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An Elementary Proof of Takagi's Theorem on the Differential Composition of Polynomials

Alan D. Sokal

Abstract. I give a short and completely elementary proof of Takagi's 1921 theorem on the zeros of a composite polynomial $f(d/dz)\,g(z)$.

Many theorems in the analytic theory of polynomials [2, 8, 10, 11] are concerned with locating the zeros of composite polynomials. More specifically, let f and g be polynomials (with complex coefficients) and let h be a polynomial formed in some way from f and g; under the assumption that the zeros of f (respectively, g) lie in a subset S (respectively, T) of the complex plane, we wish to deduce that the zeros of h lie in some subset U. The theorems are distinguished by the nature of the operation defining h, and the nature of the subsets S, T, U under consideration.

Here we shall be concerned with *differential composition*: $h(z) = f(d/dz)\,g(z)$, or $h = f(D)\,g$ for short. In detail, if $f(z) = \sum_{i=1}^{m} a_i z^i$ and $g(z) = \sum_{j=1}^{n} b_j z^j$, then $h(z) = \sum_{i=1}^{m} a_i\, g^{(i)}(z)$; and D denotes the differentiation operator, i.e., $Dg = g'$. The following important result was found by Takagi [13] in 1921, subsuming many earlier results:[1]

Theorem 1 (Takagi). *Let f and g be polynomials with complex coefficients, with $\deg f = m$ and $\deg g = n$. Let f have an r-fold zero at the origin ($0 \le r \le m$), and let the remaining zeros (with multiplicity) be $\alpha_1, \ldots, \alpha_{m-r} \neq 0$. Let K be the convex hull of the zeros of g. Then either $f(D)\,g$ is identically zero, or its zeros lie in the set $K + \sum_{i=1}^{m-r} [0, n-r]\alpha_i^{-1}$.*

Here we have used the notations $A + B = \{a + b\colon\ a \in A \text{ and } b \in B\}$ and $AB = \{ab\colon\ a \in A \text{ and } b \in B\}$.

Takagi's proof was based on Grace's apolarity theorem [3], a fundamental but somewhat enigmatic result in the analytic theory of polynomials.[2] This proof is also given in the books of Marden [8, Section 18], Obrechkoff [10, pp. 135–136], and Rahman

[1] See Honda [4], Iyanaga [5, 6], Kaplan [7], and Miyake [9] for biographies of Teiji Takagi (高木貞治, *Takagi Teiji*, 1875–1960). Takagi's papers published in languages other than Japanese (namely, English, German, and French) have been collected in [14].

[2] For discussion of Grace's apolarity theorem and its equivalents—notably Walsh's coincidence theorem and the Schur–Szegő composition theorem—see Marden [8, Chapter IV], Obrechkoff [10, Chapter VII], and especially Rahman and Schmeisser [11, Chapter 3].

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and Schmeisser [11, Sections 5.3 and 5.4]. Here I give a short and completely elementary proof of Takagi's theorem.

The key step—as Takagi [13] observed—is to understand the case of a degree-1 polynomial $f(z) = z - \alpha$:

Proposition 2 (Takagi). *Let g be a polynomial of degree n, and let K be the convex hull of the zeros of g. Let $\alpha \in \mathbb{C}$, and define $h = g' - \alpha g$. Then either h is identically zero, or all the zeros of h are contained in K if $\alpha = 0$, and in $K + [0, n]\alpha^{-1}$ if $\alpha \neq 0$.*

The case $\alpha = 0$ is the celebrated theorem of Gauss and Lucas [8, Section 6], [10, Chapter V], and [11, Section 2.1], which is the starting point of the modern analytic theory of polynomials. My proof for general α will be modeled on Cesàro's [1] 1885 proof of the Gauss–Lucas theorem [11, pp. 72–73], with a slight twist to handle the case $\alpha \neq 0$.

Proof of Proposition 2. Clearly, h is identically zero if and only if either (a) $g \equiv 0$ or (b) g is a nonzero constant and $\alpha = 0$. Moreover, if g is a nonzero constant and $\alpha \neq 0$, then the zero set of h is empty. So we can assume that $n \geq 1$.

Let $\beta_1, \ldots, \beta_n$ be the zeros of g (with multiplicity), so that $g(z) = b_n \prod_{i=1}^{n} (z - \beta_i)$ with $b_n \neq 0$. If $z \notin K$, then $g(z) \neq 0$, and we can consider

$$\frac{h(z)}{g(z)} = \frac{g'(z) - \alpha g(z)}{g(z)} = \sum_{i=1}^{n} \frac{1}{z - \beta_i} - \alpha.$$

If this equals zero, then by taking complex conjugates we obtain

$$0 = \sum_{i=1}^{n} \frac{1}{\bar{z} - \bar{\beta}_i} - \bar{\alpha} = \sum_{i=1}^{n} \frac{z - \beta_i}{|z - \beta_i|^2} - \bar{\alpha},$$

which can be rewritten as $z = \sum_{i=1}^{n} \lambda_i \beta_i + \kappa \bar{\alpha}$ where

$$\lambda_i = \frac{|z - \beta_i|^{-2}}{\sum_{j=1}^{n} |z - \beta_j|^{-2}}, \qquad \kappa = \frac{1}{\sum_{j=1}^{n} |z - \beta_j|^{-2}}.$$

Then $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, so $\sum_{i=1}^{n} \lambda_i \beta_i \in K$; and of course $\kappa > 0$. Moreover, by the Schwarz inequality we have

$$|\alpha|^2 = \left| \sum_{i=1}^{n} \frac{1}{z - \beta_i} \right|^2 \leq n \sum_{i=1}^{n} |z - \beta_i|^{-2} = \frac{n}{\kappa},$$

so $\kappa \leq n|\alpha|^{-2}$. This implies that $\kappa \bar{\alpha} \in [0, n]\alpha^{-1}$ and hence that $z \in K + [0, n]\alpha^{-1}$. ■

We can now handle polynomials f of arbitrary degree by iterating Proposition 2:

Proof of Theorem 1. From $f(z) = a_m \left(\prod_{i=1}^{m-r} (z - \alpha_i) \right) z^r$ it is easy to see that $f(D) = a_m \left(\prod_{i=1}^{m-r} (D - \alpha_i) \right) D^r$. We first apply D^r to g, yielding a polynomial of degree $n - r$ whose zeros also lie in K (by the Gauss–Lucas theorem); then we repeatedly apply (in any order) the factors $D - \alpha_i$, using Proposition 2. ■

Remark. When $\alpha = 0$, the zeros of $h = g'$ lie in K; so one might expect that when α is small, the zeros of $h = g' - \alpha g$ should lie near K. But when α is small and nonzero, the set $K + [0, n]\alpha^{-1}$ arising in Proposition 2 is in fact very *large*. What is going on here?

Here is the answer: Suppose that $\deg g = n$. When $\alpha = 0$, the polynomial $h = g'$ has degree $n - 1$; but when $\alpha \neq 0$, the polynomial $h = g' - \alpha g$ has degree n. So, in order to make a proper comparison of their zeros, we should consider the polynomial g' corresponding to the case $\alpha = 0$ as also having a zero "at infinity." This zero then moves to a value of order α^{-1} when α is small and nonzero.

This behavior is easily seen by considering the example of a quadratic polynomial $g(z) = z^2 - \beta^2$. Then the zeros of $g' - \alpha g$ are

$$\begin{aligned} z &= \frac{1 \pm \sqrt{1 + \alpha^2\beta^2}}{\alpha} \\ &= -\frac{\beta^2}{2}\alpha + O(\alpha^3), \quad 2\alpha^{-1} + O(\alpha). \end{aligned}$$

So there really is a zero of order α^{-1}, as Takagi's theorem recognizes.

In the context of Proposition 2, one expects that $g' - \alpha g$ has one zero of order α^{-1} and $n - 1$ zeros near K (within a distance of order α). More generally, in the context of Theorem 1, one would expect that h has $m - r$ zeros of order α^{-1}, with the remaining zeros near K. It is a very interesting problem — and one that is open, as far as I know — to find strengthenings of Takagi's theorem that exhibit these properties. There is an old result that goes in this direction [**8**, Corollary 18.1], [**11**, Corollary 5.4.1(ii)], but it is based on a disc D containing the zeros of g, which might in general be much larger than the convex hull K of the zeros.

Postscript. A few days after finding this proof of Proposition 2, I discovered that an essentially identical argument is buried in a 1961 paper of Shisha and Walsh [**12**, pp. 127–128 and 147–148] on the zeros of infrapolynomials. I was led to the Shisha–Walsh paper by a brief citation in Marden's book [**8**, pp. 87–88, Exercise 11]. So the proof given here is not new; but it deserves to be better known.

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A Generalization of Euler's Limit

Euler's limit is defined as $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e$. We establish a generalization of this limit in the following proposition.

Proposition. *Let A_n be a strictly increasing sequence of positive numbers satisfying the asymptotic formula $A_{n+1} \sim A_n$, and let $d_n = A_{n+1} - A_n$. Then*

$$\lim_{n\to\infty}\left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{d_n}} = e. \tag{1}$$

Proof. Let us consider the function $\ln x$ on the interval $[A_n, A_{n+1}]$ for all $n \in \mathbb{N}$. By the mean value theorem, we have $\ln A_{n+1} - \ln A_n = \frac{1}{c}(A_{n+1} - A_n)$ for some c with $A_n < c < A_{n+1}$. Hence (since $\frac{1}{A_{n+1}} < \frac{1}{c} < \frac{1}{A_n}$)

$$\frac{A_{n+1} - A_n}{A_{n+1}} < \ln A_{n+1} - \ln A_n < \frac{A_{n+1} - A_n}{A_n}.$$

Since $A_{n+1} \sim A_n$, we have

$$1 \leftarrow \frac{A_n}{A_{n+1}} < \frac{\ln A_{n+1} - \ln A_n}{\frac{A_{n+1}-A_n}{A_n}} < 1;$$

that is,

$$\lim_{n\to\infty} \ln\left(\frac{A_{n+1}}{A_n}\right)^{\frac{A_n}{A_{n+1}-A_n}} = 1.$$

This completes the proof. ■

It can be seen that generalization (1) gives Euler's limit when $A_n = n$.

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