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Citation for published version:

Durcik, P & Roos, J 2021, 'Averages of simplex Hilbert transforms', *Proceedings of the American Mathematical Society*, vol. 149, pp. 633-647. <https://doi.org/10.1090/proc/15196>

Digital Object Identifier (DOI):

[10.1090/proc/15196](https://doi.org/10.1090/proc/15196)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Proceedings of the American Mathematical Society

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AVERAGES OF SIMPLEX HILBERT TRANSFORMS

POLONA DURCIK AND JORIS ROOS

ABSTRACT. We study a multilinear singular integral obtained by taking averages of simplex Hilbert transforms. This multilinear form is also closely related to Calderón commutators and the twisted paraproduct. We prove L^p bounds in dimensions two and three and give a conditional result valid in all dimensions.

1. INTRODUCTION

In this paper we study the multilinear singular integral $\Lambda^{(n)}(F_0, \dots, F_n)$ defined by

$$\int_{\mathbb{R}^n} p.v. \int_{\mathbb{R}} \int_{[0,1]^{n-1}} F_0(x) \left(\prod_{j=1}^{n-1} F_j(x + t\alpha_j e_j) \right) F_n(x + te_n) d\alpha \frac{dt}{t} dx, \quad (1.1)$$

where e_1, \dots, e_n are the standard unit vectors in \mathbb{R}^n . We prove the following result.

Theorem 1. *Let n equal two or three. Let $p_0 \in (2^{n-1}, \infty)$, $p_1, \dots, p_n \in (1, \infty)$, and $\sum_{j=0}^n p_j^{-1} = 1$. If $n = 3$, assume in addition that $p_1^{-1} + p_2^{-1} > 2^{-1}$, $p_2^{-1} + p_3^{-1} > 2^{-1}$, and $p_1^{-1} + p_3^{-1} > 2^{-1}$. Then there exists a constant $c \in (0, \infty)$ such that*

$$|\Lambda^{(n)}(F_0, \dots, F_n)| \leq c \prod_{j=0}^n \|F_j\|_{p_j}. \quad (1.2)$$

The constant c depends on p_0, p_1, \dots, p_n . The estimate fails if $p_j = \infty$ for some $j = 1, \dots, n$.

If we drop the average over α in (1.1) and fix, say $\alpha = (1, \dots, 1)$, then we obtain the *simplex Hilbert transform*

$$\int_{\mathbb{R}^n} p.v. \int_{\mathbb{R}} F_0(x) \left(\prod_{j=1}^{n-1} F_j(x + te_j) \right) F_n(x + te_n) \frac{dt}{t} dx. \quad (1.3)$$

Note that for $n = 1$, both (1.1) and (1.3) reduce to a dualized form of the classical Hilbert transform. For $n \geq 2$, it is currently not known whether the multilinear form (1.3) satisfies any L^p estimates. This is a major open problem in harmonic analysis. In the case $n = 2$, the form (1.3) is also called *triangular Hilbert transform* and some partial progress has been made in [KTZ15], where bounds for a Walsh model are obtained, when one of the three functions takes a special form. L^p bounds for the triangular Hilbert transform would unify

Date: May 29, 2020.

2010 Mathematics Subject Classification. Primary 42B20; Secondary 42B15.

Key words and phrases. multilinear singular integrals, simplex Hilbert transform, paraproducts.

several known results in time-frequency analysis, such as uniform bounds for the bilinear Hilbert transform and bounds for Carleson's operator, which controls pointwise almost-everywhere convergence of Fourier series. More generally, bounds for the n -simplex Hilbert transform would imply bounds for the n -linear Hilbert transform

$$\int_{\mathbb{R}} p.v. \int_{\mathbb{R}} f_0(x) f_1(x+t) f_2(x+2t) \cdots f_n(x+nt) \frac{dt}{t} dx. \quad (1.4)$$

We refer to [KTZ15, Appendix B] and [Zor17, Appendix A] for the details of this implication. If $n = 2$, then the previous display represents a trilinear form dual to the bilinear Hilbert transform, which was first shown to satisfy L^p bounds by Lacey and Thiele [LT97], [LT99]. If $n \geq 3$, then it is not known if (1.4) satisfies any L^p bounds and this is considered to be a difficult open problem. An averaged form of the 3-linear Hilbert transform was studied by Palsson [Pal12]. Tao [Tao16] showed that truncations of the multilinear form (1.4) exhibit some non-trivial cancellation by using regularity lemmas from arithmetic combinatorics. For the more general simplex Hilbert transform (1.3), a similar result was obtained by Zorin-Kranich [Zor17] and this was strengthened subsequently in [DKT16].

The multilinear form (1.1) is closely related to the forms dual to *Calderón commutators*, which can be written as

$$\int_{\mathbb{R}} p.v. \int_{\mathbb{R}} \int_{[0,1]^{n-1}} f_0(x) \left(\prod_{j=1}^{n-1} f_j(x+t\alpha_j) \right) f_n(x+t) d\alpha \frac{dt}{t} dx \quad (1.5)$$

for a suitable choice of the functions f_j . In fact, L^p bounds for (1.1) imply the same L^p bounds for (1.5). This follows in the same way as bounds for the simplex Hilbert transform would imply bounds for (1.4). Calderón commutators played a key role in the resolution of the problem of bounding the Cauchy integral along Lipschitz curves by Coifman, McIntosh and Meyer [CMM82]. More recently, Muscalu [Mus14a], [Mus14b] developed an alternative approach for bounding Calderón commutators, which also inspired our proof of Theorem 1. We remark that in view of the connection with the Cauchy integral, the focus for bounding commutators often lies in obtaining L^∞ bounds and a constant that depends polynomially on n . In sharp contrast to this, the estimate (1.2) fails if $p_j = \infty$ for some $j \in \{1, \dots, n\}$. We prove this in §4.

Another object that is intimately related to (1.1) is the *twisted paraproduct*

$$\int_{\mathbb{R}^4} F_0(x, y) F_1(x-t, y) F_2(x, y-s) K(t, s) d(t, s, x, y), \quad (1.6)$$

where K is a Calderón–Zygmund kernel on \mathbb{R}^2 . This operator first arose as a degenerate case of the two-dimensional bilinear Hilbert transform that did not fall within the scope of Demeter and Thiele's time-frequency techniques [DT10]. L^p bounds were obtained by Kovač [Kov12], who used a novel argument involving repeated use of the Cauchy–Schwarz inequality and a telescoping identity. These techniques have since been further developed and applied to other problems (including the application to the simplex Hilbert transform [DKT16] which we have already mentioned).

Let m be a bounded function on $\mathbb{R}^n \setminus \{0\}$. Then we define

$$\Lambda_m(F_0, \dots, F_n) = \int_{(\mathbb{R}^n)^n} \widehat{F_0}(-(\tau_1 + \dots + \tau_n)) \left(\prod_{j=1}^n \widehat{F_j}(\tau_j) \right) m(\tau_{11}, \dots, \tau_{nn}) d\tau, \quad (1.7)$$

where $\tau = (\tau_1, \dots, \tau_n) \in (\mathbb{R}^n)^n$ and $\tau_j = (\tau_{j1}, \dots, \tau_{jn}) \in \mathbb{R}^n$. The key property here is that the multiplier only depends on the diagonal frequency variables $\tau_{11}, \dots, \tau_{nn}$. The form (1.6) is obtained by specializing $n = 2$. If we set

$$\mu_n(\xi) = \int_{[0,1]^{n-1}} \operatorname{sgn}(\alpha_1 \xi_1 + \dots + \alpha_{n-1} \xi_{n-1} + \xi_n) d\alpha, \quad (1.8)$$

then Λ_{μ_n} coincides with our multilinear form (1.1) (up to a multiplicative constant). The symbol (1.8) also coincides with the symbol of the Calderón commutator (1.5). The function μ_n is continuous on $\mathbb{R}^n \setminus \{0\}$ but fails to be differentiable on several hyperplanes. Some details and further comments on the function μ_n are contained in §5.

We call a bounded function $m \in C^\infty(\mathbb{R}^n \setminus \{0\})$ a *standard smooth symbol* if for every $\alpha \in \mathbb{N}_0^n$ there exists $c_\alpha \in (0, \infty)$ such that for every $\xi \in \mathbb{R}^n \setminus \{0\}$ we have

$$|\partial^\alpha m(\xi)| \leq c_\alpha |\xi|^{-|\alpha|}. \quad (1.9)$$

Notice that μ_n is not a standard smooth symbol. Moreover, note that even if m is a standard smooth symbol, the multilinear Fourier multipliers of the form (1.7) do not belong to the multilinear Calderón–Zygmund paradigm as set forth in [GT02].

Our proof of Theorem 1 relies on the estimate

$$|\Lambda_m(F_0, \dots, F_n)| \leq c \prod_{j=0}^n \|F_j\|_{p_j} \quad (1.10)$$

for standard smooth symbols m with the constant c only depending on m . In fact, we prove the following conditional result.

Theorem 2. *Let $n \geq 2$. If the estimate (1.10) holds for all standard smooth symbols m and some fixed exponents $p_0, \dots, p_n \in (1, \infty)$, then (1.2) holds (for the same exponents).*

In the case $n = 2$, Kovač [Kov12] showed that (1.10) holds for all $p_0 \in (2, \infty), p_1, p_2 \in (1, \infty)$ satisfying $p_0^{-1} + p_1^{-1} + p_2^{-1} = 1$.

In the case $n = 3$ we are still able to use the techniques from [Kov12] to prove (1.10) for all $p_0, p_1, p_2, p_3 \in (1, \infty)$ satisfying $p_0^{-1} + p_1^{-1} + p_2^{-1} + p_3^{-1} = 1, p_1^{-1} + p_2^{-1} > 2^{-1}, p_2^{-1} + p_3^{-1} > 2^{-1}$, and $p_1^{-1} + p_3^{-1} > 2^{-1}$. Indeed, the proof is by reduction to a result for a certain dyadic form considered in [Kov12], combined with a transition from the dyadic to the continuous case and fiber-wise Calderón–Zygmund decomposition by Bernicot [Ber12]. These arguments are well-known, but for convenience of the reader some details are given in §3.

We currently do not have the estimate (1.10) for any $n \geq 4$ (for any tuple of exponents $(p_j)_j$). This problem is closely related to the question of extending the range of exponents for the twisted paraproduct (1.6), as well as proving L^p estimates for the integrals studied in [DT18], in the case when the index set is an arbitrary subset of the cube in \mathbb{R}^n . However, in §2 we show that (1.2) holds provided that (1.10) holds.

Organization of the paper. In §2 we prove Theorem 2. In §3 we prove the estimate (1.10) for $n = 3$. In §4 we give a counterexample to (1.2) if $p_j = \infty$ for some $j \in \{1, \dots, n\}$. In §5 we make some comments about the symbol μ_n .

Acknowledgment. We thank Vjeko Kovač for helpful comments on a preliminary draft and Lenka Slavíková for pointing out an error in a related work that was relevant to a previous version of this paper.

2. MAIN ARGUMENT

In this section we fix exponents $p_0, \dots, p_n \in (1, \infty)$ and prove that (1.2) holds under the assumption that for all standard smooth symbols m we have

$$|\Lambda_m(F_0, \dots, F_n)| \lesssim \prod_{j=0}^n \|F_j\|_{p_j}, \quad (2.1)$$

where Λ_m is defined in (1.7).

First we make some comments about terminology. We say that a function ψ of one real variable is *of ψ -type* if it is a smooth function compactly supported in an interval not containing the origin. The letter ψ will always denote a ψ -type function supported on $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ such that $\sum_{k \in \mathbb{Z}} \psi(2^{-k}\eta) = 1$ for all $\eta \neq 0$. We also reserve the letters $\tilde{\psi}, \psi^1, \psi^2, \dots$ for other ψ -type functions, which may change throughout the text.

Similarly, a function φ of one real variable is *of φ -type* if it is a smooth function that is compactly supported. The letters $\varphi, \tilde{\varphi}, \varphi^1, \varphi^2, \dots$ are reserved for φ -type functions, which may change at different stages of the argument.

If φ is a function and $k \in \mathbb{Z}$ an integer, then we write $\varphi_k(\eta) = \varphi(2^{-k}\eta)$.

We start by decomposing \mathbb{R}^n into a family of Whitney boxes. Then we have

$$1 = \sum_{k_1, \dots, k_n \in \mathbb{Z}} \psi_{k_1}(\xi_1) \cdots \psi_{k_n}(\xi_n)$$

for almost every $\xi \in \mathbb{R}^n$. We split the sum over $(k_1, \dots, k_n) \in \mathbb{Z}^n$ according to which of the components k_j is the largest:

$$1 = \sum_{k_1 \geq k_2, \dots, k_n} (\cdots) + \sum_{k_2 \geq k_1, k_3, \dots, k_n, k_2 \neq k_1} (\cdots) + \cdots + \sum_{k_n > k_1, \dots, k_{n-1}} (\cdots). \quad (2.2)$$

For each of the n terms in the previous display, we perform a further decomposition. Here we have that k_1 is no smaller than k_2, \dots, k_n . We want to distinguish the case that k_1 is *much* larger than all of the remaining integers k_2, \dots, k_n from the case that k_1 is roughly equal to at least one of the numbers k_2, \dots, k_n . To be more precise, we fix a large integer $\ell_0 \gg 1$. For example, $\ell_0 = 100 \log(n)$ will do. Then decompose the corresponding summation as

$$\sum_{k_1 \geq k_2, \dots, k_n} (\cdots) = \sum_{k_1 - \ell_0 \geq k_2, \dots, k_n} (\cdots) + \sum_{\substack{k_1 \geq k_2, \dots, k_n, \\ \exists j \in \{2, \dots, n\} \text{ s.t. } k_1 - \ell_0 < k_j}} (\cdots). \quad (2.3)$$

We say that the first of these two terms is *of ψ - φ -type*. This is because we can sum in k_2, \dots, k_n to obtain

$$\begin{aligned} \sum_{k_1 - \ell_0 \geq k_2, \dots, k_n} \psi_{k_1}(\xi_1) \cdots \psi_{k_n}(\xi_n) &= \sum_{k_1 \in \mathbb{Z}} \psi_{k_1}(\xi_1) \prod_{j=2}^n \sum_{k_j \leq k_1 + \ell_0} \psi_{k_j}(\xi_j) \\ &= \sum_{k \in \mathbb{Z}} \psi_k(\xi_1) \varphi_k(\xi_2) \cdots \varphi_k(\xi_n). \end{aligned}$$

Similarly, the second term in (2.3) can be written as a sum over $O(n)$ terms of the form

$$\sum_{k \in \mathbb{Z}} \psi_k^1(\xi_1) \psi_k^{j_1}(\xi_{j_1}) \prod_{j \neq j_1} \varphi_k^j(\xi_j),$$

where ψ^j, φ^j are ψ - and φ -type functions, respectively. We refer to such a term as being *of ψ - ψ -type*.

Summarizing, we obtain a partition of unity into $O(n^2)$ terms each of which is either of ψ - φ -type or of ψ - ψ -type. Now we write our multilinear form as

$$\int_{(\mathbb{R}^n)^n} \widehat{F}_0(-(\tau_1 + \cdots + \tau_n)) \widehat{F}_1(\tau_1) \cdots \widehat{F}_n(\tau_n) \mu_n(\tau_{11}, \dots, \tau_{nn}) d(\tau_1, \dots, \tau_n), \quad (2.4)$$

and split the symbol μ_n into $O(n^2)$ pieces according to the aforementioned partition of unity. At this point our argument splits into three cases.

Case 1: the ψ - ψ case. Symbols of ψ - ψ -type can be handled using an argument similar to Muscalu's alternative proof [Mus14b] of the Coifman–McIntosh–Meyer theorem. This is detailed in §2.1.

Case 2: the smooth ψ - φ case. For the ψ - φ -type pieces we distinguish two separate cases. The first scenario is that the ψ falls onto the n th frequency variable (corresponding to the last term in (2.2)). Here we are concerned with the symbol

$$\sum_{k \in \mathbb{Z}} \mu_n(\xi) \varphi_k(\xi_1) \cdots \varphi_k(\xi_{n-1}) \psi_k(\xi_n), \quad (2.5)$$

where φ is supported in $[-2^{-\ell_0+1}, 2^{-\ell_0+1}]$. Recalling that ψ is supported in $[-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ and ℓ_0 is large, observe that the support of (2.5) is disjoint from each of the bad hyperplanes

$$\{\xi \in \mathbb{R}^n : \alpha_1 \xi_1 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n = 0\} \quad (\alpha \in \{0, 1\}^{n-1}).$$

Thus, μ_n is smooth on the support of (2.5) (see Lemma 5.1) and therefore the desired bounds for the multilinear form with symbol (2.5) follow from (2.1).

Case 3: the rough ψ - φ case. It remains to consider ψ - φ -type pieces where ψ falls onto one of ξ_1, \dots, ξ_{n-1} . By symmetry, we may restrict our attention to the symbol

$$\sum_{k \in \mathbb{Z}} \mu_n(\xi) \psi_k(\xi_1) \varphi_k(\xi_2) \cdots \varphi_k(\xi_n), \quad (2.6)$$

where φ is supported in a small neighborhood of the origin. The support of this symbol is contained in a narrow cone around the ξ_1 -axis and it has significant intersection with the singularities of μ_n . In this case the idea is to reduce to a symbol that vanishes on the ξ_1 -axis

and then consider an appropriate lacunary decomposition with respect to that axis, where each of the pieces is a ψ - ψ -type symbol which can be handled as in Case 1. This argument is detailed in §2.2. We note that by inspection of the multiplier a simpler argument is available if $n = 2$.

2.1. The ψ - ψ case. In this section we bound the multilinear form corresponding to a symbol of the form

$$\sum_{k \in \mathbb{Z}} \mu_n(\xi) \psi_k^{j_0}(\xi_{j_0}) \psi_k^{j_1}(\xi_{j_1}) \prod_{j \neq j_0, j_1} \varphi_k^j(\xi_j),$$

where $1 \leq j_0 < j_1 \leq n$ and the ψ^j, φ^j are of ψ - and φ -type, respectively.

Based on the symmetries of the function μ_n , we need to distinguish two cases: $(j_0, j_1) = (1, 2)$ and $(j_0, j_1) = (1, n)$ (they are different only if $n > 2$). However, the arguments we use for these two cases are, *mutatis mutandis*, identical. Thus, we will restrict our attention here to the case $(j_0, j_1) = (1, n)$. To improve readability we also suppose that the various ψ - and φ -type functions are identical. This is again no loss of generality. With these reductions in mind, we are now left with the symbol

$$\sum_{k \in \mathbb{Z}} \mu_n(\xi) \psi_k(\xi_1) \varphi_k(\xi_2) \cdots \varphi_k(\xi_{n-1}) \psi_k(\xi_n),$$

where ψ is a smooth function supported in $[1/2, 2] \cup [-2, -1/2]$ and φ is a smooth function supported in $[-2, 2]$. Recall that

$$\mu_n(\xi) = \int_{[0,1]^{n-1}} \operatorname{sgn}(\alpha_1 \xi_1 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n) d\alpha$$

Our proof follows the argument of Muscalu [Mus14b]. The argument simplifies because we do not need to make use of the non-compact Littlewood–Paley projections used in [Mus14b]. The key observation that we will use (due to [Mus14b]) is that

$$\mu_n(\xi) = \int_{[0,1]^{n-2}} \mu_2(\xi_1, \alpha_2 \xi_2 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n) d(\alpha_2, \dots, \alpha_{n-1}).$$

This will allow us to make use of the fact that the Fourier coefficients of μ_2 , when restricted to a Whitney box, decay quadratically (see Lemma 2.1 below). For μ_n we seem to get only linear decay (see [Mus14a]). We introduce the variable

$$\eta = \eta_{\alpha, \xi} = \alpha_2 \xi_2 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n$$

and rewrite the quantity $\mu(\xi)$ as

$$\int_{[0,1]^{n-2}} \sum_{k \in \mathbb{Z}} \rho(2^{-k}(\xi_1, \eta)) \psi_k(\xi_1) \varphi_k(\xi_2) \cdots \varphi_k(\xi_{n-1}) \psi_k(\xi_n) d(\alpha_2, \dots, \alpha_{n-1}), \quad (2.7)$$

where we have set

$$\rho(\xi_1, \eta) = \mu_2(\xi_1, \eta) \tilde{\psi}(\xi_1) \tilde{\varphi}(\eta).$$

By the Fourier inversion formula we have

$$\rho(\xi_1, \eta) = \int e^{-2\pi i(\xi_1 u + \eta v)} \check{\rho}(u, v) d(u, v). \quad (2.8)$$

At this point we record the following estimate for the function $\check{\rho}$.

Lemma 2.1. *For every $N \in \mathbb{N}$, $u, v \in \mathbb{R}$ we have*

$$|\check{\rho}(u, v)| \lesssim_N (1 + |v|)^{-2} ((1 + |v - u|)^{-N} + (1 + |u|)^{-N}). \quad (2.9)$$

This is obtained by writing $\check{\rho}(u, v) = \int e^{2\pi i(\xi_1 u + \eta v)} \mu_2(\xi_1, \eta) \tilde{\psi}(\xi_1) \tilde{\varphi}(\eta) d(\xi_1, \eta)$ and first integrating by parts twice with respect to η and then N times with respect to ξ_1 . For the details we refer to [Mus14a, Lemma 2.4]. Alternatively, the reader may look ahead at the details of our proof of (2.19) below, which is a slight variant of this estimate.

Note that the right hand side of (2.9) is integrable in $u, v \in \mathbb{R}$. Let us for the moment fix $\alpha_2, \dots, \alpha_{n-1} \in [0, 1]$, $u, v \in \mathbb{R}$ and $k \in \mathbb{Z}$. After expanding $\rho(\xi_1, \eta)$ according to (2.8), the corresponding contribution to our multilinear form (2.4) becomes

$$\begin{aligned} & \int_{(\mathbb{R}^n)^n} \left(\widehat{F}_1(\tau_1) \psi_k(\tau_{11}) e^{-2\pi i 2^{-k} \tau_{11} u} \right) \prod_{j=2}^{n-1} \left(\widehat{F}_j(\tau_j) \varphi_k(\tau_{jj}) e^{-2\pi i 2^{-k} \alpha_{jj} \tau_{jj} v} \right) \\ & \left(\widehat{F}_n(\tau_n) \psi_k(\tau_{nn}) e^{-2\pi i 2^{-k} \tau_{nn} v} \right) \widehat{F}_0(-(\tau_1 + \dots + \tau_n)) d(\tau_1, \dots, \tau_n), \end{aligned} \quad (2.10)$$

Expanding the Fourier transform \widehat{F}_0 , the previous display becomes

$$\begin{aligned} & \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \widehat{F}_1(\tau_1) \psi_k(\tau_{11}) e^{-2\pi i(2^{-k} \tau_{11} u + \tau_{11} x)} d\tau_1 \right) \prod_{j=2}^{n-1} \left(\int_{\mathbb{R}^n} \widehat{F}_j(\tau_j) \varphi_k(\tau_{jj}) e^{-2\pi i(2^{-k} \alpha_{jj} \tau_{jj} v + \tau_{jj} x)} d\tau_j \right) \\ & \left(\int_{\mathbb{R}^n} \widehat{F}_n(\tau_n) \psi_k(\tau_{nn}) e^{-2\pi i(2^{-k} \tau_{nn} v + \tau_{nn} x)} d\tau_n \right) F_0(x) dx, \end{aligned}$$

which by the Fourier inversion formula is equal to

$$\int_{\mathbb{R}^n} F_0(x) (F_1 *_1 \check{\psi}_k^{(u)})(x) \prod_{j=2}^{n-1} (F_j *_j \check{\varphi}_k^{(\alpha_j v)})(x) (F_n *_n \check{\psi}_k^{(v)})(x) dx.$$

Here $*_j$ denotes convolution in the j th variable and $f^{(u)}(x) = f(x + u)$ denotes translation. Summing in $k \in \mathbb{Z}$ we obtain the multilinear form

$$\int_{\mathbb{R}^n} F_0(x) \sum_{k \in \mathbb{Z}} (F_1 *_1 \check{\psi}_k^{(u)})(x) \prod_{j=2}^{n-1} (F_j *_j \check{\varphi}_k^{(\alpha_j v)})(x) (F_n *_n \check{\psi}_k^{(v)})(x) dx. \quad (2.11)$$

Using the triangle inequality and the Cauchy–Schwarz inequality we estimate the absolute value of (2.11) by

$$\begin{aligned} & \int_{\mathbb{R}^n} |F_0(x)| \left(\sum_{k \in \mathbb{Z}} |(F_1 *_1 \check{\psi}_k^{(u)})(x)|^2 \right)^{1/2} \prod_{j=2}^{n-1} \sup_{k \in \mathbb{Z}} |(F_j *_j \check{\varphi}_k^{(\alpha_j v)})(x)| \\ & \left(\sum_{k \in \mathbb{Z}} |(F_n *_n \check{\psi}_k^{(v)})(x)|^2 \right)^{1/2} dx \end{aligned} \quad (2.12)$$

At this point we need to make use of the following well-known bounds for *shifted maximal functions* and *shifted square-functions*.

Lemma 2.2. *Let $u \in \mathbb{R}$, φ a smooth bump function and ψ a ψ -type function. Then we have*

$$\left\| \sup_{k \in \mathbb{Z}} |f * \widetilde{\varphi}_k^{(u)}| \right\|_p \lesssim \log(2 + |u|) \|f\|_p \quad (2.13)$$

and

$$\left\| \left(\sum_{k \in \mathbb{Z}} |f * \widetilde{\psi}_k^{(u)}|^2 \right)^{1/2} \right\|_p \lesssim \log(2 + |u|) \|f\|_p \quad (2.14)$$

for every $p \in (1, \infty)$.

For the proof of (2.13) we refer to [Mus14a, Theorem 4.1] (it is also contained in [Ste93]) and for the proof of (2.14) see [Mus14a, Theorem 5.1]. Using Hölder's inequality, (2.13), (2.14) and Fubini's theorem, we see that (2.12) is bounded by a constant times

$$\prod_{j=0}^n \|F_j\|_{p_j} \quad (2.15)$$

for all tuples (p_0, \dots, p_n) satisfying $\sum_{j=0}^n p_j^{-1} = 1$, $p_j \in (1, \infty)$ ($p_j = \infty$ is also allowed unless $j \in \{1, n\}$) and the constant takes the form

$$C \log(2 + |u|) \log(2 + |v|),$$

where C is an absolute constant. Integrating over $u, v \in \mathbb{R}$, $\alpha_2, \dots, \alpha_{n-1} \in [0, 1]$ and making use of Lemma 2.1 we obtain the desired bounds for our multilinear form.

2.2. The rough ψ - φ case. In this section we deal with the symbol

$$\sum_{k \in \mathbb{Z}} \mu_n(\xi) \psi_k(\xi_1) \varphi_k(\xi_2) \cdots \varphi_k(\xi_n).$$

The difficulty is that the support of this function intersects some of the hyperplanes on which μ_n fails to be differentiable. The first step is again to use

$$\mu_n(\xi) = \int_{[0,1]^{n-2}} \mu_2(\xi_1, \alpha_2 \xi_2 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n) d(\alpha_2, \dots, \alpha_{n-1}).$$

We again consider the variable

$$\eta = \eta_{\alpha, \xi} = \alpha_2 \xi_2 + \cdots + \alpha_{n-1} \xi_{n-1} + \xi_n.$$

Fixing $\alpha_2, \dots, \alpha_{n-1} \in [0, 1]$, we now study the symbol

$$\sum_{k \in \mathbb{Z}} \mu_2(\xi_1, \eta) \psi_k(\xi_1) \prod_{j=2}^n \varphi_k(\xi_j).$$

We split $\psi = \psi^+ + \psi^-$ such that ψ^+ is supported on $\{\eta > 0\}$ and ψ^- is supported on $\{\eta < 0\}$. Note that ψ^+ and ψ^- are smooth. We split the symbol accordingly into two pieces. By symmetry it suffices to consider the symbol

$$\sum_{k \in \mathbb{Z}} \mu_2(\xi_1, \eta) \psi_k^+(\xi_1) \prod_{j=2}^n \varphi_k(\xi_j).$$

Using (2.1), we may subtract the smooth symbol $\sum_{k \in \mathbb{Z}} \psi_k^+(\xi_1) \prod_{j=2}^n \varphi_k(\xi_j)$ so that we are now concerned with

$$\sum_{k \in \mathbb{Z}} \tilde{\mu}_2(\xi_1, \eta) \psi_k^+(\xi_1) \prod_{j=2}^n \varphi_k(\xi_j),$$

where we have set

$$\tilde{\mu}(\xi_1, \eta) = \mu_2(\xi_1, \eta) - 1 = -2 \int_0^1 \mathbf{1}_{<0}(\xi_1 \alpha + \eta) d\alpha.$$

Now we perform another paraproduct decomposition. More precisely, we expand each of the $\varphi_k(\xi_j)$ into an appropriate sum of ψ -type functions and distinguish $n - 1$ terms based on which of the summation indices is the largest (as in (2.2)). With this in mind it suffices to consider the symbol

$$\sum_{\ell > \ell_0} \left(\sum_{k \in \mathbb{Z}} \tilde{\mu}(\xi_1, \eta) \psi_k^+(\xi_1) \psi_{k-\ell}(\xi_2) \prod_{j=3}^n \varphi_{k-\ell}(\xi_j) \right)$$

In the following we will fix $\ell > \ell_0$ and bound the multilinear form corresponding to the inner sum with summable decay in ℓ . Shifting the index k and using homogeneity of μ we can write the inner sum as

$$\sum_{k \in \mathbb{Z}} \tilde{\mu}(2^{-k}(\xi_1, \eta)) \psi_{k+\ell}^+(\xi_1) \psi_k(\xi_2) \prod_{j=3}^n \varphi_k(\xi_j) \quad (2.16)$$

Observe that on the support of each summand we have

$$|\eta| \leq |\xi_2| + \cdots + |\xi_{n-1}| + |\xi_n| \leq 2n \cdot 2^{-k}.$$

Choose a smooth function $\tilde{\varphi}$ such that it equals 1 on the interval $[-2n, 2n]$ and is equal to 0 on a slightly larger interval. Similarly, we choose a smooth function $\tilde{\psi}$ which is 1 on the interval $[\frac{1}{2}, 2]$ and equal to 0 on, say, $[\frac{1}{2} - \frac{1}{100}, 2 + \frac{1}{100}]$. Then (2.16) is equal to

$$\sum_{k \in \mathbb{Z}} \tilde{\mu}(2^{-k}(\xi_1, \eta)) \tilde{\psi}_{k+\ell}(\xi_1) \tilde{\varphi}_k(\eta) \psi_{k+\ell}^+(\xi_1) \psi_k(\xi_2) \prod_{j=3}^n \varphi_k(\xi_j),$$

which we rewrite as

$$\sum_{k \in \mathbb{Z}} m_\ell(2^{-k-\ell} \xi_1, 2^{-k} \eta) \psi_{k+\ell}(\xi_1) \psi_k(\xi_2) \prod_{j=3}^n \varphi_k(\xi_j), \quad (2.17)$$

where

$$m_\ell(\xi_1, \eta) = \tilde{\mu}(\xi_1, 2^{-\ell} \eta) \tilde{\psi}(\xi_1) \tilde{\varphi}(\eta).$$

Here we have used homogeneity of $\tilde{\mu}$. By the Fourier inversion formula we have

$$m_\ell(2^{-k-\ell} \xi_1, 2^{-k} \eta) = \int_{\mathbb{R}^2} e^{-2\pi i(u 2^{-k-\ell} \xi_1 + iv 2^{-k} \eta)} \tilde{m}_\ell(u, v) d(u, v). \quad (2.18)$$

Suppose for a moment that we can prove the decay estimate

$$|\tilde{m}_\ell(u, v)| \lesssim_N 2^{-\ell} (1 + |v|)^{-2} (1 + |u|)^{-N} \quad (2.19)$$

for all $\ell > \ell_0$. Then we insert (2.18) into (2.17) and fix $u, v \in \mathbb{R}$. The corresponding contribution to the multilinear form has the symbol

$$\sum_{k \in \mathbb{Z}} \left(\psi_{k+\ell}(\xi_1) e^{-2\pi i u 2^{-(k+\ell)} \xi_1} \right) \left(\psi_k(\xi_2) e^{-i v 2^{-k} \alpha_2 \xi_2} \right) \left(\prod_{j=3}^{n-1} \varphi_k(\xi_j) e^{-i v 2^{-k} \alpha_{n-1} \xi_{n-1}} \right) \left(\varphi_k(\xi_n) e^{-i v 2^{-k} \xi_n} \right).$$

Repeating the same argument as in the ψ - ψ case (more specifically, repeating the steps indicated from (2.10) to (2.15)), we obtain a bound that depends only logarithmically on u, v and does not depend on ℓ (it does not depend on ℓ , because one can shift each index k appearing in (2.12) separately). The estimate (2.19) then allows us to sum these bounds in ℓ and integrate in $u, v \in \mathbb{R}$. It now only remains to verify (2.19). We write

$$\widetilde{m}_\ell(u, v) = \int_{\mathbb{R}^2} e^{2\pi i (u \xi_1 + v \eta)} \widetilde{\mu}(\xi_1, 2^{-\ell} \eta) \widetilde{\psi}(\xi_1) \widetilde{\varphi}(\eta) d(\xi_1, \eta). \quad (2.20)$$

Note that on the support of the integrand we have $\xi_1 > 0$, $\eta < 0$ and $|2^{-\ell} \eta / \xi_1| < 1$ and for such (ξ_1, η) we have

$$\widetilde{\mu}(\xi_1, 2^{-\ell} \eta) = -2 \int_0^1 \mathbf{1}_{<0}(\alpha + 2^{-\ell} \eta / \xi_1) d\alpha = 2^{-\ell+1} \frac{\eta}{\xi_1}.$$

On the other hand, if $\eta > 0$, then $\widetilde{\mu}(\xi_1, 2^{-\ell} \eta) = 0$. Therefore (2.20) is equal to

$$2^{-\ell+1} \left(\int_{-\infty}^0 e^{2\pi i v \eta} \widetilde{\varphi}(\eta) \eta d\eta \right) \left(\int_{\mathbb{R}} e^{2\pi i u \xi_1} \widetilde{\psi}(\xi_1) \xi_1^{-1} d\xi_1 \right).$$

Integrating by parts N times in ξ_1 yields the decaying factor $(1 + |u|)^{-N}$. In the η variable we may only integrate by parts twice, which yields

$$\int_{-\infty}^0 e^{2\pi i v \eta} \widetilde{\varphi}(\eta) \eta d\eta = \frac{1}{(2\pi i v)^2} \left(-1 + \int_{-\infty}^0 e^{2\pi i v \eta} (\widetilde{\varphi}''(\eta) \eta + 2\widetilde{\varphi}'(\eta)) d\eta \right).$$

Thus we have verified (2.19).

3. AN ESTIMATE IN \mathbb{R}^3

In this section we indicate how to prove (1.10) in the case $n = 3$. The argument closely follows Kovač's proof for boundedness of the twisted paraproduct [Kov12] and here we only provide the necessary changes in his argument. By a standard cone decomposition (see [Thi06]) the symbol can be written as a certain superposition of symbols of the form

$$\begin{aligned} m_1(\xi_1, \xi_2, \xi_3) &= \sum_{k \in \mathbb{Z}} c_k \psi_k(\xi_1) \varphi_k(\xi_2) \varphi_k(\xi_3), \\ m_2(\xi_1, \xi_2, \xi_3) &= \sum_{k \in \mathbb{Z}} c_k \varphi_k(\xi_1) \psi_k(\xi_2) \varphi_k(\xi_3), \\ m_3(\xi_1, \xi_2, \xi_3) &= \sum_{k \in \mathbb{Z}} c_k \varphi_k(\xi_1) \varphi_k(\xi_2) \psi_k(\xi_3), \end{aligned}$$

where the coefficients c_k satisfy $|c_k| \leq 1$, φ is a φ -type function, and ψ a ψ -type function as in §2. Without loss of generality we will assume that $c_k = 1$ for all k (see [Kov12, §1]). Each

of m_1, m_2, m_3 is treated in the same way. We focus on m_3 . The corresponding multilinear form can be written as

$$\Lambda_{m_3}(F_0, F_1, F_2, F_3) = \langle T_c(F_1, F_2, F_3), F_4 \rangle,$$

where

$$T_c(F_1, F_2, F_3) = \sum_{k \in \mathbb{Z}} (F_1 *_1 \widetilde{\varphi}_k)(F_2 *_2 \widetilde{\varphi}_k)(F_3 *_3 \widetilde{\psi}_k),$$

where $*_j$ denotes convolution in the j th variable. We now pass to a dyadic model operator for T_c . By \mathcal{D} we denote the collection of dyadic intervals in \mathbb{R} . We define the Haar function ψ_I adapted to $I \in \mathcal{D}$ by

$$\psi_I = |I|^{-1/2}(\mathbf{1}_{I_L} - \mathbf{1}_{I_R}),$$

where I_L, I_R denote the left and right dyadic subintervals of I . Also write $\varphi_I = |I|^{-1/2}\mathbf{1}_I$. Denote dyadic martingale averages and differences by

$$\mathbb{E}_k f = \sum_{I \in \mathcal{D}, |I|=2^{-k}} \langle f, \varphi_I \rangle \varphi_I, \quad \Delta_k f = \sum_{I \in \mathcal{D}, |I|=2^{-k}} \langle f, \psi_I \rangle \psi_I = \mathbb{E}_{k+1} f - \mathbb{E}_k f.$$

Here $\langle f, g \rangle$ denotes $\int fg$. If F is a function on \mathbb{R}^3 and $j \in \{1, 2, 3\}$, we write $\mathbb{E}_k^j F, \Delta_k^j F$ to denote application of \mathbb{E}_k or Δ_k in the j th variable, respectively. We define a trilinear operator by

$$T_d(F_1, F_2, F_3) = \sum_{k \in \mathbb{Z}} (\mathbb{E}_k^1 F_1)(\mathbb{E}_k^2 F_2)(\Delta_k^3 F_3).$$

Let us suppose for a moment that we can prove

$$\|T_d(F_1, F_2, F_3)\|_{p'_0} \lesssim \prod_{j=1}^3 \|F_j\|_{p_j} \tag{3.1}$$

for (p_0, p_1, p_2, p_3) satisfying $\sum_{j=0}^3 p_j = 1$ and $p_0, p_1, p_2 \in (4, \infty)$, $p_3 \in [2, \infty)$ and p'_0 the Hölder dual exponent of p_0 . We can then use a square-function estimate due to Jones, Seeger and Wright [JSW08] comparing martingale averages with Littlewood–Paley projections to deduce the inequality

$$\|T_c(F_1, F_2, F_3)\|_{p'_0} \lesssim \prod_{j=1}^3 \|F_j\|_{p_j} \tag{3.2}$$

for (p_0, p_1, p_2, p_3) satisfying $\sum_{j=0}^3 p_j = 1$ and $p_0, p_1, p_2 \in (4, \infty)$, $p_3 \in [2, \infty)$. This argument is detailed in [Kov12, §6] in the case of the (two-dimensional) twisted paraproduct. Up to obvious modifications this argument also applies to our case, so we omit the details.

Using multilinear interpolation and the fiber-wise Calderón–Zygmund decomposition due to Bernicot [Ber12] we can extend the range of exponents in this estimate to obtain (3.2) for $p_0, p_1, p_2, p_3 \in (1, \infty)$ satisfying $\sum_{j=0}^3 p_j^{-1} = 1$, $p_0^{-1} < 2^{-2}$, $p_1^{-1} + p_2^{-1} > 2^{-2}$, $p_2^{-1} + p_3^{-1} > 2^{-1}$, and $p_1^{-1} + p_3^{-1} > 2^{-1}$. This follows in the same way as in Bernicot’s original argument up to obvious modifications, so we leave it out. By duality, this implies bounds for the form Λ_{m_3} . Arguing by symmetry, we also obtain corresponding bounds for Λ_{m_1} and Λ_{m_2} . Altogether, we obtain bounds for the form Λ_m associated with the original symbol m that are valid for

exponent tuples (p_0, p_1, p_2, p_3) that lie in the intersection of the three regions of exponents stemming from $\Lambda_{m_1}, \Lambda_{m_2}, \Lambda_{m_3}$, respectively. This region consists of all $(p_0, p_1, p_2, p_3) \in (1, \infty)^4$ so that $\sum_{j=0}^3 p_j^{-1} = 1$, $p_1^{-1} + p_2^{-1} > 2^{-1}$, $p_2^{-1} + p_3^{-1} > 2^{-1}$, and $p_1^{-1} + p_3^{-1} > 2^{-1}$.

It now only remains to address the validity of the estimate (3.1). This estimate follows from the techniques in [Kov12]. The details can be found in [Dur14] (see §4.4 there). For the reader's convenience we also sketch the estimate here in the special case that $p_3 = 2$. It can be proven in the local L^2 range $p_3 \in (2, \infty)$ by considering an appropriate local form as in [Kov12, §3]. For this purpose it is again convenient to consider a multilinear form. We write

$$\Lambda_d(F_0, F_1, F_2, F_3) = \langle T_d(F_1, F_2, F_3), F_0 \rangle.$$

Without loss of generality we may assume that the functions F_j are non-negative. This is by splitting both the real and imaginary parts into positive and negative parts each. Expanding the martingale averages and differences and using Fubini's theorem, we see that $\Lambda_d(F_0, F_1, F_2, F_3)$ is equal to

$$\sum_Q \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}} F_0(x, y, z) F_1(x', y, z) F_2(x, y', z) \psi_{I_3}(z) dz \right) \left(\int_{\mathbb{R}} F_3(x, y, z') \psi_{I_3}(z') dz' \right) \\ \varphi_{I_1}(x) \varphi_{I_1}(x') \varphi_{I_2}(y) \varphi_{I_2}(y') d(x, y, x', y'),$$

where the sum is over all dyadic cubes $Q = I_1 \times I_2 \times I_3 \subset \mathbb{R}^3$. By the Cauchy–Schwarz inequality this is no greater than

$$\left(\sum_Q \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}} F_0(x, y, z) F_1(x', y, z) F_2(x, y', z) \psi_{I_3}(z) dz \right)^2 \varphi_{I_1}(x) \varphi_{I_1}(x') \varphi_{I_2}(y) \varphi_{I_2}(y') d(x, y, x', y') \right)^{1/2} \quad (3.3)$$

times

$$\left(\sum_Q \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}} F_3(x, y, z') \psi_{I_3}(z') dz' \right)^2 \varphi_{I_1}(x) \varphi_{I_1}(x') \varphi_{I_2}(y) \varphi_{I_2}(y') d(x, y, x', y') \right)^{1/2}, \quad (3.4)$$

We will treat these two factors separately. Integrating in x', y' , (3.4) becomes

$$\left(\sum_{k \in \mathbb{Z}} \sum_{|I_3|=2^{-k}} \int_{\mathbb{R}^4} F_3(x, y, z) F_3(x, y, z') \psi_{I_3}(z) \psi_{I_3}(z') d(x, y, z, z') \right)^{1/2},$$

which equals $\|F_3\|_2$. On the other hand, by Theorem 6 from [Kov12] applied to the 6-tuple $(F_0, F_0, F_1, F_1, F_2, F_2)$ we obtain that the square of (3.3) is bounded by

$$\lesssim \|F_0\|_{p_0}^2 \|F_1\|_{p_1}^2 \|F_2\|_{p_2}^2,$$

where $\frac{1}{p_0} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$ and $p_0, p_1, p_2 \in (4, \infty)$.

4. ENDPOINT COUNTEREXAMPLE

Let $n \geq 2$ and $j_0 \in \{1, \dots, n\}$. Let $p_{j_0} = \infty$ and choose the other $(p_j)_j$ from $[1, \infty]$ such that $\sum_{j=0}^n p_j^{-1} = 1$. We claim that the estimate (1.2) fails. Let $N \gg 1$ and set

$$F_0(x) = \mathbf{1}_{(0,1)}(x_{j_0}) \prod_{j \neq j_0} \mathbf{1}_{(0,N)}(x_j), \quad F_{j_0}(x) = \mathbf{1}_{>0}(x_{j_0}),$$

$$F_j(x) = \mathbf{1}_{(0,1)}(x_{j_0}) \prod_{\ell \neq j_0} \mathbf{1}_{(-N,N)}(x_\ell) \quad \text{for } j \neq 0, j_0.$$

Clearly, $\|F_{j_0}\|_\infty = 1$ and $\|F_j\|_{p_j} \approx N^{(n-1)/p_j}$ for $j \neq j_0$. Thus, the right hand side of (1.2) equals N^{n-1} . By a straightforward computation the left hand side of (1.2) is comparable to $N^{n-1} \log(N)$. Letting $N \rightarrow \infty$ implies the claim.

5. SOME COMMENTS ON THE MULTIPLIER

In this section we are concerned with the function μ_n defined in (1.8). This function is the symbol of our multilinear form (1.1) and also coincides with the symbol of the Calderón commutator (1.5). Apart from the quadratic Fourier decay of μ_2 (see Lemma 2.1) the only additional information which we need in §2 is the following.

Lemma 5.1. *The function μ_n is smooth on the complement of the hyperplanes*

$$\{\xi \in \mathbb{R}^n : \alpha_1 \xi_1 + \dots + \alpha_{n-1} \xi_{n-1} + \xi_n = 0\} \tag{5.1}$$

where $\alpha \in \{0, 1\}^{n-1}$.

This is a routine verification. For convenience of the reader we include some details here. First, note the following explicit formula

$$\mu_n(\xi) = \frac{c_n}{\xi_1 \dots \xi_{n-1}} \sum_{\alpha \in \{0,1\}^{n-1}} (-1)^{|\alpha|} g_n \left(\xi_n + \sum_{j=1}^{n-1} \alpha_j \xi_j \right), \tag{5.2}$$

where $g_n(t) = |t|t^{n-2}$, $c_n = \frac{(-1)^{n+1}}{(n-1)!}$, and $\xi = (\xi_1, \dots, \xi_n)$, $\alpha = (\alpha_1, \dots, \alpha_{n-1})$. This formula holds if $\xi_1, \dots, \xi_{n-1} \neq 0$. If $k \geq 1$ of the numbers ξ_1, \dots, ξ_{n-1} are equal to zero, then

$$\mu_n(\xi) = \mu_{n-k}(\xi_*),$$

where ξ_* is the vector in \mathbb{R}^{n-k} obtained by removing the zero coordinates among ξ_1, \dots, ξ_{n-1} . This is proven by induction on n . Let us denote the union of the hyperplanes (5.1) by \mathbf{B} . Then $\mathbb{R}^n \setminus \mathbf{B}$ consists of a finite number of connected components each of which is a convex polytope that we call a *sector*. Each sector corresponds to a map $\sigma : \{0, 1\}^{n-1} \rightarrow \{-1, 1\}$ where $\sigma(\alpha)$ prescribes the sign of $\xi_n + \sum_{j=1}^{n-1} \alpha_j \xi_j$. On the closure of each such sector μ_n is given by a homogeneous rational function with no singularities in the closure of the sector. This in particular implies that μ_n is smooth on $\mathbb{R}^n \setminus \mathbf{B}$.

Remark. It is natural to ask how many of the $2^{2^{n-1}}$ maps σ correspond to (non-empty) sectors. To this end we observe that there is a bijection between sectors and linear boolean *threshold functions* on the $(n-1)$ -hypercube. A threshold function on the n -cube $\{-1, 1\}^n$

is a map $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ that takes the form $f(x) = \text{sgn}(a + \langle x, b \rangle)$ for some $a \in \mathbb{R}, b \in \mathbb{R}^n$. Threshold functions are of practical relevance in pattern recognition and have been studied extensively in the computer science literature [Mur71]. In particular, it has been shown that $2^{n^2+o(n^2)} \leq \#_n \leq 2^{n^2}$ (see [Mur71], [Zue89]), where $\#_n$ is the number of threshold functions on $\{-1, 1\}^n$. In view of the separating hyperplanes theorem, the number $\#_n$ also has a simple geometric interpretation: it equals the number of subsets A of vertices of the n -cube such that the convex hull of A is disjoint from the convex hull of the vertices not in A .

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