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Location Problems with Continuous Demand and Unreliable Facilities: Applications of Families of Incremental Voronoi Diagrams

Igor Averbakh* Oded Berman† Jörg Kalcsics‡ Dmitry Krass§

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Abstract

We consider conditional facility location problems with unreliable facilities that can fail with known probabilities. The demand is uniformly distributed over a convex polygon in the rectilinear plane where a number of facilities are already present, and it is required to optimally locate another facility. We analyze properties of the exponential family of incremental Voronoi diagrams associated with possible realizations of the set of operational facilities, and, based on this analysis, present polynomial algorithms for three conditional location problems. The approach can be extended to various other conditional location problems with continuous demand and unreliable facilities, under different probabilistic models including ones with correlated facility failures.

Key words: Reliable facility location, continuous location, Voronoi diagram, polynomial algorithm

1 Introduction

In this paper we present a methodology for developing polynomial algorithms for a class of reliability location models with continuous customer demand and continuous decision space.

*Department of Management, University of Toronto Scarborough, 1265 Military Trail, Toronto, Ontario M1C1A4, Canada, averbakh@utsc.utoronto.ca

†Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, Ontario M5E3E6, Canada, berman@rotman.utoronto.ca

‡School of Mathematics, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH93FD, UK, Joerg.Kalcsics@ed.ac.uk

§Rotman School of Management, University of Toronto, 105 St. George Street, Toronto, Ontario M5E3E6, Canada, krass@rotman.utoronto.ca

Location of facilities is one of the most important and strategic decisions an organization must make. For service-providing facilities, locations determine travel time, ease of access and primary service areas, which, in turn, influence customer usage, satisfaction and demand. For facilities providing logistic support within a supply chain, well-planned locations enable the efficient flow of materials leading to decreased costs and to improved customer service. An ever increasing body of literature is devoted to facility location, see, e.g., the recent book of Laporte et al. [27].

In many cases, organizations do not operate a single facility, but rather a network of facilities; the overall demand is distributed among different facilities. In case of service facilities, a common assumption is that customers tend to patronize the closest facility. In multi-facility settings, failure of a single facility may have reverberating effects throughout the network, causing customers to travel longer distances to obtain service, reducing service levels and losing potential demand. "Reliability location models" (also known as "location models with unreliable facilities") seek to design the multi-facility network so that it continues to function efficiently in the event of failure of one or more facilities. These models have been receiving increasing amount of attention in recent years, see Snyder *et. al.* [39] for a recent review.

Location models, whether they consider the possibility of facility failure or not, can be classified on two key dimensions: the representation of customer demand, and of the decision space. In both cases, the two main choices are "discrete" or "continuous".

Discrete customer demand models assume customer demand originates from a known set of discrete locations ("customer nodes"), while continuous models assume the demand is continuously distributed over a planar region according to a some spatial distribution. While discrete models are analytically simpler (in fact, most location models assume discrete demand), to maintain tractability, the number of customer nodes cannot be too large, since it strongly affects the computational complexity of the model. Since the number of individual customer locations to be served is often very large, in practice "customer nodes" in such cases are usually aggregations of customer demand (e.g., to centroids of the regions partitioning the planar region, such as census tracts), which, of course, leads to aggregation errors cf. Francis and Lowe [25]. Under the continuous demand framework, adopted in the current paper, the number of customers does not affect the computational complexity.

A similar dichotomy in modeling approaches appears with respect to the decision space, i.e., the set of potential locations for new facilities. While discrete set models restrict facility locations to a pre-determined, typically finite, set of potential locations (often coinciding with the set of customer nodes), the continuous location models allow for locations anywhere in a certain planar region. Continuity of the decision space has a number of advantages: insights of what region may be most promising for new facilities can be identified, parametric analysis of changes in optimal location(s) with respect to model parameters is possible due to continuity, the assumption that potential locations have been pre-identified may not be appropriate at the initial facility planning stage. However, continuity also comes at a price: the uncountable number of potential locations leads to serious analytical challenges.

When continuous demand distribution is coupled with continuous location space, the development of polynomial-time algorithms for the resulting models is quite challenging, even in case of perfectly reliable facilities. Recent advances in [23] and [5] focus on deriving polynomial-time algorithms for finding an optimal location of one new facility in this setting. In [23], it is assumed that only the new facility is present, and in [5] the "conditional problem" is considered: adding a single new facility to a set of pre-existing facilities.

The current paper continues this line of research by developing polynomial-time algorithms for the case where facilities may fail; to the best of our knowledge this is the first attempt to consider planar facility reliability problems in the continuous location space and continuous demand setting. We present our approach in the context of three location problems. The *conditional median problem* seeks to find an optimal location for a new facility in the presence of n existing facilities so that the total expected customer travel distance is minimized. This model is relevant when it is desirable to have facilities close to customers. The *conditional anti-median problem* seeks to maximize the expected distance; this objective arises when customers consider being near a facility to be undesirable (examples of such facilities range from landfills to busy night clubs). The final problem we consider assumes that the new facility will compete with some pre-existing facilities for customer demand, with market share "captured" by each facility consisting of all customers for whom this facility is the closest operating facility. The *conditional market share problem* is to maximize the expected customer demand "captured" by locating the new facility. The framework we develop is also applicable to a variety of other conditional location problems with unreliable facilities.

The basic mathematical construct underlying our analysis is the *Voronoi diagram* - the partition of the demand space that arises when the locations of all facilities are specified, reliability issues are ignored, and each customer is assumed to travel to the closest facility; each element of this partition, corresponding to a specific facility, is called a *Voronoi cell*. The approach developed in [5] for the case of perfectly reliable facilities is based on the analysis of the structural changes of the Voronoi diagram when the location of one of the facilities (the newly added one) varies. This leads to a partitioning of the solution space into a polynomial number of subregions such that the structure of the Voronoi diagram (and therefore the parametric representation of the objective function as a function of the coordinates of the new facility) does not change when the location of the new facility varies within a subregion. The problem restricted to any such subregion is much simpler and an optimal location over a subregion can often be found using standard continuous optimization techniques.

However, application of this approach to the case of unreliable facilities immediately runs into several difficulties. First, even computing the probability that a given customer will obtain service from a specific facility can take time exponential in the number of facilities for a general joint probability distribution of facility failures. We thus limit ourselves to probability models where some basic computations can be done in linear time. Such models, described in Section 3.1 below, include the case where failure events are independent, but also encompass certain inter-dependence patterns, thus allowing for correlated failures.

Second, in case of unreliable facilities, we no longer have a single Voronoi diagram, but rather an exponential family of Voronoi diagrams, depending on which facilities are operational and which have failed. To be able to derive a decomposition of the solution space into subregions with the required structural properties, we have to extend the analysis to the structural changes of all possible Voronoi diagrams, while avoiding the enumeration of the exponentially many Voronoi diagrams that may arise. Our polynomial algorithms are based on such an analysis, and are an illustration of its power.

Third, our decomposition of the solution space should not have cardinality that is too large if we want to obtain polynomial algorithms with complexity orders that are not prohibitively large for potential practical use. Given the complexity of the problem, this is a significant challenge. For example, as will be discussed later, more direct attempts to apply the approach of [5] would result in algorithms with complexity orders much higher than those in the paper.

The key element of our methodology in this paper is the introduction and analysis of two sets of polynomial cardinality – the set of corner-points and the set of sub-segments – that capture the minimum necessary information about the exponential family of Voronoi diagrams that is required for development of solution algorithms for the considered problems. Analysis of the structural changes in these sets and some associated entities when the location of the new facility varies results in a partition of the solution space into a polynomial number of subregions such that the problem decomposes into simpler subproblems over the subregions.

While computational complexity bounds for our algorithms involve relatively high powers of the number of existing facilities n (n^6 in case of independent facility failures, or n^7 under more general assumptions about the probability model), one should keep in mind that in many practical settings n tends to be fairly small: a network of 50 facilities is already moderately large. With modern computational speeds (300,000 MIPS and 400 GFLOPS are not uncommon), the solution based on an $O(n^6)$ algorithm for $n = 50$

can be expected in just seconds or minutes, and the computational power of computers keeps growing. As discussed earlier, the driving force behind long computational times in discrete models is typically the size of the customer set and/or the size of the potential location set, neither of which is a factor in the continuous demand / continuous location space setting we analyze. We also note that facility location decisions are strategic in nature, and in practice significant time can usually be allocated to finding good locational decisions.

The rest of the paper is organized as follows. In the next section we give an overview of the related literature. In Section 3 we define our probability model, formulate the three conditional location problem, introduce basic definitions and notations and derive a number of useful properties. In Section 4 we analyze and characterize the structural changes to all possible Voronoi diagrams and their cells that result from adding one additional location. In Section 5 we show how to utilize this analysis to derive polynomial exact algorithms for our problems. Possibilities of extensions of the results to other problems and probabilistic models are discussed in the concluding Section 6.

2 Literature Review

Our paper lies at the confluence of two bodies of literature: location models with continuous demand and solution space, and location reliability models.

With regards to the former, we direct the reader to the literature review section in Averbakh et al. [5]; only the most relevant aspects are reviewed below. In a seminal paper, Erlenkotter [22] derives closed-form expressions for the optimal supply area size and the optimal average cost for the uniform demand case and an unbounded market. He considers facility as well as transportation costs, different distance norms, and allows different shapes of supply areas for the facilities. Applications and extensions of this model can be found in [14, 15, 16, 21, 26, 36, 37]. Fekete et al. [23] consider the single facility median problem with rectilinear distances and continuous demand; the market region is not necessarily convex and may even contain holes. They develop exact polynomial algorithms for straight-line rectilinear and, in case of non-convex regions, geodesic rectilinear distances. Murat et al. [31] derive optimality conditions for the two-facility problem for Euclidean and rectilinear distances. Based on a reformulation as a two-dimensional boundary value problem, they solve the problem using a two-dimensional shooting algorithm. For a convex polygonal market region with uniform demand and rectilinear distances, Averbakh et al. [5] develop polynomial exact algorithms for five different conditional location problems: the (anti-)median, market share, maximum covering, and center problem. The solution approach is based on an analysis of structural properties of incremental Voronoi diagrams and is applicable to a variety of other facility location problems with continuous demand. The current paper further develops this approach, extending it to the unreliable facilities setting.

The research on reliability location models apparently started with Drezner [18], who introduces the reliable p-median (RMP) and p-center problems on the plane with non-uniform and not necessarily independent failure probabilities. He presents a mathematical formulation for each problem and a heuristic based on Cooper's location-allocation method; a similar framework is employed by Lee [28] for the RMP .

The majority of the literature for location reliability problems focuses on the discrete demand case. This line of research originated with Snyder and Daskin [40]. They consider the RMP and the reliable uncapacitated facility location problem. For both problems they derived integer linear programming formulations for the case of uniform failure probabilities and Lagrangean relaxation based heuristics to solve larger instances. Berman et al. [9] discuss the RMP on networks with facility dependent failure probabilities, deriving a number of structural results and developing integer programming and heuristic algorithms. The formulations and solution approaches for the discrete case have been extended and improved in Cui et al. [13], O'Hanley et al. [33], Shen et al. [38], and Aboolian et al. [1].

All of the papers discussed above lead to models that can be quite difficult to solve, especially in the presence on non-uniform or correlated failure probabilities. An alternative approach is that of Continuous Approximation (CA), where the main idea is to derive an analytical model by converting discrete data into a continuous scale. The CA approach was first discussed in Newell [32], and recently extended in Cui et al. [13]. Following the works [22, 32], among others, they represent customer demand, facility setup costs, and failure probabilities through continuous functions. Assuming an infinite market and using a hexagonal facility pattern, which was shown to be optimal for the (reliable) median problem with Euclidean distances on a plane, they derive analytic expressions that estimate the optimal size of the primary service area of a facility. Based on this analysis, they propose some guidelines to obtain a set of discrete facility locations. Li and Ouang [29] consider spatial correlation of the disruptions; they express the disruptions using the CA approach. To capture the correlated probabilities they used conditional probabilities and the beta-binomial distribution. Lim et al. [30] also study the problem with correlated disruptions, focusing on finding the impact of mis-estimating the disruption probabilities.

The continuous approximation models cannot, of course, guarantee optimality; the goal is to provide high-quality approximations for the discrete case. Exact solutions for the RMP in the continuous demand/continuous solution space case on a line are developed in Berman and Krass [8], who also discuss some optimal location patterns for larger numbers of facilities. Their approach (which also works for correlated failures) is based on reformulating the RMP as a linear combination of deterministic median problems. We are not aware of any literature on the reliable market share problem.

It is relevant to note that facility failure events considered in location reliability models are assumed to be due to random exogenous factors (ranging from power outages and machine breakdowns to catastrophic events such as earthquakes and fire), rather than failures due to attacks from an intelligent adversary; the latter are addressed by facility interdiction models - please see Laporte et al. [27] for a discussion.

The approach in this paper is based on decomposition of the solution space into sub-regions that are in some sense easier to handle. Approaches of this type have been used in other sub-areas of planar facility location, e.g. in robust planar location problems [4].

3 Problem Formulation and Properties

We represent the market area by a convex compact polygon $\mathcal{P} \subset I\!R^2$ with vertices $P_1, \ldots, P_m$ numbered in clockwise order. We assume that the demand is continuously and uniformly distributed in $\mathcal{P}$. By $\mathcal{A} = \{A_1, \ldots, A_n\}$ we represent the locations of the n already existing facilities and we assume that $A_1, \ldots, A_n \in \mathcal{P}$. To avoid trivial cases, we assume $n \geq 2$ (all results can be adapted to the case of just one existing facility with minor changes.)

For any integers $i, j,\ i \leq j$, we use $[i : j]$ to denote the set $\{i, i+1, \ldots, j\}$. For two distinct points $P = (p_x, p_y), Q = (q_x, q_y) \in \mathcal{P}$, the rectilinear (Manhattan) distance is defined as $l_1(P, Q) = |p_x - q_x| + |p_y - q_y|$. We will assume the rectilinear distance throughout the paper, commenting on the extension of our results to the Euclidean distance case in Section 6.

3.1 Facility Failures: Probabilistic Model

As described in the introduction, a central feature of our model is that facilities may fail. If the facility fails (is unavailable for service), we say that it is *non-operating*; otherwise, it is said to be *operating*. We assume that each customer travels to the closest operating facility to obtain service and that the customer has a priori information about which facilities have failed; this setting is called "Complete Information" in Berman et al. [9]. When all facilities have failed, a fixed penalty of $\beta > 0$ is charged (per customer call).

We regard each facility i for $i \in [1 : n]$ as a Bernoulli(p_i) random variable, where $0 \leq p_i \leq 1$ is the probability of failure for this facility. We will use A_i^+ (A_i^-) to represent the event that facility i is operating (is non-operating). Note that $pr(A_i^+) = 1 - p_i$.

Failure events at facilities may be correlated. Consider a partition of the facility set $\mathcal{A}$ into three subsets: $\mathcal{A}^+ \cup \mathcal{A}^- \cup \mathcal{A}^U = \mathcal{A}$, where all facilities in $\mathcal{A}^+$ are operating, all facilities in $\mathcal{A}^-$ have failed, and the status of facilities in $\mathcal{A}^U$ is not specified. Since $\mathcal{A}^U$ is completely determined once $\mathcal{A}^+, \mathcal{A}^-$ are specified, we will usually omit it from all notation.

We assume that a joint probability $pr(\mathcal{A}^+, \mathcal{A}^-)$ is specified. That is, $pr(\mathcal{A}^+, \mathcal{A}^-)$ is the probability that the facilities from $\mathcal{A}^+$ are operating and facilities from $\mathcal{A}^-$ have failed, and the facilities from $\mathcal{A}^U$ can have any status. Note that for $\mathcal{A}^+ = \{i\}$, $\mathcal{A}^- = \emptyset$, where $i \in [1:n]$, we have $pr(\mathcal{A}^+, \mathcal{A}^-) = pr(A_i^+) = 1 - p_i$. If $\mathcal{A}^+$ includes *all* operating facilities, i.e., $\mathcal{A}^- = \mathcal{A} \setminus \mathcal{A}^+$ and $\mathcal{A}^U = \emptyset$, we will refer to $\mathcal{A}^+$ as an *operational set* and use the shorthand $pr(\mathcal{A}^+) = pr(\mathcal{A}^+, \mathcal{A} \setminus \mathcal{A}^+)$. In this case, the partition of $\mathcal{A}$ into the operational set $\mathcal{A}^+$ and the *non-operational set* $\mathcal{A}^- = \mathcal{A} \setminus \mathcal{A}^+$ is called a *constellation* (with a somewhat poetic parallel between operational facilities and stars).

For a general joint distribution of n Bernulli random variables the computation of probability $pr(\mathcal{A}^+, \mathcal{A}^-)$ may take time exponential in n if $\mathcal{A}^U \neq \emptyset$. Since we are interested in polynomial algorithms in this paper, for the purpose of evaluating complexity of algorithms we make the following simplifying assumption about the distribution:

Assumption A. For any specified partition $\mathcal{A}^+, \mathcal{A}^-, \mathcal{A}^U$ of set $\mathcal{A}$ the probability $pr(\mathcal{A}^+, \mathcal{A}^-)$ can be computed in $O(n)$ time.

A simple example of the probability distribution that satisfies the above assumption is the *uniform independent probability model* where the failure events are assumed to be independent and $p_i = p$ for all $i \in [1:n]$. In this case we have

$$pr(\mathcal{A}^+, \mathcal{A}^-) = (1-p)^{|\mathcal{A}^+|} p^{|\mathcal{A}^-|}, \tag{1}$$

This probability model has been used in a number of papers, including [40] and [33].

If we drop the assumption that all failure probabilities p_i are identical, we obtain the *independent probability model*, also satisfying Assumption A above:

$$pr(\mathcal{A}^+, \mathcal{A}^-) = \prod_{i \in \mathcal{A}^+} (1 - p_i) \prod_{j \in \mathcal{A}^-} p_j \tag{2}$$

A natural extension of the independent probability model is to assume that failure events are conditionally independent, where conditioning is with respect to some background discrete random variable V with a fixed number of values and a known distribution. E.g., it may be reasonable to assume that failure probabilities vary by time of day, but failures are independent within each of several time intervals which partition the day, while V represents the interval during which a customer's call occurs. It is not hard to see that this model also satisfies Assumption A.

More generally, as discussed in Qaqish [35], specifying a complete joint distribution of n Bernoulli variables for $n \geq 15$ is impractical. For this reason, structured correlation matrices are often assumed. One useful example discussed by Qaqish is the *exchangeable* *correlation case* where the correlation matrix is assumed to have all off-diagonal elements equal to some specified $\rho \in (-1, 1)$. Using the formulas provided in [35], the probability $pr(\mathcal{A}^+, \mathcal{A}^-)$ can be computed in $O(n)$ time, once again satisfying Assumption A. Other examples provided by Qaqish (and also satisfying Assumption A) include AR(1) and MA(1) patterns of correlations [35]. Thus, while Assumption A may look restrictive, it actually allows for a fair degree of flexibility in representing probabilities of facility failures.

We also note that there is no technical difficulty in extending all results in the paper to the case where Assumption A is generalized to the probability being computable in polynomial time $O(n^k)$ for some specified $k \geq 1$ (though the polynomial complexity of the algorithms presented in the following will, of course, increase accordingly).

In the next section we formally introduce three conditional facility location problems we study in this paper to illustrate our approach.

3.2 Formulations of Three Location Problems

All problems formulated below are *conditional* in the sense that the locations of the n facilities in $\mathcal{A}$ are assumed to be known and fixed and the objective is to find the best location for a single new facility. We denote by $Z = (x, y) \in \mathcal{P}$ the location of the new facility and by $\mathcal{A}_Z = \mathcal{A} \cup \{Z\}$ the augmented facility set. For the ease of notation, we will also use $A_{n+1} = Z$.

Let $Q \in \mathcal{P}$ be a customer location. Following [9], we denote by $\pi(\cdot, Q)$ the permutation of the index set $[1\!:\!n\!+\!1]$ that sorts all facilities $A_1, \ldots, A_{n+1}$ in the order of non-decreasing distances from Q, i.e., $l_1(Q, A_{\pi(k,Q)}) \leq l_1(Q, A_{\pi(k+1,Q)})$, $1 \leq k < n$. To make sure that the permutation $\pi(\cdot, Q)$ is defined uniquely, we assume that there is a certain *tie-breaking preference order* of the facilities, according to which ties are broken if there are two or more facilities at the same distance from the customer. Also, we assume that the tie-breaking preference order is the same for all customers. These assumptions are made for convenience of presentation and to avoid ambiguity; they do not influence any results, because the set of customers for whom ties may occur (i.e., for whom there are equidistant facilities) clearly has measure 0 due to the continuous distribution of customers, and therefore does not affect the objective function for any of our problems.

For $k \in [1 : n+1]$ let $\mathcal{A}[k, Q] = \{A_{\pi(1,Q)}, \ldots, A_{\pi(k,Q)}\}$ be the set of the k closest facilities to Q, and let $\mathcal{A}[k, Q]^-$ denote the event that all facilities in that set have failed. The expected travel distance for a customer at Q is then given by

$$h(Q, \mathcal{A}_Z) \;=\; \sum_{k=1}^{n+1} pr\left(A^+_{\pi(k,Q)}, \mathcal{A}[k-1, Q]^-\right) l_1(Q, A_{\pi(k,Q)}) \;+\; \beta\, pr(\mathcal{A}[n+1, Q]^-)\,, \quad (3)$$

where the first probability term represents the probability that the customer at Q will have to travel to the k-th closest facility to obtain service, and the second probability term is the probability that all facilities have failed. As the last term is a constant and does not affect the choice of the optimal location, we will generally omit it from further discussion. We note that for the case of uniform independent probability model, the expression above simplifies to:

$$h(Q, \mathcal{A}_Z) \;=\; \sum_{k=1}^{n+1} (1-p)\, p^{k-1}\, l_1(Q, A_{\pi(k,Q)}) \;+\; \beta\, p^{n+1}\,.$$

If it does not lead to confusion, we usually drop the reference to $\mathcal{A}_Z$ from the expected distance, i.e., $h(Q) = h(Q, \mathcal{A}_Z)$. Let $\pi(Q) := (\pi(1, Q), \ldots, \pi(n+1, Q))$.

The problem of finding the location of the new facility $Z \in \mathcal{P}$ that *minimizes* the total expected travel distance

$$F_M(\mathcal{A}_Z) \;=\; \iint_{Q=(u,v)\in\mathcal{P}} h(Q)\, du\, dv\,, \tag{4}$$

is called the **Conditional Median Problem with Unreliable Facilities (CMPUF)** (this follows the terminology introduced in [9]; the "conditional" part refers to the fact that the locations of facilities $[1:n]$ are assumed to be given). The median objective arises when it is desirable to minimize travel distance between customers and facilities. This occurs in a wide variety of settings, ranging from delivered service, where facility is delivering products to customers, to location of public service facilities, such as libraries or medical clinics, to retail settings where customer demand is sensitive to travel distance. The rectilinear distance we use is particularly applicable to facilities located in the city core. We assume that facilities may "fail", i.e., become inaccessible to customers due to some exogenous events (power failures, work stoppages, etc.). As discussed [9], in contrast to the classical median problem, for a solution Z^* minimizing $F_M(\mathcal{A}_Z)$ we may have $Z \in \mathcal{A}$. This situation is called *co-location* and is often observed in problems with unreliable facilities.

The median problem assumes that customers regard proximity to facilities as desirable. However, there are many facilities which serve an important societal role, but whose proximity is considered undesirable by most residential customers. While the classical examples of these so-called obnoxious facilities, such as landfills and incineration plants, are not well suited for our rectilinear norm assumption, there are many others for which this assumption is quite natural. These may include court houses, addiction treatment clinics, half-way houses, night clubs, etc. - all of these generate traffic, congesting the streets, and are perceived to negatively impact neighborhood safety. In fact, Babawale

[6] found that even churches may fall into this category (they showed that proximity to churches had negative impact on real estate prices). This leads to the problem of finding the location $Z \in \mathcal{P}$ of the new facility that *maximizes* the total expected travel distance $F_M(\mathcal{A}_Z)$; we refer to it as the **Conditional Anti-Median Problem with Unreliable Facilities (CAMPUF)**. This problem is also known as the obnoxious median problem ([11]). We require that the distance between the new facility and each existing facility is at least $D > 0$; this requirement is often used when locating obnoxious facilities (cf. Berman and Huang [7]).

In our final problem we consider the situation where some of the existing facilities belong to one or more competitors, while the new facility and perhaps some subset of the existing facilities belong to the decision-maker (DM). In case of competition, companies are often more concerned about their market share than about travel distances. Without loss of generality, let the first $l \in [1:n]$ facilities belong to the competitors and the remaining facilities belong to the DM. We assume that the market share of a given facility is the proportion of customers for whom this is the closest operating facility (in case of equidistant facilities we assume the customer prefers DM's facilities over competitive ones). The problem of finding a location of an $(n+1)$-st facility $A_{n+1} = Z \in \mathcal{P}$ belonging to the DM that maximizes the total market share of DM's facilities is specified as follows:

$$F_{MS}(\mathcal{A}_Z) \quad = \quad \iint_{Q=(u,v)\in\mathcal{P}} \sum_{k=1}^{n+1} pr\left(A^+_{\pi(k,Q)}, \mathcal{A}[k-1,Q]^-\right) I_{\{\pi(k,Q)\geq l+1\}}\, du\, dv\,, \quad (5)$$

where $I_{\{.\}}$ is the indicator function, and is called the **Conditional Reliable Market Share Problem (CRMSP)**), cf. [20]. The term $pr\left(A^+_{\pi(k,Q)}, \mathcal{A}[k-1,Q]^-\right)$ has the same interpretation as in (3) above.

We note than none of the objective functions defined above is, in general, convex or concave over $\mathcal{P}$.

3.3 Geometric Terminology and Notation

As we will be making extensive use of geometric concepts such as “bisector”, “Voronoi Diagram”, etc., let us briefly review them.

A *bisector* $\mathcal{B}(P,Q) = \{Z \in \mathcal{P} \mid l_1(P,Z) = l_1(Q,Z)\}$ is the set of all points that are at the same distance from distinct points $P = (p_x, p_y)$ and $Q = (q_x, q_y)$ from $\mathcal{P}$. Let $\mathcal{B}^{\leq}(P,Q) := \{Z \in I\!R^2 \mid l_1(P,Z) \leq l_1(Q,Z)\}$ be the set of all points that are no farther from P than from Q.

While the bisector for the Euclidean distance is always a straight line, the shape of the rectilinear bisector depends on the position of P and Q relative to each other. If P

and Q lie on the same vertical or horizontal line, then $\mathcal{B}(P,Q)$ is a horizontal or vertical line, respectively. In most other other cases, the bisector between P and Q is piecewise linear with exactly three pieces: two vertical or horizontal half lines that are connected by a *diagonal* segment (i.e., segment with slope of 1 or -1), cf. Aurenhammer et al. [3]. We denote the two breakpoints of such a bisector by B_1 and B_2. Their coordinates are linear in the coordinates of P and Q. The case where the bisector does not consist of line segments occurs if $|p_x - q_x| = |p_y - q_y|$, i.e., P and Q are on the same diagonal line. In this case $\mathcal{B}(P,Q)$ consists of two quarter planes connected by a diagonal line segment. We call such bisectors *degenerate*.

Since bisectors between the facilities play a central role in our analysis, to avoid the inconvenience of degenerate bisectors we make the following assumption:

Assumption B. No two points of the set $\mathcal{A}$ lie on the same diagonal line.

We can make this assumption without loss of generality, because the coordinates of the corresponding facilities can always be slightly perturbed.

For a fixed point P, the vertical, horizontal, and diagonal lines through P are called *configuration lines for* P; they partition $\mathbb{R}^2$ into eight open subsets. We call the closure of such a subset a *configuration cone* rooted at P. Starting from the cone defined by $|p_x - q_x| \leq |p_y - q_y|$, $p_x \leq q_x$, and $p_y \leq q_y$, we label them in counter-clockwise order as $\mathcal{CC}^1(P), \ldots, \mathcal{CC}^8(P)$ ([5]). Configuration lines for facilities A_i, $i \in [1:n]$, will be called simply *configuration lines*.

For a set $\mathcal{A} = \{A_1, \ldots, A_n\} \subset \mathcal{P}$ of $n \geq 2$ distinct points, the *Voronoi cell* $\mathcal{V}(A_i)$ of A_i is the set of all points in $\mathcal{P}$ that are at least as close to A_i as to any other point A_j, $i \neq j$,

$$\mathcal{V}(A_i) := \{P \in \mathbb{R}^2 \mid l_1(P, A_i) \leq l_1(P, A_j)\ \forall j \in [1:n]\}.$$

A_i is called the *generator* of $\mathcal{V}(A_i)$. $\mathcal{V}(A_i)$ is a polygon that is not necessarily convex, but is always star-shaped with respect to A_i, i.e., each line segment connecting a vertex of the polygon with A_i is entirely in the polygon. We call the vertices and edges of the boundary of a Voronoi cell *Voronoi vertices* and *Voronoi edges*, respectively. The *Voronoi diagram* of $\mathcal{A}$ is then the set of all Voronoi cells $\mathcal{VD}(\mathcal{A}) := \{\mathcal{V}(A_1), \ldots, \mathcal{V}(A_n)\}$. If two Voronoi cells $\mathcal{V}(A_i)$ and $\mathcal{V}(A_j)$, $i \neq j$, are not disjoint, we call the set $\mathcal{V}(A_i) \cap \mathcal{V}(A_j)$ the *Voronoi link* of A_i and A_j. Note that $\mathcal{V}(A_i) \cap \mathcal{V}(A_j) \subseteq \mathcal{B}(A_i, A_j)$; we abbreviate the bisector $\mathcal{B}(A_i, A_j)$ by $\mathcal{B}_{ij}$. The end point of a Voronoi link is called a *Voronoi node*. We refer the reader to [3, 17, 34] for more details on Voronoi diagrams and their properties.

We denote by $int(\cdot)$, $bd(\cdot)$, and $\mu(\cdot)$ the interior, boundary, and area of a planar polygonal set, respectively. For a point $Q \in \mathbb{R}^2$, we denote by $\mathcal{K}(Q,r) = \{P \in \mathbb{R}^2 \mid l_1(Q,P) = r\}$ and $\mathcal{K}^{\leq}(Q,r) = \{P \in \mathbb{R}^2 \mid l_1(Q,P) \leq r\}$ the l_1-sphere and l_1-disc, respectively, centered at Q with radius $r \in \mathbb{R}^+$. We define the total l_1-distance from Q to all points in a polygonal set $\mathcal{U} \subset I\!\!R^2$ as

$$d(Q,\mathcal{U}) \;=\; \iint_{(u,v)\in\mathcal{U}} l_1(Q,(u,v))\, dv\, du\,.$$

3.4 Computing the Objective Functions

In this subsection, we focus on computing the objective function values for the three location models defined above for a given fixed set of facilities. While this is typically a straightforward step in most location problems, this not the case here: it is not immediately clear how to compute the integrals in the median and market share objectives, especially in view of the permutation $\pi(Q)$ that varies with Q.

If all facilities are reliable, we can reformulate the median and market share objectives using the Voronoi diagram and centroid triangulations of Voronoi cells to make them computable in polynomial time ([5]). In case of unreliable facilities, however, we no longer have THE Voronoi diagram, but rather an exponential number of different Voronoi diagrams that arise depending on which facilities are operational and which have failed. In the following, we show how to compute the objectives for all three problems formulated above in polynomial time. Our approach also naturally extends to the case where the location of the new facility is variable (not fixed).

Recall that a constellation is a partition of $\mathcal{A}$ into an operational set $\mathcal{A}^+$ and non-operational set $\mathcal{A}^-$, with $\mathcal{A}^U = \emptyset$ and $pr(\mathcal{A}^+) = pr(\mathcal{A}^+, \mathcal{A}^-)$ the probability of this constellation occurring. For a given operational set $\mathcal{A}^+$, the Voronoi cell of $A_i \in \mathcal{A}^+$ in the Voronoi diagram $\mathcal{VD}(\mathcal{A}^+)$ of the operational facilities is denoted by $\mathcal{V}(A_i, \mathcal{A}^+)$. If the context is clear, we usually abbreviate $\mathcal{V}(A_i, \mathcal{A}^+)$ by $\mathcal{V}^+(A_i)$ or $\mathcal{V}_i^+$.

3.4.1 The Conditional Median and Anti-Median Problems with Unreliable Facilities

We start with the objective function $F_M(\mathcal{A})$ of CMPUF and CAMPUF. First, we convert the customer-based representation of the objective function into a facility-based representation that avoids explicit ordering of facilities for each customer.

Lemma 1 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of $n \geq 2$ facilities satisfying Assumption B. Then*

$$F_M(\mathcal{A}) \;=\; \sum_{i=1}^{n} \sum_{\mathcal{A}^+\subseteq\mathcal{A}:A_i\in\mathcal{A}^+} pr(\mathcal{A}^+)\, d(A_i, \mathcal{V}_i^+)\,. \qquad (6)$$

Proof. Straightforward, using the definitions. □ □

To compute $d(A_i, \mathcal{V}_i^+)$, we use the concept of centroid triangulations introduced in [23]. The idea of this approach is to decompose each Voronoi cell into triangles, where one of the corners of each triangle is the facility and the other two corners are consecutive vertices on the boundary of the Voronoi cell, i.e., Voronoi vertices. For each triangle Δ of this centroid triangulation, $d(A_i, \Delta)$ can easily be computed using closed form expressions (see [23], [5]). Hence, using (6), we are able to evaluate $F_M(\mathcal{A})$. However, as we have to consider all possible operational subsets $\mathcal{A}^+$ of $\mathcal{A}$, doing this directly requires an exponential number of steps. Next, we show how to perform this task in polynomial time.

Definition 1 *For $A_i \in \mathcal{A}$, $i \in [1:n]$ let the set of* corner-points $\mathcal{CP}_i(\mathcal{A})$ *of A_i with respect to $\mathcal{A}$ be the set of all*

- *extreme points (vertices) of polygon $\mathcal{P}$,*
- *breakpoints of the bisectors $\mathcal{B}(A_i, A_j)$, $j = 1, \ldots, n$, $j \neq i$,*
- *intersection points of a bisector $\mathcal{B}(A_i, A_j)$ with the boundary of $\mathcal{P}$ or with another bisector $\mathcal{B}(A_i, A_k)$, $j, k = 1, \ldots, n$, $i \neq j$, $i \neq k$, $j \neq k$.*

If the context is clear, we usually omit the reference to $\mathcal{A}$. Note that any Voronoi vertex is a corner-point. Concerning the number of corner-points of facility A_i, there are $n-1$ bisectors $\mathcal{B}_{ij}$, $i \neq j$. A bisector $\mathcal{B}_{ij}$ intersects $bd(\mathcal{P})$ at most twice and another bisector $\mathcal{B}_{ik}$, $j \neq k$, at most once (see [5]). Hence, $|\mathcal{CP}_i(\mathcal{A})| = O(n^2 + m)$ (recall that m is the number of extreme points of $\mathcal{P}$). Next, we refine the concept of centroid triangulations.

Definition 2 *Let $\mathcal{A}^+ \subseteq \mathcal{A}$ and $A_i \in \mathcal{A}^+$ for some $i \in [1:n]$. Observe that any vertex of the Voronoi cell $\mathcal{V}(A_i, \mathcal{A}^+)$ is a corner-point. An* elementary centroid triangulation *(ECT) of the Voronoi cell $\mathcal{V}(A_i, \mathcal{A}^+)$ is a triangulation of the cell such that one of the corners of each triangle is the facility A_i (the* facility corner*), and the other two corners are consecutive corner-points on the boundary of the cell (the* non-facility corners*). The resulting triangles are called* elementary triangles*. We denote by $\mathcal{T}(A_i, \mathcal{A}^+)$ the set of elementary triangles of the ECT of $\mathcal{V}(A_i, \mathcal{A}^+)$ and by $\mathcal{T}(A_i) = \bigcup_{\mathcal{A}^+ \subseteq \mathcal{A}:\, A_i \in \mathcal{A}^+} \mathcal{T}(A_i, \mathcal{A}^+)$ the set of all elementary triangles for A_i (for all possible operational sets $\mathcal{A}^+$ that contain A_i).*

We sometimes abbreviate $\mathcal{T}(A_i, \mathcal{A}^+)$ and $\mathcal{T}(A_i)$ by $\mathcal{T}_i^+$ and $\mathcal{T}_i$, respectively.

Example 1 *Consider the convex polygon $\mathcal{P}$ with vertices $P_1, \ldots, P_7$ depicted in Figure 1 and the four facilities $A_1 = (3, 6)$, $A_2 = (6, 7)$, $A_3 = (7, 3)$, and $A_4 = (4, 3)$. The lines in* *the interior of the polygon depict the bisectors $\mathcal{B}_{1j}$ between A_1 and the other three facilities A_2, A_3, and A_4. The figure also shows the set $\mathcal{CP}_1$ of corner-points of A_1. For a bisector $\mathcal{B}_{ij}$ the (at most) two intersection points with bd($\mathcal{P}$) are labelled E_1^{ij} and E_2^{ij} and the intersection point with another bisector $\mathcal{B}_{ik}$ is denoted as E^{ijk}.*

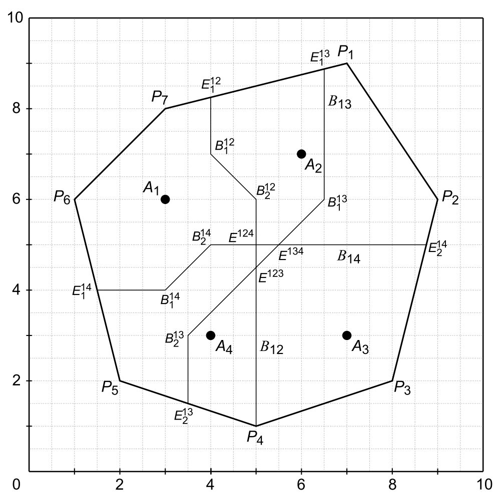

Figure 1: The set of corner-points $\mathcal{CP}_1$ of A_1.

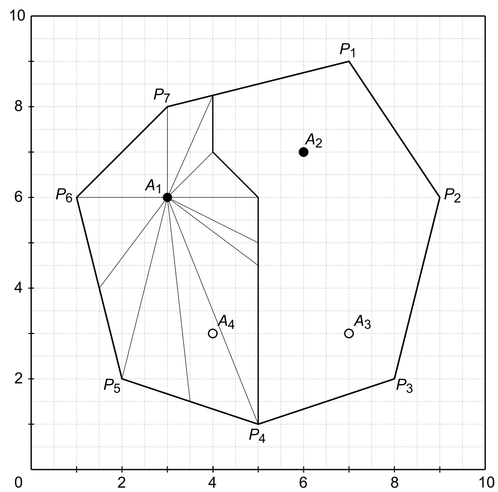

Figure 2: The elementary centroid triangulation of $\mathcal{V}_1^+ = \mathcal{V}(A_1, \{A_1, A_2\})$.

Figure 2 depicts the elementary centroid triangulation of $\mathcal{V}_1^+$ for the operational set $\mathcal{A}^+ = \{A_1, A_2\}$ (an empty circle indicates that the facility has failed).

The main idea is as follows. Instead of enumerating all constellations and computing the respective triangulations which would require an exponential number of steps, we determine for each elementary triangle for A_i the probability that it appears in an elementary centroid triangulation of $\mathcal{V}_i^+$ when $A_i \in \mathcal{A}^+$. We will show that this probability can be computed efficiently; since the number of elementary triangles is polynomial, this will allow us to compute the objective function in polynomial time.

As all Voronoi vertices of $\mathcal{V}_i^+$ are corner-points, the two non-facility corners of an elementary triangle $\Delta \in \mathcal{T}_i^+$ are consecutive corner-points either on the same edge of the boundary of $\mathcal{P}$ or on the same segment of a bisector $\mathcal{B}_{ij}$, $j \neq i$. Vice versa, any two consecutive corner-points on the same edge of $bd(\mathcal{P})$ or on the same segment of $\mathcal{B}_{ij}$ yield an elementary triangle (e.g., in the former case for $\mathcal{A}^+ = \{A_i\}$ and in the latter case for $\mathcal{A}^+ = \{A_i, A_j\}$).

Observation 1 *There is a one-to-one correspondence between elementary triangles for facility A_i and pairs of consecutive corner-points either on the same edge of the boundary of $\mathcal{P}$ or on the same segment of a bisector $\mathcal{B}_{ij}$, $i \neq j$. Thus, we obtain the following bound* *on the number of elementary triangles for facility A_i: $|\mathcal{T}_i| = O(n^2 + m)$.*

Next, we turn to the problem of determining the constellations for which a specific elementary triangle is part of the ECT of $\mathcal{V}_i^+$. Recalling Lemma 1, the contribution of A_i to the objective function can be re-written as

$$\begin{aligned}\sum_{\mathcal{A}^+\subseteq\mathcal{A}\colon A_i\in\mathcal{A}^+} pr(\mathcal{A}^+)\, d(A_i,\mathcal{V}_i^+) &= \sum_{\mathcal{A}^+\subseteq\mathcal{A}\colon A_i\in\mathcal{A}^+} pr(\mathcal{A}^+) \sum_{\Delta\in\mathcal{T}_i^+} d(A_i,\Delta)\\ &= \sum_{\Delta\in\mathcal{T}_i} d(A_i,\Delta) \underbrace{\sum_{\mathcal{A}^+\subseteq\mathcal{A}\colon A_i\in\mathcal{A}^+ \text{ and } \Delta\in\mathcal{T}_i^+} pr(\mathcal{A}^+)}_{=:\,pr(\Delta)} .\end{aligned}$$

Hereby, $pr(\Delta)$ is the probability that the elementary triangle $\Delta \in \mathcal{T}_i$ is part of the elementary centroid triangulation of $\mathcal{V}_i^+$. Next, we show how to determine $pr(\Delta)$ without enumerating all constellations. Instead of referring to elementary triangles, we refer in the remainder to the unique segment defined by the two non-facility corners of the triangle.

Definition 3 *Let $\Delta(A_iE_1E_2) \in \mathcal{T}_i$ be an elementary triangle for facility A_i where $E_1, E_2 \in \mathcal{CP}_i$. We call the closed line segment $\overline{E_1E_2}$ a* sub-segment *and we denote the set of all sub-segments of facility A_i with respect to the set $\mathcal{A}$ by $\mathcal{S}_i(\mathcal{A})$, or simply $\mathcal{S}_i$ if the context is clear.*

For $S \in \mathcal{S}_i$ ($\Delta \in \mathcal{T}_i$) we denote by $\Delta(S)$ ($S(\Delta)$) the corresponding elementary triangle from $\mathcal{T}_i$ (sub-segment from $\mathcal{S}_i$) induced by S (by Δ). Observation 1 implies that a sub-segment is either contained in an edge of the boundary of $\mathcal{P}$ or in a segment of a bisector $\mathcal{B}_{ij}$. Moreover, two sub-segments may only intersect at their endpoints, and a sub-segment is either completely included in $bd(\mathcal{V}_i^+)$ or at most one of its endpoints is in $bd(\mathcal{V}_i^+)$. We call a sub-segment $S \in \mathcal{S}_i$ *active* for a constellation with operational set $\mathcal{A}^+$, if S is part of the boundary of the Voronoi cell $\mathcal{V}_i^+$. In this case, the corresponding triangle $\Delta(S)$ is contained in the elementary centroid triangulation of $\mathcal{V}_i^+$.

Observation 2 *A sub-segment $S \in \mathcal{S}_i$ is active, iff all facilities A_j, $j \neq i$, for which $S \not\subset \mathcal{B}^{\leq}(A_i, A_j)$ have failed and, in case $S \subset \mathcal{B}(A_i, A_k)$, if A_k is operational, in addition to A_i being operational.*

Example 1 (cont.) *Consider again Figure 1 and facility A_1. The sub-segment $\overline{E^{124}E^{123}} \subset \mathcal{B}(A_1, A_2)$ is active, iff facility A_4 has failed and $A_2 \in \mathcal{A}^+$, in addition to $A_1 \in \mathcal{A}^+$. Note that the status of facility A_3 does not play a role. Similarly, for $\overline{E_2^{13}P_4}$ to* *be active, facilities A_3 and A_4 must have failed, A_1 must be active, and the status of A_2 does not matter.*

To efficiently determine the probability that a sub-segment $S \in S_i$ is active we define sets $\mathcal{A}(S)^+, \mathcal{A}(S)^-, \mathcal{A}(S)^U$ of facilities which must be operational, must have failed, and whose status is immaterial, respectively, in order for S to be active. Slightly abusing terminology, $\mathcal{A}(S)^+, \mathcal{A}(S)^-, \mathcal{A}(S)^U$ will also denote the sets of indices of these facilities. Then, $\mathcal{AL}(S) = \big(\mathcal{A}(S)^+, \mathcal{A}(S)^-, \mathcal{A}(S)^U\big)$ will be called the *active list* for the sub-segment S. Based on Observation 2 we see that

$$\mathcal{A}(S)^- = \{j \ : S \not\subset \mathcal{B}^{\leq}(A_i, A_j)\},\ \mathcal{A}(S)^+ = \begin{cases} \{i,j\} & \text{if } S \subset \mathcal{B}(A_i, A_j) \\ \{i\} & \text{if } S \subset bd(\mathcal{P}) \end{cases},$$

and $\mathcal{A}(S)^U = \mathcal{A} - \mathcal{A}(S)^+ - \mathcal{A}(S)^-$. It follows that the probability

$$pr(\Delta(S)) \ = \ pr\left(\mathcal{A}(S)^+, \mathcal{A}(S)^-\right) \tag{7}$$

is computable in $O(n)$ time by Assumption A. In case of the uniform independent probability model, this expression simplifies to

$$pr(\Delta(S)) \ = \ (1-p)^{|\mathcal{A}(S)^+|} p^{|\mathcal{A}(S)^-|} \,.$$

Example 1 (cont.)
Figure 3 depicts the active list $\mathcal{AL}(S)$ of each sub-segment $S \in \mathcal{S}_1$ for facility A_1. We represent $j \in \mathcal{A}(S)^+$ with a "+" sign, and those in $\mathcal{A}(S)^-$ with a "−" sign, omitting the index +1 which is automatically present in all active lists.

For a given facility A_i, its set $\mathcal{S}_i(\mathcal{A})$ of sub-segments can be obtained in $O(n^2{+}m)$ time, e.g., as follows. First, compute the planar arrangement of all open-ended straight lines that contain the segments of the bisectors $\mathcal{B}_{ij}$, $j \in [1:n] \setminus \{i\}$ (there are at most $3(n-1)$ such lines); this takes $O(n^2)$ time, see de Berg et al. [17]. Each edge of the arrangement will "remember" the line it belongs to and the corresponding bisector. Then, overlay this arrangement in $O(n^2+m)$ time with the planar partition $\mathcal{P}$ (Finke and Hinrichs [24]), obtaining an arrangement with $O(n^2+m)$ vertices and edges (the two planar partitions have just $O(n)$ intersection points). Each corner-point corresponds to some vertex of the overlaid arrangement, and each sub-segment is either an edge or the union of several contiguous edges of the arrangement that are on the same line. Then, all sub-segments of $\mathcal{S}_i(\mathcal{A})$ are obtained by traversing the edges of the arrangement and checking whether they belong to a bounding edge of $\mathcal{P}$ or a segment of the corresponding bisector, which takes $O(n^2+m)$ time.

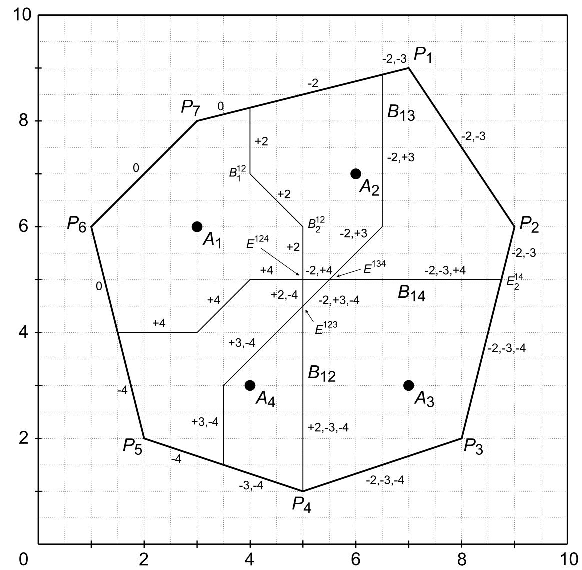

Figure 3: The set of all sub-segments $\mathcal{S}_1$ for A_1 and the respective active lists.

To construct the active lists for all sub-segments $S \in \mathcal{S}_i$, we start with sub-segments S whose elementary triangle $\Delta(S)$ is part of the Voronoi cell for the case where all facilities are operational, i.e., when $\mathcal{A}^+ = \mathcal{A}$. Then, either $S \subset \mathcal{B}_{ij}$ for some $\mathcal{B}_{ij}$ or $S \subset bd(\mathcal{P})$. In the former case we have $\mathcal{A}(S)^+ = \{i,j\}$ and in the latter, $\mathcal{A}(S)^+ = \{i\}$. In both cases, $\mathcal{A}(S)^- = \emptyset$.

Next, we show that once the active list for some sub-segment S has been constructed, the active list for any adjacent sub-segment S' can be obtained in constant time by a small modification of the active list for S. Suppose that $S \subset \mathcal{B}_{ij}$, and suppose first that S' is not on the boundary of $\mathcal{P}$ and $E \in \mathcal{CP}_i$ is the common corner-point of S and S'. If E is a breakpoint of $\mathcal{B}_{ij}$ and not an intersection point with another bisector $\mathcal{B}_{ik}$, S' has exactly the same active set as S, hence $\mathcal{AL}(S') = \mathcal{AL}(S)$.

If E is an intersection point of $\mathcal{B}_{ij}$ with $\mathcal{B}_{ik}$ and belongs to no other bisector $\mathcal{B}_{ir}$ involving A_i, we have to distinguish on which side of $\mathcal{B}_{ik}$ the sub-segment S lies and whether $S' \subset \mathcal{B}_{ij}$ or $S' \subset \mathcal{B}_{ik}$. Assume first that $S \subset \mathcal{B}^{\leq}_{ik}$. If $S' \subset \mathcal{B}_{ij}$, then A_j must still be operational for S' to be active. However, as we cross $\mathcal{B}_{ik}$, the sub-segment is now closer to A_k than to A_i. Hence, in order for it to be active, facility A_k must have failed. Thus, $\mathcal{A}(S')^+ = \mathcal{A}(S)^+$, $\mathcal{A}(S')^- = \mathcal{A}(S)^- \cup \{k\}$, and $\mathcal{A}(S')^U = \mathcal{A}(S)^U \setminus \{k\}$. If, instead, $S' \subset \mathcal{B}_{ik}$, then A_k must be operational. Moreover, if $S' \subset \mathcal{B}^{\leq}_{ji}$, then facility A_j must have failed; hence, $\mathcal{A}(S')^+ = \{i,k\} = \mathcal{A}(S)^+ \cup \{k\} \setminus \{j\}$, $\mathcal{A}(S')^- = \mathcal{A}S^- \cup \{j\}$ and $\mathcal{A}(S')^U = \mathcal{A}(S)^U \setminus \{k\}$. Otherwise, S' will be active regardless of the status of A_j; thus $\mathcal{A}(S')^+ = \{i,k\} = \mathcal{A}(S)^+ \cup \{k\} \setminus \{j\}$, $\mathcal{A}(S')^- = \mathcal{A}(S)^-$ and $\mathcal{A}(S')^U = \mathcal{A}(S)^U \cup \{j\} \setminus \{k\}$.

The remaining cases ($S \subset \mathcal{B}^{\leq}_{ki}$, E is on the boundary of $\mathcal{P}$, and $S \subset bd(\mathcal{P})$) follow analogously. Note that the changes in the active list only occur for facilities whose bisectors contain E and are thus limited to at most two facilities moving between the sets $\mathcal{A}(S)^+, \mathcal{A}(S)^-, \mathcal{A}(S)^U$ (this can be observed in Figure 3). Therefore, $\mathcal{AL}(S)$ can be modified into $\mathcal{AL}(S')$ in a constant time. If E is an intersection point of segments of several different bisectors $\mathcal{B}_{i\cdot}$, it is straightforward to adjust the computations so that the lists $\mathcal{AL}(S')$ are computed for all subsegments S' adjacent to S in a constant time per subsegment.

Thus, we can compute $\mathcal{AL}(S)$ for all sub-segments by going from already considered sub-segments to adjacent sub-segments until all sub-segments are considered. We do not need to store the active lists since they are needed only for computing the probabilities $pr(\Delta(S)$. Given $\mathcal{AL}(S)$, $pr(\Delta(S)$ can be computed in $O(n)$ time by Assumption A. In the case of independent facility failures $pr(\Delta(S))$ can be obtained in constant time from that of an adjacent sub-segment, as follows from the discussion above about the active lists. By repeating this procedure for all facilities A_i, we evaluate the objective function without having to enumerate all constellations. Since the number of sub-segments for a facility A_i is $O(n^2+m)$, there are n facilities, and for an elementary triangle $\Delta \in \mathcal{T}_i^+$ value $d(A_i, \Delta)$ can be computed in a constant time [5, 23], we obtain the following result.

Proposition 1 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of $n \geq 2$ facilities satisfying Assumption B. Then we can re-write the objective function of the CMPUF and CAMPUF problems as*

$$F_M(\mathcal{A}) \;=\; \sum_{i=1}^{n} \sum_{\Delta \in \mathcal{T}_i} pr(\Delta)\, d(A_i, \Delta)\,.$$

Moreover, assuming that Assumption A holds, $F_M(\mathcal{A})$ can be evaluated in $O(n^2\,(n^2+m))$ time. In the case of independent facility failures (and in any other case where $pr(\Delta(S))$ can be obtained in constant time from that of an adjacent sub-segment), the complexity reduces to $O(n\,(n^2+m))$.

3.4.2 Conditional Reliable Market Share Problem

Next, we turn to the objective function of the CRMSP, given by (5). We start by a facility-focused re-statement of the objective. Similar to Lemma 1, we have the following result.

Lemma 2 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of $n \geq 2$ facilities satisfying Assumption B. Then*

$$F_{MS}(\mathcal{A}) \;=\; \sum_{i=l+1}^{n} \sum_{\mathcal{A}^+ \subseteq \mathcal{A}: A_i \in \mathcal{A}^+} pr(\mathcal{A}^+)\, \mu(\mathcal{V}_i^+)\,. \tag{8}$$

Proof. Straightforward, using the definitions. □ □

Using the elementary centroid triangulation introduced in Section 3.4.1, similarly to Proposition 1 we obtain:

Proposition 2 *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of $n \geq 2$ facilities satisfying Assumption B. Then we can re-write the objective function of the CRMSP as*

$$F_{MS}(\mathcal{A}) \;=\; \sum_{i=l+1}^{n} \sum_{\Delta \in \mathcal{T}_i} pr(\Delta)\, \mu(\Delta)\,;$$

moreover, under Assumption A, $F_{MS}(\mathcal{A})$ can be evaluated in $O(n^2\,(n^2+m))$ time, which reduces to $O(n\,(n^2+m))$ time in the case of independent facility failures.

4 Parametric Representation and Structure-Preserving Partition of the Polygon

Having introduced the sets of corner-points and sub-segments for a fixed set of facility locations, we next characterize the changes to these sets after adding one new facility. Our goal is to extend the approach developed in [5] for problems with reliable facilities (based on decomposition of the solution space into areas over which the problem is simple) to our problems CMPUF, CAMPUF, and CRMSP. Some general difficulties were mentioned in Section 1; more specific challenges and issues posed by unreliable facilities will become clear as we present our approach.

Before delving into details, we first map out some key ideas. The objective functions of the median and market share problems are given by

$$F_M(\mathcal{A}_Z) \;=\; \sum_{i=1}^{n+1} \sum_{\Delta \in \mathcal{T}(\mathcal{A}_Z)_i} pr(\Delta)\, d(A_i, \Delta) \quad \text{and} \quad F_{MS}(\mathcal{A}_Z) \;=\; \sum_{i=l+1}^{n+1} \sum_{\Delta \in \mathcal{T}(\mathcal{A}_Z)_i} pr(\Delta)\, \mu(\Delta)\,,$$

where we use notation $\mathcal{T}(\mathcal{A}_Z)_i$ instead of $\mathcal{T}_i$ to emphasize that we are dealing with the extended set $\mathcal{A}_Z$ rather than $\mathcal{A}$. Each elementary triangle $\Delta \in \mathcal{T}(\mathcal{A}_Z)_i$ has a facility corner at A_i and non-facility corners E_1, E_2 which are two consecutive corner-points in set $\mathcal{CP}_i(\mathcal{A}_Z)$ for $i \in [1 : n+1]$. Since the corner-points are induced by the bisectors $\mathcal{B}_{ij}$ (as well as the vertices and edges of $\mathcal{P}$), and since the parameters of each bisector $\mathcal{B}(A_i, Z)$, $i \in [1 : n]$, are linear functions of the coordinates of the new facility $Z = (x, y)$, the non-facility corners E_1, E_2 have either constant or linear representation in x, y. In [5, 23], closed-form expressions to calculate $d(A_i, \Delta)$ and $\mu(\Delta)$ are derived; locally they are (at most) cubic and quadratic functions of x, y, respectively. Moreover, as discussed in the previous section, the expression for $pr(\Delta)$ depends only on the active list for $S(\Delta)$. It follows that both $F_M(\mathcal{A}_Z)$ and $F_{MS}(\mathcal{A}_Z)$ can be written (locally) as, respectively, cubic and quadratic functions of x, y.

Suppose that $F(x, y)$ is the (compact) explicit representation of the objective function ($F_M(\mathcal{A}_Z)$ or $F_{MS}(\mathcal{A}_Z)$) near $Z = (x, y)$ as a cubic or quadratic function of x, y. Then, $F(x, y)$ can be optimized efficiently over any polygonal area using the same methodology as in [5] (see Section 5 for further discussion). Consider another point $Z' = (x', y') \in \mathcal{P}$, and suppose that value $F(x', y')$ is better than $F(x, y)$. Can we assert that Z' is a better location than Z? In general, the answer is "no" - function $F(x', y')$ may no longer be a correct representation of the objective function at Z', because the geometry of Voronoi partitions depends on the location of the new facility, and thus all elements of the expressions above (the elementary triangulation, the active lists, the representation of contributions of elementary triangles) may be different at Z'. Thus, the answer is "yes" if the following conditions hold:

- *Structural Equivalence (SE)* For every $i \in [1 : n + 1]$, each triangle $\Delta \in \mathcal{T}(\mathcal{A}_Z)_i$ must be in one-to-one correspondence with a triangle $\Delta' \in \mathcal{T}(\mathcal{A}_{Z'})_i$. Moreover, the parametric representation of the corners of Δ in terms of x, y must be valid for Δ'.
- *Active List Equivalence (ALE)* For any such corresponding triangles Δ and Δ', the active lists $\mathcal{AL}(S(\Delta))$ and $\mathcal{AL}(S(\Delta'))$ must be the same to ensure that $pr(\Delta) = pr(\Delta')$.
- *Parametric Representation Equivalence (PRE)* For any such corresponding triangles Δ and Δ', the parametric representation expressions for $d(A_i, \Delta)$ and $\mu(\Delta)$ in terms of x, y must still be valid for Δ'. As we will see below, the condition (SE) is not sufficient for (PRE) to hold.

In the remainder of this section, we derive a partition of $\mathcal{P}$ into polygonal cells such that all three conditions above hold within each cell. Thus, it will remain to efficiently compute the parametric representations of the objective function over all cells and to optimize them over all cells.

For an existing facility A_i, $i \in [1 : n]$, the new set of corner-points, $\mathcal{CP}_i(\mathcal{A}_Z)$, is given by $\mathcal{CP}_i(\mathcal{A})$ plus all intersection points of $\mathcal{B}(A_i, Z)$ with bisectors $\mathcal{B}_{ij}$, $j \in [1 : n]$, $j \neq i$, or with the boundary of $\mathcal{P}$, plus the breakpoints of $\mathcal{B}(A_i, Z)$. Hence, it is easy to compute incrementally the sets $\mathcal{CP}_i(\mathcal{A}_Z)$, $\mathcal{S}_i(\mathcal{A}_Z)$, and $\mathcal{AL}(S)$, $S \in \mathcal{S}_i(\mathcal{A}_Z)$, based on the respective sets for $\mathcal{A}$ and the bisector $\mathcal{B}_{iZ}$. However, the changes will be different for different locations of Z; and even more so for the sets $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{S}_Z(\mathcal{A}_Z)$ of the new facility itself. In the following, we call $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_Z)$ the *incremental sets of* *corner-points* and *sub-segments*, respectively, and for any operational set $\mathcal{A}_Z^+$, $\mathcal{VD}(\mathcal{A}_Z^+)$ is called *incremental Voronoi diagram*.

Below, we derive a partition of the market area into cells such that the structure of the incremental sets of corner-points and sub-segments of all facilities is identical for all locations Z of the new facility in a cell of the partition. Let us first give a definition of identical structure. Let $Z, Z' \in \mathcal{P} \setminus \mathcal{A}$, $Z \neq Z'$, be two distinct locations, and let Z, Z' be in the interior of the same configuration cones of the existing facilities. Then, for any $i \in [1:n]$, the bisectors $\mathcal{B}(A_i, Z)$ and $\mathcal{B}(A_i, Z')$ have the same, nondegenerate shape and there is a natural correspondence between similar segments and breakpoints of $\mathcal{B}(A_i, Z)$ and $\mathcal{B}(A_i, Z')$. Moreover, $\mathcal{A}_Z$ and $\mathcal{A}_{Z'} := \mathcal{A} \cup \{Z'\}$ satisfy Assumption B.

Definition 4 *For $i \in [1:n]$, two corner-points $E \in \mathcal{CP}_i(\mathcal{A}_Z)$ and $E' \in \mathcal{CP}_i(\mathcal{A}_{Z'})$ are called* structurally identical*, if they correspond to the same corner-point of $\mathcal{CP}_i(A)$, or to similar breakpoints of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$, or to the intersection points of similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ with the same sub-segment of $\mathcal{S}_i(\mathcal{A})$. Two corner-points $E \in \mathcal{CP}_Z(\mathcal{A}_Z)$ and $E' \in \mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$ are called* structurally identical*, if they correspond to the same vertex of $\mathcal{P}$, or to similar breakpoints of $\mathcal{B}_{jZ}$ and $\mathcal{B}_{jZ'}$, $j \in [1:n]$, or to the intersection points of similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ with the same edge of bd($\mathcal{P}$) or with similar segments of $\mathcal{B}_{jZ}$ and $\mathcal{B}_{jZ'}$, $i, j \in [1:n]$, $i \neq j$. The sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$, $i \in [1:n+1]$, are called* structurally identical*, if there is a one-to-one correspondence between corner-points of $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ such that all corner-points of $\mathcal{CP}_i(\mathcal{A}_Z)$ are paired with their structurally identical counterparts in $\mathcal{CP}_i(\mathcal{A}_{Z'})$.*

For $i \in [1:n+1]$, two sub-segments $S \in \mathcal{S}_i(\mathcal{A}_Z)$ and $S' \in \mathcal{S}_i(\mathcal{A}_{Z'})$ are called structurally identical*, if their endpoints are pairwise structurally identical. $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ are called* structurally identical*, if there is a one-to-one correspondence between sub-segments of $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ such that all sub-segments of $\mathcal{S}_i(\mathcal{A}_Z)$ are paired with their structurally identical counterparts in $\mathcal{S}_i(\mathcal{A}_{Z'})$.*

Example 2 *For two alternative locations Z and Z' of the new facility, Figure 4 depicts the bisectors $\mathcal{B}(A_1, Z)$ (dashed line) and $\mathcal{B}(A_1, Z')$ (dotted line) and some of the additional corner-points generated by them. Whereas the vertical segments of $\mathcal{B}_{1Z}$ and $\mathcal{B}_{1Z'}$ intersect identical sub-segments of $\mathcal{S}_1(\mathcal{A})$, this does not hold for the diagonals. Hence, $\mathcal{CP}_1(\mathcal{A}_Z)$ and $\mathcal{CP}_1(\mathcal{A}_{Z'})$ are not structurally identical. As a result, also the incremental sets of sub-segments are different for Z and Z' as well as their active lists. For example, the activity of the sub-segments incident to E^{124} (the intersection of bisectors $\mathcal{B}_{12}$ and $\mathcal{B}_{14}$) depends on the status of the additional facility if it is located at Z, but does not depend on the status of the additional facility if it is located at Z'.*

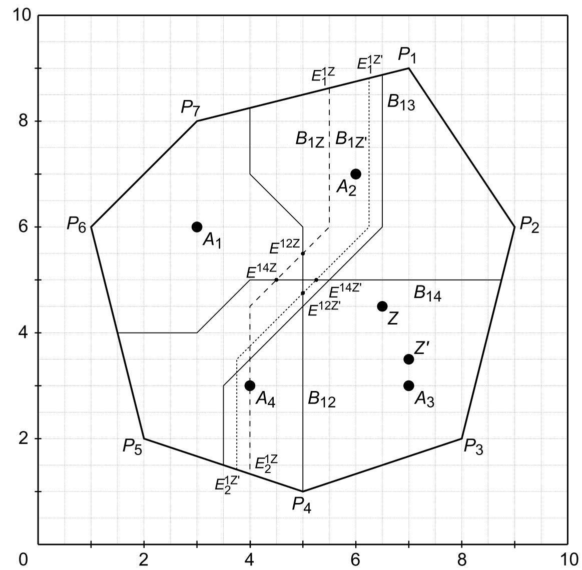

Figure 4: The bisectors for two distinct locations Z and Z' of the new facility.

Before we show how to ensure structural identity, we note that it ensures that the (SE) condition above holds.

Lemma 3 *Let $i \in [1 : n+1]$ and $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ be structurally identical. Then the coordinates of structurally identical corner-points have the same constant or linear representation in the coordinates x and y of the additional facility.*

Proof. The result follows from the definitions and the parametric representation of the segments of a bisector $\mathcal{B}(A_i, Z)$ (see Section 3.3). □

The next result establishes a *sufficient* criterion for the structural identity of the incremental sets of corner-points and sub-segments for the *already existing* facilities.

Lemma 4 *Let $Z, Z' \in \mathcal{P} \setminus \mathcal{A}$, $Z \neq Z'$, belong to the interiors of the same configuration cones of the existing facilities. Let $i \in [1 : n]$. The sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ of corner-points and the sets $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ of sub-segments are structurally identical, if similar segments of $\mathcal{B}(A_i, Z)$ and $\mathcal{B}(A_i, Z')$ intersect the same sub-segments of $\mathcal{S}_i(\mathcal{A})$ at interior points.*

Proof. Let $i \in [1 : n]$. As Z and Z' belong to the interiors of the same configuration cones of the existing facilities, $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ have the same, non-degenerate shape. If similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ intersect the same sub-segments of $\mathcal{S}_i(\mathcal{A})$ at interior points, then by definition the additional corner-points induced by the two bisectors are pairwise structurally identical in $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$. As corner-points of $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ corresponding to corner-points of $\mathcal{CP}_i(\mathcal{A})$ or breakpoints of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ obviously yield a one-to-one correspondence of structurally identical corner-points, the first result follows.

Concerning the sub-segments, if similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ intersect the same sub-segment $S \in \mathcal{S}_i(\mathcal{A})$ at interior points, then S is split both times into two sub-segments that are pairwise structurally identical in $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$. Moreover, the additional sub-segments in $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ induced by $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ are pairwise structurally identical, if their intersection points with sub-segments of $\mathcal{S}_i(\mathcal{A})$ appear in the same order along $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$, respectively. But this will always be the case, because the sub-segments of $\mathcal{S}_i(\mathcal{A})$ are fixed and the bisector $\mathcal{B}(A_i, Z)$ is a continuous mapping in Z as long as Z remains in the interior of the same configuration cone of A_i. As sub-segments of $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ corresponding to sub-segments of $\mathcal{S}_i(\mathcal{A})$ naturally yield a one-to-one correspondence of structurally identical sub-segments, also the second result follows. □

Whereas structural identity is relatively easy to ensure for the incremental sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_Z)$ of an existing facility $i \in [1:n]$, this is not so obvious for the new facility.

Example 3 *In Figure 5, we depict for two alternative locations Z and Z' of the new facility the bisectors $\mathcal{B}(A_i, Z)$ (on the left hand side) and $\mathcal{B}(A_i, Z')$, respectively, $i \in [1:4]$. Each corner-point of $\mathcal{CP}_Z(\mathcal{A}_Z)$ has a structurally identical counterpart in $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$. For example, similar segments of $\mathcal{B}(A_2, \cdot)$ and $\mathcal{B}(A_4, \cdot)$ intersect, or similar segments of $\mathcal{B}(A_3, \cdot)$ intersect the same edge of $\mathcal{P}$. As a result, $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$ are structurally identical. However, the incremental sets of sub-segments are not structurally identical because, for example, the order of intersections of $\mathcal{B}(A_1, \cdot)$ and $\mathcal{B}(A_4, \cdot)$ with $\mathcal{B}(A_2, \cdot)$ is different along the latter bisector for Z and Z', yielding different elementary triangles.*

As illustrated by Example 3, to ensure structural identity of $\mathcal{S}_Z(\mathcal{A}_Z)$ when Z varies, we need to ensure that the order of corner-points of Z along any bisector $\mathcal{B}_{iZ}$ stays the same, in addition to structural identity of the set of corner-points $\mathcal{CP}_Z(\mathcal{A}_Z)$. However, it turns out that we can derive a *sufficient* criterion for the structural identity of the corner-points and sub-segments of the new facility based only on the structural identity of the corner-points of the existing facilities, which greatly simplifies our task.

Lemma 5 *Let $Z, Z' \in \mathcal{P} \setminus \mathcal{A}$, $Z \neq Z'$, belong to the interiors of the same configuration cones of the existing facilities. If the sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ are structurally identical for all $i \in [1:n]$, then the sets of corner-points $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$ and the sets of sub-segments $\mathcal{S}_Z(\mathcal{A}_Z)$ and $\mathcal{S}_{Z'}(\mathcal{A}_{Z'})$ are structurally identical.*

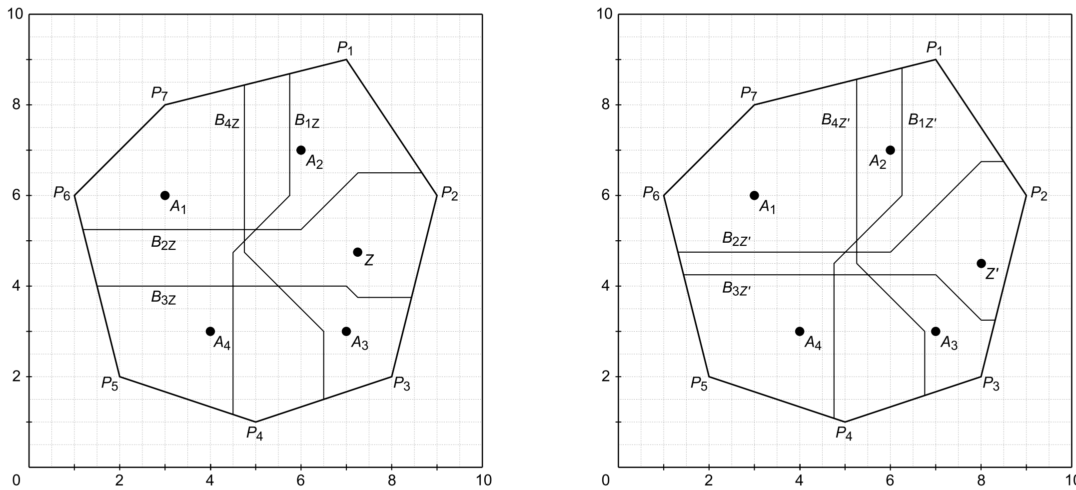

Figure 5: The bisectors for two distinct locations Z and Z' of the new facility.

Proof. Let $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ be structurally identical for all $i \in [1:n]$. First, we show that each corner-point of $\mathcal{CP}_Z(\mathcal{A}_Z)$ has a structurally identical counterpart in $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$. Let $E \in \mathcal{CP}_Z(\mathcal{A}_Z)$. If E corresponds to a vertex of $\mathcal{P}$ or breakpoint of $\mathcal{B}_{iZ}$, E has by definition a structurally identical counterpart in $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$ (the latter because $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ have the same, non-degenerate shape). Next, let E be an intersection point of $\mathcal{B}_{iZ}$ with an edge of $\mathcal{P}$, $i \in [1:n]$. As E is then also a corner-point in $\mathcal{CP}_i(\mathcal{A}_Z)$, it has a structurally identical counterpart $E' \in \mathcal{CP}_i(\mathcal{A}_{Z'})$ which corresponds to the intersection point of the same segment of $\mathcal{B}_{iZ'}$ with the same edge of $\mathcal{P}$. Thus, E and E' are also structurally identical in $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$. Finally, let E be an intersection point between $\mathcal{B}_{iZ}$ and $\mathcal{B}_{jZ}$, $i,j \in [1:n]$, $i \neq j$. As E is also on $\mathcal{B}_{ij}$, it is again a corner-point in $\mathcal{CP}_i(\mathcal{A}_Z)$. Thus, similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ must intersect the same sub-segment of $\mathcal{S}_i(\mathcal{A})$. Therefore, also $\mathcal{B}_{iZ'}$ intersects $\mathcal{B}_{ij}$ in a structurally identical corner-point $E' \in \mathcal{CP}_i(\mathcal{A}_{Z'})$ and this is the intersection point of $\mathcal{B}_{iZ'}$ with $\mathcal{B}_{jZ'}$. An analogous reasoning for j shows that similar segments of $\mathcal{B}_{jZ}$ and $\mathcal{B}_{jZ'}$ intersect $\mathcal{B}_{ij}$ in E and E'. Hence, similar segments of $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ intersect similar segments of $\mathcal{B}_{jZ}$ and $\mathcal{B}_{jZ'}$. Therefore, E and E' are structurally identical in $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$. Summing up, there is a one-to-one correspondence between structurally identical corner-points of $\mathcal{CP}_Z(\mathcal{A}_Z)$ and $\mathcal{CP}_{Z'}(\mathcal{A}_{Z'})$ and thus the two sets are structurally identical.

Next, we prove that the sets of sub-segments of the additional facility are also structurally identical. Recall that sub-segments of $\mathcal{S}_Z(\mathcal{A}_Z)$ are contained in edges of $\mathcal{P}$ or bisectors $\mathcal{B}_{iZ}$, $i \in [1:n]$. First, consider an edge e of $bd(\mathcal{P})$. As the sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ are structurally identical for all $i \in [1:n]$, the same (segments of) bisectors $\mathcal{B}_{iZ}$, $i \in [1:n]$ intersect e for Z and Z'. If they intersect e also in the same order along the edge, the resulting sub-segments are clearly structurally identical. Therefore, assume that

$\mathcal{B}_{iZ}$ and $\mathcal{B}_{jZ}$ are two bisectors that intersect e in a different order along the edge for Z and Z', $i,j \in [1:n]$, $i \neq j$. A bisector $\mathcal{B}(Q,Z)$, $Q \in I\!R^2$, is a continuous mapping in Z as long as Z remains in the interior of same configuration cone of Q, and the coordinates of the intersection point of $\mathcal{B}_{iZ}$ (or $\mathcal{B}_{jZ}$) with e depend linearly on Z. Hence, there must be a point $\hat{Z}$ "between" Z and Z', i.e., on the straight line between Z and Z', such that the two bisectors intersect the edge at the same point R. As R is on both bisectors, R is also the intersection point of $\mathcal{B}_{ij}$ with the edge and thus a corner-point of $\mathcal{CP}_i(\mathcal{A})$ (and $\mathcal{CP}_j(\mathcal{A})$). This means that the order of intersection points along e can only change at a corner-point of A_i and not in the interior of a sub-segment of $\mathcal{S}_i(\mathcal{A})$. But then $\mathcal{B}_{iZ}$ must intersect different sub-segments of A_i for Z and Z', yielding structurally different corner-points in $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$, which is a contradiction with the assumption that $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ are structurally identical.

Now, we consider a bisector $\mathcal{B}_{iZ}$, $i \in [1:n]$. Again, the same (segments of) bisectors $\mathcal{B}_{jZ}$, $i \neq j \in [1:n]$, must intersect $\mathcal{B}_{iZ}$ for Z and Z'. If the order of these intersection points along $\mathcal{B}_{iZ}$ is the same for Z and Z', then the resulting sub-segments are again structurally identical. Therefore, assume that $\mathcal{B}_{jZ}$ and $\mathcal{B}_{kZ}$, $i \neq j,k \in [1:n]$, are two bisectors that intersect $\mathcal{B}_{iZ}$ and $\mathcal{B}_{iZ'}$ with similar segments but in a different order. Similar to the previous case, there must be a point $\hat{Z}$ on the straight line between Z and Z' such that $\mathcal{B}_{j\hat{Z}}$ and $\mathcal{B}_{k\hat{Z}}$ intersect $\mathcal{B}_{i\hat{Z}}$ at the same point R, which is then the common intersection point of the three bisectors $\mathcal{B}_{ij}$, $\mathcal{B}_{ik}$, and $\mathcal{B}_{jk}$ of existing facilities. Consider now the bisector $\mathcal{B}_{ij}$. The intersection point of $\mathcal{B}_{jZ}$ with $\mathcal{B}_{iZ}$ is on $\mathcal{B}_{ij}$ and it continuously moves on $\mathcal{B}_{ij}$ towards the intersection point of $\mathcal{B}_{jZ'}$ with $\mathcal{B}_{iZ'}$ when moving from Z to Z'. As it passes R along this path, $\mathcal{B}_{iZ}$ must intersect $\mathcal{B}_{ij}$ in a different sub-segment for Z and Z', implying again that sets $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ are structurally different, which is a contradiction.
□

An immediate consequence of the previous results is that structural identity implies the condition (ALE) above.

Corollary 1 *Let $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ be structurally identical for all $i \in [1:n]$. Then any two structurally identical sub-segments of $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$, $i \in [1:n+1]$ have the same active lists.*

Proof. The result follows from the proofs of Lemmas 4 and 5. As the additional corner-points appear in the same order along a bisector $\mathcal{B}_{iZ}$ or an edge of $\mathcal{P}$, corresponding sub-segments in $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ must have the same active lists. □

Finally, we discuss how to partition the polygon into subsets (cells) such that the corner-points and sub-segments are structurally identical for any location $Z = (x,y)$ of the additional facility in the interior of such a subset. According to Lemmas 4 and 5, it is sufficient to ensure that for any $i \in [1:n]$, any segment of the bisector $\mathcal{B}(A_i, Z)$ intersects any specific sub-segment of $\mathcal{S}_i(\mathcal{A})$ at interior points either for all Z in the subset or for none at all. We do this using the technique developed in [5] which we sketch below with necessary modifications.

Let $i \in [1:n]$ and a specific configuration cone $\mathcal{CC}_i^k$, $1 \le k \le 8$, be given. The representation of the bisector $\mathcal{B}(A_i, Z)$ is the same for each $Z \in int(\mathcal{CC}_i^k)$ and $\mathcal{B}_{iZ}$ intersects another bisector $\mathcal{B}_{ij}$, $j \in [1:n]$, at most once and $bd(\mathcal{P})$ at most twice ([5]). Let L be a line segment of $\mathcal{B}_{iZ}$ and $S = \overline{E_1E_2} \in \mathcal{S}_i(\mathcal{A})$ be a sub-segment defined by two adjacent corner-points $E_1, E_2 \in \mathcal{CP}_i(\mathcal{A})$. First, we assume that S and L are not parallel. We denote by R the intersection point of the open-ended straight lines underlying S and L, respectively. S and L intersect if R lies on both segments, that is, if the first (second) coordinate of R lies between the first (second) coordinates of the endpoints of S and L. These conditions yield in total four inequalities that together guarantee the intersection between S and L. As the representation of the bisector segment as well as the coordinates of R are linear in x and y (Section 3.3), the four inequalities are linear in the coordinates of Z. If L is vertical or horizontal, it is a half-line and we can simply discard the inequality that has infinity in the right-hand side. Each of the remaining inequalities defines a half-space, and a straight line if considered as equality; these straight lines will be called *sub-intersection lines* (the prefix "sub-" stands for "sub-segment" and is used to emphasize the difference from the smaller set of *intersection lines* used in [5] and to stress that, unlike [5], the focus is on intersections of bisectors with sub-segments rather than with edges of Voronoi diagram). If S and L are parallel, they cannot have common interior points, just possibly a common endpoint since $\mathcal{B}_{iZ}$ intersects another bisector $\mathcal{B}_{ij}$ or a segment of the boundary of $\mathcal{P}$ at most once.

Summing up, each combination of a segment of $\mathcal{B}(A_i, Z)$ and a sub-segment $S \in \mathcal{S}_i(\mathcal{A})$ defines at most four sub-intersection lines. The set of all configuration and sub-intersection lines then defines a partition of $\mathcal{P}$ into cells such that for any point Z in the interior of a cell, the bisector $\mathcal{B}_{iZ}$ intersects the same sub-segments of $\mathcal{S}_i(\mathcal{A})$ at interior points.

As we have at most three segments for a specific bisector, the number of sub-intersection lines for a given $i \in [1:n]$ is linear in the number of sub-segments. Hence, there are in total $O(n(n^2+m))$ sub-intersection lines induced by all existing facilities. There are four configuration lines for each facility. Each cell of the partition of the polygon $\mathcal{P}$ defined by the set of all configuration and sub-intersection lines of the existing facilities is a convex polygon bounded by segments of configuration or sub-intersection lines or edges of $\mathcal{P}$. Since there are $O(n)$ configuration lines and $O(n(n^2+m))$ sub-intersection lines, this partition has $O(n^2(n^2+m)^2)$ vertices, edges, and cells (Edelsbrunner [19]). This is also the time required to compute this partition, because all sets of sub-segments $\mathcal{S}_i(\mathcal{A})$, $i \in [1:n]$, can be derived in $O(n(n^2+m))$ time (see Section 3) and the effort required to determine all sub-intersection lines is proportional to their number. The discussion above implies the following result.

Theorem 1 *The set of configuration and sub-intersection lines induces a partition of $\mathcal{P}$ into $O(n^2(n^2+m)^2)$ cells such that for any two distinct locations Z, Z' of the additional facility in the interior of the same cell, the sets of corner-points $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ and the sets of sub-segments $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ are structurally identical for all $i \in [1 : n+1]$. Moreover, any two structurally identical sub-segments of $\mathcal{S}_i(\mathcal{A}_Z)$ and $\mathcal{S}_i(\mathcal{A}_{Z'})$ have the same active lists.*

While the previous result derives a partition of $\mathcal{P}$ which ensures that conditions (SE) and (ALE) are satisfied, it is not sufficient to ensure that condition (PRE) - the equivalence of parametric representation - holds as well. Consider an elementary triangle $\Delta = \Delta\, A_i E_1 E_2 \in \mathcal{T}_i$ for some $i \in [1 : n+1]$. The closed-form expressions derived in [23] and [5] to calculate $d(A_i, \Delta)$ and $\mu(\Delta)$ as, respectively, cubic and quadratic functions of the coordinates of the corners of the triangle remain valid only as long as E_1 and E_2 do not cross the horizontal and vertical lines through A_i, i.e., as long as E_1 and E_2 remain in the same quadrants with respect to A_i. In [5] this was ensured by introducing the concept of *quadrant identity*:

Definition 5 *Let $Z, Z' \in \mathcal{P} \setminus \mathcal{A}$, $Z \neq Z'$, and Z, Z' belong to the interiors of the same configuration cones of the existing facilities. Moreover, let $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ be structurally identical for all $i \in [1 : n+1]$. We call $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$* quadrant identical, *if any corner-point of $\mathcal{CP}_i(\mathcal{A}_Z)$ lies in the same quadrant with respect to A_i as its structurally identical counterpart in $\mathcal{CP}_i(\mathcal{A}_{Z'})$.*

In the following, the horizontal and vertical lines through the vertices of $\mathcal{P}$ are called *quadrant lines*. We now add the set of quadrant lines to the set of all configuration and sub-intersection lines and consider the resulting partition $\mathcal{U}$ of $\mathcal{P}$.

Lemma 6 *For any Z, Z' from the interior of the same cell of the partition $\mathcal{U}$, $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ are structurally and quadrant identical for all $i \in [1 : n+1]$.*

Proof. As each sub-segment is contained in the same segment of a bisector or edge of $bd(\mathcal{P})$, the statement follows analogously to Theorem 2 in [5] using Theorem 1. □

Note that each cell is a convex polygon. For two structurally identical sub-segments $S \in \mathcal{S}_i(\mathcal{A}_Z)$ and $S' \in \mathcal{S}_i(\mathcal{A}_{Z'})$, we call $\Delta(S) \in \mathcal{T}(\mathcal{A}_Z)_i$ and $\Delta(S') \in \mathcal{T}(\mathcal{A}_{Z'})_i$ *matching* elementary triangles. Observe that S and S' must have the same slope. This gives us the following result that ensures that condition (PRE) holds for partition $\mathcal{U}$.

Lemma 7 *Let $Z, Z' \in \mathcal{P} \setminus \mathcal{A}$, $Z \neq Z'$, and $Z, Z' \in int(\mathcal{CC}^{k_i}(A_i))$, $1 \leq k_i \leq 8$, $i = 1, \ldots, n$. Moreover, let $\mathcal{CP}_i(\mathcal{A}_Z)$ and $\mathcal{CP}_i(\mathcal{A}_{Z'})$ be structurally and quadrant identical, $i \in [1 : n+1]$. If $\Delta \in \mathcal{T}(\mathcal{A}_Z)_i$ and $\Delta' \in \mathcal{T}(\mathcal{A}_{Z'})_i$ are matching elementary triangles, then the total distance $d(A_i, \Delta)$ and area $\mu(\Delta)$ have the same parametric representation in the coordinates of the new facility as the total distance and area, respectively, for Δ'.*

Proof. Follows from Definition 5, Lemma 3 and the representation of $d(A_i, \Delta)$ and $\mu(\Delta)$ from [5] as a function of the coordinates of the corners of Δ and the intersection points between the line containing $S(\Delta)$ with the horizontal and vertical lines through A_i. □

Summing up, we have:

Theorem 2 *Let $\mathcal{U}$ be the partition of $\mathcal{P}$ into cells induced by the set of all configuration, sub-intersection, and quadrant lines. Let $\mathcal{C}$ be a cell of $\mathcal{U}$. Then $\mathcal{C}$ is a convex polygon satisfying conditions (SE), (ALE) and (PRE). Thus, the functions $F_M(\mathcal{A}_Z)$ and $F_{MS}(\mathcal{A}_Z)$ are a cubic and, respectively, a quadratic polynomials in the coordinates of Z that are the same for any $Z \in int(\mathcal{C})$.*

Proof. The claim follows from Lemmas 6 and 7 using the arguments similar to those in the proof of Theorem 3 in [5]. □

Remark 1 *We note that the obtained partition has larger cardinality than the partition used in [5] since instead of vertices and edges of a single incremental Voronoi diagram we have to work with corner-points and sub-segments that capture the necessary information about the whole family of potential incremental Voronoi diagrams. We also note that more direct attempts to use the approach of [5] would result in partitions and complexities much larger than those in our paper. If in the case of reliable facilities their Voronoi diagram characterizes customer allocation, then in the case of unreliable facilities a similar role is played by the ordered order-n Voronoi diagram (OOnVD) which is the partition of the polygon by the bisectors of all pairs of different facilities. The subregions (cells) defined by this partition have the following property: for all customers within a subregion, the order of the facilities according to their distances from the customers is the same. This means that regardless of which facilities will be operational, all customers from a subregion will patronize the same facility. So, a more direct attempt to extend the approach of [5] to the case of unreliable facilities would be to work with the OOnVD instead of the family of* *possible Voronoi diagrams. However, the OOnVD has a much larger size than a Voronoi diagram, as the OOnVD has $O(n^4+m)$ edges and vertices, as opposed to $O(n+m)$ for the Voronoi diagram. Moreover, this is greatly amplified when we try to decompose the solution space into subregions over which the new facility can vary without changing structurally the OOnVD: the number of such subregions will be much larger than the cardinality of the partition we use. Focusing instead on the family of potential incremental Voronoi diagrams, even though this family is exponential, we are able to work only with the sets of corner-points and sub-segments, since these sets capture all information about the family that is needed for our purposes. Conceptually, this is the main idea of the approach of this paper.*

We will discuss the solution procedure for the conditional median, anti-median and market share problems with unreliable facilities in the next section.

5 Solving the Conditional Location Problems

In the previous section we laid the groundwork for solution algorithms for the three conditional location problems introduced in Section 3. We are now ready to discuss the algorithms and their complexity in more detail.

5.1 Solving the Conditional Median Problem with Unreliable Facilities (CMPUF)

Let $\mathcal{C}$ be a cell of the partition $\mathcal{U}$. From Theorem 2 we know that $F_M(\mathcal{A}_Z)$ is a cubic polynomial in the coordinates of $Z = (x, y)$ that has the same representation everywhere in the interior of $\mathcal{C}$. Since $F_M(A_Z)$ is continuous in Z, the parametric representation also extends to the boundary of $\mathcal{C}$.

A minimum of $F_M(\cdot)$ over $\mathcal{C}$ is now either a solution of the first-order conditions or a point on the boundary of $\mathcal{C}$. As $F_M(\mathcal{A}_Z)$ is a cubic polynomial in the coordinates of Z, the partial derivatives of $F_M(\mathcal{A}_Z)$ at $Z = (x, y) \in int(\mathcal{C})$ are quadratic polynomials in x and y. Hence, we can solve the resulting first-order conditions by finding the zeros of the determinant of the Sylvester's matrix ([12]). The determinant of the Sylvester's matrix is a quartic polynomial in y with at most four real-valued roots that can be derived in closed form ([2]). Given a zero of the determinant, we substitute its value for y in one of the two first-order conditions to obtain at most two solutions for x, and then check whether the obtained points belong to $\mathcal{C}$. This gives us at most eight candidate points for local minima in the interior of each cell of $\mathcal{U}$.

Concerning the minima on the boundary of a cell $\mathcal{C} \in \mathcal{U}$, when Z moves along an edge of $\mathcal{C}$, we can express x or y as a linear function of the other variable. Then, $F_M(\mathcal{A})$ reduces to a cubic single-variable polynomial whose minimum is derived in closed form.

The best of the obtained candidate points in the interiors and the boundaries of all cells of $\mathcal{U}$ will be an optimal solution for CMPUF.

Computational complexity analysis.

If the explicit representation of $F_M(\mathcal{A}_Z)$ over a cell $\mathcal{C} \in \mathcal{U}$ is given, it takes a constant time to find the at most eight candidate points for local minima in $int(\mathcal{C})$ as well as the minima on any edge of $\mathcal{C}$ as described above. Because $\mathcal{U}$ has $O(n^2(n^2+m)^2)$ cells, edges, and vertices, a global optimum can be found in $O(n^2(n^2+m)^2)$ time if explicit representations of $F_M(\mathcal{A}_Z)$ over all cells of $\mathcal{U}$ are given. Since the representation of the objective function $F_M(\mathcal{A}_Z)$ does not change over the interior of a cell $\mathcal{C} \in \mathcal{U}$, we can find its explicit representation by choosing an arbitrary point $Z \in int(\mathcal{C})$ as a proxy. Thus, using the approach described in Section 3.4, Proposition 1 to evaluate the objective function for each cell, the total effort to solve the conditional reliable median problem would amount to $O(n^4(n^2+m)^3)$, and $O(n^3(n^2+m)^3)$ in the case of independent facility failures.

However, it is possible to do much better, if after computing the explicit representation of $F_M(\mathcal{A}_Z)$ for a single cell $\mathcal{C} \in \mathcal{U}$ we use this representation to derive the representation of $F_M(\mathcal{A}_Z)$ for a neighboring cell $\mathcal{C}'$ that shares an edge e with $\mathcal{C}$. For simplicity, we discuss the effect of crossing the lines that define $\mathcal{U}$ assuming that they do not coincide; when several such lines coincide, the arguments are still valid since the effects of crossing the coinciding lines would add up. If Z crosses a sub-intersection or quadrant line, only a constant number of corner-points are affected. In the former case, the corresponding bisector now intersects an adjacent sub-segment, affecting one corner-point and three elementary triangles. In the latter case, only the representation of the two triangles which have the corresponding vertex of $\mathcal{P}$ as a non-facility corner changes. If Z crosses a horizontal or vertical configuration line of a facility A_i, the representation of the bisector $\mathcal{B}(A_i, Z)$ changes. However, the bisector remains vertical or horizontal and intersects the same sub-segments as for $\mathcal{C}$. As $\mathcal{B}(A_i, Z)$ intersects another bisector $\mathcal{B}_{ij}$ at most once and $bd(\mathcal{P})$ at most twice, $O(n)$ corner-points and elementary triangles are affected by this change. Finally, if Z crosses a diagonal configuration line of an existing facility A_i, the bisector $\mathcal{B}(A_i, Z)$ changes in a major way, switching between a horizontal and a vertical representation. This, however, can affect only the $O(n^2+m)$ corner-points and elementary triangles of A_i and Z.

A change affecting one elementary triangle in the parametric representation of $F_M(\mathcal{A}_Z)$ will be called an *elementary change*. Updating the parametric representation of $F_M(\mathcal{A}_Z)$ after an elementary change can be done in $O(n)$ time under Assumption A and in $O(1)$ time in case of independent facility failures. Since the partition $\mathcal{U}$ is defined by $O(n(n^2+m))$

lines, each diagonal configuration line is crossed at most $O(n(n^2+m))$ times, resulting in at most $O(n(n^2+m)^2)$ elementary changes for all crossings of a diagonal configuration line of a facility A_i. As each existing facility has four configuration lines, we conclude from the previous discussion that the total number of elementary changes required for obtaining explicit representations of $F_M(\mathcal{A}_Z)$ for all cells of $\mathcal{U}$ can be limited by $O(n^2(n^2+m)^2)$. Summing up, we obtain

Theorem 3 *The conditional median problem with unreliable facilities can be solved in $O(n^3(n^2+m)^2) = O(n^7 + n^3m^2)$ time for a probability model satisfying Assumption A, and in $O(n^2(n^2+m)^2) = O(n^6 + n^2m^2)$ time in the case of independent facility failures.*

5.2 Solving the Conditional Anti-Median Problem with Unreliable Facilities

The objective functions of the median and anti-median problem are identical. The only differences between the two problems are that for the latter the distance between the additional facility and each existing facility should be at least $D > 0$ and that we seek the maximum instead of the minimum. To ensure the distance restriction, we add to our partition all bounding lines of the l_1-discs $\mathcal{K}^{\leq}(A_i, D)$. Then, we simply skip all cells of the partition that are contained in one of the discs when searching for the optimal solution. As this adds just $O(n)$ lines to the partition and the parametric representation of the objective function does not change when crossing such a line, we obtain

Theorem 4 *The CAMPUF can be solved in $O(n^7 + n^3m^2)$ time for a probability model satisfying Assumption A, and in $O(n^6 + n^2m^2)$ time in the case of independent facility failures.*

5.3 Solving the Conditional Reliable Market Share Problem

After replacing $d(A_i, \Delta)$ by $\mu(\Delta)$, we can use essentially the same solution approach for the conditional market share problem as for the conditional median problem, with one exception. In contrast to the median problem, the objective function is no longer continuous if Z reaches a diagonal configuration line of an existing facility of a competitor. The bisector then includes two quarter-planes. Recall that we assumed that in case of distance ties customers prefer our company to competitors, and the new facility belongs to our company. As a result, the value of the objective function $F_{MS}(\mathcal{A})$ at a point on a diagonal configuration line is never smaller than the limiting objective value when Z approaches this point from within the interior of the cell. Hence, the maximum over a cell exists and is either at a vertex of the cell, or at a point in the interior of an edge of the cell, or at a point in the interior of the cell. To find it, we start by solving the first order conditions to determine local maxima in the interior of the cell, if any exist. As $\mu(\Delta)$ is a quadratic polynomial in x and y, the conditions are just a system of two linear equations with two unknowns. Concerning the maxima on the edges of the cell, if the edge is not on a diagonal configuration line, we can express x or y as a linear (or constant) function of the other variable to obtain an expression for $F_{MS}(\cdot)$ on an edge as a quadratic single-variable polynomial. Then, it is easy to find a maximum on the edge. If the edge is contained in a diagonal configuration line, we replace the bisector by the boundary of the set $\{Q \in \mathcal{P} \mid l_1(Q, Z) \leq l_1(Q, A_i)\}$ which consists of a horizontal and a vertical half lines that are joined by a diagonal line. The effort to compute the new representation of the objective function is $O(n^2(n^2 + m))$ (including updating the probability terms) under Assumption A, and $O(n(n^2+m))$ in the case of independent facility failures, and is part of the effort to update the parametric representation of the objective function when crossing this configuration line to an adjacent cell.

Hence, using arguments similar to those used for the CMPUF, we obtain:

Theorem 5 *The CRMSP can be solved in $O(n^7 + n^3m^2)$ time when the probability model satisfies Assumption A, and in $O(n^6 + n^2m^2)$ time in the case of independent facility failures.*

6 Extensions, discussion and concluding remarks

In this paper, we presented a general exact optimization approach to conditional location problems with continuous planar demand, rectilinear distance, continuous location space and unreliable facilities, in the context of three specific location problems. Our approach is to extend the general methodology of [5], based on analyzing the structure of incremental Voronoi diagrams, to the case of unreliable facilities. The difficulty of the setting with unreliable facilities is that instead of a single Voronoi diagram, we have to deal with an exponential family of Voronoi diagrams that correspond to different possible realizations of the set of operational facilities. Thus, we need to work with an exponential family of incremental Voronoi diagrams. As in the case of [5], our approach is based on partitioning of the convex polygonal market region of the rectilinear plane into convex polygonal cells such that the elements of the family of incremental Voronoi diagrams that are important for computing the objective function remain structurally identical when the location of the new facility varies within one cell. This allowed us to decompose the problems into simpler subproblems within the cells of the partition, which resulted in polynomial exact algorithms. Since we have to work with a family of incremental Voronoi diagrams rather than with a single incremental Voronoi diagram, we have to introduce new objects such as corner-points and sub-segments instead of vertices and edges of the Voronoi diagram used in the case of reliable facilities [5]. We note that the partition that we obtained for the median, anti-median and the market share problems with unreliable facilities has larger cardinality than the partition used in [5] for problems with reliable facilities. In the process of deriving and justifying the partition, we obtained some structural results that are interesting and important on their own, for example that the structural identity of the incremental sets of corner-points of the existing facilities is sufficient for the structural identity of both the incremental set of corner-points and the incremental set of sub-segments of the new facility (Lemma 5).

An important take-away from the analysis is that the sets of corner-points and sub-segments introduced in the paper capture all necessary information about the exponential family of incremental Voronoi diagrams that is required for our algorithms. This allows us to avoid having unreasonably high orders of complexity, and justifies our use of the family of Voronoi diagrams instead of the ordered order-n Voronoi diagram which would result in a much finer partition of the polygon (see the discussion in Remark 1 after Theorem 2), and, respectively, much higher complexity.

In the paper, the problems were considered under a rather general probabilistic model where the probability of a specified partition of the set of facilities based on the operational status of the facilites (active, inactive and unrestricted) can be computed in linear time. This model allows for correlations between failure events at different facilities and, as discussed earlier, encompasses several practically important special cases (such as when failures of different facilities are independent or conditionally independent with respect to some background variable). Moreover, the main structural results we derive are independent of the probability model used, as they pertain to the analysis of the family of incremental Voronoi diagrams defined by all subsets of the set of facilities. For example, the partition $\mathcal{U}$ of the polygon $\mathcal{P}$ (defined by the set of all configuration, sub-intersection, and quadrant lines) would be relevant for any probabilistic model that unambiguously defines probabilities of different constellations, and Theorem 2 is valid for any such model. However, to use our approach for development of efficient algorithms for the considered conditional location problems, the probabilistic model should allow for an "efficient" computation of the coefficients of the polynomials mentioned in Theorem 2; our Assumption A is one, but by no means the only, mechanism for assuring this efficiency.

We also note that a variety of different objectives in the context of unreliable facilities can be handled using the framework of our approach. For example, the method and the obtained partition could be used for most problems where the objective is the integral of individual contributions of all customers, each customer uses the closest facility, and triangulation approach is applicable for computing the objective. However, for more complex objectives the subproblems over the cells may no longer be solvable analytically, and numerical methods would have to be applied. The same observation applies to extension of our results to the case of Euclidean distances. While the basic ideas of triangulation and partitioning should extend to these cases, the resulting non-linear optimization problems within each cell of the partition no longer admit an analytical solution and can only be solved numerically. Thus, exact polynomial algorithms do not appear to be possible in these cases.

The assumption that the demand is distributed uniformly is a limitation of the method. While the main geometric constructions of Sections 3 and 4 do not depend on the distribution of the demand, they become less useful for more complicated demand distributions. The reason is that for an elementary triangle Δ, values $\mu(\Delta)$ and $d(A_i, \Delta)$ may no longer have compact and simple parametric representation in terms of the coordinates of the new facility, which was the basis for the triangulation approach. Hence, the objective function within the cells of the partition may no longer be simple. The general framework could still be applicable for simple continuous demand distributions (e.g., when the demand distribution is uniform but with different densities within some polygonal subareas of the market area), but the details would require additional theory which is beyond the scope of this paper. We note that for the case of continuous non-uniform demand distributions, before investigating location problems with unreliable facilities, first problems with reliable facilities should be investigated, and there is no such a study at present. Hence, the case of continuous non-uniform demand distributions may be a direction for future research.

To close, we mention several interesting open problems. As mentioned in the introduction, actual measured customer demand distribution is usually discrete with very large cardinality of the support set. The typical approach in location models with discrete demand space is to aggregate individual demand points to obtain a much smaller support set. The alternative approach adopted in the current paper is to assume a continuous spatial distribution of demand. Clearly, both approaches lead to errors in approximating the actual demand distribution. A focused examination of these errors and a comparison between discrete and continuous demand space approaches would be of interest. Another implicit assumption in the current paper is that customers are aware of whether a facility has failed before starting their journey. An alternative mechanism is to assume that customers have to search for an operating facility - a model of this type (with discrete demand and location spaces) is described in Berman et al. [10]. It would be interesting to extend this modeling framework to the continuous demand and location space framework of the current paper.

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