SEMICLASSICAL RESOLVENT BOUNDS FOR COMPACTLY SUPPORTED RADIAL POTENTIALS

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Abstract. We employ separation of variables to prove weighted resolvent estimates for the semiclassical Schrödinger operator $-h^2\Delta + V(|x|) - E$ in dimension $n \geq 2$, where h, $E > 0$, and $V : [0, \infty) \to \mathbb{R}$ is L^∞ and compactly supported. We show that the weighted resolvent estimate grows no faster than $\exp(Ch^{-1})$, and prove an exterior weighted estimate which grows $\sim h^{-1}$.

1. Introduction and statement of results

Let $\Delta := \sum_{j=1}^n \partial_{x_j}^2$ be the Laplacian on $\mathbb{R}^n$, $n \geq 2$. We consider the semiclassical Schrödinger operator on $L^2(\mathbb{R}^n)$ given by

$$Pu = P(h)u := -h^2\Delta u + V(|x|)u,$$

where $V\colon [0,\infty) \to \mathbb{R}$ is L^∞ and compactly supported, and $h > 0$ is a semiclassical parameter. Then $P\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is self-adjoint, and the resolvent $(P - z)^{-1}$ is bounded on $L^2(\mathbb{R}^n)$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Throughout the article, we let $r := |x|$, and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Our first result is an exponential upper bound on the limiting absorption resolvent.

Theorem 1. *Let $n \geq 2$. Fix $[E_{\min}, E_{\max}] \subseteq (0, \infty)$ and $1/2 < s \leq 1$. There exist C, $h_0 > 0$, such that*

$$\|\langle x \rangle^{-s}(P - E - i\varepsilon)^{-1}\langle x \rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq e^{C/h}, \tag{1.1}$$

for all $\varepsilon > 0$, $h \in (0, h_0]$, and $E \in [E_{\min}, E_{\max}]$.

In addition, we prove a 'non-trapping' type estimate for the resolvent in the exterior of a large ball. Let

$$R_0 = R_0(V) := \sup\{r \in [0,\infty) \,:\, r \in \operatorname{ess\,supp} V\},$$

and define

$$\begin{aligned} &M_0 = M_0(V, E) := \\ &\quad \inf\{m > 0 \mid V(r) + mr^{-2} - E \geq 0, \text{ for almost all } r \text{ in a neighborhood of } (0, R_0(V)]\}. \end{aligned} \tag{1.2}$$

For example, if V is the characteristic function of the interval $(0, R_0]$, then $M_0 = ER_0^2$. If V is continuous at R_0, $M_0 = \operatorname{ess\,sup}_{[0,R_0]} r^2(E - V)$. Note that always $M_0 \geq ER_0^2$ because we require $M_0 r^{-2} - E \geq 0$ for some $r > R_0$. Finally, put

$$R_1 = R_1(V, E) := \sqrt{M_0(V, E)/E}. \tag{1.3}$$

Theorem 2. *Let $n \geq 2$. Fix $[E_{\min}, E_{\max}] \subseteq (0,\infty)$, $1/2 < s \leq 1$ and $R > \sup_{E\in[E_{\min},E_{\max}]} R_1(V, E)$. There exist C, $h_0 > 0$, such that*

$$\|\langle x \rangle^{-s}\mathbf{1}_{\geq R}(P - E - i\varepsilon)^{-1}\mathbf{1}_{\geq R}\langle x \rangle^{-s}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \leq \frac{C}{h}, \tag{1.4}$$

for all $\varepsilon > 0$, $h \in (0, h_0]$, and $E \in [E_{\min}, E_{\max}]$, where $\mathbf{1}_{\geq R}$ is the characteristic function of $\{x \in \mathbb{R}^n \colon |x| \geq R\}$.

Theorem 2 is optimal in the sense that [DaJi20, Theorem 3] shows that (1.4) is false in general when $R < \sqrt{M_0/E}$. For example, if $V \in C_0^\infty([0,1))$, $E < \max(-r^2 V)$, and $R < \sqrt{M_0/E}$, then (by [DaJi20, (2.9) and (4.10)]) the left hand side of (1.4) is bounded below by $e^{1/Ch}$ for h tending to zero along a sequence of positive values.

The novelty of Theorems 1 and 2 is that they bound the weighted resolvent for an arbitrary compactly supported, radial L^∞ potential. The h-dependencies on the right sides of (1.4) and (1.1) are sharp in general, see [DDZ15] and [DaJi20] for exponential lower bounds, and recall that the free resolvent ($V \equiv 0$) has a Ch^{-1} lower bound (to see this, consider $u = e^{i\sqrt{E}h^{-1}x_1}\chi(x)$ for some $0 \not\equiv \chi \in C_0^\infty(\mathbb{R}^n)$). Vodev's work [Vo21] shows, for dimension $n \geq 3$, a bound like (1.1) still holds for radial potentials decaying like $\langle r\rangle^{-\delta}$, $\delta > 2$, except with the right side replaced by $e^{Ch^{-4/3}}$ (bounds with additional losses hold for V decaying more slowly). For $V \in L^\infty(\mathbb{R}^n;\mathbb{R})$ not necessarily radial, $n \geq 2$, with $V = O(\langle r\rangle^{-\delta})$, $\delta > 2$, the best known weighted resolvent upper bound is $e^{Ch^{-4/3}\log(h^{-1})}$ [GaSh20]. In dimension $n \geq 2$, it is an open problem to determine the optimal h-dependence of the resolvent for $V \in L^\infty$. In contrast, when $n = 1$, an $e^{Ch^{-1}}$ bound holds even if $V \in L^1(\mathbb{R};\mathbb{R})$ [DaSh20]. As far as the authors are aware, Theorem 2 is the first exterior estimate for any class of L^∞ potentials in dimension higher than one.

Proofs of semiclassical resolvent estimates have a long history and are an active research topic. Burq [Bu98] was the first to show an $e^{Ch^{-1}}$ bound for smooth perturbations of the Laplacian on $\mathbb{R}^n$. Several extensions followed [Vo00, Bu02, Sj02, CaVo02]. The exterior bound (1.4) was first proved by Cardoso and Vodev [CaVo02], refining a preliminary estimate of Burq [Bu02]. More recent works on resolvent estimates in lower regularity include [Da14, Vo14, RoTa15, KlVo19, Sh19, Vo19, GaSh20, GaSh21, Sh20, Vo20a, Vo20b, Vo20c, Vo21].

Stronger bounds on the resolvent are known when $V \in C_0^\infty(\mathbb{R}^n;\mathbb{R})$ and conditions are imposed on the classical flow $\Phi(t) = \exp t(2\xi\partial_x - \partial_x V(x)\partial_\xi)$ (note that $\Phi(t)$ may be undefined in our case). The key dynamical object is the *trapped set* $\mathcal{K}(E)$ at energy $E > 0$, defined as the set of $(x,\xi) \in T^*\mathbb{R}^n$ such that $|\xi|^2 + V(x) = E$ and $|\Phi(t)(x,\xi)|$ is bounded as $|t| \to \infty$. If $\mathcal{K}(E) = \emptyset$, that is, if E is *nontrapping*, Robert and Tamura [RoTa87] showed the weighted resolvent is bounded by Ch^{-1}. We may thus think of (1.4) as a low regularity analog; it says that applying cutoffs supported sufficiently far from zero removes the losses from (1.1) due to trapping.

Theorem 1 allows one to obtain, at high frequency, bounds on and resonance free regions for, the meromorphic continuation of the cutoff resolvent of the operator $-c^2(|x|)\Delta$ on $L^2(\mathbb{R}^n, c^{-2}dx)$. Here $c \in L^\infty([0,\infty);(0,\infty))$ is called the *wavespeed* and satisfies

$$\operatorname{supp}(1-c) \text{ is compact}, \tag{1.5}$$

$$c_0 < c(r) < c_1 \text{ for some } c_0, c_1 > 0 \text{ and for all } r \in [0,\infty). \tag{1.6}$$

More precisely, one obtains,

Theorem 3. *Let $n \geq 2$, and suppose $c \in L^\infty([0,\infty);(0,\infty))$ obeys (1.5) and (1.6). For each $\chi \in C_0^\infty(\mathbb{R}^n)$, there exist constants $C_1, C_2, M > 0$ such that the the cutoff resolvent $\chi R(\lambda)\chi := \chi(-c^2\Delta - \lambda^2)^{-1}\chi$ continues analytically from $\operatorname{Im}\lambda > 0$ into the set $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > M, \operatorname{Im}\lambda > -e^{C_2|\operatorname{Re}\lambda|}\}$, where it satisfies the bound*

$$\|\chi R(\lambda)\chi\|_{L^2(\mathbb{R}^n,\, c^{-2}dx)\to L^2(\mathbb{R}^n,\, c^{-2}dx)} \leq e^{C_1|\operatorname{Re}\lambda|}. \tag{1.7}$$

The proof of Theorem 3 is the same as the proof of [Sh18, Proposition 5.1], and is seen by identifying $V = 1 - c^{-2}$, $h = |\operatorname{Re}\lambda|^{-1}$ and applying (1.1).

Theorem 3 implies logarithmic local energy decay for the wave equation

$$\begin{cases} (\partial_t^2 - c^2(x)\Delta)u(x,t) = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty),\ n \geq 2, \\ u(x,0) = u_0(x) \in H^2(\mathbb{R}^n), \\ \partial_t u(x,0) = u_1(x) \in H^2(\mathbb{R}^n), \end{cases} \tag{1.8}$$

where the initial data are compactly supported. Such a decay rate was first proved by Burq [Bu98] for c smooth, and allowing for a smooth Dirichlet obstacle.

The assumption that $c = 1$ outside of a compact set is necessary not only to establish (1.7), but also to study the low-frequency behavior of the cutoff resolvent. Under this assumption, one can see that this behavior is exactly the same as the case $c \equiv 1$ [Sh18, Proposition 4.1], which in turn is well-known (see, e.g., [Vo01, Section 1.1]). With both the low and high frequency behavior of the $\chi R(\lambda)\chi$ illuminated, a logarithmic local energy decay rate for the solution of (1.8) follows exactly as in [Sh18, Sections 6 and 7].

1.1. **Ideas from the proofs.** When $V : \mathbb{R}^n \to \mathbb{R}$ has limited regularity, proofs of resolvent estimates typically proceed by a modified positive commutator strategy, see e.g., [GaSh20, Section 2]. The potential is treated as a perturbation because it cannot be differentiated in some or all directions. The issue one needs to overcome is that, in the positive commutator scheme, the potential appears in a term without an apparent sign (even though V is real-valued), which in turn is difficult to control as $h \to 0^+$. Controlling this term results in an estimate from above by $e^{Ch^{-4/3}\log(h^{-1})}$.

Due to the work of Meshkov [Me92] on the Landis conjecture, one knows that the exponent $h^{-4/3}$ is optimal for compactly supported complex valued potentials (see Appendix C for an explanation). Therefore, any improvement upon the exponent $h^{-4/3}$ in the resolvent estimate for an L^∞ compactly supported potential must involve additional assumptions on the potential V (e.g. reality/radiality).

The radial symmetry of the potential is used in two ways. First, it allows us to study the high angular momenta separately from the low angular momenta. In particular, we use a spherical energy type estimate to obtain resolvent estimates for low frequencies. Second, decomposition by angular momentum enables us to take advantage of reality of V using ODE techniques. In particular, for large enough angular momenta m_j, reality of V yields useful monotonicity properties of certain solutions u_0 and u_1 to $(-h^2\partial_r^2 + V + m_j r^{-2} - E)u = 0$ (see their construction in Section 2). Control of u_0 and u_1 gives a bound on the integral kernel of $(-h^2\partial_r^2 + V + mr^{-2} - E - i0)^{-1}$, which together with WKB and Bessel function asymptotics, yields the sharp semiclassical resolvent estimates.

We remark that, in the setting of Theorem 1, our ODE methods do not yield estimates on u_0 and u_1 for small m_j. However, we are able to prove good enough resolvent estimates for these frequencies. In doing so, we prove the following proposition.

Proposition 1.1. *Fix $[E_{\min}, E_{\max}] \subseteq (0, \infty)$ and $1/2 < s \le 1$. There exist C, $h_0 > 0$ such that for $0 \le \varepsilon \le 1$, $h \in (0, h_0]$, $m \ge -h^2/4$, and $E \in [E_{\min}, E_{\max}]$,*

$$\|\langle r\rangle^{-s}(-h^2\partial_r^2 + V + mr^{-2} - E - i\varepsilon)^{-1}\langle r\rangle^{-s}\|_{L^2(0,\infty)\to L^2(0,\infty)} \le e^{C(1+|m|^{1/2})/h}. \tag{1.9}$$

Proposition 1.1 is motivated by [Vo21, Proposition 3.1] where a similar estimate is proved for $m \ge 0$. However, in order to handle the case of $n = 2$, we need to extend these resolvent estimates to $m \ge -h^2/4$. The proof of Proposition 1.1 utilizes methods inspired by the b-calculus from microlocal analysis to estimate u by $(-h^2\partial_r^2 + V + mr^{-2} - E)u$ near 0, and then employs a spherical energy method to handle the region away from zero.

We expect Theorems 1 and 2 still hold for potentials V which are radial, real and non-compactly supported, with sufficient decay toward infinity. A difficulty with treating this case is finding a suitable replacement for the WKB and Bessel function asymptotics we use in Section 3.

The organization of the paper is as follows. In Section 2 we give an overview of the plan to prove Theorems 1 and 2 via separation of variables. In Section 3 we use WKB and Bessel function asymptotics to prove Theorem 2. In Section 4 we use the Mellin transform and energy estimates to prove an exponential estimate which is optimal for low angular momenta but not for high ones. In Section 5 we use ODE analysis to remove the losses for high angular momenta and complete the proof of Theorem 1.

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2. Reduction to a family of one dimensional resovlent estimates

In this section, we use separation of variables to reduce the proofs of Theorems 1 and 2 to a family of one dimensional resolvent estimates. We begin by recalling the conjugation

$$r^{(n-1)/2}(-\Delta)r^{-(n-1)/2} = -\partial_r^2 - \frac{\Delta_{\mathbb{S}^{n-1}}}{r^2} + \frac{(n-1)(n-3)}{4r^2}.$$

We then put

$$m_j = h^2(\sigma_j + 4^{-1}(n-1)(n-3)),$$

where $0 = \sigma_0 < \sigma_1 = \sigma_2 \le \sigma_3 \le \cdots$ are the eigenvalues of the nonnegative Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{n-1} \subseteq \mathbb{R}^n$, repeated according to multiplicity. (Recall that the eigenvalues of the unit sphere are $k^2+(n-2)k$, $k = 0, 1, \dots$.) Denote by $\mathbf{Y}_0, \mathbf{Y}_1, \dots$ a corresponding sequence of orthonormal real eigenfunctions. Also define

$$P_m \coloneqq -h^2\partial_r^2 + V(r) + mr^{-2}, \qquad m \ge -\frac{h^2}{4}.$$

The operator P_m, acting on $L^2(\mathbb{R}_+)$, $\mathbb{R}_+ \coloneqq (0,\infty)$, with domain $C_0^\infty(\mathbb{R}_+)$, is symmetric. As P_m commutes with complex conjugation, by von Neumann's theorem it has equal defect indices, and thus has self adjoint extensions. Elements in the domain $\mathcal{D}(P_m^*)$ are precisely those functions $u \in L^2(\mathbb{R}_+)$ having u and $\partial_r u$ absolutely continuous along with $P_m u \in L^2(\mathbb{R}_+)$. Moreover, since $\|r^{-1}\varphi\|_{L^2(\mathbb{R}_+)} \le 2\|\partial_r\varphi\|_{L^2(\mathbb{R}_+)}$ for any $\varphi \in C_0^\infty(\mathbb{R}_+)$, P_m is a semibounded operator, thus a particular self-adjoint extension of $(P_m, C_0^\infty(\mathbb{R}_+))$ is its Friedrichs extension (see [ReSi, Theorem X.23]), which we denote by $(P_m, \mathcal{D}_m)$; this is the self-adjoint extension of $(P_m, C_0^\infty(\mathbb{R}_+))$ that we work with throughout the paper.

We start by studying the resolvent kernel for P_m. To do this, we construct two convenient linearly independent solutions, u_0, and u_1 to

$$P_m u_j = (E + i\varepsilon)u_j, \qquad j \in \{0,1\}. \tag{2.1}$$

To define these solutions, let

$$\varphi_J(r) = r^{1/2}J_\nu(\lambda r), \qquad \varphi_Y(r) = r^{1/2}Y_\nu(\lambda r),$$

$$m \ge -\frac{h^2}{4} \ge 0, \quad \nu = h^{-1}\big(m + \frac{h^2}{4}\big)^{1/2} \ge 0, \quad \lambda = \frac{\sqrt{E+i\varepsilon}}{h}, \quad E \in [E_{\min}, E_{\max}], \quad \varepsilon \ge 0.$$

where $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind, respectively [DLMF, Section 10.2]. Note that

$$(-h^2\partial_r + mr^{-2} - E - i\varepsilon)\varphi_{J/Y} = 0.$$

Next, define $\varphi_0 = \varphi_J$ and, inductively,

$$\varphi_{n+1}(r) = (2\pi h^2)^{-1}\int_0^r (\varphi_Y(r)\varphi_J(r') - \varphi_J(r)\varphi_Y(r'))V(r')\varphi_n(r')dr', \qquad n \ge 0.$$

Finally, put

$$u_0(r) \coloneqq \sum_{n=0}^{\infty} \varphi_n(r). \tag{2.2}$$

In Appendix B, we prove

Lemma 2.1. *The series (2.2) and its first derivative converge uniformly for (r, E, ε) in compact subsets of*
$\mathbb{R}_+ \times [E_{\min}, E_{\max}] \times [0, \infty)$. *In addition, $u_0(r) > 0$ near $r = 0$,*

$$u_0(r) = \varphi_0(r) + (2\pi h^2)^{-1} \int_0^r (\varphi_Y(r)\varphi_J(r') - \varphi_J(r)\varphi_Y(r'))V(r')u_0(r')dr', \tag{2.3}$$

$$(-h^2\partial_r^2 + V(r) + mr^{-2} - E - i\varepsilon)u_0 = 0, \tag{2.4}$$

$$\lim_{r\to 0^+} \big(\tfrac{n-1}{2r}u_0(r) - u_0'(r)\big)r^{(n-1)/2} = 0, \tag{2.5}$$

and for all $r^* > 0, \nu \geq 0$, there is C_{ν,r^*} such that

$$|u_0(r)| \leq C_{\nu,r^*}r^{\nu+\frac{1}{2}}, \qquad r \in (0, r^*], \tag{2.6}$$

Next, put

$$u_1(r) = \varphi_J(r) + i\varphi_Y(r), \qquad r > R_0, \tag{2.7}$$

and extend u_1 by requiring that it solve (2.1). By (2.7) and [Ol, Theorem 2.1 in Section 5.2.1], u_1 depends continuously on (r, E, ε) varying in $\mathbb{R}_+ \times [E_{\min}, E_{\max}] \times [0, \infty)$. Also, from [DLMF, 10.17.5], for all $r^* \geq 1$, $\nu \geq 0$, there is $C_{r^*,\nu} > 0$ such that

$$|u_1(r)| \leq C_{r^*,\nu}e^{-\operatorname{Im}\lambda r}, \qquad r \geq r^*. \tag{2.8}$$

Lemma 2.2. *The functions u_0 in (2.2) and u_1 in (2.7) are linearly independent for all $\varepsilon \geq 0$.*

Proof. First consider $\varepsilon > 0$ and suppose u_1 and u_0 are linearly dependent. Then, since
$u_0 \in L^2((0, 1])$ by (2.6) and $u_1 \in L^2([1, \infty))$ by (2.8), we would have $u_1 \in L^2([0, \infty))$. In particular, u_1 would be an L^2-solution of $(P_m - E - i\varepsilon)u = 0$ and thus must vanish identically. Next, when $\varepsilon = 0$, u_0 and u_1 are linearly independent since u_0 is real-valued while u_1 assumes non-real values. □

We can now define the resolvent kernel for P_m,

$$\begin{gathered} K(r, r') = K(r, r'; m, E, \varepsilon, h) = -\frac{u_0(r)u_1(r')}{h^2W}, \\ r \leq r',\ m \geq -\frac{h^2}{4},\ E \in [E_{\min}, E_{\max}],\ \varepsilon \geq 0, \end{gathered} \tag{2.9}$$

and for $r' < r$, $K(r, r') = K(r', r)$, where $W = u_0u_1' - u_0'u_1$ is the Wronskian of u_0 and u_1.

Lemma 2.3. *For $E, \varepsilon > 0$, $u \in L^2(\mathbb{R}^n)$ and $v \in L^2(\mathbb{R}_+)$,*

$$\begin{aligned} (P_m - E - i\varepsilon)^{-1}v &= \int_0^\infty K(r, r'; m, E, \varepsilon, h)v(r')dr'. \\ &= -h^{-2}W^{-1}\Big(u_1(r)\int_0^r u_0(r')v(r')dr' + u_0(r)\int_r^\infty u_1(r')v(r')dr'\Big), \end{aligned} \tag{2.10}$$

$$(P - E - i\varepsilon)^{-1}u = \sum_{j=0}^\infty \mathbf{Y}_j \int_0^\infty \int_{\mathbb{S}^{n-1}} r^{-(n-1)/2}K(r, r'; m_j, E, \varepsilon, h)(r')^{(n-1)/2}u(r', \theta)\mathbf{Y}_j(\theta)d\theta dr'. \tag{2.11}$$

Proof. To prove (2.10), it suffices to work with $v \in C_0^\infty(\mathbb{R}_+)$. We check that the right side of (2.10) belongs to $\mathcal{D}_m$ and, that applying $P_m - E - i\varepsilon$ yields v. The latter is a direct computation, while the former follows from (2.6), (2.8), and the fact that a characterization of $\mathcal{D}_m$ is

$$\mathcal{D}_m = \big\{f \in L^2(\mathbb{R}_+) : P_mf \in L^2(\mathbb{R}_+),\ r^{-\nu-\frac{1}{2}}f \in L^\infty \text{ near } r = 0\big\}. \tag{2.12}$$

See [NiZe92, Section 6, Equation (4.14)].

We need only verify (2.11) for $u = \mathbf{Y}_j v$ with $v \in C_0^\infty(\mathbb{R}_+)$, as such functions have dense linear span in $L^2(\mathbb{R}^n)$. In this case, the right side of (2.11) reduces to

$$\mathbf{Y}_j \int_0^\infty r^{-(n-1)/2} K(r,r';m_j,E,\varepsilon,h)(r')^{(n-1)/2} v(r')dr'. \tag{2.13}$$

To show (2.13) and $(P-E-i\varepsilon)^{-1}\mathbf{Y}_j v$ coincide, we check that applying $P-E-i\varepsilon$, respectively Δ, to (2.13) in the sense of distributions, results in $\mathbf{Y}_j v$, respectively some function in $L^2(\mathbb{R}^n)$. Both computations are handled by integrating by parts in polar coordinates, on domains of the form $\{x \in \mathbb{R}^n : |x| > \delta > 0\}$, and sending $\delta \to 0$. All boundary terms that appear in the calculation vanish as $\delta \to 0$, thanks to (2.5)and (2.6). We leave the remaining details to the reader.

□

We next consider the limit as $\varepsilon \to 0^+$. Recall that [Ag75, Theorem 4.2] for any $s > 1/2$, and $E \in [E_{\min}, E_{\max}]$, the limit

$$\langle x\rangle^{-s}(P-E-i0)^{-1}\langle x\rangle^{-s} := \lim_{\varepsilon\to 0^+}\langle x\rangle^{-s}(P-E-i\varepsilon)^{-1}\langle x\rangle^{-s} \tag{2.14}$$

exists in the uniform topology $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. By (2.11), (2.10), and because $K(r,r';m_j,E,\varepsilon,h)$ depends continuously on (r,r',E,ε) varying in compact subsets of $\mathbb{R}_+\times\mathbb{R}_+\times[E_{\min},E_{\max}]\times[0,\infty)$, for each j and $u, v \in C_0^\infty(\mathbb{R}_+)$,

$$\begin{aligned}
&\langle \mathbf{Y}_j u, \langle x\rangle^{-s}(P-E-i0)^{-1}\langle x\rangle^{-s}\mathbf{Y}_j v\rangle_{L^2} \\
&\quad= \lim_{\varepsilon\to0^+}\langle r^{(n-1)/2}u, \langle r\rangle^{-s}\int_0^\infty K(r,r';m_j,E,\varepsilon,h)\langle r'\rangle^{-s}(r')^{(n-1)/2}v(r')dr'\rangle_{L^2(\mathbb{R}_+)} \\
&\quad= \lim_{\varepsilon\to0^+}\langle r^{(n-1)/2}u, \langle r\rangle^{-s}(P_{m_j}-E-i\varepsilon)^{-1}\langle r'\rangle^{-s}(r')^{(n-1)/2}v(r')\rangle_{L^2(\mathbb{R}_+)} \\
&\quad= \langle r^{(n-1)/2}u, \langle r\rangle^{-s}\int_0^\infty K(r,r';m_j,E,0,h)\langle r'\rangle^{-s}(r')^{(n-1)/2}v(r')dr'\rangle_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

Thus, as functions $\{\mathbf{Y}_j v : j \in \mathbb{N}_0,\, v \in C_0^\infty(\mathbb{R}_+)\}$ have dense linear span in $L^2(\mathbb{R}^n)$, to bound the norm of (2.14) on $L^2(\mathbb{R}^n)$, it suffices to bound, uniformly in j, the kernel $K(r,r';m_j,E,0,h)$, or alternatively, $\|\langle r\rangle^{-s}(P_{m_j}-E-i\varepsilon)^{-1}\langle r\rangle^{-s}\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)}$ for $\varepsilon > 0$ small.

If the bounds (1.1) and (1.4) are established for $\varepsilon = 0$, it is well-known that one can use resolvent identities to show they hold for $\varepsilon > 0$ as well. We omit the proof of this but refer the reader to the relevant results, see [BrPe00, Proposition 3] and [Vo14, Theorem 1.5].

3. Exterior estimates

Theorem 2 follows from the stronger statement that, for any $R > \sup_{E\in[E_{\min},E_{\max}]} R_1(V,E)$ (see (1.3)), we have

$$|K(r,r')| = |K(r,r',m,E,0,h)| \le \frac{C}{h}, \tag{3.1}$$

uniformly for r and r' in $[R,\infty)$ obeying $r \le r'$, $m \ge -h^2/4$, h small, and $E \in [E_{\max}, E_{\min}]$. We prove (3.1) in this section, over the course of two lemmas. The first lemma establishes (3.1) for $m \le M_+$, where

$$M_+ = M_+(E) := M_0 + \frac{E(R^2-R_1^2)}{2}. \tag{3.2}$$

In the second lemma, we prove (3.1) for $m > M_+$.

Lemma 3.1. *There are C and h_0 such that (3.1) holds for all r and r' in $[R,\infty)$ obeying $r \le r'$, $h \in (0,h_0]$, $E \in [E_{\min},E_{\max}]$ and for all $m \in [-h^2/4, M_+]$.*

Proof. The proof is essentially the same as the one in [DaJi20, Lemma 1]. It is based on WKB or Liouville-Green asymptotics for solutions to an ODE with real coefficients in a region with no turning point. First, observe that if $r \geq R$ and $m \leq M_+$, then

$$mr^{-2} \leq \frac{2M_0 + E(R^2 - R_1^2)}{2r^2} \leq \frac{2ER_1^2 + E(R^2 - R_1^2)}{2R^2} = E - \frac{E(R^2 - R_1^2)}{2R^2}.$$

Thus $E - mr^{-2}$ is bounded below by a positive constant uniformly in E, r and m.

By [Ol, Section 6.2.4], there are real numbers $A = A(h)$ and $B = B(h)$ such that, for $r \geq R$ we have

$$u_0(r) = \frac{1}{\sqrt[4]{E - mr^{-2}}} \Big(\sum_{\pm} (A \pm iB) \exp\Big(\pm \frac{i}{h} \int_R^r \sqrt{E - ms^{-2}} ds \Big) (1 + \varepsilon_\pm(r)) \Big),$$

where ε_+ and ε_- satisfy

$$|\varepsilon_\pm(r)| + h|\varepsilon_\pm'(r)| \lesssim \frac{h}{r^3}.$$

By [Ol, Section 6.2.4], there is a constant $D = D(h)$ so that for $r \geq R$,

$$u_1(r) = \frac{D}{\sqrt[4]{E - mr^{-2}}} \exp\Big(\frac{i}{h} \int_R^r \sqrt{E - ms^{-2}} ds \Big) (1 + \varepsilon_+(r)),$$

where we rule out the presence of an $\exp\Big(- \frac{i}{h} \int_R^r \sqrt{E - ms^{-2}} ds \Big)$ term by using large-argument Bessel function asymptotics [DLMF, 10.17.5 and 10.17.11] to show that

$$\partial_r u_1 - i\frac{\sqrt{E}}{h} u_1 \to 0, \qquad \text{as } r \to \infty.$$

Next we compute the Wronskian

$$W = \frac{D}{\sqrt{E - mr^{-2}}} \Big(A - iB \Big) \frac{2i}{h} \sqrt{E - mr^{-2}} \Big(1 + O(hr^{-3}) \Big) = \frac{2D(B + iA)}{h},$$

where we dropped the remainder because W is independent of r. Plugging these formulas for u_0, u_1, and W into the formula (2.9) for K gives the conclusion.

□

Lemma 3.2. *There are C and h_0 such that* (3.1) *holds for all r and r' in $[R_0, \infty)$ obeying $r \leq r'$, $h \in (0, h_0]$, $E \in [E_{\min}, E_{\max}]$, and for all $m > M_+$.*

Proof. By the Bessel function differential equation (A.1), there are real numbers A and B such that, for $r \geq R_0$,

$$u_0(r) = r^{1/2}(AJ_\nu(\nu z) + BY_\nu(\nu z)),$$

where

$$\nu = h^{-1}(m + \tfrac{1}{4}h^2)^{1/2}, \qquad z = (m + \tfrac{1}{4}h^2)^{-1/2} E^{1/2} r.$$

(Note that the constants A and B here are analogous to but different from the ones from Lemma 3.1.) Recall from (2.7) that

$$u_1(r) = r^{1/2}(J_\nu(\nu z) + iY_\nu(\nu z)).$$

By the Bessel function Wronskian formula (A.2),

$$W = 2\pi^{-1}(iA - B).$$

To bound B in terms of A we use the fact that $u_0(R_0)$, $u_0'(R_0) \geq 0$ which follows from $V + mr^{-2} - E \geq 0$ on $(0, R_0]$; see Lemma B.1. By the Bessel function bounds (A.7), for h small enough, we have

$$B \leq \frac{J_\nu(\nu z_0)}{-Y_\nu(\nu z_0)} A \lesssim e^{-2\nu\xi_0} A, \qquad -B \leq \frac{J_\nu(\nu z_0) + 2R_0 J_\nu'(\nu z_0)}{Y_\nu(\nu z_0) + 2R_0 Y_\nu'(\nu z_0)} A \lesssim e^{-2\nu\xi_0} A$$

where

$$\xi_0 = \int_{z_0}^1 t^{-1}(1-t^2)^{1/2}dt, \qquad z_0 = (m + \tfrac{1}{4}h^2)^{-1/2}E^{1/2}R_0.$$

Note that $z_0 < 1$ because

$$z_0 \le M_+^{-1/2}E^{1/2}R_0 = \left(\frac{M_0}{ER_0^2} + \frac{R^2 - R_0^2}{2R_0^2}\right)^{-1/2} \le \left(1 + \frac{R^2 - R_0^2}{2R_0^2}\right)^{-1/2}.$$

Then, when $R_0 \le r \le r'$, letting $z' = (m + \frac{1}{4}h^2)^{-1/2}E^{1/2}r'$ and inserting $|B| \lesssim e^{-2\nu\xi_0}A$ and the Bessel function bound (A.10) into (2.9) gives

$$\begin{aligned} |K(r,r')| &\lesssim h^{-2}(rr')^{1/2}(J_\nu(\nu z) + e^{-2\nu\xi_0}|Y_\nu(\nu z)|)|J_\nu(\nu z') + iY_\nu(\nu z')| \\ &\lesssim h^{-2}\nu^{-1}(rr')^{1/2}\langle z\rangle^{-1/2}\langle z'\rangle^{-1/2} \lesssim h^{-1}, \end{aligned} \tag{3.3}$$

as desired.

□

4. One dimensional resolvent estimates for low angular momenta

We now study the equation

$$(P_m - E - i\varepsilon)u = (-h^2\partial_r^2 + V(r) + mr^{-2} - E - i\varepsilon)u = f, \qquad m \ge -\frac{h^2}{4},$$

on $L^2 = L^2(\mathbb{R}_+) := L^2(0,\infty)$. Recall that the self-adjoint extension of $(P_m, C_0^\infty(\mathbb{R}_+))$ we employ is its Friedrichs extension $(P_m, \mathcal{D}_m)$. Put $\mathcal{D}_{\mathrm{comp},m} := \{u \in \mathcal{D}_m : \operatorname{supp} u \text{ is compact in } [0,\infty)\}$.

Lemma 4.1. *Let $[E_{\min}, E_{\max}] \subseteq (0,\infty)$ and $1/2 < s \le 1$. Then there is C, $h_0 > 0$ such that for all $E \in [E_{\min}, E_{\max}]$, $h \in (0, h_0]$, $m \ge -h^2/4$, $0 \le \varepsilon \le 1$, and $u \in \mathcal{D}_{\mathrm{comp},m}$,*

$$\|\langle r\rangle^{-s}u\|_{L^2(\mathbb{R}_+)} \le e^{C(1+|m|^{1/2})/h}\|\langle r\rangle^s(P_m - E - i\varepsilon)u\|_{L^2(\mathbb{R}_+)}.$$

To prove Lemma 4.1, we start by studying the operator

$$Q_m := -\partial_r^2 + mh^{-2}r^{-2}, \qquad m \ge -\frac{h^2}{4},$$

on functions compactly supported in $[0,\infty)$. We recall the Mellin transform and its inverse:

$$\mathcal{M}(u)(\sigma) := \int_0^\infty r^{i\sigma}u(r)\frac{dr}{r}, \qquad \mathcal{M}_t^{-1}(v)(r) := \frac{1}{2\pi}\int_{\mathbb{R}} r^{-i\sigma}v(\sigma)d\tau, \qquad \sigma = \tau + it,$$

where the definitions hold initially, e.g., for $u \in C_0^\infty(\mathbb{R}_+)$ and $v((\cdot)+it) \in L^1_\tau(\mathbb{R}) \cap L^2_\tau(\mathbb{R})$, and then extend by density to bounded operators $\mathcal{M} : L^2(0,\infty; r^{-2t-1}dr) \to L^2_\tau(\mathbb{R})$, $\mathcal{M}_t^{-1} : L^2_\tau(\mathbb{R}) \to L^2(0,\infty; r^{-2t-1}dr)$. Moreover, since $\mathcal{M}(u)(\tau+it) = 2\pi\mathcal{F}^{-1}(e^{-tx}u(e^x))(\tau)$, $x \in \mathbb{R}$, and $\mathcal{M}_t^{-1}(v)(r) = r^t\mathcal{F}(v)(\log r)/2\pi$, where $\mathcal{F}$ denotes Fourier transform,

$$\begin{aligned} \|\mathcal{M}(u)(\tau+it)\|_{L^2_\tau(\mathbb{R})} &= (2\pi)^{1/2}\|r^{-t-1/2}u\|_{L^2(\mathbb{R}_+)}, \\ \|r^{-t-1/2}\mathcal{M}_t^{-1}(v)(r)\|_{L^2(\mathbb{R}_+)} &= (2\pi)^{-1/2}\|v\|_{L^2_\tau(\mathbb{R})}. \end{aligned} \tag{4.1}$$

Let

$$t_\pm = t_\pm(m) := \frac{1 \pm \sqrt{1 + 4mh^{-2}}}{2}, \qquad \Lambda(t,m) := |t^2 - t - h^{-2}m|^{-1}, \qquad t \ne t_\pm.$$

Lemma 4.2. *There is $C > 0$ such that for $m \ge -h^2/4$, $N \in \mathbb{R}$, $t_0 \in \mathbb{R} \setminus \{t_+(m), t_-(m)\}$, and $u \in r^{-N}L^2_{\mathrm{comp}}[0,\infty)$ with $Q_m u \in r^{t_0 - \frac{3}{2}}L^2$, we have*

$$u = \Pi_{t_0}(r^2Q_mu) + E_{t_0}(r^2Q_mu), \tag{4.2}$$

where

$$\|r^{-t_0-\frac{1}{2}}E_{t_0}v\|_{L^2} \le C\Lambda(t_0,m)\|r^{-t_0-\frac{1}{2}}v\|_{L^2(\mathbb{R}_+)}, \tag{4.3}$$

and

$$\Pi_{t_0}v = \begin{cases} 0 & t_0 < t_-, \\ r^{\frac{1}{2}}\log r\mathcal{M}(v)(\frac{i}{2}) - r^{\frac{1}{2}}\mathcal{M}(\log rv)(\frac{i}{2}) & t_- < t_0,\, m = -\frac{h^2}{4}, \\ \frac{r^{t_-}\mathcal{M}(v)(it_-)}{t_--t_+} & t_- < t_0 < t_+,\, m > -\frac{h^2}{4}, \\ \frac{r^{t_-}\mathcal{M}(v)(it_-)}{t_--t_+} + \frac{r^{t_+}\mathcal{M}(v)(it_+)}{t_+-t_-} & t_+ < t_0,\, m > -\frac{h^2}{4}. \end{cases}$$

Remark: Applying $(r\partial_r)^j$, $j = 1, 2$, to the expression for $E_{t_0}v$, see (4.6) below, yields a strengthening of (4.3), namely

$$\|r^{-t_0-\frac{1}{2}}(r\partial_r)^jE_{t_0}v\|_{L^2} \le \Lambda_j(t_0,m)\|r^{-t_0-\frac{1}{2}}v\|_{L^2(\mathbb{R}_+)}, \qquad j = 0,\, 1,\, 2,$$

However, we omit the proof since in the sequel we do not need the estimates for $j = 1, 2$.

Proof. Without loss of generality, we take N positive and large enough so that $-N < t_0, t_+, t_-$. Since $u \in r^{-N}L^2_{\text{comp}}$ and $r^2Q_mu \in r^{t_0+\frac{1}{2}}L^2_{\text{comp}}$, $\mathcal{M}(u)(\sigma)$ is holomorphic in $\operatorname{Im}\sigma < -N-1/2$, while $\mathcal{M}(r^2Q_mu)(\sigma)$, is holomorphic in $\operatorname{Im}\sigma < t_0$ and extends continuously (as an L^2 function) to $\operatorname{Im}\sigma = t_0$. In addition,

$$\mathcal{M}(r^2Q_mu)(\sigma) = (\sigma^2 - i\sigma + mh^{-2})\mathcal{M}(u)(\sigma), \qquad \operatorname{Im}\sigma < -N - \frac{1}{2}. \tag{4.4}$$

In particular, (4.4) implies $\mathcal{M}u(\sigma) \in L^1_\tau(\mathbb{R}) \cap L^2_\tau(\mathbb{R})$ for $\operatorname{Im}\sigma < -N-1/2$, so by Fourier inversion,

$$u(r) = \frac{1}{2\pi}\int_{\operatorname{Im}\sigma=-N-1} \frac{r^{-i\sigma}\mathcal{M}(r^2Q_mu)(\sigma)}{\sigma^2 - i\sigma + mh^{-2}}d\sigma. \tag{4.5}$$

We now deform the contour to $\operatorname{Im}\sigma = t_0 - \epsilon$ and the send $\epsilon \to 0$. From (4.5),

$$u(r) = \frac{1}{2\pi}\lim_{R\to\infty}\int_{\gamma_{R,-N-1}} \frac{r^{-i\sigma}\mathcal{M}(r^2Q_mu)(\sigma)}{\sigma^2 - i\sigma + mh^{-2}}d\sigma, \qquad \gamma_{R,t} = \{\tau + it \,:\, \tau \in [-R,R]\}.$$

Next, observe that since, in $\operatorname{Im}\sigma < t_0$, $\|\mathcal{M}(r^2Q_mu)(\sigma)\|_{L^\infty_\tau(\mathbb{R})}$ is indepdendent of $\operatorname{Re}\sigma$,

$$\Big|\int_{\gamma_{\pm,R,-N,\epsilon}} \frac{r^{-i\sigma}\mathcal{M}(r^2Q_mu)(\sigma)}{\sigma^2 - i\sigma + mh^{-2}}d\sigma\Big| \le \frac{C_\epsilon}{R^2} \to 0, \qquad \text{as } R \to \infty.$$

where

$$\gamma_{\pm,R,-N,\epsilon} := \{\pm R + it \,:\, t \in [-N-1, t_0-\epsilon]\}.$$

In particular, using $t_0 \neq t_\pm(m)$ and that $\mathcal{M}(r^2Q_mu)(\tau+it)$ varies continuously in $L^2_\tau(\mathbb{R})$ for $t \le t_0$, we send $\epsilon \to 0$ to obtain

$$\begin{aligned} u(r) &= \frac{1}{2\pi}\int_{\operatorname{Im}\sigma=t_0} \frac{r^{-i\sigma}\mathcal{M}(r^2Q_mu)(\sigma)}{\sigma^2 - i\sigma + mh^{-2}}d\sigma + i\sum_{\substack{\sigma^2-i\sigma+mh^{-2}=0\\ \operatorname{Im}\sigma<t_0}} \operatorname{Res}\Big(r^{-i\sigma}\frac{\mathcal{M}(r^2Q_mu)(\sigma)}{\sigma^2 - i\sigma + mh^{-2}}\Big) \\ &=: E_{t_0}(r^2Q_mu) + \Pi_{t_0}(r^2Q_mu). \end{aligned} \tag{4.6}$$

The formula for Π_{t_0} follows from calculating residues at $t_\pm(m)$, while the bound on E_{t_0} follows from minimizing the modulus

$$|(\tau + it_0)^2 - i(\tau + it_0) + mh^{-2}|^2 = (-t_0^2 + t_0 + mh^{-2} + \tau^2)^2 + \tau^2(2t_0 - 1)^2,$$

with respect to τ.

□

Lemma 4.3. *Suppose $m > 0$ and $u \in \mathcal{D}_m$. Then $r^{-1}u \in L^2(\mathbb{R}_+)$.*

Proof. This follows immediately from the characterization (2.12) of $\mathcal{D}_m$. $\square$

Lemma 4.2 will allow us to control the behavior of solutions, u to $(P_m - E - i\varepsilon)u = f$ near 0.

Proof of Lemma 4.1. To shorten notation, we set $P_{m,E,\varepsilon} = P_m - E - i\varepsilon$. Since $u \in \mathcal{D}_{\mathrm{comp},m}$ and $u \in L^2$, for any compact $K \subseteq (0,\infty)$, there exists $C_{K,h} > 0$ so that

$$\|\partial_r^2 u\|_{L^2(K)} \le C_{K,h}(\|u\|_{L^2(K)} + \|P_m u\|_{L^2(K)}).$$

In particular, $u \in H^2_{\mathrm{loc}}(0,\infty)$.

Next, set

$$\delta = \delta(m) := \delta_0 h \langle h^{-2} m\rangle^{1/2}, \qquad \delta_1 = \delta_1(m) := \max(\delta(m), \tfrac{1}{2}), \tag{4.7}$$

for some $0 < \delta_0 \ll 1$ independent of h, m and $E \in [E_{\min}, E_{\max}]$, to be chosen. Let $\chi \in C_0^\infty[0,2)$ with $\chi \equiv 1$ near $[0,1]$. Set $\chi_{\delta_1} = \chi(\delta_1^{-1} r)$. Then

$$\begin{aligned} Q_m \chi_{\delta_1} u &= h^{-2} P_{m,E,\varepsilon} \chi_{\delta_1} u - h^{-2}(V - E - i\varepsilon)\chi_{\delta_1} u \\ &= h^{-2}\chi_{\delta_1} P_m u + h^{-2}[P_m, \chi_{\delta_1}]u - h^{-2}\chi_{\delta_1}(V - E - i\varepsilon)u. \end{aligned} \tag{4.8}$$

In particular, since $u \in H^2_{\mathrm{loc}}(0,\infty)$, $P_m u \in L^2$, and χ_{δ_1} is constant near zero, $r^2 Q_m \chi_{\delta_1} u \in r^2 L^2$. Setting

$$t_0 = t_0(m) := \begin{cases} -\frac{1}{2} & -\frac{h^2}{4} \le m \le \frac{h^2}{4}, \\ 1 & \frac{h^2}{4} < m, \end{cases}$$

observe that

$$\Lambda(t_0, m) = |t_0^2 - t_0 - h^{-2} m|^{-1} \le c\langle h^{-2} m\rangle^{-1} \tag{4.9}$$

for some $c > 0$ independent of h and m. Note also that with this definition of $t_0(m)$, $t_0(m) < t_-(m)$ for $-h^2/4 \le m \le h^2/4$, and $t_-(m) < t_0(m) < t_+(m)$ for $h^2/4 < m$. Therefore, by Lemma 4.2, since $u \in L^2$, $r^2 Q_m \chi_{\delta_1} u \in r^2 L^2$, and $t_0(m) \le 3/2$,

$$\chi_{\delta_1} u = E_{t_0}(r^2 Q_m \chi_{\delta_1} u) + \Pi_{t_0}(r^2 Q_m \chi_{\delta_1} u),$$

where

$$\Pi_{t_0}(r^2 Q_m \chi_{\delta_1} u) = \begin{cases} 0 & -\frac{h^2}{4} \le m \le \frac{h^2}{4}, \\ \frac{r^{t_-}}{t_- - t_+}\mathcal{M}(r^2 Q_m \chi_{\delta_1} u)(it_-) & \frac{h^2}{4} < m, \end{cases}$$

and

$$\|r^{-t_0 - \frac{1}{2}} E_{t_0} v\|_{L^2} \le C\Lambda(t_0, m)\|r^{-t_0 - \frac{1}{2}} v\|_{L^2}.$$

Next, by Lemma 4.3, for $m > 0$, $r^{-1}u \in L^2$. Thus, both $\mathcal{M}(\chi_{\delta_1} u)(\sigma)$ and $\mathcal{M}(r^2 Q_m \chi_{\delta_1} u)(\sigma)$ are holomorphic in $\operatorname{Im}\sigma < 1/2$. As $t_- < 1/2$ when $m > 0$, by (4.4),

$$\mathcal{M}(r^2 Q_m \chi_{\delta_1} u)(it_-) = ((it_-)^2 - i(it_-) - h^{-2} m)\mathcal{M}(\chi_{\delta_1} u)(it_-) = 0.$$

In particular, for any m and $u \in \mathcal{D}_{\mathrm{comp},m}$, $\Pi_{t_0}(r^2 Q_m \chi_{\delta_1} u) = 0$. Thus, by (4.8) and $\delta_1 \ge 1/2$,

$$\begin{aligned} \|r^{-t_0 - \frac{1}{2}} u\|_{L^2(0,\delta_1)} &\le \|r^{-t_0 - \frac{1}{2}}\chi_{\delta_1} u\|_{L^2(0,\delta_1)} \le C\Lambda(t_0, m)\|r^{\frac{3}{2} - t_0} Q_m \chi_{\delta_1} u\|_{L^2} \\ &\le C\Lambda(t_0, m)\Big(h^{-2}\|r^{\frac{3}{2} - t_0} P_{m,E,\varepsilon} u\|_{L^2(0,2\delta_1)} \\ &\quad + h^{-1}\|r^{\frac{3}{2} - t_0} h u'\|_{L^2(\delta_1, 2\delta_1)} + h^{-2}\|r^{\frac{3}{2} - t_0} u\|_{L^2(0,2\delta_1)}\Big). \end{aligned} \tag{4.10}$$

We now estimate part of the last term on the right side of (4.10), using (4.7) and (4.9):

$$\begin{aligned} C\Lambda(t_0, m) h^{-2}\|r^{\frac{3}{2} - t_0} u\|_{L^2(0,\delta)} &\le C\Lambda(t_0, m) h^{-2}\delta^2\|r^{-t_0 - \frac{1}{2}} u\|_{L^2(0,\delta)} \\ &\le Cc\delta_0\|r^{-t_0 - \frac{1}{2}} u\|_{L^2(0,\delta)}. \end{aligned}$$

Choosing δ_0 small enough, this term may be absorbed into the left side of (4.10), so we find

$$\begin{aligned}\|r^{-t_0-\frac{1}{2}}u\|_{L^2(0,\delta_1)} \leq Ch^{-2}\langle h^{-2}m\rangle^{-1}\Big(\|r^{\frac{3}{2}-t_0}P_{m,E,\varepsilon}u\|_{L^2(0,2\delta_1)}+ \\ h\|r^{\frac{3}{2}-t_0}hu'\|_{L^2(\delta_1,2\delta_1)}+\|r^{\frac{3}{2}-t_0}u\|_{L^2(\delta,2\delta_1)}\Big).\end{aligned} \tag{4.11}$$

We now employ the energy method to study the region $[\delta(m),\infty)$. Let $s>1/2$, $\eta=\min\{1,E\}/2$, and $\phi_j\in C_0^\infty([0,1);[0,1])$, $j=0,1,2$, with $\phi_0\equiv 1$ on $[0,1/2]$ and $\phi_j\equiv 1$ on $\operatorname{supp}\phi_{j-1}$. Let

$$F(r):=|hu'(r)|^2+E|u(r)|^2, \qquad w(r):=\int_0^r(1-\phi_1(r'/\delta))\phi_2(r'/\delta)dr'e^{\psi(r)/h},$$

with

$$\psi:=\int_0^r\eta^{-1}\big(|V(r')|+2|m|(1-\phi_0(r'/\delta))(r')^{-2}\big)+\langle r'\rangle^{-2s}dr'\leq\eta^{-1}\|V\|_{L^1(\mathbb{R}_+)}+C(1+\eta^{-1}|m|^{1/2}).$$

Here, we have used

$$|m|\int_{\frac{1}{2}\delta(m)}^r(r')^{-2}dr'=|m|\Big((\tfrac{1}{2}\delta(m))^{-1}-r^{-1}\Big)\lesssim|m|^{1/2}.$$

Then,

$$(wF)'(r)=-2\operatorname{Re}w\langle P_{m,E,\varepsilon}u,u'\rangle+2\varepsilon w\operatorname{Im}\langle u,u'\rangle+w'(|hu'|^2+E|u|^2)+2w\operatorname{Re}\langle Vu+mr^{-2},u'\rangle,$$

where $\langle z,z_1\rangle:=z\overline{z_1}$, $z,z_1\in\mathbb{C}$. Since $\phi_0(r/\delta)>0$ implies $w(r)=0$, we have

$$w'=\frac{\psi'}{h}w+(1-\phi_1(r/\delta))\phi_2(r/\delta))e^{\psi/h}\geq\frac{\psi'}{h}w\geq h^{-1}w[\eta^{-1}|V|+\langle r\rangle^{-2s}+\eta^{-1}|m|r^{-2}]. \tag{4.12}$$

So,

$$\begin{aligned}(wF)'(r)&\geq-2\operatorname{Re}w\langle P_{m,E,\varepsilon}u,u'\rangle+2\varepsilon w\operatorname{Im}\langle u,u'\rangle+w'(|hu'|^2+E|u|^2)+2w|\operatorname{Re}\langle(V+mr^{-2})u,u'\rangle|\\&\geq-h^{-1}\eta^{-1}\langle r\rangle^{2s}w|P_{m,E,\varepsilon}u|^2-h^{-1}\eta w\langle r\rangle^{-2s}|hu'|^2\\&\quad-h^{-1}(|V|+\varepsilon+|m|r^{-2})w(|u|^2+|hu'|^2)+\min\{1,E\}w'(|hu'|^2+|u|^2)\\&\geq-h^{-1}\eta^{-1}\langle r\rangle^{2s}w|P_{m,E,\varepsilon}u|^2-\varepsilon h^{-1}w(|u|^2+|hu'|^2)+\frac{\min\{1,E\}}{2}w'(|hu'|^2+E|u|^2).\end{aligned}$$

We integrate from 0 to ∞, and use the facts that $w(r)=0$ near zero and $u\in H^2_{\mathrm{loc}}((0,\infty))$ is compactly supported in $[0,\infty)$:

$$\begin{aligned}&\|(w')^{1/2}(hu')\|^2_{L^2(0,\infty)}+\|(w')^{1/2}u\|^2_{L^2(0,\infty)}\\&\qquad\leq Ch^{-1}\int\langle r\rangle^{2s}w(r)|P_{m,E,\varepsilon}u(r)|^2dr+\varepsilon h^{-1}\int w(|u(r)|^2+|hu'(r)|^2)dr.\end{aligned} \tag{4.13}$$

Next, observe that

$$\operatorname{Re}\int wP_{m,E,\varepsilon}u\bar{u}dr=\int w|hu'|^2dr+\operatorname{Re}\int hw'hu'\bar{u}dr+\int w(V-E+mr^{-2})|u|^2dr.$$

So, using (4.12)

$$\begin{aligned}\|(w)^{1/2}hu'\|_{L^2} &\le \frac{\gamma h}{2}\|(w')^{1/2}hu'\|_{L^2}^2 + \frac{h\gamma^{-1}}{2}\|(w')^{1/2}u\|_{L^2}^2 + C\|(w)^{1/2}u\|_{L^2}^2 \\ &\quad + C\|w^{1/2}|m|^{1/2}r^{-1}u\|_{L^2}^2 + \frac{1}{2}\|w^{1/2}\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2} + \frac{1}{2}\|w^{1/2}\langle r\rangle^{-s}u\|_{L^2} \\ &\le \frac{\gamma h}{2}\|(w')^{1/2}hu'\|_{L^2}^2 + C(1+\gamma^{-1})h\|(w')^{1/2}u\|_{L^2}^2 \\ &\quad + \frac{1}{2}\|w^{1/2}\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2} + C\|(w)^{1/2}u\|_{L^2}.\end{aligned}$$

Plugging this into the right side of (4.13), taking γ small enough, subtracting the corresponding term to the left-hand side, and using,

$$\begin{gathered} c\delta(m)\langle r\rangle^{-2s}/h \le w'(r), \qquad r \ge \delta(m), \\ |w(r)|, |w'(r)| \le e^{C(1+|m|^{1/2})/h}, \qquad r \ge 0, \\ \operatorname{supp} w \subseteq (\tfrac{1}{2}\delta(m), \infty), \end{gathered}$$

and $\varepsilon \le 1$, we have

$$\begin{aligned} &\|\langle r\rangle^{-s}(hu')\|_{L^2(\delta,\infty)} + \|\langle r\rangle^{-s}u\|_{L^2(\delta,\infty)} \\ &\le e^{C(1+|m|^{1/2})/h}\|\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2(0,\infty)} + \varepsilon^{1/2}e^{C(1+|m|^{1/2})/h}\|u\|_{L^2(\frac{1}{2}\delta,\infty)}. \end{aligned} \tag{4.14}$$

Next, observe that $-1/2 \le t_0(m) \le 3/2$ implies

$$\begin{gathered} \|r^{\frac{3}{2}-t_0}v\|_{L^2(0,2\delta_1)} \le C\delta_1^{t_0-\frac{3}{2}}\|v\|_{L^2(0,2\delta_1)}, \\ \|v\|_{L^2(0,\delta)} \le C\delta^{t_0+\frac{1}{2}}\|r^{-t_0-\frac{1}{2}}v\|_{L^2(0,\delta)}, \\ \|r^{\frac{3}{2}-t_0}v\|_{L^2(\delta_1,2\delta_1)} \le C\delta_1^{t_0-\frac{3}{2}+s}\|\langle r\rangle^{-s}v\|_{L^2(\delta_1,2\delta_1)}. \end{gathered}$$

Combining this with (4.11) and (4.14), and that $\delta_1 \le C\langle m\rangle^{1/2}$, we have

$$\|\langle r\rangle^{-s}u\|_{L^2(0,\infty)} \le e^{C(1+|m|^{1/2})/h}\|\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2(0,\infty)} + \varepsilon^{1/2}e^{C(1+|m|^{1/2})/h}\|u\|_{L^2(\frac{1}{2}\delta,\infty)}.$$

Finally, observe that, since $u \in \mathcal{D}_m$, for any $\gamma > 0$,

$$\varepsilon\|u\|_{L^2}^2 = |\operatorname{Im}\langle P_{m,E,\varepsilon}u, u\rangle| \le \|\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2}\|\langle r\rangle^{-s}u\|_{L^2} \le \tfrac{1}{2\gamma}\|\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2}^2 + \tfrac{\gamma}{2}\|\langle r\rangle^{-s}u\|_{L^2}^2,$$

and hence

$$\|\langle r\rangle^{-s}u\|_{L^2(0,\infty)} \le e^{C(1+|m|^{1/2})/h}\|\langle r\rangle^s P_{m,E,\varepsilon}u\|_{L^2(0,\infty)}.$$

This completes the proof of Lemma 4.1.

□

Proof of Proposition 1.1. Let $1/2 < s < 1$, and $W_s \in C^\infty$ with $W_s \equiv 1$ near 0 and $c\langle r\rangle^s \le W_s(r) \le C\langle r\rangle^s$, and $W_s = C\langle r\rangle^s$ on $r \ge 2$.

First, recall that for any $\chi \in C_0^\infty[0,\infty)$ with $\chi \equiv 1$ near 0,

$$\|u\|_{\mathcal{D}_m} \sim \|\chi u\|_{\mathcal{D}_m} + \|(1-\chi)u\|_{H^2}.$$

Thus, since $[W_s, P_m](W_s)^{-1} : H^2 \to H^1$ is supported away from zero, for any $v \in \mathcal{D}_m$ such that $W_s v \in \mathcal{D}_m$,

$$\|W_s(P_m - E - i\varepsilon)v\|_{L^2} \le \|[W_s, P_m](W_s)^{-1}W_s v\|_{L^2} + \|(P_m - E - i\varepsilon)W_s v\|_{L^2} \le C_{\varepsilon,h}\|W_s v\|_{\mathcal{D}_m}. \tag{4.15}$$

Next, we claim that for $\varepsilon > 0$ and $f \in L^2(\mathbb{R}_+)$,

$$W_s(P_m - E - i\varepsilon)^{-1}\langle r\rangle^{-s}f \in \mathcal{D}_m.$$

Put

$$v := (P_m - E - i\varepsilon)^{-1}\langle r\rangle^{-s} f \in \mathcal{D}_m.$$

We shall show $W_s v \in \mathcal{D}_m$ by demonstrating that

$$W_s v = (P_m - E - i\varepsilon)^{-1}\langle r\rangle^{-s}(W_s\langle r\rangle^{-s} f + g) \tag{4.16}$$

for suitable $g \in L^2(\mathbb{R}_+)$.

Let $\chi \in C_0^\infty([0,\infty);[0,1])$ with $\chi \equiv 1$, and put $\chi_k = \chi(k^{-1}r)$, $k \in \mathbb{N}$. Almost everywhere on $\mathbb{R}_+$,

$$(P_m - E - i\varepsilon)\chi_k W_s v = \chi_k W_s \langle r\rangle^{-s} f + [P_m, \chi_k W_s]v.$$

So in particular $P_m \chi_k W_s v \in L^2(\mathbb{R}_+)$, hence $\chi_k W_s v \in \mathcal{D}(P_m^*)$. This implies that, for each k,

$$\chi_k W_s v = (P_m - E - i\varepsilon)^{-1}(\chi_k W_s \langle r\rangle^{-s} f + [P_m, \chi_k W_s]v). \tag{4.17}$$

We complete the proof of (4.16) by sending $k \to \infty$ in (4.17) and noting that the $[P_m, \chi_k W_s]$, $[P_m, W_s]$ are zero on a fixed neighborhood of $r = 0$ (independent of k) and that they are uniformly bounded $H^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ because $s \le 1$.

Now, let $v_k = \chi(k^{-1}r)W_s v \in \mathcal{D}_{\mathrm{comp},m}$, $k \in \mathbb{N}$, and note that $v_k \to W_s v$ in $\mathcal{D}_m$. Observe that

$$\|\langle r\rangle^{-s}((W_s)^{-1}v_k - v)\|_{L^2} \le \|(v_k - W_s v)\|_{L^2} \to 0, \tag{4.18}$$

and by (4.15),

$$\begin{aligned}\|\langle r\rangle^{s}(P - E - i\varepsilon)((W_s)^{-1}v_k - v)\|_{L^2} &\le C\|W_s(P_m - E - i\varepsilon)((W_s)^{-1}v_k - v)\|_{L^2}\\ &\le C_{\varepsilon,h}\|v_k - W_s v\|_{\mathcal{D}_m} \to 0.\end{aligned} \tag{4.19}$$

Finally, applying Lemma 4.1,

$$\|\langle r\rangle^{-s}(W_s)^{-1}v_k\|_{L^2} \le Ce^{C(1+|m|^{1/2})/h}\|\langle r\rangle^{s}(P_m - E - i\varepsilon)(W_s)^{-1}v_k\|_{L^2}.$$

Sending $k \to \infty$ and using (4.18), (4.19), and the definition of v, this implies

$$\|\langle r\rangle^{-s}(P_m - E - i\varepsilon)^{-1}\langle r\rangle^{-s} f\|_{L^2} \le Ce^{C(1+|m|^{1/2})/h}\|f\|_{L^2},$$

completing the proof of the proposition. □

5. Exponential estimates for the resolvent

In this section we prove Theorem 1. As before, we use separation of variables to reduce estimating the resolvent of P to estimating the resolvent of each P_m. By Proposition 1.1, for every $m \ge -h^2/4$ and $0 \le \varepsilon \le 1$

$$\|\langle r\rangle^{-s}(P_m - E - i\varepsilon)^{-1}\langle r\rangle^{-s}\|_{L^2(\mathbb{R}_+)\to L^2(\mathbb{R}_+)} \le e^{C(|m|^{1/2}+1)/h}.$$

Thus it is enough to prove the following lemma.

Lemma 5.1. *There are C and h_0 such that*

$$|K(r,r')| = |K(r,r',m,E,0,h)| \le \frac{C}{h^2}$$

holds whenever $0 < r \le r'$, $m \ge M_+$, and $0 < h < h_0$.

Proof. We use the fact that $u_0(0) = 0$ and $u_0'(r) \ge 0$ when $r \le R_0$, and consider separately three cases.

(1) Suppose that $R_0 \le r \le r'$. Then the result follows from the stronger estimate proved in Lemma 3.2.

(2) Suppose that $r \le R_0 \le r'$. Then

$$|K(r,r')| = u_0(r)\left|\frac{u_1(r')}{h^2W}\right| \le u(R_0)\frac{|u_1(r')|}{h^2|W|} = |K(R_0,r')| \lesssim h^{-1},$$

where we used $u_0' \ge 0$ on $(0,R_0)$ and then (3.3).

(3) Suppose that $r \le r' \le R_0$. Dividing the definition of the Wronskian, $u_0u_1' - u_0'u_1 = W$, by u_0^2 and integrating both sides gives

$$u_1(r) = u_0(r)\Big(\frac{u_1(R_0)}{u_0(R_0)} + \int_{R_0}^r \frac{W}{u_0(s)^2}ds\Big),$$

thus, using again $u_0' \ge 0$, on $(0,R_0)$

$$|u_0(r)u_1(r')| \le u_0(R_0)|u_1(R_0)| + (R_0 - r')|W|.$$

Plugging into (2.9), using (3.3) on the first term, and estimating the second term directly gives the result.

□

Appendix A. Bessel functions

In this paper we use the Bessel functions $J_\nu(z)$ and $Y_\nu(z)$ [DLMF, Section 10.2], where $\nu \ge 0$ and $z > 0$, and below we review some standard facts about them. By [DLMF, Equation 10.13.1], the differential equation

$$\partial_r^2 w + \left(\lambda^2 - r^{-2}(\nu^2 - \tfrac{1}{4})\right) w = 0, \tag{A.1}$$

is solved by $w_1 = r^{1/2}J_\nu(\lambda r)$ and $w_2 = r^{1/2}Y_\nu(\lambda r)$, for any $\lambda > 0$. By [DLMF, Equation 10.5.2], we have the Wronskian formula

$$w_1\partial_r w_2 - w_2\partial_r w_1 = 2\pi^{-1}. \tag{A.2}$$

We use upper and lower bounds for J and Y derived from Olver's uniform asymptotics for large values of ν. To state them, we use the notation $a \asymp b$ to mean $a \lesssim b$ and $b \lesssim a$. We define a decreasing bijection $(0,\infty) \ni z \mapsto \zeta(z) \in \mathbb{R}$ by

$$\zeta = \begin{cases} \frac{3}{2}\left(\int_z^1 t^{-1}(1-t^2)^{1/2}dt\right)^{2/3}, & z \le 1, \\ -\frac{3}{2}\left(\int_1^z t^{-1}(t^2-1)^{1/2}dt\right)^{2/3}, & z \ge 1, \end{cases} \tag{A.3}$$

and use the Airy functions

$$\begin{aligned} \mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\tfrac{t^3}{3} + xt\right)dt, \\ \mathrm{Bi}(x) &= \frac{1}{\pi}\int_0^\infty \left(e^{-\frac{t^3}{3}+xt} + \sin\left(\tfrac{t^3}{3} + xt\right)\right)dt. \end{aligned} \tag{A.4}$$

Then, by [DLMF, Section 10.20] and [AbSt, Sections 9.3.35–46], we have

$$\begin{aligned} J_\nu(\nu z) &\asymp \left(\frac{\zeta}{1-z^2}\right)^{1/4}\left(\nu^{-1/3}\mathrm{Ai}(\nu^{2/3}\zeta) + \nu^{-5/3}B_0(\zeta)\mathrm{Ai}'(\nu^{2/3}\zeta)\right), \\ -zJ_\nu'(\nu z) &\asymp \left(\frac{1-z^2}{\zeta}\right)^{1/4}\left(\nu^{-2/3}\mathrm{Ai}'(\nu^{2/3}\zeta) + \nu^{-4/3}C_0(\zeta)\mathrm{Ai}(\nu^{2/3}\zeta)\right), \\ -Y_\nu(\nu z) &\asymp \left(\frac{\zeta}{1-z^2}\right)^{1/4}\left(\nu^{-1/3}\mathrm{Bi}(\nu^{2/3}\zeta) + \nu^{-5/3}B_0(\zeta)\mathrm{Bi}'(\nu^{2/3}\zeta)\right), \\ zY_\nu'(\nu z) &\asymp \left(\frac{1-z^2}{\zeta}\right)^{1/4}\left(\nu^{-2/3}\mathrm{Bi}'(\nu^{2/3}\zeta) + \nu^{-4/3}C_0(\zeta)\mathrm{Bi}(\nu^{2/3}\zeta)\right), \end{aligned} \tag{A.5}$$

uniformly for $\nu \gg 1$ and $z > 0$, where B_0 and C_0 are positive smooth functions of ζ obeying

$$B_0(\zeta) \asymp \begin{cases} \zeta^{-1/2}, & \zeta \ge 1, \\ \zeta^{-2}, & \zeta \le -1, \end{cases} \qquad C_0(\zeta) \asymp \begin{cases} \zeta^{1/2} & \zeta \ge 1, \\ -\zeta^{-1}, & \zeta \le -1. \end{cases}$$

These bounds become simpler when z is small. More specifically, by [DLMF, Section 9.7], we have

$$\begin{aligned} \mathrm{Ai}(x) &\asymp x^{-1/4}\exp(-\tfrac{2}{3}x^{3/2}), & -\mathrm{Ai}'(x) &\asymp x^{1/4}\exp(-\tfrac{2}{3}x^{3/2}), \\ \mathrm{Bi}(x) &\asymp x^{-1/4}\exp(\tfrac{2}{3}x^{3/2}), & \mathrm{Bi}'(x) &\asymp x^{1/4}\exp(\tfrac{2}{3}x^{3/2}), \end{aligned} \tag{A.6}$$

for $x \gg 1$. Hence, for any $z_0 \in (0,1)$, we have

$$\begin{aligned} J_\nu(\nu z) &\asymp \nu^{-1/2}e^{-\nu\xi}, & J_\nu'(\nu z) &\asymp z^{-1}J_\nu(\nu z), \\ -Y_\nu(\nu z) &\asymp \nu^{-1/2}e^{\nu\xi}, & Y_\nu'(\nu z) &\asymp -z^{-1}Y_\nu(\nu z), \end{aligned} \tag{A.7}$$

uniformly for $z \in (0, z_0]$ and $\nu \gg 1$, where

$$\xi = \int_z^1 t^{-1}(1-t^2)^{1/2}dt. \tag{A.8}$$

The bounds also become simpler when z is large. More specifically, by by [DLMF, Section 9.7], we have

$$\begin{aligned} \mathrm{Ai}(-x) &\asymp x^{-1/4}\left(\cos(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4}) + \tfrac{5}{48}x^{-3/2}\sin(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4})\right), \\ -\mathrm{Ai}'(-x) &\asymp x^{1/4}\left(\sin(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4}) - \tfrac{7}{48}x^{-3/2}\cos(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4})\right), \\ \mathrm{Bi}(-x) &\asymp x^{-1/4}\left(-\sin(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4}) + \tfrac{5}{48}x^{-3/2}\cos(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4})\right), \\ -\mathrm{Bi}'(-x) &\asymp x^{1/4}\left(\cos(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4}) + \tfrac{7}{48}x^{-3/2}\sin(\tfrac{2}{3}x^{3/2}-\tfrac{\pi}{4})\right), \end{aligned} \tag{A.9}$$

for $x \gg 1$. In particular, combining (A.6) and (A.9) gives

$$\begin{aligned} \exp(\tfrac{2}{3}x_+^{3/2})|\mathrm{Ai}(x)| + \exp(-\tfrac{2}{3}x_+^{3/2})|\mathrm{Bi}(x)| &\lesssim \langle x\rangle^{-1/4}, \\ \exp(\tfrac{2}{3}x_+^{3/2})|\mathrm{Ai}'(x)| + \exp(-\tfrac{2}{3}x_+^{3/2})|\mathrm{Bi}'(x)| &\lesssim \langle x\rangle^{1/4}, \end{aligned}$$

uniformly for $x \in \mathbb{R}$, where $x_+ = \max(x,0)$. Hence

$$\begin{aligned} |J_\nu(\nu z)| &\lesssim \left|\frac{\zeta}{1-z^2}\right|^{1/4} e^{-\nu\xi_+}\left(\nu^{-1/2}\langle\zeta\rangle^{-1/4} + \nu^{-3/2}\langle\zeta\rangle^{-1/4}\right) \lesssim \nu^{-1/2}\langle z\rangle^{-1/2}e^{-\nu\xi_+}, \\ |Y_\nu(\nu z)| &\lesssim \left|\frac{\zeta}{1-z^2}\right|^{1/4} e^{\nu\xi_+}\left(\nu^{-1/2}\langle\zeta\rangle^{-1/4} + \nu^{-3/2}\langle\zeta\rangle^{-1/4}\right) \lesssim \nu^{-1/2}\langle z\rangle^{-1/2}e^{\nu\xi_+}, \end{aligned} \tag{A.10}$$

uniformly for $\nu \gg 1$ and $z > 0$, where $\xi_+ = \xi$ as given by (A.8) when $z < 1$, and $\xi_+ = 0$ when $z \ge 1$.

Appendix B. Properties of u_0

In this Appendix, we prove Lemma 2.1, following [Ya10, Chapter 4, Section 1.1]. Recall our notation from Section 2,

$$\varphi_J(r) = r^{1/2}J_\nu(\lambda r), \qquad \varphi_Y(r) = r^{1/2}Y_\nu(\lambda r),$$
$$\nu = h^{-1}\big(m + \frac{h^2}{4}\big)^{1/2}, \quad m \ge -\frac{h^2}{4}, \quad \lambda = \frac{\sqrt{E+i\varepsilon}}{h}, \quad E \in [E_{\min}, E_{\max}], \quad \varepsilon \ge 0.$$

Then put $\varphi_0 = \varphi_J$ and define inductively

$$\varphi_{n+1}(r) = (2\pi h^2)^{-1}\int_0^r (\varphi_Y(r)\varphi_J(r') - \varphi_J(r)\varphi_Y(r'))V(r')\varphi_n(r')dr'. \tag{B.1}$$

We shall prove suitable estimates on the φ_n to be able to put

$$u_0(r) = \sum_{n=0}^{\infty} \varphi_n(r), \tag{B.2}$$

with the series and its first derivative converging uniformly as (r, E, ε) vary in compact subsets of $\mathbb{R}_+ \times [E_{\min}, E_{\max}] \times [0, \infty)$.

Proof of Lemma 2.1. To prove the lemma, we recall several estimates on φ_Y and φ_J. Given $r^* > 0$, by [DLMF, Sections, 10.6 (ii), 10.7(i)] there is C_0 depending continuously on $r^* > 0$, $\nu \geq 0$, $\delta > 0$, and $\lambda \neq 0$ such that

$$\begin{aligned} &|\varphi_J(r)| r^{-\nu-\frac{1}{2}} + |\varphi_Y(r)| r^{\nu+\delta-\frac{1}{2}} \leq C_0 \\ &|\varphi_J'(r)| r^{-\nu+\frac{1}{2}} + |\varphi_Y'(r)| r^{\nu+\delta+\frac{1}{2}} \leq C_0, \qquad r \in (0, r^*]. \\ &|\varphi_J''(r)| r^{-\nu+\frac{3}{2}} + |\varphi_Y''(r)| r^{\nu+\delta+\frac{3}{2}} \leq C_0 \end{aligned} \tag{B.3}$$

Next,

$$|\varphi_0(r)| \leq C_0 r^{\nu+\frac{1}{2}}, \qquad |\varphi_0'(r)| \leq C_0 r^{\nu-\frac{1}{2}}, \qquad |\varphi_0''(r)| \leq C_0 r^{\nu-\frac{3}{2}}, \qquad r \in (0, r^*]. \tag{B.4}$$

To see that the series (B.2) converges uniformly on compact sets, we claim that

$$|\varphi_n(r)| \leq C_0 \cdots C_n r^{\nu+n(2-\delta)+\frac{1}{2}}, \qquad r \in (0, r^*]. \tag{B.5}$$

where

$$C_n = C_0^2 \sup |V| \frac{2\nu + (2n-1)(2-\delta) + 2}{2\pi h^2 (2-\delta) n (2\nu + (n-1)(2-\delta) + 2)}. \tag{B.6}$$

Once we have (B.5), since $\lim C_n = 0$, the Weierstrass and ratio tests shows that the convergence of (B.2) is uniform as (r, E, ε) vary in compact subsets of $\mathbb{R}_+ \times [E_{\min}, E_{\max}] \times [0, \infty)$. Moreover, since $\varphi_0 \sim r^{\nu+\frac{1}{2}}$ as $r \to 0$, $u_0 > 0$ near zero.

To obtain (B.5), we start with $n = 1$,

$$\begin{aligned} &\frac{2\pi h^2}{C_0^3 \sup |V|} |\varphi_1(r)| \\ &\quad \leq r^{-\nu-\delta+\frac{1}{2}} \int_0^r (r')^{\nu+\frac{1}{2}} (r')^{\nu+\frac{1}{2}} dr' + r^{\nu+\frac{1}{2}} \int_0^r (r')^{-\nu-\delta+\frac{1}{2}} (r')^{\nu+\frac{1}{2}} dr' \\ &\quad = (2\nu+2)^{-1} r^{\nu+\frac{5}{2}-\delta} + (2-\delta)^{-1} r^{\nu+\frac{5}{2}-\delta} = \frac{2\nu+4-\delta}{(2\nu+2)(2-\delta)} r^{\nu+\frac{5}{2}-\delta}, \qquad r \in (0, r^*], \end{aligned}$$

or

$$|\varphi_1(r)| \leq C_0 C_1 r^{\nu+\frac{5}{2}-\delta}, \qquad r \in (0, r^*],$$

where $C_1 = C_0^2 \sup |V| (2\nu + 4 - \delta)/(2\pi h^2 (2\nu+2)(2-\delta))$.

Now, suppose the claim (B.5) holds with $n = k$ for some $k \geq 1$, with C_k given by (B.6). Then,

$$\begin{aligned} &\frac{2\pi h^2}{C_0 C_1 C_2 \ldots C_k C_0^2 \sup |V|} |\varphi_{k+1}(r)| \\ &\leq r^{-\nu+\frac{1}{2}-\delta} \int_0^r (r')^{\nu+\frac{1}{2}} (r')^{\nu+k(2-\delta)+\frac{1}{2}} dr' + r^{\nu+\frac{1}{2}} \int_0^r (r')^{-\nu+\frac{1}{2}-\delta} (r')^{\nu+k(2-\delta)+\frac{1}{2}} dr' \\ &= (2\nu + k(2-\delta) + 2)^{-1} r^{\nu+2(k+1)+\frac{1}{2}} + ((k+1)(2-\delta))^{-1} r^{\nu+(k+1)(2-\delta)+\frac{1}{2}}, \qquad r \in (0, r^*], \end{aligned}$$

or

$$|\varphi_{k+1}(r)| \leq C_0 C_1 \ldots C_{k+1} r^{\nu+(k+1)(2-\delta)+\frac{1}{2}}, \qquad r \in (0, r^*],$$

where

$$C_{k+1} = C_0^2 \sup |V| \frac{2\nu + (2k+1)(2-\delta) + 2}{2\pi h^2 (2-\delta)(k+1)(2\nu + k(2-\delta) + 2)}.$$

In particular, (B.5) holds by induction.

To see that the first derivative of (B.2) converges uniformly on compact sets, observe that for $n \geq 0$,

$$\begin{aligned}|\varphi'_{n+1}(r)| &= (2\pi h^2)^{-1}\Big|\int_0^r (\varphi'_Y(r)\varphi_J(r') - \varphi'_J(r)\varphi_Y(r'))V(r')\varphi_n(r')dr'\Big| \\ &\leq (2\pi h^2)^{-1}C_0 \dots C_n C_0^2\Big(r^{-\nu-\frac{1}{2}-\delta}\int_0^r (r')^{2\nu+n(2-\delta)+1})dr' + r^{\nu-\frac{1}{2}}\int_0^r (r')^{n(2-\delta)+1-\delta}dr'\Big) \\ &\leq (2\pi h^2)^{-1}C_0 \dots C_n C_0^2 r^{\nu+n(2-\delta)+\frac{3}{2}-\delta}\Big(\frac{1}{2\nu+n(2-\delta)+2} + \frac{1}{(n+1)(2-\delta)}\Big) \\ &\leq C_0 \cdots C_{n+1} r^{\nu+(n+1)(2-\delta)-\frac{1}{2}}, \qquad r \in (0, r^*].\end{aligned} \tag{B.7}$$

Now (2.3) and (2.4) follow from (B.5) and (B.7) by summing (B.1) from $n = 0$ to $n = N$ and sending $N \to \infty$.

Next, to see that (2.5) holds, observe that for any $\delta > 0$,

$$\sum_{n=1}^{\infty} \varphi_n(r) \leq C_{\nu,\delta} r^{\nu+\frac{5}{2}-\delta}, \qquad \sum_{n=1}^{\infty} \varphi'_n(r) \leq C_{\nu,\delta} r^{\frac{3}{2}-\delta}.$$

Therefore,

$$\begin{aligned}\lim_{r\to 0^+}\big(\tfrac{n-1}{2r}u_0(r) - u'_0(r)\big)r^{-\nu+1/2} &= \lim_{r\to 0^+}\big(\tfrac{n-1}{2r}\varphi_0(r) - \varphi'_0(r)\big)r^{-\nu+1/2} \\ &= \lim_{r\to 0^+}\big(\tfrac{n-2}{2}J_\nu(\lambda r)r^{-1/2} - \lambda J'_\nu(\lambda r)r^{1/2}\big)r^{-\nu+1/2} \\ &= \frac{\lambda^\nu}{2^\nu\Gamma(\nu+1)}\big(\frac{n-2}{2} - \nu\big).\end{aligned}$$

Since $\nu \geq 0$, (2.5) holds.

□

Now that we have Lemma 2.1, we show that u_0 has useful monotonicity properties when m is large enough. Recall the defining property (1.2) of M_0 and consider $m \geq M_0$.

Lemma B.1. *Let M_0 as in (1.2) and R_1 as in (1.3). Then for $m \geq M_0$, $u_0(r), u'_0(r) \geq 0$ for $r \in (0, R_0]$.*

Proof. First, by Lemma 2.1, we have

$$u''_0 = -h^{-2}(E - V - mr^{-2})u_0.$$

Using $m \geq M_0$ and the defintion of R_0, we have definition, $E - V - mr^{-2} \leq 0$ on $(0, R_0]$. Hence $u_0(r)u''_0(r) \geq 0$ on $(0, R_0]$. Since also $u_0(0) = 0$ and $u_0(r) > 0$ for $r > 0$ small enough, the proof is completed by the following lemma.

□

Lemma B.2. *Suppose $f, f' \in AC_{\text{loc}}(a, b)$, $f \in C([a, b]; \mathbb{R})$, $f(a) = 0$, $f' \in C(a, b]$, and $f''(t)f(t) \geq 0$, for almost every in $t \in (a, b)$ in the sense of Lebesgue measure. Then f has a fixed sign in $[a, b]$ and $f(b)$, $f'(b)$ have the same sign.*

Proof. If f is identically 0 on $[a, b]$, then the claim is trivially true. Therefore, we will assume f is not identically 0. Assume f attains its extremum $L \neq 0$ at an interior point of (a, b). Replacing f by $-f$ if necessary, we may assume $L > 0$. Set

$$x^* = \inf\{x \in (a, b) : f(x) = L\}.$$

Then $f > 0$ near x^*, hence $f'' \geq 0$ almost everywhere, in a neighborhood of x^*. Also $f'(x^*) = 0$. Now, for x sufficiently close to but less than x^*,

$$-f'(x) = \int_x^{x^*} f''(s)ds \geq 0 \implies f'(x) \leq 0.$$

But then, by the mean value theorem, for $a < x < x^*$ and some $\overline{x} \in (x, x^*)$,

$$0 < f(x^*) - f(x) = f'(\overline{x})(x^* - x) \leq 0,$$

a contradiction.

So the extrema of f must occur at the endpoints: $f(b)$ is the maximum or minimum of f on $[a, b]$, and $f(a) = 0$ is the minimum or maximum, respectively. So f has a fixed sign, hence also f'' has this sign almost everywhere.

Using the mean value theorem again,

$$\frac{f(b) - f(a)}{b - a} = \frac{f(b)}{b - a} = f'(\overline{x}), \qquad \text{some } \overline{x} \in (a, b).$$

So $f(b)$ and $f'(\overline{x})$ have the same sign, so

$$f'(b) = f'(\overline{x}) + \int_{\overline{x}}^b f''(s)ds$$

has the same sign as $f(b)$ as well.

□

Appendix C. Consequences of Meshkov's example for resolvent estimates

Recall from Fredholm theory [DyZw19, Theorems 3.8 and equation 6.0.2] that, for $V \in L^\infty_{\text{comp}}(\mathbb{R}^n; \mathbb{C})$, the cutoff resolvent $\chi(-h^2\Delta + V - z)^{-1}\chi$, $\chi \in C_0^\infty(\mathbb{R}^n)$, is meromorphic in $(E_{\min}/2, 2E_{\max}) + i(0, \infty)$ and continues meromorphically to $(E_{\min}/2, 2E_{\max}) + i\mathbb{R}$.

To see that the optimal power in a resolvent estimate for L^∞, compactly supported, complex valued potentials is $h^{-4/3}$, recall that [Me92] constructs a *complex-valued* potential $V \in L^\infty(\mathbb{R}^2)$ and a function u such that

$$(-\Delta + V)u = 0, \quad \text{in } \mathbb{R}^2, \qquad |u(x)| \leq C\exp(-C|x|^{4/3}), \qquad \|u\|_{L^2(\mathbb{R}^n)} = 1.$$

Changing variables, $y = hx$, and putting $w(y) = u(h^{-1}y)$, $V_h(y) = V(h^{-1}y)$, we have
$\|V_h\|_{L^\infty} \leq C < \infty$,

$$(-h^2\Delta + V_h)w = 0, \qquad |w(y)| \leq C\exp(-C|y|h^{-4/3}).$$

Let $\chi \in C_0^\infty(B(0, 3))$ with $\chi \equiv 1$ near $B(0, 2)$ and $\tilde{\chi} \in C_0^\infty(B(0, 3))$ with $\tilde{\chi} \equiv 1$ on $\operatorname{supp}\chi$. Then, for $E \in [E_{\min}, E_{\max}]$,

$$\begin{aligned}
\|(-h^2\Delta + \tilde{\chi}(V_h + E) - E)(\chi w)\|_{L^2(\mathbb{R}^n)} &= \|[-h^2\Delta, \chi]w\|_{L^2(\mathbb{R}^n)} \\
&\leq Ch\|w\|_{H_h^1(B(0,3)\setminus B(0,2))} \\
&\leq Ch(\| - h^2\Delta w\|_{L^2(B(0,4)\setminus B(0,1))} + \|w\|_{L^2(B(0,4)\setminus B(0,1))}) \\
&\leq Ch(\|V_h w\|_{L^2(B(0,4)\setminus B(0,1))} + \|w\|_{L^2(B(0,4)\setminus B(0,1))}) \leq Che^{-Ch^{-4/3}},
\end{aligned}$$

For h small enough,

$$\|\chi w\|_{L^2} \geq h - h\|u\|_{L^2(\mathbb{R}^2\setminus B(0,h^{-1})} \geq \tfrac{h}{2}.$$

Therefore, if the cutoff resolvent $\chi(-h^2\Delta + \tilde{\chi}(V_h + E) - E - i0)^{-1}\chi$ exists, it must have

$$\|\tilde{\chi}(-h^2\Delta + \tilde{\chi}(V_h - E) - E - i0)^{-1}\tilde{\chi}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \geq C\exp(Ch^{-4/3}).$$

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