) \to \mathfrak{sp}(2), \quad g \mapsto g^{-1}\xi g$. Denote the image of $c$ by $\mathcal{O}^\xi$. Note that $c$ descends to a diffeomorphism from ${}_{\mathbb{T}^2}\backslash^{\mathrm{Sp}(2)}$ onto its image. Quotienting both spaces by the right action of $S^1 \times S^3$ gives a homeomorphism between ${}_{\mathbb{T}^2}\backslash^{\mathbb{CP}^3}$ and $\mathcal{O}^\xi / S^1 \times S^3$. Denote the restriction of this map to $M_F$ by $c_F$ and observe that $c_F$ maps $M_F$ diffeomorphically onto

$$c_F(M_F) = \mathcal{O}^\xi_F / S^1 \times S^3$$

for $\mathcal{O}^\xi_F = \mathcal{O}^\xi \cap \mathfrak{sp}(2)_F$.

$\Pi$

If $x \in \mathfrak{sp}(2)_F$ then the orbit of $x$ under the action of $S^1 \times S^3$ intersects $\mathfrak{r}_F$ in a $K_F$ orbit. In other words, there is an injective map $\Pi\colon \mathfrak{sp}(2)_F / S^1 \times S^3 \to \mathfrak{r}_F / K_F$. Moreover, the inclusion $\mathfrak{r}_F \subset \mathfrak{sp}(2)_F$ induces $\iota\colon \mathfrak{r}_F/K_F \to \mathfrak{sp}(2)_F/S^1 \times S^3$ and $\Pi \circ \iota = \mathrm{Id}$. This implies that $\Pi$ is in fact a diffeomorphism from $\mathfrak{sp}(2)_F/S^1 \times S^3$ to $\mathfrak{r}_F/K_F$.

$\bar{\zeta}$

Consider the map

$$\bar{\zeta}\colon \mathfrak{r}_F \to \mathbb{R}^6, \quad \begin{pmatrix} \frac{-i}{2}r_- + j\bar{\lambda}\sqrt{v_-} & -\frac{\bar{\lambda}\sqrt{h}}{\sqrt{2}} \\ \frac{\lambda\sqrt{h}}{\sqrt{2}} & \frac{i}{2}r_+ + j\lambda\sqrt{v_+} \end{pmatrix} \mapsto (v_-, v_+, r_-, r_+, h, \mu = \mathrm{Re}(\lambda^4)),$$

which is $K_F$ invariant. On $\mathbb{R}^3_+ \times \mathbb{R}^2 \times [-1,1] \ni (v_-, v_+, r_-, r_+, h, \mu)$, motivated by eq. (4.6.11), we define the functions

$$\begin{aligned} C^2 &= h\sqrt{v_-v_+} \\ H_1 &= 4h + 4v_+ + 4v_- + r_+^2 + r_-^2 \\ H_2 &= (r_-^2 - r_+^2 + 4v_- - 4v_+)^2 + 8h((r_+ - r_-)^2 + 4(v_+ + v_-)). \end{aligned}$$

Denote by $\bar{D} \subset \mathbb{R}^6$ the set defined by the equations

$$\begin{aligned} H_1 &= 4(k^2 + m^2) \\ H_2 + 64C^2\mu &= 16(k^2 - m^2)^2. \end{aligned} \tag{4.6.21}$$

The image of $\mathfrak{r}_F$ under $\bar{\zeta}$ is $\bar{D}$. This follows because eq. (4.6.21) describe how eigenvalues of elements in $\mathfrak{r}$ are calculated, i.e. eq. (4.6.12). Furthermore, since $\mathfrak{sp}(2)$ is semisimple, conjugacy classes are uniquely characterised by their eigenvalues. By our previous discussion, $\bar{\zeta}$ descends to a double cover $\mathfrak{r}/K_F$, branched over $\mathbb{R}^3_+ \times \mathbb{R}^2 \times \{-1, 1\}$. The two preimages are obtained by switching between $\lambda$ and $\overline{\lambda}$.

pr

Since eq. (4.6.21) can be solved uniquely for $h$ and $\mu$ the projection pr from $\mathbb{R}^3_+ \times \mathbb{R}^2 \times [0,1]$ to the first four components maps $\bar{D}$ diffeomorphically onto its image

$$\begin{aligned} D = \{(v_-, v_+, r_-, r_+) \in (\mathbb{R}^{>0})^2 \times \mathbb{R}^2 \mid & h = 4(k^2+m^2) - 4v_+ - 4v_- - r_+^2 - r_-^2 > 0 \\ & \frac{16(k^2-m^2)^2 - H_2}{64C^2} \in [-1,1]\}. \end{aligned} \tag{4.6.22}$$

We have shown that eq. (4.6.20) restricts to

$$M_F \xrightarrow{c_F} \mathcal{O}^\xi_F/S^1 \times S^3 \xrightarrow{\Pi} (\mathcal{O}^\xi \cap \mathfrak{r}_F)/K_F \xrightarrow{\bar{\zeta}} \bar{D} \subset \mathbb{R}^6 \xrightarrow{\mathrm{pr}} D \subset \mathbb{R}^4$$

and that $\bar{\zeta}$ is a branched double cover while each other map is a diffeomorphism. Hence, $p$ is a branched double cover. By eq. (4.6.7) and lemma 4.6.2, $p = (v_-, v_+, r_-, r_+)$. Observing that $p$ stays invariant under the map $\delta\colon [Z_0, Z_1, Z_2, Z_3] \mapsto [\overline{Z_0}, \overline{Z_1}, \overline{Z_2}, \overline{Z_3}]$ completes the proof.

□

The proof of theorem 4.6.13 gives an explicit description of the branch locus of $\zeta$. The involution $\delta$ preserves the metric and reverses the almost complex structure on $\mathbb{CP}^3$. The fixed point set of $\delta$ is the standard $\mathbb{RP}^3$. This implies the following lemma which was aleady been shown [Xu06].

**Lemma 4.6.14.** *The set $\mathbb{RP}^3$ is special Lagrangian for the nearly Kähler structure on $\mathbb{CP}^3$.*

### 4.6.4 A Torus Fibration of $D$

The image of $p$ is equal to $D$ and explicitly described by eq. (4.6.22). To understand the set more conceptually, we show that $D$ itself admits a two-torus fibration $u$ over a rectangle $\mathcal{R}$ in $\mathbb{R}^2$.

**Definition 4.6.15.** Let $u = (H_2, 64C^2)$, viewed as a map from $D$ to $\mathbb{R}^2$.

We will see that the torus fibres degenerate over two edges of the rectangle. Recall that $(ik, im)$ are purely imaginary and equal to

$$\frac{1}{2\sqrt{2}}\sqrt{H_1 \pm \sqrt{H_2 + 64C^2\mu}}.$$

Since $\mu \in [-1, 1]$ we obtain the inequalities

$$H_2 - 64C^2 \leq 16(k^2 - m^2)^2 \leq H_2 + 64C^2, \quad H_2 \geq 64C^2. \tag{4.6.23}$$

Now proposition 4.6.7 implies

$$H_2 + 64C^2 \leq 16(k^2 + m^2)^2$$

and also the following propostion.

**Proposition 4.6.16.** *The image of $u$ is equal to the following rectangle, with one corner point missing*

$$\begin{aligned}\mathcal{R} = \{(H_2, 64C^2)\ |H_2 - 64C^2 \leq 16(k^2 - m^2)^2 \leq H_2 + 64C^2 \\ H_2 \geq 64C^2, \quad H_2 + 64C^2 \leq 16(k^2 + m^2)^2\} \\ \setminus \{(H_2, 64C^2) = (16(k^2 - m^2)^2, 0)\} \subset \mathbb{R}^2.\end{aligned} \tag{4.6.24}$$

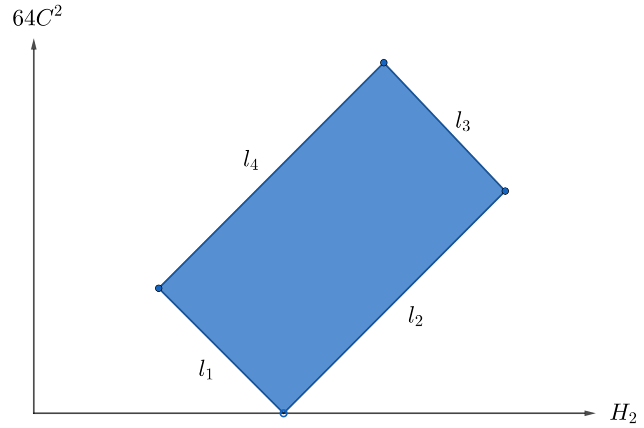


Figure 4.4: The image of $u\colon D \to \mathbb{R}^2$. theorem 4.6.17 expresses how the preimage of the boundary points is related to special subsets of $\mathbb{CP}^3$, compare with [FP90, Fig. 1].

Note that $\mathcal{R}$ is bounded by the following lines

$$l_1 = \{H_2 + 64C^2 = 16(k^2 - m^2)^2\} \qquad l_2 = \{H_2 = 16(k^2 - m^2)^2 + 64C^2\}$$
$$l_3 = \{H_2 + 64C^2 = 16(k^2 + m^2)^2\} \qquad l_4 = \{64C^2 = H_2\}.$$

The map $u \circ p$ extends to the singular set, i.e. to the map

$$P\colon {}_{\mathbb{T}^2}\backslash^{\mathbb{CP}^3} \to \bar{\mathcal{R}} = \mathcal{R} \cup \{(H_2, 64C^2) = (16(k^2 - m^2)^2, 0)\}$$

and $P$ maps ${}_{\mathbb{T}^2}\backslash^{\mathcal{S}}$ to the point $(16(k^2 - m^2)^2, 0)$.

Let $\mathcal{R}_0 = \mathcal{R} \setminus \{l_1 \cup \cdots \cup l_4\}$ be the interior of $\mathcal{R}$ and $D_0 = u^{-1}(\mathcal{R}_0)$. As a consequence of proposition 4.6.7, the fibres of $u$ are diffeomorphic to two-tori away from the boundary edges $l_3$ and $l_4$. So the fibres over $\mathcal{R}_0$ are regular and it is to be expected that the preimages of the boundary of $\mathcal{R}$ correspond to subsets with special geometric properties. In section 4.6.2, we considered the distinguished subsets $\mathcal{S}, \mathcal{T}, \mathcal{C}$ and $\mathcal{B}$. The following theorem is visualised in fig. 4.4 and summarises how the preimages of $l_1 \cup \cdots \cup l_4$ are related to these subsets.

**Theorem 4.6.17.** *The fibre of $u$ from definition 4.6.15 degenerates over the edges $l_3$ and $l_4$. The preimages $u^{-1}(l_3)$ and $u^{-1}(l_4)$ are homeomorphic to closed two disks. The flowlines of $JK^\xi$ are closed in $u^{-1}(l_3)$ and in $u^{-1}(l_4)$ (see fig. 4.5).*

*The special subsets $\mathcal{S}, \mathcal{T}$ and $\mathcal{B}$ are related to the edges of $\mathcal{R}$ by*

$$u^{-1}(l_1 \cup l_2) = p(\mathcal{B} \setminus \mathcal{S}), \quad u^{-1}(l_4) = p(\mathcal{T}).$$

*The corner points bounding $l_4$ come from the Clifford and Lawson torus, i.e.*

$$p(\tau_{k,m}) = u^{-1}(l_4 \cap l_1), \quad p(\mathcal{C}) = u^{-1}(l_4 \cap l_3).$$

*Moreover, both $u^{-1}(l_4 \cap l_1)$ and $u^{-1}(l_4 \cap l_3)$ are homeomorphic to $S^1$.*

*In the case $k = m$ the rectangle $\mathcal{R}$ degenerates to the line $l_4$. If $m = 0$ then $\mathcal{R}$ degenerates to the line $l_3$.*

*Proof.* Note that the preimage of $l_1 \cup l_2$ in $\bar{D}$ are exactly all points with $\mu = 1$ or $\mu = -1$ respectively. Hence, the preimage of $l_1 \cup l_2$ is equal to the branch locus of $\zeta$, which equals $\nu^{-1}(0)$.

Note that on $D$ we have

$$C^2 = (k^2 + m^2 - v_- v_+ - 1/4r_-^2 - 1/4r_+^2)(\sqrt{v_+ v_-}). \tag{4.6.25}$$

The maximum value of $C^2$ on $D$ is attained for

$$r_- = r_+ = 0, \quad v_- = v_+ = \frac{1}{4}(k^2 + m^2) \tag{4.6.26}$$

resulting in $C^2_{\max} = \frac{1}{8}(k^2 + m^2)^2$. Furthermore, we have $h = \frac{1}{2}(k^2 + m^2)$ implying $\alpha_-^2 = \alpha_+^2 = \frac{1}{2}$ which is consistent with lemma 4.5.4. These solutions describe the two $\mathbb{T}^2$ invariant Clifford tori. The value $C^2_{\max}$ is only attained at $l_4 \cap l_3$ and we have shown

$$p(\mathcal{C}) = u^{-1}(l_4 \cap l_3).$$

As a consequence of proposition 4.6.7 we have

$$u^{-1}(l_4) = \{v_- = v_+, \quad r_- = r_+\},$$
$$u^{-1}(l_3) = \{(v_-, v_+, r_-, r_+) \in D \mid 2C^2 = -r_+ r_- \sqrt{v_- v_+} + 4v_+ v_-, \quad r_+\sqrt{v_-} + \sqrt{v_+} r_- = 0\}.$$

By 3.2.3, twistor lifts of surfaces that lie in a totally geodesic $S^3$ satisfy $v_- = v_+$ everywhere and hence also $r_- = r_+$, which implies

$$p(\mathcal{T}) = u^{-1}(l_4).$$

All U(1) invariant minimal tori in $S^3$ belong to the family $T_{k,m,a}$, which is bounded by the Clifford and Lawson torus. We have already identified the intersection point $l_4 \cap l_3$ as the image of the Clifford torus so it follows that

$$p(\tau_{k,m}) = u^{-1}(l_4 \cap l_1).$$

The only non-trivial inequality from eq. (4.6.24) describing $\mathcal{R}$ on $l_4$ is $16(k^2 - m^2)^2 \leq$

$H_2 + 64C^2 = 128C^2$. Inserting $v_- = v_+$ and $r_- = r_+$ into this gives

$$(k^2 - m^2)^2 \leq 8C^2 \quad \Leftrightarrow \quad (k^2 - m^2)^2 \leq 8v_-(k^2 + m^2 - 2v_- - 1/2r_-^2)$$

which describes a region homeomorphic to a closed disk in the $(v_-, r_-)$ coordinates. On $u^{-1}(l_3)$, one has the equations

$$2C^2 = -r_+r_-\sqrt{v_-v_+} + 4v_+v_-, \quad r_+\sqrt{v_-} + \sqrt{v_+}r_- = 0.$$

They are uniquely solved by

$$v_+ = \frac{2C^2}{E_-}, \quad r_+ = -\frac{\sqrt{2}Cr_-}{\sqrt{v_-E_-}}$$

and from 4.6.25 we can express $C^2$ as

$$C^2 = \frac{v_-(2\sqrt{k^2 + m^2} - \sqrt{E_-})^2}{2}.$$

On $l_3$ the only non-trival inequality defining $\mathcal{R}$ is $H_1^2 - 128C^2 = H_2 - 64C^2 \leq 16(k^2 - m^2)$ which simplifies to

$$\frac{1}{2}k^2m^2 \leq 8C^2 \quad \Leftrightarrow \quad \frac{1}{2}k^2m^2 \leq \frac{v_-(2\sqrt{k^2 + m^2} - \sqrt{E_-})^2}{2}.$$

This region is again homeomorphic to a closed two-disk in the coordinates $v_-$ and $r_-$.

In both $u^{-1}(l_3)$ and $u^{-1}(l_4)$ the flow of $JK^\xi$ is restricted to a two dimensional subset where $C^2$ is a preserved quantity, whose differential only vanishes at the point described in eq. (4.6.26). The closed flowlines of $JK^\xi$ are the level sets of $C^2$. For $u^{-1}(l_4)$, they are plotted in fig. 4.5.

The last statement follows from eq. (4.6.24). □

The fact that the image of $u$ degenerates to $u^{-1}(l_4)$ is also a manifestation of lemma 4.6.10. On $u^{-1}(l_3)$, one can also write the identities in a more symmetric way

$$h = \frac{1}{2}\sqrt{E_-}\sqrt{E_+}, \quad H_1 = (\sqrt{E_-} + \sqrt{E_+})^2$$
$$2C^2 = E_+v_- = E_-v_+ \quad r_-\sqrt{E_+} + r_+\sqrt{E_-} = 0.$$

### 4.6.5 Relation between $p$ and Moment Maps

If one is just interested in a homeomorphism or cover ${}_{\mathbb{T}^2}\backslash^{\mathbb{CP}^3} \to \mathbb{R}^4$ the functions $(v_-, v_+, r_-, r_+)$ are an unnecessarily complicated choice from a topological point of

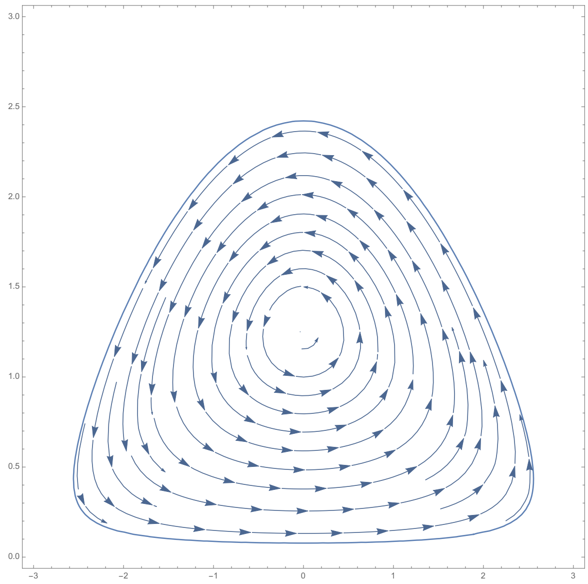


Figure 4.5: Flow lines of $JK^\xi$ in $u^{-1}(l_4)$. The boundary of the region is equal to $u^{-1}(l_4 \cap l_1)$ while the zero of $JK^\xi$ is the preimage of $u^{-1}(l_4 \cap l_3)$.

view. The reason for using these functions is that $JK^\xi$ takes a simple form

$$2(r_-v_-\frac{\partial}{\partial v_-} + r_+v_+\frac{\partial}{\partial v_+} + (h-2v_-)\frac{\partial}{\partial r_-} + (h-2v_+)\frac{\partial}{\partial r_+}).$$

To get an idea of what the functions $(v_-, v_+, r_-, r_+, h)$ look like in homogeneous coordinates consider the following set of $\mathbb{T}^2$ invariant functions

$$\begin{aligned} f_1 &= |Z|^{-2}(|Z_0|^2 + |Z_1|^2), \quad f_2 = |Z|^{-2}(|Z_0|^2 - |Z_1|^2) \\ f_3 &= |Z|^{-2}(|Z_2|^2 + |Z_3|^2), \quad f_4 = |Z|^{-2}(|Z_2|^2 - |Z_3|^2) \\ f_5 &= |Z|^{-4}\mathrm{Re}(Z_0Z_1\overline{Z_2Z_3}). \end{aligned} \tag{4.6.27}$$

Clearly, $1 = f_1 + f_3$ and the functions are also invariant under $\delta$. Note that the Kähler structure on $\mathbb{CP}^3$ admits a $\mathbb{T}^3$ action and after choosing an appropriate basis for $\mathfrak{t}^{3\vee}$, $(f_1, f_2, f_4)$ are multiples of the symplectic moment map on $\mathbb{CP}^3$.

We can furthermore deduce the relation between the functions $f_1, \dots, f_5$ and $\nu$. Note that $\nu$ is not invariant under $\delta$ but satisfies $\nu \circ \delta = -\nu$ and can thus not be expressed in terms of $f_1, \dots, f_5$. However, observe that the square of $\nu$ can be expressed in terms of the functions $f_i$ via

$$(12\nu)^2 + f_5^2 = \frac{1}{16}(f_1^2 - f_2^2)(f_3^2 - f_4^2). \tag{4.6.28}$$

Denote by $D_f$ the image of $(f_1, f_2, f_4, f_5)$ as a map from $\mathbb{CP}^3 \backslash \mathcal{S}$ to $\mathbb{R}^4$. By writing down an explicit section one can prove that away from the branch locus $\mathcal{B}$ the $\mathbb{T}^2 \times \{\pm 1\}$ principal bundle $(f_1, f_2, f_4, f_5)\colon \mathbb{CP}^3 \to \mathbb{R}^4$ is trivial.

Our aim is to express $p = (v_-, v_+, r_-, r_+)$ in terms of the functions $f_1, \ldots, f_5$. For a quaternion $q = z + jw \in \mathbb{C} \oplus j\mathbb{C}$ let $q_{\mathbb{C}} = z$ and $q_{j\mathbb{C}} = w$ denote the $\mathbb{C}$ and $j\mathbb{C}$ part.

Recall from the proof of theorem 4.6.13 that $p$ is defined as a composition of the maps $c_F$, $\zeta \circ \Pi$ and pr. Since pr is just a projection map, it remains to compute $c_F$ and $\zeta \circ \Pi$.

**Lemma 4.6.18.** *The map* $(v_-, r_-, v_+, r_+) = \zeta \circ \Pi \colon \mathfrak{sp}(2)_F \to D$ *is given by*

$$\sigma = \begin{pmatrix} q_1 & -\overline{q_2} \\ q_2 & q_3 \end{pmatrix} \mapsto (|(q_1)_{j\mathbb{C}}|^2, 2i(q_1)_{\mathbb{C}}, |Q_{3j\mathbb{C}}|^2, -2iQ_{3\mathbb{C}}) \tag{4.6.29}$$

*where* $Q_3 = q_2^{-1} q_3 q_2$.

*Proof.* The functions $(v_-, r_-, v_+, r_+)$ can be computed by conjugating $\sigma$ with an element in $S^1 \times S^3$ to an element in $\mathfrak{r}$. Note that $\mathfrak{r}_F / K_F = \mathfrak{f}^1 / K$ for

$$\mathfrak{f} = \left\{ \begin{pmatrix} q_1 & -\overline{q_2} \\ q_2 & q_3 \end{pmatrix} \mid q_2 \in \mathbb{C} \right\} \subset \mathfrak{sp}(2), \quad \mathfrak{f}^1 = \mathfrak{f} \cap \mathfrak{sp}(2)_F.$$

Furthermore, eq. (4.6.29) is invariant under the group $K$ so it suffices to find $g \in \mathrm{Sp}(2)$ such that $g^{-1}\sigma g \in \mathfrak{f}^1$. Hence, we can conjugate $\sigma$ with the element $\mathrm{diag}(1, \frac{q_3}{|q_3|})$ and the result follows from the definition of $\bar{\zeta}$. $\square$

To compute $c_F$ one picks a (local) section $s$ and then evaluates $\sigma = s^{-1}\xi s$. Then $p$ is computed for $\sigma$ in eq. (4.6.29). We choose $s$ such that $\sigma$ already lies in $\mathfrak{f}$, which simplifies eq. (4.6.29). To this end, define the section

$$s \colon \mathbb{CP}^3 \setminus (L_1 \cup L_6) \to \mathrm{Sp}(2), \quad [Z_0, Z_1, Z_2, Z_3] \mapsto \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}$$

with

$$\begin{aligned} h_1 &= |Z|^{-1}(Z_0 + jZ_1), \quad h_2 = |Z|^{-1}(Z_2 + jZ_3) \\ k_1 &= \frac{1}{f_1}\sqrt{\frac{1}{f_1} + \frac{1}{f_3}}|Z|^{-1}(Z_0 + jZ_1), \quad k_2 = \frac{-1}{f_3}\sqrt{\frac{1}{f_1} + \frac{1}{f_3}}|Z|^{-1}(Z_2 + jZ_3). \end{aligned}$$

Consequently, $s$ can be used to compute $c_F$,

$$c_F([Z_0, Z_1, Z_2, Z_3]) = \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}^{-1} \begin{pmatrix} ik & 0 \\ 0 & im \end{pmatrix} \begin{pmatrix} h_1 & k_1 \\ h_2 & k_2 \end{pmatrix}. \tag{4.6.30}$$

Let

$$E_\pm = r_\pm^2 + 4v_\pm.$$

Combining eq. (4.6.30) with lemma 4.6.18 yields after a tedious calculation

$$
\begin{aligned}
h &= -2((-1+f_1)f_1k^2 + 2f_2f_4km + (-1+f_1)f_1m^2 + 8kmf_5) \\
r_- &= -2(f_4k + f_2m) \\
E_- &= 4((-1+f_1)^2k^2 + 2f_2f_4km + f_1^2m^2 + 8kmf_5) \\
r_+ &= -2(f_4k + f_2m) - \frac{2(k-m)(k+m)(f_1f_4k + (-1+f_1)f_2m)}{(-1+f_1)f_1(k^2+m^2) + 2km(f_2f_4 + 4f_5)} \\
E_+ &= 4(f_1^2k^2 + (-1+f_1)^2m^2 + 2km(f_2f_4 + 4f_5)).
\end{aligned}
\tag{4.6.31}
$$

Denote by $\pi_{\mathbb{T}^2}$ the quotient map $\mathbb{CP}^3 \setminus \mathcal{S} \to M_F$ by the $\mathbb{T}^2$-action. Eq. 4.6.31 can in fact be solved for $(f_1, f_2, f_4, f_5)$. This implies together with the triviality of the bundle $f\colon \mathbb{CP}^3 \setminus \mathcal{S} \to D_f$ the following proposition.

**Proposition 4.6.19.** *There is a homeomorphism $D_f \to D$ which is a diffeomorphism in the smooth points such that the following diagram commutes*

$$
\begin{tikzcd}
 & \mathbb{CP}^3 \setminus \mathcal{S} \arrow[dl, "f"'] \arrow[dr, "p\circ\pi_{\mathbb{T}^2}"] & \\
D_f \arrow[rr] & & D
\end{tikzcd}
.
$$

*In particular $p \circ \pi_{\mathbb{T}^2}$ is a trivial $\mathbb{T}^2 \times \{\pm 1\}$ bundle when restricted to $\mathbb{CP}^3 \setminus (\mathcal{S} \cup \mathcal{B})$.*

Note that eq. (4.6.28) implies the following corollary of proposition 4.6.19.

**Corollary 4.6.20.** *The square of the multi-moment map $\nu$ is a function of $(v_-, v_+, p_-, p_+)$.*

For Clifford tori we have $f_1 = 1/2, f_2 = 0, f_4 = 0, f_5 = 0$ and in particular $\frac{1}{16}(f_1^2 - f_2^2)(f_3^2 - f_4^2) - f_5^2 = \frac{1}{256}$ which means $\nu = \pm 3/4$. This implies that the two Clifford tori are equal to $\nu^{-1}(\pm 3/4)$ which are the extremal values of $\nu$.

**Lemma 4.6.21.** *The set $\mathcal{T}$ is explicitly given by $\{f_1 = 1/2,\ f_4k = f_2m\} \subset \mathbb{CP}^3$.*

*Proof.* This follows from eq. (4.6.31) by setting $E_- = E_+$ and $r_- = r_+$. □

We also get an explicit description of the torus quotient of $\mathcal{T} \cap \nu^{-1}(0)$, a set which contains $\tau_{k,m}$, as the set of points $[X_0, X_1, X_2, X_3] \in \mathbb{RP}^3/\pm 1$ that satisfy $f_1 = 1/2$ and $f_4k = f_2m$, i.e.

$$X_0^2 + X_1^2 = X_2^2 + X_3^2, \quad k(X_2^2 - X_3^2) = m(X_0^2 - X_1^2).$$

### 4.6.6 Superminimal $\mathrm{U}(1)$ invariant Curves

The focus of this chapter has been on transverse $J$-holomorphic curves. But the computations carried out in section 4.6.5 give a framework for classifying $\mathrm{U}(1)$ invariant superminimal curves too. From eq. (4.6.31) we can deduce that $v_- = 4|kZ_0Z_1 + mZ_2Z_3|^2$ which implies the following lemma.

**Lemma 4.6.22.** *The vector field $K^\xi_x$ is horizontal in $x \in \mathbb{CP}^3$ if and only if $x$ lies on the quadric $\mathcal{Q}$ defined in eq.* (4.6.19).

In the case $k = m$ this has already been observed in [ABS20, Corollary 6.3] where this quadric is described in more detail. Remarkably, $\mathcal{Q}$ is constructed from $\xi$ in three different ways, via $\mathfrak{sp}(2) \otimes \mathbb{C} \cong S^2(\mathbb{C}^4)$, via the anti-selfdual part of $\nabla(K^\xi)$, as explained after eq. (4.6.18), and via lemma 4.6.22. If a $\rho$ invariant $J$ holomorphic curve intersects $\mathcal{Q}$ it will lie in $\mathcal{Q}$ entirely. In other words, $\mathcal{Q}$ is invariant under $\rho$ and $JK^\xi$ is tangent to $\mathcal{Q}$. Furthermore, it follows that $\mathcal{Q}$ is traced out by superminimal $\rho$ invariant $J$-holomorphic curves.

**Proposition 4.6.23.** *If $k > m$, all superminimal curves invariant under $\rho$ are given as the intersection of $\mathcal{Q}$ and*

$$\mathcal{P}_c\colon (\frac{-m}{k-m})^m Z_0^m Z_3^k - (\frac{k}{k-m})^k c^{k-m} Z_1^m Z_2^k = 0.$$

*for $c \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. They are equal to $L_3, L_5$ or locally parametrised by*

$$\varphi_c\colon \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \to \mathbb{CP}^3, \quad z \mapsto [1, \frac{-cm}{k-m} z^{2k}, z^{k-m}, \frac{kc}{k-m} z^{k+m}]. \tag{4.6.32}$$

To prove the proposition one checks that for each $\varphi_C$ the holomorphic contact form on $\mathbb{CP}^3$ vanishes which means $\varphi_C$ is superminimal. Clearly each $\varphi_C$ is $\rho$ invariant. The projective lines $L_2 \cup \dots L_5$ are the set where $v_+$ also vanishes.

## 4.7 The Degenerate Action

Recall that $k$ and $m$ are weights of the torus action on $\mathbb{CP}^3$. We discuss the case when $k = m$ in more detail. The other degenerate case, when $k$ or $m$ is equal to zero is similar but slightly more complicated from a computational point of view. So, from now on we will assume $k = m$ unless stated otherwise. By U(1)-action we will simply refer to the action defined in eq. (4.6.16) for $k = m$.

The key difference to the non-degenerate case is that the action now commutes with a larger subgroup, namely $\mathrm{U}(2) \subset \mathrm{Sp}(2)$. This makes the map $\zeta$ invariant under U(2) which means it can no longer be a submersion. In fact the computations simplify significantly due to the additional symmetry. We have seen in lemma 4.6.10 that the assumption $k = m$ implies $v_- = v_+$ and $r_- = r_+$ everywhere so we will simply denote these functions by $v$ and $r$ in this subsection. The other simplification is that $H_2 = 64C^2$ by eq. (4.6.23). We choose the normalisation $k = m = \frac{1}{2}$ and let $s = \frac{1}{2}(1-r)$. So $H_1 = 2$ which gives

$$h + 2v = 2s(1-s), \quad C^2 \leq \frac{1}{32}.$$

This means the functions take values in the subset $D_{\mathrm{deg}} = \{v, s \geq 0, \quad s(1-s) \geq v\}$. In fact we have an analogous result to theorem 4.6.13.

**Proposition 4.7.1.** *The image of the map $(v, s)$ is $D_{\mathrm{deg}}$, it is invariant under* $\mathrm{U}(2)$ *and $\delta$ and a submersion over the interior of $D_{\mathrm{deg}}$.*

Recall the definition of the projective lines $L_i$ from eq. (4.6.17) and note that $L_4$ and $L_2$ are fixed points of the U(1)-action. They are related by $j$ and map onto a totally geodesic two-sphere $S_0^2 \subset S^4$. Under the double cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ the U(1) action corresponds to the SO(2) subgroup fixing a linear $\mathbb{R}^3 \subset \mathbb{R}^5$ that contains $S_0^2$.

Eq. 4.6.2 simplifies to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v &= 2v(1-2s) \\ \frac{\mathrm{d}}{\mathrm{d}t} s &= 2v - h = 4v - 2s(1-s). \end{aligned} \tag{4.7.1}$$

The quantity $C^2 = hv = 2s(1-s) - 2v$ is preserved. From eq. (4.6.31) we furthermore get the explicit expressions

$$\begin{aligned} v &= |Z|^{-4}|Z_0Z_1 + Z_2Z_3|^2, \quad h = 2|Z|^{-4}|Z_0\overline{Z_3} - \overline{Z_1}Z_2|^2 \\ s &= |Z|^{-2}(|Z_1|^2 + |Z_3|^2). \end{aligned} \tag{4.7.2}$$

For $k \neq m$ we have seen that U(1) invariant superminimal curves trace out the holomorphic quadric $\mathcal{Q}$, which depends on $k$ and $m$. If $k = m$ then every superminimal curve is automatically totally geodesic by eq. (3.2.5) since $v_- = 0$ implies $v_+ = 0$. In this case, the decomposition of $\mathcal{Q}$ into U(1) invariant curves is the Segre embedding $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathcal{Q}$.

**Proposition 4.7.2.** *If $k = m$ the all superminimal* U(1) *invariant $J$-holomorphic curves in $\mathbb{CP}^3$ are projective lines given by*

$$M_c = \{[vw, -uz, uw, vz] | [z, w] \in \mathbb{CP}^1\}$$

*with $c = [u, v] \in \mathbb{CP}^1$. All* U(1) *invariant twistor lines are given by*

$$K_c = \{[su, t\bar{u}, vs, \bar{v}t], [s, t] \in \mathbb{CP}^1\}.$$

In fact, the family $K_c$ traces out the non-holomorphic quadric $Z_0\overline{Z_3} - \overline{Z_1}Z_2 = 0$, see eq. (4.7.2). There is no analogous statement in the non-degenerate case where there are only two $\mathrm{U}(1)$ invariant twistor lines and the expression for $h$ is more complicated.

Each member of $K_c$ is invariant under the involution $j$ and projects to a point in $S^2_0 \subset S^4$. On the other hand, $j$ maps $M_c$ to $M_{-\bar{c}^{-1}}$, where $[u,v]^{-1} = [v,u]$ and the family $M_c$ is projected to a $\mathbb{RP}^2$ family of $\mathrm{U}(1)$ invariant totally geodesic spheres in $S^4$. They are explicitly given by $(\mathbb{R}^2 \oplus \mathbb{R}v) \cap S^4$ where $v \in \mathbb{RP}^2$.

In fact, we have two maps $m, k\colon L_4 \to L_2$ where $m([u,0,v,0]) = [0,-v,0,u]$ and $k([u,0,v,0]) = [0,\bar{u},0,\bar{v}]$. For every point $x$ in $L_4$ there is a unique projective line in each of the families $M_c$ and $K_c$ going through $x$. This is unique the projective line going through $x$ and $k(x)$ or $m(x)$ respectively. Observe that $k(x)$ and $m(x)$ are antipodal points and that $k$ is just the restriction of $j$ to $L_4$.

Let $A \in \mathrm{SU}(2) \subset \mathrm{Sp}(2)$, then $A$ leaves the lines $L_4$ and $L_2$ invariant and maps $M_c$ and $K_c$ to $M_{Ac^{-1}}$ and $K_{Ac^{-1}}$ respectively, using the standard action of $\mathrm{SU}(2)$ on $\mathbb{CP}^1$.

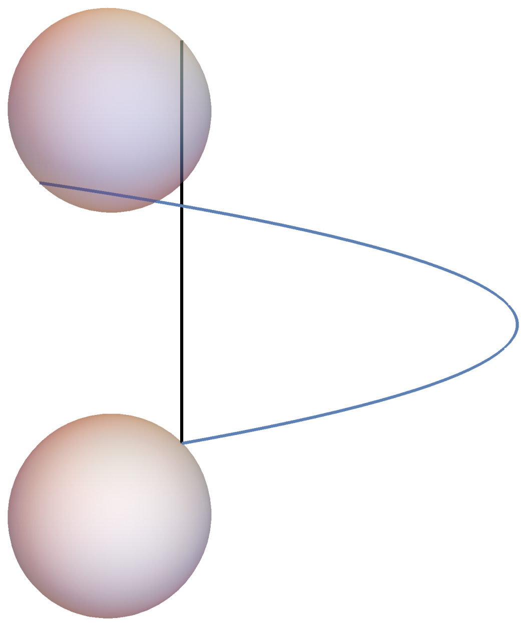

Figure 4.6: For each point $x$ in $L_4$, there are two $\mathrm{U}(1)$ invariant $J$-holomorphic spheres containing $x$. One of them is the twistorline (black), the other is the unique horizontal projective line (blue) passing through $x$. Both of these projective lines intersect $L_2$ in one point and the resulting intersection points are antipodal. The action of $\mathrm{SU}(2) \subset \mathrm{Sp}(2)$ rotates $L_4$ and $L_2$ and preserves this construction by acting on the horizontal and vertical lines accordingly.

We turn back to the general case of $\mathrm{U}(1)$ invariant $J$-holomorphic curves which are not superminimal. They are described by eq. (4.7.1) and we make the following observations.

**Lemma 4.7.3.** *The properties $h = 0$ and $v = 0$ are stable under the flow of $JK^\xi$. On $D_{\deg}$, the fixed points $(s,v)$ of $JK^\xi$ are $p = (1/2, 1/8)$, $q_1 = (1,0)$, $q_2 = (0,0)$. Any*

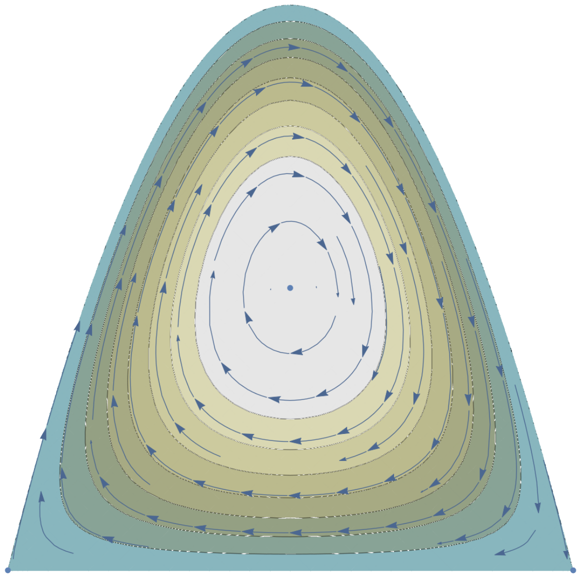

Figure 4.7: Different values of $C^2 = 2v(s(1-s)-v)$ visualised by different colours along with the flow lines of $JK^\xi$ which are equal to the level sets of $C^2$. Dots represent fixed points of the flow.

*point $(s,0) \neq q_1$ converges to $q_2$ under the flow of $JK^\xi$, while any point $(s,v) \neq q_2$ with $v = s(1-s)$, i.e. $h = 0$, converges to $q_1$. Because of the existence of the conserved quantity $C^2$, all points in the interior of $D_{\text{deg}}$ belong to closed flow lines. The minimum value $C^2 = 0$ is attained on the boundary while the maximum value $C^2 = 1/32$ is only attained on $p$. The flow lines of $JK^\xi$ satisfy the implicit equation*

$$C^2 = hv = 2v(s(1-s)-v) = \text{const.} \tag{4.7.3}$$

Note that this result is in accordance with our earlier discussions about U(1) invariant spheres and is similar to the dynamics on $u^{-1}(l_3)$ and on $u^{-1}(l_4)$. Values where $v = 0$ describe superminimal U(1) invariant curves, those with $h = 0$ describe U(1) invariant twistor lines.

The point $p$ is the unique point in $D_{\text{deg}}$ where both $h$ and $v$ stay constant. Consequently, these values must describe a Clifford torus. Because of the existence of the preserved quantity $C^2$, the functions $h, v$ are periodic. From 4.7.1 we can in fact compute the period $T$.

**Lemma 4.7.4.** *The period $T$ and the integral over $v$ and $h$ of each orbit can be expressed by elliptic integrals in terms of $C^2$ in the following way*

$$T(C^2) = \int_{a_-}^{a_+} \frac{1}{2\sqrt{-x(x-a_-)(x-a_+)}}\mathrm{d}x - \ln\left(\frac{a_+}{a_-}\right)$$
$$\int_0^T h(t)\mathrm{d}t = 2\int_0^T v(t)\mathrm{d}t = \int_{a_-}^{a_+} \frac{\sqrt{x}}{\sqrt{-(x-a_-)(x-a_+)}}\mathrm{d}x,$$

*with* $a_\pm = \frac{1}{8}(1 \pm \sqrt{1 - 32C^2})$.

*Proof.* For a given $C^2$ consider the flow line starting at $t = 0$ at the point $(s, v) = (1/2, a_+)$. Due to symmetry, the path will pass $(1/2, a_-)$ at time $t = T/2$

$$0 = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}(\ln(v))\mathrm{d}t = 2\int_0^T (1 - 2s)\mathrm{d}t = 4\int_0^{T/2} (1 - 2s)\mathrm{d}t.$$

By eq. (4.7.3), the function $s$ can be parametrised in terms of $v$ for $0 \leq s \leq 1/2$ by

$$s(v) = \frac{1}{2} - \frac{\sqrt{-2C^2 v + v^2 - 4v^3}}{2v}.$$

As a result,

$$\begin{aligned}\frac{T}{4} &= \int_0^{T/2} s(t)\mathrm{d}t = \int_0^{T/2} \frac{s(t)}{\dot{v}(t)}\mathrm{d}(v(t)) = \int_{v(0)}^{v(T/2)} \frac{s(v)}{2v(1 - 2s(v))}\mathrm{d}v \\ &= \frac{1}{4}\int_{a_-}^{a_+} \frac{1}{\sqrt{v\left(-2C^2 - 4v^2 + v\right)}} - \frac{1}{v}\mathrm{d}v.\end{aligned}$$

The integral over $v$ can be computed in the same manner

$$2\int_0^T v\mathrm{d}t = 2\int_0^{T/2} \frac{\dot{v}}{1 - 2s}\mathrm{d}t = 2\int_{a_-}^{a_+} \frac{1}{1 - 2s(v)}\mathrm{d}v = 2\int_{a_-}^{a_+} \frac{v}{\sqrt{v\left(-2C^2 - 4v^2 + v\right)}}\mathrm{d}v,$$

as required. □

By integrating the equation for $\dot{s}$ we see that every U(1) torus must necessarily satisfy $\mathrm{vol} = 2\mathrm{vol}^{\mathcal{V}}$ since $s$ is invariant with respect to the action $\rho$. By proposition 3.2.5, the Euler number of these tori in $S^4$ vanishes.

**Lemma 4.7.5.** *The size of the* U(1) *orbit is maximal if* $s = 1/2$ *and* $v \leq 1/8$ *and minimal if* $s = 1/2$ *and* $v \geq 1/8$.

*Proof.* Up to a multiple, the size of the orbit is $h + v = C^2/v + v$ with $v \in [a_-, a_+]$. Observe that $v + C^2/v$ is monotone on this interval so the extremal values are attained on the boundary. □

Consequently, any non-horizontal or non-vertical U(1) invariant $J$-holomorphic curve intersects the hypersurface $s = 1/2$ at two different times within the period $T$, when the U(1) orbit has minimal or maximal volume. Horizontal and vertical curves intersect the hypersurface at a single time which is when the U(1)-orbit has maximal volume.

A closed integral curve in $D_{\mathrm{deg}}$ is a necessary condition for the integral curve of $JK^\xi$ to close up in $\mathbb{CP}^3$. However, this is not a sufficient condition. Up to an action of a discrete group, the map $(v, s)$ is a $\mathrm{U}(2)$ principal bundle over the interior of $D_{\mathrm{deg}}$. We can declare the horizontal subspace $\mathbf{H}$ to be spanned by the vector fields $\mathrm{grad}(C^2)$ and $JK^\xi$. Then the closeness condition of $JK^\xi$ in $\mathbb{CP}^3$ is translated to whether the holonomy element has finite order, which only depends on $C^2$. Away from boundary points, there is a map $\mathrm{ord}\colon \mathbb{CP}^3 \to \mathbb{Z} \cup \{\infty\}$ which maps each element in $\mathbb{CP}^3$ to the order of the holonomy of the curve $C^2 = \mathrm{const}$. One observes that $\mathbf{H}$ has non-vanishing curvature. So one expects that ord takes finite values on a dense set in $\mathbb{CP}^3$.

This is indeed the case, by lemma 4.6.10 every $J$-holomorphic curve invariant under the action considered here corresponds to minimal surface which is contained in a totally geodesic $S^3$. In [HL71, Theorem 8] it is shown that minimal surfaces in $S^3$ invariant under $\mathrm{U}(1)$ are parametrised by an element in a rational interval, where the boundary values correspond to the Clifford surface, in our case $C = \frac{1}{32}$, or a totally geodesic two-sphere, in our case $C = 0$.

# Chapter 5

# Deformations of $J$-holomorphic Curves

Given a special submanifold one key question is whether it can be deformed to other special submanifolds. This is not only a potential method to construct new examples of such submanifolds but is also crucial when trying to understand their moduli space. In nearly Kähler manifolds, the deformation theory of $J$-holomorphic curves can also be used to compare them with holomorphic curves in complex or Kähler manifolds.

From this perspective, twistor lifts of superminimal surfaces in $S^4$ and $\mathbb{CP}^2$ have the special property that they are both $J_2$ and $J_1$ holomorphic. This has been exploited in [MU97] to study the deformation problem of superminimal surfaces by using holomorphic data. For complex submanifolds of complex manifolds, Kodaira's work is foundational. One of his results is that deformations of such submanifolds can be understood in terms of cohomology groups of the normal bundle. In particular, the deformation space of a complex submanifold is always a complex manifold.

In contrast, we have seen in example 3.2.1 and example 3.2.8 that the deformation spaces of totally geodesic $J$-holomorphic curves in $S^6$ or of twistor lines in $\mathbb{CP}^3$ do not admit almost complex structures. On the infinitesimal level, we establish in section 5.1 that deformations are described by eigensections of a twisted Dirac operator $\bar{D}$, which is a complex anti-linear operator on the normal bundle. In fact, using a complex anti-linear map, $\bar{D}$ can be identified with the $\bar{\partial}$ operator on the normal bundle.

Just as in the case of associative submanifolds, not much can be said about the dimension of the infinitesimal deformation space in general. The computation of the deformation space simplifies significantly for $J$-holomorphic homogeneous tori, see lemma 5.2.6. Such tori exist in any nearly Kähler manifold with a two-torus symmetry at the extrema of the multi-moment map by lemma 2.4.1.

We show in section 5.2 that homogenous tori in $S^6$ and in $\mathbb{CP}^3$ are rigid up to the action of the automorphism group. The homogenous tori in $\mathbb{CP}^3$ are twistor lifts of

Clifford tori in $S^4$ and deformations of $J$-holomorphic curves in $\mathbb{CP}^3$ translate into deformations of minimal surfaces in $S^4$.

For the ambient space $S^3$, the rigidity of the Clifford torus has been deduced, for example, from Hitchin's powerful treatment of minimal tori in $S^3$ [Hit90]. The Clifford torus is the unique minimal surface where the (normalised) spectral curve has genus 0. Our approach is more hands-on, we compute the infinitesimal deformation space explicitly for homogenous tori.

## 5.1 The Infinitesimal Deformation Operator

In this section we derive the operator $\bar{D}$ describing infinitesimal deformations of $J$-holomorphic curves. Then we identify this operator as both the $\bar{\partial}$-operator on the normal bundle and a twisted Dirac operator. We derive a Weitzenböck formula for $\bar{D}$ and show that it is self-adjoint, so it has index zero. We begin by defining a product of tangent vectors in $TM$ using the nearly Kähler three-form.

Let $M$ be a nearly Kähler six-manifold and for the sake of simplification assume that $\varphi\colon X \to M$ is an embedded $J$-holomorphic curve such that $X$ can be identified with its image in $M$ which we will also denote by $X$. Denote the normal bundle of $X$ in $TM|_X$ by $\nu$. Let $\nabla$ and $\overline{\nabla}$ be the Levi-Civita and the nearly Kähler connection, respectively. They are related by

$$\overline{\nabla}_v w = \nabla_v w + \frac{1}{2}(\nabla_v J)(Jw).$$

Define

$$\times\colon TM \otimes TM \to TM, \quad g(v \times w, u) = \operatorname{Re}\psi(v, w, u).$$

**Lemma 5.1.1.** *The product $\times$ has the following properties*

- $(Jv) \times w = v \times (Jw) = -J(v \times w)$
- $v \times w = (\nabla_v J)(w)$
- *It induces a map $TX \otimes \nu \to \nu$ which satisfies the Clifford condition, i.e.* $v \times (v \times w) = -|v|^2 w$.

The first and third point can be checked by writing $\operatorname{Re}\psi$ in a special unitary basis, for the second statement see [MNS08]. Note that lemma 5.1.1 implies in particular

$$\overline{\nabla}^{\perp}_v w = \nabla^{\perp}_v w - \frac{1}{2}J(v \times w). \tag{5.1.1}$$

**Lemma 5.1.2.** *Let*

$$\alpha\colon \Lambda^1_{\mathbb{C}} X \otimes \nu_{\mathbb{C}} \to \nu_{\mathbb{C}}$$

*defined by complexifying* $\times \circ g^\flat$. *Then* $\alpha$ *is the zero map on* $\Lambda^{0,1}X \otimes \nu^{0,1}$ *and on* $\Lambda^{1,0}X \otimes \nu^{1,0}$. *Besides, it induces isomorphisms* $\Lambda^{0,1}X \otimes \nu^{1,0} \to \nu^{0,1}$ *and* $\Lambda^{1,0}X \otimes \nu^{0,1} \to \nu^{1,0}$.

*Proof.* Note that $g^\flat_{\mathbb{C}}$ maps $T^{1,0}X$ to $\Lambda^{0,1}X$ and $T^{0,1}X$ to $\Lambda^{1,0}X$. Then the first part of the lemma follows from the fact that for $v \in T^{1,0}X$ and $w \in \nu^{0,1}$

$$i(v \times w) = (Jv) \times w = v \times (Jw) = -i(v \times w).$$

The second part follows because $\times$ is surjective as a map between real vector bundles, due to the Clifford relation. $\square$

This result is consistent with the Riemann-Roch theorem, the first Chern class of a nearly Kähler manifold vanishes. The Todd class $\mathrm{td}(X)$ of $X$ is $1 + \frac{1}{2}c_1(X)$ and the Chern character of $\mathrm{ch}(\nu^{1,0})$ equals $2 + c_1(\nu^{1,0})$. Combining all of these observations implies

$$\begin{aligned} 0 = \int_X c_1(TM) &= \int_X c_1(\nu^{1,0} \oplus T^{1,0}X) = \int \mathrm{td}(X)\mathrm{ch}(\nu^{1,0}) \\ &= h^0(\nu^{1,0}) - h^1(\nu^{1,0}) = h^0(\nu^{1,0}) - h^0(\nu^{0,1} \otimes \Lambda^{1,0}). \end{aligned}$$

McLean developed the deformation theory of calibrated submanifolds in different ambient geometries [McL98]. All deformations of a submanifold are locally parametrised by sections in the normal bundle. Deformations as calibrated submanifolds are then typically solutions to a non-linear PDE on the normal bundle. Usually, there is little hope to solve this PDE, so one instead passes to its linearisation.

Solution to this linearised equation are called infinitesimal deformations. If the linearisation is surjective one can apply the inverse function theorem to construct smooth deformations from infinitesimal ones. This is the case for special Lagrangian submanifolds in Calabi-Yau manifolds or for coassociative submanifolds of $G_2$ manifolds.

However, the linearisation of the deformation operator is not necessarily surjective for associative submanifolds of $G_2$ manifolds. Usually, one can identify a $k$-dimensional space of deformations geometrically, for example from actions of the automorphism group. The infinitesimal deformation equation then has a $n \geq k$-dimensional space of solutions. If $n = k$ then all infinitesimal deformations are explained geometrically. If $n > k$ one can proceed and compare higher order approximations of the deformation operator. These ideas are used quite explicitly implemented by Kawai in [Kaw17; Kaw18].

$J$-holomorphic curves in nearly Kähler manifolds are in general not calibrated. However, a similar strategy can be used to study their deformations. To set up the deformation operator as in [McL98], it is necessary to define a differential form which vanishes exactly on $J$-holomorphic curves. Observe that $\chi(u,v) = u \times v$ is a $TM$-valued two-form with such a property. This is similar to associative submanifolds and indeed they are related via the cone construction.

Infinitesimal deformations of associative submanifolds are governed by a twisted Dirac operator on the normal bundle. The following proposition establishes an analogous result for $J$-holomorphic curves in nearly Kähler manifolds. The appearance of a zero-order term is similar to infinitesimal deformations of associatives in nearly parallel $G_2$-manifolds [Kaw17].

**Proposition 5.1.3.** *Let $X \subset M$ be an embedded $J$-holomorphic curve in a nearly Kähler manifold and let $V$ be a section in the normal bundle $\nu$. Then $V$ corresponds to an infinitesimal deformation of $X$ as an embedded $J$-holomorphic curve in $M$ if*

$$0 = e_1 \times \overline{\nabla}_{e_1} V + e_2 \times \overline{\nabla}_{e_2} V + 2JV$$

*where $\{e_1, Je_1 = e_2\}$ is a local orthonormal frame of $X$.*

*Proof.* Let $V$ be a section in $\nu$ of sufficiently small norm, such that

$$\exp_V \colon X \to M, \quad x \mapsto \exp(V_x)$$

is a diffeomorphism onto its image which will be denoted by $X'$. Denote the normal bundle of $X'$ in $TM$ by $\nu'$. Let $P_V$ be the parallel transport map along $V$ which gives a map $\Gamma(X,\nu) \to \Gamma(X',\nu')$. Then $F(V) = P_V^{-1}(\exp_V^* \chi|_{X'})$ is an element in $\Omega^2(X,\nu)$ and vanishes if and only $X'$ is $J$-holomorphic. Infinitesimal deformations of $X$ are elements in the kernel of the operator $V \mapsto \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(tV)$.

To compute the linearisation of $F$ let $e_1, Je_1 = e_2$ be an orthonormal frame of $TX$ and $\{\eta_1, \eta_2, \eta_3, \eta_4\}$ be an orthonormal frame of $\nu$. Let $X_0 = \{x \in X \mid V_x \neq 0\}$ and $t_0 > 0$ such that

$$f \colon (-t_0, t_0) \times X_0 \to M, \quad (t,x) \mapsto \exp_x(V_x t)$$

is a diffeomorphism onto its image. Via this identification, the vector fields $V, e_i, \eta_j$ extend along $V$. The extended vector fields will be denoted with the same letters. The vector fields $e_i$ and $V$ commute and $\nabla_V V = 0$. Let $f_t = f(t,\cdot)$ and $X_t$ be the image $f_t(X_0)$. On the manifold $X_t$, let $\chi|_{X_t} = \chi_t^j \otimes \eta_j$ where $\chi_t^j \in \Omega^2(X_t)$, then

$$F(tV) = \exp_{tV}^* \chi_t^j \otimes P_{tV}^{-1} \eta_j \quad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} F(tV) = \mathcal{L}_V \chi_t^j \otimes \eta_j + \chi_0^j \nabla_V \eta_j.$$

Throughout this calculation we sum over $j = 1, \ldots, 4$ whenever $j$ appears in an equation and suppress the notation for summation. The forms $\chi_0^j$ vanish since $X$ is $J$-holomorphic. Write $\chi_t^j = \chi_t^j(e_1, e_2)\mathrm{vol}_X$ and note that $\mathcal{L}_V(\mathrm{vol}_X) = 0$ since $[e_i, V] = 0$. Hence, $V$ is an infinitesimal deformation if

$$\mathcal{L}_V(\chi_t^j(e_1, e_2)) \otimes \eta_j = \mathcal{L}_V(\mathrm{Re}\,\psi(e_1, e_2, \eta_j)) \otimes \eta_j$$

vanishes. Since $\mathrm{Re}\,\psi(e_1, e_2, \eta_j)$ is just a function, we can compute its Lie derivative using a covariant derivative. It proves worthwhile to work with $\overline{\nabla}$ as this connection preserves $\psi$

$$\begin{aligned}
\mathcal{L}_V(\mathrm{Re}\,\psi(e_1, e_2, \eta_j)) \otimes \eta_j &= \overline{\nabla}_V(\mathrm{Re}\,\psi(e_1, e_2, \eta_j)) \otimes \eta_j \\
&= (\mathrm{Re}\,\psi(\overline{\nabla}_V e_1, e_2, \eta_j) + \mathrm{Re}\,\psi(e_1, \overline{\nabla}_V e_2, \eta_j)) \otimes \eta_j \\
&= (\mathrm{Re}\,\psi(\overline{\nabla}_{e_1} V, e_2, \eta_j) + \mathrm{Re}\,\psi(e_1, \overline{\nabla}_{e_2} V, \eta_j)) \otimes \eta_j - 2V \\
&= -e_2 \times \overline{\nabla}_{e_1} V + e_1 \times \overline{\nabla}_{e_2} V - 2V \\
&= J(e_1 \times \overline{\nabla}_{e_1} V + e_2 \times \overline{\nabla}_{e_2} V + 2JV).
\end{aligned}$$

In the second last step we have used equation 5.1.1 to infer the torsion-relation

$$\overline{\nabla}^{\perp}_{e_i}(V) = \overline{\nabla}^{\perp}_V(e_i) - J(e_i \times V).$$

□

Define $\bar{D} = e_1 \times \overline{\nabla}_{e_1} V + e_2 \times \overline{\nabla}_{e_2} V$ such that a section $V \in \Gamma(X, \nu)$ corresponds to an infinitesimal deformation if and only if

$$\bar{D}V = -2JV. \tag{5.1.2}$$

*Remark* 5.1.4. Up to composition with $J$ this operator agrees with the one studied by Lotay in [Lot11a] for $M = S^6$ to study asymptotically conical associative submanifolds of $\mathbb{R}^7$.

Since $\overline{\nabla}^{\perp}$ preserves $J$, it can be regarded as a connection on $\nu^{1,0}$. The induced $\bar{\partial}$-operator is the composition

$$\Gamma(\nu^{1,0}) \to \Gamma(\Lambda^1_{\mathbb{C}} X \otimes \nu^{1,0}) \to \Gamma(\Lambda^{0,1} X \otimes \nu^{1,0}).$$

On the other hand, $\bar{D}$ can also be extended complex linearly and is then a map $\bar{D}\colon \Gamma(X, \nu^{1,0}) \to \Gamma(X, \nu^{0,1})$ since $\bar{D}$ and $J$ anti-commute. But $\bar{D}$ is equal to $\alpha \circ \overline{\nabla}^{\perp}$, so lemma 5.1.2 implies the following.

**Lemma 5.1.5.** *Under the identification $\Lambda^{0,1} X \otimes \nu^{1,0} \cong \nu^{0,1}$ we have that $\bar{D} = \bar{\partial}_{\overline{\nabla}^{\perp}}$.*

Let $V$ be a solution to the equation $\bar{D}(V) = -2JV$. We can write $V = V^{1,0} + V^{0,1} = V^{1,0} + \overline{V^{1,0}}$. By comparing components in $\nu^{0,1}$ we get

$$\bar{\partial}_{\overline{\nabla}^{\perp}}(V^{1,0}) = -2i\overline{V^{1,0}}. \tag{5.1.3}$$

Besides the identification with the $\bar{\partial}$ operator, $\bar{D}$ can also be viewed as a twisted Dirac operator. Indeed, lemma 5.1.1 implies that $\nu_{\mathbb{C}}$ carries the structure of a complex Clifford module bundle over $X$. By [BGV03, Proposition 3.35], there is a $\mathbb{Z}/2\mathbb{Z}$ graded (complex) rank two bundle $W$ over $X$ such that $W \otimes \mathbb{S} = \nu_{\mathbb{C}}$ as $\mathbb{Z}/2\mathbb{Z}$-graded vector bundles. Here, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ denotes the spinor bundle over $X$. The splitting $\nu_{\mathbb{C}} = \nu^{1,0} \oplus \nu^{0,1}$ defines the $\mathbb{Z}/2\mathbb{Z}$ grading on $\nu_{\mathbb{C}}$. Since $\overline{\nabla}^{\perp}$ preserves the grading in $\nu_{\mathbb{C}}$ it is a Clifford superconnection in the sense of Quillen [Qui85]. By [BGV03, Proposition 3.40] this implies that there is a connection $A$ on $W$ preserving the grading such that

$$\bar{D} = \times \circ \overline{\nabla}^{\perp} = \mathcal{D}^A$$

where $\mathcal{D}^A$ is the Dirac operator twisted by $A$. With this in mind we compute the square of $\bar{D}$.

**Proposition 5.1.6.** *We have the following Weitzenböck-type formula*

$$\bar{D}^2 = \bar{\Delta} + J \star \bar{R}^{\perp}. \tag{5.1.4}$$

*Here, $\bar{R}^{\perp}$ is the curvature form of the connection $\overline{\nabla}^{\perp}$, viewed as a* $\mathrm{End}(\nu)$ *valued two-form on $X$ and $\bar{\Delta}$ is the rough Laplacian $\overline{\nabla}^*\overline{\nabla}$.*

*Proof.* To compute $\bar{D}^2$, we fix a point $x \in X$ and choose an orthonormal frame $\{e_1, e_2\}$ such that $\nabla_{e_j}(e_i) = 0$ in the point $x$. Since $\mathrm{Re}\,\psi$ is parallel with respect to $\overline{\nabla}$ and restricts to zero on $\nu$, we have

$$\overline{\nabla}^{\perp}_{e_k}(e_l \times \overline{\nabla}^{\perp}_{e_l} V) = e_l \times \overline{\nabla}^{\perp}_{e_k}\overline{\nabla}^{\perp}_{e_l} V.$$

Consequently,

$$\begin{aligned}
\bar{D}^2 V &= \sum_{k,l=1}^{2} e_k \times e_l \times (\overline{\nabla}^{\perp}_{e_k}\overline{\nabla}^{\perp}_{e_l} V) \\
&= e_1 \times (e_2 \times (\overline{\nabla}^{\perp}_{e_1}\overline{\nabla}^{\perp}_{e_2} V - \overline{\nabla}^{\perp}_{e_2}\overline{\nabla}^{\perp}_{e_1} V)) + \sum_k e_k \times (e_k \times \overline{\nabla}^{\perp}_{e_k}\overline{\nabla}^{\perp}_{e_k} V) \\
&= (J\bar{R}^{\perp}(e_1, e_2) + \overline{\Delta})V = (\bar{\Delta} + J \star \bar{R}^{\perp})V
\end{aligned}$$

□

Denote by $\nabla^t$ the connection $(1-t)\nabla + t\overline{\nabla}$. Define $D^t = e_1 \times \nabla^t_{e_1} + e_2 \times \nabla^t_{e_2} = (1-t)D^0 + t\bar{D}$ such that eq. (5.1.1) implies

$$D^t = \bar{D} + (1-t)J. \tag{5.1.5}$$

In particular, infinitesimal deformations of $X$ lie in the kernel of $D^{-1}$.

The following lemma shows that eq. (5.1.5) is the splitting of $D^t$ into self-adjoint and anti self-adjoint part, it is analogous to [Kaw17, Lemma 3.7.].

**Lemma 5.1.7.** *The operator $\bar{D}$ is elliptic. Let $V, V' \in \Gamma(X, \nu)$. There is a vector field $\xi$ on $X$ such that*

$$g(\bar{D}V, V') = g(V, \bar{D}V') + \mathrm{div}(\xi).$$

*In particular, if $X$ is compact, $\bar{D}$ is formally self-adjoint and* $\mathrm{ind}(\bar{D}) = 0$*.*

*Proof.* The ellipticity of $\bar{D}$ is a corollary of the Weitzenböck identity. Observe that $\xi = V \times V'$ defines a vector field on $X$ since $\nu \times \nu \subset TX$

$$\begin{aligned} g(\bar{D}V, V') &= g(e_i \times \overline{\nabla}_{e_i} V, V') = -g(\overline{\nabla}_{e_i} V, e_i \times V') \\ &= -e_i(g(V, e_i \times V')) + g(V, \nabla_{e_i} e_i \times V') + g(V, \bar{D}V') \\ &= \mathrm{div}(\xi) + g(V, \bar{D}V'). \end{aligned}$$

Here, we have used $\overline{\nabla}_{e_i} e_i = \nabla_{e_i} e_i$ which follows from eq. (5.1.1). By standard functional analysis arguments the closure of $\bar{D}$ in $L^2(X, \nu)$ is self-adjoint, see [LM16, p. 116] and hence $\mathrm{ind}(\bar{D}) = 0$. ☐

Eq. 5.1.2 is not formally an eigensection equation for the real operator $\bar{D}$ since $J$ appears on the right-hand side. However, observe that if $V$ is a solution to eq. (5.1.2) then $V - JV$ satisfies

$$\bar{D}(V - JV) = \bar{D}V + J\bar{D}V = -2JV + 2V = 2(V - JV).$$

Since $V \mapsto (V - JV)$ is an isomorphism from $\Gamma(X, \nu)$ to itself we from now on work with the equation

$$\bar{D}V = 2V \tag{5.1.6}$$

whose solutions correspond to infinitesimal deformations of $X$. We want to show that squaring this equation is a way to complexify the solution space. To that end, we introduce symbols for the solution spaces.

**Definition 5.1.8.** Denote by $R_\mu$ the $\mu$-eigenspace of $\bar{D}$ and $T_\mu$ be the $\mu$-eigenspace of $\bar{D}^2$.

**Lemma 5.1.9.** *For any $\mu \in \mathbb{R}$ we have*

$$R_\mu \oplus R_{-\mu} = T_{\mu^2}.$$

*Proof.* Since $X$ is compact and $\bar{D}$ is self-adjoint and elliptic there is an orthonormal basis of $L^2(X, \nu)$ consisting of eigensections $(V_n)_{n\in\mathbb{Z}}$, i.e. $\bar{D}V_n = \lambda_n V_n$ [LM16, Theorem 5.8]. Let $V = \sum a_n V_n$ then

$$V \in T_{\mu^2} \quad \Leftrightarrow \quad a_n = 0 \text{ if } \lambda_n^2 \neq \mu^2 \quad \Leftrightarrow \quad V \in R_\mu \oplus R_{-\mu},$$

as required. $\square$

We can compare the infinitesimal deformations of $J$-holomorphic curves in nearly Kähler manifolds with the complex situation. The space of infinitesimal deformations of a complex submanifold is isomorphic to $H^0(X, \nu)$ [Kod62]. Even though a $\bar{\partial}$ operator makes an appearance here as well, solutions to eq. (5.1.3) do not admit an obvious multiplication by complex numbers. In fact, lemma 5.1.9 suggests that $T_4$ can be considered the complexification of $R_2$. It remains an open problem whether the space of infinitesimal deformations $R_2$ or the space $T_4$ can be related to a cohomology group of $X$.

## 5.2 Deformations of Homogeneous Tori

Even though eq. (5.1.6) is linear, it is usually very challenging to compute the (dimension of) the full infinitesimal deformation space. Since the index of $\bar{D}$ is zero, $J$ holomorphic curves in nearly Kähler manifolds are generically rigid. However, at least for the homogeneous nearly Kähler examples, this statement is not very useful. They all have large symmetry groups and one cannot perturb these structures to other nearly Kähler structures [MS10; Fos17].

However, we will see that if $X$ is a $J$-holomorphic homogeneous torus then the deformation problem becomes tractable if one has sufficient knowledge of $\overline{\nabla}^\perp$. We first investigate the space of deformations coming from isometries.

**Definition 5.2.1.** Let $M$ be a nearly Kähler manifold and $X \subset M$ be an embedded $J$-holomorphic curve in $M$. Denote by $G_M$ the nearly Kähler automorphism group of $M$ and by $G_X$ the subset of elements in $G_M$ acting on $X$, i.e.

$$G_X = \{g \in G_M \mid gX \subset X\}.$$

Denote by $\mathfrak{g}_X$ and $\mathfrak{g}_M$ the Lie algebras of $G_M$ and $G_X$.

Every element in $\mathfrak{g}_M$ gives rise to a one-parameter family of deformations of $X$ and in particular an infinitesimal deformation. However, if $X$ admits smooth symmetries, i.e. if the group $G_X$ has positive dimension then elements in $\mathfrak{g}_X$ result in trivial (infinitesimal) deformations of $X$. In general, the automorphism group gives rise to a $\dim(G_M) - \dim(G_X)$ dimensional space of (infinitesimal) deformations of $X$. In particular, if $\dim(G) - \dim(G_X) = \dim(R_2)$ then $X$ is rigid up to automorphisms.

The aim of this section is to show that homogenous tori in $\mathbb{CP}^3$ and $S^6$ are rigid up to isometries. For both of these spaces, the automorphism group has rank two. We will first show that this implies $\dim(G_X) = 2$. We first state a well-known auxiliary lemma.

**Lemma 5.2.2.** *A semi-simple non-compact Lie group $K$ is never a subgroup of a compact Lie group $G$.*

*Proof.* Assume $K \subset G$. Since $G$ is compact it has a faithful finite-dimensional unitary representation $\rho$. By restriction, we obtain such a representation of $K$. It is known that any finite-dimensional representation of a semi-simple Lie group has closed image in $\mathrm{Gl}_n(\mathbb{R})$. Hence, $\rho\colon K \to \mathrm{U}(n)$ has closed image in $\mathrm{U}(n)$ which is a contradiction since $\rho$ is a homeomorphism onto its image but $K$ is non-compact. $\square$

The automorphism groups of the nearly Kähler $S^6$ and $\mathbb{CP}^3$ both have rank two. So the homogeneous $J$-holomorphic tori are orbits of the maximal torus $K$. The group of automorphisms preserving this orbit is a Lie group $G \supset K$. The following proposition will be used to show that $G$ is just $K$.

**Proposition 5.2.3.** *Let $G$ be a connected compact Lie group with maximal torus $K$. Assume that $G$ acts on a two-torus $\mathbb{T}^2$ such that $K \subset G$ acts freely on $\mathbb{T}^2$, then $G = K$.*

*Proof.* Assume that $G \neq K$, then $G$ contains a subgroup $H$ with Lie-algebra $\mathfrak{su}(2)$ and $H \cap K$ is non-trivial. By lemma 5.2.2, $H$ is compact and must be covered by $\mathrm{SU}(2)$. This gives an action of $\mathrm{SU}(2)$ on $\mathbb{T}^2$, which is non-trivial since $H \cap K$ is non-trivial. Consider a point $x \in \mathbb{T}^2$ with non-trivial stabiliser $R$ and let $R_0$ be its identity component. Then $R$ is a one-dimensional subgroup of $\mathrm{SU}(2)$ which means that $\mathbb{T}^2 \cong \mathrm{SU}(2)/R$ by invariance of domain. This yields a contradiction since $\mathrm{SU}(2)/R$ is covered by $\mathrm{SU}(2)/R_0$ which is diffeomorphic to a two-sphere. $\square$

**Corollary 5.2.4.** *Let $G$ be a compact Lie group of rank two with maximal torus $K$. Let $G$ act on a space $M$ and let $x \in M$ such that $\mathrm{Stab}(x) \cap K$ is trivial and denote by $\mathcal{O}_x$ the orbit of $x$ under $K$. Then the identity component of $G_K = \{g \in G \mid g\mathcal{O}_x \subset \mathcal{O}_x\}$ is equal to $K$.*

*Proof.* The group $G_K$ is clearly closed and hence compact. Furthermore, $K$ is the maximal torus of $G_K$. The statement follows from the proposition above since $K$ acts freely on $K_x$, which is a two-torus. $\square$

Finally, we can compute $\dim(G_X)$ for homogeneous tori. Note that the corollary does not only apply to $\mathbb{CP}^3$ and $S^6$ but also to the flag manifold.

**Corollary 5.2.5.** *If $X$ is a homogenous torus in $M$ and $G_M$ has rank two then the previous corollary implies that* $\dim(G_X) = 2$.

If $X$ is any $J$-holomorphic torus in a nearly Kähler manifold then there is a global orthonormal frame $\{e_1, e_2 = Je_1\}$ on $X$ and $I_i = e_i \times \cdot$ are almost complex structures on the normal bundle. Together with $J$ they satisfy the quaternionic relationships, so $\nu$ carries a $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ structure. If $X$ is flat then $e_1$ and $e_2$ can be chosen to be parallel which makes $I_1$ and $I_2$ parallel almost complex structures and $\overline{\nabla}^{\perp}$ has holonomy contained in $\mathrm{SU}(2)$, which is consistent with $\Lambda^2(\nu) \cong TX$.

The following lemma shows that two eigensections with particular properties make it possible to compute the full spectrum of $\bar{D}$ if $X$ is a torus.

**Lemma 5.2.6.** *Assume there are two sections $V_1$ and $V_2$ such that*

- *their complex span is $\nu$ in every point*
- *they are eigensections of $\bar{D}$ with eigenvalues $\lambda_1$ and $\lambda_2$*
- *$e_1 = V_1 \times V_2$ and $e_2 = Je_1$ commute, have constant norm and periodic orbits in $X$ with periods $T_1$ and $T_2$.*

*Then the complexification of $R_\mu$ is isomorphic to $\bigoplus_{k_1 k_2 \in \mathbb{Z}} \ker A_{k_1 k_2}$ where*

$$A_{k_1k_2} = \begin{pmatrix} \lambda_1 - \mu & 0 & -2\pi i k_2/T_2 & -2\pi i k_1/T_1 \\ 0 & -\lambda_1 - \mu & -2\pi i k_1/T_1 & 2\pi i k_2/T_2 \\ 2\pi i k_2/T_2 & 2\pi i k_1/T_1 & \lambda_2 - \mu & 0 \\ 2\pi i k_1/T_1 & -2\pi i k_2/T_2 & 0 & -\lambda_2 - \mu \end{pmatrix}.$$

*Proof.* We can assume that $V_1$ and $V_2$ both have constant norm equal to one. So $\{e_1, e_2\}$ is a global orthonormal frame on $X$. By assumption, the flows of $e_1$ and $e_2$ give rise to an isometry between $X$ and $\mathbb{R}^2/\Lambda$ where $\Lambda$ is the lattice generated by $(T_1, 0)$ and $(0, T_2)$. We will use coordinates $(x_1, x_2)$ for $X \cong \mathbb{R}^2/\Lambda$ such that $e_i = \partial/\partial x_i$. Observe that for any $V \in \Gamma(X, \nu)$ and $a \in C^\infty(X, \mathbb{R})$ we have

$$\bar{D}(aV) = \sum e_i \times e_i(a)V + a\bar{D}V. \tag{5.2.1}$$

The vectors $\{W_1, W_2 = JW_1, W_3 = V_2, W_4 = JW_3\}$ are a global $\mathrm{Sp}(1) = \mathrm{SU}(2)$-frame and eq. (5.2.1) gives that

$$\bar{D} = \begin{pmatrix} \lambda_1 & 0 & -e_2 & -e_1 \\ 0 & -\lambda_1 & -e_1 & e_2 \\ e_2 & e_1 & \lambda_2 & 0 \\ e_1 & -e_2 & 0 & -\lambda_2 \end{pmatrix} \tag{5.2.2}$$

with respect to the frame $\{W_1, \ldots, W_4\}$.

This equation can also be derived from the fact that $\beta \circ \bar{\partial} = \bar{D}$ on $V^{1,0}$. Any section in $\nu \otimes \mathbb{C}$ can be written as $\sum_i a_i W_i$ with $a_i$ complex-valued functions on $X$. Write $a_i$ as a Fourier-series, i.e.

$$a_i = \sum_{k_1, k_2 \in \mathbb{Z}} a_{ik_1k_2} f_{k_1k_2}, \text{ for } f_{k_1k_2} = \exp(2\pi i(x_1/T_1 k_1 + x_2/T_2 k_2)).$$

Then $\bar{D}V = \mu V$ if and only if $(a_{1k_1k_2}, a_{2k_1k_2}, a_{3k_1k_2}, a_{4k_1k_2})$ lies in the kernel of $A_{k_1k_2}$ for every $k_1, k_2 \in \mathbb{Z}$. $\square$

The conditions on $V_1$ and $V_2$ are rather restrictive, in particular $X$ must be flat. An important case is if $X$ is given as the orbit of a torus action on $M$. Then the normal bundle has a torus action which covers the action on $X$. In particular, there is a notion of $\mathbb{T}^2$ invariant sections which we will denote by $\Gamma(X, \nu)^{\mathbb{T}^2}$. The space $\Gamma(X, \nu)^{\mathbb{T}^2}$ is a real four-dimensional vector space equipped with an inner product.

The operator $\bar{D}$ restricts to a self-adjoint endomorphism on $\Gamma(X, \nu)^{\mathbb{T}^2}$. Hence, $\Gamma(X, \nu)^{\mathbb{T}^2}$ has a basis of eigensections of $\bar{D}$. Because $J$ anticommutes with $\bar{D}$, there is a basis of eigenvectors $\Gamma(X, \nu)^{\mathbb{T}^2}$ of the form $(V_1, JV_1, V_2, JV_2)$ with real eigenvalues $(\lambda_1, -\lambda_1, \lambda_2, -\lambda_2)$. The assumptions of lemma 5.2.6 are satisfied if $V_1 \times V_2$ and $J(V_1 \times V_2)$ have periodic orbits in $X$. We will call $(\lambda_1, \lambda_2)$ the invariant spectrum of $X$.

In order to use lemma 5.2.6 in a specific situation one has to determine $T_1, T_2$ and the invariant spectrum of $X$. We will do this for homogeneous tori in $S^6$ and in $\mathbb{CP}^3$. In $S^6$, the computation is rather ad-hoc. We identify the homogeneous tori as extrema of the multi-moment map and use explicit coordinates to compute the periods $T_i$ and the invariant spectrum.

In $\mathbb{CP}^3$, we benefit from results of chapter 4, as we have already identified homogeneous tori in $\mathbb{CP}^3$ as twistor lifts of Clifford tori which have constant angle functions. The frame adaption for transverse $J$-holomorphic curves is suitable to compute the invariant spectrum.

Recall definition 5.2.1 and definition 5.1.8. We will show that all infinitesimal deformations come from automorphisms, which amounts to verifying the condition

$$\dim(G_M) - \dim(G_X) = \dim(R_2).$$

For $S^6$ we have $\dim(G_M) = \dim(G_2) = 14$ and for $\mathbb{CP}^3$ it is $\dim(G_M) = \dim(\mathrm{Sp}(2)) = 10$. As shown in corollary 5.2.5, $\dim(G_X) = 2$ holds in both cases. ´

### 5.2.1 The Invariant Spectrum in $S^6$

We identify the homogeneous $J$-holomorphic tori in $S^6$ as extrema of the multi-moment map for the two-torus action. This has been computed by Russo in [Rus20], so we follow the conventions of this paper when defining the nearly Kähler structure and two-torus action on $S^6$.

Let $G_2$ be the stabiliser of

$$\varphi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

in $\mathrm{SO}(7)$. By choosing a two-torus in $G_2$ we obtain a torus action on $S^6$. Since $G_2$ has rank two all such tori will be conjugated so it suffices to consider one particular choice of torus. Consider the splitting $\mathbb{R}^7$ as $\mathbb{C}^3 \oplus \mathbb{R}$. The stabiliser of $(0, \ldots, 1)$ in $G_2$ is isomorphic so $\mathrm{SU}(3)$ and $S^6 \cong G_2/\mathrm{SU}(3)$. By considering the subgroup of diagonal elements in $\mathrm{SU}(3)$ we obtain $\mathbb{T}^2 \subset \mathrm{SU}(3) \subset G_2$. More explicitly, $(e^{i\vartheta_1}, e^{i\vartheta_2})$ acts as $A_{\vartheta_1,\vartheta_2} = \mathrm{diag}(e^{i\vartheta_1}, e^{i\vartheta_2}, e^{-i(\vartheta_1+\vartheta_2)})$ on $\mathbb{C}^3$ and trivially on the $\mathbb{R}$ component.

**Proposition 5.2.7.** *[Rus20, Proposition 3.2.] The multi-moment map on $S^6$ is given by*

$$\nu_{S^6}(x) = 3\mathrm{Re}(z_1 z_2 z_3).$$

*The set $\{x \in S^6 \mid \nu(x) \neq 0, \quad \mathrm{d}\nu(x) = 0\}$ is equal to the union of the $\mathbb{T}^2$ orbits of the points $p = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}, 0)$ and $-p$. The map $\nu$ attains its maximum at $p$ and its minimum at $-p$.*

By lemma 2.4.1, the orbits of $p$ and $-p$ are $J$-holomorphic curves. We can explain the symmetry of $\nu$ by the existence of the antipodal map $j$ on $S^6$. This map does not lie in $G_2$ but preserves the metric anyway. It also commutes with the torus action but flips the orientation and satisfies $j^*\omega = -\omega$. In particular, $\nu \circ j = -\nu$. This also means that it suffices to consider the orbit of the point $p$ which we will denote by $X$.

We will now compute the operator $\bar{D}$ on $\Gamma(X, \nu)^{\mathbb{T}^2}$. Because of the chosen $\varphi_0$-convention identify $\mathbb{C}^3$ with $\mathbb{R}^6$ by $z_1 = x_1 + ix_6, z_2 = x_5 + ix_2, z_3 = x_4 + ix_3$. Let

$$\xi_i = \frac{\mathrm{d}}{\mathrm{d}\vartheta_i}|_{\vartheta_1=0,\vartheta_2=0} A_{\vartheta_1,\vartheta_2} \in \mathfrak{su}(3) \subset \mathfrak{g}_2$$

and reagrd tangent vectors in $S^6$ as elements in $\mathbb{R}^7$ so they can be acted upon with matrices. At $p$, the Killing vector fields corresponding to the $\mathbb{T}^2$-action are given by matrix vector products as

$$\begin{aligned}K^{\xi_1} &= \xi_1 p = (-x_6, 0, -x_4, x_3, 0, x_1, 0)^T \\ K^{\xi_2} &= \xi_2 p = (0, x_5, -x_4, x_3, -x_2, 0, 0)^T.\end{aligned}$$

On $X$, define furthermore

$$e_1 = K^{\xi_2}\sqrt{3/2}, \quad e_2 = (K^{\xi_1} - \frac{1}{2}K^{\xi_2})\sqrt{2} \tag{5.2.3}$$

Then $Je_1 = e_2$ and $\{e_1, e_2\}$ defines an orthonormal frame of $TX$. The periods of the flows of $K^{\xi_1}$ and $K^{\xi_2}$ are both equal to $2\pi$. Eq. 5.2.3 shows that the flows of $e_1$ and $e_2$ also have periodic orbits and we get $T_1 = 2\pi\sqrt{2/3}$ and $T_2 = 2\pi\sqrt{2}$.

It remains to compute the invariant spectrum before we can apply lemma 5.2.6. Consider the vectors

$$\begin{aligned}n_1 &= (0,0,0,0,0,0,1)^T, \quad n_2 = -\frac{1}{\sqrt{3}}(0,1,1,0,0,1,0)^T \\ n_3 &= \frac{1}{\sqrt{2}}(1,0,0,0,-1,0,0)^T, \quad n_4 = \frac{1}{\sqrt{6}}(-1,0,0,2,-1,0,0)^T.\end{aligned}$$

They form a special unitary frame at $p$ and we can make them a global SU(2)-frame by the $\mathbb{T}^2$ action such that they are a basis of $\Gamma(X, \nu)^{\mathbb{T}^2}$.

Let

$$E_1 = \xi_2\sqrt{3/2}, \quad E_2 = (\xi_1 - \frac{1}{2}\xi_2)\sqrt{2} \tag{5.2.4}$$

be the matrices in $\mathfrak{g}_2$ such that $E_i p = e_i$ and denote by $\pi_{TS^6}$ the projection of $S^6 \times \mathbb{R}^7$ to $TS^6$. Observe that if $V$ is in $\Gamma(X, TS^6|_X)^{\mathbb{T}^2}$ the covariant derivative can be computed by a matrix multiplication

$$\nabla_{e_i} V|_p = \pi_{TS^6}(\mathrm{d}V(e_i)|_p) = \pi_{TS^6}(\mathrm{d}V(\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\exp(tE_i)p)) = \pi_{TS^6}(E_i V|_p) = E_i V|_p.$$

The nearly Kähler connection $\overline{\nabla}$ can be obtained from the Levi-Civita connection $\nabla$ via eq. (5.1.1)

$$\overline{\nabla}^{\perp}_{e_i}(n_j) = \nabla^{\perp}_{e_i}(n_j) - \frac{1}{2}J(e_i \times n_j) = (E_i n_j)^{\perp} - \frac{1}{2}J(e_i \times n_j). \tag{5.2.5}$$

Let $B_i = g(\overline{\nabla}^{\perp}_{e_i}(n_j), n_k))_{jk}$ which is the matrix representing $\overline{\nabla}^{\perp}_{e_i}$ in the basis $(n_1, \ldots, n_4)$.

Combining eq. (5.2.4) with eq. (5.2.5) yields

$$\begin{aligned}\overline{\nabla}^{\perp}_{e_1} &= B_1 = \frac{1}{4}\begin{pmatrix} 0 & -\sqrt{3}+i \\ \sqrt{3}+i & 0 \end{pmatrix} \\ \overline{\nabla}^{\perp}_{e_2} &= B_2 = \frac{1}{4}\begin{pmatrix} 0 & -1-\sqrt{3}i \\ 1-\sqrt{3}i & 0 \end{pmatrix},\end{aligned} \tag{5.2.6}$$

where we have used the standard embedding $\mathfrak{u}(2) \subset \mathfrak{so}(4)$. The curvature of $\overline{\nabla}^{\perp}$ also acts on $\Gamma(X,\nu)^{\mathbb{T}^2}$ and is expressed in the frame $\{n_1,\dots,n_4\}$ as

$$\bar{R}^{\perp}(e_1,e_2) = B_1B_2 - B_2B_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Finally, the operator $\bar{D}$ is not complex linear so it is represented as a real $4\times 4$ matrix as follows

$$\bar{D} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.2.7}$$

In particular, the eigenvalues of $\bar{D}$ on $\Gamma(X,\nu)^{\mathbb{T}^2}$ are $(1,-1,0,0)$. The sections $n_3$ and $n_4$ are holomorphic and span a holomorphically trivial line bundle $L \subset \nu$. The torsion, in the sense of [Bry82b], of $X$ does not vanish since the matrices $B_1$ and $B_2$ have off-diagonal entries.

From eq. (5.2.6) we can compute the second fundamental form $\mathrm{II}_{S^6}$ of $X$ in $S^6$. Since $\mathrm{II}_{S^6}$ is complex-linear and $\mathbb{T}^2$ equivariant, it suffices to compute

$$\mathrm{II}_{S^6}(e_1,e_1) = (E_1^2 p)^{\perp} = (0,0,-\sqrt{\frac{3}{8}},-\sqrt{\frac{1}{8}})^T$$

in the basis $\{n_1,\dots,n_4\}$. In particular, the values of $\mathrm{II}_{S^6}$ have constant coefficients in $n_3$ and $n_4$, which means that $\mathrm{II}_{S^6}$ is holomorphic. In fact, for the ambient space $S^6$ a $J$-holomorphic curve always has holomorphic second fundamental form.

We are now in the position to use lemma 5.2.6 and determine the spectrum of $\bar{D}$. Let $V_1 = n_1$ and $V_2 = n_3$ such that $e_1 = V_1 \times V_2$. We can apply 5.2.6 with $\lambda_1 = 1, \lambda_2 = 0$ and compute

$$\det(A_{k_1k_2}) = \frac{1}{4}(k_1^2 + 3k_2^2 - 2\mu - 2\mu^2)(k_1^2 + 3k_2^2 + 2\mu - 2\mu^2).$$

The presence of the two factors corresponds to the fact that the spectrum is symmetric around 0. We see that $\mu$ lies in the spectrum of $\bar{D}$ if and only if there are $k_1, k_2 \in \mathbb{Z}$ such that

$$3k_1^2 + k_2^2 = 2(\mu^2 \pm \mu). \tag{5.2.8}$$

The space of infinitesimal deformations of $X$ is the eigenspace of $\mu = 2$. Observe that

$$\{(k_1, k_2) \in \mathbb{Z}^2 \mid 3k_1^2 + k_2^2 \in \{4, 12\}\} = \{(\pm 1, \pm 3), (\pm 1, \pm 1), (0, \pm 2), (\pm 2, 0)\}.$$

One can check that for any such choice of $(k_1, k_2)$ the kernel of $A_{k_1 k_2}$ is complex one-dimensional. This shows that the space of infinitesimal deformations of $X$ is 12-dimensional, which equals $\dim(G_2) - \dim(G_X) = 14 - 2$ from definition 5.2.1.

**Proposition 5.2.8.** *Homogeneous $J$-holomorphic tori in $S^6$ are rigid up to isometries.*

### 5.2.2 The Invariant Spectrum in $\mathbb{CP}^3$

In $\mathbb{CP}^3$, we have seen that homogeneous $J$-holomorphic tori are exactly twistor lifts of Clifford tori. This torus is unique up to an action of isometries, is characterised by having constant angle functions $\alpha_\pm = \frac{1}{\sqrt{2}}$ and in particular is a transverse $J$-holomorphic curve. Recall from section 4.3 that the bundle $\mathrm{Sp}(2)|_X$ restricts to a $\mathbb{Z}_8$ bundle over $X$. This bundle is characterised by the following equations

$$\omega_2 = 0, \quad \omega_3 = \tau = \frac{1}{\sqrt{2}}\omega_1, \quad \rho_1 = \rho_2 = 0,$$

see lemma 4.3.6. This means that the Maurer-Cartan form restricted to $R_X$ is

$$\frac{1}{\sqrt{2}} \begin{pmatrix} j\bar{\omega_1} & -\bar{\omega_1} \\ \omega_1 & j\omega_1 \end{pmatrix}. \tag{5.2.9}$$

Recall from section 4.3 that the bundle reduction $R_X$ gives an SU(3)-frame $f_1, f_2, f_3$ of $T\mathbb{CP}^3|_X$. Then $f_1$ is a complex valued vector field on $X$ of norm $\sqrt{1 + \alpha_-^2} = \sqrt{\frac{3}{2}}$, so we define $f_1 = \sqrt{\frac{2}{3}}(e_1 + ie_2)$. The vector fields $e_1$ and $e_2$ are an orthonormal frame on $X$. By acting with isometries we can assume that $R_X$ contains the identity element in $\mathrm{Sp}(2)$ such that by eq. (5.2.9) the vector fields are Killing vector fields $e_i = K^{\xi_i}$ with

$$\xi_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} j & -1 \\ 1 & j \end{pmatrix}, \quad \xi_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -ji & +i \\ i & ji \end{pmatrix}.$$

By computing matrix exponentials in $\mathfrak{sp}(2) \subset \mathfrak{su}(4)$ we see that the curves $\gamma_i \colon \mathbb{R} \to \mathbb{CP}^3, \quad t \mapsto \exp(t\xi_i)[1,0,0,0]$ for $i = 1,2$ are

$$\gamma_1(t) = [1+\cos(2t'), \sin(2t'), \sin(2t'), 1-\cos(2t')],$$
$$\gamma_2(t) = [\cos(t')^2, -i\cos(t')\sin(t'), i\cos(t')\sin(t'), -\sin^2(t')]$$

for $t' = \frac{1}{\sqrt{3}}t$, so the periods are $T_1 = T_2 = \sqrt{3}\pi$.

It remains to compute the invariant spectrum. In the U(2)-frame $f_2, f_3$ of $\nu$ we have by eq. (4.4.3)

$$\overline{\nabla}_{e_1} = \frac{1}{3}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \overline{\nabla}_{e_2} = \frac{1}{3}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

From this we can compute $\bar{D}$, since $\{f_1, f_2, f_3\}$ is an SU(3)-frame,

$$\bar{D} = e_1 \times \overline{\nabla}_{e_1} + e_2 \times \overline{\nabla}_{e_2} = \operatorname{diag}(0,0,2/3,-2/3).$$

This means that $\lambda_1 = 0$ and $\lambda_2 = 2/3$. So $f_2$ is a $\mathbb{T}^2$ invariant holomorphic section equal to $f_2$, which is consistent with lemma 4.4.6. Note that the values of the second fundamental form of $X$ are proportional to $f_3$ by 4.4.1. We have computed the invariant spectrum $\lambda_1, \lambda_2$ and the periods $T_1, T_2$ so we are now in the position to compute the full spectrum from lemma 5.2.6.

The determinant of $A_{k_1k_2}$ for $\lambda_1 = 0, \lambda = 2/3, T_1 = T_2 = \sqrt{3}\pi$ is equal to

$$\frac{1}{9}(4k_1^2 + 4k_2^2 - 2\mu - 3\mu^2)(4k_1^2 + 4k_2^2 + 2\mu - 3\mu^2).$$

For $\mu = 2$ it vanishes for the integer pairs $(k_1, k_2) = (\pm 2, 0), (0, \pm 2), (\pm 1, \pm 1)$. In each case, the kernel of $A_{k_1k_2}$ is one-dimensional, which means that the $\mu = 2$ eigenspace of $\bar{D}$ has real-dimension 8. This is equal to $\dim(\mathrm{Sp}(2)) - \dim(G_X) = 10 - 2$ from definition 5.2.1.

**Proposition 5.2.9.** *The Clifford Torus is rigid as a $J$-holomorphic curve in $\mathbb{CP}^3$.*

We can summarise our results in the following table.

| $M$ | $(T_1/2\pi, T_2/2\pi)$ | Invariant spectrum | Full spectrum | $\dim(T_2)$ | $\dim(G_M)$ |
|---|---|---|---|---|---|
| $\mathbb{CP}^3$ | $(\sqrt{3/8}, \sqrt{3/8})$ | $(2/3, -2/3, 0, 0)$ | $8k_1^2 + 8k_2^2 = 3\mu^2 \pm 2\mu$ | 8 | 10 |
| $S^6$ | $(\sqrt{2/3}, \sqrt{2})$ | $(1, -1, 0, 0)$ | $3k_1^2 + k_2^2 = 2(\mu^2 \pm \mu)$ | 12 | 14 |

In each case the dimension of the space of infinitesimal deformations equals $\dim(G_M) - \dim(G_X) = \dim(G_M) - 2$. This number is equal to the dimension space of infinitesimal deformations coming from automorphisms.

**Theorem 5.2.10.** *Homogeneous $J$-holomorphic tori in $S^6$ and $\mathbb{CP}^3$ are rigid up to the action of automorphisms.*

The integer pairs $(k_1, k_2)$ giving rise to infinitesimal deformation in $\mathbb{CP}^3$ span the root system of $\mathfrak{sp}(2)$. However, this seems to be a coincidence, as for $S^6$ they are not a root system at all. The normal bundle of the homogeneous tori in both $S^6$ and in $\mathbb{CP}^3$ has a holomorphic section. In $S^6$ this is the second fundamental form while in $\mathbb{CP}^3$ the holomorphic section is always orthogonal to the second fundamental form.

# Chapter 6

# Special Lagrangian Submanifolds in Nearly Kähler $\mathbb{CP}^3$

One peculiarity of Lagrangian submanifolds of nearly Kähler manifolds is that they are automatically special Lagrangian for the three-form $\operatorname{Re}\psi$. Because of their simple definition they are natural objects to study in nearly Kähler geometry.

Just as for $J$-holomorphic curves in nearly Kähler manifolds there are two additional lines of motivation to study Lagrangian submanifolds. The first one comes from Riemannian geometry, for any special Lagrangian in a nearly Kähler manifold is minimal. The second comes from special holonomy, for the cone of a special Lagrangian is coassociative in the $G_2$-cone of $M$.

In the last few decades, many constructions for special Lagrangian submanifolds of $S^6$ have been found and various subclasses of special Lagrangians have been classified, see for example [Vra03; Lot11b]. More recently, the ambient spaces $\mathbb{F}$ and $S^3 \times S^3$ have received attention, for example in [Bek+19; Sto20b]. This chapter is dedicated to the ambient nearly Kähler space $M = \mathbb{CP}^3$.

In section 6.2 we introduce an angle function $\theta\colon L \to [0, \frac{\pi}{4}]$ which describes the Lagrangian on a tangent level. Generically, $L$ intersects every twistor fibre transversally. The points where $\theta = \frac{\pi}{4}$ are those where this is not the case and the intersection is then diffeomorphic to $S^1$. We identify Lagrangians with $\theta \equiv \frac{\pi}{4}$ as circle bundles over superminimal surfaces in $S^4$, a construction discovered in [Sto20a] and in [Kon17]. We classify all Lagrangians where $\theta$ takes the boundary value 0. In fact, there are just two such examples and they are both homogeneous. We describe another somewhat surprising homogeneous example with $\theta \equiv \frac{1}{2}\arccos(\frac{7\sqrt{2}}{5\sqrt{5}})$ arising from the irreducible representation of $\mathrm{SU}(2)$ on $S^3(\mathbb{C}^2)$. We also show that the standard $\mathbb{RP}^3$ in $\mathbb{CP}^3$ is the only totally geodesic Lagrangian submanifold of $\mathbb{CP}^3$.

In section 6.3 we introduce $\mathrm{SU}(2)$ moment maps in nearly Kähler geometry. They encode the symmetry of the nearly Kähler manifold in a set of $\mathrm{SU}(2)$ equivariant functions $M \to \mathbb{R}^3 \oplus \mathbb{R}$. We use these moment maps to show a general existence result of special Lagrangians with SU(2) symmetry in corollary 6.3.3 and to classify special Lagrangians admitting an action of a SU(2) group of automorphisms in theorem 6.3.11. We show that they are in fact all homogeneous and describe the examples found in section 6.2 from the moment-map perspective.

Section 6.4 consists of two more ansatzes for constructing special Lagrangians. The first ansatz is to impose a $\mathbb{T}^2$-symmetry instead of SU(2)-symmetry. We show that the zero locus of the torus multi-moment map $\nu$ has an integrable special Lagrangian distribution. Integral submanifolds of this distribution correspond to Reeb orbits on a non-compact contact three-manifold. Certain elements in the Weyl group of the automorphism group of $M$ give rise to closed Reeb orbits. This is made explicit for the ambient spaces $M = S^6$ and $M = \mathbb{CP}^3$. The second ansatz is to construct Lagrangians from twistor lifts of surfaces in $S^4$. We give an example of a singular surface in $\mathbb{CP}^3$ which can be locally thickened to a special Lagrangian. Lastly, we discuss U(2) moment maps. Remarkably, there is a distinguished frame in this setting and we derive formulae of the moment maps with respect to this frame.

## 6.1 Background

Let $M$ be a strictly nearly Kähler six manifold. A three-dimensional submanifold $L$ of a nearly Kähler manifold is called Lagrangian if $\omega|_L = 0$. Just as in the case for $J$-holomorphic curves, we could work with the more general definition of a smooth map $\iota\colon L \to M$ such that $\iota^*\omega = 0$. However, the singularities of such a map are not nearly as well understood. So it is usually more sensible to work with special Lagrangian submanifolds. However, in a few cases we will encounter submanifolds with isolated singularities.

Because of the nearly Kähler identity $\mathrm{d}\omega = 3\,\mathrm{Re}\,\psi$, Lagrangian submanifolds are automatically special Lagrangian. Special Lagrangians in nearly Kähler geometry share some important general properties with special Lagrangians in Calabi-Yau manifolds. We revise these results in this subsection.

Every special Lagrangian in $M$ is minimal and orientable, see for example [VS+19]. Even more can be said about the second fundamental form. The following proposition is well known for $M = S^6$ and we prove it for a general nearly Kähler manifold.

**Proposition 6.1.1.** *Let $L$ be a special Lagrangian submanifold of a nearly Kähler manifold $M$ with second fundamental form $\mathrm{II}_M$. Then the cubic form*

$$C(X, Y, Z) = \omega(\mathrm{II}_M(X, Y), Z)$$

*is fully-symmetric, i.e. an element of $\Gamma(S^3(T^*L))$. It is also traceless when contracted* *in any two components.*

*Proof.* We have the equation

$$g(\overline{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}\operatorname{Re}\psi(X, Y, JZ).$$

Since $\operatorname{Re}\psi$ vanishes on $L$ this means that if $X, Y, Z$ are tangent to $L$ then

$$C(X, Y, Z) = \omega(\bar{\mathrm{I\!I}}_M(X, Y), Z)$$

where $\bar{\mathrm{I\!I}}_M$ is the second fundamental form of $L$ w.r.t. $\overline{\nabla}$. The connection $\overline{\nabla}$ preserves $\omega$ which results in

$$\omega(\bar{\mathrm{I\!I}}(X, Y), Z) = \omega(\overline{\nabla}^{\perp}_X Y, Z) = \omega(\overline{\nabla}_X Y, Z) = -\omega(Y, \overline{\nabla}_X Z) \qquad (6.1.1)$$

which proves that $C$ is fully symmetric. The fact that $C$ is also traceless follows from the fact that a special Lagrangian in $M$ is always minimal. □

The cubic form $C$ is also called the fundamental cubic of $L$. This result is remarkable because the second fundamental form is an extrinsic quantity but $C$ takes values in the intrinsic bundle $\Gamma(S^3(T^*L))$. This means that in order to study special Lagrangians where $C$ satisfies special properties one does not need to specify the normal bundle of $L$. One such special property would be that $C$, or equivalently the second fundamental form, is a parallel section. However, it turns out that this assumption is rather restrictive. Any such Lagrangian is automatically totally geodesic [Zha+16, Theorem 1.1].

Another special property of $C$ is that it admits symmetries. This approach has been developed in [Bry06b] for special Lagrangian submanifolds of $\mathbb{C}^3$. By picking a frame in a point $x \in L$ we regard $C$ as a homogeneous polynomial of degree three in three variables. Since $C$ is traceless, this polynomial is harmonic, i.e. an element of $\mathcal{H}_3(\mathbb{R}^3)$, which is a seven-dimensional vector space. Changing the frame at $x \in L$ amounts to applying an element in SO(3) to the harmonic polynomial. So one studies the space $\mathcal{H}_3(\mathbb{R}^3)$ as an SO(3) module. A generic element in $\mathcal{H}_3(\mathbb{R}^3)$ does not have any symmetries in SO(3). The possible symmetry groups are classified in the following proposition.

**Proposition 6.1.2.** *[Bry06b, Proposition 1] The* SO(3)*-stabilizer of $h \in \mathcal{H}_3(\mathbb{R}^3)$ is non-trivial if and only if $h$ lies on the* SO(3) *orbit of exactly one of the following polynomials*

1. $0 \in \mathcal{H}_3(\mathbb{R}^3)$*, whose stabilizer is* SO(3).

2. $r\,(2z^3 - 3zx^2 - 3zy^2)$ *for some $r > 0$, whose stabilizer is* SO(2).

3. $6s\,xyz$ *for some* $s > 0$*, whose stabilizer is the subgroup* $A_4 \subset \mathrm{SO}(3)$ *of order* 12 *generated by the rotations by an angle of* $\pi$ *about the* $x$*-,* $y$*-, and* $z$*-axes and by rotation by an angle of* $\frac{2}{3}\pi$ *about the line* $x = y = z$*.*

4. $s\,(x^3 - 3xy^2)$ *for some* $s > 0$*, whose stabilizer is the subgroup* $S_3 \subset \mathrm{SO}(3)$ *of order* 6 *generated by the rotation by an angle of* $\pi$ *about the* $x$*-axis and the rotation by an angle of* $\frac{2}{3}\pi$ *about the* $z$*-axis.*

5. $r\,(2z^3 - 3zx^2 - 3zy^2) + 6s\,xyz$ *for some* $r, s > 0$ *satisfying* $s \neq r$*, whose stabilizer is the* $\mathbb{Z}_2$ *subgroup of* $\mathrm{SO}(3)$ *generated by rotation by an angle of* $\pi$ *about the* $z$*-axis.*

6. $r\,(2z^3 - 3zx^2 - 3zy^2) + s\,(x^3 - 3xy^2)$ *for some* $r, s > 0$ *satisfying* $s \neq r\sqrt{2}$*, whose stabilizer is the* $\mathbb{Z}_3$ *subgroup of* $\mathrm{SO}(3)$ *generated by rotation by an angle of* $\frac{2}{3}\pi$ *about the* $z$*-axis.*

This classification gives a natural ansatz for finding special Lagrangian submanifolds. Impose one of the pointwise symmetries above to every point in $L$. This ansatz has led to the construction of new special Lagrangians in the Calabi-Yau $\mathbb{C}^3$ in [Bry06b] and in the nearly Kähler $S^6$ [Vra03; Lot11b]. For $\mathbb{CP}^3$ however, this ansatz is less fruitful since the curvature tensor is more complicated so we do not have SO(3)-freedom to change frames as we will see later. However, this framework gives us a way to categorise examples of special Lagrangian that are constructed in different ways.

The following result is known for Calabi-Yau manifolds and the nearly Kähler $S^6$ but it holds for any nearly Kähler manifold.

**Proposition 6.1.3.** *Every real analytic surface on which* $\omega$ *vanishes can locally be uniquely thickened to a special Lagrangian submanifold in* $M$*. Special Lagrangian submanifolds in a nearly Kähler manifold locally depend on two functions of two variables.*

*Proof.* See [Lot11b], the proof is based on the fact that the Cartan test holds and thus holds for any SU(3) structure. □

Infinitesimal deformations of nearly Kähler manifolds correspond to eigensections of a rotation operator on $L$ [Kaw17]. It is shown in [VS+19], that the moduli space of smooth Lagrangian deformations of special Lagrangians is a finite dimensional analytic variety. All formally unobstructed infinitesimal deformations are smoothly unobstructed.

### 6.1.1 Structure Equations for Special Lagrangians

The structure equations for a special Lagrangian manifold in Calabi-Yau $\mathbb{C}^3$ were established in [Bry06b] and for nearly Kähler $S^6$ in [Lot11b]. We generalise the equations to the setting of a general nearly Kähler manifold. The main difference is the appearance of an extra curvature term. We characterise nearly Kähler manifolds by differential identities on the frame bundle, as done in [Bry06a]. If an index appears on the right-hand side but not on the left-hand side of an equation, summation over the index set $\{1, 2, 3\}$ is implicit.

Let $M^6$ be a nearly Kähler manifold and consider the SU(3)-frame bundle $P_{\mathrm{SU}(3)}$. Let $(\zeta_1, \zeta_2, \zeta_3) \in \Omega^1(P_{\mathrm{SU}(3)}, \mathbb{C}^3)$ be the tautological one-forms on $P_{\mathrm{SU}(3)}$ and let $\phi \in \Omega^1(P, \mathfrak{su}(3))$ be the nearly Kähler connection one-form on $P_{\mathrm{SU}(3)}$, giving the torsion relation

$$\mathrm{d}\begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix} = -\phi \wedge \begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{pmatrix} + \begin{pmatrix}\overline{\zeta_2} \wedge \overline{\zeta_3}\\ \overline{\zeta_3} \wedge \overline{\zeta_1}\\ \overline{\zeta_1} \wedge \overline{\zeta_2}\end{pmatrix} \tag{6.1.2}$$

and the curvature identity

$$\mathrm{d}\phi_{ij} = -\phi_{ik} \wedge \phi_{kj} + K_{ijpq}\zeta_q \wedge \overline{\zeta_p}. \tag{6.1.3}$$

In particular, the curvature of $\overline{\nabla}$ is always of type $(1, 1)$. In [Bry06a] it is remarked that the tensor $K$ can be written as sum

$$K_{ijpq} = K'_{ijpq} + \frac{3}{4}\delta_{pi}\delta_{qj} - \frac{1}{4}\delta_{ij}\delta_{pq}$$

where $K'$ has the following symmetries

$$K'_{ijpq} = K'_{pjiq} = K'_{iqpj} = \overline{K'_{jiqp}}, \qquad \sum_i K'_{iipq} = 0.$$

The tensor $K'$ vanishes exactly when $M$ is the round six-sphere. The nearly Kähler forms are expressed in terms of $\zeta_i$ by

$$\omega = \frac{i}{2}\sum_i \zeta_i \wedge \overline{\zeta_i}, \quad \psi = -i\zeta_1 \wedge \zeta_2 \wedge \zeta_3. \tag{6.1.4}$$

Note the difference from [Bry06a] in the convention for $\psi$ in order to satisfy the standard nearly Kähler integrability equations.

The torsion-relation eq. (6.1.2) and curvature-relation eq. (6.1.3) yield differential identities for the connection-one form and tautological-one form on the frame bundle $P_{\mathrm{SU}(3)}$. If $L$ is a special Lagrangian submanifold in $M$ then one obtains more differential identities because the frame bundle $P_{\mathrm{SU}(3)}$ admits a natural reduction to an SO(3) bundle over $L$. The reason for this is that, on the tangent level, a Lagrangian subspace looks like $\mathbb{R}^3$ in $\mathbb{C}^3$, which defines the restriction

$$P_{\mathrm{SO}(3)} = \{p\colon \mathbb{C}^3 \to TM, \quad p \in P_{\mathrm{SU}(3)}|_L \mid p(\mathbb{R}^3) = TL\}.$$

If $\mathrm{d}z_1, \mathrm{d}z_2, \mathrm{d}z_3$ are the standard complex-valued one-forms on $\mathbb{C}^3$ then $\mathbb{R}^3 \subset \mathbb{C}^3$ is characterised as the three-dimensional subspace of $\mathbb{C}^3$ on which the imaginary parts of $\mathrm{d}z_i$ vanish. Similarly, our aim is to describe the reduction $P_{\mathrm{SO}(3)}$ as the vanishing set of one forms on $P_{\mathrm{SU}(3)}$. To that end, split the forms $\zeta_i = \sigma_i + i\eta_i$ and $\phi = \alpha + i\beta$ into real and imaginary part. The bundle $P_{\mathrm{SO}(3)}$ is now defined by imposing the condition $\eta_i = 0$.

This characterisation implies more differential identities. From the torsion-relation we get

$$\begin{aligned}
\mathrm{d}\sigma_i &= -\alpha_{ij} \wedge \sigma_j + \beta_{ij} \wedge \eta_j + \sigma_k \wedge \sigma_l - \eta_k \wedge \eta_l \\
\mathrm{d}\eta_i &= -\beta_{ij} \wedge \sigma_j - \alpha_{ij} \wedge \eta_j - \sigma_k \wedge \eta_l - \eta_k \wedge \sigma_l
\end{aligned}$$

where $(i,k,l)$ is an even permutation of $(1,2,3)$. The condition $\eta_i = 0$ implies $\beta_{ij} \wedge \sigma_j = 0$. By Cartan's lemma, we have $\beta_{ij} = h_{ijk}\sigma_k$ or $\beta = h\sigma$ where $h$ is a fully symmetric three-tensor. In fact, this tensor corresponds to the fundamental cubic up to a factor, just as in the case of special Lagrangians in $\mathbb{C}^3$ or in $S^6$.

On the reduced bundle, we split $K$ into real and imaginary part,

$$K_{ijpq}\zeta_q \wedge \overline{\zeta_p} = K_{ijpq}\sigma_q \wedge \sigma_p = (R_{ijpq} + iS_{ijpq})\sigma_q \wedge \sigma_p = (-R_{ijpq} - iS_{ijpq})\sigma_p \wedge \sigma_q.$$

This also allows us to split the curvature identity into real imaginary part

$$\mathrm{d}\alpha_{ij} = -\alpha_{ik} \wedge \alpha_{kj} + \beta_{ik} \wedge \beta_{kj} - R_{ijpq}\sigma_p \wedge \sigma_q \tag{6.1.5}$$

$$\mathrm{d}\beta_{ij} = -\beta_{ik} \wedge \alpha_{kj} - \alpha_{ik} \wedge \beta_{kj} - S_{ijpq}\sigma_p \wedge \sigma_q. \tag{6.1.6}$$

To write these equations more invariantly, let

$$[\sigma] = \begin{pmatrix} 0 & \sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & \sigma_1 \\ \sigma_2 & -\sigma_1 & 0 \end{pmatrix}.$$

We can summarise the equations on the reduced bundle over $L$ in tensor notation

$$\beta \wedge \sigma = 0 \tag{6.1.7}$$

$$\mathrm{d}\sigma = -\alpha \wedge \sigma - \frac{1}{2}[\sigma] \wedge \sigma \tag{6.1.8}$$

$$\mathrm{d}\alpha = -\alpha \wedge \alpha + \beta \wedge \beta - R\sigma \wedge \sigma \tag{6.1.9}$$

$$\mathrm{d}\beta = -\beta \wedge \alpha - \alpha \wedge \beta - S\sigma \wedge \sigma \tag{6.1.10}$$

where $(\sigma \wedge \sigma)_{pq} = \sigma_p \wedge \sigma_q$. The matrix of one forms $\beta$ is completely defined by the symmetric tensor $h$. The advantage to work with $h$ is that its components are not one-forms but functions, allowing us to rewrite eq. (6.1.7),eq. (6.1.9) and eq. (6.1.10)

$$\begin{aligned} \beta &= h\sigma \\ \mathrm{d}\alpha &= -\alpha \wedge \alpha + h\sigma \wedge h\sigma + \frac{3}{4}\sigma \wedge \sigma - R\sigma \wedge \sigma \\ 0 &= (\mathrm{d}h + ((h\alpha + \frac{1}{2}h[\sigma])) + S\sigma) \wedge \sigma. \end{aligned}$$

The Levi-Civita connection one-form of the induced metric on $L$ is $\alpha + \frac{1}{2}[\sigma]$. Note that the forms $\sigma$ differ by a factor 2 from the orthonormal one forms considered in [Lot11b].

If $M = G/H$ is one of the homogeneous nearly Kähler manifolds then a special Lagrangian submanifold can locally be recovered from a solution to eq. (6.1.7)-eq. (6.1.9), which we will make precise now. There is a splitting

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

such that $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$. The nearly Kähler structure then yields an $\mathrm{Ad}(H)$ invariant special unitary basis $\omega_1, \omega_2, \omega_3$ on $\mathfrak{m} \cong \mathbb{C}^3$. Up to a cover, $G$ embeds into the SU(3)-frame bundle $P_{\mathrm{SU}(3)}$ via the adjoint action $H \to \mathrm{SU}(\mathfrak{m})$. Under this identification

$$\psi + (\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{h} \oplus \mathbb{C}^3 \cong \mathfrak{h} \oplus \mathfrak{m}$$

is the Maurer-Cartan form $\omega_G$ on $G$. In other words, the nearly Kähler connection is equal to the canonical homogeneous connection on $G \to M$, see [CH16]. The following proposition guarantees that for the homogeneous nearly Kähler manifolds we can locally recover the special Lagrangian from a solution of the structure equations. Since $\alpha$ and $\beta$ determine the first and second fundamental form, this can be viewed a Bonnet-type theorem.

**Proposition 6.1.4.** *Let $M = G/H$ be a homogeneous nearly Kähler manifold, $L^3$ be a simply-connected three manifold and $\sigma \in \Omega^1(L, \mathbb{R}^3)$, defining a linearly independent co-*

*frame at each point, $\alpha \in \Omega^1(L, \mathfrak{so}(3))$ and $\beta \in \Omega^1(L, S^2(\mathbb{R}^3))$ satisfying the equations 6.1.7-6.1.10. Then there is a special Lagrangian immersion $L \to M$, unique up to isometries, with $\alpha, \beta$ determining the metric and second fundamental form of $L$ in $M$.*

*Proof.* Define the form $\gamma = \alpha + i\beta + (\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{h} \oplus \mathfrak{m} \cong \mathfrak{g}$. Since $\sigma, \alpha, \beta$ satisfy the equations 6.1.7-6.1.10 we have $\mathrm{d}\gamma + [\gamma, \gamma] = 0$. The statement now follows, as in the proof of lemma 4.5.1, from Cartan's theorem. □

*Remark* 6.1.5. Note that the tautological one form $(\zeta_1, \zeta_2, \zeta_3)$ can also be regarded as an element in $\Gamma(P, \mathrm{End}(\mathbb{C}^3))$. With this identification, a local section $s$ of $L \supset U \to P_{\mathrm{SO}(3)}$ gives a section $\Gamma(U, (T^\vee M)|_L \otimes \mathbb{C}^3) \cong \Omega^1(U, \mathbb{C}^3)$. Then $s^*\eta_i$ vanishes on $TL$ while $s^*\sigma$ vanishes on the normal bundle.

## 6.2 An Angle Function for Special Lagrangians

Since twistor fibres are $J$-holomorphic they can never be contained in a special Lagrangian submanifold. Generically, a special Lagrangian intersects every twistor fibre transversally. However, there is a special class of special Lagrangians which are circle bundles over superminimal surfaces in $S^4$. We review this construction and define an angle function $L \to [0, \frac{\pi}{4}]$ which has value $\frac{\pi}{4}$ if $L$ intersects a twistor fibre non-transversally. We use a gauge transformation, which depends on $\theta$, to use the moving frame setup from the previous section for special Lagrangians in $\mathbb{CP}^3$. We identify special solutions to the resulting structure equations, all of which turn out to be homogeneous.

### 6.2.1 Some Linear Algebra

We start with the study of Lagrangian subspaces in a twistor space on the tangent level. The space of special Lagrangian subspaces of $\mathbb{C}^n$ is identified with the homogeneous space $\mathrm{SU}(n)/\mathrm{SO}(n)$. Twistor nearly Kähler spaces have the property that the holonomy of the nearly Kähler connection reduces to $\mathrm{U}(2) \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$. The two-form splits into a horizontal and vertical part $\omega = \omega_{\mathcal{H}} + \omega_{\mathcal{V}}$. So, in order to understand how frames can be adapted further to a special Lagrangian of a twistor space, we study the linear problem first.

Let $(b_1, b_2, b_3)$ denote the standard basis of $\mathbb{C}^3$ with dual basis $(\omega_1, \omega_2, \omega_3)$ and let $\omega_{\mathcal{H}} = \frac{i}{2}(\omega_1 \wedge \bar{\omega}_1 + \omega_2 \wedge \bar{\omega}_2)$ as well as $\omega_{\mathcal{V}} = \frac{i}{2}(\omega_3 \wedge \bar{\omega}_3)$. Let $H \cong \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ be the stabiliser of $\omega_{\mathcal{V}}$ inside $\mathrm{SU}(3)$. Let also $\psi = \mathrm{Re}\,\psi + i\,\mathrm{Im}\,\psi$ be the complex-valued three form $-i\omega_1 \wedge \omega_2 \wedge \omega_3$ on $\mathbb{C}^3$. We have abused notation slightly here, since $\omega, \psi$ are forms on the nearly Kähler manifold but also denote their linear models on $\mathbb{C}^3$.

For a complex subspace $W \subset \mathbb{C}^3$ denote by $\mathrm{SLag}(W)$ the set of all special Lagrangian subspaces of $W$. By $\mathbb{C}^2 \subset \mathbb{C}^3$ we refer to the subspace spanned by $b_2$ and $b_3$. Note that $\mathrm{SLag}(\mathbb{C}^2) \cong S^2$ and that $\mathrm{U}(1) \subset \mathrm{SU}(2)$ acts from the left on this space. The quotient is an interval and the following lemma gives a description of each representative.

**Lemma 6.2.1.** *Under the action of* $\mathrm{U}(1) \cong \{\mathrm{diag}(e^{i\varphi}, e^{-i\varphi}\} \subset \mathrm{SU}(2)$ *any element in* $\mathrm{SLag}(\mathbb{C}^2)$ *has a unique representative of the form* $V_\theta = \mathrm{span}(-ie^{-i\theta}b_2 - e^{-i\theta}b_3, e^{i\theta}b_2 + ie^{i\theta}b_3)$ *for* $0 \leq \theta \leq \pi/2$.

*Proof.* Special Lagrangian planes in $\mathbb{C}^2$ are parametrised by $\mathrm{SU}(2)/\mathrm{SO}(2)$. Thus, we have to find a unique representative of the action $(A,B)X = AXB^{-1}$ of $K = \mathrm{U}(1) \times \mathrm{SO}(2)$ on $\mathrm{SU}(2)$, which is the action of a maximal torus in $\mathrm{SO}(4)$ acting on $S^3$. The standard torus $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{U}(2) \subset \mathrm{SO}(4)$ acting on $S^3$ admits unique representatives of the form $(\cos(\theta), 0, \sin(\theta), 0)$ for $0 \leq \theta \leq \pi/2$. The statement follows by conjugating the action of $K$ to the standard torus action. $\square$

For any subspace $W \subset \mathbb{C}^3$ denote by $K_W$ the kernel of the projection onto $\mathrm{span}(b_3)$ and by $n_W$ its dimension. Let

$$T_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{ie^{-i\theta}}{\sqrt{2}} & \frac{e^{i\theta}}{\sqrt{2}} \\ 0 & -\frac{e^{-i\theta}}{\sqrt{2}} & \frac{ie^{i\theta}}{\sqrt{2}} \end{pmatrix} \tag{6.2.1}$$

and $W_\theta$ be the image of $T_\theta$ when applied to the standard $\mathbb{R}^3$ in $\mathbb{C}^3$, i.e.

$$W_\theta = \mathrm{span}(b_1, -ie^{-i\theta}b_2 - e^{-i\theta}b_3, e^{i\theta}b_2 + ie^{i\theta}b_3).$$

**Proposition 6.2.2.** *Any special Lagrangian subspace* $W \subset \mathbb{C}^3$ *admits a unique representative* $W_\theta$ *for* $0 \leq \theta \leq \pi/4$ *under the action of* $H$. *Furthermore,* $n_W = 2$ *if and only if* $\theta = \pi/4$.

*Proof.* Since $W$ is Lagrangian, $n_W \geq 1$. If $n_W = 2$ then $W$ is represented by the standard $\mathbb{R}^3$ in $\mathbb{C}^3$ and $n_{W_\theta} = 2$ if and only if $\theta = \pi/4$. So from now on we assume that $n_W = 1$. Consider the map $l\colon \mathrm{Gr}_2(\mathbb{C}^2) \to \mathrm{Gr}_3(\mathbb{C}^3)$, $\quad V \mapsto \mathrm{span}(b_1, V)$. Note that $W_\theta = l(V_\theta)$ and that $l$ descends to a map

$$\hat{l}\colon \mathrm{Lag}(\mathbb{C}^2)/\mathrm{U}(1) \to \mathrm{Lag}(\mathbb{C}^3)/H.$$

To show surjectivity observe that for $W \in \mathrm{Lag}(\mathbb{C}^3)$ we have $K_W \subset \mathrm{span}(b_1, b_2)$. So by acting with $H$ we can achieve that $K_W$ is spanned by $b_1$. Furthermore, observe that

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} T_\theta \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = T_{\pi/2-\theta}$$

which means that $W_\theta = W_{\theta'}$ for $\theta + \theta' = \pi/2$. We have shown that any element in $\mathrm{Lag}(\mathbb{C}^3)$ is represented by a $W_\theta$ for $0 \le \theta \le \pi/4$. The uniqueness follows from the observation that $\omega_{\mathcal{V}}$ has norm $\frac{1}{2}|\cos(2\theta)|$ when restricted to the vector space $W_\theta$. $\square$

If $w_1, w_2, w_3$ is a basis of $W$ such that $w_1 \in K_W$ and $w_2, w_2 \in K_W^\perp$ then $\theta$ can be computed by the formula

$$\frac{1}{2\|w_1\|}|\cos(2\theta)|\psi^-(w_1, w_2, w_3) = \omega_{\mathcal{V}}(w_2, w_3). \tag{6.2.2}$$

Motivated by the existence of the two almost complex structures $J_1$ and $J_2$ on the twistor space, consider the almost complex structure

$$J' \colon (b_1, b_2, b_3) \mapsto (ib_1, ib_2, -ib_3) \tag{6.2.3}$$

on $\mathbb{C}^3$. Any special Lagrangian subspace $W$ in $\mathbb{C}^3$ splits as $K_W \oplus K_W^\perp$.

**Lemma 6.2.3.** *If $\theta \neq \frac{\pi}{4}$ then $J(K_W)$ is orthogonal to $W$. The subspace $K_W^\perp$ is invariant under $J'$ if and only if $\theta = 0$.*

*Proof.* The endomorphism $J'$ commutes with the action of $H$ on $\mathbb{C}^3$, so it suffices to prove the statement for $W_\theta$. If $\theta = \frac{\pi}{4}$ then $K_W^\perp$ is one-dimensional so it cannot be invariant under $J'$. Otherwise, $K_W$ is spanned by $b_1$ and $K_W^\perp$ equals $V_\theta$. Clearly $Jb_1 = ib_1$ is orthogonal to $W$. The statement follows by observing that $V_\theta$ is invariant under the endomorphism $(b_2, b_3) \mapsto (ib_2, -ib_3)$ if and only if $\theta = 0$. $\square$

The following lemma can be proven by a computation in $\mathrm{SU}(3)$ and is important for adapting frames on special Lagrangians in twistor spaces.

**Lemma 6.2.4.** *Let $H_\theta = T_\theta^{-1} H T_\theta \cap \mathrm{SO}(3)$ be the stabiliser group of $W_\theta$ in $H$ with*

*Lie algebra* $\mathfrak{h}_\theta$ *then*

$$\mathfrak{h}_\theta = \begin{cases} \operatorname{span}_{\mathbb{R}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \theta = \pi/4 \\ \{0\} \oplus \mathfrak{so}(2) & \theta = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

$$T_\theta \mathfrak{h}_\theta T_\theta^{-1} = \begin{cases} \operatorname{span}_{\mathbb{R}} \begin{pmatrix} 0 & -1+i & 0 \\ 1+i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \theta = \pi/4 \\ \{0\} \oplus \mathfrak{s}(\mathfrak{u}(1) \oplus \mathfrak{u}(1)) & \theta = 0 \\ \{0\} & \textit{otherwise} \end{cases}$$

*Then* $H_\theta$ *is generated by* $\exp(\mathfrak{h}_\theta)$ *and the element* $\operatorname{diag}(1,-1,-1)$*. In particular,* $H_\theta$ *is isomorphic to* $\mathrm{O}(2)$ *if* $\theta = \pi/4$*, to* $\mathrm{SO}(2)$ *if* $\theta = 0$ *and to* $\mathbb{Z}_2$ *otherwise.*

The action of $H$ on $\operatorname{Lag}(\mathbb{C}^3)$ is a smooth cohomogeneity one action. The orbit at $W_\theta$ is diffeomorphic to $H/(T_\theta H_\theta T_\theta^{-1})$ and is singular for $\theta = 0, \pi/4$ and of principal type otherwise. The principal orbits are diffeomorphic to $H/\langle \operatorname{diag}(1,-1,-1)\rangle \cong \mathrm{U}(2)/\langle \operatorname{diag}(1,-1)\rangle$. The orbit of $W_0$ is diffeomorphic to $H/(\{1\} \times S(\mathrm{U}(1) \times \mathrm{U}(1))) \cong \mathrm{U}(2)/(\{1\} \times \mathrm{U}(1)) \cong S^3$. Observe that $A_{\pi/4} H_{\pi_4} T_\theta^{-1}$ is conjugate to the $\mathrm{O}(2)$ subgroup generated by

$$S(\mathrm{U}(1) \times \mathrm{U}(1)) \text{ and } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

This subgroup is equal to the preimage of $[([1,0],1)$ of the map

$$\mathrm{U}(2) \to (\mathbb{CP}^1 \times S^1)/\mathbb{Z}_2, \quad A \mapsto [[A(1,0)^T], \det(A)].$$

Here $\mathbb{Z}_2$ acts as the antipodal map on both $\mathbb{CP}^1 \cong S^2$ and on $S^1$. Hence, the orbit of $W_{\pi/4}$ is diffeomorphic to $(S^2 \times S^1)/\mathbb{Z}_2$. The following lemma summarises these observations.

**Lemma 6.2.5.** *The action of* $H$ *on* $\operatorname{Lag}(\mathbb{C}^3)$ *is of cohomogeneity one. The principal orbit is diffeomorphic to* $\mathrm{U}(2)/\mathbb{Z}_2$*, two singular orbits occur at* $\theta = 0$ *and* $\theta = \frac{\pi}{4}$*. The orbit* $W_0$ *is diffeomorphic to* $S^3$ *and that of* $W_{\pi/4}$ *to* $(S^2 \times S^1)/\mathbb{Z}_2$*.*

### 6.2.2 Adapting Frames

We now assume that $M$ is a nearly Kähler twistor space over a Riemannian four manifold $N$. In other words, $M$ is either $\mathbb{CP}^3$ or the flag manifold. Lagrangian submanifolds of the latter have been studied in [Sto20b] so our interest is in $\mathbb{CP}^3$ in this chapter. Before using the explicit description of $\mathbb{CP}^3$ we give a few general statements that could be useful for generalisations to other spaces, such as non-nearly Kähler twistor spaces.

Given a special Lagrangian submanifold $L \subset M$, we clearly have $T_xL \in \mathrm{Lag}(T_xM)$ for $x \in L$. Since the frame bundle reduces to $H$ there is a map $\mathrm{Lag}(TM|_L) \to \mathrm{Lag}(\mathbb{C}^3)/H$. Hence, $\theta$ can be understood as a map from $L$ to the interval $[0, \frac{\pi}{4}]$ and $T_\theta$ from $L$ to $\mathrm{SU}(3)$. We now apply our knowledge of the action of $H$ on $\mathrm{Lag}(\mathbb{C}^3)$ to obtain a further frame reduction for special Lagrangian submanifolds in nearly Kähler twistor spaces. In that case, the holonomy of the nearly Kähler connection on $M$ reduces to $H$, so $P_{\mathrm{SU}(3)}$ reduces to an $H$-bundle and we can assume $\phi_{13} = \phi_{23} = \phi_{31} = \phi_{32} = 0$. This means that there are two different reductions of $P|_L$: The first is to an $H$-bundle $P_H = \{p\colon \mathbb{C}^3 \to TM, p \in P_{\mathrm{SU}(3)}|_L \mid p(b_3) \in \mathcal{V}\}$, simply because $P_{\mathrm{SU}(3)}$ itself reduces to an $H$ bundle. The second reduction is to an $\mathrm{SO}(3)$-bundle $P_{\mathrm{SO}(3)} = \{p\colon \mathbb{C}^3 \to TM, p \in P_{\mathrm{SU}(3)}|_L \mid p(\mathbb{R}^3) = TL\}$ or equivalently by imposing $\eta_i = 0$ as in section 6.1.1.

If $TL \cap \mathcal{V}$ is a rank one bundle, or equivalently $\theta \equiv \frac{\pi}{4}$, then the intersection $P_{\mathrm{SO}(3)} \cap P_H$ is a $H \cap \mathrm{SO}(3)$ bundle. We will derive its structure equations in section 6.2.3. If $\theta$ avoids the value $\frac{\pi}{4}$ then the intersection $TL \cap \mathcal{V}$ is trivial and $P_{\mathrm{SO}(3)} \cap P_H = \emptyset$ which precludes the existence of a distinguished frame. However, by lemma 6.2.4 we can apply a gauge transformation to guarantee a non-empty intersection.

For $x \in L$ there is a frame in $P_H$ which maps $W_\theta$ to $TL$. Such a frame is unique up to the action of the stabiliser of $W_\theta$ in $H$, which is computed in lemma 6.2.4. This means that

$$Q = P_H T_\theta \cap P_{\mathrm{SO}(3)} \neq \emptyset. \tag{6.2.4}$$

This is a principal bundle over $L$ with structure group given by 6.2.4 if $\theta$ is either equal to 0 or $\frac{\pi}{4}$ everywhere or if $\theta$ avoids these values altogether. In the latter case, the structure group is discrete. We first describe all special Lagrangians where $\theta$ is constant and equal to one of the boundary values everywhere. If $\theta \equiv \frac{\pi}{4}$ then $L$ intersects every twistor fibre in a circle and maps to a surface in $N$.

### 6.2.3 Lagrangians with $\theta \equiv \frac{\pi}{4}$

There is a general construction for Lagrangian submanifolds in the twistor space $Z$ of an arbitrary Riemannian four-manifold $N$ due to Storm [Sto20a] and Konstantinov [Kon17]. To make sense of how a Lagrangian submanifold in $Z$ is defined, recall that $Z$ carries two almost complex structures $J_1, J_2$ and metrics $g_\lambda$ for $\lambda \in \mathbb{R}_{\geq 0}$. For a surface $X \subset N$ define the circle bundle $L_X \subset Z(N)$ with fibre over $x \in X$ equal to $\{J \in Z_x(N) \mid J(T_xX) = \nu_x\}$. Geometrically, the fibre of $L_X$ at $x \in X$ is the equator in each twistor fibre, which is diffeomorphic to $S^2$, relative to the twistor lift of $X$ at $x$. It turns out that this construction gives a lot of examples of Lagrangians in twistor spaces.

**Proposition 6.2.6.** *[Sto20a] The submanifold $L_X$ is Lagrangian in $Z$ for both $J_1$ and $J_2$ and every $g_\lambda$ if $X$ is superminimal. Conversely, if $L_X$ is Lagrangian for any $J_a$ and $g_\lambda$ then $X$ is superminimal.*

Assume $L$ is Lagrangian with $\theta \equiv \frac{\pi}{4}$ so $TL \cap \mathcal{H}$ and $TL \cap \mathcal{V}$ are a rank two and a rank one bundle and

$$TL = TL \cap \mathcal{H} \oplus TL \cap \mathcal{V}.$$

So $L$ is also Lagrangian for $J_1$ and $L$ arises via the construction above. In this case the intersection $P_{\mathrm{O}(2)} = P_H \cap P_{\mathrm{SO}(3)}$ is an $S(\mathrm{O}(2) \times O(1))$ bundle which is defined by imposing $\eta_i = 0$ for $i = 1, \ldots, 3$ on $P_H$. Since $\beta_{32} = \beta_{23} = \beta_{31} = \beta_{13} = 0$ the equation $\beta \wedge \sigma = 0$ implies that $\beta_{33}$ lies in the span of $\sigma_3$ and $\beta_{11}, \beta_{22}$ lie in the span of $\sigma_1$ and $\sigma_2$. Since $\mathrm{Tr}(\phi) = 0$ this implies that $\beta_{33} = 0 = \beta_{11} + \beta_{22}$, i.e. $\phi$ takes values in $\mathfrak{su}(2)$ when restricted to $P_{\mathrm{O}(2)}$.

We can view $(\sigma_1, \sigma_2, \sigma_3, \eta_1, \eta_2, \eta_3)$ locally as an orthonormal co-frame on $TM|_L$, see remark 6.1.5. The forms $\sigma_i$ vanish on the normal bundle while $\eta_i$ vanish on $TL$. The form $\sigma_3$ is dual to the unit vector field tangent along the fibres of $L \to X$. Since $\beta_{33} = 0$ this means that the fibres of $L \to X$ are in fact geodesics. Since twistor fibres are totally geodesic $\mathbb{CP}^1 \subset M$ these geodesics are great circles in the twistor fibres.

Since $\beta_{3i} = 0$ for $i = 1, 2, 3$ this implies $h_{3ij} = 0$ so the fundamental cubic is of the form

$$a(x_1^3 - 3x_1x_2^2) + b(x_2^3 - 3x_2x_1^2).$$

We have shown.

**Proposition 6.2.7.** *The fundamental cubic of $L_X$ in a nearly Kähler twistor space either vanishes or has stabiliser $S_3$.*

We can also recover the result that $X$ is superminimal by showing that the second fundamental form $X$ in $N$ is complex-linear and using [MU97, Proposition 1c].

*Remark* 6.2.8. Bryant considers special Lagrangians of the form $C^1 \times \Sigma^2 \subset \mathbb{C} \oplus \mathbb{C}^2 = \mathbb{C}^3$. The cubic form of such submanifolds is always stabilised by $S_3$. These examples are somewhat analogous to horizontal Lagrangians whose fundamental cubic also admits an $S_3$ symmetry.

From now on, we will work specifically with $M = \mathbb{CP}^3$. We have seen that $L_X$ is either totally geodesic or its fundamental cubic has stabiliser $S_3$. If $L_X$ is homogeneous then $X$ is a homogeneous superminimal surface in $S^4$. Such a surface is equal to a totally geodesic $S^2 \subset S^4$ or the Veronese curve in $S^4$. Hence, there are only two different examples of homogeneous special Lagrangian submanifolds with $\theta \equiv \frac{\pi}{4}$. Both of them are well known for the Kähler structure on $\mathbb{CP}^3$.

**Example 6.2.9** (The standard $\mathbb{RP}^3$)**.** *The standard* $\mathbb{RP}^3 \subset \mathbb{CP}^3$ *is a totally geodesic special Lagrangian submanifold. It fibres over a totally geodesic* $S^2$ *in* $S^4$ *under the twistor fibration. It is the orbit of* $\{\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{R} \oplus j\mathbb{R}, \quad |a|^2 + |b|^2 = 1\} \cong \mathrm{SU}(2)$ *on* $[1, 0, 0, 0]$.

The second example was discovered in [Chi04] and is described in [Kon17] in terms of the twistor fibration.

**Example 6.2.10** (Chiang Lagrangian)**.** *The* $\mathrm{SU}(2)$ *subgroup of* $\mathrm{Sp}(2)$ *which comes from the irreducible representation of* $\mathrm{SU}(2)$ *on* $\mathbb{C}^4 = S^3(\mathbb{C}^2)$ *has a special Lagrangian orbit at* $[1, 0, 0, 1] \in \mathbb{CP}^3$. *This example is known as the Chiang Lagrangian and fibres over the Veronese surface in* $S^4$. *The* $\mathrm{SU}(2)$ *subgroup acts with stabiliser* $S_3$ *on* $[1, 0, 0, 1]$. *The stabiliser subgroup induces the full symmetry group of the fundamental cubic since the Chiang Lagrangian is not totally geodesic.*

Since superminimal curves in $S^4$ have an explicit Weierstraß parametrisation one can produce a lot of (explicit) examples of special Lagrangians in $\mathbb{CP}^3$. However, our focus is on exploring special Lagrangians which do not arise from superminimal surfaces.

### 6.2.4 Changing the Gauge

If one expresses the nearly Kähler structure on $\mathbb{CP}^3$ in terms of local coordinates one can work out a system of PDE's which, at least locally, describes special Lagrangian submanifolds. However, this approach is not very likely to succeed since local coordinates on $\mathbb{CP}^3$ are not an elegant way to define its nearly Kähler structure. Of more geometric importance are the first and second fundamental form and proposition 6.1.4 shows that locally they contain all information about the submanifold. We use a gauge transformation, which depends on the function $\theta$ to describe the structure equations for a special Lagrangian in $\mathbb{CP}^3$.

The bundle Sp(2) embeds into the frame bundle of $\mathbb{CP}^3$ via the adjoint action of $S^1 \times S^3$ on $\mathfrak{m}$ which factors through the double cover $S^1 \times S^3 \to \mathrm{U}(2)$. So, on the level of structure equations we identify $P_H$ with Sp(2). We apply the gauge transformation $T_\theta$ to Sp(2), which defines the bundle $Q$ as in eq. (6.2.4). This bundle has a reduced structure group, depending on the behaviour of the function $\theta$, which is made precise in lemma 6.2.4. For example, if $\theta$ avoids the values 0 and $\frac{\pi}{4}$ then the structure group of $Q$ is $\mathbb{Z}_2$.

Recall from section 2.2 that Sp(2) is an $S^1 \times S^3$ principal bundle over $\mathbb{CP}^3$. A local unitary frame for the nearly Kähler structure on $\mathbb{CP}^3$ is obtained by pulling back the forms $(\omega_1, \omega_2, \omega_3)$, which are components of the Maurer-Cartan form on Sp(2). We can realise the bundle $Q$ by setting $T_\theta^{-1}(\omega_1, \omega_2, \omega_3) = (\zeta_1, \zeta_2, \zeta_3)$, where $T_\theta$ is defined in eq. (6.2.1), and imposing the equations

$$\eta_1 = 0, \quad \eta_2 = 0, \quad \eta_3 = 0. \tag{6.2.5}$$

Our aim is to compute the differentials of the one forms $\zeta_i$ and also of $\rho_i$ and $\tau$ on the reduced bundle $Q$. We will achieve this by first computing the connection and curvature form in the transformed frame and then applying eq. (6.1.7)-eq. (6.1.10). We begin by applying the transformation formula for a connection-one form under the gauge transformation $T_\theta$

$$\phi = T_\theta^{-1} A_\omega T_\theta + T_\theta^{-1} dT_\theta. \tag{6.2.6}$$

Here $A_\omega$ is the connection form defined on Sp(2), see eq. (2.2.2). Since $T_\theta$ lies in SU(3) the torsion transforms trivially and we have

$$\mathrm{d}\zeta = -\phi \wedge \zeta - [\zeta] \wedge \zeta$$

by eq. (6.1.2). So in order to compute the differentials of $\zeta$ we compute the transformed connection one-form $\phi$ from eq. (6.2.6). We split $\phi$ into real and imaginary part $\phi = \alpha + i\beta$ to get

$$\alpha = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \operatorname{Re}(i \exp(i\theta)\bar{\tau}) & \frac{-1}{\sqrt{2}} \operatorname{Re}(\exp(-i\theta)\bar{\tau}) \\ \frac{1}{\sqrt{2}} \operatorname{Re}(i \exp(i\theta)\tau) & 0 & \frac{1}{2}(3\rho_1 + \rho_2)\cos(2\theta) \\ \frac{1}{\sqrt{2}} \operatorname{Re}(\exp(-i\theta)\tau) & \frac{-1}{2}(3\rho_1 + \rho_2)\cos(2\theta) & 0 \end{pmatrix} \tag{6.2.7}$$

and

$$\beta = \begin{pmatrix} -\rho_1+\rho_2 & \frac{1}{\sqrt{2}}\operatorname{Im}(i\exp(i\theta)\bar{\tau}) & \frac{-1}{\sqrt{2}}\operatorname{Im}(\exp(-i\theta)\bar{\tau}) \\ \frac{1}{\sqrt{2}}\operatorname{Im}(i\exp(i\theta)\tau) & \frac{1}{2}(\rho_1-\rho_2-2\mathrm{d}\theta) & \frac{1}{2}(3\rho_1+\rho_2)\sin(2\theta) \\ \frac{1}{\sqrt{2}}\operatorname{Im}(\exp(-i\theta)\tau) & \frac{1}{2}(3\rho_1+\rho_2)\sin(2\theta) & \frac{1}{2}(\rho_1-\rho_2+2\mathrm{d}\theta) \end{pmatrix}. \tag{6.2.8}$$

To furthermore obtain expressions for the differentials of $\rho_i$ and $\tau$ we use the curvature equation 6.1.10. The curvature tensor $(R+iS)\sigma\wedge\sigma$ transforms tensorially under gauge transformations yielding the explicit expressions

$$R\sigma\wedge\sigma = \begin{pmatrix} -\cos(2\theta)\sigma_2\wedge\sigma_3 & -\frac{1}{2}\cos(2\theta)\sigma_1\wedge\sigma_3 & \frac{1}{2}\cos(2\theta)\sigma_1\wedge\sigma_2 \\ -\frac{1}{2}\cos(2\theta)\sigma_1\wedge\sigma_3 & \frac{1}{2}\cos(2\theta)\sigma_2\wedge\sigma_3 & \frac{5}{4}\sin(4\theta)\sigma_2\wedge\sigma_3 \\ \frac{1}{2}\cos(2\theta)\sigma_1\wedge\sigma_2 & \frac{5}{4}\sin(4\theta)\sigma_2\wedge\sigma_3 & \frac{1}{2}\cos(2\theta)\sigma_2\wedge\sigma_3 \end{pmatrix} \tag{6.2.9}$$

$$S\sigma\wedge\sigma = \frac{1}{2}\begin{pmatrix} 0 & \sigma_1\wedge(\sigma_2-\sin(2\theta)\sigma_3) & \sigma_1\wedge(\sigma_3-\sin(2\theta)\sigma_2) \\ \sigma_1\wedge(\sin(2\theta)\sigma_3-\sigma_2) & 0 & 5\cos^2(2\theta)\sigma_2\wedge\sigma_3 \\ \sigma_1\wedge(\sin(2\theta)\sigma_2-\sigma_3) & -5\cos^2(2\theta)\sigma_2\wedge\sigma_3 & 0 \end{pmatrix}. \tag{6.2.10}$$

Finally, combining the explicit expressions of $\phi, R, S$ with eq. (6.1.7)-eq. (6.1.10) results in the following differential identities

$$\begin{aligned} \mathrm{d}\rho_1 &= \frac{3}{2}\cos(2\theta)\sigma_2\wedge\sigma_3 \\ \mathrm{d}\rho_2 &= \frac{1}{2}\cos(2\theta)\sigma_2\wedge\sigma_3 + i\tau\wedge\bar{\tau} \\ \mathrm{d}\tau &= -2i\tau\wedge\rho_2 + \frac{1}{\sqrt{2}}\sigma_1\wedge(i\sigma_2\exp(-i\theta)-\sigma_3\exp(i\theta)) \\ \mathrm{d}\sigma_1 &= (\delta_1\wedge\sigma_2+\delta_2\wedge\sigma_3)+\sigma_2\wedge\sigma_3 \\ \mathrm{d}\sigma_2 &= -\frac{1}{2}\cos(2\theta)(3\rho_1+\rho_2)\wedge\sigma_3 - \delta_1\wedge\sigma_1 - \sigma_1\wedge\sigma_3 \\ \mathrm{d}\sigma_3 &= \frac{1}{2}\cos(2\theta)(3\rho_1+\rho_2)\wedge\sigma_2 - \delta_2\wedge\sigma_1 + \sigma_1\wedge\sigma_2 \end{aligned} \tag{6.2.11}$$

with $\delta_1 = \frac{i}{2\sqrt{2}}(\exp(i\theta)\tau - \exp(-i\theta)\bar{\tau})$ and $\delta_2 = \frac{1}{2\sqrt{2}}(\exp(-i\theta)\tau + \exp(i\theta)\bar{\tau})$.

These differential identities are satisfied on any special Lagrangian in $\mathbb{CP}^3$. Conversely, a special Lagrangian submanifold can locally be reconstructed from such a solution. There is no hope to work out all solutions of eq. (6.2.11). Instead, one typically imposes additional conditions and then tries to classify all special Lagrangians satisfying the extra condition. For example, one can already see that for $\theta = 0$ the equations simplify considerably.

The structure equations have a $\mathbb{Z}_2$-symmetry coming from the involution $j$ from eq. (3.3.1). Consider the map $\Gamma\colon \mathrm{Sp}(2) \to \mathrm{Sp}(2)$ which is defined by right multiplication of the element $\operatorname{diag}(\exp(ji\pi/4), -\exp(ji\pi/4))$. Note that $\Gamma$ covers the map $j$ on $\mathbb{CP}^3$ and that

$$\Gamma^*\rho_i = -\rho_i, \quad \Gamma^*\tau = i\bar{\tau}, \quad \Gamma^*(\omega_1) = -\bar{\omega_1}, \quad \Gamma^*\omega_j = -i\bar{\omega_j} \tag{6.2.12}$$

for $i = 1, 2$ $j = 2, 3$.

**Lemma 6.2.11.** *If $L$ is a special Lagrangian with adapted frame bundle $Q_L$ then $L' = j(L)$ is special Lagrangian with angle function $\theta' = \theta \circ j$ and adapted frame bundle $\Gamma^{-1}(Q_L)$. In the structure equations eq. (6.2.11), $j(L)$ represents a solution with*

$$\rho_1' = -\rho_i, \quad \tau' = i\bar{\tau}, \quad \sigma_1' = -\sigma_1, \quad \sigma_2' = \sigma_3, \quad \sigma_3' = \sigma_2. \tag{6.2.13}$$

*If $h' \in \Gamma(S^3(T^*L'))$ denotes the fundamental cubic of $L'$ then $j^*(h) = -h'$.*

*Proof.* Note that $j$ is an isometry and $j^*\omega_{\mathcal{V}} = -\omega_{\mathcal{V}}$ and $j^*\omega_{\mathcal{H}} = -\omega_{\mathcal{H}}$ which implies that $j(L)$ is special Lagrangian with angle function $\theta' = \theta \circ j$. Furthermore, $\Gamma$ maps $\mathrm{Sp}(2)|_{L'}$ to $\mathrm{Sp}(2)|_L$ and $\Gamma'(\zeta_1, \zeta_2, \zeta_3) = (-\bar{\zeta_1}, \bar{\zeta_3}, \bar{\zeta_2})$ since $\theta' = \theta \circ j$. Together with eq. (6.2.12) this implies eq. (6.2.13) and that $\Gamma^{-1}(P_L)$ is the adapted frame bundle of $j(L)$. Finally, note that since $j$ preserves the metric on $\mathbb{CP}^3$ as well as $\operatorname{Im}\psi$ and flips the sign of $J, \omega$ and $\operatorname{Re}\psi$ it preserves the connection $\overline{\nabla}$ which implies the last statement. ☐

If $\theta \neq \pi/4$ everywhere there is a splitting $TL = E \oplus E^\perp$ where $E^\perp$ is the kernel of the projection $TL \to \mathcal{V}$. Recall that the standard complex structure $J_1$ on $\mathbb{CP}^3$ agrees with the nearly Kähler structure $J_2$ on $\mathcal{H}$ and differs by a sign on $\mathcal{V}$.

**Proposition 6.2.12.** *The distribution $E$ is invariant under the standard complex structure $J_1$ on $\mathbb{CP}^3$ if and only if $\theta \equiv 0$. In that case, $L$ is a CR submanifold for the Kähler structure on $\mathbb{CP}^3$ with $E$ being the $J_1$-invariant distribution on $L$.*

*Proof.* In each point $x \in L$ we can pick a frame $p : T_xM \to \mathbb{C}^3$ such that $TL$ is identified with $W_{\theta(x)}$, $\mathcal{V}$ with $\operatorname{span}(b_3)$ and $J_1$ with $J'$ from eq. (6.2.3). The statement follows from 6.2.3. If $\theta = 0$ then $E$ is invariant under $J_1$ and $J_1(E^\perp)$ is orthogonal to $TL$, as required. ☐

The splitting $TL = E \oplus E^\perp$ gives an ansatz for Lagrangians arising as a product $X^2 \times S^1$ such that $TX = E$. Indeed, we will give such an example for $\theta \equiv 0$ later. However, we first show that this ansatz fails when $\theta \neq 0$ and $X$ is compact. Note that

$$\omega_{\mathcal{V}} = \frac{i}{2}(\omega_3 \wedge \bar{\omega_3}) = \frac{1}{2}\cos(2\theta)\sigma_2 \wedge \sigma_3$$

and that $\mathrm{d}\omega_{\mathcal{V}}$ is a multiple of $\operatorname{Re}\psi$, which vanishes on $L$. This implies the following.

**Lemma 6.2.13.** *If $\theta \neq \pi/4$ is constant then $\frac{2}{\cos(2\theta)}\omega_{\mathcal{V}}$ defines a calibration on $L$. The fibres of $E$ are the calibrated subspaces of $\frac{2}{\cos(2\theta)}\omega_{\mathcal{V}}$.*

Since $\omega_{\mathcal{V}}$ is closed on $L$ it defines a cohomology class in $H^2(L, \mathbb{R})$. We have that when pulled back to $\mathrm{Sp}(2)$, this class vanishes since $\mathrm{d}\rho_1 = 2\omega_{\mathcal{V}} - \omega_{\mathcal{H}} = 3\omega_{\mathcal{V}}$. If $\theta$ takes values in $(0, \frac{\pi}{4})$ the structure group of $Q$ is just $\mathbb{Z}_2$, generated by $\mathrm{diag}(1, -1, -1)$ in $H$ which corresponds to $\mathrm{diag}(i, i) \in S^1 \times S^3 \subset \mathrm{Sp}(2)$. This element leaves $\rho_1$ and $\rho_2$ invariant so in particular $\rho_1$ reduces to a form on $L$ and we have shown.

**Proposition 6.2.14.** *If $\theta$ takes values in $(0, \pi/4)$ then $[\omega_{\mathcal{V}}] = 0$. In particular, in this case $L$ does not have any compact two-dimensional submanifold which is tangent to $E$.*

### 6.2.5 Lagrangians with $\theta = 0$

The structure equations simplify significantly under the assumption $\theta \equiv 0$. Indeed, plugging $\theta = 0$ into eq. (6.2.11) yields $\beta_{32} = 0$ and $\mathrm{d}\theta = 0$ implies $\beta_{22} = \beta_{33}$. As a consequence, there is a single function $f\colon L \to \mathbb{R}$ such that

$$\beta_{22} = \beta_{33} = -1/2\beta_{11} = 1/2f\sigma_1, \beta_{21} = 1/2f\sigma_2, \beta_{31} = 1/2\sigma_3.$$

The fundamental cubic is equal to $f(-x_1^3 + 3/2x_2^2x_1 + 3/2x_3^2x_1)$ which has stabiliser $\mathrm{SO}(2)$. Let $\gamma = \rho_1 + \rho_2$, then

$$\alpha_{21} = -\frac{1}{2}f\sigma_3, \quad \alpha_{31} = \frac{1}{2}f\sigma_2, \quad \alpha_{32} = -\gamma - \frac{1}{2}f\sigma_1.$$

The structure equations are then equivalent to

$$-1 + f + 2f^2 = 0, \quad d\gamma_1 = \frac{1}{2}(5 - f)\sigma_2 \wedge \sigma_3. \tag{6.2.14}$$

Hence, there are two examples of special Lagrangian submanifolds $L_f$ with $\theta = 0$, both of which are homogeneous and in particular compact. Neither of them is totally geodesic. Note that, as a subset of $\mathrm{Sp}(2)$, the adapted frame bundle is defined by the equations

$$f\sigma_1 = \rho_1 - \rho_2, \quad \tau = f\frac{\sigma_2 + i\sigma_3}{\sqrt{2}}. \tag{6.2.15}$$

They are both orbits of a Lie group $G_f$ with Lie algebra given as the span of

$$\mathfrak{m}_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \mathfrak{m}_2 = \begin{pmatrix} if/\sqrt{2} & -1 \\ 1 & -if/\sqrt{2} \end{pmatrix} \tag{6.2.16}$$

$$\mathfrak{m}_3 = \begin{pmatrix} -j & -j\frac{i}{\sqrt{2}} \\ -j\frac{i}{\sqrt{2}} & jf \end{pmatrix}, \quad \mathfrak{m}_4 = \begin{pmatrix} -ji & j\frac{1}{\sqrt{2}} \\ j\frac{1}{\sqrt{2}} & jif \end{pmatrix}. \tag{6.2.17}$$

Note that eq. (6.2.15) are invariant under the transformation eq. (6.2.13) which means that $j(L_f)$ is isometric to $L_f$. Eq. 6.2.14 shows that $f$ is in fact constant and must be equal to either $-1$ or $\frac{1}{2}$. So, there are two distinct examples of special Lagrangians with $\theta = 0$. We describe the geometry of each of them.

**Example 6.2.15.** *There is a unique special Lagrangian with $\theta = 0$ and $f = -1$. This submanifold is described by*

$$\mathrm{d}\sigma_1 = 0, \quad \mathrm{d}\sigma_2 = -\gamma \wedge \sigma_3, \quad \mathrm{d}\sigma_3 = \gamma \wedge \sigma_2, \quad \mathrm{d}\gamma = 3\sigma_2 \wedge \sigma_3.$$

*These are the structure equations of $S^1 \times S^2$ where $S^2$ carries a metric of constant curvature* 3.

Setting $f = \frac{1}{2}$ results in the equations

$$\mathrm{d}\sigma_1 = \frac{3}{2}\sigma_2 \wedge \sigma_3, \quad \mathrm{d}\sigma_2 = -(\gamma + \frac{3}{2}\sigma_1) \wedge \sigma_3, \quad \mathrm{d}\sigma_3 = (\gamma + \frac{3}{2}\sigma_1) \wedge \sigma_2, \quad \mathrm{d}\gamma = \frac{9}{4}\sigma_2 \wedge \sigma_3.$$

The isotropy subgroup of $G_{1/2}$ acting on $[1,0,0,0]$ is the intersection $G_{1/2} \cap S^1 \cap S^3$ whose Lie algebra is defined by imposing the additional equations $\sigma_1 = \sigma_2 = \sigma_3 = 0$. Note that this Lie algebra has a complement $\mathfrak{k}$ inside $\mathrm{Lie}(G_f)$ which is characterised by the equation $3/2\sigma_1 - \gamma = 0$. Since $\mathrm{d}(3/2\sigma_1 - \gamma) = 0$ the space $\mathfrak{k}$ is indeed a Lie algebra which is isomorphic to $\mathfrak{su}(2)$ and spanned by $\frac{3}{4}\mathfrak{m}_1 + \frac{1}{\sqrt{2}}\mathfrak{m}_2, \mathfrak{m}_3$ and $\mathfrak{m}_4$.

**Example 6.2.16.** *There is a unique special Lagrangian with $\theta = 0$ and $f = \frac{1}{2}$. This submanifold is described by*

$$\mathrm{d}\sigma_1 = \frac{3}{2}\sigma_2 \wedge \sigma_3, \quad \mathrm{d}\sigma_2 = -3\sigma_1 \wedge \sigma_3, \quad \sigma_3 = +3\sigma_2 \wedge \sigma_3$$

*which are the structure equations of a Berger sphere.*

### 6.2.6 Lagrangians avoiding Boundary Values

From now on we assume that $\theta$ is not constant to any of the boundary values and denote by $L^*$ the open set where $\theta$ avoids these values. On $L^*$ the frame bundle reduces to a discrete bundle. We let $\mathrm{d}\theta = t_1\sigma_1 + t_2\sigma_2 + t_3\sigma_3$ and $x = h_{221}, y =$

$h_{222}, z = h_{322}, w = h_{321}$ such that $\beta$ is entirely determined by $\theta$ and these quantities. The symmetry eq. (6.2.12) then translates into

$$t_1' = -t_1, \quad t_2' = t_3, \quad t_3 = t_2, \quad x' = x+2t_1, \quad y' = -z-2t_3, \quad z' = -y-2t_2, \quad w' = w.$$

Clearly all of these functions are constant on orbits of Lie subgroups of Sp(2), the converse is also true.

**Proposition 6.2.17.** *Any solution with $x, y, z, w$ and $\theta \in (0, \pi/4)$ constant is an orbit of a Lie group.*

*Proof.* Let $L$ be the special Lagrangian corresponding to this solution with adapted frame bundle $\hat{L}$. Then $\hat{L}$ is an integral submanifold of the EDS generated by $\eta_i, \beta - h\sigma$. By assumption, $h$ has constant coefficients which means that the equations $\eta_i = 0, \beta = h\sigma$ describe a linear subspace of $\mathfrak{sp}(2)$ and hence $\hat{L}$ is a Lie group and $\hat{L} \to L$ is a double cover of Lie groups. $\square$

In principle, we could derive a set of PDE's for $\theta, x, y, z$ and $t_i$ from the structure equations but this is not practical in full generality. However, there is a somewhat surprising homogeneous example.

**Example 6.2.18.** *Setting*

$$x = -\sqrt{2/5}, \quad y = 0, \quad z = 0, \quad w = -\frac{3}{5}\sqrt{3/2}, \quad \theta = \frac{1}{2}\arccos\left(\frac{7\sqrt{2}}{5\sqrt{5}}\right)$$

*is a solution to eq.* (6.2.11) *and hence corresponds to a unique special Lagrangian in* $\mathbb{CP}^3$. *The fundamental cubic of this example is given by*

$$\sqrt{\frac{2}{5}}(2x_1^3 - 3x_1x_2^2 - 3x_1x_3^3) - \frac{9}{5}\sqrt{6}x_1x_2x_3,$$

*whose orientation preserving symmetry group is* $\mathbb{Z}_2$ *coming from* $(x_2, x_3) \mapsto (-x_2, -x_3)$. *The Ricci curvature is diagonal in the (dual) frame* $\sigma_1, \sigma_2 + \sigma_3, \sigma_2 - \sigma_3$ *in which*

$$\mathrm{Ric} = \mathrm{diag}(-99/50, -27/50(-2+\sqrt{15}), 27/50(2+\sqrt{15})).$$

By proposition 6.2.17, this example is homogeneous. We have found examples by imposing conditions on $\theta$. All of them have non-trivial symmetries in terms of the classification in proposition 6.1.2. The structure equations eq. (6.2.11) only hold in a fixed gauge. This makes it difficult to classify special Lagrangians where the fundamental cubic has a symmetry everywhere. We do not have the gauge freedom to bring them into the standard form in proposition 6.1.2. However, this poses no problem for the totally geodesic case.

**Proposition 6.2.19.** *Up to isometries, the standard $\mathbb{RP}^3$ is the unique totally geodesic special Lagrangian in $\mathbb{CP}^3$.*

*Proof.* There is no totally geodesic Lagrangian which lies in $\theta \in [0, \frac{\pi}{4})$. This is because in that case $\beta = 0$ forces $\rho_1 = 0$ but this is a contradiction to the first equation of 6.2.11.

If $L$ is a totally geodesic Lagrangian with $\theta \equiv \frac{\pi}{4}$ then the adapted frame bundle $Q$ is a four-dimensional submanifold of $\mathrm{Sp}(2)$ on which $\eta_i$ and $\beta$ vanish. If $\theta \equiv \frac{\pi}{4}$ then $S$ vanishes on the adapted bundle $Q$ and by eq. (6.1.10) the ideal generated by $\eta_i$ and $\beta_{ij}$ is closed under differentials. By Frobenius' theorem, there is a unique maximal submanifold on which these forms vanish that passes through the identity $e \in \mathrm{Sp}(2)$. Hence, up to isometries, there is a unique totally geodesic special Lagrangian in $\mathbb{CP}^3$ with $\theta = \frac{\pi}{4}$. We have already found this example, it is the standard $\mathbb{RP}^3 \subset \mathbb{CP}^3$. $\square$

In fact if $\theta \equiv \frac{\pi}{4}$ we have

$$\beta = \begin{pmatrix} \rho_2 - \rho_1 & (\frac{1}{4} + \frac{i}{4})(\tau - i\bar{\tau}) & (-\frac{1}{4} - \frac{i}{4})(\tau - i\bar{\tau}) \\ (\frac{1}{4} + \frac{i}{4})(\tau - i\bar{\tau}) & \frac{\rho_1 - \rho_2}{2} & \frac{1}{2}(3\rho_1 + \rho_2) \\ (-\frac{1}{4} - \frac{i}{4})(\tau - i\bar{\tau}) & \frac{1}{2}(3\rho_1 + \rho_2) & \frac{\rho_1 - \rho_2}{2} \end{pmatrix}$$

so on $Q$ over $\mathbb{RP}^3$ we have that $\rho_1 = 0, \rho_2 = 0$ and $\bar{\tau} = -\tau$. The forms $\sigma_1, \sigma_2, \sigma_3$ and $\alpha_{21}$ constitute a co-frame on $Q$, which is an orbit of $\mathrm{U}(2) \subset \mathrm{Sp}(2)$.

## 6.3 Classifying $\mathrm{SU}(2)$ invariant Special Lagrangians

Instead of imposing symmetries on the fundamental cubic we impose them on the special Lagrangian itself. We have already encountered examples of homogoneous special Lagrangians.

There are examples of special Lagrangians admitting a cohomogeneity one action of $\mathrm{SU}(2)$ in both $S^6$ and $\mathbb{C}^3$. In $S^6$, there is a unique example of this type, the squashed three-sphere [Lot11b, Example 6.4]. In $\mathbb{C}^3$, the Harvey-Lawson examples [Bry06b; HL82]

$$L_c = \{(s + it)u \mid u \in S^2 \subset \mathbb{R}^3, t^3 - 3s^2 t = c^3\}$$

admit a cohomogeneity one action of $\mathrm{SO}(3)$ for $c \neq 0$.

The situation in $\mathbb{CP}^3$ is different. We show in this section that all special Lagrangians that admit an action of an $\mathrm{SU}(2)$ group of automorphism are in fact homogeneous and have already been described in the previous section. We introduce $\mathrm{SU}(2)$ moment maps to prove this classification. A similar approach is to impose $\mathbb{T}^2$-symmetry, this is the subject of section 6.4.

### 6.3.1 SU(2) Moment Maps

Assume that SU(2) acts effectively on $M$ with three-dimensional principal orbits and by nearly Kähler automorphisms. Let $\{\xi_1, \xi_2, \xi_3\}$ be a basis of $\mathfrak{su}(2)$ such that $[\xi_i, \xi_j] = -\epsilon_{ijk}\xi_k$. Denote the corresponding fundamental vector fields by $K^{\xi_i}$. The map $\xi \to K^{\xi}$ is an anti Lie algebra homomorphism. Hence, the vector fields $K^{\xi_i}$ obey the standard Pauli commutator relationships $[K^{\xi_i}, K^{\xi_j}] = \epsilon_{ijk}K^{\xi_k}$. Consider the map

$$\sigma = (\sigma_1, \sigma_2, \sigma_3) = (\omega(K^{\xi_2}, K^{\xi_3}), \omega(K^{\xi_3}, K^{\xi_1}), \omega(K^{\xi_1}, K^{\xi_2})).$$

Then $\sigma \colon M \to \mathbb{R}^3$ is an SU(2) equivariant map with respect to the action of SU(2) on $\mathbb{R}^3$ coming from the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Let furthermore

$$\mu = \operatorname{Im}\psi(K^{\xi_1}, K^{\xi_2}, K^{\xi_3})$$

which is invariant under the action of $SU(2)$. The general strategy to obtain moment-type maps is to contract Killing vector fields with the nearly Kähler forms. The following lemma shows that all such combinations are exhausted by $\sigma$ and $\mu$.

**Lemma 6.3.1.** *The form* $\operatorname{Re}\psi$ *vanishes on* $SU(2)$ *orbits, i.e.* $\operatorname{Re}\psi(K^{\xi_1}, K^{\xi_2}, K^{\xi_3}) = 0$.

*Proof.* Let $\mathcal{O}$ be a three-dimensional orbit of $SU(2)$. Since $SU(2)$ acts by isometries on $M$ we have that $\mathrm{vol}_{\mathcal{O}}$ is a $SU(2)$ invariant form on $\mathcal{O}$. The same holds for $\operatorname{Re}\psi|_{\mathcal{O}}$. So there is $\lambda \in \mathbb{R}$ such that $\operatorname{Re}\psi|_{\mathcal{O}} = \lambda \mathrm{vol}_{\mathcal{O}}$. Since $\operatorname{Re}\psi$ is exact

$$\lambda \mathrm{vol}(\mathcal{O}) = \int_{\mathcal{O}} \operatorname{Re}\psi = 0$$

i.e. $\lambda = 0$. □

Since $\psi = \operatorname{Re}\psi + i\operatorname{Im}\psi$ is non-degenerate this means that $\mu$ vanishes if and only if $K^{\xi_1}, K^{\xi_2}, K^{\xi_3}$ are linearly dependent over $\mathbb{C}$.

By Cartan's formula and the nearly Kähler structure equations we get

$$\mathrm{d}\mu = 2\sum_l \sigma_l \omega(K^{\xi_l}, \cdot) \tag{6.3.1}$$

$$\mathrm{d}\sigma_k = -\omega(K^{\xi_k}, \cdot) + 3\operatorname{Re}\psi(K^{\xi_i}, K^{\xi_j}, \cdot) \tag{6.3.2}$$

where $(i, j, k)$ is a positive permutation of $(1, 2, 3)$. The following proposition is somewhat similar to lemma 2.4.1 and means that we can identify $SU(2)$ invariant special Lagrangians orbits by the values of the maps $\sigma$ and $\mu$..

**Proposition 6.3.2.** *The orbit of a point* $x \in M$ *is special Lagrangian if and only if* $\mu(x) \neq 0$ *and* $\sigma(x) = 0$. *The set* $\mu^{-1}(0) \cap \sigma^{-1}(0)$ *is a union of fixed points of the*

*$SU(2)$ action and two-spheres on which $\omega$ vanishes. If $\chi(M) \neq 0$ then $0$ lies in the image of $\mu$. The function $\mu$ is not constant and the set of points in which* $\mathrm{d}\mu = 0$ *and $\mu \neq 0$ consists of special Lagrangian orbits.*

*Proof.* By the definition of $\sigma$, the two-form $\omega$ vanishes on the SU(2) orbit of $x$ if and only if $\sigma(x) = 0$. If $\mu(x) \neq 0$ then $K^{\xi_1,\xi_2,\xi_3}$ are linearly independent at $x$ and the orbit at $x$ is three dimensional, which implies the first statement.

If $\mu(x) = 0$ then the orbit has dimension less than three and the second statement follows from the fact that lower-dimensional SU(2) orbits must be points or two-spheres.

Eq. 6.3.1 implies that if $\mu$ is constant then $(K^{\xi_1}, K^{\xi_2}, K^{\xi_3})$ are linearly dependent everywhere which contradicts the principal orbit type being three-dimensional. If $\chi(M) \neq 0$ then any vector field $K^{\xi_i}$ must have a zero, which forces $\mu$ to vanish. Finally, consider a point $x$ in which $\mathrm{d}\mu = 0$ and $\mu \neq 0$. We want to show that $\sigma(x) = (0,0,0)$. Using the action of $SU(2)$ we can assume that $\sigma_2(x), \sigma_3(x) = 0$. Then $0 = JK^{\xi_1}\mu = -2\|K^{\xi_1}\|^2\sigma_1$. But $\mu \neq 0$ and hence $\sigma_1(x) = 0$. □

Since either the maximum or minimum of $\mu$ is not zero this implies an existence result for special Lagrangians.

**Corollary 6.3.3.** *If $M$ is compact then the $SU(2)$ action has a special Lagrangian orbit.*

If $L$ is a special Lagrangian submanifold on which a SU(2) subgroup acts then $L$ will lie in the vanishing set of $\sigma$. So we can classify all SU(2) invariant special Lagrangian submanifolds of $\mathbb{CP}^3$ by computing the vanishing set of $\sigma$ for every SU(2) subgroup of Sp(2).

**Definition 6.3.4.** Define the three SU(2) subgroups of Sp(2) as $G_{(1)} = \{1\} \times \mathrm{Sp}(1)$, $G_{(2)} = \mathrm{SU}(2)$, arising from the inclusion $\mathbb{C}^2 \subset \mathbb{H}^2$, and $G_{(3)}$ which comes from the irreducible representation of SU(2) on $S^3(\mathbb{C}^2) \cong \mathbb{C}^4$.

Any three-dimensional subgroup of Sp(2) is conjugate to one of $G_{(1)}, G_{(2)}, G_{(3)}$.

*Remark* 6.3.5. Note that SO(4) contains two SU(2) subgroups that do not stabilise a vector in $\mathbb{R}^4$. They are not conjugated to each other and, on the Lie algebra level, correspond to the splitting of $\Lambda^2(\mathbb{R}^4)$ into self-dual and anti-self-dual two forms. However, in SO(5), these two Lie algebras are conjugated to each other, for example via the element $(x_4, x_5) \mapsto (-x_4, -x_5)$. Since $\mathrm{Sp}(2) = \mathrm{Spin}(5)$ the same holds true for the corresponding SU(2) subgroups in Sp(2).

The groups $G_{(i)}$ naturally act on $S^4$ through the double cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$. The double cover is defined via constructing the following five-dimensional real representation of Sp(2). View $\mathbb{H}^2$ as the complex vector space $\mathbb{C}^4$ equipped with the quaternionic structure $j$, i.e. $j^2 = -1$. Then $\Lambda^2(\mathbb{C}^4)$ carries the real structure $j \otimes j$. The group $\mathrm{Sp}(2)$ is the stabiliser of a complex-valued two form on $\mathbb{C}^4$, so the representation of $\mathrm{Sp}(2)$ on $\Lambda^2(\mathbb{C}^4)$ splits into a trivial one-dimensional and a five dimensional component. Since the action of $\mathrm{Sp}(2)$ is also compatible with the real structure the action descends to real five dimensional representation, defining the double cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$.

**Lemma 6.3.6.** *Consider the restriction of the action of* $\mathrm{Sp}(2)$ *on* $\mathbb{R}^5$ *to the subgroups* $G_{(i)}$. *The group* $G_{(1)}$ *acts via* $\mathrm{SU}(2) \subset \mathrm{SO}(4)$, *the group* $G_{(2)}$ *via the double cover* $\mathrm{SU}(2) \to \mathrm{SO}(3)$ *leaving a plane in* $\mathbb{R}^5$ *invariant and* $G_{(3)}$ *acts irreducibly on* $\mathbb{R}^5$ *and factors through* $\mathrm{SO}(3)$.

*Proof.* We decompose the action of $G_{(i)}$ on $\Lambda^2(\mathbb{C}^4)$ into irreducible components. Let $(V_k, \rho_k)$ be the irreducible representation of $\mathrm{SU}(2)$ of dimension $k$. For odd $k$ the representation is real and factors through $\mathrm{SO}(3)$. In other words, there is a representation on a real vector space $W_k$ such that $V_k = W_k \otimes \mathbb{C}$. Let $k' = \frac{1}{2}(k-1)$ and $m' = \frac{1}{2}(m-1)$. Then $V_{k'} \otimes V_{m'}$ with $k' \geq m'$ splits into the irreducible components [Hal+03, Theorem D.1.]

$$V_{k'+m'} \oplus V_{k'+m'-1} \oplus \cdots \oplus V_{k'-m'}.$$

In particular $V_2 \otimes V_1 = V_3 \oplus V_1$ and

$$V_4 \otimes V_4 = V_7 \oplus V_5 \oplus V_3 \oplus V_1$$

which means that $S^2(V_4) = V_7 \oplus V_3$ and $\Lambda^2(V_4) = V_5 \oplus V_1$. The subgroup $G_{(1)}$ acts as $\rho_2 \oplus \rho_1 \oplus \rho_1$ on $\mathbb{C}^4$ and

$$\Lambda^2(V_2 \oplus V_1 \oplus V_1) = \Lambda^2(V_2) \oplus V_2 \otimes V_1 + V_2 \otimes V_1 \oplus V_1 \otimes V_1 = V_2 \oplus V_2 \oplus V_1 \oplus V_1.$$

So, the action on $\mathbb{R}^5$ is $V_2 \oplus W_1$, where $V_2$ is viewed as a real four-dimensional representation.

Similarly $G_{(2)}$ acts as $\rho_2 \oplus \rho_2$ on $\mathbb{C}^4$ and

$$\Lambda^2(V_2 \oplus V_2) = \Lambda^2(V_2) \oplus V_2 \otimes V_2 \oplus \Lambda^2(V_2) = V_3 \oplus V_1 \oplus V_1 \oplus V_1.$$

So $G_{(2)}$ acts on $\mathbb{R}^5$ by $W_3 \oplus W_1 \oplus W_1$. Lastly, $G_{(3)}$ acts as $\rho_4$ on $\mathbb{C}^4$ and

$$\Lambda^2(V_4) = V_5 \oplus V_1,$$

so $G_{(3)}$ acts on $\mathbb{R}^5$ by $W_5$. $\square$

To relate the group invariant examples to those found in the previous section we compute the function $\theta$ for group orbits, for which we use eq. (6.2.2). To that end it makes sense to define $\sigma_{\mathcal{V}} = (\omega_{\mathcal{V}}(K^{\xi_2}, K^{\xi_3}), \omega_{\mathcal{V}}(K^{\xi_3}, K^{\xi_1}), \omega_{\mathcal{V}}(K^{\xi_1}, K^{\xi_2}))$.

The Killing vector fields corresponding to the subgroups $G_{(i)}$ admit quite simple expressions in local coordinates. So, to express $\mu$ for $G_{(i)}$ in homogeneous coordinates we need to do so for the nearly Kähler form $\omega = \frac{i}{2}\sum_i \omega_i \wedge \bar{\omega}_i$. This is the essence of eq. (2.2.3), where the forms $\omega_i$ are pulled back to a chart in $\mathbb{CP}^3$ by a local section.

It will be challenging to compute $\mu$ for $G_{(2)}$ and $G_{(3)}$, so we first establish representation theoretic results to simplify the computations. In [GDV02], it is shown that given an irreducible finite-dimensional continuous real representation of a compact Lie group $G$, the intersection of any hyperplane and any group orbit is non-empty. The authors of [GDV02] pose the question whether the same statement holds for complex representations, in particular irreducible representations of SU(2). There is a general framework to relate this question to the existence of nowhere vanishing sections in bundles over the flag manifold $G/T$ [AD10]. The following result follows a similar strategy and gives a direct proof for $G = \mathrm{SU}(2)$.

**Lemma 6.3.7.** *Let $(V, \rho)$ be a finite dimensional unitary representation of $G = \mathrm{SU}(2)$ with all weights non-zero and $H$ be a hyperplane which is invariant under the maximal torus $\mathrm{U}(1) \subset \mathrm{SU}(2)$. Then $H$ intersects every $G$ orbit.*

*Proof.* Since $H$ is U(1) invariant there is a linear U(1) equivariant map $f\colon V \to \mathbb{C}$ such that $\ker(f) = H$. Assume that there is an $x \in V$ such that $G.x \cap H = \emptyset$. Then $s\colon g \mapsto f(gx)$ is a non-vanishing U(1) equivariant map $\mathrm{SU}(2) \to \mathbb{C}$. Restricting this map to $\mathrm{U}(1) \subset \mathrm{SU}(2)$ gives a representation $\tau$ of U(1) on $\mathbb{C}$ of weight $k \in \mathbb{Z}$.

Note that the principal bundle $\mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{U}(1) = S^2$ is the Hopf fibration and that $s$ gives rise to a nowhere vanishing section of the associated bundle $E = \mathrm{SU}(2) \times_\tau \mathbb{C}$ over $S^2$. Since the Hopf fibration has non-trivial Chern class, the complex line bundle $E$ is trivial which forces $k = 0$. This is a contradiction because $f$ restricts to an equivariant isomorphism from $H^\perp$ to $\mathbb{C}$, so $H^\perp$ is a zero-weight subspace. □

Note that in the situation above, $H$ is invariant under U(1) and the action of U(1) on $H$ splits into one-dimensional components. Then every $G$ orbit also intersects the set $H' \subset H$ where one of the $\mathbb{C}$ components is restricted to the set $\mathbb{R}_{\geq 0}$.

All the actions of $G_i$ on $\mathbb{CP}^3$ factor through an action of SU(4) on $\mathbb{C}^4$. The irreducible action $\rho_k$ of SU(2) on $S^k(\mathbb{C}^2)$ has weights $(k, k-2, \ldots, -k+2, -k)$. The action of $G_{(2)}$ on $\mathbb{CP}^3$ factors through $\rho_1 \oplus \rho_1$ on $\mathbb{C}^4$ and $G_{(3)}$ through $\rho_3$ on $\mathbb{C}^4$. In particular neither has a zero weight, so lemma 6.3.7 applies to these cases.

$G_{(1)}$

Recall $G_{(1)} = \{1\} \times \mathrm{Sp}(1)$, we compute the Killing vector fields on the chart $\mathbb{A}_0 = \{Z_0 \neq 0\}$

$$K^{\xi_1} = -\operatorname{Im}(Z_2\frac{\partial}{\partial Z_2} - Z_3\frac{\partial}{\partial Z_3}), \quad K^{\xi_2} = \operatorname{Re}(Z_3\frac{\partial}{\partial Z_2} - Z_2\frac{\partial}{\partial Z_3}), \quad K^{\xi_3} = \operatorname{Im}(Z_3\frac{\partial}{\partial Z_2} + Z_2\frac{\partial}{\partial Z_3}).$$

We contract these vector fields with the nearly Kähler forms $\omega$ and $\psi$ in homogeneous coordinates from eq. (2.2.3), which gives

$$\begin{aligned}
\mu &= -|Z|^{-6}\frac{1}{2}(|Z_0|^2 + |Z_1|^2)(|Z_2|^2 + |Z_3|^2)^2 \\
\sigma_1 &= |Z|^{-2}(|Z_3|^2 - |Z_2|^2)f \\
\sigma_2 + i\sigma_3 &= 2i|Z|^{-2}Z_2\overline{Z_3}f \\
f &= \frac{1}{4}|Z|^{-2}(-2(|Z_0|^2 + |Z_1|^2) + (|Z_2|^2 + |Z_3|^2)).
\end{aligned}$$

Hence, $\mu$ vanishes on the line of fixed points $\{Z_2 = Z_3 = 0\}$ or when $f = 0$. Note that $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is the centraliser of $G_{(1)}$ in $\mathrm{Sp}(2)$, acts with cohomogeneity one on $\mathbb{CP}^3$ and the orbits of that action are the level sets of $f$. In particular $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts transitively on $f = 0$ which means that up to isometries there is a unique special Lagrangian on which $G_{(1)}$ acts. Hence, for simplification we consider the orbit $\mathcal{O}_{11}$ at the point $P_{11} = [1, 0, \sqrt{2}, 0]$. At this point, $K^{\xi_1}$ annihilates $\omega_{\mathcal{V}}$ which means that evaluating eq. (6.2.2) at $P_{11}$ yields

$$\frac{1}{2}|\cos(2\theta)| = \|K^{\xi_1}\|\left|\frac{\omega_{\mathcal{V}}(K^{\xi_2}, K^{\xi_3})}{\nu}\right| = \frac{1}{2}.$$

Hence $\theta = 0$ and $\mathcal{O}_{11}$ is diffeomorphic to $S^3$. It is also the orbit of the larger group $S^1 \times \mathrm{Sp}(1)$.

**Lemma 6.3.8.** *The unique special Lagrangian invariant under $G_{(1)}$ is $\mathcal{O}_{11}$ which is identified with example 6.2.16.*

$G_{(2)}$

The group $G_{(2)}$ lies inside $\mathrm{U}(2) \subset \mathrm{Sp}(2)$. Let $\xi_0 = \operatorname{diag}(i, i) \in \mathfrak{sp}(2)$, which commutes with all elements in the Lie algebra of $G_{(2)}$. Again, we compute

$$\begin{aligned}
K^{\xi_0} &= 2\operatorname{Re}(iZ_1\frac{\partial}{\partial Z_1}) \qquad K^{\xi_1} = 2\operatorname{Re}(-2i(Z_1\frac{\partial}{\partial Z_1} + Z_2\frac{\partial}{\partial Z_2})), \\
K^{\xi_2} + iK^{\xi_3} &= 2(Z_3 - Z_1Z_2\frac{\partial}{\partial Z_1} - (1 + Z_2^2)\frac{\partial}{\partial Z_2} - (Z_1 + Z_2Z_3)\frac{\partial}{\partial Z_3}).
\end{aligned}$$

For $G_{(2)}$, the map $\mu$ is equal to

$$-8\frac{|Z_0Z_1 + Z_2Z_3|^2}{|Z|^2}$$

by eq. (2.2.3). We apply lemma 6.3.7 and compute $\sigma$ on the set $Z_2 = 0$ and $Z_1 = r \geq 0$, and w.l.o.g we assume $Z_0 = 1$. Then we have

$$\begin{aligned}\sigma_1 &= -2|Z|^{-4}(-1 + r^4 + 4|Z_3|^2 - |Z_3|^4)\\ \sigma_2 - i\sigma_3 &= -4i|Z|^{-4}rZ_3(-2 + r^2 + |Z_3|^2).\end{aligned}$$

The set $\mu = 0$ is a $J_1$-holomorphic quadric and hence diffeomorphic to $S^2 \times S^2$. The action of U(2) on this quadric is of cohomogeneity one. The principal orbit is $S^2 \times S^1$ and the singular orbit $S^2$.

If $\nu = 0$ then $\mu$ vanishes if $r = 0$ and $|Z_3| = \sqrt{2 + \sqrt{3}}$. Denote this point by $P_{21}$, the U(2) orbit $\mathcal{O}_{21}$ is special Lagrangian and $K^{\xi_0}$ is horizontal on $\nu^{-1}(0)$. We compute via eq. (6.2.2)

$$\frac{1}{2}|\cos(2\theta)| = \|K^{\xi_0}\|\frac{|\omega_{\mathcal{V}}(K^{\xi_2}, K^{\xi_3})|}{|\operatorname{Im}\psi(K^{\xi_0}, K^{\xi_2}, K^{\xi_3})|} = \frac{1}{2},$$

i.e. $\theta = 0$.

If $\nu \neq 0$ then $r \neq 0$ and $\mu = 0$ only occurs for $Z_3 = 0$ and $r = 1$. Denote this point by $P_{22}$ and note $\mu_{\mathcal{V}}$ vanishes on $P_{22}$. Hence, $\theta = \pi/4$ and the orbit $\mathcal{O}_{22}$ is diffeomorphic to $\mathbb{RP}^3$.

**Lemma 6.3.9.** *All special Lagrangians that admit a $G_{(2)}$ action are $\mathcal{O}_{21}, \mathcal{O}_{22}$ which corresponds to example 6.2.15 and example 6.2.9 respectively.*

$G_{(3)}$

To compute the Killing vector fields for $G_{(3)}$ we need the explicit description of $G_{(3)} \subset$ SU(4)

$$G_{(3)} = \left\{ \begin{pmatrix} a^3 & -\sqrt{3}a^2\overline{b} & \sqrt{3}a\overline{b}^2 & -\overline{b}^3 \\ \sqrt{3}a^2b & a(|a|^2 - 2|b|^2) & -\overline{b}(2|a|^2 - |b|^2) & \sqrt{3}\overline{a}\overline{b}^2 \\ \sqrt{3}ab^2 & b(2|a|^2 - |b|^2) & \overline{a}(|a|^2 - 2|b|^2) & -\sqrt{3}\overline{a}^2\overline{b} \\ b^3 & \sqrt{3}\overline{a}b^2 & \sqrt{3}\overline{a}^2b & \overline{a}^3 \end{pmatrix} \middle| \ (a, b) \in S^3 \subset \mathbb{C}^2 \right\},$$

see for example [Kaw18]. Now we can compute the Killing vector fields for $G_{(3)}$ on $\mathbb{A}_0$

$$\begin{aligned}
K^{\xi_1} =& 2\operatorname{Im}(3Z_1\frac{\partial}{\partial Z_1}+2Z_2\frac{\partial}{\partial Z_2}+Z_3\frac{\partial}{\partial Z_3})\\
K^{\xi_2} =& \operatorname{Re}(-\sqrt{3}(Z_1Z_3+Z_2)\frac{\partial}{\partial Z_1}+(\sqrt{3}Z_1-(2+\sqrt{3}Z_2)Z_3)\frac{\partial}{\partial Z_2}\\
&+(2Z_2-\sqrt{3}(1+Z_3^2))\frac{\partial}{\partial Z_3})\\
K^{\xi_3} =& \operatorname{Im}(\sqrt{3}(-Z_1Z_3+Z_2)\frac{\partial}{\partial Z_1}+(\sqrt{3}Z_1+(2-\sqrt{3}Z_2)Z_3)\frac{\partial}{\partial Z_2}\\
&+(2Z_2-\sqrt{3}(-1+Z_3^2))\frac{\partial}{\partial Z_3}).
\end{aligned}$$

Again, we apply lemma 6.3.7 and restrict ourselves to compute $\nu$ and $\mu$ for $Z_0 = 1, Z_2 = r > 0$ and $Z_3 = 0$. Let furthermore $Z_1 = \exp(i\phi)s$, then by eq. (2.2.3)

$$\begin{aligned}
\sigma_1 &= 2|Z|^{-4}\left(5r^4-4r^2s^2-16r^2-3s^4+3\right)\\
\sigma_2 &= |Z|^{-4}4rs\sin(\phi)\left(r(\sqrt{3}r-9)+\sqrt{3}\left(s^2-8\right)\right)\\
\sigma_3 &= |Z|^{-4}4rs\cos(\phi)\left(r(-\sqrt{3}r-9)-\sqrt{3}\left(s^2-8\right)\right)\\
\mu &= |Z|^{-3}8\left(4r^4\left(s^2-5\right)-12\sqrt{3}r^3s^2\cos(2\phi)+3r^2\left(s^2+4\right)-9\left(s^4+s^2\right)\right).
\end{aligned}$$

Hence, the only solutions of $\sigma = (0,0,0)$ are $(r,s) \in \{(0,1),(\sqrt{3},0),(1/\sqrt{5},0)\}$. The solutions with $r = 0$ are in the U(1) orbit of the point $P_{31} = [1,0,1,0]$. The point $[1,\sqrt{3},0,0]$ is also in the same $G_{(3)}$ orbit as $P_{31}$. So, it suffices to consider the points $P_{31}$ and $P_{32} = [1,1/\sqrt{5},0,0]$.

Note that $\mu(P_{31}) = -18$ and $\mu(P_{32}) = 200/27$ which must hence be the minimum and maximum of $\mu$ respectively. The map $\sigma_{\mathcal{V}}$ vanishes at $P_{31}$ and hence the orbit $\mathcal{O}_{31}$ satisfies $\theta = 0$ and is in fact the Chiang Lagrangian.

Furthermore, $\sigma_{\mathcal{V}}(P_{32}) = (-\frac{14}{9},0,0)$ which means that $K^{\xi_1}$ is horizontal at $P_{32}$. By eq. (6.2.2) we have

$$\frac{1}{2}|\cos(2\theta)| = \|K^{\xi_1}\|\frac{|\mu_{\mathcal{V}}|}{|\nu|} = \frac{7}{5\sqrt{10}}, \quad \theta = \frac{1}{2}\arccos(\frac{7\sqrt{2}}{5\sqrt{5}}) \approx 0.24$$

on $\mathcal{O}_{32}$.

**Lemma 6.3.10.** *All $G_{(3)}$ invariant special Lagrangians are given by the orbits $\mathcal{O}_{31}$ and $\mathcal{O}_{32}$, which correspond to example 6.2.10 and example 6.2.18 respectively.*

### 6.3.2 The Classification

Combining all results of this section results in the following theorem.

**Theorem 6.3.11.** *Every Special Lagrangian in* $\mathbb{CP}^3$ *that admits an action of a* $\mathrm{SU}(2)$ *subgroup of* $\mathrm{Sp}(2)$ *is homogeneous and one of the following orbits.*

| *Example* | *Properties* | $\theta$ | *Group orbit* | *Stabiliser group of* $C$ |
|---|---|---|---|---|
| *6.2.16* | *Berger Sphere* | *0* | $G_{(1)}$ | $\mathrm{SO}(2)$ |
| *6.2.15* | $S^1 \times S^2$ | *0* | $\mathrm{U}(2) \supset G_{(2)}$ | $\mathrm{SO}(2)$ |
| *6.2.9* | *standard* $\mathbb{RP}^3$ | $\pi/4$ | $G_{(2)}$ | $\mathrm{SO}(3)$ *(tot. geodesic)* |
| *6.2.10* | *Chiang Lagrangian* | $\pi/4$ | $G_{(3)}$ | $S_3$ |
| *6.2.18* | *distinct* Ric *e'values* | $\approx 0.24$ | $G_{(3)}$ | $\mathbb{Z}_2$ |

For the definition of the $\mathrm{SU}(2)$ subgroups $G_{(i)}$ see definition 6.3.4.

## 6.4 Further Directions

We present two further possible constructions for special Lagrangians. At the current stage, they yield incomplete examples. It is a direction for future work to refine them to constructions of new compact examples. We discuss $\mathrm{U}(2)$ moment maps in the last subsection.

### 6.4.1 Special Lagrangians with two-Torus Symmetry

As in section 2.4 we consider a general nearly Kähler manifold $M$ which admits an effective two-torus action of automorphisms. Let $\xi_0$ and $\xi_1$ be a basis of $\mathfrak{t}^2$ and denote the corresponding infinitesimal symmetries by $K^{\xi_0}$ and $K^{\xi_1}$. Recall the multi-moment map $\nu = \omega(K^{\xi_0}, K^{\xi_1})$ and

$$M^* = \{x \in M \mid \mathrm{d}_x\nu \neq 0\} = \{x \in M \mid K^{\xi_0}(x) \text{ and } K^{\xi_1}(x) \text{ are lin. independent over } \mathbb{C}\}.$$

On $M^*$, there is a standard frame given by $\{K^{\xi_1}, K^{\xi_2}, JK^{\xi_1}, JK^{\xi_2}, K^{\xi_1} \times K^{\xi_2}, J(K^{\xi_1} \times K^{\xi_2})\}$. We are interested in special Lagrangians $L$ in $M^6$ that are invariant under the torus action. If $L \subset M^*$ then invariance means that the tangent space of $L$ at every point must contain the span of $K^{\xi_1}$ and $K^{\xi_2}$. This forces $L \subset \nu^{-1}(0)$. The tangent space of $L$ also contains a vector $v$ in $\mathrm{span}(K^{\xi_1} \times K^{\xi_2}, JK^{\xi_1} \times K^{\xi_2})$. Since $\mathrm{d}\nu(K^{\xi_1} \times K^{\xi_2}) \neq 0$ on $M^* \cap \nu^{-1}(0)$ but $\mathrm{d}\nu(J(K^{\xi_1} \times K^{\xi_2})) = 0$ it follows that $v$ must be a multiple of $J(K^{\xi_1} \times K^{\xi_2})$. Since $K^{\xi_1}$ and $K^{\xi_2}$ preserve the three-form $\psi$ we have that $[K^{\xi_i}, J(K^{\xi_1} \times K^{\xi_2})]$ for $i = 1, 2$. By Frobenius' theorem, we have shown that there is an integrable special Lagrangian distribution on $\nu^{-1}(0) \cap M^*$.

**Lemma 6.4.1.** *The distribution on $M^*$ defined by* $\operatorname{span}(K^{\xi_1}, K^{\xi_2}, J(K^{\xi_1} \times K^{\xi_2}))$ *is integrable. The leaves lie in the level sets of $\nu$ and are special Lagrangian if and only if $\nu = 0$. Furthermore, any special torus invariant Lagrangian in $M^*$ is of that form.*

The integral curves of $J(K^{\xi_1} \times K^{\xi_2})$ are geodesics of the special Lagrangian. This follows from the following well-known fact.

**Proposition 6.4.2.** *The integral curves of a vector field that is orthogonal to the orbits of an isometric cohomogeneity one action are geodesics.*

The crucial question is when the leaves are compact. Consider the quotient $Z = \nu^{-1}(0)/\mathbb{T}^2$ which is smooth on the set $Z^* = \nu^{-1}(0) \cap M^*/\mathbb{T}^2$, a three-dimensional manifold. By [Rus20], the set $Z \setminus Z^*$ has the structure of a trivalent graph and is in particular of dimension one. The smooth locus $Z^*$ inherits a metric from $M$ and the vector fields $\{JK^{\xi_1}, JK^{\xi_2}, J(K^{\xi_1} \times K^{\xi_2})\}$ descend to vector fields on $Z^*$ which we will denote by $T_1^Z, T_2^Z$ and $R^Z$. They are a globally defined frame on $Z^*$.

Preimages of flow lines of $R^Z$ correspond to special Lagrangians in $M^*$ under the projection $M^* \cap \nu^{-1}(0) \to Z^*$. The special Lagrangian $L$ is compact if the flow line is a closed curve in $Z^*$. It can also be compact if the flow line converges to two points in $Z \setminus Z^*$. In the first case $L$ is diffeomorphic to $\mathbb{T}^3$ while in the second case it is diffeomorphic to $S^2 \times S^1$ or a Lens space. This follows from the classification [Neu68] and the fact the $\mathbb{T}^2$ action on $M$ does not have points with discrete stabiliser.

We establish commutator relationships between the vector fields that constitute the frame on $M^*$ using the following fact.

**Lemma 6.4.3.** *The Nijenhuis tensor of a nearly Kähler manifold is equal to a multiple of the torsion of $\overline{\nabla}$*

$$N_J(U, V) = -4T_{\overline{\nabla}}(U, V) = -4J(U \times V).$$

Lemma 6.4.3 also implies

$$\begin{aligned} [JK^{\xi_1}, JK^{\xi_2}] &= -4J(K^{\xi_1} \times K^{\xi_2}) \\ [J\xi_i, J(K^{\xi_1} \times K^{\xi_2})] &= J([J\xi_i, K^{\xi_1} \times K^{\xi_2}]). \end{aligned} \tag{6.4.1}$$

This shows in particular that the distribution spanned by $T_1^Z, T_2^Z$ is a contact distribution on $Z^*$. The form $\tilde{\alpha} = \psi^-(K^{\xi_1}, K^{\xi_2}, \cdot)$ on $\nu^{-1}(0)$ reduces a form $\alpha$ on $Z^*$ and is the corresponding contact one-form. The Reeb vector field is proportional to $R^Z$. These statements can also be shown by using [RS19, Proposition 4.2.]. As a result, we can relate torus invariant special Lagrangians to three-dimensional contact topology.

**Proposition 6.4.4.** *The nearly Kähler structure induces a contact structure on $Z^*$*

*such that (closed) Reeb orbits correspond to (compact) torus invariant special Lagrangian submanifolds.*

Since $Z^*$ is only compact if the action of $\mathbb{T}^2$ is free one cannot use the Weinstein conjecture to guarantee the existence of a closed Reeb orbit in $Z^*$.

Smooth $\mathbb{T}^2$ invariant special Lagrangians that intersect the singular set $S$ can only do so in at most two points and they cannot intersect fixed points at all since the $\mathbb{T}^2$ action is of cohomogeneity one. The following observation is useful to obtain closed integral curves from symmetries.

**Lemma 6.4.5.** *Let $\gamma$ be an involution on $Z^*$ preserving $R^Z$ whose fixed-point set has a one-dimensional connected component $C$. Then $C$ is an integral curve of $R^Z$.*

If $\mathbb{T}^2$ is a maximal torus in $G$ then the Weyl group $W(G)$ of $G$ acts on the quotient $M/\mathbb{T}^2$ and leaves the vector field $R^Z$ invariant up to a sign. The action of $W(G)$ on $\mathfrak{t}^2 \subset \mathfrak{g}$ preserves a volume element on $\mathfrak{t}^2$ up to a sign. Denote by $W_+(G)$ the elements in $W(G)$ that preserve the orientation. Since the map $\mathfrak{t}^2 \otimes \mathfrak{t}^2 \to \Gamma(TM), \quad \xi_1 \otimes \xi_2 \mapsto J(K^{\xi_1} \times K^{\xi_2})$ descends to map from $\Lambda^2(\mathfrak{t}^2)$ every element in $W_+(G)$ preserves $R^Z$. The other elements in $W(G)$ flip the sign of $R^Z$. Hence, if $G$ has rank two then elements in $W_+(G)$ can be used to construct integral curves of $R^Z$ by applying lemma 6.4.5.

Another source of examples is an involution $j$ that commutes with $G$, flips the sign of $J$ but preserves $g$. Then $j$ descends to an involution on $Z^*$ that preserves $R^Z$. This will be used to construct an example of a torus invariant special Lagrangian in $\mathbb{CP}^3$.

For the homogeneous examples of $M$, there is a smooth three-manifold $\hat{Z} \subset \nu^{-1}(0)$ and a finite group $K \subset G$ such that $\hat{Z}/K = Z$. Topologically, $Z$ has the structure of an orbifold in these cases. The vector field $R^Z$ vanishes at the non-smooth points. The form $\tilde{\alpha}$ restricts to a form $\hat{\alpha}$ on $\hat{Z}$ which is the pull-back of $\alpha$. The vector field $J(K^{\xi_1} \times K^{\xi_2})$ is not necessarily tangent to $\hat{Z}$ but since $\hat{Z}$ covers $Z$ one gets a vector field $\hat{R}$ from $R^Z$. Our focus is on the space $M = \mathbb{CP}^3$. For illustration, we also treat the ambient space $M = S^6$, which is generally the best understood amongst the homogeneous examples.

$M = S^6$

We follow the notation of section 5.2.1 and let $\hat{Z} = \{x_3 = 0, x_5 = 0, x_6 = 0\}$ and $(u_1, u_2, u_3, u_4) = (x_1, x_2, x_4, x_7)$. Then $\hat{Z}$ is a totally geodesic special Lagrangian submanifold and $Z$ is equal to the quotient of $\hat{Z}$ by the group $K$ consisting of real elements in a maximal torus in SU(3). The group $K$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, g_1, g_2, g_3\}$ and acts on $\hat{Z}$ as follows. For $(i, j, k)$ a permutation of $(1, 2, 3)$, the element $g_i$ swaps the signs of $u_j$ and $u_k$ and leaves $u_i$ invariant.

The Weyl group of $G_2$ is isomorphic to $D_6$. It is generated by the Weyl group of SU(3), which is isomorphic to $D_3 \subset D_6$, and an involution $\gamma$ which acts by $\gamma(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)$. Clearly, $\gamma$ lies in $\mathrm{SO}(4) \subset G_2$. The subgroup $D_3$ of the Weyl group lifts to an action on $\hat{Z}$ by permutation of $(u_1, u_2, u_3)$. The element $\gamma$ acts by $\gamma(u_1, u_2, u_3, u_4) = (u_1, u_2, -u_3, -u_4)$. The group $W_+(G_2)$ is generated by $\gamma$ and cyclic permutations of $(u_1, u_2, u_3)$.

On $\hat{Z}$, $\tilde{\alpha}$ restricts to

$$\begin{aligned}\hat{\alpha} =&(u_1u_2^2 - u_1u_3^2 - u_2u_3u_4)\mathrm{d}u_1 + (-u_1^2u_2 + u_2u_3^2 - u_1u_3u_4)\mathrm{d}u_2 \\ &+ (u_1^2u_3 - u_2^2u_3 - u_1u_2u_4)\mathrm{d}u_3 + 3u_1u_2u_3\mathrm{d}u_4\end{aligned}$$

which is invariant under $K$ and $W_+(G_2)$. Let $S_1 = \{u_1 = u_2 = 0\}$, $S_2 = \{u_2 = u_3 = 0\}$ and $S_3 = \{u_1 = u_3 = 0\} \subset \hat{Z}$. Then $\hat{\alpha}$ vanishes on $S = S_1 \cup S_2 \cup S_3$. Each of $S_i$ is a great circle of which two intersect in the points $u_4 = \pm 1$, this picture agrees with the trivalent graph described in [Rus20]. We can verify that $\hat{\alpha}$ defines a contact form on the smooth locus. For example, in the chart $u_4 = \sqrt{1 - x_1^2 - x_2^2 - x_4^2}$ we compute

$$\hat{\alpha} \wedge \mathrm{d}\hat{\alpha} = \frac{4\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \mathrm{d}u_3\,(u_1^2\ u_2^2 + u_1^2u_3^2 + u_2^2u_3^2)}{\sqrt{-u_1^2 - u_2^2 - u_3^2 + 1}}$$

which only vanishes on $S$. Furthermore, we can compute that the vector field

$$\begin{aligned}\hat{R} =&(u_1u_2^2 - u_1u_3^2 - u_2u_3u_4)\partial_{u_1} + (u_1^2(-u_2) - u_1u_3u_4 + u_2u_3^2)\partial_{u_2} \\ &+ (u_1^2u_3 - u_1u_2u_4 - u_2^2u_3)\partial_{u_3} + (3u_1u_2u_3)\partial_{u_4}\end{aligned}$$

annihilates $\mathrm{d}\hat{\alpha}$ and is tangent to $\hat{Z}$ and is hence a multiple of the Reeb vector field for $\hat{\alpha}$. The equations take the form

$$\begin{aligned}\dot{u}_1 &= u_1(u_2^2 - u_3^2) - u_2u_3u_4 \\ \dot{u}_2 &= u_2(u_3^2 - u_1^2) - u_1u_3u_4 \\ \dot{u}_3 &= u_3(u_1^2 - u_2^2) - u_1u_2u_4 \\ \dot{u}_4 &= 3u_1u_2u_3.\end{aligned} \tag{6.4.2}$$

Torus invariant special Lagrangian submanifolds in $S^6$ are in one-to-one correspondence to torus invariant conical coassociative submanifolds of $\mathbb{R}^7$ and eq. (6.4.2) reduces the ODE of [Lot07, Theorem 6.4] from $S^6$ to $S^3$.

The solutions are invariant under cyclic permutations of $(u_1, u_2, u_3)$ and to sign changes of $u_4$ and one of $(u_1, u_2, u_3)$. This is a manifestation of $W_+(G_2)$ leaving the flow equations invariant. In fact $W_+(G_2)$ has two conjugacy classes in $W(G_2)$. They are represented by the element $\gamma$ and a cyclic permutation of $u_1, u_2, u_3$. Each of these elements gives distinct examples of solutions to eq. (6.4.2) by applying 6.4.5, which we describe explicitly.

We find that the conditions $u_4 = 0$ and $u_3 = 0$ are stable. In the coordinates $u_1, u_2$ there are four stationary points $P^1 = (1,0), P^2 = (0,1), P^3 = (-1,0), P^4 = (0,-1)$ yielding four solutions which connect the points $P^i$ and $P^{i+1}$. Each of the integral curves lies in the same orbit of $K$, hence only giving rise to one integral curve in $Z$. The corresponding $\mathbb{T}^2$ invariant special Lagrangian submanifold is the totally geodesic three sphere given by $x_3, x_4, x_7 = 0$.

Two other solutions are of the form $u_1 = u_2 = u_3$, which is a stable condition. They connect the fixed points $u_4 = \pm 1$ with each other. Under the action of $K$ they correspond to the same integral curve. Hence, the corresponding special Lagrangian in $S^6$ is not smooth at these fixed points. We have shown that

$$\{(e^{i\vartheta}x, e^{i\varphi}x, e^{i(-\vartheta-\varphi)}x, y) \mid (x,y) \in S^1 \subset \mathbb{R}^2\} \subset S^6 \subset \mathbb{C}^3 \oplus \mathbb{R}$$

is torus invariant special Lagrangian with two non-smooth points. This example is by no means new. It yields a conical coassociative submanifold in $\mathbb{R}^7$ which is a product of the Harvey-Lawson example $M_0$ from [HL82, Theorem 3.1] with the real line. In particular, as a coassociative submanifold it can be desingularised.

$M = \mathbb{CP}^3$

As in section 4.6.2 we consider the torus action on $\mathbb{CP}^3$ coming from the standard $\mathbb{T}^2 \subset \mathrm{U}(2) \subset \mathrm{Sp}(2)$. The Weyl group of Sp(2) is the symmetry group of square, i.e. $D_4$, which is generated by the elements

$$\gamma_1 = \begin{pmatrix} 0 & -j \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} j & 0 \\ 0 & 1 \end{pmatrix}.$$

The element $\gamma_1$ generates the subgroup $W_+(\mathrm{Sp}(2))$ of positive elements in the Weyl group and acts as a rotation. In $\mathbb{CP}^3$, we can define $\hat{Z}$ to be the standard $\mathbb{RP}^3 \subset \mathbb{CP}^3$. The group $K$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $[X_0, X_1, X_2, X_3] \mapsto [-X_0, -X_1, X_2, X_3]$ and $[X_0, X_1, X_2, X_3] \mapsto [X_0, -X_1, X_2, -X_3]$. We can construct an integral curve of $R^Z$ by using lemma 6.4.5. The elements in $W_+(\mathrm{Sp}(2))$ all have discrete stabilisers. However, the map $j$ preserves $R^Z$. On $\hat{Z}$, it is given by $[X_0, X_1, X_2, X_3] \mapsto [-X_1, X_0, -X_3, X_2]$. Let

$$\hat{C} = \{[x, x, y, y] \mid x \geq 0, y \geq 0\} \subset \hat{Z}.$$

The projection $\hat{Z} \to Z$ is one-to-one on $\hat{C}$ and the image of $\hat{C}$ in $Z$ is the fixed point set of $j$ on $Z$. The set $\hat{C}$ is diffeomorphic to a real interval whose endpoint are mapped to $[1, 1, 0, 0]$ and $[0, 0, 1, 1]$ where they intersect the twistor fibres $L_0$ and $L_6$. We have shown that

$$L_T = \mathbb{T}^2\hat{C} = \{[U,\bar{U},W,\bar{W}] \mid U,W \in \mathbb{C}\} \cong \mathbb{RP}^3$$

is a $\mathbb{T}^2$ invariant special Lagrangian submanifold. The twistor fibre going through a point $[U,\bar{U},W,\bar{W}]$ is equal to $\{[UX,\bar{U}X,WY,\bar{W}Y, \mid [X,Y] \in \mathbb{CP}^1\}$. In particular the intersection of $L_T$ with a twistor fibre is either empty or diffeomorphic to $S^1$. In fact, $\pi(L_T)$ is a totally geodesic two-sphere and is isometric to example 6.2.9.

By using eq. (2.2.3) we can compute $\alpha$ in the affine chart $X_0 \neq 0$

$$\begin{aligned}\hat{\alpha} = |X|^{-6}((&2X_1X_2^2 - X_2X_3 + 3X_1^2X_2X_3 + X_2^3X_3 - 2X_1X_3^2 + X_2X_3^3)\mathrm{d}X_1 + \\ &(-X_1X_3 - X_1^3X_3 - 3X_1X_2^2X_3 + 2X_2X_3^2 - 2X_1^2X_2X_3^2 + X_1X_3^3)\mathrm{d}X_2 + \\ &(-X_1X_2 - X_1^3X_2 + X_1X_2^3 - 2X_2^2X_3 + 2X_1^2X_2^2X_3 - 3X_1X_2X_3^2)\mathrm{d}X_3).\end{aligned} \tag{6.4.3}$$

The resulting system of ODE's is

$$\begin{aligned}\dot{X}_1 =& 3X_1X_2^2 + 3X_1^3X_2^2 - 2X_2X_3 + 2X_1^4X_2X_3 + X_2^3X_3 \\ &- X_1^2X_2^3X_3 - 3X_1X_3^2 - 3X_1^3X_3^2 + X_2X_3^3 - X_1^2X_2X_3^3 \\ \dot{X}_2 =& 3X_1^2X_2^3 - X_1X_3 - X_1^3X_3 - 8X_1X_2^2X_3 + 2X_1^3X_2^2X_3 \\ &- X_1X_2^4X_3 + 3X_2X_3^2 - 6X_1^2X_2X_3^2 + 2X_1X_3^3 - X_1X_2^2X_3^3 \\ \dot{X}_3 =& - X_1X_2 - X_1^3X_2 + 2X_1X_2^3 - 3X_2^2X_3 + 6X_1^2X_2^2X_3 \\ &- 8X_1X_2X_3^2 + 2X_1^3X_2X_3^2 - X_1X_2^3X_3^2 - 3X_1^2X_3^3 - X_1X_2X_3^4.\end{aligned}$$

This system has the aforementioned solution ($X_1 = 1, X_2 = X_3$). However, the local expression for the Reeb vector field seems too involved to obtain more complete solutions from it.

### 6.4.2 Lagrangians from Twistor Lifts

The construction of proposition 6.2.6 only applies to special Lagrangians with $\theta \equiv \frac{\pi}{4}$. Any other special Lagrangian maps to a three-manifold in $S^4$ under the twistor fibration. This raises the question which three manifolds in $S^4$ admit a special Lagrangian lift in $\mathbb{CP}^3$. To construct a natural twistor lift of three-manifolds one needs additional data of a rank-two sub-bundle of the tangent bundle of the three-manifold. However, it is more natural to consider the twistor lift of a surface. Every surface on which $\omega$ vanishes can locally be thickened to a special Lagrangian in $\mathbb{CP}^3$ by proposition 6.1.3.

**Proposition 6.4.6.** *A surface $f\colon X \to S^4$ satisfies $K + K^N = 3$ if and only if its twistor lift $\varphi\colon X \to \mathbb{CP}^3$ satisfies $\varphi^*\omega = 0$.*

*Proof.* In $S^4$, we have the curvature identities $R_{1221} = 1$ and $R_{1234} = 0$. The statement

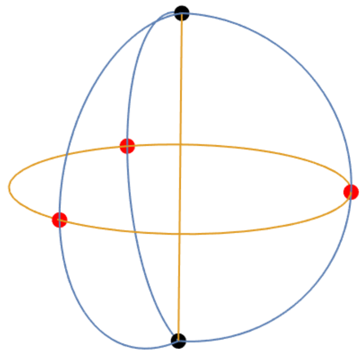

Figure 6.1: The action of $\mathbb{T}^2$ on $S^6$ has two fixed points (black) and three torus invariant $J$-holomorphic spheres, connecting both fixed points (blue). There is also a singular torus invariant special Lagrangian (orange) connecting both fixed points and three special Lagrangian three-spheres (orange) intersecting each $J$-holomorphic sphere in a circle (red). Compare with [Rus20, Figure 1].

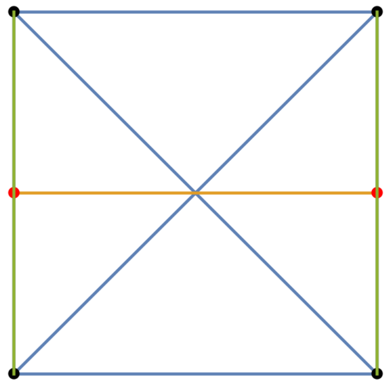

Figure 6.2: The action of $\mathbb{T}^2$ on $\mathbb{CP}^3$ has four fixed points (black). There are four superminimal torus invariant spheres (blue) and two torus invariant twistor lines (green). The standard $\mathbb{RP}^3$ is a torus invariant special Lagrangian submanifold (orange) and intersects both torus invariant twistor fibres in a circle (red). Compare with [Rus20, Figure 3].

follows from lemma 3.2.2, eq. (3.2.6) and the Gauß equation $K - 1 = |G|^2$. □

This gives a way to, at least locally, construct examples of Lagrangians in $\mathbb{CP}^3$. Start with a surface satisfying $K + K^N = 3$ in $S^4$. The twistor lift can locally be thickened to a special Lagrangian in $\mathbb{CP}^3$. However, not every special Lagrangian arises in that way, since it need not contain the twistor lift of a surface.

Note that a surface that lies in a totally geodesic $S^3 \subset S^4$ satisfies $K^N = 0$. So any surface with constant curvature $K = 3$ in $S^3$ will give an example for proposition 6.4.6. However, any complete hypersurface with constant curvature in $S^3$ is the intersection of $S^3$ with an affine hyperplane by Liebmann's theorem [Gál09]. In fact, the projection of example 6.2.15 to $S^4$ is

$$\{(v, w) \in \mathbb{R}^3 \oplus \mathbb{R}^2 \mid \|v\| = \frac{1}{\sqrt{3}}, \quad \|w\| = \frac{\sqrt{2}}{\sqrt{3}}\}$$

i.e. an $S^1$-family of two-spheres contained in a totally geodesic $S^3$ with curvature 3.

We give an example of a hypersurface in $S^3$ with constant curvature 3 but two isolated singularities. Consider the ansatz for a parametrisation

$$S^1 \times \mathbb{R} \to S^3 \subset \mathbb{C}^2, \quad (\theta, t) \mapsto (\exp(i\theta)\sqrt{1 - r^2(t)}, \exp(i\phi(t))r(t))$$

for two functions $\phi, r$ with $0 \leq r(t) \leq 1$. The induced metric on the cylinder is

$$\begin{pmatrix} 1 - r(t)^2 & 0 \\ 0 & -\frac{(r')^2}{-1+(r')^2} + (r')^2(\phi')^2 \end{pmatrix}.$$

We make the ansatz that both diagonal entries should be equal to a function $u(t)$, such that $r(t)$ and $\phi(t)$ are isothermal coordinates. The condition $K \equiv 3$ translates into the ODE

$$(6u^2 + u'')u = u'(t).$$

One solution is given by $u(t) = 1/12(1 - \tanh(t/2)^2)$. This induces the constant curvature metric from the branched $3:1$ cover $S^2 \to S^2$. In fact, it results in

$$r(t) = \sqrt{1 + \frac{1}{12}(-1 + \tanh(t/2)^2)}, \quad \phi(t) = \int_{-\infty}^{t} \sqrt{\frac{13 + 9\cosh(t')}{2(5 + 6\cosh(t'))^2}} \mathrm{d}t'. \qquad (6.4.4)$$

Both functions converge for $t \to -\infty$ and $t \to \infty$.

**Example 6.4.7.** *The functions $\phi$ and $r$ from eq.* (6.4.4) *define a* $\mathrm{U}(1)$ *invariant smooth map $S^2 \to S^3$ with two non-immersive points and constant curvature* 3. *The twistor lift is a surface with $\omega = 0$, which means it can be locally thickened to a unique special Lagrangian in $\mathbb{CP}^3$.*

We have seen in section 3.3 that the twistor lift construction might resolve isolated singularities of the surface. A direction for future work could be to investigate if this happens for example 6.4.7, if it can be extended to a complete or compact special Lagrangian and what is the function $\theta$ on this example.

### 6.4.3 U(2) Moment Maps

In the presence of a three-torus symmetry, there is an $\mathbb{R}^3 \oplus \mathbb{R}$-valued moment map $\Psi$ which encodes information about the toric geometry of $M$. For example, $\Psi$ is a homeomorphism onto $S^3 \subset \mathbb{R}^3 \oplus \mathbb{R}$ [Dix21] and locally, nearly Kähler structures with three-torus symmetry are parametrised by solutions of a Monge–Ampère-type equation on $\mathbb{R}^3$ as shown by Moroianu and Nagy [MN19]. However, of the known complete examples, only the homogeneous $S^3 \times S^3$ admits a three-torus symmetry.

In contrast, all the known complete nearly Kähler manifolds, including the inhomogeneous examples, admit a cohomogeneity-two action of $S^1 \times S^3$. Using Cartan-Kähler theory, Madnick has shown that if a Lie group $G$ acts with cohomogeneity-two on a complete nearly Kähler manifold then $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and that such structures locally depend on two functions of one variable [Mad18]. This suggests that there is a PDE on the quotient $M/G$, where we assume from now on $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, whose solutions locally parametrise $G$ invariant nearly Kähler structures, which is a possible direction for a future project.

The Lie algebra $\mathfrak{u}(2)$ splits as $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$. Let $\xi_0$ lie in $\mathfrak{u}(1)$ and $\xi_i \in \mathfrak{su}(2)$ as before, i.e. $[\xi_0, \xi_i] = 0$. Via $\mathrm{SU}(2) \subset \mathrm{U}(2)$ we get the maps $\mu$ and $\sigma$ as in section 6.3.1. In addition, we consider $\nu, \tau \colon M \to \mathbb{R}^3$ with

$$\nu_i = \omega(K^{\xi_0}, K^{\xi_i}), \quad \tau_i = \operatorname{Im}\psi(K^{\xi_0}, K^{\xi_j}, K^{\xi_k}).$$

All in all, we have a map

$$(\mu, \sigma, \nu, \tau) \colon M \to \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$$

which is invariant under $\mathrm{U}(1) \subset \mathrm{U}(2)$ and equivariant under $\mathrm{SU}(2) \subset \mathrm{U}(2)$.

*Remark* 6.4.8. The cone $C(M)$ of a nearly Kähler manifold $M$ admits a (generally singular) torsion-free $G_2$-structure which induces an $\mathrm{SU}(3)$ structure on the $\mathrm{U}(1)$ quotient of $C(M)$. For $M = \mathbb{CP}^3$, the resulting quotient has been described by Acharya, Bryant and Salamon [ABS20] by constructing a $\mathrm{U}(1)$ invariant map $Q \colon \mathbb{CP}^3 \times \mathbb{R}^{>0} \to \mathbb{R}^3 \times \mathbb{R}^3$ which is $\mathrm{SU}(2)$ equivariant. This raises the question whether $Q$ can be expressed in terms of $(\mu, \sigma, \nu, \tau)$.

The differentials of $\mu$ and $\sigma$ were computed in section 6.3.1. Similarly, we get

$$\mathrm{d}\nu_i = 3\operatorname{Re}\psi(K^{\xi_0}, K^{\xi_i}, \cdot), \quad \mathrm{d}\tau_k = 2\omega \wedge \omega(K^{\xi_0}, K^{\xi_i}, K^{\xi_j}, \cdot) - \operatorname{Im}\psi(K^{\xi_0}, K^{\xi_k}, \cdot).$$

Let $\boldsymbol{\tau} = (-\mu, \tau) \colon M \to \mathbb{R} \times \mathbb{R}^3$ and $\boldsymbol{\nu} = (0, \nu)$ and let $\boldsymbol{h}$ denote the matrix with component $\boldsymbol{h}_{ij} = g(K^{\xi_i}, K^{\xi_j})$ for $i, j = 0, \ldots, 3$ and $h$ the submatrix where $i, j = 1, \ldots, 3$ and the vector $h_0 = (\boldsymbol{h}_{0i})_{i=1,\ldots 3}$.

**Lemma 6.4.9.** *We have the following equations*

$$\begin{aligned}
\langle \sigma, \nu \rangle &= 0 \\
\nu_i &= 3\operatorname{Re}\psi(K^{\xi_0}, K^{\xi_j}, K^{\xi_k}) \\
\boldsymbol{h}\boldsymbol{\tau} &= \frac{1}{3}(|\nu|^2, -\sigma \times \nu)^T \\
\boldsymbol{h}\boldsymbol{\nu} &= 3(-\langle \nu, \tau \rangle, \sigma \times \tau - \mu\nu)^T \\
\det(h) &= \mu^2 + \sigma^T h \sigma.
\end{aligned}$$

*Proof.* It is clear that $K^{\xi_0}$ preserves $\mu$ and evaluating $0 = \mathrm{d}\mu(K^{\xi_0})$ implies the first equation. Similarly, the second equation follows from the fact that $K^{\xi_0}$ preserves $\sigma$. From now on we assume $\nu = (|\nu|, 0, 0)$. The forms $\psi^{\pm} \wedge \omega$ vanish on $M$ and the last two equation follow by plugging $(K^{\xi_0}, K^{\xi_1}, K^{\xi_2}, K^{\xi_3}, JK^{\xi_i})$ for $i = 0, \ldots, 3$ into

$\psi^{\pm} \wedge \omega$. The last equation is shown by plugging in $(\xi_1, J\xi_1, \xi_2, J\xi_2, \xi_3, J\xi_3)$ into the identity $2\omega^3 = 3\psi_+ \wedge \psi_-$. $\square$

For $\boldsymbol{v} = (v_0, v)\colon M \to \mathbb{R} \times \mathbb{R}^3$ consider the vector field $K_{\boldsymbol{v}} = \sum_{i=0}^{3} v_i K^{\xi_i}$ and let $a = \det(h) - \mu^2$ and $b = -\mu\nu^2$, $c = -\langle \sigma \times \nu, \tau \rangle$. The equations imply that

$$g(3K_{\boldsymbol{\tau}}+JK_{\boldsymbol{\nu}}, 3K_{\boldsymbol{\tau}}+JK_{\boldsymbol{\nu}}) = 9\boldsymbol{\tau}^T\boldsymbol{h}\boldsymbol{\tau}+\boldsymbol{\nu}^T\boldsymbol{h}\boldsymbol{\nu}-6\omega(K_{\boldsymbol{\tau}}, K_{\boldsymbol{\nu}}) = 3(b+c)+3(b+c)-6(b+c) = 0$$

and hence we have

$$K_{\boldsymbol{\tau}} = -\frac{1}{3}JK_{\nu}.$$

This means that $(V_1 = K^{\xi_0}, V_2 = K_\sigma, V_3 = K_\nu, V_4 = JK^{\xi_0}, V_5 = JK_\sigma, V_6 = JK_\nu)$ is a natural choice of vector fields. One computes

$$\operatorname{Re}\psi(V_1, V_2, V_3) = 0, \quad \operatorname{Im}\psi(V_1, V_2, V_3) = c$$

so $V_i$ constitute a frame on $M$ when $c \neq 0$. Using the previous equations we can now compute the differentials of the moment maps in the direction of the vector fields $V_i$.

| | $K_\sigma$ | $JK_\sigma$ | $K_\nu$ | $JK_\nu$ | $JK^{\xi_0}$ |
|---|---|---|---|---|---|
| $\mathrm{d}\sigma$ | $0$ | $-3\mu\sigma - h\sigma$ | $\sigma \times \nu$ | $3\tau \times \sigma$ | $-3\tau - h_0$ |
| $\mathrm{d}\nu$ | $\nu \times \sigma$ | $3\tau \times \sigma$ | $0$ | $3\tau \times \nu$ | $0$ |
| $\mathrm{d}\tau$ | $\tau \times \sigma$ | $\frac{1}{3}\sigma \times \nu + 4\sigma\langle\sigma, h_0\rangle + 4\nu \times (h\sigma)$ | $\tau \times \nu$ | $0$ | $4(\nu \times h_0 + h_{00}\sigma)$ |
| $\mathrm{d}\mu$ | $0$ | $2a$ | $0$ | $0$ | $2\langle h_0, \sigma\rangle$ |

By lemma 6.4.9, $\langle 3\tau - h_0, \nu\rangle = 0$ and $\nu h \sigma = 0$ which is consistent with the above computation and $\mathrm{d}\langle \nu, \sigma\rangle = 0$. We also observe that $\mathrm{d}\langle \tau, \nu\rangle = 0$. If $M$ is compact then $\nu$ has a zero and $\tau$ and $\nu$ are orthogonal everywhere.

**Proposition 6.4.10.** *The map $\nu$ is a submersion on the set $\{c \neq 0\}$. If a fibre of $\nu$ lies inside $\{c \neq 0\}$ then its connected components are covered by three-tori.*

*Proof.* If $c \neq 0$ then the vectors $\mathrm{d}\nu(K_\sigma) = \nu \times \sigma, \mathrm{d}\nu(JK_\sigma), \mathrm{d}\nu(JK_\nu)$ are linearly independent in $\mathbb{R}^3$ which means $\nu$ is a submersion. Furthermore, the vector fields $V_i$ are a frame and $K_\nu, K^{\xi_0}, JK^{\xi_0}$ are a basis for the tangent space at each fibre that lies inside $\{c \neq 0\}$. These vector fields commute. The metric defined by declaring these vector fields to be orthonormal is flat which implies the statement. $\square$

Consider a fibre $\nu^{-1}(x)$ that lies inside the set $c \neq 0$. Then the span of $K^{\xi_0}$ and $JK^{\xi_0}$ defines an integrable distribution on $\nu^{-1}(x)$. The leaves are $J$-holomorphic curves. One expects that there is a dense set of $x$ in $M$ such that the $J$-holomorphic leaves in $\nu^{-1}(x)$ are compact. *Can the compactness condition of the $J$-holomorphic leaves of $\nu^{-1}(x)$ be given in terms of the values of the moment maps?*

Just as in the case of three-torus symmetry in [MN19] one expects that the moment maps and $\boldsymbol{h}$ satisfy a PDE on the quotient $M/\mathrm{U}(2)$. The vector fields $V_i$ are a frame on an open subset of $M$. This is not an SU(3) frame but one can compute an SU(3) co-frame $\zeta_1, \zeta_2, \zeta_3$ from it, which involves coefficients of $\boldsymbol{h}$. The nearly Kähler forms $\omega$ and $\psi$ are then expressed in terms of $\zeta_i$ by eq. (6.1.4). The nearly Kähler identities are then likely to impose equations on $\boldsymbol{h}$ and the moment-maps. If successful, this approach would also yield a reconstruction process.

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