Quantitative Strongest Post

A Calculus for Reasoning about the Flow of Quantitative Information

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We present a novel **strongest-postcondition-style calculus for quantitative reasoning** about non-deterministic programs with loops. Whereas existing quantitative weakest pre allows reasoning about the value of a quantity *after* a program terminates on a given *initial state*, quantitative strongest post allows reasoning about the value that a quantity had *before* the program was executed and reached a given *final state*. We show how strongest post enables reasoning about the *flow of quantitative information through programs*.

Similarly to weakest *liberal* preconditions, we also develop a *quantitative strongest liberal post*. As a byproduct, we obtain the entirely unexplored notion of *strongest liberal postconditions* and show how these foreshadow a potential new program logic — *partial incorrectness logic* — which would be a more liberal version of O'Hearn's recent incorrectness logic.

CCS Concepts: • Theory of computation \rightarrow Logic and verification; Programming logic; Axiomatic semantics; Pre- and post-conditions; Program verification; Program analysis.

Additional Key Words and Phrases: Incorrectness Logic, Quantitative Verification, Strongest Postcondition, Weakest Precondition

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1 INTRODUCTION

Partial Correctness. Already in one of the earliest works on program verification, Turing [1949] separates reasoning about partial correctness and termination. Partial correctness means that the program is correct, *if* it terminates. Nontermination is in that sense deemed "correct" behavior. *Hoare triples* [Hoare 1969] capture partial correctness formally: Given program C and predicates G, F, we say that $\langle G \rangle C \langle F \rangle$ is *valid for partial correctness*, if from every state σ satisfying precondition G, C either terminates in some state satisfying postcondition $F, O \cap C$ does *not* terminate on σ .

A different approach to partial correctness are the *weakest liberal preconditions* of Dijkstra [1975]: Given program C and postcondition F, the weakest liberal precondition is the weakest (largest) predicate $\mathsf{wlp}[\![C]\!](F)$, such that starting from any state σ satisfying the precondition $\mathsf{wlp}[\![C]\!](F)$, C either terminates in some state satisfying the postcondition F, or C does not terminate on σ . $\mathsf{wlp}[\![C]\!](\underline{\hspace{0.5cm}})$ is a called a backward-moving *predicate transformer semantics*, because it transforms a postcondition (a predicate) F into a precondition (another predicate) $\mathsf{wlp}[\![C]\!](F)$.

A different predicate transformer semantics are the forward-moving *strongest postconditions* of Dijkstra and Scholten [1990]: they transform a precondition G into the strongest (smallest) predicate sp $\llbracket C \rrbracket$ (G), such that sp $\llbracket C \rrbracket$ (G) contains all states that can be reached by executing C on

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some state satisfying the precondition *G*. Hoare triples, weakest liberal preconditions, and strongest postconditions are strongly related by the following well-known fact:

$$\langle G \rangle C \langle F \rangle$$
 is valid for part. corr. iff $G \Longrightarrow \mathsf{wlp} \llbracket C \rrbracket (F)$ iff $\mathsf{sp} \llbracket C \rrbracket (G) \Longrightarrow F$.

Having a choice between wlp and sp is beneficial because sometimes the partial correctness proof can be easier in the, say, forward direction than in the backward direction.

Quantitative Verification. Backward-moving predicate transformers have been generalized to real-valued-function transformers, first by Kozen [1985], in order to reason about probabilistic programs, e.g. about the probability that some postcondition will be satisfied after program termination. For the forward direction, Jones [1990] presented a counterexample to the existence of probabilistic strongest postconditions. While we also cannot handle probabilistic programs, we will in this paper develop a quantitative strongest post transformer for reasoning about nondeterministic programs.

Intuitively, quantitative predicate-transformer-style calculi lift reasoning

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from predicates F: \text{States} \to \{\text{true}, \text{false}\} to quantities f: \text{States} \to \mathbb{R}^{\pm \infty},
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i.e. functions f that associate a $real\ number\ (\text{or}\ +\infty\ \text{or}\ -\infty)$ to each state. Given a post quantity f associating a number to final states, our backward-moving weakest liberal pre transformer $\mathsf{wlp}[\![C]\!]\ (f)$: States $\to \mathbb{R}^{\pm\infty}$ associates numbers to initial states, so that $\mathsf{wlp}[\![C]\!]\ (f)\ (\sigma)$ anticipates what value f will have after C terminates on σ (and $\mathsf{wlp}\ anticipates\ +\infty$ if C does not terminate on σ).

For example, what is the anticipated value of 2x after executing the assignment x := x + 1? Our quantitative weakest liberal pre calculus will push the "assertion" 2x backward through the program, obtaining the annotations on the right (read from bottom to top). Indeed, given an *initial* value $x_{\sigma} = 5$ for the program variable x, the *final* value of the expression 2x will be $2x_{\sigma} + 2 = 2 \cdot 5 + 2 = 12$.

While counterintuitive — since wlp moves backwards —, wlp acts like a weather *forecast*: Given the current state σ of the global atmosphere, a function f mapping atmosphere state to the temperature in Auckland, and an (algorithmic) description C of how the atmosphere evolves within 24 hours, wlp $[C](f)(\sigma)$ anticipates *now* what the temperature in Auckland will be *tomorrow*.

In this paper, we develop a *quantitative strongest post transformer* sp with as strong a connection (more precisely: a *Galois connection*) to quantitative wlp as in the qualitative case, namely

$$g \le \mathsf{wlp} \llbracket C \rrbracket (f)$$
 iff $\mathsf{sp} \llbracket C \rrbracket (g) \le f$.

Dually to wlp, our forward-moving strongest post transformer acts like a weather *backcast*: Given the current global atmosphere state τ , sp $\llbracket C \rrbracket$ (f) (τ) retrocipates now what the temperature in Auckland was *yesterday*. Speaking in terms of programs and quantities, given a *pre*quantity f associating a number to *initial* states, sp $\llbracket C \rrbracket$ (f): States $\to \mathbb{R}^{\pm\infty}$ associates numbers to *final* states, such that sp $\llbracket C \rrbracket$ (f) (τ) retrocipates what value f had in an initial state before C terminated in τ (and sp retrocipates $-\infty$ if τ is not reachable by executing C on some initial state).

For example, what is the retrocipated value of 2x before the assignment x := x+1? Our quantitative strongest post calculus will push the "assertion" 2x forward through the program, obtaining the annotations on the right (read from top to bottom). Indeed, given a *final* value $x_{\tau} = 5$ for the program variable x, the *initial* value of the expression 2x must have been $2x_{\tau} - 2 = 2 \cdot 5 - 2 = 8$.

Notably, our quantitative strongest post transformer provides some notion of *flow of quantitative information through the program*: If we start the above program with initial value $x_{\sigma} = 4$ for x, then we have initially $2x_{\sigma} = 2 \cdot 4 = 8$. After the execution of the program, the final value of x is $x_{\tau} = 5$. The expression 2x - 2 evaluated in x_{τ} is again $2 \cdot x_{\tau} = 2 \cdot 5 - 2 = 8$. In that sense, our quantitative sp

takes a quantity — for instance: a secret value — and propagates through the program an expression which *preserves* the value of the initial quantity. Given some final state, we can hence read off what the quantity was initially and so reason about quantitative flow and leakage of information.

Contributions and Organization

Not being our main contribution, we present in Sec. 3 quantitative wp and wlp. Differently from [Batz et al. 2018; Kaminski 2019; McIver and Morgan 2005a], our quantitative transformers act on *signed* unbounded quantities in $\mathbb{R}^{\pm\infty}$, whereas traditional probabilistic wlp act on [0, 1] and wp on $\mathbb{R}^{\infty}_{>0}$.

In Section 4, we present our *main contribution*: a novel quantitative strongest post transformer sp as described above. Moreover, we provide a quantitative strongest *liberal* post transformer slp, which gives a different value than sp to *unreachable* states (whereas wlp gives a different value than wp to *nonterminating* states). We study essential properties of all our transformers in Section 5 and show how they embed reasoning about predicates à la Dijkstra and Scholten [1990].

In Section 6, we show that slp has as tight a (Galois) connection to wp as sp to wlp, namely

$$\operatorname{wp} \left[\!\!\left[C \right]\!\!\right](f) \ \le \ g \qquad \text{iff} \qquad f \ \le \ \operatorname{slp} \left[\!\!\left[C \right]\!\!\right](g) \ .$$

When restricting to predicates, our slp transformer yields the novel notion of *strongest liberal postconditions*, which is entirely unexplored in the literature. While it is known that strongest postconditions are tightly connected with the recent *incorrectness logic* of O'Hearn [2019], we show how slp foreshadows a new program logic — *partial incorrectness logic*. We also hint at two further new program logics: one of *necessary liberal preconditions* and one of *necessary liberal postconditions*.

In Section 7, we present proof rules for loops for all four quantitative transformers. In Section 8 we demonstrate efficacy of sp and slp for reasoning about the flow of quantitative information.

2 NONDETERMINISTIC PROGRAMS

The syntax of the nondeterministic guarded command language (nGCL) à la Dijkstra is given by

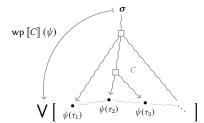
$$C := x := e \mid C \ \ C \mid \{C\} \ \ | \ \ \text{if} \ (\varphi) \ \{C\} \ \text{else} \ \{C\} \mid \ \ \text{while} \ (\varphi) \ \{C\} \ .$$

where $x \in \mathsf{Vars}$ is a variable, e is an arithmetic expression and φ is a predicate. A program state σ is a function that assigns an integer to each program variable. The set of program states is given by $\Sigma = \{ \sigma \mid \sigma : \mathsf{Vars} \to \mathbb{Z} \}$. Given a program state σ , we denote by $\sigma(\xi)$ the evaluation of an arithmetic or Boolean expression ξ in σ , i.e. the value that is obtained by evaluating ξ after replacing any occurrence of any variable x in ξ by the value $\sigma(x)$. Moreover, we denote by $\sigma[x/v]$ a new state that is obtained from σ by setting the valuation of the variable $x \in \mathsf{Vars}$ to $v \in \mathbb{Z}$. Formally: $\sigma[x/v]$ (y) = v, if y = x; and $\sigma[x/v]$ $(y) = \sigma(y)$, otherwise.

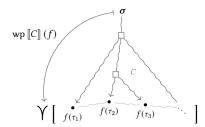
We assign meaning to our nondeterministic nGCL-statements in terms of a denotational *collecting* semantics (as is standard in program analysis, see [Cousot and Cousot 1977; Hecht 1977; Rival and Yi 2020]), i.e. we have as input a *set of initial states* and as output the *set of reachable states*.

Definition 2.1 (Collecting Semantics for nGCL Programs). Let Conf = $\mathcal{P}(\Sigma)$ be the set of program configurations, i.e. a single configuration is a set of program states; and let $[\![\varphi]\!]S = \{\sigma \mid \sigma \in S \land \sigma \models \varphi\}$ be a filtering of a program configuration to only those states where the predicate φ holds.

The collecting semantics $[\![C]\!]$: Conf \to Conf of an nGCL program C is defined inductively by



(a) **Weakest preconditions:** Given initial state σ , wp $\llbracket C \rrbracket$ (ψ) determines all final states τ_i reachable from executing C on σ , evaluates ψ in those states, and returns the disjunction (\vee) over all these truth values.



(b) **Quantitative weakest pre**: Given initial state σ , wp $\llbracket C \rrbracket$ (f) determines all final states τ_i reachable from executing C on σ , evaluates f in those states, and returns the supremum (Υ) over all these quantities.

Fig. 1. (Angelic) weakest preconditions and quantitative weakest pres.

$$[\{C_1\} \square \{C_2\}]S = [C_1]S \cup [C_2]S.$$
 (nondeterministic choice)

By slight abuse of notation, we write $[\![C]\!](\sigma)$ for $[\![C]\!](\sigma)$. For more details, see Appendix A.

3 WEAKEST PRE

We develop novel weakest (liberal) pre calculi á la Dijkstra [1975] for *quantitative reasoning* about nondeterministic programs. While we repeat that the weakest pre calculi are not our main contribution (that being the quantitative strongest post calculi), we believe that weakest pre calculi are easier to understand and provide the necessary intuition for moving from the Boolean to the quantitative realm. We first shortly recap Dijkstra's classical weakest preconditions before we lift them to a quantitative setting. Thereafter, we lift weakest *liberal* preconditions to quantities.

3.1 Classical Weakest Preconditions

Dijkstra's weakest precondition calculus employs predicate transformers of type

$$\operatorname{\sf wp}[\![C]\!]\colon \quad \mathbb{B} \ \to \ \mathbb{B} \ , \qquad \text{where} \quad \mathbb{B} \ = \ \{0,\ 1\}^\Sigma \ ,$$

which associate to each nondeterministic program C a mapping from predicates (sets of program states) to predicates. Somewhat less common, we consider here an *angelic* setting, where the nondeterminism is resolved to our advantage. Specifically, the *angelic* weakest precondition transformer wp $[\![C]\!]$ maps a *post*condition ψ over final states to a *pre*condition wp $[\![C]\!]$ (ψ) over initial states, such that executing the program C on an initial state satisfying wp $[\![C]\!]$ (ψ) guarantees that C *can*² terminate in a final state satisfying ψ . More symbolically, recalling that $[\![C]\!]$ (σ) is the set of all final states reachable after termination of C on σ ,

$$\sigma \models \mathsf{wp} \llbracket C \rrbracket (\psi) \quad \text{iff} \quad \exists \tau \in \llbracket C \rrbracket (\sigma) \colon \quad \tau \models \psi .$$

While the above is a *set perspective* on wp, an equivalent perspective on wp is a *map perspective*, see Figure 1a: The postcondition $\psi \colon \Sigma \to \{0, 1\}$ maps program states to truth values. The predicate wp $\llbracket C \rrbracket (\psi)$ is then a map that takes as input an initial state σ , determines for each reachable final

¹Considering an angelic setting allows us not only to show that our transformers enjoy several properties, but also to provide tight connections between quantitative weakest preconditions and quantitative strongest postconditions.

 $^{^{2}}$ Recall that C is a nondeterministic program. For the (standard) demonic setting as well as for deterministic programs, we can replace "can" by "will".

state $\tau \in [\![C]\!](\sigma)$ the (truth) value $\psi(\tau)$, takes a disjunction over all these truth values, and finally returns the truth value of that disjunction. More symbolically,

$$\operatorname{wp} \llbracket C \rrbracket \left(\psi \right) \left(\sigma \right) \qquad = \qquad \bigvee_{\tau \in \llbracket C \rrbracket \left(\sigma \right)} \psi(\tau) \; .$$

It is this map perspective which we will now gradually lift to a quantitative setting. For that, we first need to leave the realm of Boolean valued predicates and move to *real-valued* functions.

3.2 Quantities

For our development here, we are interested in *signed* quantities. Such quantities form — just like first-order logic for weakest preconditions — the *assertion "language"* of our quantitative calculi.

Definition 3.1 (Quantities). The set of all quantities is defined by

$$\mathbb{A} = \left\{ f \mid f \colon \Sigma \to \mathbb{R}^{\pm \infty} \right\}$$

i.e. the set of all functions $f: \Sigma \to \mathbb{R}^{\pm \infty}$ associating an *extended real* (i.e. either a proper real number, or $-\infty$, or $+\infty$) to each program state. The point-wise order

$$f \leq g$$
 iff $\forall \sigma \in \Sigma$: $f(\sigma) \leq g(\sigma)$

renders $\langle \mathbb{A}, \leq \rangle$ a complete lattice with join \wedge and meet \vee , given point-wise by

$$f \wedge g = \lambda \sigma \cdot \min\{f(\sigma), g(\sigma)\}$$
 and $f \vee g = \lambda \sigma \cdot \max\{f(\sigma), g(\sigma)\}$.

Joins and meets over arbitrary subsets exist. When we write $a \lor b \land c$, we assume that \land binds stronger than \lor , so we read that as $a \lor (b \land c)$.

Remark 3.2 (Signed Quantities). Kozen [1985] also considers signed functions for reasoning about probabilistic programs. However, Kozen's induction rule for while loops only applies to non-negative functions, see [Kozen 1985, page 168]. Kaminski and Katoen [2017] have rules for probabilistic loops and signed functions, but their machinery is quite involved and their rule for loops is more involved than simple induction. Our development in this paper is — on the plus-side — comparatively simple, but — as a trade-off — we cannot handle probabilistic programs. \triangleleft

3.3 Quantitative Weakest Pre

We now define a calculus á la Dijkstra for formal reasoning about the value of a quantity $f \in \mathbb{A}$ after execution of a nondeterministic program. For that, we generalize the *map perspective* of weakest preconditions to quantities. Instead of a post*condition*, we now have a post *quantity* $f : \Sigma \to \mathbb{R}^{\pm \infty}$ mapping (final) program states to extended reals. wp $[\![C]\!](f) : \Sigma \to \mathbb{R}^{\pm \infty}$ is then a function that takes as input an initial state σ , determines all final states τ reachable from executing C on σ , evaluates the postquantity $f(\tau)$ in each final state τ , and finally returns the supremum over all these so-determined quantities, see Figure 1b. If the program is completely *deterministic* and if C terminates on input σ , then wp $[\![C]\!](f)(\sigma)$ <u>anticipates</u> the single possible value that f will have, evaluated in the final state that is reached after executing C on σ .

One of the main advantages of Dijkstra's calculus is that the weakest preconditions can be defined by induction on the program structure, thus allowing for *compositional reasoning*. Indeed, the same applies to our quantitative setting.

Definition 3.3 (Quantitative Weakest Pre). The weakest pre transformer

wp:
$$nGCL \rightarrow (\mathbb{A} \rightarrow \mathbb{A})$$

С	$wp\left[\!\left[C\right]\!\right]\left(f ight)$	$wlp \llbracket C rbracket(f)$	
diverge	-∞	+∞	
$x \coloneqq e$	f[x/e]	f[x/e]	
$C_1 dip C_2$	$wp\left[\!\left[C_1 ight]\!\right]\left(wp\left[\!\left[C_2 ight]\!\right]\left(f ight) ight)$	$wlp\llbracket C_1 rbracket \left(wlp \llbracket C_2 rbracket \left(f \right) \right)$	
$\{C_1\} \square \{C_2\}$	$wp \llbracket C_1 \rrbracket (f) \forall wp \llbracket C_2 \rrbracket (f)$	$wlp\llbracket C_1 \rrbracket \left(f \right) \ \land \ wlp\llbracket C_2 \rrbracket \left(f \right)$	
$\text{if}\;(\varphi)\;\{C_1\}\;\text{else}\;\{C_2\}$	$[\varphi] \land wp [\![C_1]\!] (f) \forall [\neg \varphi] \land wp [\![C_2]\!] (f)$	$[\varphi] \land wlp \llbracket C_1 \rrbracket \left(f \right) \ \lor \ \llbracket \neg \varphi \rrbracket \land wlp \llbracket C_2 \rrbracket \left(f \right)$	
$\mathtt{while}(\varphi)\{C'\}$	$lfpX. \ \left[\neg\varphi\right] \land f \ \lor \ \left[\varphi\right] \land wp\left[\!\!\left[\mathcal{C}'\right]\!\!\right](X)$	$\operatorname{gfp} X \text{.} \left[\neg \varphi \right] \land f \ \lor \ \left[\varphi \right] \land \operatorname{wlp} \llbracket C' \rrbracket \left(X \right)$	

Table 1. Rules for wp and wlp. If $p \cdot \Phi(q)$ and gfp $q \cdot \Phi(q)$ denote the least and greatest fixed point of Φ .

is defined inductively according to the rules in Table 1 (middle column). We call the function

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wp} [\![C]\!](X) ,$$

whose *least* fixed point defines the weakest pre wp [while $(\varphi) \{C\}$] (f), the wp-characteristic function (of while $(\varphi) \{C\}$ with respect to f).

Let us show for some of the rules how the quantitative weakest pre semantics can be developed and understood analogously to Dijkstra's classical weakest preconditions.

Assignment. The weakest precondition of an assignment is given by

$$\text{wp} [x := e] (\psi) = \psi [x/e],$$

where $\psi\left[x/e\right]$ is the replacement of every occurrence of variable x in the postcondition ψ by the expression e. For *quantitative* weakest pre, we can do something completely analogous, except that we do not have a syntax like first-order logic for the postquantities at hand.³ Still, we can define semantically what it means to "syntactically replace" every "occurrence" of x in f by e — and with it the quantitative weakest pre of an assignment — as follows:

$$\operatorname{wp} \left[\!\!\left[x \coloneqq e\right]\!\!\right](f) \ = \ f\left[x/e\right] \ \coloneqq \ \lambda \, \sigma \boldsymbol{.} \, f\!\left(\sigma\left[x \mapsto \sigma(e)\right]\right).$$

So what is the value of f in the final state reached after executing the assignment x := e on initial state σ ? It is precisely f, but evaluated at the final state $\sigma[x \mapsto \sigma(e)]$ — the state obtained from σ by updating variable x to value $\sigma(e)$.

Nondeterministic Choice. When "executing" the nondeterministic choice $\{C_1\} \square \{C_2\}$ on some initial state σ , *either* C_1 *or* C_2 will be executed, chosen nondeterministically. Hence, the execution will reach either a final state in which executing C_1 on σ terminates or a final state in which executing C_2 on σ terminates (or no final state if both computations diverge).

Denotationally, the *angelic* weakest pre*condition* of $\{C_1\} \square \{C_2\}$ is given by

$$\operatorname{wp}\left[\!\left\{\,C_{1}\,\right\}\, \boxminus\, \left\{\,C_{2}\,\right\}\right]\!\right]\left(\psi\right) \;\; = \; \operatorname{wp}\left[\!\left[C_{1}\right]\!\right]\left(\psi\right) \;\; \vee \;\; \operatorname{wp}\left[\!\left[C_{2}\right]\!\right]\left(\psi\right) \;\; .$$

Indeed, whenever an initial state σ satisfies the precondition wp $[\![C_1]\!]$ $(\psi) \vee$ wp $[\![C_2]\!]$ (ψ) , then — either by executing C_1 or by executing C_2 — it is possible that the computation will terminate in some final state satisfying the postcondition ψ .

Quantitatively, what is the anticipated value of f after termination of either C_1 or C_2 ? Since C_1 and C_2 could both terminate but very well yield different values for f, we need to accommodate

 $[\]overline{}^3$ For probabilistic programs, an expressive and relatively complete (with respect to taking weakest preexpectations) syntax for expressing functions (expectations) of type $\Sigma \to \mathbb{R}^{\infty}_{\geq 0}$ has been presented in [Batz et al. 2021].

for two different numbers. In the maximizing spirit of *angelic* wp, we also maximize and select as *quantitative* weakest pre of $\{C_1\} \square \{C_2\}$ the *largest* possible final value of f via the meet

$$\operatorname{wp} [\![\{C_1\} \Box \{C_2\}]\!] (f) = \operatorname{wp} [\![C_1]\!] (f) \vee \operatorname{wp} [\![C_2]\!] (f)$$
.

Diverge. diverge is a shorthand for while (true) { skip } — the certainly diverging loop. Denotationally, the weakest precondition of diverge is given by

$$\text{wp} [\text{diverge}] (\psi) = \text{false}.$$

As there is *no* initial state that satisfies false, this simply tells us that there is no initial on which diverge could possibly terminate in any final state satisfying ψ .

Note that the predicate false is the *least element* in the Boolean lattice. When lifting this to a quantitative setting, we also assign the least element. Hence,

$$\operatorname{wp} [\operatorname{diverge}](f) = -\infty.$$

Another explanation goes by considering again the *angelic*, i.e. maximizing, aspect of quantitative weakest pre: What is the maximal value that we can anticipate for f after diverge has terminated? Since diverge does not terminate at all (but we are still forced to assign some "number" to this situation), the largest value that we can possibly anticipate is the absolute minimum: $-\infty$.

Remark 3.4 (Quantitative Weakest Pre and Nontermination). In some sense, $-\infty$ is the value of nontermination in quantitative wp. Note that it is more tedious to detect nontermination by standard weakest preconditions: Consider e.g. the program diverge and postcondition "x is odd". Then

$$\text{wp} [\text{diverge}] (x \text{ is odd}) = \text{false}.$$

On the other hand, for the *terminating* program $x := 2 \cdot x$, we also have

$$wp [x := 2 \cdot x] (x \text{ is odd}) = 2 \cdot x \text{ is odd} = \text{false}.$$

Thus, wp $\llbracket C \rrbracket$ (ψ) (σ) = false is not a sufficient criterion for detecting nontermination of C on σ . false merely tells us that the program *either* does not terminate σ it fails to establish the postcondition. To distinguish the two cases, one needs to check, *additionally*, whether σ terminates, i.e., whether wp $\llbracket C \rrbracket$ (true) (σ) holds.

In *our* quantitative wp calculus, given any non-infinite postquantity f our wp transformer distinguishes whether the program terminates or not *in one go*. Indeed, if $-\infty \le f \le +\infty$ and wp $[\![C]\!]$ (f) $(\sigma) = 0$, then definitely C terminates on σ and f assumes value 0 after termination of C on σ . For instance, for postquantity x we have

$$\text{wp} [\text{diverge}](x) = -\infty$$
 and $\text{wp} [x := 2 \cdot x](x) = 2 \cdot x$,

and can thus read off that the program diverge indeed does not terminate, whereas, since $x > -\infty$, we can see that $x := 2 \cdot x$ does always terminate.

Conditional Choice. When executing if (φ) $\{C_1\}$ else $\{C_2\}$ on some initial state σ , the branch C_1 is executed σ satisfies the predicate φ and otherwise C_2 is executed.

Denotationally, the weakest precondition of if (φ) { C_1 } else { C_2 } is given by

$$\mathsf{wp} \llbracket \mathsf{if} (\varphi) \{ C_1 \} \mathsf{else} \{ C_2 \} \rrbracket (\psi) = \varphi \wedge \mathsf{wp} \llbracket C_1 \rrbracket (\varphi) \vee \neg \varphi \wedge \mathsf{wp} \llbracket C_2 \rrbracket (\varphi) ,$$

where — as usual — \land binds stronger than \lor . Indeed, whenever an initial state σ satisfies the above precondition then *either* $\sigma \models \varphi$ and then — since then σ must also satisfy wp $\llbracket C_1 \rrbracket (\psi)$ — executing C_1 can terminate in a final state satisfying φ , or $\sigma \not\models \varphi$ and then — since then σ must also satisfy wp $\llbracket C_2 \rrbracket (\psi)$ — executing C_2 can terminate in a final state satisfying φ .

In order to mimic the above in a quantitative setting, we make use of so called *Iverson brackets* [Knuth 1992]. Usually, these turn a predicate φ into an indicator function $[\varphi]_{\text{std}}: \Sigma \to \{0, 1\}$,

which map a state σ to 1 or 0, depending on whether $\sigma \models \varphi$ or not. In our extended real setting, however, we need to slightly adapt the Iverson brackets as follows:

Definition 3.5 (Extended Iverson Brackets). For a predicate φ , we define the extended Iverson bracket $[\varphi]: \Sigma \to \{-\infty, +\infty\}$ by

 $[\varphi] (\sigma) = \begin{cases} +\infty & \text{if } \sigma \models \varphi \\ -\infty & \text{otherwise.} \end{cases}$

Intuitively, this choice is motivated by the fact that $-\infty$, $+\infty$ are respectively the bottom and top element of the lattice, and equipped with \vee , \wedge , they behave exactly as the boolean values true, false with \vee , \wedge . Using these Iverson brackets, we define the *quantitative* weakest pre of conditional choice by

$$\mathsf{wp} \llbracket \mathsf{if} (\varphi) \{ C_1 \} \mathsf{else} \{ C_2 \} \rrbracket (f) = \llbracket \varphi \rrbracket \land \mathsf{wp} \llbracket C_1 \rrbracket (f) \lor \llbracket \neg \varphi \rrbracket \land \mathsf{wp} \llbracket C_2 \rrbracket (f) .$$

(Recall that \land binds stronger than \lor .) If the current program state σ satisfies φ , then $[\varphi]$ evaluates to $+\infty$ — the *greatest* element of \mathbb{A} . Taking a minimum (\land) with wp $[\![C_1]\!]$ (f) will thus yield exactly wp $[\![C_1]\!]$ (f). $[\![\neg\varphi]\!]$, on the other hand, then evaluates to $-\infty$ — the *smallest* element of \mathbb{A} . Taking a minimum with any other lattice element will again yield $-\infty$. Finally, we then take a maximum (\lor) between wp $[\![C_1]\!]$ (f) and $-\infty$, yielding wp $[\![C_1]\!]$ (f). This is precisely the quantity that we would expect to anticipate for f, if $\sigma \models \varphi$, because then C_1 is executed and wp $[\![C_1]\!]$ (f) anticipates the value of f after execution of C. The situation for $\sigma \not\models \varphi$ is completely dual, yielding wp $[\![C_2]\!]$ (f). Indeed, depending on whether an initial state satisfies φ or not, the quantitative weakest pre anticipates *either* wp $[\![C_1]\!]$ (f) or wp $[\![C_2]\!]$ (f).

Remark 3.6. We note that our wp rule for conditional choice is different from e.g. [Kaminski 2019; Kozen 1985; McIver and Morgan 2005b], who use standard instead of extended Iverson brackets, multiplication instead of minimum, and summation instead of maximum, i.e.

$$\mathsf{wp}\left[\!\left[\mathsf{if}\left(\varphi\right)\left\{C_{1}\right\}\right]\mathsf{else}\left\{C_{2}\right\}\right]\!\left[f\right) = \left[\varphi\right]_{\mathsf{std}} \cdot \mathsf{wp}\left[\!\left[C_{1}\right]\!\right]\left(f\right) + \left[\neg\varphi\right]_{\mathsf{std}} \cdot \mathsf{wp}\left[\!\left[C_{2}\right]\!\right]\left(f\right)\right],$$

This rule, however, would fail in our context of signed quantities because of issues with $+\infty \cdot -\infty$.

Looping. The quantitative weakest pre of a loop while (φ) { C} is defined as a least fixed point of the wp-characteristic function $\Phi_f : \mathbb{A} \to \mathbb{A}$. This function is chosen in a way so that iterating Φ_f on the least element of the lattice $-\infty$ essentially yields an ascending chain of loop unrollings

$$\begin{split} &\Phi_f(-\infty) \;=\; \text{wp} \, \big[\![\text{if}(\varphi)\{\text{diverge}\}\big]\!] \, (f) \\ &\Phi_f^2(-\infty) \;=\; \text{wp} \, \big[\![\text{if}(\varphi)\{C\, \mathring{\circ}\, \text{if}(\varphi)\{\text{diverge}\}\}\big]\!] \, (f) \\ &\Phi_f^3(-\infty) \;=\; \text{wp} \, \big[\![\text{if}(\varphi)\{C\, \mathring{\circ}\, \text{if}(\varphi)\{C\, \mathring{\circ}\, \text{if}(\varphi)\{\text{diverge}\}\}\}\big]\!] \, (f) \end{split}$$

and so on, whose supremum is the least fixed point of Φ_f .

THEOREM 3.7 (Soundness of wp). For all programs C and initial states σ ,

$$\operatorname{wp} \left[\!\!\left[C \right]\!\!\right](f)\left(\sigma\right) \quad = \quad \bigvee_{\tau \in \left[\!\!\left[C \right]\!\!\right](\sigma)} f(\tau) \; .$$

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Intuitively, for a given postquantity f and initial state σ , wp $\llbracket C \rrbracket$ (f) (σ) is the *supremum* over all the values that f can assume measured in the final states reached after successful termination of the program C on initial state σ . In case of no terminating state, i.e. $\llbracket C \rrbracket (\sigma) = \emptyset$, that supremum becomes $-\infty$ — the absolute minimal value. In particular, if $\forall \tau \colon f(\tau) > -\infty$, then wp $\llbracket C \rrbracket$ (f) $(\sigma) = -\infty$ unambiguously indicates *nontermination* of C on input σ .

3.4 Weakest Liberal Pre

Besides weakest preconditions, Dijkstra also defines weakest *liberal* preconditions. The *weakest liberal* precondition transformer is again of type

$$\mathsf{wlp}[\![C]\!]\colon \ \mathbb{B} \to \mathbb{B}$$
,

associating to each nondeterministic program C a mapping from predicates to predicates. For reasons of duality, we now consider a *demonic* setting, where the nondeterminism is resolved to our *dis*advantage. The difference from *non*liberal weakest preconditions, however, is that *nonterminating* behavior is deemed *good behavior* (i.e. as if the program terminated in a state satisfying the post-condition). Specifically, the *demonic* weakest liberal precondition transformer $\mathsf{wlp}[\![C]\!]$ maps a *post*-condition ψ over final states to a *pre*condition $\mathsf{wlp}[\![C]\!]$ (ψ) over initial states, such that executing C on an initial state satisfying $\mathsf{wlp}[\![C]\!]$ (ψ) guarantees that C will either not terminate, or terminate in a final state satisfying ψ . More symbolically, recalling that $[\![C]\!]$ (σ) is the set of all final states reachable after termination of C on σ ,

$$\sigma \models \mathsf{wlp}[\![C]\!](\psi) \quad \text{iff} \quad \forall \tau \in [\![C]\!](\sigma) \colon \quad \tau \models \psi,$$

where the right-hand-side of the implication is vacuously true if $\llbracket C \rrbracket(\sigma) = \emptyset$, i.e. if C does not terminate on σ . From the *map perspective*, $\mathsf{wlp}\llbracket C \rrbracket(\psi)$ is a function that takes as input an initial state σ , determines for each reachable final state $\tau \in \llbracket C \rrbracket(\sigma)$ the (truth) value $\psi(\tau)$, and returns a *conjunction* over all these truth values. More symbolically,

$$\mathsf{wlp}[\![C]\!]\left(\psi\right)\left(\sigma\right) \qquad = \qquad \bigwedge_{\tau \in [\![C]\!]\left(\sigma\right)} \psi(\tau) \; ,$$

where the conjunction over an empty set is - as is standard - given by true.

Just like a conjunction in some sense minimizes truth values, our quantitative weakest liberal pre should also minimize, while at the same time assigning a maximal value to nontermination. This is captured by the following transformer:

Definition 3.8 (Quantitative Weakest Liberal Pre). The quantitative weakest liberal pre transformer

wlp:
$$nGCL \rightarrow (\mathbb{A} \rightarrow \mathbb{A})$$

is defined inductively according to the rules in Table 1 (right column). We call the function

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp} \llbracket C \rrbracket(X) ,$$

whose *greatest* fixed point defines the weakest liberal pre wlp[while $(\varphi) \{C\}$] (f), the wlp-characteristic function (of while $(\varphi) \{C\}$ with respect to f).

The rules for assignments, sequential composition, and conditional choice are the same as for wp. This is unsurprisingly so, since those rules pertain neither to nontermination nor to nondeterminism. Let us thus go over the rules for the language constructs, where the rules for wlp and wp differ.

Diverge. Since diverge is certainly nonterminating and liberal preconditions deem this good behavior, the weakest liberal precondition of diverge is given by

$$wlp[diverge](\psi) = true.$$

Note that true is the *greatest element* in the Boolean lattice. When moving to quantities, we also assign to nonterminating behavior the greatest element, i.e.

$$\mathsf{wlp}[\mathsf{diverge}](f) = +\infty$$
.

Remark 3.9 (Quantitative Weakest Liberal Pre and Nontermination). Analogously to $-\infty$ being the the value of nontermination in wp (see Remark 3.4), $+\infty$ is the value of nontermination in wlp.

Nondeterministic Choice. Since weakest *liberal* pre is *demonic*, we ask in wlp for the *minimal* anticipated value of f after termination of C_1 or C_2 . Hence the rule is dually given by the meet

$$\mathsf{wlp}[\![\{C_1\} \, \square \, \{C_2\}]\!]\,(f) \ = \ \mathsf{wlp}[\![C_1]\!]\,(f) \ \land \ \mathsf{wlp}[\![C_2]\!]\,(f) \ .$$

Notice that if either C_1 or C_2 yield $+\infty$ because of nontermination, the wlp above will select as value the respective other branch if that one terminates.

Looping. The weakest liberal pre of a loop while (φ) $\{C\}$ is defined as a greatest fixed point of the wlp-characteristic function $\Phi_f \colon \mathbb{A} \to \mathbb{A}$. This function is chosen in a way so that iterating Φ_f on the greatest element of the lattice $+\infty$ essentially yields a descending chain of loop unrollings

$$\begin{array}{ll} \Phi_f(+\infty) \; = \; \mathsf{wlp} \llbracket \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \rrbracket \, (f) \\ \Phi_f^2(+\infty) \; = \; \mathsf{wlp} \llbracket \mathsf{if}(\varphi) \{ C\, \mathring{\circ}\, \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \} \rrbracket \, (f) \\ \Phi_f^3(+\infty) \; = \; \mathsf{wlp} \llbracket \mathsf{if}(\varphi) \{ C\, \mathring{\circ}\, \mathsf{if}(\varphi) \{ C\, \mathring{\circ}\, \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \} \} \rrbracket \, (f) \end{array}$$

and so on, whose infimum is the greatest fixed point of Φ_f .

Theorem 3.10 (Soundness of wlp). For all programs C and states $\sigma \in \Sigma$,

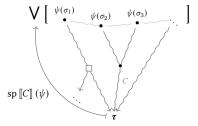
$$\mathsf{wlp}[\![C]\!](f)(\sigma) \quad = \quad \bigwedge_{\tau \in [\![C]\!](\sigma)} f(\tau) \; .$$

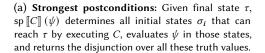
Intuitively, for a given postquantity f and initial state σ , the quantitative weakest liberal pre $\mathsf{wlp}[\![C]\!](f)(\sigma)$ is the $\mathit{infimum}$ over all values that f can assume measured in the final states after termination of the program C on initial state σ . In case of no terminating state, i.e. $[\![C]\!](\sigma) = \emptyset$, that infimum automatically becomes $+\infty$ — the absolute maximal value. In particular, if $\forall \tau \colon f(\tau) < +\infty$, then $\mathsf{wlp}[\![C]\!](f)(\sigma) = +\infty$ unambiguously indicates $\mathit{nontermination}$ of C on input σ .

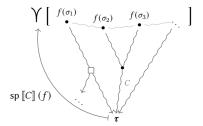
4 STRONGEST POST

We now present our main contribution: A lifting of the strongest postcondition calculus of Dijkstra and Scholten [1990] to quantities and a completely novel (quantitative) strongest *liberal* post calculus. To the best of our knowledge, a strongest liberal post(condition) has never been proposed before, not even in the *qualitative* setting. We again start by recapping the classical calculus.

⁴Although some authors do use the term "strongest liberal postcondition", see Section 9 for a comparison.







(b) **Quantitative strongest post**: Given final state τ , sp $\llbracket C \rrbracket$ (f) determines all initial states σ_i that can reach τ by executing C, evaluates f in those states, and returns the supremum (\vee) over all these quantities.

Fig. 2. Angelic strongest postconditions and quantitative strongest posts.

4.1 Classical Strongest Postconditions

Dijkstra and Scholten's strongest postcondition calculus employs predicate transformers of type

$$\operatorname{sp}[\![C]\!]: \mathbb{B} \to \mathbb{B}, \quad \text{where} \quad \mathbb{B} = \Sigma \to \{0, 1\},$$

which associate to each nondeterministic program C a mapping from predicates (sets of program states) to predicates. Strongest post transformers, analogously to the collecting semantics, characterize the set states that can be reached, so that an angelic setting is chosen to resolve nondeterminism to our advantage. Concretely, the angelic strongest postcondition transformer $\operatorname{sp}[\![C]\!]$ maps a pre-condition ψ over initial states to a postcondition $\operatorname{sp}[\![C]\!]$ (ψ) over final states, such that every state in the postcondition is reachable from some initial state satisfying ψ . This corresponds exactly with the definition of the collecting semantics $[\![C]\!]$ (σ): In fact,

$$\tau \models \operatorname{sp} \llbracket C \rrbracket (\psi) \quad \text{iff} \quad \exists \sigma \text{ with } \tau \in \llbracket C \rrbracket (\sigma) \colon \quad \sigma \models \psi .$$

As we did for weakest pre, let us provide a *map perspective* on strongest postconditions, see Figure 2a. From this perspective, the precondition $\psi \colon \Sigma \to \{0, 1\}$ maps program states to truth values. The predicate sp $\llbracket C \rrbracket (\psi)$ is then a map that takes as input a *final* state τ , determines for all initial states σ that can reach τ the (truth) value $\psi(\sigma)$, and returns the disjunction (\vee) over all these truth values:

$$\mathsf{sp} \, \llbracket C \rrbracket \, (\psi) \, (\tau) \qquad = \qquad \bigvee_{\sigma \text{ with } \tau \in \llbracket C \rrbracket \, (\sigma)} \psi (\sigma) \; .$$

In other words: Given a final state τ , sp $\llbracket C \rrbracket$ (ψ) (τ) retrodicts whether before executing C the predicate ψ could have been true. In the following, we define quantitative strongest post and strongest liberal post calculi which retrocipate values of signed quantities before the execution of a nondeterministic program (whereas wp and wlp anticipate values after the execution).

4.2 Quantitative Strongest Post

Let us generalize the *map perspective* of strongest postconditions to quantities. Instead of a pre*condition*, we now have a pre*quantity* $f: \Sigma \to \mathbb{R}^{\pm \infty}$. sp $[\![C]\!](f): \Sigma \to \mathbb{R}^{\pm \infty}$ is then a function that takes as input a *final* state τ , determines all initial states σ that can reach τ by executing C, evaluates the prequantity $f(\sigma)$ in each of those initial states σ , and finally returns the supremum over all these so-determined quantities, see Figure 2b. As a transformer, we obtain the following:

Definition 4.1 (Quantitative Strongest Post). The strongest post transformer

sp:
$$nGCL \rightarrow (\mathbb{A} \rightarrow \mathbb{A})$$

C	sp [[C]] (f)	$slp \llbracket C rbracket (f)$
diverge	-∞	+∞
$x \coloneqq e$	$a: [x = e[x/\alpha]] \land f[x/\alpha]$	$\boldsymbol{\zeta}\alpha\colon [x\neq e\ [x/\alpha]] \ \ \forall \ f\ [x/\alpha]$
$C_1 \stackrel{\circ}{\circ} C_2$	$sp \left[\!\left[C_2 \right]\!\right] \left(sp \left[\!\left[C_1 \right]\!\right] \left(f \right) \right)$	$slp\llbracket C_2 rbracket \left(slp\llbracket C_1 rbracket \left(f ight) ight)$
$\{C_1\} \square \{C_2\}$	$\operatorname{sp} \llbracket C_1 rbracket (f) \forall \operatorname{sp} \llbracket C_2 rbracket (f)$	$slp\llbracket C_1 \rrbracket \left(f \right) \ \land \ slp\llbracket C_2 \rrbracket \left(f \right)$
$\texttt{if}\;(\varphi)\;\{\mathit{C}_1\}\;\texttt{else}\;\{\mathit{C}_2\}$	$\operatorname{sp} \llbracket C_1 \rrbracket \left(\llbracket \varphi \rrbracket \land f \right) \ \lor \ \operatorname{sp} \llbracket C_2 \rrbracket \left(\llbracket \neg \varphi \rrbracket \land f \right)$	$slp\llbracket C_1 \rrbracket \left(\llbracket \neg \varphi \rrbracket \vee f \right) \ \land \ slp\llbracket C_2 \rrbracket \left(\llbracket \varphi \rrbracket \vee f \right)$
while $(\varphi) \{C'\}$	$[\neg\varphi] \mathrel{\wedge} \left(lfp \mathrel{Y} \ldotp f \mathrel{\vee} sp \left[\!\!\left[C' \right]\!\!\right] ([\varphi] \mathrel{\wedge} Y)\right)$	$[\varphi] \vee \big(gfp Y \boldsymbol{.} f \wedge slp \llbracket C' \rrbracket (\llbracket \neg \varphi \rrbracket \vee Y) \big)$

Table 2. Rules for sp and slp. Ifp $g \cdot \Psi(g)$ and gfp $g \cdot \Psi(g)$ denote the least and greatest fixed point of Φ . $\mathcal{L}\alpha$: $f(\alpha)$ and $\mathcal{L}\alpha$: $f(\alpha)$ denote the infimum and supremum of $f(\alpha)$ ranging over all values of α .

is defined inductively according to the rules in Table 2 (middle column). We call the function

$$\Psi_f(X) = f \vee \operatorname{sp} [\![C]\!] ([\varphi] \wedge X) ,$$

whose *least* fixed point is used to define sp [while $(\varphi) \{C\}$] (f), the sp-characteristic function of while $(\varphi) \{C\}$ with respect to f.

Again, let us go over some of the rules for quantitative sp and show how they can be developed and understood analogously to strongest postconditions.

Assignment. Dijkstra and Scholten's strongest postcondition of an assignment is given by

$$\operatorname{sp} \left[\!\!\left[x \coloneqq e\right]\!\!\right] (\psi) \qquad = \qquad \exists \, \alpha \colon \underbrace{\quad x = e\left[x/\alpha\right]}_{\text{(1)}} \, \wedge \, \underbrace{\psi\left[x/\alpha\right]}_{\text{(2)}}.$$

Intuitively, the quantified α represents an *initial* value that x could have had *before* executing the assignment. (If at all possible), the α is chosen in a way so that

- (1) x has in the final state the value of expression e but evaluated using x's initial value α , and
- (2) the precondition ψ was true in the initial state where x had value α .

For quantities, we note that, regarding (1), there could have been multiple valid initial values α for x; for instance, before the execution of x := 10, any initial value α is valid. Our intuition is that, in order to preserve backward compatibility, we substitute the *existential* quantifier with a *supremum* (denoted by the \mathbf{Z} "quantifier", cf. [Batz et al. 2021]), thus obtaining the supremum of $f[x/\alpha]$ ranging over all valid initial values α of x:

$$\operatorname{sp} [\![x \coloneqq e]\!] (f) = \operatorname{\mathsf{Z}} \alpha \colon [x = e [x/\alpha]] \wedge f [x/\alpha].$$

Let us consider a few examples. First, consider

$$\operatorname{sp} \llbracket x \coloneqq x+1 \rrbracket (x) = \operatorname{\mathsf{Z}} \alpha \colon \llbracket x = \alpha+1 \rrbracket \ \land \ \alpha = \operatorname{\mathsf{Z}} \alpha \colon \llbracket \alpha = x-1 \rrbracket \ \land \ \alpha = x-1 \ .$$

For a final state $\tau(x) = 10$, this gives us $\tau(x) - 1 = 10 - 1 = 9$ which is indeed *the* initial value that the prequantity x must have had if the final state after executing x := x + 1 is $\tau(x) = 10$.

As another example, consider

$$\operatorname{sp} [\![x \coloneqq 10]\!] (x) = \operatorname{\mathbf{Z}} \alpha \colon [x = 10] \wedge \alpha = [x = 10] \wedge \infty = [x = 10] .$$

For the final state $\tau(x) = 10$, this gives us $[10 = 10] = [\text{true}] = +\infty$ which is indeed the *least upper bound* (angelic!) on the initial value of x if the final state after executing x := 10 is τ . In other words: by evaluating [x = 10] in τ , we know that τ was reachable, but we have no information on what maximal value x could have had initially, which is sensible because x := 10 forgets any initial value of x. For final state $\tau'(x) = 9$, on the other hand, we get $[9 = 10] = [\text{false}] = -\infty$

which is the *value of unreachability* in sp (cf. also the next paragraph on divergence). Indeed, the final state after executing x := 10 cannot ever be τ' .

Diverge. The strongest postcondition of diverge is given by

$$sp [diverge] (\psi) = false,$$

the *least element* in the Boolean lattice. Since there is *no* state that satisfies false, this simply tells us that there is no final state reachable by executing diverge.

For quantities, we also assign the least element and hence get

$$sp [diverge](f) = -\infty$$
.

Another explanation goes by considering again the *angelic*, i.e. maximizing, aspect of strongest post: What is the maximal value that we can retrocipate for f before diverge has terminated in some final state τ ? Since diverge does not terminate at all and hence no such τ could have been reached (but we are still forced to assign some "number" to this situation), the largest value that we can possibly retrocipate is the absolute minimum: $-\infty$.

Remark 4.2 (Quantitative Strongest Post and Unreachability). Dually to values of nontermination in w(l)p (see Remarks 3.4 and 3.9), $-\infty$ is in that sense the value of unreachability in sp.

Nondeterministic Choice. The angelic strongest postcondition of $\{C_1\} \square \{C_2\}$ is given by

$$\operatorname{sp} \llbracket \{ C_1 \} \Box \{ C_2 \} \rrbracket (\psi) = \operatorname{sp} \llbracket C_1 \rrbracket (\psi) \vee \operatorname{sp} \llbracket C_2 \rrbracket (\psi) .$$

Indeed, the set of reachable states starting from initial states satisfying ψ is the union of the reachable set after executing C_1 and the ones after executing C_2 .

In a quantitative setting, where we want to retrocipate the value of a quantity f before executing either C_1 or C_2 , we *angelically* maximize between the two retrocipated quantities:

$$\operatorname{sp} [\![\{ C_1 \} \Box \{ C_2 \} \!]\!] (f) = \operatorname{sp} [\![C_1]\!] (f) \vee \operatorname{sp} [\![C_2]\!] (f)$$
.

Conditional Choice. The strongest postcondition of if (φ) $\{C_1\}$ else $\{C_2\}$ is given by

$$\mathsf{sp}\left[\!\left[\mathsf{if}\left(\varphi\right)\left\{C_{1}\right\}\right.\mathsf{else}\left\{\left.C_{2}\right\}\right]\!\right]\left(\psi\right)\right. = \left.\mathsf{sp}\left[\!\left[C_{1}\right]\!\right]\left(\varphi\wedge\psi\right)\right. \lor \left.\mathsf{sp}\left[\!\left[C_{2}\right]\!\right]\left(\neg\varphi\wedge\psi\right)\right.,$$

So to determine the set of reachable states starting from precondition ψ , we split the precondition into two disjoint ones $-\varphi \wedge \psi$ assumes that the guard is true and we execute C_1 , whereas $\neg \varphi \wedge \psi$ assumes the guard to be false and we execute C_2 . Thereafter, we union the so-obtained reachable sets.

Similarly for our quantitative strongest post calculi, we make use of the extended Iverson brackets and thus, the denotational strongest post of the conditional choice is:

$$\operatorname{sp} \llbracket \operatorname{if} (\varphi) \{ C_1 \} \operatorname{else} \{ C_2 \} \rrbracket (f) = \operatorname{sp} \llbracket C_1 \rrbracket (\llbracket \varphi \rrbracket \wedge f) \vee \operatorname{sp} \llbracket C_2 \rrbracket (\llbracket \neg \varphi \rrbracket \wedge f) .$$

Intuitively, sp $[\![C_1]\!]$ ($[\![\varphi]\!] \land f$) is the supremum of f measured in all initial states before the execution of C_1 satisfying φ ; and analogously for sp $[\![C_2]\!]$ ($[\![\neg \varphi]\!] \land f$). By then taking \lor , we finally obtain the maximum initial quantity that f could have had before the execution of the conditional choice.

Looping. The strongest post of a loop while (φ) { C } is characterized using the least fixed point of the so-called sp-characteristic function $\Psi_f \colon \mathbb{A} \to \mathbb{A}$. As for weakest pre, the function is chosen so that by Kleene's fixpoint theorem, the least fixed point corresponds to iterating on the least element of the lattice $-\infty$, which yields an ascending chain of loop unrollings

$$\begin{split} & \left[\neg \varphi \right] \land \Psi_f(-\infty) \; = \; \operatorname{sp} \left[\operatorname{if}(\varphi) \{ \operatorname{diverge} \} \right] (f) \\ & \left[\neg \varphi \right] \land \Psi_f^2(-\infty) \; = \; \operatorname{sp} \left[\operatorname{if}(\varphi) \{ C \, \mathring{\circ} \, \operatorname{if}(\varphi) \{ \operatorname{diverge} \} \} \right] (f) \\ & \left[\neg \varphi \right] \land \Psi_f^3(-\infty) \; = \; \operatorname{sp} \left[\operatorname{if}(\varphi) \{ C \, \mathring{\circ} \, \operatorname{if}(\varphi) \{ \operatorname{diverge} \} \} \right] (f) \end{split}$$

and so on, where the guard is needed to filter only those states that exit the loop; we finally obtain as strongest post

$$\operatorname{sp} \left[\operatorname{while} \left(\, \varphi \, \right) \, \left\{ \, \, C \right] \right] \, \left(f \right) \quad = \quad \left[\, \neg \varphi \right] \; \wedge \; \operatorname{lfp} \, \Psi_f \; .$$

Theorem 4.3 (Soundness of sp). For all programs C and final states τ ,

$$\operatorname{sp} \llbracket C \rrbracket \left(f \right) \left(\tau \right) \quad = \bigvee_{\sigma \text{ with } \tau \in \llbracket C \rrbracket \left(\sigma \right)} f(\sigma) \; .$$

Intuitively, for a given prequantity f and final state τ , $\operatorname{sp}[\![f]\!](\tau)$ is the *supremum* over all the values that f can assume in those initial states σ from which executing C terminates in τ . In case that the final state τ is *unreachable*, i.e. $\forall \sigma \colon \tau \notin [\![C]\!](\sigma)$, that supremum automatically becomes $-\infty$ — the absolute minimal value. In particular, if $\forall \sigma \colon f(\sigma) > -\infty$, then $\operatorname{sp}[\![C]\!](f)(\tau) = -\infty$ unambiguously indicates *unreachability* of τ by executing C on any input σ .

4.3 Quantitative Strongest Liberal Post

Although Dijkstra does not define strongest *liberal* postconditions, we believe that a reasonable choice for a quantitative strongest liberal post transformer is to take the *infimum* over all prequantities. Restricting to predicates, we thereby also obtain a novel *strongest liberal postcondition* transformer of type $slp[C]: \mathbb{B} \to \mathbb{B}$ associating to each nondeterministic program C a mapping from predicates to predicates. Since slp is associated with the infimum, we will consider a *demonic* setting, where the nondeterminism is resolved to our *dis*advantage. Whereas weakest liberal pre, in contrast to the *non*-liberal transformers, deems *non-termination* good behavior, strongest liberal post deems *unreachability* good behavior.

Specifically, the *demonic* strongest liberal postcondition transformer $slp[\![C]\!]$ maps a *pre*condition ψ over initial states to a *post*condition $slp[\![C]\!]$ (ψ) over final states, such that for a given final state τ satisfying $slp[\![C]\!]$ (ψ), all initial states that can reach τ satisfy the precondition ψ . More symbolically, recalling that $[\![C]\!]$ (σ) is the set of all final states reachable after termination of σ 0 or σ 5.

$$\tau \models \mathsf{slp}\llbracket C \rrbracket (\psi) \quad \text{iff} \quad \forall \sigma \text{ with } \tau \in \llbracket C \rrbracket (\sigma) \colon \quad \sigma \models \psi,$$

where the right-hand-side of the implication is vacuously true if τ is unreachable. From a *map* perspective on slp, the predicate $\text{slp}[\![C]\!]$ (ψ) is a function that takes as input a final state τ , determines for each initial state σ that can reach τ , i.e., $\tau \in [\![C]\!]$ (σ), the (truth) value $\psi(\sigma)$, takes a *conjunction* over all these truth values, and finally returns the truth value of that conjunction. More symbolically,

where the conjunction over an empty set is defined — as is standard — as true. For quantities, we essentially replace \land by \land and define the following quantitative strongest liberal post transformer:

Definition 4.4 (Quant. Strongest Liberal Post). The quantitative strongest liberal post transformer

slp:
$$nGCL \rightarrow (\mathbb{A} \rightarrow \mathbb{A})$$

is defined inductively according to the rules in Table 2 (right column). We call the function

$$\Psi_f(X) = f \wedge \operatorname{sp} [\![C]\!] ([\neg \varphi] \vee X) ,$$

whose *greatest* fixed point is used to define $slp[while(\varphi)\{C\}](f)$, the slp-characteristic function of $while(\varphi)\{C\}$ with respect to f.

Let us thus go over the language constructs where the rules for slp and sp differ and explain both strongest liberal postconditions and quantitative strongest liberal post.

Assignment. The strongest liberal postcondition of an assignment is given by

$$\mathsf{slp}[\![x \coloneqq e]\!] \, (\psi) \qquad = \qquad \forall \, \alpha \colon \underbrace{\quad x \neq e \, [x/\alpha] \quad}_{(1)} \, \vee \, \underbrace{\psi \, [x/\alpha]}_{(2)} \, .$$

Intuitively, the quantified α represents *candidates for initial values* of x *before* executing the assignment. For each such candidate α , it must be true that

- (1) α is in fact *not* a valid initial value for x, i.e. x does *not* have in the final state the value of expression e evaluated using the candidate value α for x, or
- (2) α is valid and the precondition ψ was true in the initial state where x had value α .

Intuitively, (1) captures that strongest liberal postconditions deem unreachability good behavior, because if some state is not reachable by executing x := e, then $x \neq e [x/\alpha]$ is true for all α and hence the strongest liberal post evaluates to true.

For quantities, dually to the strongest *non-liberal* post, we now substitute the *universal* quantifier with an *infimum* (denoted by the ζ "quantifier" [Batz et al. 2021]) and the \vee with a \vee , thus obtaining

$$slp[x := e](f) = \alpha : [x \neq e[x/\alpha]] \lor f[x/\alpha]$$

Let us again consider a few examples. First, one can convince oneself that

$$slp[x := x + 1](x) = x - 1 = sp[x := x + 1](x)$$
.

slp = sp is not surprising in this case, because *every* state $\tau(x) = \beta$ is reachable by executing x := x + 1, namely by starting from initial state $\sigma(x) = \beta - 1$. As another example, consider

$$\mathsf{slp}[\![x \coloneqq 10]\!](x) \quad = \quad \boldsymbol{\zeta} \, \alpha \colon [x \neq 10] \quad \forall \ \alpha \quad = \quad [x \neq 10] \quad \forall \ \infty \quad = \quad [x \neq 10] \ .$$

For the final state $\tau(x)=10$, this gives us $[10 \neq 10]=[{\sf false}]=-\infty$ which is indeed the *greatest lower bound* (demonic!) on the initial value of x if the final state after executing $x\coloneqq 10$ is τ . In other words: by evaluating $[x\neq 10]$ in τ , we know that τ was reachable, but we have no information on what minimal value x could have had initially, which is sensible because $x\coloneqq 10$ forgets any initial value of x. For final state $\tau'(x)=9$, on the other hand, we get $[9\neq 10]=[{\sf true}]=+\infty$ which is the *value of unreachability* in slp (cf. also the next paragraph on divergence). Indeed, the final state after executing $x\coloneqq 10$ cannot ever be τ' .

Diverge. Since diverge is certainly nonterminating, i.e. it reaches no final state, and since liberal post deems nonreachability good behavior, the quantitative strongest liberal post assigns the greatest element, i.e. $slp[diverge](f) = +\infty$.

Remark 4.5 (Quantitative Strongest Liberal Post and Unreachability). Analogously to $-\infty$ being the value of unreachability in sp (cf. Remark 4.2), $+\infty$ is the value of unreachability in slp.

Nondeterministic Choice. The *demonic* strongest liberal postcondition of $\{C_1\} \square \{C_2\}$ is

$$\mathsf{slp}\llbracket \{ \, C_1 \, \} \, \Box \, \{ \, C_2 \, \} \rrbracket \, (\psi) \ = \ \mathsf{slp}\llbracket C_1 \rrbracket \, (\psi) \ \wedge \ \mathsf{slp}\llbracket C_2 \rrbracket \, (\psi) \ .$$

Indeed, $\mathsf{slp}\llbracket C_i \rrbracket (\psi)$ contains all final states τ such that all initial states σ that can reach τ by executing C_i satisfy ψ . By intersecting $\mathsf{sp}\llbracket C_1 \rrbracket (\psi)$ and $\mathsf{sp}\llbracket C_2 \rrbracket (\psi)$ we ensure the *stronger* requirement that all initial states σ that can reach τ by executing C_1 or C_2 satisfy ψ .

In a quantitative setting, where we want to retrocipate the value of a quantity f before executing C_1 or C_2 , we *demonically* minimize the possible initial value and hence take as strongest post

$$\mathsf{slp}[\![\{\mathit{C}_1\} \ \square \ \{\mathit{C}_2\}]\!]\,(f) \ = \ \mathsf{slp}[\![\mathit{C}_1]\!]\,(f) \ \land \ \mathsf{slp}[\![\mathit{C}_2]\!]\,(f) \ .$$

Conditional Choice. The *demonic* strongest liberal postcondition of $\{C_1\} \square \{C_2\}$ is given by

$$\mathsf{slp}[\![\mathsf{if}\;(\,\varphi)\;\{\,C_1\,\}\;\mathsf{else}\;\{\,C_2\,\}]\!]\,(\psi) \;\;=\; \mathsf{slp}[\![C_1]\!]\,(\neg\varphi\vee\psi) \;\;\wedge\; \mathsf{slp}[\![C_2]\!]\,(\varphi\vee\psi) \;\;,$$

Indeed, since the disjunction can be seen as an implication, $\operatorname{slp}[\![C_1]\!]$ $(\neg \varphi \lor \psi)$ contains all final states τ such that, all initial states that satisfy φ (sic!) and that can reach τ by executing C_1 do also satisfy ψ . Similarly, $\operatorname{slp}[\![C_2]\!]$ $(\varphi \lor \psi)$ contains all final states τ such that, all initial states that satisfy $\neg \varphi$ (sic!) and that can reach τ by executing C_2 do also satisfy ψ . By intersecting the postconditions $\operatorname{slp}[\![C_1]\!]$ $(\neg \varphi \lor \psi)$ and $\operatorname{slp}[\![C_2]\!]$ $(\varphi \lor \psi)$, we obtain exactly all those final states τ such that, all initial states that, either satisfy φ and can reach τ by executing C_1 , or satisfy $\neg \varphi$ and can reach τ by executing C_2 do also satisfy the precondition ψ .

Similarly for our quantitative strongest post calculi, we make use of the *extended Iverson brackets* and thus, the quantitative strongest liberal post of the conditional choice is

$$\mathsf{slp}\llbracket\mathsf{if}\;(\,\varphi\,)\;\{\,C_1\,\}\;\mathsf{else}\;\{\,C_2\,\}\rrbracket\,(f)\;\;=\;\;\mathsf{slp}\llbracket C_1\rrbracket\,([\neg\varphi]\,\vee\,f)\;\;\wedge\;\;\mathsf{slp}\llbracket C_2\rrbracket\,([\varphi]\,\vee\,f)\;\;.$$

Intuitively, $\operatorname{slp}[\![C_1]\!]$ ($[\neg \varphi] \lor f$) characterizes the infimum of f measured in all initial states before the execution of C_1 satisfying φ ; and analogously for $\operatorname{slp}[\![C_2]\!]$ ($[\varphi] \lor f$). By taking \land , we obtain exactly the minimum initial quantity that f could have had before executing the conditional choice.

Looping. For a loop while (φ) { C }, slp is characterized using the greatest fixed point of the so-called slp-characteristic function $\Psi_f \colon \mathbb{A} \to \mathbb{A}$. As for weakest liberal pre, the function is chosen so that by Kleene's fixpoint theorem, the greatest fixed point corresponds to iterating on the top element of the lattice $+\infty$, which yields a descending chain of loop unrollings

$$\begin{split} & [\varphi] \vee \Psi_f(+\infty) \; = \; \mathsf{slp} \llbracket \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \rrbracket \, (f) \\ & [\varphi] \vee \Psi_f^2(+\infty) \; = \; \mathsf{slp} \llbracket \mathsf{if}(\varphi) \{ C \, \mathring{\circ} \, \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \} \rrbracket \, (f) \\ & [\varphi] \vee \Psi_f^3(+\infty) \; = \; \mathsf{slp} \llbracket \mathsf{if}(\varphi) \{ C \, \mathring{\circ} \, \mathsf{if}(\varphi) \{ C \, \mathring{\circ} \, \mathsf{if}(\varphi) \{ \mathsf{diverge} \} \} \} \rrbracket \, (f) \end{split}$$

and so on. Since our strongest liberal postcondition considers unreachability as "good behavior", we join the Kleene's iterates with all the final states where the guard still hold and obtain as strongest liberal post:

$$\mathsf{slp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(f) \quad = \quad [\,\varphi\,] \quad \forall \ \mathsf{gfp}\,\,\Psi_{\!f}\,\,.$$

Theorem 4.6 (Soundness of slp). For all programs C and states $\tau \in \Sigma$,

$$\mathsf{slp}\llbracket C \rrbracket \left(f \right) \left(\tau \right) \quad = \quad \quad \int_{\sigma \text{ with } \tau \in \llbracket C \rrbracket \sigma} f(\sigma)$$

Intuitively, for a given prequantity f and final state τ , the $\text{slp}[\![C]\!](f)(\tau)$ is the *infimum* over all values that f can assume measured in the initial states σ , so that executing C on σ terminates in τ . In case that the final state τ is *unreachable*, i.e. $\forall \sigma \colon \tau \notin [\![C]\!](\sigma)$, that infimum becomes

 $+\infty$ — the absolute maximum value. In particular, if $\forall \sigma \colon f(\sigma) < +\infty$, then sp $[\![C]\!]$ (f) $(\tau) = +\infty$ unambiguously indicates *unreachability* of τ by executing C on any input σ .

5 HEALTHINESS PROPERTIES OF QUANTITATIVE TRANSFORMERS

Our quantitative transformers enjoy of several so-called *healthiness properties*, some of which are analogous to Dijkstra's, Kozen's, or McIver & Morgan's calculi. We furthermore present several dualities between our transformers and how to embed classical into quantitative reasoning.

5.1 Healthiness Properties

Theorem 5.1 (Healthiness Properties of Quantitative Transformers). For all programs C, the non-liberal transformers wp[C] and sp[C] satisfy the following properties:

(1) Quantitative universal conjunctiveness: For any set of quantities $S \subseteq \mathbb{A}$,

$$\operatorname{wp} \llbracket C \rrbracket (\forall S) = \operatorname{V} \operatorname{wp} \llbracket C \rrbracket (S) \text{ and } \operatorname{sp} \llbracket C \rrbracket (\forall S) = \operatorname{V} \operatorname{sp} \llbracket C \rrbracket (S)$$
.

(2) Strictness: $\operatorname{wp} [\![C]\!] (-\infty) = -\infty$ and $\operatorname{sp} [\![C]\!] (-\infty) = -\infty$

The liberal transformers $wlp[\![C]\!]$ and $slp[\![C]\!]$ satisfy the following properties:

(3) Quantitative universal disjunctiveness: For any set of quantities $S \subseteq \mathbb{A}$,

$$\mathsf{wlp}[\![C]\!](\Lambda S) = \Lambda \; \mathsf{wlp}[\![C]\!](S) \quad \text{and} \quad \mathsf{slp}[\![C]\!](\Lambda S) = \Lambda \; \mathsf{slp}[\![C]\!](S) \; .$$

(4) Costrictness: $\operatorname{wlp}[\![C]\!](+\infty) = +\infty$ and $\operatorname{slp}[\![C]\!](+\infty) = +\infty$

All quantitive transformers are monotonic, i.e.

$$f \, \leq \, g \qquad \text{implies} \qquad \operatorname{ttt} \, \llbracket C \rrbracket \, (f) \, \, \leq \, \operatorname{ttt} \, \llbracket C \rrbracket \, (g) \, \, , \quad \text{for ttt} \in \{ \operatorname{wp, \, wlp, \, sp, \, slp} \} \, .$$

Quantitative universal conjunctiveness of wp/sp as well as disjunctiveness of wlp are quantitative analogues to Dijkstra and Scholten's original calculi, whereas disjunctiveness of slp is novel (since slp is novel) and fits well into this picture of duality. Note that quantitative universal conjunctiveness (disjunctiveness) implies ω -(co)continuity, which in turn ensures that Kleene's fixed point theorem guarantees the existence of least (greatest) fixed points for defining weakest/strongest (liberal) pre/post of loops. Monotonicity (implied by continuity) also ensures existence of fixed points but fixed point iteration may stabilize only at ordinals higher than ω for non-(co)continuous functions.

Strictness of wp, i.e. wp $[\![C]\!]$ $(-\infty) = -\infty$, says that the anticipated value of $-\infty$ after executing C is $-\infty$ if the program terminates, and otherwise yields wp's value of nontermination: $-\infty$. Strictness of sp, i.e. sp $[\![C]\!]$ $(-\infty) = -\infty$, says that $-\infty$ retrocipates the value of $-\infty$ if the final state is reachable, and otherwise yields sp's value of unreachability: $-\infty$. Explanations for costrictness are analogous.

The predicate interpretation of (co)strictness is also preserved: Since $-\infty = [false]$ and $+\infty = [true]$ and hence wp $[\![C]\!]$ ([false]) = [false] and wlp $[\![C]\!]$ ([true]) = [true], strictness of quantitative wp $[\![C]\!]$ means that C cannot terminate in some $\tau \in \emptyset$; strictness of sp $[\![C]\!]$ that no τ is reachable by executing C on any $\sigma \in \emptyset$; costrictness of wlp $[\![C]\!]$ that on all states C either terminates or not; and costrictness of slp $[\![C]\!]$ (novelly) that all states are either reachable by executing C or unreachable.

Sub- and superlinearity have been studied by Kozen, McIver & Morgan, and Kaminski for probabilistic w(I)p transformers. Our transformers similarly also obey linearity.

THEOREM 5.2 (LINEARITY). For all programs C, wp $[\![C]\!]$ and sp $[\![C]\!]$ are sublinear, and wlp $[\![C]\!]$ and slp $[\![C]\!]$ are superlinear, i.e. for all $f, g \in \mathbb{A}$ and non-negative constants $r \in \mathbb{R}_{\geq 0}$,

$$\begin{split} \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) & \leq r \cdot \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (g) \;, \\ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) & \leq r \cdot \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (g) \;, \\ r \cdot \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (g) & \leq \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) \;, \quad \text{and} \\ r \cdot \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (g) & \leq \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) \;. \end{split}$$

5.2 Relationship between Qualitative and Quantitative Transformers

Our calculi subsume both the classical ones of Dijkstra and Scholten [1990] and our definition of strongest liberal postcondition for predicates by means of our extended Iverson brackets:

Theorem 5.3 (Embedding Classical into Quantitative Transformers). For all <u>deterministic</u> programs C and predicates ψ , we have

$$\operatorname{wp} \llbracket C \rrbracket (\llbracket \psi \rrbracket) = [\operatorname{wp} \llbracket C \rrbracket (\psi)] \quad \text{and} \quad \operatorname{wlp} \llbracket C \rrbracket (\llbracket \psi \rrbracket) = [\operatorname{wlp} \llbracket C \rrbracket (\psi)],$$

and for all programs C and predicates ψ , we have

$$\operatorname{sp} \llbracket C \rrbracket \left(\llbracket \psi \rrbracket \right) \ = \ \left[\operatorname{sp} \llbracket C \rrbracket \left(\psi \right) \right] \quad \text{and} \quad \operatorname{slp} \llbracket C \rrbracket \left(\llbracket \psi \right] \right) \ = \ \left[\operatorname{slp} \llbracket C \rrbracket \left(\psi \right) \right] \ .$$

From a predicate perspective, sp $[\![C]\!]$ (ψ) contains final states τ that are reachable from at least one initial state satisfying ψ , whereas slp $[\![C]\!]$ (ψ) requires that every initial state that may end in τ satisfies ψ . Hence, we have a fundamentally dual meaning of the word *liberal*:

- wlp, differently from wp, provides *preconditions* containing all *diverging* initial states, but contains no state that can terminate outside the postcondition.
- slp, differently from sp, provides *postconditions* containing all *unreachable* final states, but contains no state that can be reached from outside the precondition.

Let us also consider two other examples: sp $[\![C]\!]$ ([true]) is the indicator function of the reachable states. If sp $[\![C]\!]$ ([true]) = [false] (i.e. sp $[\![C]\!]$ ($+\infty$) = $-\infty$), no state is reachable and hence C diverges on every input. Similarly, slp $[\![C]\!]$ ([false]) is the indicator function of all states that are either reachable from an initial state satisfying false (of which there are none) or which are unreachable. Thus, if slp $[\![C]\!]$ ([false]) = [true] (i.e. slp $[\![C]\!]$ ($-\infty$) = $+\infty$) then all states are unreachable, meaning C diverges on every input. Put shortly,

$$\operatorname{sp} [\![C]\!] (+\infty) = -\infty \quad \text{iff} \quad \operatorname{slp} [\![C]\!] (-\infty) = +\infty.$$

Finally, we note that the *quantitative weakest pre calculi* of Kaminski [2019, Section 2.3], restricted to *deterministic non-probabilistic programs* are even simply subsumed by the fact that we consider a larger lattice, namely quantities of type $f: \Sigma \to \mathbb{R}^{\pm \infty}$ instead of $f: \Sigma \to \mathbb{R}^{\infty}_{\geq 0}$.

5.3 Relationship between Liberal and Non-liberal Transformers

THEOREM 5.4 (LIBERAL-NON-LIBERAL DUALITY). For any program C and quantity f, we have

$$\operatorname{wp} [\![C]\!] (f) = -\operatorname{wlp} [\![C]\!] (-f) \quad \text{and} \quad \operatorname{sp} [\![C]\!] (f) = -\operatorname{slp} [\![C]\!] (-f) .$$

The duality for weakest pre is very similar to wp $[\![C]\!]$ $(\psi) = \neg \text{wlp}[\![C]\!]$ $(\neg \psi)$ in Dijkstra's classical calculus and wp $[\![C]\!]$ $(f) = 1 - \text{wlp}[\![C]\!]$ (1 - f) for 1-bounded functions f in Kozen's and McIver & Morgans development for probabilistic programs.

When considering only *deterministic* programs C (i.e. *syntactically* without nondeterministic choices), then executing C on initial state σ will either terminate in a *single* final state (i.e. $[C](\sigma)$ =

 $\{\tau\}$, for some τ), or diverge (i.e. $[\![C]\!](\sigma) = \emptyset$), meaning that $[\![C]\!](\underline{\hspace{0.4cm}})$ becomes a proper (partial) function. Hence, in case of termination, supremum and infimum of the final values of f coincide:

COROLLARY 5.5. If a deterministic program C terminates on an input σ , then for all quantities f,

$$\operatorname{wp} [\![C]\!] (f) (\sigma) = \operatorname{wlp} [\![C]\!] (f) (\sigma),$$

and otherwise

$$\operatorname{wp} [\![C]\!] (f) (\sigma) = -\infty \quad \text{and} \quad \operatorname{wlp} [\![C]\!] (f) (\sigma) = -\infty.$$

As a direct consequence of Corollary 5.5, for postquantities everywhere smaller than $+\infty$ (which is not restrictive since values of program variables are finite), we can precisely detect whether a given initial state has terminated or not. Kaminski [2019, Remark 2.12], in contrast, cannot easily distinguish whether a certain initial state does not terminate, or whether the anticipated value is 0.

Note that dual results for sp and slp do *not* hold since even for deterministic programs the fiber of the concrete semantics is not a function: multiple initial states can terminate in a single final state τ .

6 CORRECTNESS AND INCORRECTNESS REASONING

6.1 Galois Connections between Weakest Pre and Strongest Post

The classical strongest postcondition is the left adjoint to the weakest liberal precondition [Dijkstra and Scholten 1990, Section 12], i.e. the transformers wlp and sp form the Galois connection

$$G \implies \mathsf{wlp}[\![C]\!](F) \quad \text{iff} \quad \mathsf{sp}[\![C]\!](G) \implies F,$$
 (†)

which intuitively is true because $G \implies \mathsf{wlp} \, \llbracket C \rrbracket \, (F)$ means that starting from G the program C will either diverge or terminate in a state satisfying F, and $\mathsf{sp} \, \llbracket C \rrbracket \, (G) \implies F$ means that starting from G any state reachable by executing C satisfies F.

The above Galois connection is preserved in our quantitative setting; in fact, by substituting the partial order \implies on predicates with the partial order \le on \mathbb{A} we obtain:

THEOREM 6.1 (GALOIS CONNECTION BETWEEN wlp and sp). For all $C \in nGCL$ and $q, f \in A$:

$$q \le \text{wlp}[C](f)$$
 iff $\text{sp}[C](q) \le f$.

As wlp is for partial correctness, Theorem 6.1 shows that sp is also suitable for partial correctness. One may now wonder whether there exists a strongest post transformer that is tightly related to wp, and hence, to total correctness. Unfortunately, Dijkstra and Scholten [1990, Section 12] show that there cannot exist a predicate transformer stp - a "strongest total postcondition" - such that

$$G \implies \operatorname{wp} \llbracket C \rrbracket (F) \quad \text{iff} \quad \operatorname{stp} \llbracket C \rrbracket (G) \implies F.$$

Categorically, that negative result is a consequence of the fact that we are requiring wp to be a *right adjoint functor*, and a necessary condition for that is to preserve all infima, but this is not true since wp is not costrict. Despite this negative result, since wp preserves all suprema (cf. Theorem 5.1 (1)), we argue that wp is instead a *left adjoint functor* and show that its right adjoint is exactly slp:

Theorem 6.2 (Galois Connection between wp and slp). For all $C \in nGCL$ and $g, f \in A$:

$$\operatorname{wp} [\![C]\!] (f) \leq g \quad \text{iff} \quad f \leq \operatorname{slp} [\![C]\!] (g)$$

Let us provide an intuition on this connection, for simplicity only with "predicates" [F] and [G]: $[F] \leq \mathsf{slp}[\![C]\!]$ ([G]) means that every final state satisfying F is either reached only by states satisfying F or unreachable. This is equivalent to saying that all initial states terminating in F must satisfy F0, which is precisely expressed by $\mathsf{wp}[\![C]\!]$ ([F]1) $\leq [F]$ 2.

6.2 Resolving Nondeterministic Choice: Angelic vs. Demonic

Our choices of how to resolve nondeterminism are motivated by establishing dualities between weakest pre and strongest post presented in Section 6.1. The only thing we take for granted is that the standard definition of sp is angelic, thus characterizing the "set of reachable states". Indeed, if sp is angelic, then we are (provably) also forced to make wp angelic, and both wlp and slp demonic – otherwise, duality would break. We can also come up with an intuition for these choices: Both, angelic wp and demonic wlp transformers try to *avoid nontermination*, if at all possible, whereas angelic sp and demonic slp try to *avoid unreachability*.

By dualizing all resolutions of nondeterminism one would obtain the following intuition: Demonic wp and angelic wlp transformers try to *drive the execution towards nontermination* (more standard for both wp and wlp), whereas demonic sp and angelic slp try to *establish unreachability* (less standard for sp, whereas slp is novel anyway). We leave it as future work to study whether this dual situation would also preserve the Galois connections of Section 6.1.

6.3 Strongest Post and Incorrectness Logic

A Hoare triple $\langle G \rangle C \langle F \rangle$ is valid for partial correctness iff $G \Longrightarrow \mathsf{wlp}[\![C]\!](F)$ or (equivalently, see (†) in Section 6.1) sp $[\![C]\!](G) \Longrightarrow F$ holds. Somewhat recently, a different kind of triples have been proposed, first by de Vries and Koutavas [2011] under the name reverse Hoare logic for studying reachability specifications. A few years, O'Hearn [2019] rediscovered those triples under the name incorrectness logic and used them for explicit error handling. Bruni et al. [2021] provide a logic parametrized by an abstract interpretation that, through a notion of local completeness, can prove both correctness and incorrectness.

In this section we show, first, the relationship between our strongest post transformer and incorrectness triples [de Vries and Koutavas 2011; O'Hearn 2019]; then, more importantly, we argue that such triples deal with *total incorrectness* and hint at novel *partial incorrectness* triples.

(Total) Incorrectness. In the sense of de Vries and Koutavas [2011], an incorrectness triple

$$\llbracket G \rrbracket C \llbracket F \rrbracket$$
 is valid iff $\forall \tau \models F \exists \sigma \text{ with } \tau \in \llbracket C \rrbracket (\sigma) \colon \sigma \models G$.

In other words, the set of states F is an underapproximation of the set of states reachable by executing C on some state in G, i.e., $F \subseteq \operatorname{sp} \llbracket C \rrbracket$ (G) [O'Hearn 2019, Definition 1]. The term incorrectness logic originates from the fact that if $\llbracket G \rrbracket C \rrbracket F \rrbracket$ is valid and F contains an error state, then this error state is guaranteed to be reachable from G. Since our quantitative strongest post transformer subsumes the classical one, we can (re)define incorrectness triples by substituting predicates with extended Iverson brackets and obtain the following equivalent definition:

Definition 6.3 (Incorrectness Triples). For predicates G, F and program C, the incorrectness triple

$$[G] C [F]$$
 is valid for (total) incorrectness iff $[F] \leq \operatorname{sp} [C] ([G])$.

Partial Incorrectness. We argue that the aforementioned triples deal with *total incorrectness* by providing novel triples for *partial incorrectness*. Recall that a Hoare triple $\langle G \rangle C \langle F \rangle$ is valid for total correctness if $G \implies \text{wp} \llbracket C \rrbracket (F)$. By replacing wp with wlp, we can define partial correctness triples: $\langle G \rangle C \langle F \rangle$ is valid for partial correctness if $G \implies \text{wlp} \llbracket C \rrbracket (F)$. By mimicking the above, we define *partial incorrectness* by replacing sp with slp in Definition 6.3:

Definition 6.4 (Partial Incorrectness). For predicates G, F and program C, the incorrectness triple

$$[G] C [F]$$
 is valid for partial incorrectness iff $[F] \leq slp[C]([G])$.

By definition of slp,

$$[G] C [F]$$
 is val. for part. incorr. iff $\forall \tau \models F \ \forall \sigma \text{ with } \tau \in [\![C]\!](\sigma) \colon \sigma \models G$.

In other words, only if the state τ is *reachable*, then the triple guarantees that τ is reached only from initial states σ that satisfy G. Note that this is dual to the relationship between total and partial correctness: with partial incorrectness, to have *full* information on *initial states* we require an additional *proof of reachability* on *final states* (whereas with partial correctness, to obtain full information on *final states* we require an additional *proof of termination* on *initial states*).

We also note that, due to the Galois between wp and slp (Theorem 6.2) we have

$$[G] C [F]$$
 is valid for partial incorrectness iff wp $[C] ([F]) \leq [G]$.

This implies that G is an overapproximation of the set of states that end up in F, and corresponds to the notion of *necessary preconditions* studied by Cousot et al. [2013]. In particular, if an initial state $\sigma \not\models G$, then σ is guaranteed to not terminate in F (σ could also diverge).

Other Triples. We note that the naming conventions correctness and incorrectness may not necessarily always be appropriate. First of all, we argue that incorrectness triples [de Vries and Koutavas 2011; O'Hearn 2019] can be used to prove good behavior: for instance, a triple [G] C [F] where F contains good states, ensures that every (good) state in F is reachable from precondition G. Rather than correctness

implication			defines
G	\Longrightarrow	$\operatorname{wp} \llbracket C \rrbracket (F)$	total correctness
G	\Longrightarrow	$wlp\llbracket C \rrbracket (F)$	partial correctness
$wp\llbracket C\rrbracket\;(F)$	\Longrightarrow	G	partial incorrectness
$wlp[\![C]\!](F)$	\Longrightarrow	G	???
F	\Longrightarrow	$\operatorname{sp} \llbracket C \rrbracket (G)$	(total) incorrectness
F	\Longrightarrow	$slp\llbracket C rbracket(G)$	partial incorrectness
$\operatorname{sp} \llbracket C rbracket(G)$	\Longrightarrow	F	partial correctness
$slp[\![C]\!]\;(G)$	\Longrightarrow	F	555

versus *incorrectness*, we believe that the fundamental difference between the triples is that correctness triples provide information on the behavior of *initial* states satisfying *preconditions*, whereas incorrectness triples guarantee reachability properties on *final* states satisfying *postconditions*.

Secondly, note that our transformers can define two additional triples other than total (partial) correctness (incorrectness), for which the current naming conventions are insufficient. So far, we have the picture depicted in the table above. The two blue and the two orange lines define the same notion due to the Galois connections between wlp/sp and wp/slp. For ??? and ¿¿¿, however, there are no appropriate names (let alone program logics) yet. We can say, however, that ??? gives rise to a notion of necessary liberal preconditions, in the sense that (1) G contains all initial states σ that diverge, and (2) whenever $\sigma \not\models G$, then σ is guaranteed to terminate in a state $\tau \not\models F$. ¿¿¿, on the other hand, provides necessary liberal postconditions, meaning that (1) F contains all unreachable states, and every final state $\tau \not\models F$ is guaranteed to be reachable from some initial state $\sigma \not\models G$.

Following the terminology from above, which is inspired from the naming *necessary preconditions* of Cousot et al. [2013], we can state that

- total correctness triples provide *sufficient preconditions*;
- total incorrectness triples provide *sufficient postconditions*;
- partial correctness triples provide sufficient liberal preconditions (or necessary postconditions);
- partial incorrectness triples provide sufficient liberal postconditions (or necessary preconditions).

We also note that even the terminology for the predicate transformers, *strongest* post- and *weakest* precondition, might be imprecise. Indeed, as pointed by O'Hearn [2019], such terminology is tied with the classical aim of Hoare logic to find either the smallest (strongest) set of necessary (overapproximating) postconditions or the largest (weakest) set of sufficient (underapproximating) preconditions. The strongest postcondition can be seen also as the *weakest sufficient postcondition*,

whereas the weakest precondition is the *strongest necessary precondition*. Switching to our liberal predicate transformers, our strongest liberal post computes the *strongest necessary liberal postcondition* or, equivalently, the *weakest sufficient liberal postcondition*. Finally, our weakest liberal precomputes the *weakest sufficient liberal precondition* or the *strongest necessary liberal precondition*.

Duality. As a consequence of the liberal-non-liberal duality of Theorem 5.4, we have

$$G \implies \operatorname{wp} \llbracket C \rrbracket (F) \quad \text{iff} \quad \operatorname{wlp} \llbracket C \rrbracket (\neg F) \implies \neg G.$$

In other words, the triples connected to ??? are the contrapositive of total correctness triples. Similarly, ¿¿¿ is the contrapositive of total incorrectness, whereas partial incorrectness is the contrapositive of partial correctness. This implies (interestingly) that only three kind of triples fundamentally cannot be stated in terms of other triples. Nevertheless, we would argue that it is still useful to work with, e.g. ??? triples, depending on the verification aim, especially in the context of *explainable verification*: For example, if one is interested in inferring necessary preconditions, it would certainly appear easier and more natural to work and think directly with partial incorrectness, instead of complementing both the *sufficient liberal preconditions* obtained via partial correctness and the original postcondition. The resulting proof and annotations, directly in terms of *necessary preconditions*, will be much easier to understand for a working programmer.

7 LOOPS RULES

THEOREM 7.1 (INDUCTION RULES FOR LOOPS). The following proof rules for loops are valid:

$$\frac{g \leq i \leq [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp}[\![C]\!](i)}{g \leq \mathsf{wlp}[\![\mathsf{while}(\varphi) \{C\}]\!](f)} \text{ while-wlp}$$

$$\frac{g \lor \mathsf{sp}[\![C]\!]([\varphi] \land i) \leq i \quad \mathsf{and} \quad [\neg \varphi] \land i \leq f}{\mathsf{sp}[\![\mathsf{while}(\varphi) \{C\}]\!](g) \leq f} \text{ while-sp}$$

$$\frac{[\neg \varphi] \land f \lor [\varphi] \land \mathsf{wp}[\![C]\!](i) \leq i \leq g}{\mathsf{wp}[\![\mathsf{while}(\varphi) \{C\}]\!](f) \leq g} \text{ while-wp}$$

$$\frac{i \leq g \land \mathsf{slp}[\![C]\!]([\neg \varphi] \lor i) \quad \mathsf{and} \quad f \leq [\varphi] \lor i}{f \leq \mathsf{slp}[\![\mathsf{while}(\varphi) \{C\}]\!](g)} \text{ while-slp}$$

The rule while—sp is novel. The while—wlp rule has already been investigated in [Kaminski 2019, Section 5] in a probabilistic setting, but in a more restricted lattice where quantities map to the unit interval. Our definition of wlp is not probabilistic but for a more general lattice of unbounded signed quantities. Notice that while—wlp and while—sp are tightly connected by a Galois connection (cf. Theorem 6.1), and by taking g = [G] and f = [F] for predicates G, F, we conclude for both rules the validity of the Hoare triple $\langle G \rangle$ while (φ) { C } $\langle F \rangle$ for partial correctness. Indeed, as standard in literature, the rule while—wlp requires to find an invariant that satisfy two conditions:

- (1) $[G] \leq [I]$, meaning that whenever precondition G holds, then the invariant I also holds.
- (2) $[I] \leq [\neg \varphi] \land [F] \lor [\varphi] \land \mathsf{wlp}[C]$ ([I]), meaning that whenever I holds, either the loop guard φ does *not* hold, but then postcondition F holds; or φ does hold, but then I still holds after one iteration of the loop body (or the loop body itself diverges (think: nested loops)).

By induction, (2) ensures that, starting from I and no matter how many loop iterations are executed, I can only terminate in states that again satisfy I. Assuming termination, eventually $\neg \varphi$ will hold and thus I implies the postcondition F. (1) guarantees that the initial precondition G implies I.

Hence any state initially satisfying *G* and on which the loop eventually terminates will do so in a final state satisfying postcondition *F*. The rule while–sp is analogous, but for forward reasoning.

The rule while—wp has also been investigated by Kaminski [2019] in a probabilistic setting but again in a more restricted lattice where quantities map to unsigned positive extended reals. The rule while—slp is completely novel (since slp is novel). Again, by Galois connection and by taking as quantities the Iverson bracket of predicates G, F, we obtain for the last two rules as conclusion the validity of the triple [G] while (φ) $\{C\}$ [F] for partial incorrectness in the sense of Definition 6.4. As for an intuition, recall that validity for partial incorrectness means here that G is a necessary precondition to end in a final state satisfying F after termination of while (φ) $\{C\}$. For proving this, the rule while—wp requires to find an invariant I, such that:

- (1) $[I] \leq [G]$, meaning that whenever invariant I holds, then the precondition G also holds.
- (2) $[\neg \varphi] \land [F] \leq [I]$, meaning that if the loop has terminated in postcondition F, then I holds;
- (3) $[\varphi] \land \text{wp } [\![C]\!] ([I]) \leq [I]$, meaning that if the loop is in some state σ in which the loop guard holds (i.e. the loop is about to be executed once more) and one loop iteration will terminate in some state where I holds again, then I holds for σ .

By induction, (2) and (3), which represent the first premise of while—wp, imply that I is a necessary precondition for the loop to terminate in F. Indeed, starting from the base case (2), for the inductive step we assume that I overapproximates those states terminating in F after n loop iterations. By (3), I also contains $[\varphi] \land \text{wp}[C]([I])$, i.e., an overapproximation of those states terminating in F after n+1 iterations. (1) guarantees that the precondition G contains I and hence G is a necessary precondition for the loop to terminate in F. Again, the rule while—slp is analogous, but forward.

Example 7.2 (Inductive Reasoning). Consider the loop while (x < 10) { x := x + 4 }. In order to show that $x \mid 4$ (read: x is divisible by 4) is a necessary precondition to terminate in postcondition x = 12, it is sufficient to prove the partial incorrectness triple [x = 12] $\leq \text{slp}[\![C]\!]$ ([$x \mid 4$]). If we apply the inductive rule we obtain:

$$\frac{i \leq [x \mid 4] \land \mathsf{slp}[x \coloneqq x + 4] \ ([x \geq 10] \lor i) \qquad \mathsf{and} \qquad [x = 12] \leq [x < 10] \lor i}{[x = 12] \qquad \leq \qquad \mathsf{slp}[\mathsf{while} \ (x < 10) \ \{ \ x \coloneqq x + 4 \}] \ ([x \mid 4])} \ \mathsf{while} - \mathsf{slp}$$

Now take as invariant $i = [x \mid 4]$. As for the right premise, we can easily convince ourselves that $[x = 12] \le [x < 10] \lor [x \mid 4]$ holds. As for the left premise, we have

$$[x \mid 4] \land slp[[x \coloneqq x + 4]] ([x \ge 10] \lor [x \mid 4]) = [x \mid 4] \land ([x - 4 \ge 10] \lor [x - 4 \mid 4])$$

$$= [x \mid 4] \land ([x \ge 14] \lor [x \mid 4])$$

$$= [x \mid 4] \land ([x \ge 14] \lor [x \mid 4])$$

$$= [x \mid 4] \land ([x \ge 14] \lor [x \mid 4])$$

Hence we can infer the conclusion of while–slp and we have proven that $[x \mid 4]$ is a necessary precondition for the loop to terminate in [x = 12].

The forward transformers sp and slp come with an additional induction rule: under certain premises, it allows to immediately conclude that the fixpoint of the characteristic function for a quantity f is precisely f itself, i.e. the second Kleene iterate.

Proposition 7.3. The following proof rules for loops are valid:

$$\frac{\sup \llbracket C \rrbracket \left(f \right) \quad \leq \quad f}{\sup \llbracket \text{while} \left(\varphi \right) \left\{ C \right\} \rrbracket \left(f \right) \quad = \quad \llbracket \neg \varphi \rrbracket \wedge f} \qquad \frac{f \quad \leq \quad \sup \llbracket C \rrbracket \left(f \right)}{\sup \llbracket \text{while} \left(\varphi \right) \left\{ C \right\} \rrbracket \left(f \right) \quad = \quad \llbracket \varphi \rrbracket \vee f \rbrace}$$

An intuition of Proposition 7.3 for sp is the following: for a loop while (φ) { C }, the premise sp $[\![C]\!]$ (f) $\leq f$ means that the value of f retrocipated for one iteration is lower than the original value of f. By induction, retrocipating f for any number of iterations leads to a decreasing quantity.

So what is the *maximum initial value* that f could have had? It is the initial quantity f, i.e. sp "gets away" with not even entering the loop. The guard $[\neg \varphi]$ in the conclusion is needed to ensure reachability. For slp, retrocipating the execution of the loop increases the initial quantity f - and hence the *minimum initial value* of f is again f itself.

Example 7.4. Consider the loop $C = \text{while}(x < 10) \{ \{ x := x + 1 \} \square \{ x := x + 2 \} \}$ and the precondition $x \ge 0$. To determine the set of states reachable from precondition $x \ge 0$, i.e. to determine $\text{sp}[\![C]\!]([x \ge 0])$, we first check the premise

$$\begin{split} & \sup \left[\!\left\{\,x \coloneqq x + 1\,\right\} \, \Box \, \left\{\,x \coloneqq x + 2\,\right\}\right] \left(\left[x \ge 0\right]\right) \\ & = \quad \left[x - 1 \ge 0\right] \, \lor \, \left[x - 2 \ge 0\right] \quad = \quad \left[x \ge 1\right] \, \lor \, \left[x \ge 2\right] \quad = \quad \left[x \ge 1\right] \quad \le \quad \left[x \ge 0\right] \end{split}$$

and thus conclude by Proposition 7.3 that

$$sp [C] ([x \ge 0]) \le [x \ge 10] \land [x \ge 0] = [x \ge 10]$$

This allows to include immediately that $x \ge 10$ is the *strongest necessary postcondition* or, equivalently, the *weakest sufficient postcondition*. In particular, this result verifies that *precisely* those final states with $x \ge 10$ are *reachable* from initial states with $x \ge 0$.

8 CASE STUDIES

In this section, we demonstrate the efficacy of quantitative strongest (liberal) post reasoning. We use the annotation style on the right to express that $g = \operatorname{sp} \llbracket C \rrbracket (f)$ (or that $g = \operatorname{slp} \llbracket C \rrbracket (f)$, depending on the context) and furthermore that g' = g.

Full calculations of strongest posts are provided in Appendix G.

8.1 Quantitative Information Flow — Loop Free

Consider the program $C_{flow} = \text{if } (hi > 7) \{ lo := 99 \} \text{ else } \{ lo := 80 \}$. As usual in quantitative information flow, hi is a secret and we want to ensure that, by observing the variable lo, one cannot infer information about hi. Below, we show sp (left) and slp (right) annotations for prequantity hi, i.e. we indeed show how the initial value of hi flows from the top to the bottom of the computation.

```
//// hi
                                                                                             //// hi
if (hi > 7) {
                                                                                             if (hi > 7) {
      /// [hi > 7] A hi
                                                                                                    /// [hi ≤ 7] Y hi
      lo := 99
                                                                                                   lo := 99
       /// 2\alpha: [lo = 99] \wedge [hi > 7] \wedge hi
                                                                                                    //// \ell \alpha: [lo \neq 99] \forall [hi \leq 7] \forall hi
      =//// [lo = 99] \wedge [hi > 7] \wedge hi
                                                                                                   =//// [lo ≠ 99] \vee [hi > 7] \vee hi
}else {
      /// [hi ≤ 7] ∧ hi
                                                                                                    /// [hi > 7] \vee hi
      /// 2\alpha: [lo = 80] \wedge [hi \leq 7] \wedge hi
                                                                                                    /// \ell\alpha\colon [\mathrm{lo}\neq 80] Y [hi > 7] Y hi
      =[/// [lo = 80] ∧ [hi ≤ 7] ∧ hi
                                                                                                   = [lo \neq 80] \vee [hi > 7] \vee hi
 /\!\!/ ([\log 99] \land [\operatorname{hi} > 7] \land \operatorname{hi}) \lor ([\log 80] \land [\operatorname{hi} \le 7] \land \operatorname{hi}) 
 /\!\!/ ([\log \neq 99] \lor [\operatorname{hi} \le 7] \lor \operatorname{hi}) \land ([\log \neq 80] \lor [\operatorname{hi} > 7] \lor \operatorname{hi})
```

Let us first note that we can precisely infer the set of states that are reachable after executing C_{flow} by recalling that for a prequantity f strictly larger than $-\infty$, sp $[\![C]\!]$ (f) $(\tau) = -\infty$ if and only if τ is unreachable. When does the (left) expression $([lo = 99] \land [hi > 7] \land hi) \lor ([lo = 80] \land [hi \le 7] \land hi)$ evaluate to something larger than $-\infty$? This is precisely the case if either the final value of lo is 99

and hi is larger than 7, or if lo is 80 and hi smaller or equal 7. The reachable states are thus given by

$$\{\tau \mid \operatorname{sp} \llbracket C \rrbracket \left(hi \right) \left(\tau \right) \neq -\infty \} \ = \ \{\tau \mid \tau(lo) = 99 \land \tau(hi) > 7 \ \lor \ \tau(lo) = 80 \land \tau(hi) \leq 7 \} \ .$$

The same insight could have been achieved with slp by computing $\{\tau : \text{slp}[\![C]\!] (hi) (\tau) \neq +\infty\}$.

Secondly, we can — in a principled way — construct from the sp and slp annotations a function ξ that, given the final value of only the observable variable lo (which we denote lo'), returns the set containing an overapproximation of all possible *initial* values of the quantity hi, namely:

$$\begin{split} \xi(lo') &= \left\{ \alpha \ \middle| \ \tau \in \Sigma, \quad \tau(lo) = lo', \quad \text{slp} \llbracket C_{flow} \rrbracket \left(hi \right) \left(\tau \right) \, \leq \, \alpha \, \leq \, \text{sp} \, \llbracket C_{flow} \rrbracket \left(hi \right) \left(\tau \right) \, \right\} \\ &= \left\{ \left\{ \alpha \ \middle| \ 7 < \alpha \right\}, \quad \text{if } lo' = 99 \\ \left\{ \alpha \ \middle| \ \alpha \leq 7 \right\}, \quad \text{if } lo' = 80 \\ \emptyset, \qquad \qquad \text{otherwise}. \end{split} \right. \end{split}$$

Now, what can we infer about the secret initial value of hi by observing only the final value lo'? If lo' = 99, then hi must be larger than 7; if lo = 90, then hi must be smaller or equal 7, and otherwise this state was actually unreachable (and hence such a situation could have not been observed in the first place). Hence, observing the final value of lo leaks information about the secret hi. In fact, by having used both sp and slp, the above gave us precisely the entire information that is leaked about hi from observing the final value of lo.

8.2 Quantitative Information Flow for Loops

Consider the program $C_{while} = hi := hi + 5 \degree$ while (lo < hi) {lo := lo + 1}. Again, we show below the sp (left) and slp (right) annotations for prequantity hi.

For sp and slp of the loop, the Kleene iteration stabilizes after 2 iterations, see Appendix G for detailed computations. There is no need for invariant, nor reasoning about limits, or anything alike. Even more conveniently, we can alternatively apply Proposition 7.3: indeed, for instance for sp we have sp [lo := lo + 1] $(hi - 5) = hi - 5 \le hi - 5$ and thus Proposition 7.3 yields that sp of the loop is *precisely* $[lo \ge hi] \land (hi - 5)$.

We construct (again) the function ξ that, given the final value lo' of the variable lo, returns an overapproximation of all possible initial values of the quantity hi, and obtain $\xi(lo') = \{\alpha \mid \alpha \leq lo' - 5\}$. Hence, by observing only the final value lo' we infer that hi must be at most lo' - 5. In fact, any of such value $\alpha \leq lo' - 5$ after being incremented by 5 leads to a value that $\alpha' \leq lo'$, so without entering the loop, C_{while} terminates with the correct final value lo'. Again, using both sp and slp, we obtain precisely the entire information that is leaked about hi from observing the final value of lo.

Quantitative Information Flow for Loops using wp. The set $\xi(lo')$ could have alternatively been determined with classical weakest preconditions: In fact, wp $[\![C]\!]$ ([lo=lo']) is the set of all initial states that will end with a final state where lo=lo', and by projecting only to the values of the variable hi we obtain all initial values of hi. However, aside from a (perhaps subjective) elegance perspective, we point out that the computation of wp $[\![C]\!]$ ([lo=lo']) is actually more

involved: the Kleene's iterates of the loop for wp stabilize only at ω – not 2:

$$\begin{split} &\Phi(\mathsf{false}) \ = \ [lo \geq hi] \land [lo = lo'] \\ &\Phi^2(\mathsf{false}) \ = \ [lo \geq hi] \land [lo = lo'] \lor [lo' - 1 < hi \leq lo'] \land [lo = lo' - 1] \\ &\Phi^3(\mathsf{false}) \ = \ [lo \geq hi] \land [lo = lo'] \lor [lo' - 1 < hi \leq lo'] \land [lo = lo' - 1] \\ & \lor \ [lo' - 1 < hi \leq lo'] \land [lo = lo' - 2] \\ & \vdots \\ &\Phi^\omega(\mathsf{false}) \ = \ [hi \leq lo] \land [lo = lo'] \lor \left(\bigvee_{n=1}^\omega [lo' - 1 < hi \leq lo'] \land [lo = lo' - n]\right) \end{split}$$

Reasoning about this requires some form of creativity or advanced technique: either reasoning about the limit, or finding an invariant plus a termination prove. Only after determining Φ^{ω} (false), one can perform the wp for the assignment, which again results in a huge formula. For sp and slp, the Kleene's iterates stabilize after 2 iterations (Appendix G): no need for invariant nor reasoning about limits nor projections of huge formulas.

8.3 Automation

Our calculi, in their full generality, cannot be fully automated, which is not surprising since our calculi can express both termination and reachability properties for a Turing-complete computational model – both of which are well known to be undecidable [Rice 1953; Turing 1936]. Nevertheless, we believe that our calculi are at least syntactically mechanizable. For this aim, we plan to investigate an expressive "assertion" language for quantities, such as the one proposed by Batz et al. [2021] for quantitative reasoning about probabilistic programs. This would allow showing *relative completeness* in the sense of Cook [1978], i.e., decidability modulo checking whether $g \le f$ holds, where g, f may contain suprema and infima. Similar problems (decidability modulo checking a logical implication) exist for classical predicate transformers and Hoare logic [Cook 1978].

We also point out that the main goal of our calculi is to provide a framework, on which future tools for (partially) automating quantitative wlp/sp/slp proofs can ground. For example, it may well be possible to fully automate the transformers for some syntactic (e.g. linear) fragments of nGCL.

8.4 Partial Incorrectness Reasoning

We now show an application of partial incorrectness triples and, hence, of our strongest liberal post-conditions. Consider a program/system C_{login} that takes as input a variable password. If password contains the correct password, say "oopsla2022", then C_{login} terminates in a final state containing a boolean variable "access" storing the value true; otherwise, the program terminates with value access = false. Now, recall that

$$slp[C_{login}]$$
 ([password = "oopsla2022"])

is a predicate characterizing those final states which are reached *only* by initial states σ with the correct password, i.e. initial states with $\sigma(password)$ = "oopsla2022". If the partial incorrectness triple [access = true] C_{loqin} [password = "oopsla2022"], which translates to

[
$$access = true$$
] $\implies slp[C]([password = "oopsla2022"]),$

holds, then knowing the correct password is a *necessary precondition* to access the system. In other words, validity of the partial incorrectness triple guarantees that no user without knowledge of the correct password can end up in a final state τ where $\tau(access) = \text{true}$.

We also note that, by the Galois Connection of Theorem 6.2, one can check whether the partial incorrectness triple holds also by employing wp:

$$wp [C] ([access = true]) \implies [password = "oopsla2022"]$$

However, reasoning with slp may well (1) be more feasible in practice (as demonstrated in Section 8.2) as well as (2) more intuitive when reasoning about *necessary preconditions* to access a system.

9 RELATED WORK

More General Predicate Transformers. Aguirre and Katsumata [2020] focus on an abstract theory of wp for loop-free programs. In particular, our w(l)p, restricted to the fragment of loop-free programs, can be derived by instantiating their Corollary 4.6 (for details, see Appendix H). Aguirre and Katsumata [2020, Section 4.1] also define an abstract strongest postcondition as a left adjoint of their weakest precondition (without constructing it); we believe that, due to our Theorem 6.2, an abstract strongest liberal post can be defined dually as a right adjoint of their weakest precondition. On the other hand, our definition of strongest post is explicitly given by induction on the program structure and not implicitly as an adjoint. The difficulties with finding strongest posts for probabilistic programs demonstrate that an explicit definition of a strongest post is more than desirable.

Strongest Liberal Post. The term "strongest <u>liberal</u> postcondition" is sometimes used in the literature for the original <u>non-liberal</u> strongest postcondition, see e.g. [Back 1988, Section 2.2], [Jacobs and Gries 1985, Section 0], or [Wulandari and Plump 2020, Definition 8]. In fact, [Back 1988, Section 2.2] argues that the strongest postcondition is often denoted also as strongest liberal postcondition due to the relationship between weakest liberal pre. However, since wlp "allows" nontermination whereas wp does not, and analogously slp "allows" unreachability whereas sp does not, we believe that our naming convention of slp and sp is more appropriate and natural.

Information Flow Analysis. Some previous work on information flow analysis use type systems [Ørbæk and Palsberg 1997; Volpano and Smith 1997]. However, these are imprecise and may reject safe programs such as lo := hi : lo := 0 due to a potential flow from hi to lo [Amtoft and Banerjee 2004]. A Hoare-like logic combined with abstract interpretation has been proposed by Amtoft and Banerjee [2004], but fails for simple programs such as [Amtoft and Banerjee 2004, Section 9], which instead can be easily detected with our s(l)p analysis. Other abstract interpretationbased techniques focus on the trace semantics [Cousot 2019; Urban and Müller 2018]. Urban et al. [2019] verify dependency fairness of neural networks by applying a backward analysis to compute the set of input values that lead to a certain outur value; this approach is similar to a wp-based calculus with ghost variables, as shown in Example 8.2, and we speculate that sp-based approaches could also be applied and potentially lead to better performances (as shown in Example 8.2). In Security Concurrent Separation logic [Ernst and Murray 2019] the authors provide an extension of concurrent separation logic [O'Hearn 2004; Reynolds 2002] by adding sensitivity assertions which, roughly, assigns to a certain variable a certain degree of security; however, their proof system deals only with partial correctness and restricts to conditional statements and loops that cannot use sensitive variables, so that our examples from Section 8 cannot be covered by their logic. Differently from the aforementioned works, our framework provides quantitative details about the amount of information flow, instead of a single boolean output, see [Smith 2009] for an overview.

10 CONCLUSION & FUTURE WORK

We have presented a novel *quantitative strongest post calculus* that subsumes classical strongest postconditions. Moreover, we developed a novel *quantitative strongest* <u>liberal</u> post calculus. Restricted to a Boolean setting, we obtain the – to the best of our knowledge – unexplored notion of

strongest liberal postconditions which ultimately lead to our definition of partial incorrectness. The latter connection is justified by the fundamental Galois connection between slp and wp, and the strong duality between total and partial correctness, but where we replace nontermination with unreachability. Finally, we notice that there are three additional Hoare-style triples that can be naturally defined using our transformers, and we identify a precise connection between partial incorrectness and the so-called necessary preconditions [Cousot et al. 2013].

As future work, we plan to investigate the newly observed Hoare triples and to provide novel proof systems for them. We also plan to extend our quantitative strongest calculi with heap manipulation, similarly to the work of [Batz et al. 2018] for weakest pre calculi; this could lead to connections with incorrectness separation logic [Raad et al. 2020].

Finally, we plan to deepen the applications of quantitative strongest post calculi to quantitative information flow, perhaps by establishing connections with abstract interpretation [Cousot and Cousot 1977]. In fact, we believe that our s(l)p transformers can be viewed as sound approximations of the fiber of the concrete semantics. Examples 8.1, 8.2 go into this direction after-all, since the combination of our strongest and strongest liberal post calculi can be viewed as an *interval abstraction* [Cousot and Cousot 1976] of the possible initial values of a certain pre-quantity.

REFERENCES

Alejandro Aguirre and Shin-ya Katsumata. 2020. Weakest Preconditions in Fibrations. In MFPS.

Torben Amtoft and Anindya Banerjee. 2004. Information Flow Analysis in Logical Form. In *Static Analysis*, Roberto Giacobazzi (Ed.). Springer Berlin Heidelberg, Berlin, Heidelberg, 100–115.

R. J. R. Back. 1988. A Calculus of Refinements for Program Derivations. Acta Inf. 25, 6 (Aug. 1988), 593–624. https://doi.org/10.1007/BF00291051

Kevin Batz, Benjamin Lucien Kaminski, Joost-Pieter Katoen, and Christoph Matheja. 2021. Relatively complete verification of probabilistic programs: an expressive language for expectation-based reasoning. *Proc. ACM Program. Lang.* 5, POPL (2021), 1–30.

Kevin Batz, Benjamin Lucien Kaminski, Joost-Pieter Katoen, Christoph Matheja, and Thomas Noll. 2018. Quantitative Separation Logic. CoRR abs/1802.10467 (2018). arXiv:1802.10467 http://arxiv.org/abs/1802.10467

Roberto Bruni, Roberto Giacobazzi, Roberta Gori, and Francesco Ranzato. 2021. A Logic for Locally Complete Abstract Interpretations. In 2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). 1–13. https://doi.org/10.1109/LICS52264.2021.9470608

Stephen A. Cook. 1978. Soundness and Completeness of an Axiom System for Program Verification. SIAM J. Comput. 7 (1978), 70–90.

Patrick Cousot. 2019. Abstract Semantic Dependency. In SAS (Lecture Notes in Computer Science, Vol. 11822). Springer, 389–410.

P. Cousot and R. Cousot. 1976. Static determination of dynamic properties of programs. In *Proceedings of the Second International Symposium on Programming*. Dunod, Paris, France, 106–130.

Patrick Cousot and Radhia Cousot. 1977. Abstract Interpretation: A Unified Lattice Model for Static Analysis of Programs by Construction or Approximation of Fixpoints. In Conference Record of the Fourth ACM Symposium on Principles of Programming Languages, Los Angeles, California, USA, January 1977, Robert M. Graham, Michael A. Harrison, and Ravi Sethi (Eds.). ACM, 238–252. https://doi.org/10.1145/512950.512973

Patrick Cousot, Radhia Cousot, Manuel Fähndrich, and Francesco Logozzo. 2013. Automatic Inference of Necessary Preconditions. In *Verification, Model Checking, and Abstract Interpretation*, Roberto Giacobazzi, Josh Berdine, and Isabella Mastroeni (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 128–148.

Edsko de Vries and Vasileios Koutavas. 2011. Reverse Hoare Logic. In Software Engineering and Formal Methods, Gilles Barthe, Alberto Pardo, and Gerardo Schneider (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 155–171.

Edsger Wybe Dijkstra. 1975. Guarded Commands, Nondeterminacy and Formal Derivation of Programs. 18, 8 (1975), 453–457.

Edsger W. Dijkstra and Carel S. Scholten. 1990. *Predicate Calculus and Program Semantics*. Springer-Verlag, Berlin, Heidelberg. Gidon Ernst and Toby Murray. 2019. SecCSL: Security Concurrent Separation Logic. In *Computer Aided Verification*, Isil Dillig and Serdar Tasiran (Eds.). Springer International Publishing, Cham, 208–230.

Matthew S. Hecht. 1977. Flow Analysis of Computer Programs. Elsevier.

C. A. R. Hoare. 1969. An Axiomatic Basis for Computer Programming. Commun. ACM 12, 10 (Oct. 1969), 576–580. https://doi.org/10.1145/363235.363259 Dean Jacobs and David Gries. 1985. General Correctness: A Unification of Partial and Total Correctness. *Acta Inf.* 22, 1 (April 1985), 67–83. https://doi.org/10.1007/BF00290146

Claire Jones. 1990. Probabilistic Non-Determinism. Ph.D. Dissertation. University of Edinburgh, UK.

Benjamin Lucien Kaminski. 2019. Advanced weakest precondition calculi for probabilistic programs. Ph.D. Dissertation. RWTH Aachen University, Germany.

Benjamin Lucien Kaminski and Joost-Pieter Katoen. 2017. A weakest pre-expectation semantics for mixed-sign expectations. In *LICS*. IEEE Computer Society, 1–12.

Donald E. Knuth. 1992. Two Notes on Notation. Am. Math. Monthly 99, 5 (May 1992), 403–422. https://doi.org/10.2307/

Dexter Kozen. 1985. A Probabilistic PDL. J. Comput. System Sci. 30, 2 (1985), 162-178.

Annabelle McIver and Carroll Morgan. 2005a. Abstraction, Refinement and Proof for Probabilistic Systems. Springer. https://doi.org/10.1007/b138392

Annabelle McIver and Carroll Morgan. 2005b. Abstraction, Refinement and Proof for Probabilistic Systems. Springer.

Peter W. O'Hearn. 2004. Resources, Concurrency and Local Reasoning. In *CONCUR 2004 - Concurrency Theory*, Philippa Gardner and Nobuko Yoshida (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 49–67.

Peter W. O'Hearn. 2019. Incorrectness Logic. Proc. ACM Program. Lang. 4, POPL, Article 10 (Dec. 2019), 32 pages. https://doi.org/10.1145/3371078

P. Ørbæk and J. Palsberg. 1997. Trust in the λ -Calculus. J. Funct. Program. 7, 6 (Nov. 1997), 557–591. https://doi.org/10. 1017/S0956796897002906

David Michael Ritchie Park. 1969. Fixpoint Induction and Proofs of Program Properties, Vol. 5. Machine intelligence.

Azalea Raad, Josh Berdine, Hoang-Hai Dang, Derek Dreyer, Peter O'Hearn, and Jules Villard. 2020. Local Reasoning About the Presence of Bugs: Incorrectness Separation Logic. In *Computer Aided Verification*, Shuvendu K. Lahiri and Chao Wang (Eds.). Springer International Publishing, Cham, 225–252.

J.C. Reynolds. 2002. Separation logic: a logic for shared mutable data structures. In *Proceedings 17th Annual IEEE Symposium on Logic in Computer Science*. 55–74. https://doi.org/10.1109/LICS.2002.1029817

H.G. Rice. 1953. Classes of recursively enumerable sets and their decision problems. *Trans. Amer. Math. Soc.* 74 (1953), 358–366. https://doi.org/10.2307/1990888

Xavier Rival and Kwangkeun Yi. 2020. Introduction to Static Analysis – An Abstract Interpretation Perspective. MIT Press. Geoffrey Smith. 2009. On the foundations of quantitative information flow. In International Conference on Foundations of Software Science and Computational Structures. Springer, 288–302.

Alan Turing. 1936. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society* 42, 1 (1936), 230–265. https://doi.org/10.2307/2268810

Alan Mathison Turing. 1949. Checking a Large Routine. In Report of a Conference on High Speed Automatic Calculating Machines. Univ. Math. Lab., Cambridge, 67–69.

Caterina Urban, Maria Christakis, Valentin Wüstholz, and Fuyuan Zhang. 2019. Perfectly Parallel Fairness Certification of Neural Networks. CoRR abs/1912.02499 (2019). arXiv:1912.02499 http://arxiv.org/abs/1912.02499

Caterina Urban and Peter Müller. 2018. An Abstract Interpretation Framework for Input Data Usage. In ESOP. 683-710.

Dennis M. Volpano and Geoffrey Smith. 1997. A Type-Based Approach to Program Security. In *Proceedings of the 7th International Joint Conference CAAP/FASE on Theory and Practice of Software Development (TAPSOFT '97)*. Springer-Verlag, Berlin, Heidelberg, 607–621.

Gia S. Wulandari and Detlef Plump. 2020. Verifying Graph Programs with First-Order Logic. *Electronic Proceedings in Theoretical Computer Science* 330 (Dec 2020), 181–200. https://doi.org/10.4204/eptcs.330.11

APPENDIX

A COLLECTING SEMANTICS OF WHILE-LOOPS

Let us explain the semantics of while (φ) { C }. Let S again be the set of input states. First, we denote by F_S the function

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X$$
,

i.e. F_S first applies the filtering with respect to the loop guard φ to its input X, then applies the semantics of the loop body C to the filtered set, and finally unions that result with the given set of input states S. Using F_S , the standard collecting semantics for while loops can be expressed as

$$[\text{while}(\varphi) \{C\}] S = [\neg \varphi] (\text{Ifp } X \cdot F_S(X)),$$

where the least fixed point above is understood with respect to the partial order of set inclusion, which renders the structure $\langle Conf, \subseteq \rangle$ a complete lattice with least element \emptyset . The least fixed point above filtered by $\neg \varphi$ expresses exactly the set $\llbracket while(\varphi) \{C\} \rrbracket S$ of final states reachable after termination of while(φ) $\{C\}$ starting from any initial state in S. We remark that to determine the least fixed point of the continuous function F_S , it is sufficient to apply Kleene's fixpoint theorem and, as a result, we have that the infinite ascending chain $\emptyset \subseteq F_S^1(\emptyset) \subseteq F_S^2(\emptyset) \subseteq \dots F_S^\omega(\emptyset)$, where $F_S^{i+1}(X) = F_S(F_S^i(X))$, converges in at most ω iterations.

Example A.1 (Standard Collecting Semantics of While Loops). Assume there is only a single program variable x and consider the configuration $S = \{\{x \mapsto 0\}, \{x \mapsto 8\}\}$. We now want to execute the loop while (x > 5) $\{x := x + 1\}$ on this configuration and collect the reachable states. By our construction above, we have

$$\begin{split} & \|\text{while}\,(\,x > 5\,)\,\{\,x \coloneqq x + 1\,\} \|S \,=\, \|x \le 5\| \big(\text{lfp}\,X \boldsymbol{.}\, F_S(X)\big)\,, \quad \text{where} \\ & F_S(X) \,=\, S\,\cup\, \big(\|C\| \circ \|\varphi\| \big) X \,=\, \{\{x \mapsto 0\}, \{x \mapsto 8\}\} \,\cup\, \, \{\,\sigma\,[x/x + 1] \mid \sigma \in X, \,\, \sigma(x) > 5\,\}\,\,, \end{split}$$

and the Kleene iterates are:

$$F(\emptyset) = \{ \{x \mapsto 0\}, \{x \mapsto 8\} \} \cup \emptyset$$

$$F^{2}(\emptyset) = \{ \{x \mapsto 0\}, \{x \mapsto 8\} \} \cup \{ \{x \mapsto 9\} \}$$

$$F^{2}(\emptyset) = \{ \{x \mapsto 0\}, \{x \mapsto 8\} \} \cup \{ \{x \mapsto 9\}, \{x \mapsto 10\} \}$$

$$\vdots$$

$$F^{\omega}(\emptyset) = \{ \{x \mapsto 0\} \} \cup \{ \{x \mapsto i\} \mid i \geq 9 \}$$

After filtering $F^{\omega}(\emptyset)$ by the negation of the loop guard, we obtain the loop's collecting semantics

$$[\![\mathsf{while}\,(\,x > 5\,)\,\{\,x \coloneqq x + 1\,\}]\!]S \,=\, [\![x \le 5]\!]\big(F^\omega(\emptyset)\big) \,=\, \big\{\{x \mapsto 0\}\big\}\,\,. \qquad \, \lhd$$

B PROOFS OF SECTION 3

B.1 Proof of Soundness for wp, Thereom 3.7

THEOREM 3.7 (SOUNDNESS OF wp). For all programs C and initial states σ ,

$$\operatorname{wp} \left[\!\!\left[C \right]\!\!\right](f) \left(\sigma \right) \quad = \quad \bigvee_{\tau \in \left[\!\!\left[C \right]\!\!\right](\sigma)} f(\tau) \; .$$

PROOF. We prove Theorem 3.7 by induction on the structure of *C*. For the induction base, we have the atomic statements:

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The effectless program skip: We have

$$\begin{split} \operatorname{wp} \left[\!\!\left[\operatorname{skip} \right]\!\!\right](f)\left(\sigma\right) &= f(\sigma) \\ &= \sup_{\tau \in \left[\!\!\left[\operatorname{skip} \right]\!\!\right](\sigma)} f(\tau) \,. \end{split}$$

The assignment x := e: We have

$$\begin{split} \operatorname{wp} \left[\!\!\left[x \coloneqq e\right]\!\!\right](f)\left(\sigma\right) &= f\left[x/e\right]\left(\sigma\right) \\ &= f(\sigma\left[x/\sigma(e)\right]\right) \\ &= \sup_{\tau \in \left\{\sigma\left[x/\sigma(e)\right]\right\}} f(\tau) \\ &= \sup_{\tau \in \left[x \coloneqq e\right]\!\!\left[\sigma\right)} f(\tau) \;. \end{split}$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , we proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^\circ_2 C_2$: We have

$$\begin{split} \operatorname{wp} \left[\!\!\left[C_1 \right]\!\!\right] \left(f \right) \left(\sigma \right) &= \operatorname{wp} \left[\!\!\left[C_1 \right]\!\!\right] \left(\operatorname{wp} \left[\!\!\left[C_2 \right]\!\!\right] \left(f \right) \right) \left(\sigma \right) \\ &= \sup_{\tau' \in \left[\!\!\left[C_1 \right]\!\!\right] \left(\sigma \right)} \operatorname{wp} \left[\!\!\left[C_2 \right]\!\!\right] \left(f \right) \left(\tau' \right) \\ &= \sup_{\tau' \in \left[\!\!\left[C_1 \right]\!\!\right] \left(\sigma \right) \wedge \tau \in \left[\!\!\left[C_2 \right]\!\!\right] \left(\tau' \right)} f(\tau) \\ &= \sup_{\tau \in \left[\!\!\left[C_2 \right]\!\!\right] \left(\left[\!\!\left[C_1 \right]\!\!\right] \left(\sigma \right) \right)} f(\tau) \\ &= \sup_{\tau \in \left[\!\!\left[C_1 \right]\!\!\right] \left(\sigma \right)} f(\tau) \,. \end{split}$$

The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{split} & \text{wp} \, \llbracket \text{if} \, \left(\varphi \right) \, \{ \, C_1 \, \} \, \text{else} \, \{ \, C_2 \, \} \rrbracket \, \left(f \right) \, \left(\sigma \right) \\ & = \, \left(\left[\varphi \right] \wedge \, \text{wp} \, \llbracket C_1 \rrbracket \, \left(f \right) \, \vee \, \left[\neg \varphi \right] \wedge \, \text{wp} \, \llbracket C_2 \rrbracket \, \left(f \right) \, \right) (\sigma) \\ & = \, \begin{cases} & \text{wp} \, \llbracket C_1 \rrbracket \, \left(f \right) \, \left(\sigma \right) & \text{if} \, \sigma \, \vDash \, \varphi \\ & \text{wp} \, \llbracket C_2 \rrbracket \, \left(f \right) \, \left(\sigma \right) & \text{otherwise} \end{cases} \\ & = \, \begin{cases} & \sup_{\tau \in \llbracket C_1 \rrbracket \, \left(\sigma \right)} f(\tau) & \text{if} \, \sigma \, \vDash \, \varphi \\ & \sup_{\tau \in \llbracket C_1 \rrbracket \, \left(\sigma \right)} f(\tau) & \text{otherwise} \end{cases} \\ & = \, \sup_{\tau \in \llbracket C_1 \rrbracket \, \left(\sigma \right) \, \left(\sigma \right) \cup \left(\llbracket C_2 \rrbracket \, \left(\sigma \right) \, \left(\sigma \right) \right) \left(\sigma \right)} f(\tau) \\ & = \, \sup_{\tau \in \llbracket \text{if} \, \left(\varphi \right) \, \left\{ \, C_1 \, \right\} \, \text{else} \, \left\{ \, C_2 \, \right\} \, \left[\sigma \right)} f(\tau) \, . \end{split}$$

The nondeterministic choice $\{C_1\} \square \{C_2\}$: We have

$$\operatorname{wp} \left[\!\left\{ C_{1} \right\} \, \middle \mid \left\{ C_{2} \right\} \right]\!\right] (f) (\sigma) = \left(\operatorname{wp} \left[\!\left[C_{1} \right]\!\right] (f) \, \vee \, \operatorname{wp} \left[\!\left[C_{2} \right]\!\right] (f) \right) (\sigma)$$

$$= \sup_{\tau \in \left[\!\left[C_{1} \right]\!\right] (\sigma)} f(\tau) \, \vee \, \sup_{\tau \in \left[\!\left[C_{2} \right]\!\right] (\sigma)} f(\tau) \qquad \text{(by I.H. on } C_{1}, C_{2})$$

$$= \sup_{\tau \in [\![C_1]\!](\sigma) \cup [\![C_2]\!](\sigma)} f(\tau)$$

$$= \sup_{\tau \in [\![C_1]\!] \cup [\![C_2]\!](\sigma)} f(\tau).$$

The loop while (φ) { C }: Let

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \text{wp} [\![C]\!](X) ,$$

be the wp-characteristic functions of the loop while (φ) { C } with respect to postanticipation f and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. We now prove by induction on n that, for all $\sigma \in \Sigma$

$$\Phi_f^n(-\infty)(\sigma) = \sup_{\tau \in \llbracket -\varphi \rrbracket F_{\{\sigma\}}^n(\emptyset)} f(\tau) . \tag{1}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Phi_f^0(-\infty)(\sigma) &= -\infty \\ &= \sup \emptyset \\ &= \sup_{\tau \in \emptyset} f(\tau) \\ &= \sup_{\tau \in \llbracket \neg \varphi \rrbracket F^0_{\{\sigma\}}(\emptyset)} f(\tau) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n and all $\sigma \in \Sigma$,

$$\Phi_f^n(-\infty)(\sigma) = \sup_{\tau \in \llbracket \neg \varphi \rrbracket F_{\{\sigma\}}^n(\emptyset)} f(\tau) .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} & \Phi_{f}^{n+1}(-\infty)(\sigma) \\ & = ([\neg \varphi] \land f)(\sigma) \lor ([\varphi] \land \operatorname{wp} [\![C]\!] \left(\Phi_{f}^{n}(-\infty)\right))(\sigma) \\ & = ([\neg \varphi] \land f)(\sigma) \lor \sup_{\tau \in [\![C]\!] (\sigma) \land \sigma \models \varphi} \Phi_{f}^{n}(-\infty)(\tau) \\ & = \begin{cases} \sup_{\tau \in [\![C]\!] (\sigma)} \Phi_{f}^{n}(-\infty)(\tau) & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \sup_{\tau \in [\![C]\!] (\sigma)} \sup_{\tau' \in [\![\neg \varphi]\!]} F_{\{\tau\}}^{n}(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \sup_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!] (\sigma)}^{n}(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \sup_{\tau' \in [\![\neg \varphi]\!]} F_{([\![C]\!] \circ [\![\varphi]\!] (\sigma)}^{n}(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \sup_{\tau' \in [\![\neg \varphi]\!]} F_{([\![C]\!] \circ [\![\varphi]\!] (\sigma)}^{n}(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \sup_{\tau' \in [\![\neg \varphi]\!]} F_{([\![C]\!] \circ [\![\varphi]\!] (\sigma)}^{n}(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \end{split}$$

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$$= \sup_{\tau \in \llbracket \neg \varphi \rrbracket F_{\{\sigma\}}^{n+1}(\emptyset)} f(\tau) .$$

This concludes the induction on n. Now we have:

$$\begin{split} \operatorname{wp} \left[\!\!\left[\operatorname{while}\left(\varphi\right)\left\{C\right\}\!\right]\!\!\right] (f) (\sigma) &= \left(\operatorname{lfp} X \boldsymbol{.} \left[\neg\varphi\right] \wedge f \vee \left[\varphi\right] \wedge \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (X)\right) (\sigma) \\ &= \sup_{n \in \mathbb{N}} \Phi_f^n(-\infty)(\sigma) \qquad \text{(By Kleene's fixpoint theorem)} \\ &= \sup_{n \in \mathbb{N}} \sup_{\tau \in \left[\!\!\left[\neg\varphi\right]\!\!\right] F_{\{\sigma\}}^n(\emptyset)} f(\tau) \qquad \text{(by Equation 1)} \\ &= \sup_{\tau \in \cup_{n \in \mathbb{N}} \left(\left[\!\!\left[\neg\varphi\right]\!\!\right] F_{\{\sigma\}}^n(\emptyset)\right)} f(\tau) \\ &= \sup_{\tau \in \left[\!\!\left[\neg\varphi\right]\!\!\right] \left(\cup_{n \in \mathbb{N}} F_{\{\sigma\}}^n(\emptyset)\right)} f(\tau) \qquad \text{(by continuity of } \left[\!\!\left[\neg\varphi\right]\!\!\right]\right)} \\ &= \sup_{\tau \in \left[\!\!\left[\neg\varphi\right]\!\!\right] \left(\operatorname{lfp} X \boldsymbol{.} \left\{\sigma\right\} \cup \left(\left[\!\!\left[C\right]\!\!\right] \circ \left[\!\!\left[\varphi\right]\!\!\right]\right) X\right)} f(\tau) \\ &= \sup_{\tau \in \left[\!\!\left[\neg\varphi\right]\!\!\right] \left(\operatorname{lfp} X \boldsymbol{.} \left\{\sigma\right\} \cup \left(\left[\!\!\left[C\right]\!\!\right] \circ \left[\!\!\left[\varphi\right]\!\!\right]\right) X\right)} f(\tau) \\ &= \sup_{\tau \in \left[\!\!\left[\operatorname{while}\left(\varphi\right)\!\!\right] \left\{\sigma\right\}\!\!\right] \left(\sigma\right)} f(\tau) \,, \end{split}$$

and this concludes the proof.

B.2 Proof of Soundness for wlp, Thereom 3.10

Theorem 3.10 (Soundness of wlp). For all programs C and states $\sigma \in \Sigma$,

$$\mathsf{wlp}\llbracket C \rrbracket \left(f \right) \left(\sigma \right) \quad = \quad \bigwedge_{\tau \in \llbracket C \rrbracket \left(\sigma \right)} f(\tau) \; .$$

PROOF. We prove Theorem 3.10 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \mathsf{wlp}[\![\mathsf{skip}]\!] \left(f \right) \left(\sigma \right) &= f(\sigma) \\ &= \inf_{\tau \in \{\sigma\}} f(\tau) \\ &= \inf_{\tau \in [\![\mathsf{skip}]\!] \left(\sigma \right)} f(\tau) \;. \end{split}$$

The assignment x := e: We have

$$\begin{aligned} \mathsf{wlp}[\![x \coloneqq e]\!] \left(f\right) \left(\sigma\right) &= f\left[x/e\right] \left(\sigma\right) \\ &= f\left(\sigma\left[x/\sigma(e)\right]\right) \\ &= \inf_{\tau \in \{\sigma\left[x/\sigma(e)\right]\}} f(\tau) \\ &= \inf_{\tau \in [\![x \coloneqq e]\!] \left(\sigma\right)} f(\tau) \;. \end{aligned}$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , we proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^\circ_2 C_2$: We have

$$\begin{aligned} \operatorname{wlp} \llbracket C_1 \circ C_2 \rrbracket \left(f \right) \left(\sigma \right) &= \operatorname{wlp} \llbracket C_1 \rrbracket \left(\operatorname{wlp} \llbracket C_2 \rrbracket \left(f \right) \right) \left(\sigma \right) \\ &= \inf_{\tau' \in \llbracket C_1 \rrbracket \left(\sigma \right)} \operatorname{wlp} \llbracket C_2 \rrbracket \left(f \right) \left(\tau' \right) & \text{(by I.H. on } C_1 \right) \\ &= \inf_{\tau' \in \llbracket C_1 \rrbracket \left(\sigma \right) \wedge \tau \in \llbracket C_2 \rrbracket \left(\tau' \right)} f(\tau) & \text{(by I.H. on } C_2 \right) \\ &= \inf_{\tau \in \llbracket C_2 \rrbracket \left(\llbracket C_1 \rrbracket \left(\sigma \right) \right) } f(\tau) & \\ &= \inf_{\tau \in \llbracket C_1 \rrbracket \left(\sigma \right) } f(\tau) & \text{.} \end{aligned}$$

The conditional branching if (φ) { C_1 } else { C_2 }: We have

The nondeterministic choice $\{C_1\} \square \{C_2\}$: We have

$$\begin{aligned} \mathsf{wlp}[\![\{C_1\} \, \Box \, \{C_2\}]\!] \, (f) \, (\sigma) &= \, \left(\mathsf{wlp}[\![C_1]\!] \, (f) \, \wedge \, \mathsf{wlp}[\![C_2]\!] \, (f) \, \right) (\sigma) \\ &= \, \inf_{\tau \in [\![C_1]\!] \, (\sigma)} f(\tau) \, \wedge \, \inf_{\tau \in [\![C_2]\!] \, (\sigma)} f(\tau) \\ &= \, \inf_{\tau \in [\![C_1]\!] \, (\sigma) \cup [\![C_2]\!] \, (\sigma)} f(\tau) \\ &= \, \inf_{\tau \in [\![C_1]\!] \cup \{C_2\}\!] \, (\sigma)} f(\tau) \, . \end{aligned}$$
 (by I.H. on C_1, C_2)

The loop while (φ) { C }: Let

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp}[\![C]\!](X) ,$$

be the wlp-characteristic functions of the loop while (φ) { C } with respect to postanticipation f and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) $\{C\}$ with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. We now prove by induction on n that, for all $\sigma \in \Sigma$

$$\Phi_f^n(+\infty)(\sigma) = \inf_{\tau \in \llbracket \neg \varphi \rrbracket F_{\{\sigma\}}^n(\emptyset)} f(\tau) . \tag{2}$$

For the induction base n = 0, consider the following:

$$\Phi_f^0(+\infty)(\sigma) = +\infty$$

$$= \inf_{\tau \in \emptyset} 0$$

$$= \inf_{\tau \in \emptyset} f(\tau)$$

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$$= \inf_{\tau \in \llbracket \neg \varphi \rrbracket F^0_{\{\sigma\}}(\emptyset)} f(\tau) \ .$$

As induction hypothesis, we have for arbitrary but fixed n and all $\sigma \in \Sigma$,

$$\Phi_f^n(+\infty)(\sigma) = \inf_{\tau \in \llbracket \neg \varphi \rrbracket F_{f\sigma}^n(\emptyset)} f(\tau) .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} & \Phi_f^{n+1}(+\infty)(\sigma) \\ & = ([\neg \varphi] \land f)(\sigma) \lor ([\varphi] \land \text{wlp}[\![C]\!] \left(\Phi_f^n(+\infty)\right))(\sigma) \\ & = ([\neg \varphi] \land f)(\sigma) \lor [\varphi](\sigma) \land \inf_{\tau \in [\![C]\!](\sigma)} \Phi_f^n(+\infty)(\tau) \\ & = \begin{cases} \inf_{\tau \in [\![C]\!](\sigma)} \Phi_f^n(+\infty)(\tau) & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \inf_{\tau \in [\![C]\!](\sigma)} \inf_{\tau' \in [\![\neg \varphi]\!]} F_{\{\tau\}}^n(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \inf_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!](\sigma)}^n(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \inf_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!](\sigma)}^n(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \begin{cases} \inf_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!](\sigma)}^n(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \inf_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!](\sigma)}^n(\emptyset) f(\tau') & \text{if } \sigma \models \varphi \\ f(\sigma) & \text{otherwise} \end{cases} \\ & = \inf_{\tau' \in [\![\neg \varphi]\!]} F_{[\![C]\!](\sigma)}^n(\emptyset) f(\tau') \end{cases} \end{split}$$

This concludes the induction on n. Now we have:

$$\begin{split} \mathsf{wlp}[\![\mathsf{while}\,(\varphi)\,\{C\}]\!]\,(f)\,(\sigma) &= \big(\mathsf{gfp}\,X\boldsymbol{\cdot}\big[\neg\varphi] \land f \ \lor \ [\varphi] \land \mathsf{wlp}[\![C]\!]\,(X)\big)(\sigma) \\ &= \inf_{n\in\mathbb{N}}\Phi_f^n(+\infty)(\sigma) \qquad \qquad \text{(by Kleene's fixpoint theorem)} \\ &= \inf_{n\in\mathbb{N}}\inf_{\tau\in[\![\neg\varphi]\!]F_{\{\sigma\}}^n(\emptyset)}f(\tau) \qquad \qquad \text{(by Equation 2)} \\ &= \inf_{\tau\in[\![\neg\varphi]\!]\,(\cup_{n\in\mathbb{N}}F_{\{\sigma\}}^n(\emptyset))}f(\tau) \\ &= \inf_{\tau\in[\![\neg\varphi]\!]\,(\cup_{n\in\mathbb{N}}F_{\{\sigma\}}^n(\emptyset))}f(\tau) \qquad \qquad \text{(by continuity of } [\![\neg\varphi]\!]) \\ &= \inf_{\tau\in[\![\neg\varphi]\!]\,(\mathsf{lfp}\,X\boldsymbol{\cdot}\{\sigma\}\cup([\![C]\!]\circ[\![\varphi]\!])X)}f(\tau) \\ &= \inf_{\tau\in[\![\neg\psi]\!]\,(\mathsf{lfp}\,X\boldsymbol{\cdot}\{\sigma\}\cup([\![C]\!]\circ[\![\varphi]\!])X)}f(\tau) \\ &= \inf_{\tau\in[\![\neg\psi]\!]\,(\mathsf{lfp}\,X\boldsymbol{\cdot}\{\sigma\}\cup([\![C]\!]\circ[\![\varphi]\!])X)}f(\tau) \\ &= \inf_{\tau\in[\![\neg\psi]\!]\,(\mathsf{lfp}\,X\boldsymbol{\cdot}\{\sigma\}\cup([\![C]\!]\circ[\![\varphi]\!])X)}f(\tau) \end{split}$$

and this concludes the proof.

C PROOFS OF SECTION 4

C.1 Proof of Soundness for sp, Thereom 4.3

Theorem 4.3 (Soundness of sp). For all programs C and final states τ ,

PROOF. We prove Theorem 4.3 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \sup & [\![\mathsf{skip}]\!] \left(f \right) \left(\tau \right) \; = \; f(\tau) \\ & = \; \sup_{\sigma \in \Sigma, \tau \in \{\sigma\}} f(\sigma) \\ & = \; \sup_{\sigma \in \Sigma, \tau \in [\![\mathsf{skip}]\!] \left(\sigma \right)} f(\sigma) \; . \end{split}$$

The assignment x := e: We have

$$sp [x := e] (f) (\tau) = (2\alpha: [x = e[x/\alpha]] \land f[x/\alpha])(\tau)
= (\sup_{\alpha} [x = e[x/\alpha]] \land f[x/\alpha])(\tau)
= \sup_{\alpha: \tau(x) = \tau(e[x/\alpha])} (f[x/\alpha])(\tau)
= \sup_{\alpha: \tau(x) = \tau(e[x/\alpha])} f(\tau[x/\alpha])
= \sup_{\alpha: \tau[x/\alpha][x/\tau(e[x/\alpha])] = \tau} f(\tau[x/\alpha])
= \sup_{\alpha: \tau[x/\alpha][x/\tau[x/\alpha](e)] = \tau} f(\tau[x/\alpha])
= \sup_{\alpha: \tau[x/\alpha][x/\tau[x/\alpha](e)] = \tau} f(\tau[x/\alpha])
= \sup_{\alpha: \tau[x/\alpha][x/\tau[x/\alpha](e)] = \tau} f(\sigma)
= \sup_{\sigma \in \Sigma, \sigma[x/\sigma(e)] = \tau} f(\sigma)
= \sup_{\sigma \in \Sigma, \tau \in \{\sigma[x/\sigma(e)]\}} f(\sigma)
= \sup_{\sigma \in \Sigma, \tau \in [x:=e](\sigma)} f(\sigma) .$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , we proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^\circ_9 \ C_2$: We have

$$\operatorname{sp} \left[\left[C_{2} \right] C_{1} \right] \left(f \right) \left(\tau \right) &= \operatorname{sp} \left[\left[C_{2} \right] \left(f \right) C_{1} \right] \left(f \right) \left(\tau \right) \\
&= \sup_{\sigma' \in \Sigma, \tau \in \left[\left[C_{2} \right] \right] \left(\sigma' \right)} \operatorname{sp} \left[\left[C_{1} \right] C_{1} \right] \left(f \right) \left(\sigma' \right) \\
&= \sup_{\sigma \in \Sigma, \tau \in \left[\left[C_{2} \right] \left(\sigma' \right) \wedge \sigma' \in \left[\left[C_{1} \right] C_{1} \right] \left(\sigma \right)} f(\sigma) \\
&= \sup_{\sigma \in \Sigma, \tau \in \left[\left[C_{2} \right] \left(\left[\left[C_{1} \right] C_{1} \right] C_{1} \right) \right] \left(\sigma \right)} f(\sigma) \\
&= \sup_{\sigma \in \Sigma, \tau \in \left[\left[C_{1} \right] C_{1} \right] C_{1} \left(\sigma \right)} f(\sigma) .$$

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The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{split} & \sup \left[\operatorname{if} \left(\varphi \right) \left\{ C_{1} \right\} \operatorname{else} \left\{ C_{2} \right\} \right] (f) \left(\tau \right) \\ & = \left(\operatorname{sp} \left[\left[C_{1} \right] \left(\left[\varphi \right] \wedge f \right) \, \vee \, \operatorname{sp} \left[\left[C_{2} \right] \left(\left[\neg \varphi \right] \wedge f \right) \right) (\tau) \right) \\ & = \sup_{\sigma \in \Sigma, \tau \in \left[\left[C_{1} \right] \left(\sigma \right) \, } \left(\left[\varphi \right] \wedge f \right) (\sigma) \, \vee \, \sup_{\sigma \in \Sigma, \tau \in \left[\left[C_{2} \right] \left(\sigma \right) \, } \left(\left[\neg \varphi \right] \wedge f \right) (\sigma) \right) \\ & = \sup_{\sigma \in \Sigma, \tau \in \left(\left[\left[C_{1} \right] \right] \circ \left[\varphi \right] \right) (\sigma) \, \vee \, \sup_{\sigma \in \Sigma, \tau \in \left(\left[\left[C_{2} \right] \circ \left[\neg \varphi \right] \right) (\sigma) \, } f(\sigma) \\ & = \sup_{\sigma \in \Sigma, \tau \in \left(\left[\left[C_{1} \right] \circ \left[\varphi \right] \right) \left(\sigma \right) \cup \left(\left[\left[C_{2} \right] \circ \left[\neg \varphi \right] \right) (\sigma) \, } f(\sigma) \, . \end{split}$$

$$& = \sup_{\sigma \in \Sigma, \tau \in \left[\left[\inf \left(\varphi \right) \right\} \left\{ C_{1} \right\} \operatorname{else} \left\{ C_{2} \right\} \right] (\sigma) \, } f(\sigma) \, . \end{split}$$

The nondeterministic choice $\{C_1\} \square \{C_2\}$: We have

$$\operatorname{sp}\left[\!\left\{\left.C_{1}\right.\right\} \, \middle| \, \left\{\left.C_{2}\right.\right\}\right]\!\right]\left(f\right)\left(\tau\right) = \left(\operatorname{sp}\left[\!\left[C_{1}\right]\!\right]\left(f\right) \, \vee \, \operatorname{sp}\left[\!\left[C_{2}\right]\!\right]\left(f\right)\right)\left(\tau\right)$$

$$= \sup_{\sigma \in \Sigma, \tau \in \left[\!\left[C_{1}\right]\!\right]\left(\sigma\right)} f(\sigma) \, \vee \, \sup_{\sigma \in \Sigma, \tau \in \left[\!\left[C_{2}\right]\!\right]\left(\sigma\right)} f(\sigma) \quad \text{(by I.H. on } C_{1}, C_{2})$$

$$= \sup_{\sigma \in \Sigma, \tau \in \left[\!\left[C_{1}\right]\!\right]\left(\sigma\right) \cup \left[\!\left[C_{2}\right]\!\right]\left(\sigma\right)} f(\sigma)$$

$$= \sup_{\sigma \in \Sigma, \tau \in \left[\!\left\{\left.C_{1}\right.\right\}\right] \cap \left\{\left.C_{2}\right.\right\}\right]\left(\sigma\right)} f(\sigma) \, .$$

The loop while (φ) { C }: Let

$$\Psi_f(X) = f \vee \operatorname{sp} \llbracket C \rrbracket (\llbracket \varphi \rrbracket \wedge X) ,$$

be the sp-characteristic functions of the loop while (φ) { C } with respect to preanticipation f and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. We now prove by induction on n that, for all $\tau \in \Sigma$

$$\Psi_f^n(-\infty)(\tau) = \sup_{\sigma \in \Sigma, \tau \in F_{\{\sigma\}}^n(\emptyset)} f(\sigma) . \tag{3}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Psi_f^0(-\infty)(\tau) &= -\infty \\ &= \sup \emptyset \\ &= \sup_{\sigma \in \Sigma, \tau \in \emptyset} f(\sigma) \\ &= \sup_{\sigma \in \Sigma, \tau \in F^0_{\{\sigma\}}(\emptyset)} f(\sigma) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n and all $\tau \in \Sigma$

$$\Psi^n_f(-\infty)(\tau) = \sup_{\sigma \in \Sigma, \tau \in F^n_{\{\sigma\}}(\emptyset)} f(\sigma) \; .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Psi_f^{n+1}(-\infty)(\tau) \\ &= \left(f \ \lor \ \text{sp} \left[\mathbb{C} \right] \left(\left[\varphi \right] \land \Psi_f^n(-\infty) \right) \right)(\tau) \\ &= f(\tau) \ \lor \ \sup_{\sigma \in \Sigma, \tau \in \left[\mathbb{C} \right] \left(\sigma \right)} \left(\left[\varphi \right] \land \Psi_f^n(-\infty) \right)(\sigma) \end{split} \tag{by I.H. on } C) \end{split}$$

$$= f(\tau) \vee \sup_{\sigma \in \Sigma, \tau \in \llbracket C \rrbracket(\sigma)} \sup_{\sigma' \in \Sigma, \sigma \in \llbracket \varphi \rrbracket F_{\{\sigma'\}}^n(\emptyset)} f(\sigma')$$
 (by I.H. on n)
$$= f(\tau) \vee \sup_{\sigma' \in \Sigma, \tau \in (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) F_{\{\sigma'\}}^n(\emptyset)} f(\sigma')$$

$$= \sup_{\sigma' \in \Sigma, \tau \in (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) F_{\{\sigma'\}}^n(\emptyset) \cup \{\sigma'\}} f(\sigma')$$

$$= \sup_{\sigma \in \Sigma, \tau \in F_{\{\sigma^1\}}^{n+1}(\emptyset)} f(\sigma).$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \operatorname{sp} \left[\operatorname{while} \left(\varphi \right) \left\{ C \right\} \right] (f) (\tau) &= \left(\left[\neg \varphi \right] \wedge \left(\operatorname{lfp} X. \ f \ \vee \ \operatorname{sp} \left[C \right] \left(\left[\varphi \right] \wedge X \right) \right) (\tau) \\ &= \left(\left[\neg \varphi \right] \wedge \sup_{n \in \mathbb{N}} \ \Psi_f^n (-\infty) \right) (\tau) \qquad \text{(by Kleene's fixpoint theorem)} \\ &= \sup_{n \in \mathbb{N}} \left(\left[\neg \varphi \right] \wedge \Psi_f^n (-\infty) \right) (\tau) \qquad \text{(by continuity of } \lambda X. \ \left[\neg \varphi \right] \wedge X \right) \\ &= \sup_{n \in \mathbb{N}} \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] F_{\{\sigma\}}^n (\emptyset)} f(\sigma) \qquad \qquad \text{(by Equation 3)} \\ &= \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] \left(\cup_{n \in \mathbb{N}} F_{\{\sigma\}}^n (\emptyset) \right)} f(\sigma) \\ &= \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] \left(\cup_{n \in \mathbb{N}} F_{\{\sigma\}}^n (\emptyset) \right)} f(\sigma) \qquad \qquad \text{(by continuity of } \left[\neg \varphi \right] \right) \\ &= \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] \left(\operatorname{lfp} X. \bullet \left\{ \sigma \right\} \cup \left(\left[C \right] \circ \left[\varphi \right] \right) X \right)} f(\sigma) \\ &= \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] \left(\operatorname{lfp} X. \bullet \left\{ \sigma \right\} \cup \left(\left[C \right] \circ \left[\varphi \right] \right) X \right)} f(\sigma) \\ &= \sup_{\sigma \in \Sigma, \tau \in \left[\neg \varphi \right] \left(\operatorname{lfp} X. \bullet \left\{ \sigma \right\} \cup \left(\left[C \right] \circ \left[\varphi \right] \right) X \right)} f(\sigma), \end{split}$$

and this concludes the proof.

C.2 Proof of Soundness for slp, Thereom 4.6

Theorem 4.6 (Soundness of slp). For all programs C and states $\tau \in \Sigma$,

$$\mathsf{slp} \llbracket C \rrbracket \left(f \right) \left(\tau \right) \quad = \quad \quad \bigwedge_{\sigma \text{ with } \tau \in \llbracket C \rrbracket \sigma} \quad f(\sigma)$$

PROOF. We prove Theorem 4.6 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \mathsf{slp}[\![\mathsf{skip}]\!] (f) (\tau) &= f(\tau) \\ &= \inf_{\sigma \in \Sigma, \tau \in \{\sigma\}} f(\sigma) \\ &= \inf_{\sigma \in \Sigma, \tau \in [\![\mathsf{skip}]\!] (\sigma)} f(\sigma) \;. \end{split}$$

The assignment x := e: We have

$$\begin{split} \mathsf{slp} \llbracket x \coloneqq e \rrbracket \left(f \right) (\tau) \; &= \; \left(\, \boldsymbol{\mathcal{L}} \, \alpha \colon \left[\, \boldsymbol{x} \neq e \left[\boldsymbol{x} / \alpha \right] \right] \, \forall \, f \left[\boldsymbol{x} / \alpha \right] \right) (\tau) \\ &= \; \left(\inf_{\alpha} \, \left[\, \boldsymbol{x} \neq e \left[\boldsymbol{x} / \alpha \right] \right] \, \forall \, f \left[\boldsymbol{x} / \alpha \right] \right) (\tau) \end{split}$$

$$= \inf_{\alpha: \tau(x) = \tau(e[x/\alpha])} (f[x/\alpha])(\tau)$$

$$= \inf_{\alpha: \tau(x) = \tau(e[x/\alpha])} f(\tau[x/\alpha])$$

$$= \inf_{\alpha: \tau[x/\alpha]} f(\tau[x/\alpha])$$

$$= \inf_{\alpha: \tau[x/\alpha][x/\tau(e[x/\alpha])] = \tau} f(\tau[x/\alpha])$$

$$= \inf_{\alpha: \tau[x/\alpha][x/\tau[x/\alpha](e)] = \tau} f(\tau[x/\alpha])$$

$$= \inf_{\alpha: \tau[x/\alpha][x/\tau[x/\alpha](e)] = \tau} f(\sigma) \qquad \text{(By taking } \sigma = \tau[x/\alpha])$$

$$= \inf_{\sigma \in \Sigma, \tau \in \{\sigma[x/\sigma(e)]\}} f(\sigma)$$

$$= \inf_{\sigma \in \Sigma, \tau \in [x:=e](\sigma)} f(\sigma).$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , we proceed with the inductive step on the composite statements.

The sequential composition $C_1 \stackrel{\circ}{\circ} C_2$: We have

$$\begin{aligned} \operatorname{slp} \llbracket C_2 \, {}^\circ_{} \, C_1 \rrbracket \, (f) \, (\tau) &= \, \operatorname{slp} \llbracket C_2 \rrbracket \, \big(\operatorname{slp} \llbracket C_1 \rrbracket \, (f) \big) \, (\tau) \\ &= \, \inf_{\sigma' \in \Sigma, \tau \in \llbracket C_2 \rrbracket \, (\sigma')} \operatorname{slp} \llbracket C_1 \rrbracket \, (f) \, (\sigma') \\ &= \, \inf_{\sigma \in \Sigma, \tau \in \llbracket C_2 \rrbracket \, (\sigma') \wedge \sigma' \in \llbracket C_1 \rrbracket \, (\sigma)} f(\sigma) \\ &= \, \inf_{\sigma \in \Sigma, \tau \in \llbracket C_2 \rrbracket \, (\llbracket C_1 \rrbracket \, (\sigma))} f(\sigma) \\ &= \, \inf_{\sigma \in \Sigma, \tau \in \llbracket C_2 \rrbracket \, (\llbracket C_1 \rrbracket \, (\sigma))} f(\sigma) \\ &= \, \inf_{\sigma \in \Sigma, \tau \in \llbracket C_1 \rrbracket \, (\sigma)} f(\sigma) \, . \end{aligned}$$

The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{split} & \mathsf{slp} \llbracket \mathsf{if} \ (\varphi) \ \{ C_1 \} \ \mathsf{else} \ \{ C_2 \} \rrbracket \ (f) \ (\tau) \\ & = \ \left(\mathsf{slp} \llbracket C_1 \rrbracket \left(\llbracket \neg \varphi \rrbracket \vee f \right) \ \land \ \mathsf{slp} \llbracket C_2 \rrbracket \left(\llbracket \varphi \rrbracket \vee f \right) \right) (\tau) \\ & = \ \inf_{\sigma \in \Sigma, \tau \in \llbracket C_1 \rrbracket (\sigma)} \left(\llbracket \neg \varphi \rrbracket \vee f \right) (\sigma) \ \land \ \inf_{\sigma \in \Sigma, \tau \in \llbracket C_2 \rrbracket (\sigma)} \left(\llbracket \varphi \rrbracket \vee f \right) (\sigma) \\ & = \ \inf_{\sigma \in \Sigma, \tau \in (\llbracket C_1 \rrbracket \circ \llbracket \varphi \rrbracket) (\sigma)} f(\sigma) \ \land \ \inf_{\sigma \in \Sigma, \tau \in (\llbracket C_2 \rrbracket \circ \llbracket \neg \varphi \rrbracket) (\sigma)} f(\sigma) \\ & = \ \inf_{\sigma \in \Sigma, \tau \in (\llbracket C_1 \rrbracket \circ \llbracket \varphi \rrbracket) (\sigma) \cup (\llbracket C_2 \rrbracket \circ \llbracket \neg \varphi \rrbracket) (\sigma)} f(\sigma) \\ & = \ \inf_{\sigma \in \Sigma, \tau \in \llbracket \mathsf{if} \ (\varphi) \ \{ C_1 \} \ \mathsf{else} \ \{ C_2 \} \rrbracket (\sigma)} f(\sigma) \ . \end{split}$$

The nondeterministic choice $\{C_1\} \square \{C_2\}$: We have

$$\begin{split} \operatorname{slp}[\![\{C_1\} \ \square \ \{C_2\}]\!](f)(\tau) &= \left(\operatorname{slp}[\![C_1]\!](f) \ \land \ \operatorname{slp}[\![C_2]\!](f)\right)(\tau) \\ &= \inf_{\sigma \in \Sigma, \tau \in [\![C_1]\!](\sigma)} f(\sigma) \ \land \ \inf_{\sigma \in \Sigma, \tau \in [\![C_2]\!](\sigma)} f(\sigma) \quad \text{(by I.H. on } C_1, C_2) \\ &= \inf_{\sigma \in \Sigma, \tau \in [\![C_1]\!](\sigma) \cup [\![C_2]\!](\sigma)} f(\sigma) \\ &= \inf_{\sigma \in \Sigma, \tau \in [\![\{C_1\}\!] \cap \{C_2\}\!](\sigma)} f(\sigma) \ . \end{split}$$

The loop while (φ) { C }: Let

$$\Psi_f(X) = f \wedge \operatorname{slp}[\![C]\!] ([\neg \varphi] \vee X) ,$$

be the slp-characteristic functions of the loop while (φ) { C } with respect to preanticipation f and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) $\{C\}$ with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. We now prove by induction on n that, for all $\tau \in \Sigma$

$$\Psi_f^n(+\infty)(\tau) = \inf_{\sigma \in \Sigma, \tau \in F_{(\sigma)}^n(\emptyset)} f(\sigma) . \tag{4}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Psi_f^0(+\infty)(\tau) &= +\infty \\ &= \inf \emptyset \\ &= \inf_{\sigma \in \Sigma, \tau \in \emptyset} f(\sigma) \\ &= \inf_{\sigma \in \Sigma, \tau \in F^0_{\{\sigma\}}(\emptyset)} f(\sigma) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n and all $\tau \in \Sigma$

$$\Psi_f^n(+\infty)(\tau) = \inf_{\sigma \in \Sigma, \tau \in F_{\{\sigma\}}^n(\emptyset)} f(\sigma) .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Psi_{f}^{n+1}(+\infty)(\tau) \\ &= \left(f \wedge \text{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_{f}^{n}(+\infty)\right)\right)(\tau) \\ &= f(\tau) \wedge \inf_{\sigma \in \Sigma, \tau \in [\![C]\!](\sigma)} \left([\neg \varphi] \vee \Psi_{f}^{n}(+\infty)\right)(\sigma) \qquad \text{(by I.H. on } C) \\ &= f(\tau) \wedge \inf_{\sigma \in \Sigma, \tau \in [\![C]\!](\sigma)} \inf_{\sigma' \in \Sigma, \sigma \in [\![\varphi]\!] F_{\{\sigma'\}}^{n}(\emptyset)} f(\sigma') \qquad \text{(by I.H. on } n) \\ &= f(\tau) \wedge \inf_{\sigma' \in \Sigma, \tau \in ([\![C]\!] \circ [\![\varphi]\!]) F_{\{\sigma'\}}^{n}(\emptyset)} f(\sigma') \\ &= \inf_{\sigma' \in \Sigma, \tau \in ([\![C]\!] \circ [\![\varphi]\!]) F_{\{\sigma'\}}^{n}(\emptyset) \cup \{\sigma'\}} f(\sigma') \\ &= \inf_{\sigma \in \Sigma, \tau \in F_{\{\sigma\}}^{n+1}(\emptyset)} f(\sigma) \,. \end{split}$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \mathsf{slp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(f)\,(\tau) &= \, \left([\varphi]\,\,\vee\,\, \left(\mathsf{gfp}\,X.\,f\,\,\wedge\,\,\mathsf{slp}[\![C]\!]\,([\neg\varphi]\,\vee\,X)\right)\right)(\tau) \\ &= \, \left([\varphi]\,\,\vee\,\,\inf_{n\in\mathbb{N}}\,\,\Psi^n_f(+\infty)\right)(\tau) \qquad \text{(by Kleene's fixpoint theorem)} \\ &= \,\,\inf_{n\in\mathbb{N}}\,\,\left([\varphi]\,\,\vee\,\,\Psi^n_f(+\infty)\right)(\tau) \qquad \text{(by co-continuity of }\lambda X.\,\,[\varphi]\,\,\vee\,X) \\ &= \,\,\inf_{n\in\mathbb{N}}\,\,\inf_{\sigma\in\Sigma,\tau\in[\![\neg\varphi]\!]\,F^n_{\{\sigma\}}(\emptyset)}f(\sigma) \qquad \qquad \text{(by Equation 4)} \\ &= \,\,\inf_{\sigma\in\Sigma,\tau\in[\![\neg\varphi]\!]\,(\cup_{n\in\mathbb{N}}F^n_{\{\sigma\}}(\emptyset))}f(\sigma) \\ &= \,\,\inf_{\sigma\in\Sigma,\tau\in[\![\neg\varphi]\!]\,(\cup_{n\in\mathbb{N}}F^n_{\{\sigma\}}(\emptyset))}f(\sigma) \qquad \qquad \text{(by continuity of }[\![\neg\varphi]\!]) \end{split}$$

$$\begin{split} &= \inf_{\sigma \in \Sigma, \tau \in \llbracket \neg \varphi \rrbracket \text{ (Ifp } X \bullet \{\sigma\} \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket)X)} f(\sigma) \\ &\qquad \qquad \text{(by Kleene's fixpoint theorem)} \\ &= \inf_{\sigma \in \Sigma, \tau \in \llbracket \text{while} (\varphi) \{C\} \rrbracket (\sigma)} f(\sigma) \,, \end{split}$$

and this concludes the proof.

D PROOFS OF SECTION 5

D.1 Proof of Healthiness Properties of Quantitative Transformers, Theorem 5.1

Each of the properties is proven individually below.

- Quantitative universal conjunctiveness: Theorem D.1, D.2;
- Quantitative universal disjunctiveness: Theorem D.3, D.4;
- Strictness: Corollary D.5, D.6;
- Costrictness: Corollary D.7, D.8;
- Monotonicity: Corollary D.9

Theorem D.1 (Quantitative universal conjunctiveness of wp). For any set of quantities $\subseteq \mathbb{A}$,

$$\operatorname{wp} [\![C]\!] (\sup S) = \sup \operatorname{wp} [\![C]\!] (S) .$$

PROOF. We prove Theorem D.1 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \operatorname{wp} \left[\operatorname{skip} \right] &(\sup S) &= \sup_{g \in S} S \\ &= \sup_{g \in S} \operatorname{wp} \left[\operatorname{skip} \right] (g) \\ &= \sup_{g \in S} \operatorname{wp} \left[\operatorname{skip} \right] (S) \; . \end{split}$$

The assignment x := e: We have

$$\operatorname{wp} [\![x \coloneqq e]\!] (\sup S) = (\sup S) [x/e]$$

$$= \left(\lambda \sigma. \sup_{g \in S} g(\sigma) \right) [x/e]$$

$$= \left(\lambda \sigma. \sup_{g \in S} g [x/e] (\sigma) \right)$$

$$= \sup_{g \in S} g [x/e]$$

$$= \sup_{g \in S} \operatorname{wp} [\![x \coloneqq e]\!] (g)$$

$$= \sup_{g \in S} \operatorname{wp} [\![x \coloneqq e]\!] (S) .$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , Theorem D.1 holds. We proceed with the inductive step on the composite statements.

$$\begin{aligned} &\operatorname{wp} \left[\! \left[C_1 \right] \circ C_2 \right] \left(\operatorname{sup} S \right) &= \operatorname{wp} \left[\! \left[C_1 \right] \left(\operatorname{wp} \left[\! \left[C_2 \right] \right] \left(\operatorname{sup} S \right) \right) \right. \\ &= \operatorname{wp} \left[\! \left[C_1 \right] \left(\operatorname{sup} \operatorname{wp} \left[\! \left[C_2 \right] \right] \left(S \right) \right) \\ &= \operatorname{sup} \operatorname{wp} \left[\! \left[C_1 \right] \left(\operatorname{wp} \left[\left[C_2 \right] \right] \left(S \right) \right) \right. \end{aligned} \end{aligned}$$
 (by I.H. on C_2)
$$&= \operatorname{sup} \operatorname{wp} \left[\! \left[C_1 \right] \circ C_2 \right] \left(S \right) .$$

The conditional branching if (φ) { C_1 } else { C_2 }: Here we reason in the reverse direction from the cases before. We have

The loop while (φ) { C }: Let

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wp} [\![C]\!](X) ,$$

be the wp-characteristic function of the loop while (φ) { C } with respect to any postanticipation $f \in \mathbb{A}$ and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. Observe that $\Phi_f(X)$ is continuous by inductive hypothesis on C and by composition of continuous functions. We now prove by induction on n that

$$\Phi^n_{\sup S}(-\infty) = \sup_{g \in S} \Phi^n_g(-\infty) . \tag{5}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Phi^0_{\sup S}(-\infty) &=& = -\infty \\ &=& \sup_{g \in S} -\infty \\ &=& \sup_{g \in S} \Phi^0_g(-\infty) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n

$$\Phi^n_{\sup S}(-\infty) \ = \ \sup_{g \in S} \Phi^n_g(-\infty) \ .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Phi^{n+1}_{\sup S}(-\infty) \\ &= \left[\neg \varphi\right] \wedge \sup S \ \lor \ \left[\varphi\right] \wedge \mathsf{wp} \left[\!\!\left[C\right]\!\!\right] \left(\!\!\left.\Phi^n_{\sup S}(-\infty)\right) \right) \\ &= \left[\neg \varphi\right] \wedge \sup S \ \lor \ \left[\varphi\right] \wedge \mathsf{wp} \left[\!\!\left[C\right]\!\!\right] \left(\!\!\sup_{g \in S} \Phi^n_g(-\infty)\right) \end{split} \tag{by I.H. on } n)$$

$$= \left[\neg \varphi \right] \land \sup S \lor \left[\varphi \right] \land \sup_{g \in S} \operatorname{wp} \left[C \right] \left(\Phi_g^n(-\infty) \right)$$
 (by I.H. on C)
$$= \sup_{g \in S} \left(\left[\neg \varphi \right] \land g \right) \lor \sup_{g \in S} \left(\left[\varphi \right] \land \operatorname{wp} \left[C \right] \left(\Phi_g^n(-\infty) \right) \right)$$

$$= \sup_{g \in S} \left(\left[\neg \varphi \right] \land g \lor \left[\varphi \right] \land \operatorname{wp} \left[C \right] \left(\Phi_g^n(-\infty) \right) \right)$$

$$= \sup_{g \in S} \Phi_g^{n+1}(-\infty) .$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \operatorname{wp} \left[\!\!\left[\operatorname{while} \left(\varphi\right)\left\{C\right\}\right]\!\!\right] &(\sup S) &= \operatorname{lfp} X. \left[\neg\varphi\right] \wedge \sup S \vee \left[\varphi\right] \wedge \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (X) \\ &= \sup_{n \in \mathbb{N}} \Phi_{\sup S}^n(-\infty) & \text{(by Kleene's fixpoint theorem)} \\ &= \sup_{n \in \mathbb{N}} \sup_{g \in S} \Phi_g^n(-\infty) & \text{(by Equation 5)} \\ &= \sup_{g \in S} \sup_{n \in \mathbb{N}} \Phi_g^n(-\infty) \\ &= \sup_{g \in S} \operatorname{wp} \left[\!\!\left[\operatorname{while} \left(\varphi\right)\left\{C\right\}\right]\!\!\right] (g) & \text{(by Kleene's fixpoint theorem)} \\ &= \sup_{g \in S} \operatorname{wp} \left[\!\!\left[\operatorname{while} \left(\varphi\right)\left\{C\right\}\right]\!\!\right] (S) \,, \end{split}$$

and this concludes the proof.

Theorem D.2 (Quantitative universal conjunctiveness of sp). For any set of quantities $\subseteq \mathbb{A}$, $\operatorname{sp} \mathbb{C} \mathbb{C} (\sup S) = \sup \operatorname{sp} \mathbb{C} \mathbb{C} (S)$.

PROOF. We prove Theorem D.2 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \operatorname{sp} \left[\operatorname{skip} \right] &(\operatorname{sup} S) &= \operatorname{sup} S \\ &= \operatorname{sup} g \\ &= \operatorname{sup} \operatorname{sp} \left[\operatorname{skip} \right] (g) \\ &= \operatorname{sup} \operatorname{sp} \left[\operatorname{skip} \right] (S) \ . \end{split}$$

The assignment x := e: We have

$$\begin{split} \operatorname{sp} \left[\!\!\left[x \coloneqq e\right]\!\!\right] (\operatorname{sup} S) &= \operatorname{\textbf{Z}} \alpha \colon \left[x = e\left[x/\alpha\right]\right] \wedge (\operatorname{sup} S) \left[x/\alpha\right] \\ &= \operatorname{\textbf{Z}} \alpha \colon \left[x = e\left[x/\alpha\right]\right] \wedge \left(\lambda \sigma \boldsymbol{\cdot} \sup_{g \in S} g(\sigma)\right) \left[x/\alpha\right] \\ &= \operatorname{\textbf{Z}} \alpha \colon \left[x = e\left[x/\alpha\right]\right] \wedge \left(\lambda \sigma \boldsymbol{\cdot} \sup_{g \in S} g\left[x/\alpha\right](\sigma)\right) \\ &= \operatorname{\textbf{Z}} \alpha \colon \left[x = e\left[x/\alpha\right]\right] \wedge \sup_{g \in S} g\left[x/\alpha\right] \\ &= \operatorname{\textbf{Z}} \alpha \colon \sup_{g \in S} \left[x = e\left[x/\alpha\right]\right] \wedge g\left[x/\alpha\right] \end{split}$$

$$= \sup_{g \in S} \mathbf{Z}\alpha \colon [x = e [x/\alpha]] \land g [x/\alpha]$$

$$= \sup_{g \in S} \operatorname{sp} [x \coloneqq e] (g)$$

$$= \sup_{g \in S} \operatorname{sp} [x \coloneqq e] (S) .$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , Theorem D.2 holds. We proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^{\circ}_{\circ} C_2$: We have

The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{split} & \operatorname{sp} \left[\!\left[\operatorname{if}\left(\varphi\right)\left\{C_{1}\right\} \operatorname{else}\left\{C_{2}\right\}\right]\left(\operatorname{sup}S\right)\right. \\ & = \left.\operatorname{sp}\left[\!\left[C_{1}\right]\!\right]\left(\left[\varphi\right] \wedge \operatorname{sup}S\right) \ \ \, \vee \ \, \operatorname{sp}\left[\!\left[C_{2}\right]\!\right]\left(\left[\neg\varphi\right] \wedge \operatorname{sup}S\right) \\ & = \left.\operatorname{sp}\left[\!\left[C_{1}\right]\!\right]\left(\operatorname{sup}\left[\varphi\right] \wedge S\right) \ \, \vee \ \, \operatorname{sp}\left[\!\left[C_{2}\right]\!\right]\left(\operatorname{sup}\left[\neg\varphi\right] \wedge S\right) \\ & = \left.\operatorname{sup}\left.\operatorname{sp}\left[\!\left[C_{1}\right]\!\right]\left(\left[\varphi\right] \wedge S\right) \ \, \vee \ \, \operatorname{sup}\left.\operatorname{sp}\left[\!\left[C_{2}\right]\!\right]\left(\left[\neg\varphi\right] \wedge S\right) \right. \end{split} \right. \tag{by I.H. on C_{1} and C_{2})} \\ & = \left.\operatorname{sup}\left.\left(\operatorname{sp}\left[\!\left[C_{1}\right]\!\right]\left(\left[\varphi\right] \wedge S\right) \ \, \vee \ \, \operatorname{sp}\left[\!\left[C_{2}\right]\!\right]\left(\left[\neg\varphi\right] \wedge S\right)\right) \right. \\ & = \left.\operatorname{sup}\left.\operatorname{sp}\left[\!\left[\operatorname{if}\left(\varphi\right) \left\{C_{1}\right\} \right] \operatorname{else}\left\{C_{2}\right\}\right]\left(S\right) \ \, . \end{split}$$

The loop while $(\varphi) \{C\}$: Let

$$\Psi_f(X) = f \vee \operatorname{sp} [\![C]\!] ([\varphi] \wedge X),$$

be the sp-characteristic function of the loop while (φ) { C } with respect to any preanticipation $f \in \mathbb{A}$ and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. Observe that $\Psi_f(X)$ is continuous by inductive hypothesis on C and by composition of continuous functions. We now prove by induction on n that

$$\Psi^n_{\sup S}(-\infty) = \sup_{g \in S} \Psi^n_g(-\infty) . \tag{6}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Psi^0_{\sup S}(-\infty) &=& = -\infty \\ &=& \sup_{g \in S} -\infty \\ &=& \sup_{g \in S} \Psi^0_g(-\infty) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n

$$\Psi^n_{\sup S}(-\infty) \ = \ \sup_{g \in S} \Psi^n_g(-\infty) \ .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Psi^{n+1}_{\sup S}(-\infty) \\ &= \sup S \ \lor \ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\left[\varphi\right] \wedge \Psi^n_{\sup S}(-\infty)\right) \\ &= \sup S \ \lor \ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\left[\varphi\right] \wedge \sup_{g \in S} \Psi^n_g(-\infty)\right) \\ &= \sup S \ \lor \ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\sup_{g \in S} \left[\varphi\right] \wedge \Psi^n_g(-\infty)\right) \\ &= \sup S \ \lor \ \sup_{g \in S} \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\left[\varphi\right] \wedge \Psi^n_g(-\infty)\right) \\ &= \sup_{g \in S} g \ \lor \ \sup_{g \in S} \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\left[\varphi\right] \wedge \Psi^n_g(-\infty)\right) \\ &= \sup_{g \in S} \left(g \ \lor \ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] \left(\left[\varphi\right] \wedge \Psi^n_g(-\infty)\right)\right) \\ &= \sup_{g \in S} \Psi^{n+1}_g(-\infty) \ . \end{split}$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \operatorname{sp} \left[\operatorname{while} \left(\varphi \right) \left\{ C \right\} \right] &(\operatorname{sup} S) &= \left[\neg \varphi \right] \wedge \left(\operatorname{lfp} X. \ \operatorname{sup} S \ \vee \ \operatorname{sp} \left[C \right] \left(\left[\varphi \right] \wedge X \right) \right) \\ &= \left[\neg \varphi \right] \wedge \sup_{n \in \mathbb{N}} \ \Psi_{\sup S}^{n} \left(-\infty \right) & \text{(by Kleene's fixpoint theorem)} \\ &= \left[\neg \varphi \right] \wedge \sup_{n \in \mathbb{N}} \sup_{g \in S} \Psi_{g}^{n} \left(-\infty \right) \\ &= \left[\neg \varphi \right] \wedge \sup_{g \in S} \sup_{n \in \mathbb{N}} \Psi_{g}^{n} \left(-\infty \right) \\ &= \left[\neg \varphi \right] \wedge \sup_{g \in S} \sup_{n \in \mathbb{N}} \Psi_{g}^{n} \left(-\infty \right) \\ &= \sup_{g \in S} \left(\left[\neg \varphi \right] \wedge \sup_{n \in \mathbb{N}} \Psi_{g}^{n} \left(-\infty \right) \right) \\ &= \sup_{g \in S} \sup_{g \in S} \left[\operatorname{while} \left(\varphi \right) \left\{ C \right\} \right] \left(g \right) & \text{(by Kleene's fixpoint theorem)} \\ &= \sup_{g \in S} \sup_{g \in S} \left[\operatorname{while} \left(\varphi \right) \left\{ C \right\} \right] \left(S \right), \end{split}$$

and this concludes the proof.

Theorem D.3 (Quantitative universal disjunctiveness of wlp). For any set of quantities $\subseteq \mathbb{A}$,

$$\mathsf{wlp} \llbracket C \rrbracket \text{ (inf } S) = \inf \mathsf{wlp} \llbracket C \rrbracket \text{ (} S) \text{ .}$$

PROOF. We prove Theorem D.3 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$wlp[skip] (inf S) = \inf S$$
$$= \inf_{g \in S} g$$

$$= \inf_{g \in S} \text{ wlp[skip]}(g)$$
$$= \inf \text{ wlp[skip]}(S).$$

The assignment x := e: We have

$$\begin{aligned} \mathsf{wlp}[\![x \coloneqq e]\!] & (\inf S) \ = \ (\inf S) \ [x/e] \\ & = \ \left(\lambda \sigma \boldsymbol{.} \inf_{g \in S} g(\sigma)\right) [x/e] \\ & = \ \left(\lambda \sigma \boldsymbol{.} \inf_{g \in S} g \left[x/e\right] (\sigma)\right) \\ & = \inf_{g \in S} g \left[x/e\right] \\ & = \inf_{g \in S} \mathsf{wlp}[\![x \coloneqq e]\!] & (g) \\ & = \inf_{g \in S} \mathsf{wlp}[\![x \coloneqq e]\!] & (S) \ . \end{aligned}$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , Theorem D.3 holds. We proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^\circ_2 C_2$: We have

$$\begin{aligned} \mathsf{wlp} \llbracket C_1 \, \mathring{\varsigma} \, C_2 \rrbracket \, (\inf S) &= \, \mathsf{wlp} \llbracket C_1 \rrbracket \, \big(\mathsf{wlp} \llbracket C_2 \rrbracket \, (\inf S) \big) \\ &= \, \mathsf{wlp} \llbracket C_1 \rrbracket \, \big(\inf \, \mathsf{wlp} \llbracket C_2 \rrbracket \, (S) \big) \\ &= \, \inf \, \mathsf{wlp} \llbracket C_1 \rrbracket \, \big(\mathsf{wlp} \llbracket C_2 \rrbracket \, (S) \big) \\ &= \, \inf \, \mathsf{wlp} \llbracket C_1 \, \mathring{\varsigma} \, C_2 \rrbracket \, (S) \, . \end{aligned}$$
 (by I.H. on C_2)
$$= \, \inf \, \mathsf{wlp} \llbracket C_1 \, \mathring{\varsigma} \, C_2 \rrbracket \, (S) \, .$$

The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{split} &\operatorname{\mathsf{wlp}}\llbracket\operatorname{\mathsf{if}}\ (\varphi)\ \{C_1\}\ \operatorname{\mathsf{else}}\ \{C_2\}\rrbracket\ (\operatorname{\mathsf{inf}}\ S) \\ &= \ [\varphi] \land \operatorname{\mathsf{wlp}}\llbracket C_1\rrbracket\ (\operatorname{\mathsf{inf}}\ S) \ \lor \ [\neg\varphi] \land \operatorname{\mathsf{wlp}}\llbracket C_2\rrbracket\ (\operatorname{\mathsf{inf}}\ S) \\ &= \ [\varphi] \land \operatorname{\mathsf{inf}}\ \operatorname{\mathsf{wlp}}\llbracket C_1\rrbracket\ (S) \ \lor \ [\neg\varphi] \land \operatorname{\mathsf{inf}}\ \operatorname{\mathsf{wlp}}\llbracket C_2\rrbracket\ (S) \end{split} \qquad \text{(by I.H. on C_1 and C_2)} \\ &= \inf \left(\left[\varphi\right] \land \operatorname{\mathsf{wlp}}\llbracket C_1\rrbracket\ (S) \right) \ \lor \ \operatorname{\mathsf{inf}}\ \left(\left[\neg\varphi\right] \land \operatorname{\mathsf{wlp}}\llbracket C_2\rrbracket\ (S) \right) \\ &= \lambda\sigma. \begin{cases} \inf \left(\operatorname{\mathsf{wlp}}\llbracket C_1\rrbracket\ (S) \right) \ \operatorname{\mathsf{if}}\ \sigma \ \models \ \varphi \\ \inf \left(\operatorname{\mathsf{wlp}}\llbracket C_2\rrbracket\ (S) \right) \ \operatorname{\mathsf{otherwise}} \end{cases} \\ &= \inf \left(\left[\varphi\right] \land \operatorname{\mathsf{wlp}}\llbracket C_1\rrbracket\ (S) \ \lor \ \left[\neg\varphi\right] \land \operatorname{\mathsf{wlp}}\llbracket C_2\rrbracket\ (S) \right) \\ &= \inf \operatorname{\mathsf{wlp}}\llbracket \operatorname{\mathsf{if}}\ (\varphi)\ \{C_1\}\ \operatorname{\mathsf{else}}\ \{C_2\} \rrbracket\ (S)\ . \end{split}$$

The loop while (φ) { C }: Let

$$\Phi_f(X) = [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp} \llbracket C \rrbracket (X) ,$$

be the wlp-characteristic function of the loop while (φ) { C } with respect to any postanticipation $f \in \mathbb{A}$ and

$$F_S(X) \ = \ S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X \,,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. Observe that $\Phi_f(X)$ is continuous by inductive hypothesis on C and by composition of continuous functions. We now prove by induction on n that

$$\Phi_{\inf S}^{n}(+\infty) = \inf_{g \in S} \Phi_{g}^{n}(+\infty) . \tag{7}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Phi^0_{\inf S}(+\infty) &= &= +\infty \\ &= &\inf_{g \in S} +\infty \\ &= &\inf_{g \in S} \Phi^0_g(+\infty) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed n

$$\Phi^n_{\inf S}(+\infty) \ = \ \inf_{g \in S} \Phi^n_g(+\infty) \ .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Phi_{\inf S}^{n+1}(+\infty) \\ &= \left[\neg \varphi \right] \wedge \inf S \ \lor \ \left[\varphi \right] \wedge \operatorname{wlp} \left[\mathcal{C} \right] \left(\Phi_{\inf S}^{n}(+\infty) \right) \\ &= \left[\neg \varphi \right] \wedge \inf S \ \lor \ \left[\varphi \right] \wedge \operatorname{wlp} \left[\mathcal{C} \right] \left(\inf_{g \in S} \Phi_{g}^{n}(+\infty) \right) \\ &= \left[\neg \varphi \right] \wedge \inf S \ \lor \ \left[\varphi \right] \wedge \inf_{g \in S} \operatorname{wlp} \left[\mathcal{C} \right] \left(\Phi_{g}^{n}(+\infty) \right) \\ &= \inf_{g \in S} \left(\left[\neg \varphi \right] \wedge g \right) \ \lor \ \inf_{g \in S} \left(\left[\varphi \right] \wedge \operatorname{wlp} \left[\mathcal{C} \right] \left(\Phi_{g}^{n}(+\infty) \right) \right) \\ &= \lambda \sigma \cdot \begin{cases} \inf_{g \in S} \left(\operatorname{wlp} \left[\mathcal{C} \right] \left(\Phi_{g}^{n}(+\infty) \right) \right) & \text{if } \sigma \ \models \varphi \\ \inf_{g \in S} \left(g \right) & \text{otherwise} \end{cases} \\ &= \inf_{g \in S} \left(\left[\neg \varphi \right] \wedge g \ \lor \ \left[\varphi \right] \wedge \operatorname{wlp} \left[\mathcal{C} \right] \left(\Phi_{g}^{n}(+\infty) \right) \right) \\ &= \inf_{g \in S} \Phi_{g}^{n+1}(+\infty) \ . \end{split}$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \mathsf{wlp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(\inf S) &= \mathsf{gfp}\,X.\,\,[\,\neg\varphi]\,\land\,\inf S\,\,\lor\,\,[\,\varphi]\,\land\,\mathsf{wlp}[\![C]\!]\,(X) \\ &= \inf_{n\in\mathbb{N}}\,\Phi^n_{\inf S}(+\infty) \qquad \qquad \text{(by Kleene's fixpoint theorem)} \\ &= \inf_{n\in\mathbb{N}}\,\inf_{g\in S}\,\Phi^n_g(+\infty) \qquad \qquad \text{(by Equation 7)} \\ &= \inf_{g\in S}\,\inf_{n\in\mathbb{N}}\,\Phi^n_g(+\infty) \\ &= \inf_{g\in S}\,\mathsf{wlp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(g) \quad \text{(by Kleene's fixpoint theorem)} \\ &= \inf\,\mathsf{wlp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(S)\,, \end{split}$$

and this concludes the proof.

Theorem D.4 (Quantitative universal disjunctiveness of slp). For any set of quantities $\subseteq \mathbb{A}$,

$$slp \llbracket C \rrbracket \text{ (inf } S) = \inf slp \llbracket C \rrbracket \text{ (}S) \text{ .}$$

PROOF. We prove Theorem D.4 by induction on the structure of C. For the induction base, we have the atomic statements:

The effectless program skip: We have

$$\begin{split} \mathsf{slp}[\![\mathsf{skip}]\!] (\inf S) &= \inf S \\ &= \inf_{g \in S} g \\ &= \inf_{g \in S} \mathsf{slp}[\![\mathsf{skip}]\!] (g) \\ &= \inf \mathsf{slp}[\![\mathsf{skip}]\!] (S) \;. \end{split}$$

The assignment x := e: We have

$$\begin{aligned} \operatorname{slp}[\![x \coloneqq e]\!] & (\inf S) &= \boldsymbol{\ell} \, \alpha \colon [x \neq e \, [x/\alpha]] \, \vee (\inf S) \, [x/\alpha] \\ &= \boldsymbol{\ell} \, \alpha \colon [x \neq e \, [x/\alpha]] \, \vee \left(\lambda \sigma . \inf_{g \in S} g(\sigma) \right) [x/\alpha] \\ &= \boldsymbol{\ell} \, \alpha \colon [x \neq e \, [x/\alpha]] \, \vee \left(\lambda \sigma . \inf_{g \in S} g \, [x/\alpha] \, (\sigma) \right) \\ &= \boldsymbol{\ell} \, \alpha \colon [x \neq e \, [x/\alpha]] \, \vee \inf_{g \in S} g \, [x/\alpha] \\ &= \boldsymbol{\ell} \, \alpha \colon \inf_{g \in S} [x \neq e \, [x/\alpha]] \, \vee g \, [x/\alpha] \\ &= \inf_{g \in S} \boldsymbol{\ell} \, \alpha \colon [x \neq e \, [x/\alpha]] \, \vee g \, [x/\alpha] \\ &= \inf_{g \in S} \operatorname{slp}[\![x \coloneqq e]\!] & (g) \\ &= \inf \operatorname{slp}[\![x \coloneqq e]\!] & (S) \, . \end{aligned}$$

This concludes the proof for the atomic statements.

Induction Hypothesis: For arbitrary but fixed programs C, C_1 , C_2 , Theorem D.4 holds. We proceed with the inductive step on the composite statements.

The sequential composition $C_1 \ ^\circ_2 C_2$: We have

$$slp\llbracket C_1 \, \, \, \, \, \, \, C_2 \rrbracket \, (\inf S) = slp\llbracket C_2 \rrbracket \, \big(slp\llbracket C_1 \rrbracket \, (\inf S) \big)$$

$$= slp\llbracket C_2 \rrbracket \, \big(\inf \, slp\llbracket C_1 \rrbracket \, (S) \big) \qquad (by I.H. on C_1)$$

$$= \inf \, slp\llbracket C_2 \rrbracket \, \big(slp\llbracket C_1 \rrbracket \, (S) \big) \qquad (by I.H. on C_2)$$

$$= \inf \, slp\llbracket C_1 \, \, \, \, \, \, \, C_2 \rrbracket \, (S) .$$

The conditional branching if (φ) { C_1 } else { C_2 }: We have

$$\begin{aligned} & \mathsf{slp}\llbracket\mathsf{if}\ (\varphi)\ \{C_1\ \}\ \mathsf{else}\ \{C_2\ \}\rrbracket\ (\mathsf{inf}\ S) \\ & = \ \mathsf{slp}\llbracketC_1\rrbracket\ ([\neg\varphi]\ \lor\ \mathsf{inf}\ S)\ \land\ \mathsf{slp}\llbracketC_2\rrbracket\ ([\varphi]\ \lor\ \mathsf{inf}\ S) \\ & = \ \mathsf{slp}\llbracketC_1\rrbracket\ (\mathsf{inf}\ [\neg\varphi]\ \lor\ S)\ \land\ \mathsf{slp}\llbracketC_2\rrbracket\ (\mathsf{inf}\ [\varphi]\ \lor\ S) \\ & = \ \mathsf{inf}\ \mathsf{slp}\llbracketC_1\rrbracket\ ([\neg\varphi]\ \lor\ S)\ \land\ \mathsf{inf}\ \mathsf{slp}\llbracketC_2\rrbracket\ ([\varphi]\ \lor\ S) \end{aligned} \qquad \text{(by I.H. on C_1 and C_2)} \\ & = \ \mathsf{inf}\ (\mathsf{slp}\llbracketC_1\rrbracket\ ([\neg\varphi]\ \lor\ S)\ \land\ \mathsf{slp}\llbracketC_2\rrbracket\ ([\varphi]\ [\neg\varphi]\ S)) \end{aligned}$$

=
$$\inf slp[if(\varphi) \{C_1\} else\{C_2\}](S)$$
.

The loop while (φ) { C }: Let

$$\Psi_f(X) = f \wedge \operatorname{slp}[\![C]\!] ([\neg \varphi] \vee X) ,$$

be the slp-characteristic function of the loop while (φ) { C } with respect to any preanticipation $f \in \mathbb{A}$ and

$$F_S(X) = S \cup (\llbracket C \rrbracket \circ \llbracket \varphi \rrbracket) X,$$

be the collecting semantics characteristic functions of the loop while (φ) { C } with respect to any input $S \in \mathcal{P}(\mathsf{Conf})$. Observe that $\Psi_f(X)$ is continuous by inductive hypothesis on C and by composition of continuous functions. We now prove by induction on n that

$$\Psi_{\inf S}^{n}(+\infty) = \inf_{g \in S} \Psi_{g}^{n}(+\infty) . \tag{8}$$

For the induction base n = 0, consider the following:

$$\begin{split} \Psi^0_{\inf S}(+\infty) &=& = +\infty \\ &=& \inf_{g \in S} +\infty \\ &=& \inf_{g \in S} \Psi^0_g(+\infty) \;. \end{split}$$

As induction hypothesis, we have for arbitrary but fixed *n*

$$\Psi^n_{\inf S}(+\infty) \ = \ \inf_{g \in S} \Psi^n_g(+\infty) \ .$$

For the induction step $n \longrightarrow n + 1$, consider the following:

$$\begin{split} &\Psi_{\inf S}^{n+1}(+\infty) \\ &= \inf S \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_{\inf S}^n(+\infty) \right) \\ &= \inf S \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \inf_{g \in S} \Psi_g^n(+\infty) \right) \\ &= \inf S \wedge \operatorname{slp}[\![C]\!] \left(\inf_{g \in S} [\neg \varphi] \vee \Psi_g^n(+\infty) \right) \\ &= \inf S \wedge \inf_{g \in S} \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \\ &= \inf_{g \in S} g \wedge \inf_{g \in S} \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \\ &= \inf_{g \in S} \left(g \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \right) \\ &= \inf_{g \in S} \left(g \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \right) \\ &= \inf_{g \in S} \left(g \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \right) \\ &= \inf_{g \in S} \left(g \wedge \operatorname{slp}[\![C]\!] \left([\neg \varphi] \vee \Psi_g^n(+\infty) \right) \right) \end{split}$$

This concludes the induction on *n*. Now we have:

$$\begin{split} \mathsf{slp}[\![\mathsf{while}\,(\,\varphi\,)\,\{\,C\,\}]\!]\,(\inf S) &= [\,\varphi\,]\,\,\vee\,\,\big(\mathsf{gfp}\,X\boldsymbol{.}\,\,\inf S\,\,\wedge\,\,\mathsf{slp}[\![\,C\,]\!]\,([\,\neg\varphi\,]\,\,\vee\,\,X)\big) \\ &= [\,\varphi\,]\,\,\vee\,\,\inf_{n\in\mathbb{N}}\,\,\Psi^n_{\inf S}(+\infty) \qquad \qquad \text{(by Kleene's fixpoint theorem)} \\ &= [\,\varphi\,]\,\,\vee\,\inf_{n\in\mathbb{N}}\,\inf_{g\in S}\,\Psi^n_g(+\infty) \qquad \qquad \qquad \text{(by Equation 8)} \\ &= [\,\varphi\,]\,\,\vee\,\inf_{g\in S}\,\inf_{n\in\mathbb{N}}\,\Psi^n_g(+\infty) \end{split}$$

$$\begin{split} &= \left[\varphi\right] \vee \inf_{g \in S} \inf_{n \in \mathbb{N}} \Psi_g^n(+\infty) \\ &= \inf_{g \in S} (\left[\varphi\right] \vee \inf_{n \in \mathbb{N}} \Psi_g^n(+\infty)) \\ &= \inf_{g \in S} \sup[\text{while } (\varphi) \left\{C\right\}] (g) \quad \text{(by Kleene's fixpoint theorem)} \\ &= \inf \sup[\text{while } (\varphi) \left\{C\right\}] (S), \end{split}$$

and this concludes the proof.

COROLLARY D.5 (STRICTNESS OF wp). For all programs C, wp $\llbracket C \rrbracket$ is strict, i.e.

$$\operatorname{wp} [\![C]\!] (-\infty) = -\infty.$$

Proof.

$$\operatorname{wp} \llbracket C \rrbracket (-\infty) = \lambda \sigma. \sup_{\tau \in \llbracket C \rrbracket (\sigma)} -\infty(\tau)$$
 (by Theorem 3.7)
= $-\infty$.

COROLLARY D.6 (STRICTNESS OF sp). For all programs C, sp $[\![C]\!]$ is strict, i.e.

$$\operatorname{sp} \llbracket C \rrbracket (-\infty) = -\infty.$$

Proof.

$$sp [\![C]\!] (-\infty) = \lambda \tau. \sup_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} -\infty(\sigma) \qquad \text{(by Theorem 4.3)}$$

$$= -\infty.$$

Corollary D.7 (Co-strictness of wlp). For all programs C, wp $[\![C]\!]$ is co-strict, i.e.

$$\operatorname{wlp}[C](+\infty) = +\infty$$
.

Proof.

$$\mathsf{wlp}[\![C]\!](+\infty) = \lambda \sigma \cdot \inf_{\tau \in [\![C]\!](\sigma)} + \infty(\tau)$$
 (by Theorem 3.10)
= $+\infty$.

Corollary D.8 (Co-strictness of slp). For all programs C, $slp[\![C]\!]$ is co-strict, i.e.

$$slp[C](+\infty) = +\infty.$$

Proof.

$$slp[\![C]\!] (+\infty) = \lambda \tau \cdot \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} + \infty(\sigma) \qquad \text{(by Theorem 4.6)}$$

$$= +\infty.$$

Corollary D.9 (Monotonicity of Quantitative Transformers). For all programs $C, f, g \in \mathbb{A}$, we have

$$f \leq g$$
 implies $\operatorname{ttt} [\![C]\!] (f) \leq \operatorname{ttt} [\![C]\!] (g)$, for $\operatorname{ttt} \in \{\text{wp, wlp, sp, slp}\}$

Proof. Direct consequence of universal conjunctiveness and universal disjunctiveness.

D.2 Proof of Linearity, Theorem 5.2

THEOREM 5.2 (LINEARITY). For all programs C, wp $[\![C]\!]$ and sp $[\![C]\!]$ are sublinear, and wlp $[\![C]\!]$ and slp $[\![C]\!]$ are superlinear, i.e. for all $f, g \in \mathbb{A}$ and non-negative constants $r \in \mathbb{R}_{>0}$,

$$\begin{split} \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) & \leq r \cdot \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{wp} \left[\!\!\left[C\right]\!\!\right] (g) \ , \\ \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) & \leq r \cdot \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (g) \ , \\ r \cdot \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (g) & \leq \operatorname{wlp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) \ , \quad \text{and} \\ r \cdot \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (f) + \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (g) & \leq \operatorname{slp} \left[\!\!\left[C\right]\!\!\right] (r \cdot f + g) \ . \end{split}$$

PROOF. For wp we have:

$$\begin{split} & \operatorname{wp} \, \llbracket C \rrbracket \, (r \cdot f + g) \\ &= \, \lambda \sigma. \quad \sup_{\tau \in \llbracket C \rrbracket \sigma} (r \cdot f + g)(\tau) \qquad \qquad \text{(by Theorem 3.7)} \\ &= \, \lambda \sigma. \quad \sup_{\tau \in \llbracket C \rrbracket \sigma} ((r \cdot f)(\tau) + g(\tau)) \\ &\leq \, \lambda \sigma. \quad \sup_{\tau \in \llbracket C \rrbracket \sigma} (r \cdot f)(\tau) + \sup_{\tau \in \llbracket C \rrbracket \sigma} g(\tau) \\ &= \, \lambda \sigma. \, r \cdot \sup_{\tau \in \llbracket C \rrbracket \sigma} f(\tau) + \sup_{\tau \in \llbracket C \rrbracket \sigma} g(\tau) \qquad \qquad \text{(sup}(r \cdot A) = r \cdot \sup A \text{ for } A \subseteq \mathbb{R}, r \in \mathbb{R}_{\geq 0}) \\ &= \, r \cdot \lambda \sigma. \quad \sup_{\tau \in \llbracket C \rrbracket \sigma} f(\tau) + \lambda \sigma. \quad \sup_{\tau \in \llbracket C \rrbracket \sigma} g(\tau) \\ &= \, r \cdot \operatorname{wp} \, \llbracket C \rrbracket \, (f) + \operatorname{wp} \, \llbracket C \rrbracket \, (g) \, . \qquad \qquad \text{(by Theorem 3.7)} \end{split}$$

For wp we have:

$$\begin{split} & \sup \left \| C \right \| (r \cdot f + g) \\ &= \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} (r \cdot f + g)(\sigma) \qquad \qquad \text{(by Theorem 4.3)} \\ &= \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} ((r \cdot f)(\sigma) + g(\sigma)) \\ &\leq \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} (r \cdot f)(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma) \\ &= \lambda \tau. \quad r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma) \qquad \text{(sup}(r \cdot A) = r \cdot \sup A \text{ for } A \subseteq \mathbb{R}, r \in \mathbb{R}_{\geq 0}) \\ &= r \cdot \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma) \\ &= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \lambda \tau. \quad \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma) \\ &= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma) \end{aligned}$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} g(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma)$$

$$= r \cdot \sup_{\sigma \in \Sigma, \tau \in \left \| C \right \| \sigma} f(\sigma) + \sup_{\sigma \in \Sigma,$$

For wlp we have:

$$r \cdot \mathsf{wlp}[\![C]\!] (f) + \mathsf{wlp}[\![C]\!] (g)$$

$$= r \cdot \lambda \sigma \cdot \inf_{\tau \in [\![C]\!] \sigma} f(\tau) + \lambda \sigma \cdot \inf_{\tau \in [\![C]\!] \sigma} g(\tau)$$

$$= \lambda \sigma \cdot r \cdot \inf_{\tau \in [\![C]\!] \sigma} f(\tau) + \inf_{\tau \in [\![C]\!] \sigma} g(\tau)$$
(by Theorem 3.10)

For slp we have:

$$\begin{split} r \cdot \mathsf{slp}[\![C]\!] & (f) + \mathsf{slp}[\![C]\!] (g) \\ &= r \cdot \lambda \tau. \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} f(\sigma) + \lambda \tau. \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} g(\sigma) \\ &= \lambda \tau. \quad r \cdot \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} f(\sigma) + \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} g(\sigma) \\ &= \lambda \tau. \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} (r \cdot f)(\sigma) + \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} g(\sigma) \quad (\inf(r \cdot A) = r \cdot \inf A \text{ for } A \subseteq \mathbb{R}, r \in \mathbb{R}_{\geq 0}) \\ &\leq \lambda \tau. \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} ((r \cdot f)(\sigma) + g(\sigma)) \\ &= \lambda \tau. \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} (r \cdot f + g)(\sigma) \\ &= \mathsf{slp}[\![C]\!] (r \cdot f + g) \; . \end{split} \tag{by Theorem 4.6}$$

D.3 Proof of Embedding Classical into Quantitative Transformers, Theorem 5.3

Theorem 5.3 (Embedding Classical into Quantitative Transformers). For all <u>deterministic</u> programs C and predicates ψ , we have

$$\mathsf{wp} \, \llbracket C \rrbracket \, (\llbracket \psi \rrbracket) \ = \ \left[\mathsf{wp} \, \llbracket C \rrbracket \, (\psi) \right] \qquad \mathsf{and} \qquad \mathsf{wlp} \llbracket C \rrbracket \, (\llbracket \psi \rrbracket) \ = \ \left[\mathsf{wlp} \, \llbracket C \rrbracket \, (\psi) \right] \, ,$$

and for all programs C and predicates ψ , we have

$$\operatorname{sp}\left[\!\left[C\right]\!\right]\left(\left[\psi\right]\right) \ = \ \left[\operatorname{sp}\left[\!\left[C\right]\!\right]\left(\psi\right)\right] \quad \text{ and } \quad \operatorname{slp}\left[\!\left[C\right]\!\right]\left(\left[\psi\right]\right) \ = \ \left[\operatorname{slp}\left[\!\left[C\right]\!\right]\left(\psi\right)\right] \ .$$

PROOF. For wp we have:

$$\operatorname{wp} \llbracket C \rrbracket (\llbracket F \rrbracket) = \lambda \sigma \cdot \begin{cases} \llbracket F \rrbracket (\tau) & \text{if } \llbracket C \rrbracket (\sigma) = \{\tau\} \\ -\infty & \text{otherwise} \end{cases}$$

$$= \lambda \sigma \cdot \begin{cases} +\infty & \text{if } \llbracket C \rrbracket (\sigma) = \{\tau\} \land \tau \models F \\ -\infty & \text{otherwise} \end{cases}$$

$$= \left[\operatorname{wp} \llbracket C \rrbracket (F) \right] .$$
(by Corollary 5.5)

For wlp we have:

$$\begin{aligned} \mathsf{wlp} \llbracket C \rrbracket \left(\llbracket F \rrbracket \right) &= \lambda \sigma. \begin{cases} \llbracket F \rrbracket \left(\tau \right) & \text{if } \llbracket C \rrbracket \left(\sigma \right) = \{ \tau \} \\ + \infty & \text{otherwise} \end{cases} \\ &= \lambda \sigma. \begin{cases} - \infty & \text{if } \llbracket C \rrbracket \left(\sigma \right) = \{ \tau \} \land \tau \not\models F \\ + \infty & \text{otherwise} \end{cases} \\ &= \left[\mathsf{wlp} \llbracket C \rrbracket \left(F \right) \right] \; . \end{aligned}$$

For sp we have:

$$\begin{split} \operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (\left[G\right]) &= \lambda \tau \text{.} \quad \sup_{\sigma \in \Sigma, \tau \in \left[\!\!\left[C\right]\!\!\right] \sigma} \left[G\right] (\sigma) \\ &= \lambda \tau \text{.} \quad \begin{cases} +\infty & \text{if } \exists \sigma \in \Sigma, \tau \in \left[\!\!\left[C\right]\!\!\right] (\sigma) \land \sigma \ \models \ G \\ -\infty & \text{otherwise} \end{cases} \\ &= \left[\operatorname{sp} \left[\!\!\left[C\right]\!\!\right] (G)\right] \,, \end{split}$$

For slp we have:

$$\begin{split} \operatorname{slp}[\![C]\!] ([G]) &= \lambda \tau \text{.} \quad \inf_{\sigma \in \Sigma, \tau \in [\![C]\!] \sigma} [G] (\sigma) & \text{(by Theorem 4.6)} \\ &= \lambda \tau \text{.} \quad \begin{cases} -\infty & \text{if } \exists \sigma \in \Sigma, \tau \in [\![C]\!] (\sigma) \land \sigma \not\models G \\ +\infty & \text{otherwise} \end{cases} \\ &= \lambda \tau \text{.} \quad \begin{cases} +\infty & \text{if } \forall \sigma \in \Sigma, \tau \notin [\![C]\!] (\sigma) \lor \sigma \models G \\ -\infty & \text{otherwise} \end{cases} \\ &= \lambda \tau \text{.} \quad \begin{cases} +\infty & \text{if } \forall \sigma \in \Sigma, \tau \in [\![C]\!] (\sigma) \implies \sigma \models G \\ -\infty & \text{otherwise} \end{cases} \\ &= [\![\operatorname{slp}[\![C]\!] (\psi)]\!] \text{.} \end{split}$$

D.4 Proof of Liberal-Non-liberal Duality, Theorem 5.4

THEOREM 5.4 (LIBERAL-NON-LIBERAL DUALITY). For any program C and quantity f, we have

$$\mathsf{wp} \, \llbracket C \rrbracket \, (f) \ = \ - \, \mathsf{wlp} \, \llbracket C \rrbracket \, (-f) \qquad \text{ and } \qquad \mathsf{sp} \, \llbracket C \rrbracket \, (f) \ = \ - \, \mathsf{slp} \llbracket C \rrbracket \, (-f) \ .$$

PROOF. For wp and wlp we have:

$$\operatorname{wp} [\![C]\!] (f) = \lambda \sigma. \sup_{\tau \in [\![C]\!] \sigma} f(\tau)$$
 (by Theorem 3.7)
$$= \lambda \sigma. - \inf_{\tau \in [\![C]\!] \sigma} -f(\tau)$$
 (sup $A = -\inf(-A)$)
$$= -\operatorname{wlp} [\![C]\!] (-f) .$$

For sp and slp we have:

$$\begin{split} \operatorname{sp} \left[\!\!\left[C\right]\!\!\right](g) &= \lambda \tau \boldsymbol{.} \quad \sup_{\sigma \in \Sigma, \tau \in \left[\!\!\left[C\right]\!\!\right] \sigma} g(\sigma) \\ &= \lambda \tau \boldsymbol{.} \quad - \inf_{\sigma \in \Sigma, \tau \in \left[\!\!\left[C\right]\!\!\right] \sigma} - g(\sigma) \\ &= -\operatorname{slp} \left[\!\!\left[C\right]\!\!\right](-g) \; . \end{split} \tag{sup } A = -\inf(-A))$$

E PROOFS OF SECTION 6

E.1 Proof of Galois Connection between wlp and sp, Theorem 6.1

Theorem 6.1 (Galois Connection between wlp and sp). For all $C \in nGCL$ and $q, f \in A$:

$$q \leq \text{wlp}[\![C]\!](f)$$
 iff $\text{sp}[\![C]\!](q) \leq f$.

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Proof.

$$\begin{split} g & \leq \ \text{wlp} \llbracket C \rrbracket \left(f \right) \iff \forall \sigma \in \Sigma \text{. } g(\sigma) \leq \ \text{wlp} \llbracket C \rrbracket \left(f \right) \left(\sigma \right) \\ & \iff \forall \sigma \in \Sigma \text{. } g(\sigma) \leq \inf_{\tau \in \llbracket C \rrbracket \left(\sigma \right)} f(\tau) \qquad \text{(by Theorem 3.10)} \\ & \iff \forall \sigma, \tau \in \Sigma \colon \tau \in \llbracket C \rrbracket \left(\sigma \right) \text{. } g(\sigma) \leq f(\tau) \\ & \iff \forall \tau \in \Sigma \text{. } \sup_{\sigma \in \Sigma, \tau \in \llbracket C \rrbracket \left(\sigma \right)} g(\sigma) \leq f(\tau) \\ & \iff \forall \tau \in \Sigma \text{. } \text{sp} \llbracket C \rrbracket \left(g \right) \left(\tau \right) \leq f(\tau) \qquad \text{(by Theorem 4.3)} \\ & \iff \text{sp} \llbracket C \rrbracket \left(g \right) \leq f \, . \end{split}$$

E.2 Proof of Galois Connection between wp and slp, Theorem 6.2

Theorem 6.2 (Galois Connection between wp and slp). For all $C \in nGCL$ and $q, f \in A$:

$$\operatorname{wp} [\![C]\!] (f) \leq g \quad \text{iff} \quad f \leq \operatorname{slp} [\![C]\!] (g)$$

Proof.

$$\begin{split} \operatorname{wp} & [\![\mathcal{C}]\!] (f) \leq g \iff \forall \sigma \in \Sigma \text{. } \operatorname{wp} [\![\mathcal{C}]\!] (f) \, (\sigma) \leq g(\sigma) \\ & \iff \forall \sigma \in \Sigma \text{. } \sup_{\tau \in [\![\mathcal{C}]\!] (\sigma)} f(\tau) \leq g(\sigma) \qquad \text{(by Theorem 3.7)} \\ & \iff \forall \sigma, \tau \in \Sigma \colon \tau \in [\![\mathcal{C}]\!] (\sigma) \text{. } f(\tau) \leq g(\sigma) \\ & \iff \forall \tau \in \Sigma \text{. } f(\tau) \leq \inf_{\sigma \in \Sigma, \tau \in [\![\mathcal{C}]\!] (\sigma)} g(\sigma) \\ & \iff \forall \tau \in \Sigma \text{. } f(\tau) \leq \operatorname{slp} [\![\mathcal{C}]\!] (g) \, (\tau) \qquad \text{(by Theorem 4.3)} \\ & \iff f \leq \operatorname{slp} [\![\mathcal{C}]\!] (g) \; . \end{split}$$

F PROOFS OF SECTION 7

F.1 Proof of Induction Rules for Loops, Theorem 7.1

THEOREM 7.1 (INDUCTION RULES FOR LOOPS). The following proof rules for loops are valid:

$$\frac{g \ \, \le \ \, i \ \, \le \ \, [\neg\varphi] \land f \ \, \lor \ \, [\varphi] \land \mathsf{wlp}[\![C]\!]\,(i)}{g \ \, \le \ \, \mathsf{wlp}[\![\mathsf{while}\,(\varphi)\,\{C\,\}]\!]\,(f)} \ \, \mathsf{while-wlp}$$

$$\frac{g \ \, \lor \ \, \mathsf{sp}[\![C]\!]\,([\varphi] \land i) \ \, \le \ \, i \ \, \mathsf{and} \ \, [\neg\varphi] \land i \ \, \le \ \, f}{\mathsf{sp}[\![\mathsf{while}\,(\varphi)\,\{C\,\}]\!]\,(g) \ \, \le \ \, f} \ \, \mathsf{while-sp}$$

$$\frac{[\neg\varphi] \land f \ \, \lor \ \, [\varphi] \land \mathsf{wp}[\![C]\!]\,(i) \ \, \le \ \, i \ \, \le \ \, g}{\mathsf{wp}[\![\mathsf{while}\,(\varphi)\,\{C\,\}]\!]\,(f) \ \, \le \ \, g} \ \, \mathsf{while-wp}$$

$$\frac{i \ \, \le \ \, g \land \ \, \mathsf{slp}[\![C]\!]\,([\neg\varphi] \lor i) \ \, \mathsf{and} \ \, f \ \, \le \ \, [\varphi] \lor i}{f \ \, \le \ \, \mathsf{slp}[\![\mathsf{while}\,(\varphi)\,\{C\,\}]\!]\,(g)} \ \, \mathsf{while-slp}$$

PROOF. We prove each rule individually.

For while-wlp we have:

$$i \leq [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp}[\![C]\!](i)$$
 (Premise of the rule)
 $\implies i \leq \mathsf{gfp} \ X. \ [\neg \varphi] \land f \lor [\varphi] \land \mathsf{wlp}[\![C]\!](X)$ (by Park's Induction [Park 1969])
 $\implies i \leq \mathsf{wlp}[\![\mathsf{while}(\varphi) \ \{C\}\!]\!](f)$ (by Definition 3.8)
 $\implies g \leq \mathsf{wlp}[\![\mathsf{while}(\varphi) \ \{C\}\!]\!](f)$ ($g \leq i$ and transitivity of $\leq i$)

For while-sp we have:

$$g \vee \operatorname{sp} \llbracket C \rrbracket (\llbracket \varphi \rrbracket \wedge i) \leq i \qquad \qquad \text{(Premise of the rule)}$$

$$\Longrightarrow \operatorname{Ifp} X. \ g \vee \operatorname{sp} \llbracket C \rrbracket (\llbracket \varphi \rrbracket \wedge X) \leq i \qquad \qquad \text{(by Park's Induction } \llbracket \operatorname{Park } 1969 \rrbracket)$$

$$\Longrightarrow \llbracket \neg \varphi \rrbracket \wedge \operatorname{Ifp} X. \ g \vee \operatorname{sp} \llbracket C \rrbracket (\llbracket \varphi \rrbracket \wedge X) \leq \llbracket \neg \varphi \rrbracket \wedge i \qquad \text{(by monotonicity of } \lambda X. \llbracket \neg \varphi \rrbracket \wedge X)$$

$$\Longrightarrow \operatorname{sp} \llbracket \operatorname{while} (\varphi) \{C\} \rrbracket (g) \leq \llbracket \neg \varphi \rrbracket \wedge i \qquad \text{(by Definition } 4.1)$$

$$\Longrightarrow \operatorname{sp} \llbracket \operatorname{while} (\varphi) \{C\} \rrbracket (g) \leq f \qquad (\llbracket \neg \varphi \rrbracket \wedge i \leq f \text{ and transitivity of } \leq)$$

For while-wp we have:

For while-slp we have:

$$\begin{split} i & \leq g \wedge \mathsf{slp}\llbracket C \rrbracket \left(\llbracket \neg \varphi \rrbracket \vee i \right) & \text{(Premise of the rule)} \\ & \Longrightarrow i \leq \mathsf{gfp} \, X. \, g \wedge \mathsf{slp}\llbracket C \rrbracket \left(\llbracket \neg \varphi \rrbracket \vee X \right) & \text{(by Park's Induction [Park 1969])} \\ & \Longrightarrow \llbracket \varphi \rrbracket \vee i \leq \llbracket \varphi \rrbracket \vee \mathsf{gfp} \, X. \, g \wedge \mathsf{slp}\llbracket C \rrbracket \left(\llbracket \neg \varphi \rrbracket \vee X \right) & \text{(by monotonicity of } \lambda X. \llbracket \varphi \rrbracket \vee X \right) \\ & \Longrightarrow \llbracket \varphi \rrbracket \vee i \leq \mathsf{slp}\llbracket \mathsf{while} \left(\varphi \right) \left\{ C \right\} \rrbracket \left(g \right) & \text{(by Definition 4.4)} \\ & \Longrightarrow f \leq \mathsf{slp}\llbracket \mathsf{while} \left(\varphi \right) \left\{ C \right\} \rrbracket \left(g \right) & \text{(} f \leq \llbracket \varphi \rrbracket \vee i \text{ and transitivity of } \leq \right) \end{split}$$

F.2 Proof of Proposition 7.3

Proposition 7.3. The following proof rules for loops are valid:

PROOF. We prove each statement individually. Let ${}^{\rm sp}\Psi_f$ and ${}^{\rm slp}\Psi_f$ be, respectively, the sp-characteristic and slp-characteristic functions of while (φ) { C }. For sp we have:

$$\begin{split} & [\varphi] \wedge f \leq f \\ & \text{sp} \, \llbracket C \rrbracket \, ([\varphi] \wedge f) \leq \text{sp} \, \llbracket C \rrbracket \, (f) \\ & \text{sp} \, \llbracket C \rrbracket \, ([\varphi] \wedge f) \leq f \end{split} \qquad \text{(by Monotonicity of sp)} \\ & f \vee \text{sp} \, \llbracket C \rrbracket \, ([\varphi] \wedge f) \leq f \\ & \text{(by hypothesis and transitivity of } \leq) \\ & \text{sp} \, \Psi_f^2(-\infty) \leq \text{sp} \, \Psi_f(-\infty) \end{split} \qquad \text{(by Definition 4.1)}$$

Hence, the Kleene's iterates have converged immediately and the least fixpoint is exactly:

If
$$p(X) \cdot {}^{sp}\Psi_f(X) = {}^{sp}\Psi_f(-\infty) = f$$
,

and thus we conclude:

$$\begin{split} \operatorname{sp} \left[\operatorname{while} \left(\, \varphi \, \right) \, \left\{ \, C \, \right\} \right] \, (f) &= \, \left[\, \neg \varphi \right] \, \wedge \, \operatorname{lfp} \, X \boldsymbol{\cdot}^{\, \operatorname{sp}} \Psi_f(X) \\ &= \, \left[\, \neg \varphi \right] \, \wedge \, f \, \, . \end{split} \qquad \qquad (\operatorname{lfp} \, X \boldsymbol{\cdot}^{\, \operatorname{sp}} \Psi_f(X) \, = \, f) \end{split}$$

For slp we have:

$$f \leq [\neg \varphi] \vee f \tag{\dagger}$$

$$\operatorname{slp}[\![C]\!](f) \leq \operatorname{slp}[\![C]\!]([\neg \varphi] \vee f) \tag{by Monotonicity of slp)}$$

$$f \leq \operatorname{slp}[\![C]\!]([\neg \varphi] \vee f) \tag{by hypothesis and transitivity of } \leq)$$

$$f \leq f \wedge \operatorname{slp}[\![C]\!]([\neg \varphi] \vee f) \tag{by Monotonicity of } \lambda X. f \wedge X)$$

$$\operatorname{slp}\Psi_f(+\infty) \leq \operatorname{slp}\Psi_f^2(+\infty) \tag{by Definition 4.4}$$

Hence, the Kleene's iterates have converged immediately and the greatest fixpoint is exactly:

gfp
$$X \cdot {}^{\text{slp}}\Psi_f(X) = {}^{\text{slp}}\Psi_f(+\infty) = f$$
,

and thus we conclude:

$$\begin{split} \operatorname{sp} \left[\operatorname{while} \left(\, \varphi \, \right) \, \left\{ \, C \, \right\} \right] \, (f) &= \, \left[\, \varphi \, \right] \, \vee \, \operatorname{gfp} \, X_{\:\raisebox{1pt}{\text{\circle*{1.5}}}} \, \operatorname{slp} \Psi_f(X) & \qquad \qquad \left(\operatorname{by} \, \operatorname{Definition} \, 4.4 \right) \\ &= \, \left[\, \varphi \, \right] \, \vee \, f \, \, . & \qquad \left(\operatorname{gfp} \, X_{\:\raisebox{1pt}{\text{\circle*{1.5}}}} \, \operatorname{slp} \Psi_f(X) \, = \, f \right) \end{split}$$

G FULL CALCULATIONS OF SECTION 8

G.1 Full calculations of Example 8.1

Example G.1. The strongest post of $C = \text{if } (hi > 7) \{ lo := 99 \} \text{ else } \{ lo := 80 \} \text{ for the pre-anticipation } hi = \lambda \sigma \cdot \sigma(hi) \text{ are:}$

and

$$\begin{split} & \mathsf{slp}[\![\mathsf{if}\ (\mathit{hi} > 7)\ \{\mathit{lo} \coloneqq 99\,\}\ \mathsf{else}\ \{\mathit{lo} \coloneqq 80\,\}]\!]\ (\mathit{hi}) \\ & = \ \mathsf{slp}[\![\mathit{lo} \coloneqq 99]\!]\ ([\mathit{hi} \le 7] \lor \mathit{hi}) \quad \land \ \mathsf{slp}[\![\mathit{lo} \coloneqq 80]\!]\ ([\mathit{hi} > 7] \lor \mathit{hi}) \\ & = \ \big(\mathcal{L}\alpha \colon [\mathit{lo} \ne 99] \lor ([\mathit{hi} \le 7] \lor \mathit{hi})\ [\mathit{lo}/\alpha] \big) \quad \land \quad \big(\mathcal{L}\alpha \colon [\mathit{lo} \ne 80] \lor ([\mathit{hi} > 7] \lor \mathit{hi})\ [\mathit{lo}/\alpha] \big) \\ & = \ \big([\mathit{lo} \ne 99] \lor [\mathit{hi} \le 7] \lor \mathit{hi} \big) \quad \land \quad \big([\mathit{lo} \ne 80] \lor [\mathit{hi} > 7] \lor \mathit{hi} \big) \ . \end{split}$$

G.2 Full calculations of Example 8.2

Example G.2. The strongest post of C = hi := hi + 5 % while $(lo < hi) \{ lo := lo + 1 \}$ for the preanticipation $hi = \lambda \sigma_{\bullet} \sigma(hi)$ are:

$$sp [C] (hi) = [lo \ge hi] \land (hi - 5)$$

$$slp[C] (hi) = [lo < hi] \lor (hi - 5)$$

In fact, we have:

$$\begin{split} &\operatorname{sp}\left[\!\!\left[hi \coloneqq hi + 5\,\right]\!\!\right] \left(hi\right) \\ &= \operatorname{sp}\left[\!\!\left[\operatorname{while}\left(\left.lo < hi\right)\right\} \left\{\left.lo \coloneqq lo + 1\right\}\right]\!\!\right] \left(\operatorname{sp}\left[\!\!\left[hi \coloneqq hi + 5\right]\!\!\right] \left(hi\right)\right) \\ &= \operatorname{sp}\left[\!\!\left[\operatorname{while}\left(\left.lo < hi\right)\right\} \left\{\left.lo \coloneqq lo + 1\right\}\right]\!\!\right] \left(\operatorname{\textbf{Z}}\alpha \colon \left[hi = \alpha + 5\right] \land \alpha\right) \\ &= \operatorname{sp}\left[\!\!\left[\operatorname{while}\left(\left.lo < hi\right)\right\} \left\{\left.lo \coloneqq lo + 1\right\}\right]\!\!\right] \left(hi - 5\right) \\ &= \left[\left.lo \ge hi\right] \land \operatorname{lfp} X \cdot \Psi_{hi - 5}(X) \\ &= \left[\left.lo \ge hi\right] \land \Psi_{hi - 5}^{\omega}(-\infty) \end{split} \qquad \text{(by Kleene's fixpoint theorem)} \end{split}$$

Let us compute some Kleene's iterates:

$$\begin{split} \Psi_{hi-5}(-\infty) &= (hi-5) \ \lor \ \text{sp} \ \llbracket lo \coloneqq lo+1 \rrbracket \ (\lceil lo < hi \rceil \land -\infty) \\ &= (hi-5) \ \lor \ \text{sp} \ \llbracket lo \coloneqq lo+1 \rrbracket \ (-\infty) \\ &= (hi-5) \ \lor \ (-\infty) \qquad \qquad \text{(by Theorem 5.1 (2))} \\ &= (hi-5) \ \lor \ \text{sp} \ \llbracket lo \coloneqq lo+1 \rrbracket \ (\lceil lo < hi \rceil \land (hi-5)) \\ &= (hi-5) \ \lor \ (2\alpha \colon [lo=\alpha+1] \land [\alpha < hi] \land (hi-5)) \\ &= (hi-5) \ \lor \ (\lceil lo < hi+1 \rceil \land (hi-5)) \qquad \qquad (\alpha = lo-1 \text{ is selected)} \\ &= (hi-5) \end{aligned}$$

The iteration sequence has converged (in just 2 iterations), so we obtain:

$$\begin{split} & \operatorname{sp} \left[hi \coloneqq hi + 5 \, \mathring{\varsigma} \, \operatorname{while} \left(\, lo < hi \, \right) \left\{ \, lo \coloneqq lo + 1 \, \right\} \right] (hi) \\ & = \left[\, lo \ge hi \right] \, \land \, \Psi^{\omega}_{hi-5} (-\infty) \\ & = \left[\, lo \ge hi \right] \, \land \, (hi-5) \end{split}$$

Similarly, for slp we have:

$$\begin{split} & \mathsf{slp}[\![hi \coloneqq hi + 5\, \S\, \, \mathsf{while}\, (\,\, lo < hi\,) \, \{\, lo \coloneqq lo + 1\, \}]\!] \, (hi) \\ & = \, \mathsf{slp}[\![\, \mathsf{while}\, (\,\, lo < hi\,) \, \{\, lo \coloneqq lo + 1\, \}]\!] \, \big(\mathsf{slp}[\![\, hi \coloneqq hi + 5]\!] \, (hi) \big) \\ & = \, \mathsf{slp}[\![\, \mathsf{while}\, (\,\, lo < hi\,) \, \{\, lo \coloneqq lo + 1\, \}]\!] \, \big(\, \mathcal{L}\, \alpha \colon \, [\, hi \neq \alpha + 5] \, \lor \, \alpha \big) \\ & = \, \, \mathsf{slp}[\![\, \mathsf{while}\, (\,\, lo < hi\,) \, \{\, lo \coloneqq lo + 1\, \}]\!] \, (hi - 5) \qquad \qquad (\alpha = hi - 5 \text{ is selected}) \\ & = \, \, [\, lo < hi\,] \, \lor \, \, \Psi_{hi - 5}^\omega(+\infty) \qquad \qquad \text{(by Kleene's fixpoint theorem)} \end{split}$$

Let us compute some Kleene's iterates:

Again, the iteration sequence has converged in 2 iterations, so we conclude:

$$\begin{split} & \sup [\![hi \coloneqq hi + 5\, \S\, \, \text{while} \, (\,lo < hi\,) \, \{\, lo \coloneqq lo + 1\,\}]\!] \, (hi) \\ & = \, [\,lo < hi\,] \, \lor \, \Psi^\omega_{hi-5}(+\infty) \\ & = \, [\,lo < hi\,] \, \lor \, (hi-5) \end{split}$$

H EXTENDED COMPARISON WITH [Aguirre and Katsumata 2020]

In this section, we show how our w(l)p, restricted to the fragment of loop-free programs, can be derived by instantiating [Aguirre and Katsumata 2020, Corollary 4.6]. Consider:

- the powerset monad \mathcal{P} ;
- the lattice of extended reals $\mathbb{R}^{\pm\infty}$;
- the Eilenberg-Moore algebra sup: $\mathcal{P}(\mathbb{R}^{\pm\infty}) \to \mathbb{R}^{\pm\infty}$.

As a consequence of [Aguirre and Katsumata 2020, Corollary 4.6], we obtain an abstract operation awp: $(A \to \mathcal{P}(B)) \to (B \to \mathbb{R}^{\pm \infty}) \to (A \to \mathbb{R}^{\pm \infty})$ such that:

$$\operatorname{awp}(C)(f)(a) = \sup_{b \in C(a)} f(b)$$

Note that awp preserves all joins in the position of f. By taking as monad the collecting semantics starting from a single state $[\![C]\!]: \Sigma \to \mathcal{P}(\Sigma)$ which maps states into set of states, for all loop-free programs $C, f \in \mathbb{A}$, $\sigma \in \Sigma$ we have:

$$\operatorname{awp}(\llbracket C \rrbracket)(f)(\sigma) = \sup_{\tau \in \llbracket C \rrbracket(\sigma)} f(\tau) = \operatorname{wp} \llbracket C \rrbracket(f) \ .$$

Similarly, if we consider the Eilenberg-Moore algebra inf, we obtain an abstract operator awlp such that:

$$\operatorname{awlp}(\llbracket C \rrbracket)(f)(\sigma) = \inf_{\tau \in \llbracket C \rrbracket(\sigma)} f(\tau) = \operatorname{wlp} \llbracket C \rrbracket(f) \ .$$