A pure view of ecumenical modalities

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Abstract. Recent works about ecumenical systems, where connectives from classical and intuitionistic logics can co-exist in peace, warmed the discussion on proof systems for combining logics. This discussion has been extended to alethic modalities using Simpson's meta-logical characterization: necessity is independent of the viewer, while possibility can be either *intuitionistic* or *classical*. In this work, we propose a *pure, label free* calculus for ecumenical modalities, nEK, where exactly one logical operator figures in introduction rules and every basic object of the calculus can be read as a formula in the language of the ecumenical modal logic EK. We prove that nEK is sound and complete w.r.t. the ecumenical birelational semantics and discuss fragments and extensions.

1 Introduction

Ecumenism can be seen as the search for *unicity*, that is, for different thoughts, ideas or points of view to coexist in harmony. In mathematical logic, ecumenical approaches for a peaceful coexistence of logical systems have been studied deeply, *e.g.* [Gir93,LM11].

More recently, Prawitz proposed a natural deduction system combining classical and intuitionistic logics [Pra15]. The fundamental question he addressed was: what makes a connective classical or intuitionistic? We will illustrate, with a simple example, some ways of answering this. Consider the following statement, where $x, y, z \in \mathbb{R}$ and $z \geq 0$:

$$\text{if } x + y = 2z \text{ then } x \geq z \text{ or } y \geq z$$

How should we interpret "if then" and "or" in this sentence for it to be valid? The answer is: it depends! If we view this sentence with the classical mathematician's eyes (CM), the intuitionistic mathematician (IM) would not see a theorem. Since intuitionists can see classical tautologies through the lens of double negation, we could embed this classical interpretation in the intuitionistic setting as:

$$\text{not (not (if } x + y = 2z \text{ then } x \geq z \text{ or } y \geq z))$$

This would indeed be a valid statement for both CM and IM.

A finer possibility for guaranteeing the validity of the sentence is to give to the implication an intuitionistic interpretation and to the disjunction a classical one. Namely, the following statement is also a theorem for both CM and IM:

$$\text{if } x + y = 2z \text{ then not (not } (x \geq z \text{ or } y \geq z))$$

Prawitz' ecumenism idea can be summarized as: pinpoint the exact places where the classical and intuitionistic views differ and signal it such that IM knows to read it with her "ecumenical glasses", i.e. through a double negation filter. The example above shows that CM and IM can consider, for example, different connectives for disjunction $\vee_c$ and $\vee_i$, respectively. Prawitz answered this question for all the first-order connectives in [Pra15] by presenting an ecumenical natural deduction system.

In [PPdP19], we justified some of Prawitz' choices via pure proof theoretical reasoning, using sequent based systems. Consider the well known classical and intuitionistic sequent systems G3c and G3i [TS96]. Since all rules in G3c are invertible, no choices have to be made during a classical proof search: one can apply any rule bottom-up in any order. This is not the case in G3i: choices may have to be made for disjunction, implication and the existential quantifier. This suggests that CM and IM would share the universal quantifier, conjunction and the constant for the absurd (hence also negation) – *the neutral connectives*, but they would each have their own existential quantifier, disjunction and implication, with different meanings.

Following this discussion, the originial statement is ecumenically translated as

$$(x + y = 2z) \to_i x \geq z \vee_c y \geq z$$

Now the classical mathematician would see everything just fine (since she cannot differentiate classical from intuitionistic), while the intuitionistic mathematician would put on her ecumenical glasses only when observing the disjunction, so they would both agree on the statement. *This* is the essence of ecumenism!

In [MPPS20], we have extended this discussion to modalities to address the question: how would CM and IM view such concepts as "necessity" and "possibility"? Using Simpson's meta-logical characterization [Sim94], the answer is that, if something is necessarily true, then it is independent of the viewer. Possibility, on the other hand, can be either *intuitionistic:* in the sense that one should have a guarantee that something will eventually be true; or *classical:* in the sense that it is not the case that necessarily something will not be true. Hence CM and IM share the necessity connective $\square$, but each would have their own possibility views, represented by $\Diamond_c$ and $\Diamond_i$, respectively.

Our solution, however, was not entirely satisfactory since the ecumenical modal calculi presented so far are not *pure* [Dum91]: the introduction rules for some connectives depend on negation and other connectives. Moreover, the ecumenical modal systems in [MPPS20] make use of labels: the basic objects used in proofs are from a more expressive language than the logic itself, which partially encodes the logic's semantics.

This paper tackles these issues, proposing a *pure label free* calculus for ecumenical modalities, where every basic object of the calculus can be read as a formula in the language of the logic. For that, we will use *nested systems* [Bull92,Kas94,Brü09,Pog09] with a *stoup* [Gir91], together with a new notion of *polarities* for ecumenical formulas. Nested systems are extensions of the sequent framework where each sequent is replaced by a tree of sequents. The stoup is a distinguished context containing a single formula. Finally, formulas can be polarized as *negative* if the main connective is classical or the negation, or as *positive* otherwise. This is unlike any other notion of polarities that we know of, but could well be related [Lau02]. The idea is that negative formulas are stored in the classical context, while positive formulas are decomposed in the stoup. This not only allows for establishing the meaning of modalities via the rules that determine their correct use (*logical inferentialism*), but also places the ecumenical system as a unifying framework for modalities of which well known modal systems are fragments.

Organization and contributions. Sec. 2 reviews the notation for modal formulas, the labeled system labEK and the ecumenical birelational semantics; Sec. 3 introduces the ecumenical nested system nEK and its normalization procedure; in Secs. 4 and 5 soundness and completeness of nEK w.r.t. the ecumenical birelational semantics are proved; Sec. 6 identifies the classical and intuitionistic fragments of nEK; Sec. 7 discusses some modal extensions; and Sec. 8 concludes the paper.

2 Preliminaries

In [MPPS20] we proposed an ecumenical version of normal modal logic, where classical and intuitionistic modalities co-exist. The system adopts Simpson's approach [Sim94], called *meta-logical characterization*, where a modal logic is characterized by the interpretation of modalities in a first-order meta-theory. We translated modalities into the ecumenical first-order logic LE [Pra15,PPdP19], justified similarly by the standard interpretation of alethic modalities in a model. The presence of classical and intuitionistic existential connectives in LE induces two *possibility* modalities, while the neutral universal quantifier in LE entails a neutral *necessity* modality.

Language $\mathcal{A}$ of ecumenical modal formulas is generated by the following grammar:

$$A ::= p_i \mid p_c \mid \bot \mid \neg A \mid A \wedge A \mid A \vee_i A \mid A \vee_c A \mid A \to_i A \mid A \to_c A \mid \Box A \mid \Diamond_i A \mid \Diamond_c A$$

We use subscript c for the classical meaning and i for the intuitionistic one, dropping such subscripts when formulas/connectives can have either meaning. A classical version p_c and an intuitionistic version p_i of each propositional variable co-exist in $\mathcal{A}$: their meanings are different but related via double negation. The neutral logical connectives $\{\bot, \neg, \wedge, \Box\}$ are common for classical and intuitionistic fragments, while $\{\to_i, \vee_i, \Diamond_i\}$ and $\{\to_c, \vee_c, \Diamond_c\}$ are restricted to intuitionistic and classical interpretations, respectively.

The meta-logical characterization naturally induces a labeled proof system [Sim94]. The language $\mathcal{L}$ of *labeled modal formulas* is determined by *labeled formulas* of the form $x : A$ with $A \in \mathcal{A}$ and *relational atoms* of the form xRy, where x, y range over a set of variables. *Labeled sequents* have the form $\Gamma \Rightarrow x : A$, where Γ is a multiset containing labeled modal formulas and relational atoms. In what follows, if L is a sequent based calculus, we use $\vdash_{\mathsf{L}} \Gamma \Rightarrow A$ to denote that there is an L-proof of $\Gamma \Rightarrow A$. The labeled ecumenical system labEK [MPPS20] is presented in Figure 1.

Example 1. Below the derivation in labEK of the distributivity of the intuitionistic diamond w.r.t the intuitionistic disjunction (see axiom k_2 in Section 5).

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{xRy, y : A \Rightarrow y : A}\ \mathsf{init}}{xRy, y : A \Rightarrow x : \Diamond_i A}\ \Diamond_i R}{xRy, y : A \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\ \vee_i R \quad \dfrac{\dfrac{\overline{xRy, y : B \Rightarrow y : B}\ \mathsf{init}}{xRy, y : B \Rightarrow x : \Diamond_i B}\ \Diamond_i R}{xRy, y : B \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\ \vee_i R}{xRy, y : A \vee_i B \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}\ \vee_i L}{\dfrac{x : \Diamond_i (A \vee_i B) \Rightarrow x : \Diamond_i A \vee_i \Diamond_i B}{\Rightarrow x : \Diamond_i (A \vee_i B) \to_i (\Diamond_i A \vee_i \Diamond_i B)}\ \to_i R}\ \Diamond_i L$$

$$\frac{}{x : p_i, \Gamma \Rightarrow x : p_i}\ \mathsf{init} \qquad \frac{}{x : \bot, \Gamma \Rightarrow z : C}\ \bot_L \qquad \frac{\Gamma \Rightarrow y : \bot}{\Gamma \Rightarrow x : A}\ \mathsf{W}$$

$$\frac{x : p_i, \Gamma \Rightarrow z : \bot}{x : p_c, \Gamma \Rightarrow z : \bot}\ L_c \qquad \frac{x : \neg p_i, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : p_c}\ R_c$$

$$\frac{x : A, x : B, \Gamma \Rightarrow z : C}{x : A \wedge B, \Gamma \Rightarrow z : C}\ \wedge L \qquad \frac{\Gamma \Rightarrow x : A \quad \Gamma \Rightarrow x : B}{\Gamma \Rightarrow x : A \wedge B}\ \wedge R \qquad \frac{x : \neg A, \Gamma \Rightarrow z : A}{x : \neg A, \Gamma \Rightarrow z : \bot}\ \neg L$$

$$\frac{x : A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \neg A}\ \neg R \qquad \frac{x : A, \Gamma \Rightarrow z : C \quad x : B, \Gamma \Rightarrow z : C}{x : A \vee_i B, \Gamma \Rightarrow z : C}\ \vee_i L \qquad \frac{\Gamma \Rightarrow x : A_j}{\Gamma \Rightarrow x : A_1 \vee_i A_2}\ \vee_i R_j$$

$$\frac{x : A, \Gamma \Rightarrow z : \bot \quad x : B, \Gamma \Rightarrow z : \bot}{x : A \vee_c B, \Gamma \Rightarrow z : \bot}\ \vee_c L \qquad \frac{\Gamma, x : \neg A, x : \neg B \Rightarrow x : \bot}{\Gamma \Rightarrow x : A \vee_c B}\ \vee_c R$$

$$\frac{x : A \rightarrow_i B, \Gamma \Rightarrow x : A \quad x : B, \Gamma \Rightarrow z : C}{x : A \rightarrow_i B, \Gamma \Rightarrow z : C}\ \rightarrow_i L \qquad \frac{x : A, \Gamma \Rightarrow x : B}{\Gamma \Rightarrow x : A \rightarrow_i B}\ \rightarrow_i R$$

$$\frac{x : A \rightarrow_c B, \Gamma \Rightarrow x : A \quad x : B, \Gamma \Rightarrow z : \bot}{x : A \rightarrow_c B, \Gamma \Rightarrow z : \bot}\ \rightarrow_c L \qquad \frac{x : A, x : \neg B, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : A \rightarrow_c B}\ \rightarrow_c R$$

$$\frac{xRy, y : A, x : \Box A, \Gamma \Rightarrow z : C}{xRy, x : \Box A, \Gamma \Rightarrow z : C}\ \Box L \qquad \frac{xRy, \Gamma \Rightarrow y : A}{\Gamma \Rightarrow x : \Box A}\ \Box R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : C}{x : \Diamond_i A, \Gamma \Rightarrow z : C}\ \Diamond_i L$$

$$\frac{xRy, \Gamma \Rightarrow y : A}{xRy, \Gamma \Rightarrow x : \Diamond_i A}\ \Diamond_i R \qquad \frac{xRy, y : A, \Gamma \Rightarrow z : \bot}{x : \Diamond_c A, \Gamma \Rightarrow z : \bot}\ \Diamond_c L \qquad \frac{x : \Box\neg A, \Gamma \Rightarrow x : \bot}{\Gamma \Rightarrow x : \Diamond_c A}\ \Diamond_c R$$

Fig. 1. Ecumenical modal system labEK. In rules $\Box R, \Diamond_i L, \Diamond_c L$, the eigenvariable y does not occur free in any formula of the conclusion. In the rule W, either $A \neq \bot$ or $x \neq y$.

2.1 Ecumenical birelational models

The ecumenical birelational Kripke semantics, which is an extension of the proposal in [PR17] to modalities, was presented in [MPPS20].

Definition 2. *A* birelational model *[PS86] is a quadruple $\mathcal{M} = (W, \leq, R, V)$ with a poset $(W, \leq)$, a binary relation $R \subset W \times W$, a monotone valuation $V : \langle W, \leq\rangle \rightarrow \langle 2^{\mathcal{P}}, \subseteq\rangle$ and*
F1. For all worlds w, v, v', if wRv and $v \leq v'$, there is a w' such that $w \leq w'$ and $w'Rv'$;
F2. For all worlds w', w, v, if $w \leq w'$ and wRv, there is a v' such that $w'Rv'$ and $v \leq v'$.

An ecumenical modal model *is a birelational model such that truth of an ecumenical formula at a point w is the smallest relation $\models_{\mathsf{E}}$ satisfying*

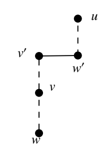
$\mathcal{M}, w \models_{\mathsf{E}} p_i$	*iff*	$p_i \in V(w)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \wedge B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *and* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_i B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} A$ *or* $\mathcal{M}, w \models_{\mathsf{E}} B$;
$\mathcal{M}, w \models_{\mathsf{E}} A \rightarrow_i B$	*iff*	*for all v such that $w \leq v$, $\mathcal{M}, v \models_{\mathsf{E}} A$ implies $\mathcal{M}, v \models_{\mathsf{E}} B$*;
$\mathcal{M}, w \models_{\mathsf{E}} \neg A$	*iff*	*for all v such that $w \leq v$, $\mathcal{M}, v \not\models_{\mathsf{E}} A$*;
$\mathcal{M}, w \models_{\mathsf{E}} \bot$		*never holds*;
$\mathcal{M}, w \models_{\mathsf{E}} \Box A$	*iff*	*for all v, w' such that $w \leq w'$ and $w'Rv$, $\mathcal{M}, v \models_{\mathsf{E}} A$.*
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_i A$	*iff*	*there exists v such that wRv and $\mathcal{M}, v \models_{\mathsf{E}} A$.*
$\mathcal{M}, w \models_{\mathsf{E}} p_c$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg p_i)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \vee_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(\neg A \wedge \neg B)$;
$\mathcal{M}, w \models_{\mathsf{E}} A \rightarrow_c B$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg(A \wedge \neg B)$;
$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A$	*iff*	$\mathcal{M}, w \models_{\mathsf{E}} \neg\Box\neg A$.

A formula A is valid *in a model $\mathcal{M} = (W, \leq, R, V)$ if for all $w \in W$, $\mathcal{M}, w \models_{\mathsf{E}} A$. A formula A is* valid *in a frame $(W, \leq, R)$ if, for all valuations V, A is valid in the model $(W, \leq, R, V)$. Finally, we say that a formula is* valid*, if it is valid in all frames.*

Since, restricted to intuitionistic and neutral connectives, $\models_{\mathsf{E}}$ is the usual birelational interpretation $\models$ for IK [Sim94], and since the classical connectives are interpreted via the neutral ones using the double-negation translation, an ecumenical birelational model coincides with the standard birelational model for intuitionistic modal logic IK. Hence the following result easily holds from the similar result for IK.

Theorem 3 ([MPPS20]). *The system labEK is sound and complete w.r.t. the ecumenical modal semantics, that is, $\vdash_{\mathsf{labEK}} x : A$ iff $\models_{\mathsf{E}} A$.*

Remark 4. It is interesting to note that the relational semantics for the classical connectives is surprisingly more complex than for the intuitionistic ones. In fact, the definition of $\models_{\mathsf{E}}$ for the classical diamond is equivalent to



$$\mathcal{M}, w \models_{\mathsf{E}} \Diamond_c A \text{ iff } \forall v \geq w. \exists u. v\, (\leq \circ R \circ \leq)\, u,\ \mathcal{M}, u \models_{\mathsf{E}} A$$

where $v\,(\leq \circ R \circ \leq)\,u$ represents that there exist $v', w' \in W$ such that $v \leq v'$, $v' R w'$ and $w' \leq u$. Although intriguing, this kind of two-level semantics also appears in the relational semantics for classical logic in [ILH10], where the forcing relation is defined on top of the primitive notion of "strong refutation".

3 A nested system for ecumenical modal logic

The two main criticisms regarding system labEK are: (i) it is not *pure*, in the sense that negation still plays an important role on interpreting classical connectives – for example, the rule $\Diamond_c R$ introduces a classical diamond via its boxed negated version; and (ii) it includes *labels* in the technical machinery, hence allowing one to write sequents that cannot always be interpreted within the ecumenical modal language.

This section is devoted to tackle these points and propose a *pure label free* calculus for ecumenical modalities, where every basic object of the calculus can be translated as a formula in the language of the logic, with no use of auxiliary negations.

The inspiration comes from Girard's notion of *stoup* [Gir91] and Straßburger's *nested system* for IK [Str13]. The main idea is to let sequents of the form $\Sigma \Rightarrow \Pi$, with Σ, Π multisets of formulas, go through a two-phase refinement: the first one is to separate the succedent Π into two parts: one that is essentially classical; and another containing a single formula, the stoup. The second one is to add nested layers to sequents, which intuitively corresponds to worlds in a relational structure [Fit14,Brü09,Pog09].

The primary key concept to distinguish which formulas are allowed or not in the stoup is the following notion of polarity.

Definition 5. *A formula is called* negative *if its main connective is classical or the negation, and* positive *otherwise (we will use N for negative and P for positive formulas).*

The structure of a nested sequent for ecumenical modal logics is a tree whose nodes are multisets of formulas, just like in [Str13], with the relationship between parent and child in the tree represented by bracketing $[\cdot]$. The difference however is that the ecumenical formulas can be *left inputs* (in the left contexts – marked with a full circle $^\bullet$), *right inputs* (in the classical right contexts – marked with a triangle $^\triangledown$) or a *single right output* (the stoup – marked with a white circle $^\circ$).

Definition 6. Ecumenical nested sequents *are defined in terms of a grammar of* input sequents *(written Λ) and* full sequents *(written Γ) where the left/right input formulas are denoted by $A^\bullet$ and $A^\triangledown$, respectively, and A° denote the output formula. When the distinction between input and full sequents is not essential or cannot be made explicit, we will use Δ to stand for either case.*

$$\Lambda \coloneqq \varnothing \mid A^\bullet, \Lambda \mid A^\triangledown, \Lambda \mid [\Lambda] \qquad \Gamma \coloneqq A^\circ, \Lambda \mid [\Gamma], \Lambda \qquad \Delta \coloneqq \Lambda \mid \Gamma$$

As usual, we allow sequents to be empty, and we consider sequents to be equal modulo associativity and commutativity of the comma.

We write $\Gamma^{\perp^\circ}$ for the result of replacing an output formula from Γ by $\perp^\circ$, while $\Lambda^{\perp^\circ}$ represents the result of adding anywhere of the input context Λ the output formula $\perp^\circ$. Finally, Δ^* is the result of erasing an output formula (if any) from Δ.

Observe that full sequents Γ necessarily contain exactly one output-like formula, having the form $\Lambda_1, [\Lambda_2, [\ldots, [\Lambda_n, A^\circ]] \ldots]$.

Example 7. The nested sequent $\Diamond_c A^\triangledown, [\neg A^\circ]$ represents a tree of sequents where $\Diamond_c A$ is in the right (classical) input context of the root sequent, while $\neg A$ is in the output context (stoup), in the leaf sequent.

The next definition (of contexts) allows for identifying subtrees within nested sequents, which is necessary for introducing inference rules in this setting.

Definition 8. *An n-ary* context $\Delta\left\{^1\right\}\cdots\left\{^n\right\}$ *is like a sequent but contains n pairwise distinct numbered* holes $\{\}$ *wherever a formula may otherwise occur. It is a* full *or a* input *context when $\Delta = \Gamma$ or Λ respectively.*

Given n sequents $\Delta_1, \ldots, \Delta_n$, we write $\Delta\{\Delta_1\}\cdots\{\Delta_n\}$ for the sequent where the i-th hole in $\Delta\left\{^1\right\}\cdots\left\{^n\right\}$ has been replaced by Δ_i (for $1 \leq i \leq n$), assuming that the result is well-formed, i.e.*, there is at most one output formula. If $\Delta_i = \varnothing$ the hole is removed.*

Given two nested contexts $\Gamma^i\{\} = \Delta^i_1, \left[\Delta^i_2, \left[\ldots, \left[\Delta^i_n, \{\}\right]\right] \ldots\right]$, $i \in \{1,2\}$, their merge[5] *is*

$$\Gamma^1 \otimes \Gamma^2\{\} = \Delta^1_1, \Delta^2_1, \left[\Delta^1_2, \Delta^1_2, \left[\ldots, \left[\Delta^1_n, \Delta^2_n, \{\}\right]\right] \ldots\right]$$

Figure 2 presents the nested sequent system nEK for ecumenical modal logic EK.

Example 9. Below left the nested proof corresponding to the labeled one in Example 1.

[5] As observed in [Pog09,Lel19], the merge is a "zipping" of the two nested sequents along the path from the root to the hole.

$$\dfrac{}{\Lambda\{p_i^\bullet, p_i^\circ\}}\ \mathsf{init} \qquad \dfrac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \dfrac{\Gamma^{\bot^\circ}}{\Gamma}\ \mathsf{W} \qquad \dfrac{\Gamma^{\bot^\circ}\{p_i^\bullet\}}{\Gamma^{\bot^\circ}\{p_c^\bullet\}}\ p_c^\bullet \qquad \dfrac{\Gamma^{\bot^\circ}\{p_i^\triangledown\}}{\Gamma^{\bot^\circ}\{p_c^\triangledown\}}\ p_c^\triangledown$$

$$\dfrac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \dfrac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ \qquad \dfrac{\Gamma^*\{\neg A^\bullet, A^\circ\}}{\Gamma^{\bot^\circ}\{\neg A^\bullet\}}\ \neg^\bullet \qquad \dfrac{\Gamma^{\bot^\circ}\{A^\bullet\}}{\Gamma^{\bot^\circ}\{\neg A^\triangledown\}}\ \neg^\triangledown$$

$$\dfrac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\ \vee_i^\bullet \qquad \dfrac{\Lambda\{A_j^\circ\}}{\Lambda\{A_1 \vee_i A_2^\circ\}}\ \vee_{i_j}^\circ \qquad \dfrac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\ \vee_c^\bullet \qquad \dfrac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\ \vee_c^\triangledown$$

$$\dfrac{\Gamma^*\{A \to_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to_i B^\bullet\}}\ \to_i^\bullet \qquad \dfrac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to_i B^\circ\}}\ \to_i^\circ \qquad \dfrac{\Gamma^*\{A \to_c B^\bullet, A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \to_c B^\bullet\}}\ \to_c^\bullet$$

$$\dfrac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \to_c B^\triangledown\}}\ \to_c^\triangledown \qquad \dfrac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \dfrac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ \qquad \dfrac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \dfrac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\ \Diamond_i^\circ$$

$$\dfrac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\ \Diamond_c^\bullet \qquad \dfrac{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [A^\triangledown, \Delta_2^{\bot^\circ}]\}}{\Delta_1^{\bot^\circ}\{\Diamond_c A^\triangledown, [\Delta_2^{\bot^\circ}]\}}\ \Diamond_c^\triangledown \qquad \dfrac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\ \mathsf{dec} \qquad \dfrac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{sto}$$

Fig. 2. Nested ecumenical modal system nEK. *P* is a *positive* formula, *N* is a *negative* formula.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A^\bullet, A^\circ]}\ \mathsf{init}}{\Diamond_i A^\circ, [A^\bullet]}\ \Diamond_i^\circ}{\Diamond_i A \vee_i \Diamond_i B^\circ, [A^\bullet]}\ \vee_i^\circ \quad \dfrac{\dfrac{\dfrac{}{[B^\bullet, B^\circ]}\ \mathsf{init}}{\Diamond_i B^\circ, [B^\bullet]}\ \Diamond_i^\circ}{\Diamond_i A \vee_i \Diamond_i B^\circ, [B^\bullet]}\ \vee_i^\circ}{\Diamond_i A \vee_i \Diamond_i B^\circ, [A \vee_i B^\bullet]}\ \vee_i^\bullet}{\Diamond_i (A \vee_i B)^\bullet, \Diamond_i A \vee_i \Diamond_i B^\circ}\ \Diamond_i^\bullet}{\Diamond_i (A \vee_i B) \to_i (\Diamond_i A \vee_i \Diamond_i B)^\circ}\ \to_i^\circ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[A^\bullet, A^\triangledown, \bot^\circ]}\ \mathsf{init_c}}{[A^\triangledown, \neg A^\circ]}\ \neg^\circ}{\Diamond_c A^\triangledown, [\neg A^\circ]}\ \Diamond_c^\triangledown}{\Box\neg A^\circ, \Diamond_c A^\triangledown}\ \Box^\circ}{\neg\Box\neg A^\bullet, \Diamond_c A^\triangledown, \bot^\circ}\ \neg^\bullet}{\neg\Box\neg A^\bullet, \Diamond_c A^\circ}\ \mathsf{sto}$$

The derivation above right shows part of the proof that $\Diamond_c$ can be defined from $\Box$ ($\Diamond_c A \equiv \neg\Box\neg A$). Note the instance of the classical general version of the initial axiom, $\mathsf{init_c}$ (see Theorem 11 in the next section). It also *illustrates* well the relationship between nestings, classical inputs, and birelational structures: reading the proof bottom-up, the sto rule is a delay on applying rules over classical connectives. It corresponds to moving the formula up w.r.t. $\leq$ in the birelational semantics. The rule $\Box^\circ$, on the other hand, slides the formula to a fresh new world, related to the former one through the relation R. Finally, rule $\neg^\circ$ also moves up the formula w.r.t. $\leq$. Compare this description with the image in Remark 4. In this paper, we will not explore *formally* the relationship between delays/negations/nestings and semantics.

3.1 Harmony

A logical connective is called *harmonious* in a certain proof system if there exists a certain balance between the rules defining it. For example, in natural deduction based systems, harmony is ensured when introduction/elimination rules do not contain insufficient/excessive amounts of information [DD20]. In sequent calculus, this property is often guaranteed by the admissibility of a general initial axiom (*identity-expansion*) and of the cut rule (*cut-elimination*).In the following, we will prove harmony, together with some intermediate results. We start with a proof theoretical result in nEK, which has a standard proof (see [PPdP19] and [MPPS20] for similar results).

Lemma 10. *1. In* nEK*, the rules* $\vee_c^\bullet, \vee_c^\triangledown, \to_c^\bullet, \to_c^\triangledown, \neg^\bullet, \neg^\circ, p_c^\bullet, p_c^\triangledown, \Diamond_c^\bullet, \Diamond_c^\triangledown$ *and* dec *are* invertible*, that is, in any application of such rules, if the conclusion is a provable nested sequent so are the premises.*

2. *The rules* $\wedge^\bullet, \wedge^\circ, \vee_i^\bullet, \to_i^\circ, \Diamond_i^\bullet, \Box^\bullet, \Box^\circ$ *and* sto *are* totally invertible*, that is, they are invertible and can be applied in any contexts.*
3. *The following rules are admissible in* nEK

$$\frac{\Gamma}{\Lambda \otimes \Gamma}\ \mathsf{W}_c \qquad \frac{\Lambda \otimes \Lambda \otimes \Gamma}{\Lambda \otimes \Gamma}\ \mathsf{C}_c$$

Proof. The proofs are by standard induction on the height of derivations. The proof of admissibility of W_c does not depend on any other result, while the admissibility of C_c depends on the invertibility results above.

The invertible but not totally invertible rules in nEK concern negative formulas, hence they can only be applied in the presence of empty stoups ($\bot^\circ$). Note also that the rules W, $\vee_i^\circ$, and $\Diamond_i^\circ$ are not invertible, while $\to_i^\bullet$ is invertible only w.r.t. the right premise.

Theorem 11. *The following rules are admissible in* nEK

$$\frac{}{\Lambda\{A^\bullet, A^\circ\}}\ \mathsf{init_i} \qquad \frac{}{\Gamma^{\bot^\circ}\{A^\bullet, A^\triangledown\}}\ \mathsf{init_c}$$

Proof. The proof of admissibility of the general initial axioms is by mutual induction. Below we show the modal cases where, by induction hypothesis, instances of the axioms hold for the premises.

$$\dfrac{\dfrac{\dfrac{}{\Gamma^{\bot^\circ}\{[A^\bullet, A^\triangledown]\}}\ \mathsf{init_c}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\triangledown, [A^\bullet]\}}\ \Diamond_c^\triangledown}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet, \Diamond_c A^\triangledown\}}\ \Diamond_c^\bullet \qquad \dfrac{\dfrac{\dfrac{}{\Lambda\{[A^\bullet, A^\circ]\}}\ \mathsf{init_i}}{\Lambda\{\Box A^\bullet, [A^\circ]\}}\ \Box^\bullet}{\Lambda\{\Box A^\bullet, \Box A^\circ\}}\ \Box^\circ$$

Proving admissibility of cut rules in sequent based systems with multiple contexts is often tricky, since the cut formulas can change contexts during cut reductions. This is the case for nEK. The proof is by mutual induction, with inductive measure (n, m) where m is the cut-height, the cumulative height of derivations above the cut, and n is the ecumenical weight of the cut-formula, defined as

$$\begin{array}{ll}
\mathsf{ew}(P_i) = \mathsf{ew}(\bot) = 0 & \mathsf{ew}(A \star B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 1 \text{ if } \star \in \{\wedge, \to_i, \vee_i\} \\
\mathsf{ew}(P_c) = 4 & \mathsf{ew}(\heartsuit A) = \mathsf{ew}(A) + 1 \text{ if } \heartsuit \in \{\neg, \Diamond_i, \Box\} \\
\mathsf{ew}(\Diamond_c A) = \mathsf{ew}(A) + 4 & \mathsf{ew}(A \circ B) = \mathsf{ew}(A) + \mathsf{ew}(B) + 4 \text{ if } \circ \in \{\to_c, \vee_c\}
\end{array}$$

Intuitively, the ecumenical weight measures the amount of extra information needed (the negations added) to define classical connectives from intuitionistic and neutral ones.

Theorem 12. *The following intuitionistic and classical cut rules are admissible in* nEK

$$\dfrac{\Lambda\{P^\circ\} \quad \Gamma\{P^\bullet\}}{\Lambda\otimes\Gamma\{\varnothing\}}\,\mathsf{cut}^\circ \qquad \dfrac{\Lambda^{\perp^\circ}\{N^\triangledown\}\quad \Gamma\{N^\bullet\}}{\Lambda\otimes\Gamma\{\varnothing\}}\,\mathsf{cut}^\triangledown$$

Proof. The dynamic of the proof is the following: cut applications either move up in the proof, i.e. the cut-height is reduced, or are substituted by simpler cuts of the same kind, i.e. the ecumenical weight is reduced, as in usual cut-elimination reductions. The cut instances alternate between intuitionistic and classical (and vice-versa) in the principal cases, where the polarity of the subformulas flip. We sketch the main cut-reductions.

– Base cases. Consider the derivation below left

$$\dfrac{\overset{\pi}{\Lambda\left\{p_i^\circ\right\}} \quad \dfrac{}{\Gamma\left\{p_i^\bullet\right\}}\,init}{\Lambda\otimes\Gamma\{\varnothing\}}\,\mathsf{cut}^\circ \qquad\qquad \dfrac{\overset{\pi}{\Lambda\left\{p_i^\circ\right\}}}{\Lambda\otimes\Gamma^*\left\{p_i^\circ\right\}}\,\mathsf{W}_c$$

If $p_i^\bullet$ is principal, then $\Gamma\left\{p_i^\bullet\right\} = \Gamma^*\left\{p_i^\circ, p_i^\bullet\right\}$ and the reduction is the one above right. If $p_i^\bullet$ is not principal, then there is an atom q for which the pair $q_i^\circ, q_i^\bullet$ appears in $\Lambda\otimes\Gamma\{\varnothing\}$ and the reduction is a trivial one. Similar analyses hold for $\mathsf{cut}^\triangledown$, when the left premise is an instance of init, and for the other axioms.

– Non-principal cases. In all the cases where the cut-formula is not principal in one of the premises, the cut moves upwards. We illustrate the most significant case, where a decide rule is applied, as in the derivation below left.

$$\dfrac{\dfrac{\overset{\pi_1}{\Lambda\{P^\triangledown,P^\circ\}\{N^\triangledown\}}}{\Lambda^{\perp^\circ}\{P^\triangledown\}\{N^\triangledown\}}\,\mathsf{dec} \quad \overset{\pi_2}{\Gamma\{N^\bullet\}}}{\Lambda\{P^\triangledown\}\otimes\Gamma\{\varnothing\}}\,\mathsf{cut}^\triangledown \qquad \dfrac{\dfrac{\overset{\pi_1}{\Lambda\{P^\triangledown,P^\circ\}\{N^\triangledown\}}\quad \overset{\pi_2}{\Gamma^{\perp^\circ}\{N^\bullet\}}}{\Lambda\{P^\triangledown,P^\circ\}\otimes\Gamma^*\{\varnothing\}}\,\mathsf{cut}^\triangledown}{\Lambda\{P^\triangledown\}\otimes\Gamma^{\perp^\circ}\{\varnothing\}}\,\mathsf{dec}$$

The cut moves upwards in the right premise until $N^\bullet$ is principal in the bottom-most step of π_2, at which point $\Gamma = \Gamma^{\perp^\circ}$ and dec can move below the cut, obtaining the derivation above right.

– Principal cases. If the cut formula is principal in both premises, then we need to be extra-careful with the polarities. We illustrate below the reduction for case where $N = P \to_c Q$, with P, Q positive.

$$\dfrac{\dfrac{\overset{\pi_1}{\Lambda^{\perp^\circ}\{P^\bullet,Q^\triangledown\}}}{\Lambda^{\perp^\circ}\{P\to_c Q^\triangledown\}}\,\to_c^\triangledown \quad \dfrac{\overset{\pi_2}{\Gamma^*\{P\to_c Q^\bullet,P^\circ\}}\quad\overset{\pi_3}{\Gamma^{\perp^\circ}\{Q^\bullet\}}}{\Gamma^{\perp^\circ}\{P\to_c Q^\bullet\}}\,\to_c^\bullet}{\Lambda\otimes\Gamma^{\perp^\circ}\{\varnothing\}}\,\mathsf{cut}_0^\triangledown$$

reduces to

$$\dfrac{\dfrac{\dfrac{\overset{\pi_3}{\Gamma^{\perp^\circ}\{Q^\bullet\}}}{\Gamma^{\perp^\circ}\{\neg Q^\triangledown\}}\,\neg^\triangledown \quad \dfrac{\dfrac{\dfrac{\overset{\pi_1}{\Lambda^{\perp^\circ}\{P^\bullet,Q^\triangledown\}}}{\Lambda^{\perp^\circ}\{P\to_c Q^\triangledown\}}\,\to_c^\triangledown \quad \overset{\pi_2}{\Gamma^*\{P\to_c Q^\bullet,P^\circ\}}}{\Lambda\otimes\Gamma^*\{P^\circ\}}\,\mathsf{cut}_2^\triangledown \quad \overset{\pi_1^\equiv}{\Lambda^{\perp^\circ}\{P^\bullet,\neg Q^\bullet\}}}{\Lambda^2\otimes\Gamma^{\perp^\circ}\{\neg Q^\bullet\}}\,\mathsf{cut}^\circ}{\Lambda^2\otimes\Gamma^*\otimes\Gamma^{\perp^\circ}\{\varnothing\}}\,\mathsf{cut}_1^\triangledown}{\Lambda\otimes\Gamma^{\perp^\circ}\{\varnothing\}}\,\mathsf{C}_c$$

where $\pi_1^{\equiv}$ is the same as π_1 where every application of the rule dec over $Q^{\triangledown}$ is substituted by an application of $\neg^{\bullet}$ over $\neg Q^{\bullet}$. Observe that the cut-formula of $\mathsf{cut}_1^{\triangledown}$ has lower ecumenical weight than $\mathsf{cut}_0^{\triangledown}$, while the cut-height of $\mathsf{cut}_2^{\triangledown}$ is smaller than $\mathsf{cut}_0^{\triangledown}$. Finally, observe that this is a non-trivial cut-reduction: usually, the cut over the implication is replaced by a cut over Q first. Due to polarities, if Q is positive, then $\neg Q$ is negative and cutting over it will add to the left context the classical information Q, hence mimicking the behavior of formulas in the right input context.

4 Soundness

In this section we will show that all rules presented in Figure 2 are sound w.r.t. the ecumenical birelational model. The idea is to prove that the rules of the system nEK preserve *validity*, in the sense that if the interpretation of the premises is valid, so is the interpretation of the conclusion.

The first step is to determine the interpretation of ecumenical nested sequents. In this section, we will present the translation of nestings to labeled sequents, hence establishing, at the same time, soundness of nEK and the relation between this system with labEK.

First of all, we observe that the entailment in ecumenical systems is intrinsically intuitionistic, in the sense that $\Gamma \Rightarrow B$ is valid iff $\bigwedge \Gamma \rightarrow_i B$ is valid [PPdP19]. Moreover, the classical connectives are defined semantically via the intuitionistic ones by sporadic double-negation. Another interesting aspect is that, in the labeled ecumenical modal system labEK, fresh world labels can be created (bottom-up) by the box operator in succedents and both diamond connectives in antecedents. Yet, once this new label is created, it is shared by all modal formulas, independently of their intuitionistic or classical nature.

This suggests the following interpretation of nested into labeled ecumenical sequents.

Definition 13. *Let $\Pi^{\bullet}, \Pi^{\triangledown}, \Pi^{\circ}$ represent that all formulas in the each multiset are respectively input left, right, or output formulas. The underlying Π will represent in all cases the corresponding multiset of unmarked formula in $\mathcal{A}$. The translation $[\![\cdot]\!]_x$ from nested into labeled sequents is defined recursively by*

$$[\![\Pi_1^{\bullet}, \Pi_2^{\triangledown}, \Pi_3^{\circ}, [\Delta_1], \ldots, [\Delta_n]]\!]_x := (\{xRx_i\}_i, x : \Pi_1, x : \neg\Pi_2 \Rightarrow x : \Pi_3) \otimes \{[\![\Delta_i]\!]_{x_i}\}_i$$

where $1 \leq i \leq n$, x_i are fresh, and the merge operation on labeled sequents is defined as

$$(\Sigma_1 \Rightarrow \Pi_1) \otimes (\Sigma_2 \Rightarrow \Pi_2) \;:=\; \Sigma_1, \Sigma_2 \Rightarrow \Pi_1, \Pi_2$$

Given $\mathcal{R}$ a set of relational formulas, we will denote by xR^*z the fact that there is a path from x to z in $\mathcal{R}$, i.e., there are $y_j \in \mathcal{R}$ for $0 \leq j \leq k$ such that $x = y_0, y_{j-1}Ry_j$ and $y_k = z$.

That is, right input formulas are translated as negated left formulas in labeled sequents, and nestings correspond to relational altoms. The next result shows that, in fact, this interpretation is correct.

Theorem 14. *Let Γ be a nested sequent and x be any label, if $\vdash_{\mathsf{nEK}} \Gamma$ then $\vdash_{\mathsf{labEK}} [\![\Gamma]\!]_x$.*

Proof. The proof is by structural induction on the proof π of Γ. We will illustrate a classical and a modal case.

- If the last rule applied in π is $\vee_c^{\triangledown}$, by induction hypothesis,

$$[\![\Gamma^{\perp^\circ}\{A^{\triangledown}, B^{\triangledown}\}]\!]_x = \mathcal{R}, \Sigma, z : \neg A, z : \neg B \Rightarrow x : \perp$$

  is provable for a set $\mathcal{R}$ of relational atoms and a multiset Σ of labeled formulas, obtained by translating $\Gamma^{\perp^\circ}\{\emptyset\}$ and such that $xR^*z \in \mathcal{R}$. Hence:

$$\dfrac{\dfrac{\mathcal{R}, \Sigma, z : \neg A, z : \neg B \Rightarrow z : \perp}{\mathcal{R}, \Sigma \Rightarrow z : A \vee_c B}\ \vee_c R}{\mathcal{R}, \Sigma, z : \neg(A \vee_c B) \Rightarrow x : \perp}\ \neg L$$

- If the last rule applied in π is $\Diamond_c^{\triangledown}$, by induction hypothesis,

$$[\![\Delta_1^{\perp^\circ}\left\{\Diamond_c A^{\triangledown}, \left[A^{\triangledown}, \Delta_2^{\perp^\circ}\right]\right\}]\!]_x = \mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), y : \neg A \Rightarrow x : \perp$$

  is provable for a set $\mathcal{R}$ and a multiset Σ of relational and labeled formulas, resp., obtained by translating sequents $\Delta_1^{\perp^\circ}$ and $\Delta_2^{\perp^\circ}$, and where $xR^*z \in \mathcal{R}$. Hence:

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), y : \neg A \Rightarrow z : \perp}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), z : \Box\neg A, y : \neg A \Rightarrow z : \perp}\ \mathsf{W}}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A), z : \Box\neg A \Rightarrow z : \perp}\ \Box L}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A) \Rightarrow z : \Diamond_c A}\ \Diamond_c R}{\mathcal{R}, zRy, \Sigma, z : \neg(\Diamond_c A) \Rightarrow x : \perp}\ \neg L$$

Due to rule W in labEK, the label assigned to $\perp$ on the right is irrelevant in both cases.

The proof above also establishes the relationship between *proofs* in nEK and labEK: the right input context stores *negative* formulas, which are in fact negated positive formulas (as in [Gir91]), and the decision rule dec in nEK is mimicked in labEK by applications of the left rule for negation. In this way, the use of nestings together with decision and store rules imposes a *discipline* on rule applications in labeled systems.

Theorems 3 and 14 immediately imply the following.

Corollary 15. *Nested system* nEK *is sound w.r.t. ecumenical birelational semantics.*

Finally, we observe that from labeled to nested sequents, on the other hand, is not a simple task, sometimes even impossible. In fact, although the relational atoms of a sequent appearing in a labEK *proof* can be arranged so as to correspond to nestings, in general, if the relational context is not tree-like [GR12], the existence of such a translation is not clear. For instance, how should the sequent $xRy, yRx, x : A \Rightarrow y : B$ be interpreted in modal systems with symmetrical relations?

Hence, we will avoid the translation method for proving completeness, which will instead be proven in Section 5 with respect to the *Hilbert system*.

However, thanks to their tree shape, it is possible to interpret nested sequents as ecumenical modal formulas, and hence prove soundness in the same way as in [Str13]. This direct interpretation of nested sequents as ecumenical formulas means that nEK is a so-called *internal proof system*. We thus finish this section by sketching an alternative proof of soundness of nEK w.r.t. the ecumenical birelational semantics.

Definition 16. *The formula translation* $\mathrm{et}(\cdot)$ *for ecumenical nested sequents is given by*

$$\begin{array}{rlrl} \mathrm{et}(\varnothing) &:= \top & \mathrm{et}(A^{\bullet},\Lambda) &:= A \wedge \mathrm{et}(\Lambda) \\ \mathrm{et}(A^{\triangledown},\Lambda) &:= \neg A \wedge \mathrm{et}(\Lambda) & \mathrm{et}([\Lambda_1],\Lambda_2) &:= \Diamond_i \mathrm{et}(\Lambda_1) \wedge \mathrm{et}(\Lambda_2) \\ \mathrm{et}(\Lambda, A^{\circ}) &:= \mathrm{et}(\Lambda) \rightarrow_i A & \mathrm{et}(\Lambda,[\Gamma]) &:= \mathrm{et}(\Lambda) \rightarrow_i \Box \mathrm{et}(\Gamma) \end{array}$$

where all occurrences of $A \wedge \top$ *and* $\top \rightarrow_i A$ *are simplified to* A*. We say a sequent is* valid *if its corresponding formula is valid.*

The following technical lemma holds in nEK, adapting the proof from NIK.

Lemma 17. *[Str13, Lemmas 4.3 and 4.4] Let* Δ *and* Σ *be input (resp. full) sequents, and* $\Gamma\{\ \}$ *be a full context (resp.* $\Lambda\{\ \}$ *be an input context). If* $\mathrm{et}(\Delta) \rightarrow_i \mathrm{et}(\Sigma)$ *is valid, then* $\mathrm{et}(\Gamma\{\Sigma\}) \rightarrow_i \mathrm{et}(\Gamma\{\Delta\})$ *and* $\mathrm{et}(\Lambda\{\Delta\}) \rightarrow_i \mathrm{et}(\Lambda\{\Sigma\})$ *are valid.*

The next theorem shows that the rules of nEK preserve validity in ecumenical modal frames w.r.t. the formula interpretation $\mathrm{et}(\cdot)$.

Theorem 18. *Let*

$$\frac{\Gamma_1 \quad \ldots \quad \Gamma_n}{\Gamma}\ r \qquad n \in \{0,1,2\}$$

be an instance of the rule r *in the system* nEK*. Then* $\mathrm{et}(\Gamma_1) \wedge \ldots \wedge \mathrm{et}(\Gamma_n) \rightarrow_i \mathrm{et}(\Gamma)$ *is valid in the birelational ecumenical semantics.*

Proof. The proof for the intuitionistic propositional and modal connectives follows the same lines as in [Str13]. For the other cases, due to Lemma 17, it is sufficient to show that the following formulas are valid; all such proofs are straightforward.

1. for W: $\bot \rightarrow_i A$
2. for $\neg^{\bullet}$: $(\neg A \rightarrow_i A) \rightarrow_i (\neg\neg A)$
3. for $\neg^{\triangledown}$: $\neg A \rightarrow_i \neg A$
4. for $\vee_c^{\bullet}$: $(\neg A \wedge \neg B) \rightarrow_i (\neg(A \vee_c B))$
5. for $\vee_c^{\triangledown}$: $(\neg(\neg A \wedge \neg B)) \rightarrow_i (A \vee_c B)$
6. for $\rightarrow_c^{\bullet}$: $(((A \rightarrow_c B) \rightarrow_i A) \wedge (\neg B)) \rightarrow_i (\neg(A \rightarrow_c B))$
7. for $\rightarrow_c^{\triangledown}$: $(\neg(A \wedge \neg B)) \rightarrow_i (A \rightarrow_c B)$
8. for $p_c^{\bullet}$: $(\neg p_i) \rightarrow_i (\neg p_c)$
9. for $p_c^{\triangledown}$: $(\neg\neg p_i) \rightarrow_i p_c$
10. for $\Diamond_c^{\bullet}$: $(\neg \Diamond_i A) \rightarrow_i (\neg \Diamond_c A)$
11. for $\Diamond_c^{\triangledown}$: $(\Box \neg A) \rightarrow_i (\neg \Diamond_c A)$

5 Completeness

Classical modal logic K is defined as propositional classical logic, extended with the *necessitation rule* (presented in Hilbert style) $A/\Box A$ and the *distributivity axiom* k : $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$.

There are, however, many variants of axiom k that induce logics that are classically, but not intuitionistically, equivalent (see [PS86,Sim94]). In fact, the following axioms follow from k via the De Morgan laws, but are intuitionistically independent

$$\begin{array}{ll} \mathsf{k}_1 :\ \Box(A \rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B) & \mathsf{k}_2 :\ \Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B) \\ \mathsf{k}_3 :\ (\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B) & \mathsf{k}_4 :\ \Diamond\bot \rightarrow \bot \end{array}$$

Combining axiom k with axioms $k_1 - k_4$ defines intuitionistic modal logic IK [PS86].

In the ecumenical setting, this discussion is even more interesting, since there are many more variants of k, depending on the classical or intuitionistic interpretation of implications and diamonds.

Theorems of *ecumenical modal logic* EK are defined as the formulas that are derivable from the axioms of intuitionistic propositional logic plus the definitions of classical operators using negation and the intuitionistic versions of the axioms $k - k_4$. We can show that all these EK axioms are provable in nEK (*e.g.* Example 9). Hence, in the presence of cut-elimination (Section 3.1), we can deduce completeness of nEK w.r.t. EK.

Theorem 19. *Every theorem of the logic* EK *is provable in* nEK.

Moreover, a formula is derivable in EK iff it is valid in all birelational frames (see [MPPS20]), which in turn implies completeness of nEK w.r.t. birelational semantics.

6 Extracting fragments

In this section, we will study pure classical and intuitionistic fragments of nEK. For the sake of simplicity, negation will not be considered a primitive connective, it will rather take its respective intuitionistic or classical form.

Definition 20. *An ecumenical modal formula C is* classical *(*intuitionistic*) if it is built from classical (intuitionistic) atomic propositions using only neutral and classical (intuitionistic) connectives but negation, which will be replaced by $A \rightarrow_c \bot$ ($A \rightarrow_i \bot$).*

The first thing to observe is that, when only pure fragments are concerned, weakening is admissible. Observe that this is not the case for the whole system nEK. In fact, $A \vee_c \neg A^{\triangledown}, C^{\circ}$ is provable in nEK for any formula C, but the proof necessarily starts with an application of the rule W if, *e.g.*, C is an atomic formula p_i.

Proposition 21. *Let* nEK_i *(*nEK_c*) be the system obtained from* nEK – W *by restricting the rules to the intuitionistic (classical) case (see Figures 3 and 4). The rule* W *is admissible in* nEK_i *and* nEK_c*.*

Proof. For the intuitionistic fragment, the proof is standard, by induction on the height of derivations (considering all possible rule applications). The classical case is more involved. The idea is that classical formulas in the *stoup* are eagerly decomposed until either an axiom is applied, or the formula is stored in the classical input context and the stoup becomes empty. This is only possible because the rules $\wedge^{\circ}$ and $\Box^{\circ}$ are totally invertible and all the other rules in nEK_c are invertible (Lemma 10). Formally, the following proof strategy is complete for nEK_c, when proving a nested sequent Γ:

i. Apply the rules $\wedge^{\bullet}, \wedge^{\circ}, \Box^{\bullet}, \Box^{\circ}$ and sto eagerly, obtaining leaves of the form $\Lambda\{\bot^{\circ}\}$.
ii. Apply any rule of nEK_c eagerly, until either finishing the proof with an axiom application or obtaining leaves of the form $\Lambda\{P^{\circ}\}$, where P is a positive formula in nEK_c, that is, having as main connective $\wedge$ or $\Box$. Start again from step (i).

$$\frac{}{\Lambda\left\{p_i^\bullet, p_i^\circ\right\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee_i B^\bullet\}}\ \vee_i^\bullet \qquad \frac{\Lambda\left\{A_j^\circ\right\}}{\Lambda\left\{A_1 \vee_i A_2^\circ\right\}}\ \vee_{i_j}^\circ \qquad \frac{\Gamma^*\{A \to_i B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \to_i B^\bullet\}}\ \to_i^\bullet \qquad \frac{\Lambda\{A^\bullet, B^\circ\}}{\Lambda\{A \to_i B^\circ\}}\ \to_i^\circ$$

$$\frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{\Diamond_i A^\bullet\}}\ \Diamond_i^\bullet \qquad \frac{\Lambda_1\{[A^\circ, \Lambda_2]\}}{\Lambda_1\{\Diamond_i A^\circ, [\Lambda_2]\}}\ \Diamond_i^\circ$$

Fig. 3. Intuitionistic fragment nEK_i.

$$\frac{}{\Gamma\{p_c^\bullet, p_c^\triangledown\}}\ \mathsf{init} \qquad \frac{}{\Gamma\{\bot^\bullet\}}\ \bot^\bullet \qquad \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}}\ \wedge^\bullet \qquad \frac{\Lambda\{A^\circ\} \quad \Lambda\{B^\circ\}}{\Lambda\{A \wedge B^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\bullet\}}\ \vee_c^\bullet \qquad \frac{\Gamma^{\bot^\circ}\{A^\triangledown, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \vee_c B^\triangledown\}}\ \vee_c^\triangledown \qquad \frac{\Gamma^*\{A \to_c B^\bullet, A^\circ\} \quad \Gamma^{\bot^\circ}\{B^\bullet\}}{\Gamma^{\bot^\circ}\{A \to_c B^\bullet\}}\ \to_c^\bullet$$

$$\frac{\Gamma^{\bot^\circ}\{A^\bullet, B^\triangledown\}}{\Gamma^{\bot^\circ}\{A \to_c B^\triangledown\}}\ \to_c^\triangledown \qquad \frac{\Delta_1\{\Box A^\bullet, [A^\bullet, \Delta_2]\}}{\Delta_1\{\Box A^\bullet, [\Delta_2]\}}\ \Box^\bullet \qquad \frac{\Lambda\{[A^\circ]\}}{\Lambda\{\Box A^\circ\}}\ \Box^\circ$$

$$\frac{\Gamma^{\bot^\circ}\{[A^\bullet]\}}{\Gamma^{\bot^\circ}\{\Diamond_c A^\bullet\}}\ \Diamond_c^\bullet \qquad \frac{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[A^\triangledown, \Delta_2^{\bot^\circ}\right]\right\}}{\Delta_1^{\bot^\circ}\left\{\Diamond_c A^\triangledown, \left[\Delta_2^{\bot^\circ}\right]\right\}}\ \Diamond_c^\triangledown \qquad \frac{\Gamma^*\{P^\triangledown, P^\circ\}}{\Gamma^{\bot^\circ}\{P^\triangledown\}}\ \mathsf{dec} \qquad \frac{\Lambda\{N^\triangledown, \bot^\circ\}}{\Lambda\{N^\circ\}}\ \mathsf{sto}$$

Fig. 4. Classical fragment nEK_c.

Observe that weakening is never applied, since a positive classical formula P° is totally decomposable into negative subformulas of the form N°, which are stored in the classical input context as $N^\triangledown$, or $\bot^\circ$.

This result clarifies the role of weakening in nEK: it serves as a bridge between intuitionistic and classical parts of a derivation and its application can be restricted to just below classical rules.

Since weakening is not present, nEK_i matches exactly the system NIK in [Str13].

Fact 22 *The intuitionistic fragment of* nEK *is Straßburger's system* NIK.

For the classical fragment, the discipline presented in the proof of Proposition 21 is interesting as it resembles focused search [LM11] in nEK_c: neutral connectives are handled in the stoup, while rules on classical connectives are applied in classical context.

Yet, this discipline does not match the focusing defined in [CMS16], since in that work diamond is considered positive and box negative, while the ecumenical system enforces the opposite polarity assignment. The task of providing a fully focused system, as well as adding polarized versions of conjunction and disjunction, as done *e.g.* in [LM11,CMS16] is left for a future work.

Axiom	**Condition**	**First-Order Formula**
$\mathsf{d}: \Box A \to \Diamond A$	Seriality	$\forall x \exists y. R(x,y)$
$\mathsf{t}: \Box A \to A \wedge A \to \Diamond A$	Reflexivity	$\forall x. R(x,x)$
$\mathsf{b}: A \to \Box\Diamond A \wedge \Diamond\Box A \to A$	Symmetry	$\forall x,y. R(x,y) \to R(y,x)$
$4: \Box A \to \Box\Box A \wedge \Diamond\Diamond A \to \Diamond A$	Transitivity	$\forall x,y,z.(R(x,y) \wedge R(y,z)) \to R(x,z)$
$5: \Box A \to \Box\Diamond A \wedge \Diamond\Box A \to \Diamond A$	Euclideaness	$\forall x,y,z.(R(x,y) \wedge R(x,z)) \to R(y,z)$

Table 1. Axioms and corresponding first-order conditions on R.

$$\frac{\Gamma\{\Box A^\bullet, A^\bullet\}}{\Gamma\{\Box A^\bullet\}}\ \mathsf{t}^\bullet \qquad \frac{\Delta_1\{[\Delta_2, \Box A^\bullet], A^\bullet\}}{\Delta_1\{[\Delta_2, \Box A^\bullet]\}}\ \mathsf{b}^\bullet \qquad \frac{\Delta_1\{[\Delta_2, \Box A^\bullet], \Box A^\bullet\}}{\Delta_1\{[\Delta_2], \Box A^\bullet\}}\ 4^\bullet \qquad \frac{\Gamma\{[\Box A^\bullet][\Box A^\bullet]\}}{\Gamma\{[\Box A^\bullet][\varnothing]\}}\ 5^\bullet$$

$$\frac{\Lambda\{A^\circ\}}{\Lambda\{\Diamond_i A^\circ\}}\ \mathsf{t}^\circ \qquad \frac{\Lambda_1\{[\Lambda_2], A^\circ\}}{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}\ \mathsf{b}^\circ \qquad \frac{\Lambda_1\{[\Lambda_2, \Diamond_i A^\circ]\}}{\Lambda_1\{[\Lambda_2], \Diamond_i A^\circ\}}\ 4^\circ \qquad \frac{\Lambda\{[\varnothing][\Diamond_i A^\circ]\}}{\Lambda\{[\Diamond_i A^\circ][\varnothing]\}}\ 5^\circ$$

$$\frac{\Gamma^{\perp^\circ}\{A^\triangledown\}}{\Gamma^{\perp^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{t}^\triangledown \qquad \frac{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}\right], A^\triangledown\right\}}{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}, \Diamond_c A^\triangledown\right]\right\}}\ \mathsf{b}^\triangledown \qquad \frac{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}, \Diamond_i A^\circ\right]\right\}}{\Delta_1^{\perp^\circ}\left\{\left[\Delta_2^{\perp^\circ}\right], \Diamond_c A^\triangledown\right\}}\ 4^\triangledown \qquad \frac{\Gamma^{\perp^\circ}\{[\varnothing][\Diamond_c A^\triangledown]\}}{\Gamma^{\perp^\circ}\{[\Diamond_c A^\triangledown][\varnothing]\}}\ 5^\triangledown$$

Fig. 5. Ecumenical modal extensions for axioms $\mathsf{d}, \mathsf{t}, \mathsf{b}, 4$ and 5.

7 Extensions

Depending on the application, several further modal logics can be defined as extensions of EK by simply restricting the class of frames we consider or, equivalently, by adding axioms over modalities. Many of the restrictions one can be interested in are definable as formulas of first-order logic, where the binary predicate $R(x,y)$ refers to the corresponding accessibility relation. Table 1 summarizes some of the most common logics, the corresponding frame property, together with the modal axiom capturing it [Sah75].

Since the intuitionistic fragment of nEK coincides with NIK, intuitionistic versions for the rules for the axioms $\mathsf{t}, \mathsf{b}, 4$, and 5 match the rules ($^\bullet$) and ($^\circ$) presented in [Str13], and are depicted in Figure 5.

For completing the ecumenical view, the classical ($^\triangledown$) rules for extensions are justified via translation to the labeled system labEK. For example, the labeled derivation on the left justifies the classical rule in the middle.

$$\dfrac{\dfrac{\dfrac{\dfrac{xRx, \mathcal{R}, \Sigma, x : \neg A \Rightarrow x : \bot}{xRx, \mathcal{R}, \Sigma, x : \Box\neg A \Rightarrow x : \bot}\ \Box L}{\mathcal{R}, \Sigma, x : \Box\neg A \Rightarrow x : \bot}\ \mathsf{T}}{\mathcal{R}, \Sigma \Rightarrow x : \Diamond_c A}\ \Diamond_c R}{\mathcal{R}, \Sigma, x : \neg\Diamond_c A \Rightarrow z : \bot}\ \neg L + \mathsf{W} \qquad \frac{\Gamma^{\perp^\circ}\{A^\triangledown\}}{\Gamma^{\perp^\circ}\{\Diamond_c A^\triangledown\}}\ \mathsf{t}^\triangledown \qquad \frac{xRx, \Gamma \vdash z : C}{\Gamma \vdash z : C}\ \mathsf{T}$$

The rule T above right is the labeled rule corresponding to the axiom t [Sim94]. The rules $\mathsf{b}^\triangledown, 4^\triangledown$ and $5^\triangledown$, shown in Figure 5, are obtained in the same manner. By mixing and matching these rules, we conjecture hat we obtain ecumenical modal systems for most logics in the S5 modal cube [BRV01], i.e. those that are not defined with axiom d.

8 Conclusion

In this paper, we have presented a pure, nested proof system nEK for the ecumenical modal logic EK, together with pure fragments and extensions. We proved soundness of nEK w.r.t. the ecumenical birelational model via a translation to the labeled ecumenical modal system labEK. For completeness, we used the fact that EK axioms are provable in nEK and we proved cut-elimination for nEK. Finally, having an ecumenical nested system allowed for extracting well known systems as fragments.

First of all, it should be noted that combining classical and intuitionistic modalities *conservatively* in the same *pure* logical system is not trivial. In fact, the labeled system in [MPPS20] makes an extensive use of negations in order to keep classical information *persistent*. We have shown that this can be avoided by having an additional classical context to store negative formulas, similarly to Girard's classical system LC [Gir91], henceforth solving the "impurity" issue. On the other hand, there seems to be no trivial solution to remove labels from intuitionistic modal sequent systems where the distributivity of the diamond w.r.t. the disjunction holds [Sim94]. The solution here was to adopt the framework to nested sequents, whose tree structure describes the corresponding path in the birelational semantics, a special case in which labels can be eliminated.

It turns out that this mix of classical context, polarities and nestings can be implosive, in the sense that adding a cut rule may lead to a collapse of the system to classical modal logic. For controlling the implosion, the cut rules must have a restricted use of polarities which, in turn, makes the cut-elimination proof non trivial.

There are many interesting ideas that can be explored for the proposed systems, axioms and semantics, as indicated throughout the text, and many lines to be pursued in this research direction. First of all, we have proposed a proof discipline for nEK, which does not correspond to focusing for modal systems as presented in [CMS16]. In fact, the presence of weakening is known to break focusing, we should threfore investigate alternative ways of having a fully focused system. Moreover, it should be interesting to study *typing* in ecumenical systems, in which fragments of already known modal type systems could be embedded. Finally, we plan to implement ecumenical provers, as well as to automate the cut-elimination proof in the L-Framework [OPR21].

Acknowledgements This work was partially financed by CNPq, CAPES and by the UK's EPSRC through research grant EP/S013008/1. We would like to thank the anonymous reviewers for their suggestions and comments.

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