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QUANTIZATION OF CONTINUUM KAC–MOODY ALGEBRAS

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ABSTRACT. Continuum Kac–Moody algebras have been recently introduced by the authors and O. Schiffmann in [ASS18]. These are Lie algebras governed by a continuum root system, which can be realized as uncountable colimits of Borcherds–Kac–Moody algebras. In this paper, we prove that any continuum Kac–Moody algebra $\mathfrak{g}$ is canonically endowed with a non–degenerate invariant bilinear form. The positive and negative Borel subalgebras form a Manin triple with respect to this pairing, which allows to define on $\mathfrak{g}$ a topological quasi–triangular Lie bialgebra structure. We then construct an explicit quantization of $\mathfrak{g}$, which we refer to as a *continuum quantum group*, and we show that the latter is similarly realized as an uncountable colimit of Drinfeld–Jimbo quantum groups.

Dedicated to Prof. Kyoji Saito on the occasion of his 75th birthday.

CONTENTS

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1. Introduction

Continuum Kac–Moody algebras have been recently introduced by the authors and O. Schiffmann in [ASS18]. Their definition is similar to that of Kac–Moody algebras, but they are controlled by a *continuum* root system, arising from the combinatorics of connected intervals living in a one–dimensional topological space. They are not Kac–Moody algebras themselves, but they can be realized as uncountable colimits of symmetric (Borcherds–)Kac–Moody algebras.

In this paper, we provide a gentle introduction to this new theory, avoiding the technicalities of [ASS18], and we push further the study of these Lie algebras, providing two main contributions. First, we prove that continuum Kac–Moody algebras have a canonical structure of (topological) Lie bialgebras, which arises, as in the classical Kac–Moody case, from the construction of a non–degenerate invariant symmetric bilinear form. Then, we construct an explicit algebraic quantization of these topological structures, which we call *continuum quantum group*: they can be similarly realized as uncountable colimits of Drinfeld–Jimbo quantum groups. Moreover, we prove that, in the simplest cases of the line and the circle, they coincide with the quantum groups constructed *geometrically* in [SS17] by the second-named author and O. Schiffmann via the theory of Hall algebras. In [AKSS19], we adopt a similar approach to show that continuum quantum groups admit analogous geometric realizations arising from Hall algebras.

In the remaining part of this introduction, we shall explain our work in more detail.

The continuum Kac–Moody algebra. The defining datum of a continuum Kac–Moody algebra is a continuum analogue of a quiver, defined as follows. Recall that the latter is just an oriented graph $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ with set of vertices $\mathcal{Q}_0$ and a set of edges $\mathcal{Q}_1$. In a *continuum* quiver, the discrete set $\mathcal{Q}_0$ is replaced by a *vertex space* X, which is, roughly, a Hausdorff topological space locally modeled over $\mathbb{R}$ (cf. Definition 3.1). Examples of vertex spaces are the line $\mathbb{R}$, the circle $S^1 = \mathbb{R}/\mathbb{Z}$, smoothings of possibly infinite trees, or combinations of these. Thus, it is possible to lift the notion of *connected interval* from $\mathbb{R}$ to X, in such a way that the set of all possible *intervals in* X, denoted $\mathsf{Int}(X)$, is naturally endowed with two *partially defined* operations, that is, a sum $\oplus$, given by concatenation of intervals, and a difference $\ominus$, given by set difference whenever the outcome is again in $\mathsf{Int}(X)$.

The set $\mathsf{Int}(X)$ comes naturally equipped with a set–theoretic non–degenerate pairing $(\cdot|\cdot) : \mathsf{Int}(X) \times \mathsf{Int}(X) \to \mathbb{Z}$, defined as follows. On the space of locally constant, compactly supported, left–continuous functions on $\mathbb{R}$, we consider a non–symmetric bilinear form given by:

$$\langle f, g \rangle := \sum_x f_-(x)(g_-(x) - g_+(x)) \, .$$

This restricts to $\mathsf{Int}(\mathbb{R})$, by identifying an interval α with its characteristic function $\mathbb{1}_\alpha$. As before, we lift it from $\mathbb{R}$ to X by decomposing every interval in X into an iterated concatenation of *elementary* intervals in $\mathbb{R}$. Finally, we define the *Euler form* $(\mathbb{1}_\alpha|\mathbb{1}_\beta) := \langle \mathbb{1}_\alpha, \mathbb{1}_\beta \rangle + \langle \mathbb{1}_\beta, \mathbb{1}_\alpha \rangle$. Then, the *continuum quiver* of the vertex space X is precisely the datum $\mathcal{Q}_X := (\mathsf{Int}(X), \oplus, \ominus, \langle \cdot, \cdot \rangle, (\cdot|\cdot))$. Henceforth, we denote by $\mathfrak{f}_X$ the span of the characteristic functions $\mathbb{1}_\alpha$, $\alpha \in \mathsf{Int}(X)$.

Given a continuum quiver $\mathcal{Q}_X$, together with O. Schiffmann, we construct in [ASS18] a Lie algebra $\mathfrak{g}_X$, which we refer to as the *continuum Kac–Moody algebra of* $\mathcal{Q}_X$, whose Cartan subalgebra is generated by the characteristic functions of the intervals of X. The definition of $\mathfrak{g}_X$ mimics the usual construction of Kac–Moody algebras, with some fundamental differences controlled by the partial operations of $\mathcal{Q}_X$. Namely, we first consider the Lie algebra $\widetilde{\mathfrak{g}}_X$ over $\mathbb{C}$, freely generated by $\mathfrak{f}_X$ and the elements $x_\alpha^\pm$, $\alpha \in \mathsf{Int}(X)$, subject to the relations:

$$[\xi_\alpha, \xi_\beta] = 0\,, \quad [\xi_\alpha, x_\beta^\pm] = \pm\,(\alpha|\beta)\cdot x_\beta^\pm\,, \quad [x_\alpha^+, x_\beta^-] = \delta_{\alpha\beta}\xi_\alpha + \mathsf{a}_{\alpha\beta}\cdot(x^+_{\alpha\ominus\beta} - x^-_{\beta\ominus\alpha})\,,$$

where $\xi_\alpha := \mathbb{1}_\alpha$ and $\mathsf{a}_{\alpha\beta} := (-1)^{\langle\alpha,\beta\rangle}\cdot(\alpha|\beta)$. Then, we set $\mathfrak{g}_X := \widetilde{\mathfrak{g}}_X/\mathfrak{r}_X$, where $\mathfrak{r}_X \subset \widetilde{\mathfrak{g}}_X$ is the sum of all two–sided graded [1] ideals having trivial intersection with $\mathfrak{f}_X$.

In [ASS18], we show that the ideal $\mathfrak{r}_X$ is generated by certain quadratic Serre relations governed by the concatenation of intervals, thus generalizing Gabber–Kac theorem for continuum Kac–Moody algebras (cf. [GK81]) and obtaining an explicit description of $\mathfrak{g}_X$ (cf. [ASS18, Thm. 5.17] or Theorem 3.11 below).

Namely, $\mathfrak{g}_X$ is freely generated by the abelian Lie algebra $\mathfrak{f}_X$ and the elements $x_\alpha^\pm$, $\alpha \in \mathsf{Int}(X)$, subject to the following defining relations:

(1) **Diagonal action:** for $\alpha, \beta \in \mathsf{Int}(X)$,

$$[\xi_\alpha, x_\beta^\pm] = \pm\,(\alpha|\beta)\cdot x_\beta^\pm\,;$$

(2) **Double relations:** for $\alpha, \beta \in \mathsf{Int}(X)$,

$$[x_\alpha^+, x_\beta^-] = \delta_{\alpha\beta}\,\xi_\alpha + \mathsf{a}_{\alpha\beta}\cdot\left(x^+_{\alpha\ominus\beta} - x^-_{\beta\ominus\alpha}\right)\,;$$

(3) **Serre relations:** for $(\alpha, \beta) \in \mathsf{Serre}(X)$,

$$[x_\alpha^\pm, x_\beta^\pm] = \pm\mathsf{a}_{\alpha,\alpha\oplus\beta}\cdot x^\pm_{\alpha\oplus\beta}\,.$$

Here, $\mathsf{Serre}(X)$ is the set of all pairs $(\alpha, \beta) \in \mathsf{Int}(X)\times\mathsf{Int}(X)$ such that one of the following occurs:

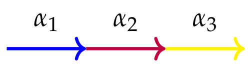
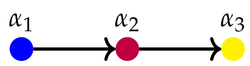
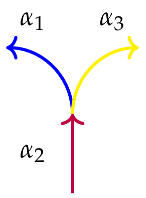
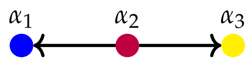
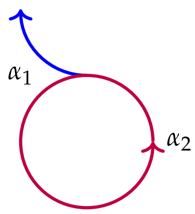
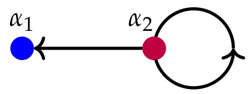
- α is contractible, and, for subintervals $\alpha' \subseteq \alpha$ and $\beta' \subseteq \beta$ with $(\beta|\beta') \neq 0$ whenever $\beta' \neq \beta$, $\alpha' \oplus \beta'$ is either undefined or non–homeomorphic to S^1;
- $\alpha \perp \beta$, *i.e.*, $\alpha\oplus\beta$ does not exist and $\alpha\cap\beta = \emptyset$.

As mentioned earlier, $\mathfrak{g}_X$ can be equivalently realized as certain continuous colimits of Borcherds–Kac–Moody algebras, further motivating our choice of the terminology. This is based on the following observation. Let $\mathcal{J} = \{\alpha_k\}_k$ be an *irreducible* finite set of intervals $\alpha_k \in \mathsf{Int}(X)$, *i.e.*,

(1) every interval is either contractible or homeomorphic to S^1;
(2) given two intervals $\alpha, \beta \in \mathcal{J}$, $\alpha \neq \beta$, one of the following mutually exclusive cases occurs:
   (a) $\alpha\oplus\beta$ exists;
   (b) $\alpha\oplus\beta$ does not exist and $\alpha\cap\beta = \emptyset$;
   (c) $\alpha \simeq S^1$ and $\beta \subset \alpha$.

Let $\mathsf{A}_\mathcal{J}$ be the matrix given by the values of $(\cdot|\cdot)$ on $\mathcal{J}$, *i.e.*, $\left(\mathsf{A}_\mathcal{J}\right)_{\alpha\beta} = (\alpha|\beta)$ for $\alpha, \beta \in \mathcal{J}$. Note that the diagonal entries of $\mathsf{A}_\mathcal{J}$ are either 2 or 0, while the only possible off–diagonal entries are $0, -1, -2$. Let $\mathcal{Q}_\mathcal{J}$ be the corresponding quiver with Cartan matrix $\mathsf{A}_\mathcal{J}$. For example, we obtain the following quivers.

[1] The gradation is with respect to $\mathfrak{f}_X$: we set $\deg(x_\alpha^\pm) = \pm\mathbb{1}_\alpha$ and $\deg(\xi_\alpha) = 0$.

Configuration of intervals	Borcherds–Cartan diagram
	
	
	

Note, in particular, that any contractible elementary interval corresponds to a vertex of $\mathcal{Q}_{\mathcal{J}}$ without loops, while any interval homeomorphic to S^1, corresponds to a vertex having exactly one loop.

There are two Lie algebras naturally associated to $\mathcal{J}$:

(1) the Lie subalgebra $\mathfrak{g}_{\mathcal{J}} \subset \mathfrak{g}_X$ generated by the elements $\{x_\alpha^\pm, \xi_\alpha \mid \alpha \in \mathcal{J}\}$;

(2) the derived Borcherds–Kac–Moody algebra $\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} := \mathfrak{g}(\mathsf{A}_{\mathcal{J}})'$.

In [ASS18, Section 5.5], we show that $\mathfrak{g}_{\mathcal{J}}$ and $\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}$ are canonically isomorphic. In particular, $\mathfrak{g}_X$ can be *covered* by Borcherds–Kac–Moody algebras. Moreover, we show that, given two *compatible* irreducible sets $\mathcal{J}, \mathcal{J}'$, there is an obvious embedding $\phi_{\mathcal{J}',\mathcal{J}} \colon \mathfrak{g}_{\mathcal{J}} \to \mathfrak{g}_{\mathcal{J}'}$, and the collection of all such ϕ's is a direct system, so that we get a canonical isomorphism of Lie algebras (cf. [ASS18, Cor. 5.18] or Corollary 3.14 below)

$$\mathfrak{g}_X \simeq \mathsf{colim}_{\mathcal{J}}\, \mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}\ .$$

Continuum Lie bialgebras. It is well–known that any symmetrisable Borcherds–Kac–Moody algebra $\mathfrak{g}$ is endowed with a symmetric non–degenerate bilinear form, inducing an isomorphism of graded vector spaces $\mathfrak{b}_+ \simeq \mathfrak{b}_-^\star$ between the positive and negative Borel subalgebras, and consequently defining a Lie bialgebra structure on $\mathfrak{g}$. Moreover, the latter is quasi–triangular with respect to the canonical element $r \in \mathfrak{b}_+ \widehat{\otimes} \mathfrak{b}_-$ corresponding to the perfect pairing $\mathfrak{b}_+ \otimes \mathfrak{b}_- \to \mathbb{C}$ (cf. Section 2).

The first contribution of this paper is the extension of these results for continuum Kac–Moody algebras.

Theorem (cf. Theorem 4.6)**.** *Let $\mathcal{Q}_X$ be a continuum quiver and $\mathfrak{g}_X$ the corresponding continuum Kac–Moody algebras.*

(1) The Euler form on $\mathfrak{f}_X$ uniquely extends to an invariant symmetric bilinear form $(\cdot|\cdot)\colon \widetilde{\mathfrak{g}}_X\otimes\widetilde{\mathfrak{g}}_X\to\mathbb{C}$ defined on the generators as follows:

$$(\xi_\alpha|\xi_\beta):=(\alpha|\beta)\ ,\quad (x_\alpha^\pm|\xi_\beta):=0\,,\quad \left(x_\alpha^\pm|x_\beta^\pm\right):=0\,,\quad \left(x_\alpha^+|x_\beta^-\right):=\delta_{\alpha\beta}\,.$$

Moreover, $\ker(\cdot|\cdot)=\mathfrak{r}_X$ *and therefore the Euler form descends to a non–degenerate invariant symmetric bilinear form on $\mathfrak{g}_X$.*

(2) There is a unique topological cobracket $\delta\colon\mathfrak{g}_X\to\mathfrak{g}_X\widehat{\otimes}\mathfrak{g}_X$ defined on the generators by

$$\delta(\xi_\alpha):=0\quad and\quad \delta(x_\alpha^\pm):=\xi_\alpha^\pm\wedge x_\alpha^\pm+\sum_{\beta\oplus\gamma=\alpha}\mathsf{a}_{\beta,\beta\oplus\gamma}\cdot x_\beta^\pm\wedge x_\gamma^\pm\,,$$

and inducing on $\mathfrak{g}_X$ a topological Lie bialgebra structure, with respect to which the positive and negative Borel subalgebras $\mathfrak{b}_X^\pm$ are Lie sub-bialgebras.

(3) The Euler form restricts to a non–degenerate pairing of Lie bialgebras $(\cdot|\cdot)\colon\mathfrak{b}_X^+\otimes(\mathfrak{b}_X^-)^{\mathrm{cop}}\to\mathbb{C}$. Then, the canonical element $r_X\in\mathfrak{b}_X^+\widehat{\otimes}\mathfrak{b}_X^-$ corresponding to $(\cdot|\cdot)$ defines a quasi–triangular structure on $\mathfrak{g}_X$.

Note however that in order to prove this result one cannot rely on the colimit realization of $\mathfrak{g}_X$ given above, since the embeddings $\phi_{\mathcal{J}',\mathcal{J}}\colon\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}\to\mathfrak{g}_{\mathcal{J}'}^{\mathrm{BKM}}$, do not respect the cobracket, as clear from their definition (cf. Corollary 3.14). Instead, our proof is based on an alternative realization of $\mathfrak{g}_X$ *by duality*, inspired by the work of G. Halbout [Hal99] which relies on a semi–classical version of techniques coming from the foundational theory of quantum groups [Dri87, Lus10].

By the result above, we can now associate to any continuum quiver $\mathcal{Q}_X$ a topological quasi–triangular Lie bialgebra $(\mathfrak{g}_X,[\cdot,\cdot],\delta)$. The second and main contribution of this paper is the algebraic explicit construction of a quantization $\mathbf{U}_q\mathfrak{g}_X$, *i.e.*, a topological quasi–triangular Hopf algebra over $\mathbb{C}[\![\hbar]\!]$ such that

(1) there exists an isomorphism of Hopf algebras $\mathbf{U}_q\mathfrak{g}_X/\hbar\mathbf{U}_q\mathfrak{g}_X\simeq\mathbf{U}\mathfrak{g}_X$;

(2) for any $x\in\mathfrak{g}_X$,

$$\delta(x)=\frac{\Delta(\widetilde{x})-\Delta^{21}(\widetilde{x})}{\hbar}\quad\mathrm{mod}\ \hbar\,,$$

where $\widetilde{x}\in\mathbf{U}_q\mathfrak{g}_X$ is any lift of $x\in\mathfrak{g}_X$.

We refer to $\mathbf{U}_q\mathfrak{g}_X$ as *the continuum quantum group of* $\mathcal{Q}_X$.

The continuum quantum group. The definition of $\mathbf{U}_q\mathfrak{g}_X$ is very similar in spirit to that of $\mathfrak{g}_X$, but it depends on two additional partial operations on $\mathsf{Int}(X)$:

(1) the *strict union of two intervals α and β*, whenever defined, is the smallest interval $\alpha\triangledown\beta\in\mathsf{Int}(X)$ for which $(\alpha\triangledown\beta)\ominus\alpha$ and $(\alpha\triangledown\beta)\ominus\beta$ are both defined;

(2) the *strict intersection of two intervals α and β*, whenever defined, is the biggest interval $\alpha\vartriangle\beta\in\mathsf{Int}(X)$ for which $\alpha\ominus(\alpha\vartriangle\beta)$ and $\beta\ominus(\alpha\vartriangle\beta)$ are both defined.

Note that $\alpha\triangledown\beta$ (resp. $\alpha\vartriangle\beta$) is defined and coincides with $\alpha\cup\beta$ (resp. $\alpha\cap\beta$) whenever it contains *strictly* α and β (resp. it is contained *strictly* in α and β).

Definition (cf. Definition 5.6)**.** Let $\mathcal{Q}_X$ be a continuum quiver. The *continuum quantum group of X* is the associative algebra $\mathbf{U}_q\mathfrak{g}_X$ generated by $\mathfrak{f}_X$ and the elements $X_\alpha^\pm$, $\alpha\in\mathsf{Int}(X)$, satisfying the following defining relations:

(1) **Diagonal action:** for any $\alpha,\beta\in\mathsf{Int}(X)$,

$$[\xi_\alpha,\xi_\beta]=0\qquad\text{and}\qquad[\xi_\alpha,X^\pm_\beta]=\pm\,(\alpha|\beta)\,X^\pm_\beta\,.$$

In particular, for $K_\alpha:=\exp(\hbar/2\cdot\xi_\alpha)$, it holds $K_\alpha X^\pm_\beta=q^{\pm(\alpha|\beta)}\cdot X^\pm_\beta K_\alpha$.

(2) **Quantum double relations:** for any $\alpha,\beta\in\mathsf{Int}(X)$,

$$[X^+_\alpha,X^-_\beta]=\delta_{\alpha\beta}\frac{K_\alpha-K_\alpha^{-1}}{q-q^{-1}}+\mathsf{a}_{\alpha\beta}\cdot\left(q^{\mathsf{c}^+_{\alpha\beta}}\,X^+_{\alpha\ominus\beta}\,K_\beta^{\mathsf{a}_{\alpha\beta}}-q^{\mathsf{c}^-_{\alpha\beta}}\,K_\alpha^{\mathsf{a}_{\alpha\beta}}\,X^-_{\beta\ominus\alpha}\right)$$
$$+\,\mathsf{b}_{\beta\alpha}\,q^{\mathsf{b}_{\beta\alpha}}\,(q-q^{-1})\,X^+_{(\alpha\triangledown\beta)\ominus\beta}\,K_{\alpha\vartriangle\beta}^{\mathsf{b}_{\alpha\beta}}\,X^-_{(\alpha\triangledown\beta)\ominus\alpha}\,.$$

(3) **Quantum Serre relations:** for any $(\alpha,\beta)\in\mathsf{Serre}(X)$,

$$X^\pm_\alpha X^\pm_\beta-q^{\mathsf{r}_{\alpha\beta}}\cdot X^\pm_\beta X^\pm_\alpha=\pm\mathsf{b}_{\alpha\beta}\cdot q^{\mathsf{s}^\pm_{\alpha\beta}}\cdot X^\pm_{\alpha\oplus\beta}+\mathsf{b}_{\alpha\beta}\cdot(q-q^{-1})\cdot X^\pm_{\alpha\triangledown\beta}X^\pm_{\alpha\vartriangle\beta}\,.$$

⊘

In the definition above, we assume that $X^\pm_{\alpha\odot\beta}=0$ whenever $\alpha\odot\beta$ is not defined, for $\odot=\oplus,\ominus,\triangledown,\vartriangle$. Moreover, the coefficients are defined as follows:

- $\mathsf{a}_{\alpha\beta}:=(-1)^{\langle\alpha,\beta\rangle}\,(\alpha|\beta)$;
- $\mathsf{b}_{\alpha\beta}:=\mathsf{a}_{\alpha,\alpha\triangledown\beta}$;
- $\mathsf{c}^+_{\alpha\beta}:=\dfrac{1}{2}\left(\mathsf{a}_{\beta,\alpha\ominus\beta}-1\right)$ and $\mathsf{c}^-_{\alpha\beta}:=\dfrac{1}{2}\left(\mathsf{a}_{\beta\ominus\alpha,\alpha}+1\right)$;
- $\mathsf{r}_{\alpha\beta}:=(1-\delta_{\alpha\beta})(-1)^{\langle\alpha,\beta\rangle}\,(\alpha|\beta)^2$;
- $\mathsf{s}^\pm_{\alpha\beta}:=\dfrac{1}{2}\left(\mathsf{a}_{\beta,\alpha\oplus\beta}\pm1\right)$.

In order to prove that $\mathbf{U}_q\mathfrak{g}_X$ is naturally endowed with a topological quasi–triangular Hopf algebra structure, we proceed as in the classical case, by showing that $\mathbf{U}_q\mathfrak{g}_X$ can be equivalently realized *by duality*. This leads to the following.

Theorem (cf. Theorem 5.11)**.** *Let $\mathcal{Q}_X$ be a continuum quiver and $\mathbf{U}_q\mathfrak{g}_X$ the corresponding continuum quantum group.*

(1) The algebra $\mathbf{U}_q\mathfrak{g}_X$ is a topological Hopf algebra with respect to the maps

$$\Delta\colon\mathbf{U}_q\mathfrak{g}_X\to\mathbf{U}_q\mathfrak{g}_X\widehat{\otimes}\mathbf{U}_q\mathfrak{g}_X\quad\text{and}\quad\varepsilon\colon\mathbf{U}_q\mathfrak{g}_X\to\mathbb{C}[\![\hbar]\!]\,,$$

defined on the generators by $\varepsilon(\xi_\alpha):=0=:\varepsilon(X^\pm_\alpha)$, $\Delta(\xi_\alpha):=\xi_\alpha\otimes1+1\otimes\xi_\alpha$, and

$$\Delta(X^+_\alpha):=X^+_\alpha\otimes1+K_\alpha\otimes X^+_\alpha+\sum_{\alpha=\beta\oplus\gamma}\mathsf{a}_{\gamma,\beta\oplus\gamma}\,\mathsf{s}^-_{\beta\gamma}\cdot q^{-1}(q-q^{-1})\,X^+_\beta K_\gamma\otimes X^+_\gamma\,,$$

$$\Delta(X^-_\alpha):=1\otimes X^-_\alpha+X^-_\alpha\otimes K_\alpha^{-1}-\sum_{\alpha=\beta\oplus\gamma}\mathsf{a}_{\gamma,\beta\oplus\gamma}\,\mathsf{s}^-_{\beta\gamma}\cdot(q-q^{-1})\,X^-_\beta\otimes X^-_\gamma K_\gamma^{-1}\,.$$

In particular, $\varepsilon(K_\alpha)=1$ and $\Delta(K_\alpha)=K_\alpha\otimes K_\alpha$. As usual, the antipode is given by the formula

$$S:=\sum_n m^{(n)}\circ(\mathsf{id}-\iota\circ\varepsilon)^{\otimes n}\circ\Delta^{(n)}\,,$$

where $m^{(n)}$ and $\Delta^{(n)}$ denote the nth iterated product and coproduct, respectively.

(2) Denote by $\mathbf{U}_q\mathfrak{b}^\pm_X$ the Hopf subalgebras generated by $\mathfrak{h}_X$ and $X^\pm_\alpha$, $\alpha\in\mathsf{Int}(X)$. Then, there exists a unique non–degenerate Hopf pairing $(\cdot|\cdot):\mathbf{U}_q\mathfrak{b}^+_X\otimes(\mathbf{U}_q\mathfrak{b}^-_X)^{\mathrm{cop}}\to\mathbb{C}(\!(\hbar)\!)$, defined on the generators by

$$(1|1):=1\,,\quad(\xi_\alpha|\xi_\beta):=\frac{1}{\hbar}\,(\alpha|\beta)\,,\quad\left(X^+_\alpha|X^-_\beta\right):=\frac{\delta_{\alpha\beta}}{q-q^{-1}}\,,$$

and zero otherwise. In particular, $\left(K_\alpha|K_\beta\right) = q^{(\alpha|\beta)}$.

(3) *Through the Hopf pairing* $(\cdot|\cdot)$*, the Hopf algebras* $(\mathbf{U}_q\mathfrak{b}_X^+, \mathbf{U}_q\mathfrak{b}_X^-)$ *give rise to a match pair of Hopf algebras. Then,* $\mathbf{U}_q\mathfrak{g}_X$ *is realized as a quotient of the double cross product Hopf algebra* $\mathbf{U}_q\mathfrak{b}_X^+ \triangleright\triangleleft \mathbf{U}_q\mathfrak{b}_X^-$ *obtained by identifying the two copies of the commutative subalgebra* $\mathfrak{f}_X$*. In particular,* $\mathbf{U}_q\mathfrak{g}_X$ *is a topological quasi–triangular Hopf algebra.*

(4) *The topological quasi–triangular Hopf algebra* $\mathbf{U}_q\mathfrak{g}_X$ *is a quantization of the topological quasi–triangular Lie bialgebra* $\mathfrak{g}_X$.

Moreover, we prove that, as in the classical case, the continuum quantum group can be realized as an uncountable colimits of Drinfeld–Jimbo quantum groups.

Theorem (cf. Corollary 5.8)**.** *Let* $\mathcal{J}, \mathcal{J}'$ *be two irreducible (finite) sets of intervals in* X.

(1) *Let* $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$ *be the Hopf subalgebra in* $\mathbf{U}_q\mathfrak{g}_X$ *generated by the elements* ξ_α *and* $X_\alpha^\pm$*, with* $\alpha \in \mathcal{J}$*. Then, there is a canonical isomorphism of algebras* $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$.

(2) *If* $\mathcal{J}' \subseteq \mathcal{J}$*, there is a canonical embedding* $\phi'_{\mathcal{J},\mathcal{J}'} \colon \mathbf{U}_q\mathfrak{g}_{\mathcal{J}'} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$ *sending generator to generator.*

(3) *If* $\mathcal{J}$ *is obtained from* $\mathcal{J}'$ *by replacing an element* $\gamma \in \mathcal{J}'$ *with two intervals* α, β *such that* $\gamma = \alpha \oplus \beta$*, there is a canonical embedding* $\phi''_{\mathcal{J},\mathcal{J}'} \colon \mathbf{U}_q\mathfrak{g}_{\mathcal{J}'} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$*, which is the identity on* $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}'\setminus\{\gamma\}} = \mathbf{U}_q\mathfrak{g}_{\mathcal{J}\setminus\{\alpha,\beta\}}$ *and sends*

$$\xi_\gamma \mapsto \xi_\alpha + \xi_\beta\,, \quad X_\gamma^\pm \mapsto \mp \mathsf{b}_{\alpha\beta}^{-1} \cdot q^{-\mathsf{s}_{\alpha\beta}^\pm} \cdot \left(X_\alpha^\pm X_\beta^\pm - q^{\mathsf{r}_{\alpha\beta}} \cdot X_\beta^\pm X_\alpha^\pm\right)\,.$$

(4) *The collection of embeddings* $\phi'_{\mathcal{J},\mathcal{J}'}, \phi''_{\mathcal{J},\mathcal{J}'}$*, indexed by all possible irreducible sets of intervals in* X*, form a direct system. Moreover, there is a canonical isomorphism of algebras*

$$\mathbf{U}_q\mathfrak{g}_X \simeq \mathsf{colim}_{\mathcal{J}}\, \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}\,.$$

The quantum groups of the line and the circle. In [SS17], the second-named author and O. Schiffmann introduced the line quantum group $\mathbf{U}_q\mathfrak{sl}(\mathbb{R})$ and the circle quantum group $\mathbf{U}_q\mathfrak{sl}(S^1)$, the latter arising from the Hall algebra of parabolic (torsion) coherent sheaves on a curve. These are the simplest examples of continuum quantum groups. Namely, we get the following.

Theorem (cf. Propositions 3.17 and 5.10)**.** *There exists a canonical isomorphism of topological Hopf algebras* $\mathbf{U}_q\mathfrak{sl}(\mathbb{R}) \to \mathbf{U}_q\mathfrak{g}_{\mathbb{R}}$*. At* $q = 1$*, it gives rise to an isomorphism of topological Lie bialgebras* $\mathfrak{sl}(\mathbb{R}) \to \mathfrak{g}_{\mathbb{R}}$.

The case of the circle is slightly more delicate. Namely, the continuum Kac–Moody algebra $\mathfrak{g}_{S^1}$ contains strictly the Lie algebra $\mathfrak{sl}(S^1)$. Their difference is reduced to the elements $x_{S^1}^\pm$ corresponding to the full circle. More precisely, let $\overline{\mathfrak{g}}_{S^1}$ be the subalgebra in $\mathfrak{g}_{S^1}$ generated by the elements $x_\alpha^\pm$, ξ_α, $\alpha \neq S^1$. Note that the elements $x_{S^1}^\pm$, ξ_{S^1}, generate a Heisenberg Lie algebra of order one in $\mathfrak{g}_{S^1}$, which we denote $\mathfrak{heis}_{S^1}$. Then, $\mathfrak{g}_{S^1} = \overline{\mathfrak{g}}_{S^1} \oplus \mathfrak{heis}_{S^1}$ and there is a canonical embedding $\mathfrak{sl}(S^1) \to \mathfrak{g}_{S^1}$, whose image is $\overline{\mathfrak{g}}_{S^1} \oplus \mathbf{k} \cdot \xi_{S^1}$. A similar relation holds for the Hopf algebras $\mathbf{U}_q\mathfrak{sl}(S^1)$ and $\mathbf{U}_q\mathfrak{g}_{S^1}$, where the role of $\mathfrak{heis}_{S^1}$ is played by the subalgebra generated in $\mathbf{U}_q\mathfrak{g}_{S^1}$ by ξ_{S^1} and $X_{S^1}^\pm$.

Future directions. In this last section, we shall outline some further directions of research, currently under investigations.

Geometric quantization. As mentioned earlier, the continuum quantum groups $\mathbf{U}_q\mathfrak{sl}(S^1)$ and $\mathbf{U}_q\mathfrak{sl}(\mathbb{R})$ originate from a Hall algebra type construction. More precisely, the *rational* circle quantum group $\mathbf{U}_q\mathfrak{sl}(\mathbb{Q}/\mathbb{Z})$ was realized in [SS17] in two different ways. That is, by the second-named author and O. Schiffmann, as the (reduced) quantum double of the spherical Hall algebra of torsion parabolic sheaves on a smooth projective curve over a finite field, and, by T. Kuwagaki, from the spherical Hall algebra of locally constant sheaves on $\mathbb{Q}/\mathbb{Z}$ with fixed singular support. The latter approach generalizes easily to $\mathbb{R}$ and S^1, and to the type D case (a smooth tree with one root, one node, and two leaves).

In [AKSS19], together with T. Kuwagaki and O. Schiffmann, we will provide two geometric realization of $\mathbf{U}_q\mathfrak{g}_X$ arising from Hall algebras associated with the following abelian categories defined over a finite field. We first consider the category of *coherent persistence modules* (extending the definition given in [SS19] for the line $\mathbb{R}$ and the circle S^1 to an arbitrary continuum quiver). Such objects can be thought of as a generalization of the usual notion of parabolic torsion sheaves on a curve, mimicking the first realization of the circle quantum group. The analogue of the second *symplectic* realization is instead obtained from the category of locally constant sheaves over the underlying vertex space.

Highest weight theory. In general, the usual combinatorics governing the highest weight theory of Borcherds–Kac–Moody algebras does not extend in a straightforward way to continuum Kac–Moody algebras, mainly due to the lack of simple roots. The appropriate tools to describe the highest weight theory of $\mathfrak{g}_X$, the corresponding continuum Weyl group, and the character formulas, are currently under study. The same difficulties arise also at the quantum level.

Nonetheless, the geometric realization of continuum quantum groups would likely help towards a better understanding of its representation theory. An inspiring example is given in [SS19], where the second-named author and O. Schiffmann define the Fock space for $\mathbf{U}_q\mathfrak{sl}(\mathbb{R})$, considering a continuum analogue of the usual combinatorial construction in the case of $\mathbf{U}_q\mathfrak{sl}(\infty)$. In addition, the quantum group $\mathbf{U}_q\mathfrak{sl}(S^1)$ act on such a Fock space, in a way similar to the *folding procedure* of Hayashi–Misra–Miwa. This construction should extend to the case of an arbitrary continuum quiver X, producing a wide class of interesting representations for the continuum quantum group $\mathbf{U}_q\mathfrak{g}_X$, and therefore for the continuum Kac–Moody algebra $\mathfrak{g}_X$.

Outline. In Section 2, we recall the basic definition of Kac–Moody algebras and Drinfeld–Jimbo quantum groups in the more general framework of quantization of Lie bialgebras. In Section 3, we provide a concise exposition of the construction of continuum Kac–Moody algebras, as introduced in [ASS18], and their realization as uncountable colimits of Borcherds–Kac–Moody algebras. In Section 4, we prove the first main result of the paper, showing that continuum Kac–Moody algebras are naturally endowed with a standard topological quasi–triangular Lie bialgebra structure. In Section 5, we define the continuum quantum group associated to a continuum quiver and show that, in the cases of $\mathbb{R}$ and S^1, it coincides with the quantum groups of the line and the circle introduced in [SS17]. Finally, in Section 5.4, we prove the second main result of the paper, showing that continuum quantum groups are topological quasi–triangular Hopf algebra, quantizing the standard Lie bialgebra structure of continuum Kac–Moody algebras.

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2. KAC–MOODY ALGEBRAS AND QUANTUM GROUPS

In this section, we recall the basic definition of Kac–Moody algebras and Drinfeld–Jimbo quantum groups in the more general framework of quantization of Lie bialgebras. The results of this section are well–known and due to [Kac90, Dri87]. We follow the exposition of [ATL18].

Henceforth, we fix a base field $\mathbf{k}$ of characteristic zero and set $\mathbf{K} := \mathbf{k}[\![\hbar]\!]$.

2.1. **Quantization of Lie bialgebras.** A *Lie bialgebra* is a triple $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ where

(1) $\mathfrak{b}$ is a discrete $\mathbf{k}$–vector space;

(2) $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ is a Lie algebra, *i.e.*, $[\cdot,\cdot]_{\mathfrak{b}}\colon \mathfrak{b}\otimes\mathfrak{b}\to\mathfrak{b}$ is anti-symmetric and satisfies the Jacobi identity
$$[\cdot,\cdot]_{\mathfrak{b}}\circ \mathsf{id}_{\mathfrak{b}}\otimes[\cdot,\cdot]_{\mathfrak{b}}\circ(\mathsf{id}_{\mathfrak{b}^{\otimes 3}}+(1\,2\,3)+(1\,3\,2))=0\,;$$

(3) $(\mathfrak{b}, \delta_{\mathfrak{b}})$ is a Lie coalgebra, *i.e.*, $\delta_{\mathfrak{b}}\colon \mathfrak{b}\to\mathfrak{b}\otimes\mathfrak{b}$ is anti-symmetric and satisfies the co–Jacobi identity
$$(\mathsf{id}_{\mathfrak{b}^{\otimes 3}}+(1\,2\,3)+(1\,3\,2))\circ\mathsf{id}_{\mathfrak{b}}\otimes\delta_{\mathfrak{b}}\circ\delta_{\mathfrak{b}}=0\,; \tag{2.1}$$

(4) the cobracket $\delta_{\mathfrak{b}}$ satisfies the *cocycle condition*
$$\delta_{\mathfrak{b}}\circ[\cdot,\cdot]_{\mathfrak{b}}=\mathsf{ad}_{\mathfrak{b}}\circ\mathsf{id}_{\mathfrak{b}}\otimes\delta_{\mathfrak{b}}\circ(\mathsf{id}_{\mathfrak{b}^{\otimes 2}}-(1\,2))\,, \tag{2.2}$$
as maps $\mathfrak{b}\otimes\mathfrak{b}\to\mathfrak{b}\otimes\mathfrak{b}$, where $\mathsf{ad}_{\mathfrak{b}}\colon\mathfrak{b}\otimes\mathfrak{b}\otimes\mathfrak{b}\to\mathfrak{b}\otimes\mathfrak{b}$ denotes the left adjoint action of $\mathfrak{b}$ on $\mathfrak{b}\otimes\mathfrak{b}$.

A *quantized enveloping algebra* (QUE) is a Hopf algebra B in $\mathrm{Vect}_{\mathbf{K}}$ such that

(1) B is endowed with the $\hbar$–adic topology, that is $\{\hbar^n B\}_{n\geq 0}$ is a basis of neighborhoods of 0. Equivalently, B is isomorphic, as topological $\mathbf{K}$–module, to $B_0[\![\hbar]\!]$, for some discrete topological vector space B_0.

(2) $B/\hbar B$ is a connected, cocommutative Hopf algebra over $\mathbf{k}$. Equivalently, $B/\hbar B$ is isomorphic to an enveloping algebra $\mathbf{U}\mathfrak{b}$ for some Lie bialgebra $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ and, under this identification,
$$\delta_{\mathfrak{b}}(b)=\frac{\Delta(\widetilde{b})-\Delta^{21}(\widetilde{b})}{\hbar}\mod\hbar\,,$$
where $\widetilde{b}\in B$ is any lift of $b\in\mathfrak{b}$.

We say that B is a *quantization* of $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$.

In Sections 2.2–2.6 we will describe the standard Lie bialgebra structure on symmetrisable Kac–Moody algebras and their quantization provided by Drinfeld–Jimbo quantum groups.

2.2. **Kac–Moody algebras.** We recall the definition from [Kac90, Chapter 1]. Fix a finite set $\mathbf{I}$ and a matrix $\mathsf{A}=(a_{ij})_{i,j\in\mathbf{I}}$ with entries in $\mathbf{k}$. Recall that a realization $(\mathfrak{h},\Pi,\Pi^\vee)$ of A is the datum of a finite dimensional $\mathbf{k}$–vector space $\mathfrak{h}$, and linearly independent vectors $\Pi:=\{\alpha_i\}_{i\in\mathbf{I}}\subset\mathfrak{h}^*$, $\Pi^\vee:=\{h_i\}_{i\in\mathbf{I}}\subset\mathfrak{h}$ such that $\alpha_i(h_j)=a_{ji}$. One checks easily that, in any realization $(\mathfrak{h},\Pi,\Pi^\vee)$, $\dim\mathfrak{h}\geqslant 2|\mathbf{I}|-\mathsf{rk}(\mathcal{R})$. Moreover, up to a (non–unique) isomorphism, there is a unique realization of minimal dimension $2|\mathbf{I}|-\mathsf{rk}(\mathcal{R})$.

For any realization $\mathcal{R}=(\mathfrak{h},\Pi,\Pi^\vee)$, let $\widetilde{\mathfrak{g}}(\mathcal{R})$ be the Lie algebra generated by $\mathfrak{h}$, $\{e_i,f_i\}_{i\in\mathbf{I}}$ with relations $[h,h']=0$, for any $h,h'\in\mathfrak{h}$, and
$$[h,e_i]=\alpha_i(h)\,e_i\,,\quad [h,f_i]=-\alpha_i(h)\,f_i\,,\quad [e_i,f_j]=\delta_{ij}\,h_i\,.$$
Set
$$\mathsf{Q}_+:=\bigoplus_{i\in\mathbf{I}}\mathbb{Z}_{\geqslant 0}\,\alpha_i\subseteq\mathfrak{h}^*\,,$$
$\mathsf{Q}:=\mathsf{Q}_+\oplus(-\mathsf{Q}_+)$, and denote by $\widetilde{\mathfrak{n}}_+$ (resp. $\widetilde{\mathfrak{n}}_-$) the subalgebra generated by $\{e_i\}_{i\in\mathbf{I}}$ (resp. $\{f_i\}_{i\in\mathbf{I}}$). Then, as vector spaces, $\widetilde{\mathfrak{g}}(\mathcal{R})=\widetilde{\mathfrak{n}}_+\oplus\mathfrak{h}\oplus\widetilde{\mathfrak{n}}_-$ and, with respect to $\mathfrak{h}$, one has the root space decomposition
$$\widetilde{\mathfrak{n}}_\pm=\bigoplus_{\substack{\alpha\in\mathsf{Q}_+\\ \alpha\neq 0}}\widetilde{\mathfrak{g}}_{\pm\alpha}$$
where $\widetilde{\mathfrak{g}}_{\pm\alpha}=\{X\in\widetilde{\mathfrak{g}}(\mathcal{R})\mid\forall h\in\mathfrak{h},[h,X]=\pm\alpha(h)X\}$. Note also that $\widetilde{\mathfrak{g}}_0=\mathfrak{h}$ and $\dim\widetilde{\mathfrak{g}}_{\pm\alpha}<\infty$.

The *Kac–Moody algebra* corresponding to the realization $\mathcal{R}$ is the Lie algebra $\mathfrak{g}(\mathcal{R}) := \widetilde{\mathfrak{g}}(\mathcal{R})/\mathfrak{r}$, where $\mathfrak{r}$ is the sum of all two–sided graded ideals in $\widetilde{\mathfrak{g}}(\mathcal{R})$ having trivial intersection with $\mathfrak{h}$. In particular, as ideals, $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$, where $\mathfrak{r}_\pm := \mathfrak{r} \cap \widetilde{\mathfrak{n}}_\pm$. [2]

Since $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$, the Lie algebra $\mathfrak{g}(\mathcal{R})$ has an induced triangular decomposition $\mathfrak{g}(\mathcal{R}) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}+$ (as vector spaces), where

$$\mathfrak{n}_\pm := \bigoplus_{\substack{\alpha\in\mathsf{Q}_+\\ \alpha\neq 0}} \mathfrak{g}_{\pm\alpha}\,, \qquad \mathfrak{g}_\alpha := \{X \in \mathfrak{g}(\mathcal{R}) \mid \forall\, h \in \mathfrak{h},\ [h, X] = \alpha(h)\, X\}\,.$$

Note that $\dim \mathfrak{g}_\alpha < \infty$. The set of positive roots is denoted $\mathsf{R}_+ := \{\alpha \in \mathsf{Q}_+ \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$.

Remark 2.1. The derived subalgebra $\mathfrak{g}(\mathcal{R})' := [\mathfrak{g}(\mathcal{R}), \mathfrak{g}(\mathcal{R})]$ has a similar and somewhat simpler description. One can show easily that $\mathfrak{g}(\mathcal{R})'$ is generated by the Chevalley generators $\{e_, f_i, h_i\}_{i\in\mathbf{I}}$ and admits a presentation similar to that of $\mathfrak{g}(\mathcal{R})$. Namely, let $\widetilde{\mathfrak{g}}'$ be the Lie algebra generated by $\{h_i, e_i, f_i\}_{i\in\mathbf{I}}$ with relations

$$[h_i, h_j] = 0\,, \quad [h_j, e_i] = \alpha_i(h_j)\, e_i\,, \quad [h_j, f_i] = -\alpha_i(h_j)\, f_i\,, \quad [e_i, f_j] = \delta_{ij}\, h_i\,.$$

Then, $\widetilde{\mathfrak{g}}'$ has a Q–gradation defined by $\deg(e_i) = \alpha_i$, $\deg(f_i) = -\alpha_i$, $\deg(h_i) = 0$, and $\widetilde{\mathfrak{g}}'_0 = \mathfrak{h}'$, where the latter is the $|\mathbf{I}|$–dimensional span of $\{h_i\}_{i\in\mathbf{I}}$. The quotient of $\widetilde{\mathfrak{g}}'$ by the sum of all two–sided graded ideals with trivial intersection with $\mathfrak{h}'$ is easily seen to be canonically isomorphic to $\mathfrak{g}(\mathcal{R})'$. △

Remark 2.2. It is sometimes convenient to consider the Kac–Moody algebras associated to a (non–minimal) *canonical realization*, which allows to obtain a presentation similar to that of the derived subalgebra (cf. [FZ85, MO12, ATL19a]). Namely, let $\overline{\mathcal{R}} = (\overline{\mathfrak{h}}, \overline{\Pi}, \overline{\Pi}^\vee)$ be the realization given by $\overline{h} \cong \mathbf{k}^{2|\mathbf{I}|}$ with basis $\{h_i\}_{i\in\mathbf{I}} \cup \{\lambda_i^\vee\}_{i\in\mathbf{I}}$, $\overline{\Pi}^\vee = \{h_i\}_{i\in\mathbf{I}}$ and $\overline{\Pi} = \{\alpha_i\}_{i\in\mathbf{I}} \subset \overline{\mathfrak{h}}^*$, where α_i is defined by

$$\alpha_i(h_j) = a_{ji} \qquad \text{and} \qquad \alpha_i(\lambda_j^\vee) = \delta_{ij}\,.$$

Then, $\widetilde{\mathfrak{g}}(\overline{\mathcal{R}})$ is the Lie algebra generated by $\{h_i, \lambda_i^\vee, e_i, f_i\}_{i\in\mathbf{I}}$ with relations

$$[h_i, h_j] = 0\,, \quad [h_i, \lambda_j^\vee] = 0\,, \quad [\lambda_i^\vee, \lambda_j^\vee] = 0\,,$$

and

$$[h_i, e_j] = a_{ij}\, e_j\,, \quad [h_i, f_j] = -a_{ij}\, f_j\,, \quad [\lambda_i^\vee, e_j] = \delta_{ij}\, e_j\,, \quad [\lambda_i^\vee, f_j] = -\delta_{ij}\, f_j\,, \quad [e_i, f_j] = \delta_{ij}\, h_i\,.$$

It is easy to check that the Kac–Moody algebra $\mathfrak{g}(\overline{\mathcal{R}})$ is just a central extension of $\mathfrak{g}(\mathcal{R}_{\min})$, *i.e.*, $\mathfrak{g}(\overline{\mathcal{R}}) \simeq \mathfrak{g}(\mathcal{R}_{\min}) \oplus \mathfrak{c}$, with $\dim \mathfrak{c} = \mathsf{rk}(\mathsf{A})$. △

Remark 2.3. In certain cases the ideal $\mathfrak{r}$ can be described explicitly. If A is a *generalised Cartan matrix* (*i.e.*, $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leqslant 0}$, $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$), then $\mathfrak{r}$ contains the ideal generated by the Serre relations

$$\mathsf{ad}(e_i)^{1-a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-a_{ij}}(f_j) \qquad i \neq j$$

and coincides with it if A is also symmetrizable [GK81].

A similar description of $\mathfrak{r}$ holds for any *Borcherds–Cartan matrix* A (*i.e.*, such that $a_{ij} \in \mathbb{Z}_{\leqslant 0}$, $i \neq j$, and $2a_{ij}/a_{ii} \in \mathbb{Z}$ whenever $a_{ii} > 0$). In this case, $\mathfrak{g}$ is called a Borcherds–Kac–Moody algebra and the corresponding maximal ideal contains the Serre relations

$$\mathsf{ad}(e_i)^{1-\frac{2}{a_{ii}}a_{ij}}(e_j) = 0 = \mathsf{ad}(f_i)^{1-\frac{2}{a_{ii}}a_{ij}}(f_j) \qquad \text{if} \quad a_{ii} > 0 \quad \text{and} \quad i \neq j$$

and

$$[e_i, e_j] = 0 = [f_i, f_j] \qquad \text{if} \quad a_{ii} \leqslant 0 \quad \text{and} \quad a_{ij} = 0\,.$$

[2]The terminology differs slightly from the one given in [Kac90] where $\mathfrak{g}(\mathcal{R})$ is called a Kac–Moody algebra if A is a generalised Cartan matrix (cf. Remark 2.3) and $\mathcal{R}$ is the minimal realization. Note also that in [Kac90, Theorem 1.2] $\mathfrak{r}$ is set to be the sum of all two–sided ideals, not necessarily graded. However, since the functionals α_i are linearly independent in $\mathfrak{h}^*$ by construction, $\mathfrak{r}$ is automatically graded and satisfies $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$ (cf. [Kac90, Proposition 1.5]).

If A is symmetrizable, $\mathfrak{r}$ is generated by the Serre relations (cf. [Bor88, Corollary 2.6]). △

If the matrix A is symmetrizable, the corresponding Kac–Moody algebra can be further endowed with a standard Lie bialgebra structure. Assume henceforth that A is symmetrizable, and fix a realization $\mathcal{R} = (\mathfrak{h}, \Pi, \Pi^\vee)$ and an invertible diagonal matrix $\mathsf{D} = \mathsf{diag}(d_i)_{i\in\mathbf{I}}$ such that DA is symmetric. Let $\mathfrak{h}' \subset \mathfrak{h}$ be the span of $\{h_i\}_{i\in\mathbf{I}}$, and $\mathfrak{h}'' \subset \mathfrak{h}$ a complementary subspace. Then, there is a symmetric, non–degenerate bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$, which is uniquely determined by $(h_i|\cdot) := d_i^{-1}\alpha_i(\cdot)$ and $(\mathfrak{h}''|\mathfrak{h}'') := 0$. The form $(\cdot|\cdot)$ uniquely extends to an invariant symmetric bilinear form on $\widetilde{\mathfrak{g}}$, and $(e_i|f_j) = \delta_{ij}d_i^{-1}$. The kernel of this form is precisely $\mathfrak{r}$, so that $(\cdot|\cdot)$ descends to a non–degenerate form on $\mathfrak{g}$.[3]

Set $\mathfrak{b}_\pm := \mathfrak{h} \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{\pm\alpha} \subset \mathfrak{g}$. One can see easily that the bilinear form induces a canonical isomorphism of graded vector spaces $\mathfrak{b}_+ \simeq \mathfrak{b}_-^\star$, where $\mathfrak{b}_-^\star := \mathfrak{h}^* \oplus \bigoplus_{\alpha\in\mathsf{R}_+} \mathfrak{g}_{-\alpha}^*$, and, more specifically, $\mathfrak{g}_\alpha \simeq \mathfrak{g}_{-\alpha}^*$.

Let $\{e_{\alpha,i} \mid i = 1, \ldots, \dim\mathfrak{g}_\alpha\}$ and $\{f_{\alpha,i} \mid i = 1, \ldots, \dim\mathfrak{g}_\alpha\}$ be bases of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$, respectively, which are dual to each other with respect to $(\cdot|\cdot)$, and set

$$r := \sum_{\alpha\in\mathsf{R}_+} \sum_{i=1}^{\dim\mathfrak{g}_\alpha} e_{\alpha,i} \otimes f_{\alpha,i} + \sum_{i=1}^{\dim\mathfrak{h}} x_i \otimes x_i\ ,$$

where $\{x_i \mid i = 1, \ldots, \dim\mathfrak{h}\}$ is an orthonormal basis of $\mathfrak{h}$.

We will show in Section 2.4 that $\mathfrak{g}$ has a natural structure of Lie bialgebra with cobracket $\delta\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by

$$\delta|_{\mathfrak{h}} := 0\ , \qquad \delta(e_i) := d_i h_i \wedge e_i\ , \qquad \delta(f_i) := d_i h_i \wedge f_i\ .$$

Moreover, it satisfies $\delta(x) = [x \otimes 1 + 1 \otimes x, r]$.

2.3. **Quasi–triangular Lie bialgebras.** A Lie bialgebra is *quasi–triangular* if there exists a tensor $r \in \mathfrak{b} \otimes \mathfrak{b}$ such that

(1) $\Omega := r + r_{21}$ is $\mathfrak{b}$–invariant, *i.e.*, $[x \otimes 1 + 1 \otimes x, \Omega] = 0$ for any $x \in \mathfrak{b}$;

(2) r is a solution of the *classical Yang–Baxter equation*, *i.e.*,

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0\ ;$$

(3) $\delta_{\mathfrak{b}} = \partial r$, *i.e.*, for any $x \in \mathfrak{b}$, $\delta_{\mathfrak{b}}(x) = [x \otimes 1 + 1 \otimes x, r]$.

It is well–known that any Lie bialgebra $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ can be canonically embedded into a quasi–triangular *topological* Lie bialgebra. We recall below three versions of this construction, in terms of *Drinfeld doubles*, *Manin triples* and *matched pairs of Lie algebras*.

2.3.1. *Drinfeld double.* Let $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ be a Lie bialgebra. The Drinfeld double $\mathfrak{g}_{\mathfrak{b}}$ is defined as follows.

- As a vector space, $\mathfrak{g}_{\mathfrak{b}} = \mathfrak{b} \oplus \mathfrak{b}^*$. The canonical pairing $(\cdot|\cdot) : \mathfrak{b} \otimes \mathfrak{b}^* \to \mathbf{k}$ extends uniquely to a symmetric non–degenerate bilinear form on $\mathfrak{g}_{\mathfrak{b}}$, with respect to which $\mathfrak{b}$ and $\mathfrak{b}^*$ are isotropic. The Lie bracket on $\mathfrak{g}_{\mathfrak{b}}$ is defined as the unique bracket which coincides with $[\cdot,\cdot]_{\mathfrak{b}}$ on $\mathfrak{b}$, with $\delta_{\mathfrak{b}}^t$ on $\mathfrak{b}^*$, and is compatible with $(\cdot|\cdot)$, *i.e.*, satisfies $([x,y]|z) = (x|[y,z])$ for all $x, y, z \in \mathfrak{g}_{\mathfrak{b}}$. The mixed bracket of $x \in \mathfrak{b}$ and $\phi \in \mathfrak{b}^*$ is then given by

$$[x, \phi] := \mathsf{ad}^*(x)(\phi) - \mathsf{ad}^*(\phi)(x)\ ,$$

  where ad^* denotes the coadjoint actions of $\mathfrak{b}$ on $\mathfrak{b}^*$ and of $\mathfrak{b}^*$ on $(\mathfrak{b}^*)^*$.

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[3] Since $(\cdot|\cdot)$ is non–degenerate on $\mathfrak{h}$, the kernel $\mathfrak{k} := \ker(\cdot|\cdot)$ is a graded ideal trivially intersecting $\mathfrak{h}$ and therefore it is contained in $\mathfrak{r}$. Conversely, for any graded ideal $\mathfrak{i} = \bigoplus_\alpha \mathfrak{i}_\alpha$ trivially intersecting $\mathfrak{h}$, one has $\mathfrak{i} \subseteq \mathfrak{k}$. More precisely, let $X \in \mathfrak{i}_\alpha$, $Y \in \widetilde{\mathfrak{g}}_\beta$ and $Z \in \mathfrak{h}$ such that $\beta(Z) \neq 0$. Then,

$$\beta(Z) \cdot (X|Y) = (X|[Z,Y]) = -([X,Y]|Z) = 0\ .$$

In particular $\mathfrak{r} \subseteq \mathfrak{k}$ and therefore $\mathfrak{r} = \mathfrak{k}$.

- We endow $\mathfrak{b}$ and $\mathfrak{b}^*$ with the discrete and the weak topology, respectively, and $\mathfrak{g}_\mathfrak{b} = \mathfrak{b} \oplus \mathfrak{b}^*$ with the product topology. It is clear that the map $[\cdot,\cdot]_\mathfrak{b}^t \colon \mathfrak{b}^* \to \mathfrak{b}^* \widehat{\otimes} \mathfrak{b}^*$, where $\widehat{\otimes}$ denotes the completed tensor product, defines on $\mathfrak{b}^*$ a topological cobracket. Similarly, $\delta := \delta_\mathfrak{b} - [\cdot,\cdot]_\mathfrak{b}^t$ defines a topological cobracket on $\mathfrak{g}_\mathfrak{b}$, which is easily seen to be compatible with $[\cdot,\cdot]$. Therefore, $(\mathfrak{g}_\mathfrak{b}, [\cdot,\cdot], \delta)$ is a topological Lie bialgebra.

- Finally, the quasi–triangular structre on $\mathfrak{g}_\mathfrak{b}$ is given by the *topological* canonical tensor $r \in \mathfrak{b}\widehat{\otimes}\mathfrak{b}^* \subset \mathfrak{g}_\mathfrak{b}\widehat{\otimes}\mathfrak{g}_\mathfrak{b}$ corresponding to the identity under the identification $\mathsf{End}(\mathfrak{b}) \simeq \mathfrak{b}\widehat{\otimes}\mathfrak{b}^*$.

Remark 2.4. If $\mathfrak{b} = \bigoplus_{n\in\mathbb{N}} \mathfrak{b}_n$ is $\mathbb{N}$–graded with finite–dimensional homogeneous components, the restricted dual $\mathfrak{b}^\star := \bigoplus_{n\in\mathbb{N}} \mathfrak{b}_n^*$ and the restricted double $\mathfrak{g}_\mathfrak{b}^{\mathrm{res}} = \mathfrak{b} \oplus \mathfrak{b}^\star$ of $\mathfrak{b}$ are also Lie bialgebras, with cobracket $\delta_\mathfrak{b} - [\cdot,\cdot]_\mathfrak{b}^t$. Moreover, $\mathfrak{g}_\mathfrak{b}^{\mathrm{res}}$ is quasi–triangular with respect to the canonical tensor $r \in \mathfrak{b}\widehat{\otimes}\mathfrak{b}^\star := \prod_{n\in\mathbb{N}} \mathfrak{b}_n \otimes \mathfrak{b}_n^*$. △

2.3.2. *Manin triples.* A Manin triple is the datum of a Lie algebra $\mathfrak{g}$ with a non–degenerate invariant symmetric bilinear form $(\cdot|\cdot)$ and two isotropic Lie subalgebras $\mathfrak{b}_\pm \subset \mathfrak{g}$ such that

(1) as a vector space $\mathfrak{g} = \mathfrak{b}_+ \oplus \mathfrak{b}_-$;

(2) the inner product defines an isomorphism $\mathfrak{b}_+ \to \mathfrak{b}_-^*$;

(3) the commutator of $\mathfrak{g}$ is continuous with respect to the topology obtained by putting the discrete and the weak topologies on $\mathfrak{b}_-$ and $\mathfrak{b}_+$ respectively.

Under these assumptions, the commutator on $\mathfrak{b}_+ \simeq \mathfrak{b}_-^*$ induces a cobracket $\delta \colon \mathfrak{b}_- \to \mathfrak{b}_- \otimes \mathfrak{b}_-$ which satisfies the cocycle condition. Therefore, $\mathfrak{b}_-$ is canonically endowed with a Lie bialgebra structure, while $\mathfrak{b}_+$ and $\mathfrak{g}$ are, in general, only topological Lie bialgebras. Moreover, $\mathfrak{g}$ is isomorphic, as a topological Lie bialgebra, to the Drinfeld double of $\mathfrak{b}_-$.

Remark 2.5. If $\mathfrak{b}$ is an $\mathbb{N}$–graded Lie bialgebra with finite–dimensional homogeneous components, one can consider *restricted* Manin triples, where the inner product induces a isomorphism $\mathfrak{b}_+ \to \mathfrak{b}_-^\star$. In this case, $\mathfrak{b}_+$ and $\mathfrak{g}$ are both Lie bialgebras and the latter is isomorphic to the restricted Drinfeld double of $\mathfrak{b}_-$. △

2.3.3. *Matched pairs of Lie algebras.* The last construction is due to S. Majid [Maj95] and it is, from a certain point of view, the most abstract, since it does not rely on a pairing. Two Lie algebras $(\mathfrak{c}, [\cdot,\cdot]_\mathfrak{c})$ and $(\mathfrak{d}, [\cdot,\cdot]_\mathfrak{d})$ form a *matched pair* if there are maps

$$\triangleright \colon \mathfrak{c} \otimes \mathfrak{d} \to \mathfrak{d} \qquad \text{and} \qquad \triangleleft \colon \mathfrak{c} \otimes \mathfrak{d} \to \mathfrak{c}$$

such that

(1) $\triangleright$ is a left action of $\mathfrak{c}$ on $\mathfrak{d}$, *i.e.*,

$$\triangleright \circ [\cdot,\cdot]_\mathfrak{c} \otimes \mathsf{id} = \triangleright \circ \mathsf{id} \otimes \triangleright \circ (\mathsf{id} - (1\,2)) \,,$$

and $\triangleleft$ is a right action of $\mathfrak{d}$ on $\mathfrak{c}$, *i.e.*,

$$\triangleleft \circ \mathsf{id} \otimes [\cdot,\cdot]_\mathfrak{d} = \triangleleft \circ \triangleleft \otimes \mathsf{id} \circ (\mathsf{id} - (2\,3)) \,;$$

(2) $\triangleleft, \triangleright$ satisfy the compatibility conditions

$$\triangleleft \circ [\cdot,\cdot]_\mathfrak{c} \otimes \mathsf{id} = [\cdot,\cdot]_\mathfrak{c} \circ \triangleleft \otimes \mathsf{id} \circ (2\,3) + [\cdot,\cdot]_\mathfrak{c} \circ \mathsf{id} \otimes \triangleleft + \triangleleft \circ \mathsf{id} \otimes \triangleright \circ (\mathsf{id} - (1\,2)) \,,$$

and

$$\triangleright \circ \mathsf{id} \otimes [\cdot,\cdot]_\mathfrak{d} = [\cdot,\cdot]_\mathfrak{d} \circ \triangleright \otimes \mathsf{id} + [\cdot,\cdot]_\mathfrak{d} \circ \mathsf{id} \otimes \triangleright \circ (1\,2) + \triangleright \circ \triangleleft \otimes \mathsf{id} \circ (\mathsf{id} - (2\,3)) \,.$$

Remark 2.6. The conditions above are equivalent to the requirement that the vector space $\mathfrak{c} \oplus \mathfrak{d}$ is endowed with a Lie bracket for which $\mathfrak{c}, \mathfrak{d}$ are Lie subalgebras and, for $X \in \mathfrak{c}$ and $Y \in \mathfrak{d}$,

$$[X, Y]_{\bowtie} = X \triangleright Y + X \triangleleft Y \,.$$

The Lie algebra $\mathfrak{c} \bowtie \mathfrak{d} = (\mathfrak{c} \oplus \mathfrak{d}, [\cdot,\cdot]_{\bowtie})$ is called the *bicross sum Lie algebra of* $\mathfrak{c}, \mathfrak{d}$. △

Example 2.7. If $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ is a Lie bialgebra, then $(\mathfrak{b}, [\cdot,\cdot]_{\mathfrak{b}})$ and $(\mathfrak{b}^*, \delta_{\mathfrak{b}}^t)$ form a matched pair with respect to the coadjoint action of $\mathfrak{b}$ on $\mathfrak{b}^*$ and the opposite coadjoint action of $\mathfrak{b}^*$ on $\mathfrak{b}$. The corresponding double cross sum Lie algebra $\mathfrak{b} \bowtie \mathfrak{b}^*$ is precisely the Drinfeld double of $\mathfrak{a}$. △

2.4. **Lie bialgebra structure on Kac–Moody algebras.** It is well–known that any symmetrisable Kac–Moody algebra has a canonical structure of (quotient of) a Manin triple, which induces on it a standard Lie bialgebra structure.

Let A be a symmetrisable Borcherds–Cartan matrix and fix an invertible diagonal matrix $\mathsf{D} = \mathsf{diag}(d_i)_{i\in\mathbf{I}}$ such that DA is symmetric. The bilinear form $(\cdot|\cdot)$ induces a canonical isomorphisms $\mathfrak{b}_\pm^\star \simeq \mathfrak{b}_\mp$, where $\mathfrak{b}_\pm^\star$ denotes the restricted dual. Consider the product Lie algebra $\mathfrak{g}^{(2)} = \mathfrak{g} \oplus \mathfrak{h}^z$, with $\mathfrak{h}^z = \mathfrak{h}$, and endow it with the inner product $(\cdot|\cdot) - (\cdot|\cdot)|_{\mathfrak{h}^z\times\mathfrak{h}^z}$. Let $\pi_0 \colon \mathfrak{g} \to \mathfrak{g}_0 := \mathfrak{h}$ be the projection, and $\mathfrak{b}_\pm^{(2)} \subset \mathfrak{g}^{(2)}$ the subalgebras

$$\mathfrak{b}_\pm^{(2)} := \{(X,h) \in \mathfrak{b}_\pm \oplus \mathfrak{h}^z |\, \pi(X) = \pm h\} \,.$$

Note that the projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ onto the first component restricts to an isomorphism $\mathfrak{b}_\pm^{(2)} \to \mathfrak{b}_\pm$ with inverse $\mathfrak{b}_\pm \ni X \mapsto (X, \pm\pi_0(X)) \in \mathfrak{b}_\pm^{(2)}$. The following is straightforward.

(1) $(\mathfrak{g}^{(2)}, \mathfrak{b}_-^{(2)}, \mathfrak{b}_+^{(2)})$ is a restricted Manin triple. In particular, $\mathfrak{b}_\mp^{(2)}$ and $\mathfrak{g}^{(2)}$ are Lie bialgebras, with cobracket $\delta_{\mathfrak{b}_\mp^{(2)}} := [\cdot,\cdot]^t_{\mathfrak{b}_\pm^{(2)}}$ and $\delta_{\mathfrak{g}^{(2)}} = \delta_{\mathfrak{b}_-^{(2)}} - \delta_{\mathfrak{b}_+^{(2)}}$.

(2) The central subalgebra $0 \oplus \mathfrak{h}^z \subset \mathfrak{g}^{(2)}$ is a coideal, so that the projection $\mathfrak{g}^{(2)} \to \mathfrak{g}$ induces a Lie bialgebra structure on $\mathfrak{g}$ and $\mathfrak{b}_\mp$.

(3) The Lie bialgebra structure on $\mathfrak{g}$ is given by

$$\delta|_{\mathfrak{h}} = 0\,, \quad \delta(e_i) = d_i h_i \wedge e_i\,, \quad \delta(f_i) = d_i h_i \wedge f_i\,.$$

2.5. **Kac–Moody algebras by duality.** We recall an alternative construction of symmetrisable Kac–Moody algebra, provided by G. Halbout in terms of matched pairs of Lie bialgebras [Hal99]. More precisely, his construction goes as follows.

- Let A be a symmetrisable Borcherds–Cartan matrix, $\mathsf{D} = \mathsf{diag}(d_i)_{i\in\mathbf{I}}$ an invertible diagonal matrix such that DA is symmetric, and $(\cdot|\cdot)$ the corresponding non–degenerate bilinear form on $\mathfrak{h}$.

- Let $\mathcal{L}_\pm$ be the free Lie algebras generated by the set $X_\pm := \{x_i^\pm, \xi^\pm \mid i \in \mathbf{I}, \xi \in \mathfrak{h}\}$. The assignment
$$\delta_\pm(\xi^\pm) := 0 \quad\text{and}\quad \delta_\pm(x_i^\pm) := \mp d_i h_i^\pm \wedge x_i^\pm$$
extends uniquely to a cobracket on $\mathcal{L}_\pm$ and induces a Lie bialgebra structure on it.

- The assignment
$$\langle x_i^+, x_j^- \rangle := d_i^{-1}\delta_{ij}\,, \quad \langle \xi^+, \zeta^- \rangle := 2\,(\xi|\zeta)\,, \quad\text{and}\quad (x_i^+|\zeta^-) := 0 =: (\xi^+|x_i^-)\,,$$
extends uniquely to Lie bialgebra pairing $\langle\cdot,\cdot\rangle \colon \mathcal{L}_+ \otimes \mathcal{L}_- \to \mathbf{k}$, *i.e.*, for $X_\pm, Y_\pm \in \mathcal{L}_\pm$,
$$\langle [X_\pm, Y_\pm], X_\mp \rangle = \langle X_\pm \otimes Y_\pm, \delta_\mp(X_\mp) \rangle\,. \tag{2.3}$$
Then, $\mathcal{L}_+$ and $\mathcal{L}_-$ naturally form a matched pair of Lie bialgebras. [4]
The pairing $\langle\cdot,\cdot\rangle$ extends to the a possibly degenerate, invariant pairing on the double cross sum Lie bialgebra $\mathcal{L}_+ \bowtie \mathcal{L}_-$.

[4] By slight abuse of notation, we impose that $\langle\cdot,\cdot\rangle$ is *symmetric*, *i.e.*, it can be considered as a function on either $\mathcal{L}_+ \otimes \mathcal{L}_-$ or $\mathcal{L}_- \otimes \mathcal{L}_+$, regardless of the order. Moreover, note that (2.3) can be equivalently restated as $\langle [X_\pm, Y_\pm], X_\mp \rangle = \langle X_\pm \wedge Y_\pm, \delta_\mp(X_\mp) \rangle$.

- The ideals generated by $[\xi^{\pm},\xi^{\pm}]$, $[\xi^{\pm},x_i^{\pm}]\mp\alpha_i(\xi)x_i^{\pm}$, $\mathsf{ad}(x_i^{\pm})^{1-\frac{2}{a_{ii}}a_{ij}}(x_j^{\pm})$ ($i\neq j$ and $a_{ii}>0$), and $[x_i^{\pm},x_j^{\pm}]$ ($a_{ii}\leqslant 0$ and $a_{ij}=0$) are orthogonal to $\mathcal{L}_{\mp}$ and are coideals. Let $\mathfrak{s}$ be the sum of these ideals. In particular, $\mathfrak{s}\subseteq\mathfrak{k}=\ker\langle\cdot,\cdot\rangle\subseteq\mathcal{L}_+\bowtie\mathcal{L}_-$.

- Finally, one observes that $\mathcal{L}_+\bowtie\mathcal{L}_-/\mathfrak{k}$ has the form $\mathfrak{g}\oplus\mathfrak{h}^z$, where $\mathfrak{h}^z$ is a central copy of $\mathfrak{h}$ and $\mathfrak{g}$ is the Borcherds–Kac–Moody algebra associated to A.[5] This implies, in particular, that $\mathfrak{k}$ coincides with s and it is a coideal. Therefore, the Lie bialgebra structure on $\mathcal{L}_+\bowtie\mathcal{L}_-$ naturally descends to $\mathfrak{g}$.

2.6. **Drinfeld–Jimbo quantum groups.** Let A be a symmetrisable Borcherds–Cartan matrix and fix an invertible diagonal matrix $\mathsf{D}=\mathsf{diag}(d_i)_{i\in\mathbf{I}}$ such that DA is symmetric. Let $\mathfrak{g}=\mathfrak{g}(\mathsf{A})$ be the corresponding Borcherds–Kac–Moody algebra with its standard Lie bialgebra structure, and set $q:=\exp(\hbar/2)$, $q_i:=\exp(\hbar/2\cdot d_i)$. The following is a straightforward generalization to Borcherds–Kac–Moody algebras of the quantum group defined by Drinfeld [Dri87] and Jimbo [Jim85] (cf. also [Kan95]).

The Drinfeld–Jimbo quantum group of $\mathfrak{g}$ is the associative algebra $\mathbf{U}_q\mathfrak{g}$ topologically generated over $\mathbf{K}$ by $\mathfrak{h}$ and the elements $E_i, F_i, i\in\mathbf{I}$ satisfying the following defining relations.

(1) **Diagonal action:** for $h,h'\in\mathfrak{h}$, $i\in\mathbf{I}$,

$$[h,h']=0\,,\quad [h,E_i]=\alpha_i(h)E_i\,,\quad [h,F_i]=-\alpha_i(h)F_i\,.$$

In particular, for $K_i:=\exp(\hbar/2\cdot d_ih_i)$, it holds $K_iE_j=q_i^{a_{ij}}\cdot E_jK_i$ and $K_iF_j=q_i^{-a_{ij}}\cdot E_jK_i$.

(2) **Quantum double relations:**

$$[E_i,F_i]=\frac{K_i-K_i^{-1}}{q_i-q_i^{-1}}\,,$$

(3) **Quantum Serre relations:** for $i,j\in\mathbf{I}$ with $a_{ij}=0$,

$$[E_i,E_j]=0=[F_i,F_j]\,,$$

and for $i,j\in\mathbf{I}$, $i\neq j$, with $a_{ii}=2$,

$$\sum_{m=0}^{1-a_{ij}}\frac{(-1)^m}{[m]_{q_i}![1-a_{ij}-m]_{q_i}!}X_i^{1-a_{ij}-m}X_jX_i^m=0\qquad (X=E,F)\,.$$

Moreover, $\mathbf{U}_q\mathfrak{g}$ has a Hopf algebra structure with counit, coproduct and antipode defined, for $h\in\mathfrak{h}$ and $i\in\mathbf{I}$, by

$$\begin{aligned}
\varepsilon(h)&=0\,, & \Delta(h)&=h\otimes 1+1\otimes h\,, & S(h)&=-h\,,\\
\varepsilon(E_i)&=0\,, & \Delta(E_i)&=E_i\otimes K_i+1\otimes E_i\,, & S(E_i)&=-E_iK_i^{-1}\,,\\
\varepsilon(F_i)&=0\,, & \Delta(F_i)&=F_i\otimes 1+K_i^{-1}\otimes F_i\,, & S(F_i)&=-K_iF_i\,.
\end{aligned}$$

The following is well–known (cf. [Dri87, Lus10, CP95]).

Theorem 2.8.

(1) The Hopf algebra $\mathbf{U}_q\mathfrak{g}$ is a quantization of the Lie bialgebra $\mathfrak{g}$.

(2) Denote by $\mathbf{U}_q\mathfrak{b}_-$ (resp. $\mathbf{U}_q\mathfrak{b}_+$) the Hopf subalgebra generated by $\mathfrak{h}$ and $\{F_i, i\in\mathbf{I}\}$ (resp. $\mathfrak{h}$ and $\{E_i, i\in\mathbf{I}\}$). Then, $\mathbf{U}_q\mathfrak{b}_-$ (resp. $\mathbf{U}_q\mathfrak{b}_+$) is a quantisation of the Lie bialgebra $\mathfrak{b}_-$ (resp. $\mathfrak{b}_+$), and

[5] Indeed, it is clear that there is a surjective Lie algebra homomorphism $\pi\colon\mathfrak{g}\to\mathfrak{d}$, where $\mathfrak{d}=\mathcal{L}_+\bowtie\mathcal{L}_-/(\mathfrak{k}\oplus\mathfrak{h}^z)$, and, since $\ker\pi\subset\mathfrak{g}$ is an ideal trivially intersecting $\mathfrak{h}$, it must be necessarily trivial.

there exists a unique non–degenerate Hopf pairing $(\cdot|\cdot)_{\mathcal{D}} : \mathbf{U}_q\mathfrak{b}_- \otimes \mathbf{U}_q\mathfrak{b}_+ \to \mathbf{k}((\hbar))$, i.e., *a non–degenerate bilinear form compatible with the Hopf algebra structure, defined on the generators by*

$$(1|1)_{\mathcal{D}} := 1\,, \quad (h|h')_{\mathcal{D}} := \frac{1}{\hbar}\,(h|h')\,, \quad (F_i|E_j)_{\mathcal{D}} := \frac{\delta_{ij}}{q-q^{-1}}\,,$$

and zero otherwise.

(3) *The Hopf pairing* $(\cdot|\cdot)_{\mathcal{D}}$ *induces an isomorphism of Hopf algebras* $\mathbf{U}_q\mathfrak{b}_- \simeq (\mathbf{U}_q\mathfrak{b}_+)^{\star}$, *which restricts to the identification* $\phi\colon \mathfrak{h} \to \mathfrak{h}^*$, $\phi(h) = -2\,(h|\cdot)$. *Moreover,* $\mathbf{U}_q\mathfrak{g}$ *can be realized as a quotient of the* restricted quantum double *of* $\mathbf{U}_q\mathfrak{b}_-$ *with respect to this identification,* i.e., $\mathcal{D}\mathbf{U}_q\mathfrak{b}_-/(\mathfrak{h} \simeq \mathfrak{h}^*) \simeq \mathbf{U}_q\mathfrak{g}$. *In particular,* $\mathbf{U}_q\mathfrak{g}$ *is a quasi–triangular Hopf algebra with R–matrix*

$$\overline{R} = q^{\sum_i u_i \otimes u_i} \cdot \sum_p X_p \otimes X^p,$$

where $\{u_i\} \subset \mathfrak{h}$ *is an orthonormal basis with respect to* $(\cdot|\cdot)$, $\{X_p\} \subset \mathbf{U}_q\mathfrak{n}_-$, $\{X^p\} \subset \mathbf{U}_q\mathfrak{n}_+$ *are dual basis with respect to the pairing* $(\cdot|\cdot)_{\mathcal{D}}$.

It is useful to notice here that the proof of the theorem and the construction of the Hopf pairing $(\cdot|\cdot)_{\mathcal{D}}$ is obtained following a quantum analogue of the procedure described in Section 2.5 (cf. [Lus10, Part I]).

3. Continuum Kac–Moody algebras

In this section, we recall the notion of continuum Kac–Moody algebras introduced in [ASS18], and their realization as continuous colimits of Borcherds–Kac–Moody algebras.

3.1. **Vertex space.**

Definition 3.1. Let X be a Hausdorff topological space. We say that X is a *vertex space* if for any $x \in X$, there exists a *chart* (U, A, ϕ) *around* x such that

(1) U is an open neighborhood of x,

(2) $A = \{A_i\}$ is a family of closed subsets $A_i \subseteq U$ containing x, such that $U = \bigcup_i A_i$,

(3) $\phi = \{\phi_i\}$ is a family of continuous maps $\phi_i\colon A_i \to \mathbb{R}$ which are homeomorphisms onto open intervals of $\mathbb{R}$, such that if the intersection between A_i and A_j strictly contains the point x, then $\phi_i|_{A_i\cap A_j} = \phi_j|_{A_i\cap A_j}$ and $\phi_i|_{A_i\cap A_j}$ induces a homeomorphism between $A_i \cap A_j$ and a closed interval of $\mathbb{R}$.

We say that x is an *regular point* if the exist a chart such that $A = \{U\}$; while, we say that x is a *critical point* if there exists a chart such that the boundary $\partial(A_i \cap A_j)$ of $A_i \cap A_j$, as a subset of U, contains x for any i, j.[6] ⊘

Remark 3.2. Let x be a critical point with a chart (U, A, ϕ) such that $x \in \partial(A_i \cap A_j)$ for any i, j. Then $x \in \partial A_i$ for any i. △

Definition 3.3. Let X be a vertex space and let α be a subset of X. We say that α is an *elementary interval* if there exists a chart (U, A, ϕ) for which $J \subset A_i$ for some i and $\phi_i(\alpha)$ is a open-closed interval of $\mathbb{R}$. A sequence of elementary intervals $(\alpha_1, \dots, \alpha_n)$, $n > 0$, is *admissible* if

(a) $(\alpha_1 \cup \dots \cup \alpha_i) \cap \alpha_{i+1} = \emptyset$ and $(\alpha_1 \cup \dots \cup \alpha_i) \cup \alpha_{i+1}$ is connected for any $i = 1, \dots, n-1$;

(b) for any $i = 1, \dots, n-1$, there exist $x \in X$ and a chart (U, A, ϕ) around x for which $U \supseteq (\alpha_1 \cup \dots \cup \alpha_i) \cup \alpha_{i+1}$ and $\big((\alpha_1 \cup \dots \cup \alpha_i) \cup \alpha_{i+1}\big) \cap A_k$ is either empty or an elementary interval for any k.

[6] Here, somehow we are following the terminology coming from the theory of persistence modules (cf. [DEHH18, Section 2.3].

An *interval* of X is a subset α of the form $\alpha_1 \cup \cdots \cup \alpha_n$, where $(\alpha_1, \ldots, \alpha_n)$ is an admissible sequence of elementary intervals. We denote by $\mathsf{Int}(X)$ the set of all intervals in X. ⊘

Example 3.4. Let $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. Then $\mathbb{K}$ is an example of a vertex space. An interval of $\mathbb{K}$ is a subset $\alpha \subset \mathbb{R}$ which is an an open–closed interval of the form $\alpha = (a, b] := \{x \in \mathbb{R} \mid a < x \leqslant b\}$ for some $\mathbb{K}$-values $a < b$. △

3.2. **Continuum quivers.** Let X be a vertex space and $\mathsf{Int}(X)$ the set of all intervals of X. Set

$$\alpha \oplus \beta := \begin{cases} \alpha \cup \beta & \text{if } \alpha \cap \beta = \emptyset \text{ and } \alpha \cup \beta \in \mathsf{Int}(X)\,, \\ \text{n.d.} & \text{otherwise}\,, \end{cases}$$

$$\alpha \ominus \beta := \begin{cases} \alpha \setminus \beta & \text{if } \alpha \cap \beta = \beta \text{ and } \alpha \setminus \beta \in \mathsf{Int}(X)\,, \\ \text{n.d.} & \text{otherwise}\,. \end{cases}$$

We call $\oplus$ the *sum of intervals*, while we call $\ominus$ the *difference of intervals*.

Remark 3.5. The elements of $\mathsf{Int}(X)$ are described as follows [ASS18, Lemma 5.5].

(1) Every contractible interval is homeomorphic to a finite oriented tree such that any vertex is the target of at most one edge.
(2) Every non–contractible interval is homeomorphic to an interval of the form

$$S^1 \oplus \bigoplus_{k=1}^{N} T_k := (\cdots(S^1 \oplus T_1) \oplus T_2) \cdots \oplus T_N)$$

for some pairwise disjoint contractible intervals T_k, with $N \geqslant 0$.

△

We denote by $\mathfrak{f}_X$ the $\mathbb{Z}$-span of the characteristic functions $\mathbb{1}_\alpha$ for all interval α of X. Note that $\mathbb{1}_{\alpha\oplus\beta} = \mathbb{1}_\alpha + \mathbb{1}_\beta$ for a given $(\alpha, \beta) \in \mathsf{Int}(X)^{(2)}_{\oplus}$. We call *support* of a function $f \in \mathfrak{f}_X$ the set $\mathsf{supp}(f) := \{p \in X \mid f(p) \neq 0\}$. It is a disjoint union of finitely many intervals of X.

Define a bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{f}_X$ in the following way. Let $f, g \in \mathfrak{f}_X$, and assume that there exists a point x with a chart (U, A, ϕ) for which the supports of f and g are contained in A_i for some i, then we set

$$\langle f, g\rangle := \sum_{p \in A_i} f_-(p)(g_-(p) - g_+(p))\,, \tag{3.1}$$

where $h_\pm(x) = \lim_{t\to 0^+} h(x \pm t)$.

Since we can always decompose an interval into a sum of elementary subintervals (and we can do similarly with supports of functions of $\mathfrak{f}_X$), we extend $\langle\cdot,\cdot\rangle$ with respect to $\oplus$ by imposing the condition that $\langle \mathbb{1}_\alpha, \mathbb{1}_\beta\rangle = 0$ for two elementary intervals α, β for which there does not exist a common A_i containing both.

As a consequence of the definition, the bilinear form $\langle\cdot,\cdot\rangle$ is compatible with the concatenation of intervals, by Remark 3.5, it is entirely determined by its values on contractible elements.

Remark 3.6. Thanks to the condition (b) of Definition 3.3, one can easily verify that if β is a non–contractible sub–interval of α, then $\langle \mathbb{1}_\alpha, \mathbb{1}_\beta\rangle = \langle \mathbb{1}_{\alpha\ominus\beta}, \mathbb{1}_\beta\rangle$, whenever $\alpha \ominus \beta$ is defined.

Moreover, whenever $\alpha \perp \beta$, *i.e.*, $(\alpha, \beta) \notin \mathsf{Int}(X)^{(2)}_{\oplus}$ and $\alpha \cap \beta = \emptyset$, then $\langle \mathbb{1}_\alpha, \mathbb{1}_\beta\rangle = 0$. Note also that

$$\langle \mathbb{1}_\alpha, \mathbb{1}_\alpha\rangle = \begin{cases} 1 & \text{if } \alpha \text{ is contractible}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

△

Set

$$(f|g) := \langle f,g\rangle + \langle g,f\rangle$$

for $f,g \in \mathfrak{f}_X$. Then, if $J,J' \in \mathsf{Int}(X)$ are contractible, then

$$(\mathbb{1}_\alpha|\mathbb{1}_\beta) = \begin{cases} 2 & \text{if } \alpha=\beta\,, \\ 1 & \text{if } (\alpha,\beta)\in\mathsf{Int}(X)^{(2)}_{\ominus} \text{ or } (\beta,\alpha)\in\mathsf{Int}(X)^{(2)}_{\ominus}\,, \\ 0 & \text{if } (\alpha,\beta)\notin\mathsf{Int}(X)^{(2)}_{\oplus} \text{ and } \alpha\cap\beta=\emptyset, \\ -1 & \text{if } (\alpha,\beta)\in\mathsf{Int}(X)^{(2)}_{\oplus} \text{ and } \alpha\oplus\beta \text{ is contractible}\,, \\ -2 & \text{if } (\alpha,\beta)\in\mathsf{Int}(X)^{(2)}_{\oplus} \text{ and } \alpha\oplus\beta \text{ is non–contractible}\,. \end{cases}$$

All other cases follow therein. Note in particular that, if α is non–contractible, $(\mathbb{1}_\alpha|\mathbb{1}_\alpha)=0$.

Henceforth, we set $\langle\alpha,\beta\rangle := \langle\mathbb{1}_\alpha,\mathbb{1}_\beta\rangle$ and $(\alpha|\beta) := (\mathbb{1}_\alpha|\mathbb{1}_\beta)$. It follows immediately from Remark 3.6 that

$$(\alpha|\alpha) = \begin{cases} 2 & \text{if } \alpha \text{ is contractible}\,, \\ 0 & \text{if } \alpha \text{ is non–contractible}\,. \end{cases}$$

Therefore, we will use *real* (resp. *imaginary*) as a synonym of contractible (resp. non–contractible) in analogy with the terminology used for the roots of a Kac–Moody algebra.

Finally, we give the following:

Definition 3.7. Let X be a space of vertices. The *continuum quiver of* X is the datum $\mathcal{Q}_X := (\mathsf{Int}(X),\oplus,\ominus,\langle\cdot,\cdot\rangle,(\cdot|\cdot))$. ⊘

3.3. **Continuum Kac–Moody algebras.** It is well–known that the set R_+ of positive roots of a Kac–Moody algebra $\mathfrak{g}$ has a standard structure of *partial semigroup*, induced by its embedding in the root lattice Q_+, and that, as Lie bialgebras, the positive and negative Borel subalgebras $\mathfrak{b}_\pm$ are graded over R_+ (cf. [ATL19b, Sec. 8]). Roughly speaking, continuum Kac–Moody algebras are obtained by replacing the semigroup of the positive roots with the continuum quiver $\mathcal{Q}_X$. Namely, to any continuum quiver $\mathcal{Q}_X$, we associate a Lie algebra $\mathfrak{g}_X$, whose definition mimics the construction of Kac–Moody algebras. Let $\widetilde{\mathfrak{g}}_X$ be the Lie algebra over $\mathbb{C}$, freely generated by $\mathfrak{f}_X$ and the elements $x^\pm_\alpha$, $\alpha\in\mathsf{Int}(X)$, subject to the relations:

$$[\xi_\alpha,\xi_\beta]=0\,,\qquad [\xi_\alpha,x^\pm_\beta]=\pm\,(\alpha|\beta)\cdot x^\pm_\beta\,,$$

$$[x^+_\alpha,x^-_\beta]=\delta_{\alpha\beta}\xi_\alpha+(-1)^{\langle\alpha,\beta\rangle}\cdot(\alpha|\beta)\cdot(x^+_{\alpha\ominus\beta}-x^-_{\beta\ominus\alpha})\,.$$

where $\xi_\alpha := \mathbb{1}_\alpha$.

Note that the relation $\xi_{\alpha\oplus\beta}=\delta_{\alpha\oplus\beta}\,(\xi_\alpha+\xi_\beta)$ holds by definition. Set

$$\mathfrak{f}^+_X := \mathsf{span}_{\mathbb{Z}_{\geqslant 0}}\{\mathbb{1}_\alpha \mid \alpha\in\mathsf{Int}(X)\}\,.$$

There is a natural $\mathfrak{f}_X$–gradation on $\widetilde{\mathfrak{g}}_X$ given by $\deg(x^\pm_\alpha)=\pm\mathbb{1}_\alpha$ and $\deg(\xi_\alpha)=0$, inducing a triangular decomposition

$$\widetilde{\mathfrak{g}}_X = \left(\bigoplus_{\phi\in\mathfrak{f}^+_X}\widetilde{\mathfrak{g}}_{+\phi}\right)\oplus\mathfrak{f}_X\oplus\left(\bigoplus_{\phi\in\mathfrak{f}^+_X}\widetilde{\mathfrak{g}}_{-\phi}\right)\,.$$

where $\widetilde{\mathfrak{g}}_{\pm\phi}$ denotes the homogeneous subspace of degree $\pm\phi$.

It is important to observe that the bilinear form $(\cdot|\cdot)$ on $\mathfrak{f}_X$ is non–degenerate unless $X=S^1$, in which case, $\ker(\cdot|\cdot)=\mathbb{Z}\cdot\mathbb{1}_{S^1}$. Therefore, whenever $X\neq S^1$, the homogeneous spaces $\widetilde{\mathfrak{g}}_{\pm\phi}$ coincide with weight spaces corresponding to the diagonal action of $\mathfrak{f}_X$. That is, we have

$$\widetilde{\mathfrak{g}}_{\pm\phi}=\{x\in\widetilde{\mathfrak{g}}_X \mid \forall\psi\in\mathfrak{f}_X \mid [\psi,x]=\pm\,(\phi|\psi)\cdot x\}\,,$$

for $\phi \in \mathfrak{f}_X^+$.

Definition 3.8. The *continuum Kac–Moody algebra of* $\mathcal{Q}_X$ is the Lie algebra $\mathfrak{g}_X := \widetilde{\mathfrak{g}}_X / \mathfrak{r}_X$, where $\mathfrak{r}_X \subset \widetilde{\mathfrak{g}}_X$ is the sum of all two–sided graded ideals with trivial intersection with $\mathfrak{f}_X$. ⊘

In particular, $\mathfrak{g}_X$ has a triangular decomposition

$$\mathfrak{g}_X = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- ,$$

where $\mathfrak{h} = \mathfrak{f}_X$ and $\mathfrak{n}_\pm$ are the Lie subalgebras generated, respectively, by the elements $x_\alpha^\pm$, $\alpha \in \mathsf{Int}(X)$.

The main result of [ASS18] is a generalization to the case of $\mathfrak{g}_X$ of the results of Gabber–Kac [GK81] and Borcherds [Bor88], showing that the ideal $\mathfrak{r}_X$ is generated by the Serre relations. In particular, this gives an explicit description of the Lie algebra $\mathfrak{g}_X$ as follows.

Definition 3.9. Let $\mathsf{Serre}(X)$ be the set of all pairs $(\alpha, \beta) \in \mathsf{Int}(X) \times \mathsf{Int}(X)$ such that one of the following occurs:

- α is contractible, and, for subintervals $\alpha' \subseteq \alpha$ and $\beta' \subseteq \beta$ with $(\beta|\beta') \neq 0$ whenever $\beta' \neq \beta$, $\alpha' \oplus \beta'$ is either undefined or non–homeomorphic to S^1;
- $\alpha \perp \beta$, *i.e.*, $\alpha \oplus \beta$ does not exist and $\alpha \cap \beta = \emptyset$.

⊘

Example 3.10. One has $\mathsf{Serre}(\mathbb{R}) = \mathsf{Int}(\mathbb{R}) \times \mathsf{Int}(\mathbb{R})$ and $\mathsf{Serre}(S^1) = \left(\mathsf{Int}(S^1) \setminus \{S^1\}\right) \times \mathsf{Int}(S^1)$. △

Set

$$\mathsf{a}_{\alpha\beta} := (-1)^{\langle \alpha, \beta \rangle} \cdot (\alpha|\beta) \quad \text{and} \quad \mathsf{b}_{\alpha\beta} := \mathsf{a}_{\alpha, \alpha\oplus\beta} . \tag{3.2}$$

Note that, if $\alpha \ominus \beta$ or $\beta \ominus \alpha$ are defined, then $\mathsf{a}_{\alpha\beta} \in \{0, \pm 1\}$, and, if $\alpha \oplus \beta$ is defined and $(\alpha, \beta) \in \mathsf{Serre}(X)$, then $\mathsf{b}_{\alpha\beta} \in \{\pm 1\}$.

Theorem 3.11 (cf. [ASS18, Theorem 5.16])**.** *The continuum Kac–Moody algebra* $\mathfrak{g}_X$ *is freely generated by the abelian Lie algebra* $\mathfrak{f}_X$ *and the elements* $x_\alpha^\pm$*,* $\alpha \in \mathsf{Int}(X)$*, subject to the following defining relations:*

(1) **Diagonal action:** *for* $\alpha, \beta \in \mathsf{Int}(X)$*,*

$$[\xi_\alpha, x_\beta^\pm] = \pm\, (\alpha|\beta) \cdot x_\beta^\pm \; ;$$

(2) **Double relations:** *for* $\alpha, \beta \in \mathsf{Int}(X)$*,*

$$[x_\alpha^+, x_\beta^-] = \delta_{\alpha\beta}\, \xi_\alpha + \mathsf{a}_{\alpha\beta} \cdot \left(x_{\alpha\ominus\beta}^+ - x_{\beta\ominus\alpha}^- \right) \; ;$$

(3) **Serre relations:** *for* $(\alpha, \beta) \in \mathsf{Serre}(X)$*,*

$$[x_\alpha^\pm, x_\beta^\pm] = \pm \mathsf{b}_{\alpha\beta} \cdot x_{\alpha\oplus\beta}^\pm \, .$$

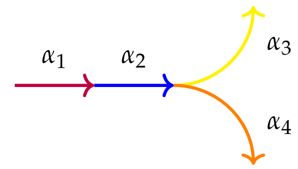
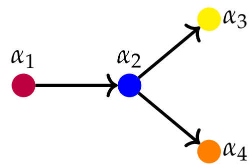
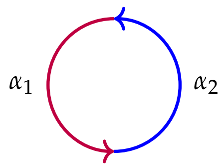
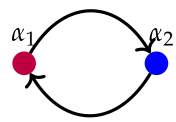
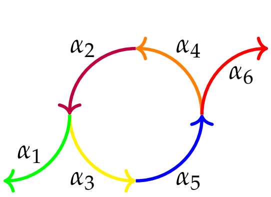
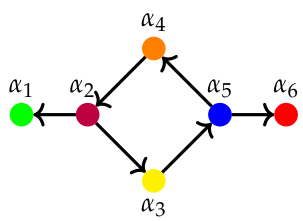
Remark 3.12. If $\beta \simeq S^1$ and $\alpha \subseteq \beta$, then $(\alpha, \beta) \in \mathsf{Serre}(X)$. Hence, by (2) above $[x_\alpha^\pm, x_\beta^\pm] = 0$. △

3.4. **Colimit realization.** One fundamental ingredient in the proof of Theorem 3.11 is the relation between $\mathfrak{g}_X$ and certain Borcherds–Kac–Moody algebras naturally arising from families of intervals. Let $\mathcal{J} = \{\alpha_k\}_k$ be a finite set of intervals $\alpha_k \in \mathsf{Int}(X)$. We say that $\mathcal{J}$ is *irreducible* if the following conditions hold:

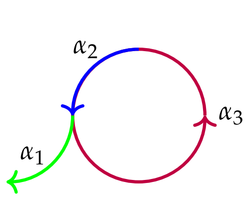
(1) every interval is either contractible or homeomorphic to S^1;

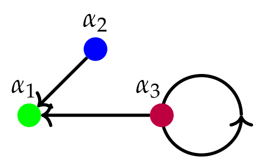
(2) given two intervals $\alpha, \beta \in \mathcal{J}$, $\alpha \neq \beta$, one of the following mutually exclusive cases occurs:
  (a) $\alpha \oplus \beta$ exists;
  (b) $\alpha \oplus \beta$ does not exist and $\alpha \cap \beta = \emptyset$;
  (c) $\alpha \simeq S^1$ and $\beta \subset \alpha$.

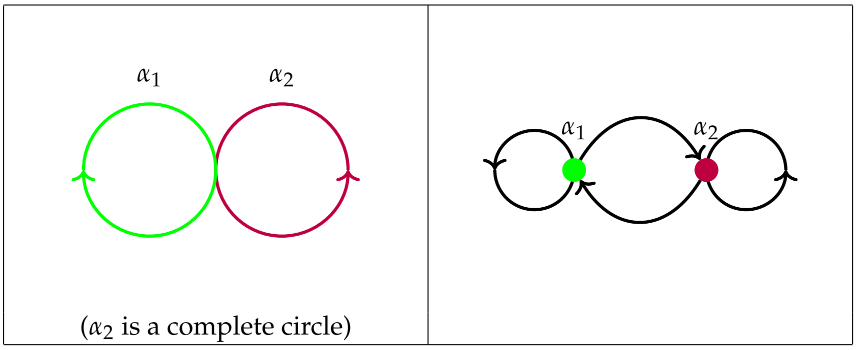
Assume henceforth that $\mathcal{J}$ is an irreducible set of intervals. Let $\mathsf{A}_{\mathcal{J}}$ be the matrix given by the values of $(\cdot|\cdot)$ on $\mathcal{J}$, *i.e.*, $(\mathsf{A}_{\mathcal{J}})_{\alpha\beta} = (\alpha|\beta)$ for $\alpha, \beta \in \mathcal{J}$. Note that the diagonal entries of $\mathsf{A}_{\mathcal{J}}$ are either 2 or 0, while the only possible off–diagonal entries are $0, -1, -2$. Let $\mathcal{Q}_{\mathcal{J}}$ be the corresponding quiver with Cartan matrix $\mathsf{A}_{\mathcal{J}}$. Note that a contractible elementary interval in $\mathcal{J}$ corresponds to a vertex of $\mathcal{Q}_{\mathcal{J}}$ without loops at it. For example, we obtain the following quivers.

<table>
<tr><th>Configuration of intervals</th><th>Borcherds–Cartan diagram</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Instead, an interval of $\mathcal{J}$ homeomorphic to S^1 corresponds in $\mathcal{Q}_{\mathcal{J}}$ to a vertex having exactly one loop at it, as in the following examples.

<table>
<tr><th>Configuration of intervals</th><th>Borcherds–Cartan diagram</th></tr>
<tr><td>

(α_3 is a complete circle)</td><td>
</td></tr>
</table>



To any irreducible set of intervals $\mathcal{J}$, we can associate two Lie algebras:

(1) the Lie subalgebra $\mathfrak{g}_{\mathcal{J}} \subset \mathfrak{g}_X$ generated by the elements $\{x_\alpha^\pm, \xi_\alpha \mid \alpha \in \mathcal{J}\}$;

(2) the derived Borcherds–Kac–Moody algebra $\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} := \mathfrak{g}(\mathsf{A}_{\mathcal{J}})'$.

We prove in [ASS18, Section 5.5] that $\mathfrak{g}_{\mathcal{J}}$ and $\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}$ are canonically isomorphic. More precisely, we have the following.

Proposition 3.13. *The assignment*

$$e_\alpha \mapsto x_\alpha^+ \,, \quad f_\alpha \mapsto x_\alpha^- \quad \text{and} \quad h_\alpha \mapsto \xi_\alpha$$

for any $\alpha \in \mathcal{J}$, defines an isomorphism of Lie algebras $\Phi_{\mathcal{J}} \colon \mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \to \mathfrak{g}_{\mathcal{J}}$.

The proof relies on the simple observation that, for $\alpha, \beta \in \mathcal{J}$, $\alpha \neq S^1, \beta$,

$$\mathsf{ad}(x_\alpha^\pm)^{1-(\alpha|\beta)}(x_\beta^\pm) = 0 \,.$$

It is then clear that $\mathfrak{g}_X$ can be constructed exclusively from Borcherds–Kac–Moody algebras. That is, we have the following.

Corollary 3.14. *Let $\mathcal{J}, \mathcal{J}'$ be two irreducible (finite) sets of intervals in X.*

(1) If $\mathcal{J}' \subseteq \mathcal{J}$, there is a canonical embedding $\phi'_{\mathcal{J},\mathcal{J}'} \colon \mathfrak{g}_{\mathcal{J}'} \to \mathfrak{g}_{\mathcal{J}}$ sending $x_\alpha^\pm \mapsto x_\alpha^\pm$, $\xi_\alpha \to \xi_\alpha$ for $\alpha \in \mathcal{J}'$.

(2) If $\mathcal{J}$ is obtained from $\mathcal{J}'$ by replacing an element $\gamma \in \mathcal{J}'$ with two intervals α, β such that $\gamma = \alpha \oplus \beta$, there is a canonical embedding $\phi''_{\mathcal{J},\mathcal{J}'} \colon \mathfrak{g}_{\mathcal{J}'} \to \mathfrak{g}_{\mathcal{J}}$, which is the identity on $\mathfrak{g}_{\mathcal{J}'\setminus\{\gamma\}} = \mathfrak{g}_{\mathcal{J}\setminus\{\alpha,\beta\}}$ and sends

$$\xi_\gamma \mapsto \xi_\alpha + \xi_\beta \,, \quad x_\gamma^\pm \mapsto \pm(-1)^{\langle\beta,\alpha\rangle}\, [x_\alpha^+, x_\beta^+] \,.$$

(3) The collection of embeddings $\phi'_{\mathcal{J},\mathcal{J}'}, \phi''_{\mathcal{J},\mathcal{J}'}$, indexed by all possible irreducible sets of intervals in X, form a direct system. Moreover,

$$\mathfrak{g}_X \simeq \mathsf{colim}_{\mathcal{J}}\, \mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \,.$$

3.5. **The Lie algebras of the line and of the circle.** In this section we recall the defining relations of the Lie algebras of the line and the circle, $\mathfrak{sl}(\mathbb{K})$ and $\mathfrak{sl}(\mathbb{K}/\mathbb{Z})$ with $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, introduced in [SS17], and their realizations as continuum Kac–Moody algebras. Indeed, these examples were the stepping stones for the definition of continuum Kac–Moody algebras.

First, we need to distinguish all relative positions of two intervals. For any two intervals $\alpha = (a, b]$ and $\beta = (a', b']$, we write

- $\alpha \to \beta$ if $b = a'$ (*adjacent*)
- $\alpha \perp \beta$ if $b < a'$ or $b' < a$ (*disjoints*)
- $\alpha \vdash \beta$ if $a = a'$ and $b < b'$ (*closed subinterval*)

- $\alpha \dashv \beta$ if $a' < a$ and $b = b'$ (*open subinterval*) [7]
- $\alpha < \beta$ if $a' < a < b < b'$ (*strict subinterval*)
- $\alpha \pitchfork \beta$ if $a < a' < b < b'$ (*overlapping*)

We are ready to give the definition of $\mathfrak{sl}(\mathbb{K})$.

Definition 3.15. Let $\mathfrak{sl}(\mathbb{K})$ be the Lie algebra generated by elements $e_\alpha, f_\alpha, h_\alpha$, with $\alpha \in \mathsf{Int}(\mathbb{K})$, modulo the following set of relations:

- **Kac–Moody type relations:** for any two intervals α, β,

$$[h_\alpha, h_\beta] = 0\,, \qquad [h_\alpha, e_\beta] = (\mathbb{1}_\alpha | \mathbb{1}_\beta)\; e_\beta\,, \qquad [h_\alpha, f_\beta] = -\,(\mathbb{1}_\alpha | \mathbb{1}_\beta)\; f_\beta\,,$$

$$[e_\alpha, f_\beta] = \begin{cases} h_\alpha & \text{if } \alpha = \beta\,, \\ 0 & \text{if } \alpha \perp \beta, \alpha \to \beta, \text{ or } \beta \to \alpha\,, \end{cases}$$

- **join relations:** for any two intervals α, β with $(\alpha, \beta) \in \mathsf{Int}(\mathbb{K})^{(2)}_{\oplus}$,

$$h_{\alpha\oplus\beta} = h_\alpha + h_\beta\,, \qquad e_{\alpha\oplus\beta} = (-1)^{\langle \mathbb{1}_\beta, \mathbb{1}_\alpha\rangle}[e_\alpha, e_\beta]\,, \qquad f_{\alpha\oplus\beta} = (-1)^{\langle \mathbb{1}_\alpha, \mathbb{1}_\beta\rangle}[f_\alpha, f_\beta]\,;$$

- **nest relations:** for any nested $\alpha, \beta \in \mathsf{Int}(\mathbb{K})$ (that is, such that $\alpha = \beta$, $\alpha \perp \beta$, $\alpha < \beta$, $\beta < \alpha$, $\alpha \vdash \beta$, $\alpha \dashv \beta$, $\beta \vdash \alpha$, or $\beta \dashv \alpha$),

$$[e_\alpha, e_\beta] = 0 \quad \text{and} \quad [f_\alpha, f_\beta] = 0\,.$$

⊘

Remark 3.16. It is easy to check that the bracket is anti–symmetric and satisfies the Jacobi identity. Note that the joint relations are consistent with anti–symmetry, since, whenever $\alpha \oplus \beta$ is defined, $(-1)^{\langle \mathbb{1}_\alpha, \mathbb{1}_\beta\rangle} = -(-1)^{\langle \mathbb{1}_\beta, \mathbb{1}_\alpha\rangle}$. Moreover, the combination of join and nest relations yields the *(type* A*) Serre relations* $(\alpha \neq \beta)$

$$[e_\alpha, [e_\alpha, e_\beta]] = 0 = [f_\alpha, [f_\alpha, f_\beta]] \quad \text{if } (\mathbb{1}_\alpha | \mathbb{1}_\beta) = -1\,,$$
$$[e_\alpha, e_\beta] = 0 = [f_\alpha, f_\beta] \qquad \text{if } (\mathbb{1}_\alpha | \mathbb{1}_\beta) = 0\,.$$

Let $\mathfrak{sl}(\mathbb{Z})$ be the subalgebra generated by the elements $e_\alpha, h_\alpha, f_\alpha$ for α of the form $(i, i+1]$, $i \in \mathbb{Z}$. Then it is clear that $\mathfrak{sl}(\mathbb{Z}) \simeq \mathfrak{sl}(\infty)$ and there are canonical strict embeddings $\mathfrak{sl}(\mathbb{Z}) \subset \mathfrak{sl}(\mathbb{Q}) \subset \mathfrak{sl}(\mathbb{R})$. △

First, note that the Cartan subalgebra of $\mathfrak{sl}(\mathbb{K})$, $\mathfrak{h} := \langle h_\alpha \mid \alpha \in \mathsf{Int}(\mathbb{K})\rangle$, is canonically isomorphic, as a Lie algebra, to the commutative algebra $\mathfrak{f}_{\mathbb{K}}$ generated by the characteristic functions $\{\xi_\alpha := \mathbb{1}_\alpha \mid \alpha \in \mathsf{Int}(\mathbb{K})\}$. In [ASS18, Corollary 2.10], we show that the set of relations satisfied by the generators of $\mathfrak{sl}(\mathbb{K})$ can be simplified, indeed we have:

Proposition 3.17. *The Lie algebra $\mathfrak{sl}(\mathbb{K})$ is isomorphic to $\mathfrak{g}_{\mathbb{K}}$.*

Let us now move to the Lie algebra of the circle.

Definition 3.18. Let $\mathfrak{sl}(\mathbb{K}/\mathbb{Z})$ be the Lie algebra generated by elements $e_\alpha, f_\alpha, h_\beta$, with $\alpha, \beta \in \mathsf{Int}(\mathbb{K}/\mathbb{Z})$ and $\alpha \neq S^1$, modulo the following set of relations:

- **Kac–Moody type relations:** for any two intervals α, β,

$$[h_\alpha, h_\beta] = 0\,, \qquad [h_\alpha, e_\beta] = (\mathbb{1}_\alpha | \mathbb{1}_\beta)\; e_\beta\,, \qquad [h_\alpha, f_\beta] = -\,(\mathbb{1}_\alpha | \mathbb{1}_\beta)\; f_\beta\,,$$

$$[e_\alpha, f_\beta] = \begin{cases} h_\alpha & \text{if } \alpha = \beta\,, \\ 0 & \text{if } \alpha \perp \beta, \alpha \to \beta, \text{ or } \beta \to \alpha\,, \end{cases}$$

[7]The symbol $\vdash$ (resp. $\dashv$) should be read as α *is a proper subinterval in* β *starting from the left (resp. right) endpoint*.

- **join relations:**
  - for any two intervals α, β with $(\alpha, \beta) \in \mathsf{Int}(\mathbb{K}/\mathbb{Z})^{(2)}_{\oplus}$, we have $h_{\alpha\oplus\beta} = h_\alpha + h_\beta$;
  - for any two intervals α, β with $(\alpha, \beta) \in \mathsf{Int}(\mathbb{K}/\mathbb{Z})^{(2)}_{\oplus}$, such that $\alpha, \beta, \alpha \oplus \beta \neq S^1$,

$$e_{\alpha\oplus\beta} = (-1)^{\langle \mathbb{1}_\beta, \mathbb{1}_\alpha \rangle} [e_\alpha, e_\beta] \,, \qquad f_{\alpha\oplus\beta} = (-1)^{\langle \mathbb{1}_\alpha, \mathbb{1}_\beta \rangle} [f_\alpha, f_\beta] \,;$$

- **nest relations:** for any nested $\alpha, \beta \in \mathsf{Int}(\mathbb{K}/\mathbb{Z})$ (that is, such that $\alpha = \beta$, $\alpha \perp \beta$, $\alpha < \beta$, $\beta < \alpha$, $\alpha \vdash \beta$, $\alpha \dashv \beta$, $\beta \vdash \alpha$, or $\beta \dashv \alpha$), with $\alpha, \beta \neq S^1$,

$$[e_\alpha, e_\beta] = 0 \quad \text{and} \quad [f_\alpha, f_\beta] = 0 \,.$$

⊘

The continuum Kac–Moody algebra $\mathfrak{g}_{S^1}$ strictly contains the Lie algebra $\mathfrak{sl}(S^1)$ and their difference is reduced to the elements $x^\pm_{S^1}$ corresponding to the entire space. More precisely, let $\overline{\mathfrak{g}}_{S^1}$ be the subalgebra in $\mathfrak{g}_{S^1}$ generated by the elements $x^\pm_\alpha$, ξ_α, $\alpha \neq S^1$. Note that the elements $x^\pm_{S^1}$, ξ_{S^1}, generate a Heisenberg Lie algebra of order one in $\mathfrak{g}_{S^1}$, which we denote $\mathfrak{heis}_{S^1}$. Then, $\mathfrak{g}_{S^1} = \overline{\mathfrak{g}}_{S^1} \oplus \mathfrak{heis}_{S^1}$ and there is a canonical embedding $\mathfrak{sl}(S^1) \to \mathfrak{g}_{S^1}$, whose image is $\overline{\mathfrak{g}}_{S^1} \oplus \mathbf{k} \cdot \xi_{S^1}$.

4. The classical continuum R–matrix

In this section, we show that continuum Kac–Moody algebras are naturally endowed with a standard *topological* quasi–triangular Lie bialgebra structure. To this end, we provide here an alternative construction of continuum Kac–Moody algebras *by duality* in the spirit of [Hal99].

Note that the results of this section rely on the non–degeneracy of the Euler form on $\mathfrak{f}_X$, which is automatic whenever $X \not\simeq S^1$. If $X = S^1$, the kernel of the Euler form is one–dimensional, spanned by the central element ξ_{S^1}. However, this can be easily corrected by extending the vector space $\mathfrak{f}_{S^1}$ with a derivation corresponding to the Heisenberg Lie algebras $\mathfrak{heis}_{S^1}$. [8] Henceforth, we will therefore assume that the Euler form is non–degenerate on $\mathfrak{f}_X$ for any vertex space X.

4.1. **Continuum free Lie algebras.** Let $\mathcal{L}_\pm$ be the free Lie algebras generated, respectively, by the sets $V_\pm = \{x^\pm_\alpha, \xi^\pm_\alpha \mid \alpha \in \mathsf{Int}(X)\}$. Let $\mathbb{1}_\alpha$ be the characteristic function corresponding to the interval $\alpha \in \mathsf{Int}(X)$, and $\mathsf{F} := \mathfrak{f}^+_X = \mathsf{span}_{\mathbb{Z}_{\geqslant 0}}\{\mathbb{1}_\alpha \mid \alpha \in \mathsf{Int}(X)\}$. We consider on $\mathcal{L}_\pm$ the natural grading over F given by $\deg(x^\pm_\alpha) = \mathbb{1}_\alpha$ and $\deg(\xi^\pm_\alpha) = 0$, thus

$$\mathcal{L}_\pm = \bigoplus_{\phi \in \mathsf{F}} \mathcal{L}_{\pm,\phi} \,.$$

Example 4.1. Let $\alpha, \beta, \gamma \in \mathsf{Int}(X)$ such that $\alpha = \beta \oplus \gamma$. Then, the elements $x^\pm_\alpha, [x^\pm_\beta, x^\pm_\gamma]$, and $[[[x^\pm_\beta, \xi^\pm_\gamma], x^\pm_\gamma], \xi^\pm_\alpha]]$ have degree $\mathbb{1}_\alpha$. △

For $N > 0$ and $\phi \in \mathsf{F}$, a N-th partition of ϕ is a tuple $\underline{\psi} = (\psi_1, \ldots, \psi_N) \in \mathsf{F}^N$ such that $\psi_1 + \cdots + \psi_N = \phi$. We denote the set of Nth partitions of ϕ by $\mathsf{F}(\phi, N)$. Then, we set

$$\mathcal{L}^{\odot}_{\pm,N} := \bigoplus_{\phi \in \mathsf{F}} \left(\prod_{\underline{\psi} \in \mathsf{F}(\phi,N)} \mathcal{L}_{\pm,\psi_1} \odot \cdots \odot \mathcal{L}_{\pm,\psi_N} \right) \,,$$

where $\odot = \otimes, \wedge$. We regard $\mathcal{L}^{\otimes}_{\pm,N}$ (resp. $\mathcal{L}^{\wedge}_{\pm,N}$) as a completion of $\mathcal{L}^{\otimes N}_\pm$ (resp. $\wedge^N \mathcal{L}_\pm$). The following is a straightforward generalization of [Hal99, Propositions 2.2, 2.3, and 2.5].

Proposition 4.2.

[8] In other words, we need to consider the canonical realization of the Cartan matrix [0] (cf. Section 2.2).

(1) *For any collection of antisymmetric elements* $u^{\pm}\colon \mathsf{Int}(X) \to \mathcal{L}_{\pm}^{\wedge 2}$, *there exist unique maps* $\delta_{\pm}\colon \mathcal{L}_{\pm} \to \mathcal{L}_{\pm}^{\wedge 2}$ *such that*

$$\delta_{\pm}(x_{\alpha}^{\pm}) = u_{\alpha}^{\pm} \quad \text{and} \quad \delta_{\pm}(\xi_{\alpha}^{\pm}) = 0$$

and the cocycle condition (2.2) *holds. Moreover, if the co–Jacobi identity* (2.1) *holds for the generators* $x_{\alpha}^{\pm}$, i.e.,

$$(\mathsf{id} + (1\,2\,3) + (1\,3\,2)) \circ \mathsf{id} \otimes \delta_{\pm}(u_{\alpha}^{\pm}) = 0\,, \tag{4.1}$$

then $(\mathcal{L}_{\pm}, [\cdot,\cdot], \delta_{\pm})$ *are topological Lie bialgebras.*

(2) *Fix two matrices* $\kappa_i\colon \mathsf{Int}(X) \times \mathsf{Int}(X) \to \mathbf{k}$, $i = 0, 1$, *and let* $\mathbf{V}_{\pm} \subset \mathcal{L}_{\pm}$ *be the subspace spanned by the set* $V_{\pm}$. *Assume that the elements* $u_{\alpha}^{\pm}$ *satisfy the condition* (4.1), *so that* $(\mathcal{L}_{\pm}, [\cdot,\cdot], \delta_{\pm})$ *are topological Lie bialgebras, and belong to* $\mathbf{V}_{\pm}^{\wedge 2}$. *Then, there exists a unique pairing of Lie bialgebras* $\langle\cdot,\cdot\rangle\colon \mathcal{L}_{+} \otimes \mathcal{L}_{-} \to \mathbf{k}$ *such that*

$\langle\cdot,\cdot\rangle$	ξ_{γ}^{-}	x_{δ}^{-}
ξ_{α}^{+}	$\kappa_0(\alpha,\gamma)$	0
x_{β}^{+}	0	$\kappa_1(\beta,\delta)$

(3) *Fix two matrices* $\kappa_i\colon \mathsf{Int}(X) \times \mathsf{Int}(X) \to \mathbf{k}$, $i = 0, 1$, *and a collection of elements* $u^{\pm}\colon \mathsf{Int}(X) \to \mathbf{V}_{\pm}^{\wedge 2}$ *satisfying the condition* (4.1), *so that* $(\mathcal{L}_{\pm}, [\cdot,\cdot], \delta_{\pm})$ *are topological Lie bialgebras with a pairing* $\langle\cdot,\cdot\rangle\colon \mathcal{L}_{+} \otimes \mathcal{L}_{-} \to \mathbf{k}$. *Let* $\mathbf{J}$ *be a set, and let*

$$U^{\pm}\colon \mathbf{J} \to \mathcal{L}_{\pm} \quad \text{and} \quad V^{\pm}\colon \mathbf{J} \times \mathbf{J} \to \mathcal{L}_{\pm}$$

be two collections of elements such that $\delta_{\pm}(U_j^{\pm}) \in \mathbf{V}^{\pm}(U_{\cdot}, V_{j,\cdot})$, *where* $\mathbf{V}^{\pm}(U_{\cdot}, V_{j,\cdot})$ *denotes the completion in* $\mathcal{L}_{\pm}^{\wedge 2}$ *of the subspace spanned by the elements* $U_{\cdot}^{\pm} \wedge V_{j,\cdot}^{\pm}\colon \mathbf{J} \to \mathcal{L}_{\pm}$. *Then, if*

$$\langle U_{\cdot}^{\pm}, x_{\alpha}^{\mp}\rangle = 0 = \langle U_{\cdot}^{\pm}, \xi_{\alpha}^{\mp}\rangle\,,$$

the ideal generated by $U_{\cdot}^{\pm}$ *is orthogonal to* $\mathcal{L}_{\mp}$ *and is a topological coideal in* $\mathcal{L}_{\pm}$.

4.2. **Orthogonal coideals.** We shall now fix a Lie bialgebra structure on $\mathcal{L}_{\pm}$ with a pairing, and show that the defining relations of continuum Kac–Moody algebras arise from orthogonal coideals. We shall use repeatedly Proposition 4.2–(3).

Henceforth, we consider the Lie bialgebra structure on $\mathcal{L}_{\pm}$ given by

$$\delta_{\pm}(\xi_{\alpha}^{\pm}) := 0 \quad \text{and} \quad \delta_{\pm}(x_{\alpha}^{\pm}) := \mp\xi_{\alpha}^{\pm} \wedge x_{\alpha}^{\pm} \mp \sum_{\beta\oplus\gamma=\alpha} \mathsf{b}_{\beta\gamma} x_{\beta}^{\pm} \wedge x_{\gamma}^{\pm}\,, \tag{4.2}$$

where $\mathsf{b}_{\beta\gamma} = (-1)^{\langle\beta,\beta\oplus\gamma\rangle}(\beta|\beta\oplus\gamma) = \mathsf{a}_{\beta,\beta\oplus\gamma}$. Then, we define a pairing $\langle\cdot,\cdot\rangle\colon \mathcal{L}_{+} \otimes \mathcal{L}_{-} \to \mathbf{k}$ by setting $\langle x_{\alpha}^{\pm}, \xi_{\beta}^{\mp}\rangle := 0$ and

$$\langle x_{\alpha}^{+}, x_{\beta}^{-}\rangle := \delta_{\alpha\beta} \quad \text{and} \quad \langle \xi_{\alpha}^{+}, \xi_{\beta}^{-}\rangle := 2\,(\alpha|\beta)\;.$$

Proposition 4.3. *Let* $\mathfrak{i}_{\pm}$ *be the ideal generated in* $\mathcal{L}_{\pm}$ *by the elements*

$$\xi_{\alpha\oplus\beta}^{\pm} - \delta_{\alpha\oplus\beta}\left(\xi_{\alpha}^{\pm} + \xi_{\beta}^{\pm}\right)\,, \quad [\xi_{\alpha}^{\pm}, \xi_{\beta}^{\pm}]\,, \quad [\xi_{\alpha}^{\pm}, x_{\beta}^{\pm}] \mp (\alpha|\beta)\, x_{\beta}^{\pm}\;.$$

Then, $\mathfrak{i}_{\pm}$ *is a coideal and it is orthogonal to* $\mathcal{L}_{\mp}$.

Proof. We show that the conditions of Proposition 4.2–(3) apply. We first observe that the elements $\xi_{\alpha\oplus\beta}^{\pm} - \delta_{\alpha\oplus\beta}\left(\xi_{\alpha}^{\pm} + \xi_{\beta}^{\pm}\right)$ are orthogonal to $\mathcal{L}_{\mp}$. Namely, if $\alpha \oplus \beta$ is defined, then $(\alpha\oplus\beta|\gamma) = (\alpha|\gamma) + (\beta|\gamma)$, and therefore

$$\langle \xi_{\alpha\oplus\beta}^{\pm}, \xi_{\gamma}^{\mp}\rangle = \langle \xi_{\alpha}^{\pm} + \xi_{\beta}^{\pm}, \xi_{\gamma}^{\mp}\rangle\,,$$

while $\langle \xi^{\pm}_{\alpha\oplus\beta}, \xi^{\mp}_{\gamma}\rangle = 0 = \langle \xi^{\pm}_{\alpha} + \xi^{\pm}_{\beta}, \xi^{\mp}_{\gamma}\rangle$. Moreover, $\delta_{\pm}(\xi^{\pm}_{\alpha}) = 0$, therefore the condition on the cobracket is trivially satisfied. Similarly, by duality, one has

$$\langle [\xi^{\pm}_{\alpha}, \xi^{\pm}_{\beta}], \xi^{\mp}_{\gamma}\rangle = 0 = \langle [\xi^{\pm}_{\alpha}, \xi^{\pm}_{\beta}], x^{\mp}_{\gamma}\rangle \,,$$

and, by Formula (4.2), $\delta_{\pm}([\xi^{\pm}_{\alpha}, \xi^{\pm}_{\beta}]) = 0$. Finally, we have

$$\langle [\xi^{\pm}_{\alpha}, x^{\pm}_{\beta}], \xi^{\mp}_{\gamma}\rangle \mp (\alpha|\beta)\, \langle x^{\pm}_{\beta}, \xi^{\mp}_{\gamma}\rangle = 0 \,,$$

and

$$\begin{aligned}
\langle [\xi^{\pm}_{\alpha}, x^{\pm}_{\beta}], x^{\mp}_{\gamma}\rangle \mp (\alpha|\beta)\, \langle x^{\pm}_{\beta}, x^{\mp}_{\gamma}\rangle &= \langle \xi^{\pm}_{\alpha} \wedge x^{\pm}_{\beta}, \delta_{\mp}(x^{\mp}_{\gamma})\rangle \mp \delta_{\beta\gamma}\,(\alpha|\beta)\\
&= \pm\delta_{\beta\gamma}\langle \xi^{\pm}_{\alpha} \wedge x^{\pm}_{\beta}, \xi^{\mp}_{\gamma} \wedge x^{\mp}_{\gamma}\rangle \mp \delta_{\beta\gamma}\,(\alpha|\beta)\\
&= \pm\delta_{\beta\gamma}\,(\alpha|\gamma) \mp \delta_{\beta\gamma}\,(\alpha|\beta) = 0 \,.
\end{aligned}$$

Moreover, since $(\alpha|\beta) = (\alpha|\gamma) + (\alpha|\gamma')$ whenever $\gamma \oplus \gamma' = \beta$, we get

$$\begin{aligned}
&\delta_{\pm}([\xi^{\pm}_{\alpha}, x^{\pm}_{\beta}] \mp (\alpha|\beta)\, x^{\pm}_{\beta})\\
&\quad = [\xi^{\pm}_{\alpha} \otimes 1 + 1 \otimes \xi^{\pm}_{\alpha}, \delta_{\pm}(x^{\pm}_{\beta})] \mp (\alpha|\beta)\, \delta_{\pm}(x^{\pm}_{\beta})\\
&\quad = \mp[\xi^{\pm}_{\alpha}, \xi^{\pm}_{\beta}] \wedge x^{\pm}_{\beta} \mp \xi^{\pm}_{\beta} \wedge [\xi^{\pm}_{\alpha}, x^{\pm}_{\beta}]\\
&\qquad \mp \sum_{\gamma\oplus\gamma'=\beta} \mathsf{b}_{\gamma\gamma'} \left([\xi^{\pm}_{\alpha}, x^{\pm}_{\gamma}] \wedge x^{\pm}_{\gamma'} + x^{\pm}_{\gamma} \wedge [\xi^{\pm}_{\alpha}, x^{\pm}_{\gamma'}]\right) \mp (\alpha|\beta)\, \delta_{\pm}(x^{\pm}_{\beta})\\
&\quad = \mp[\xi^{\pm}_{\alpha}, \xi^{\pm}_{\beta}] \wedge x^{\pm}_{\beta} \mp \xi^{\pm}_{\beta} \wedge \left([\xi^{\pm}_{\alpha}, x^{\pm}_{\beta}] \mp (\alpha|\beta)\, x^{\pm}_{\beta}\right)\\
&\qquad \mp \sum_{\gamma\oplus\gamma'=\beta} \mathsf{b}_{\gamma\gamma'} \left(\left([\xi^{\pm}_{\alpha}, x^{\pm}_{\gamma}] \mp (\alpha|\gamma)\, x^{\pm}_{\gamma}\right) \wedge x^{\pm}_{\gamma'} + x^{\pm}_{\gamma} \wedge \left([\xi^{\pm}_{\alpha}, x^{\pm}_{\gamma'}] \mp (\alpha|\gamma')\, x^{\pm}_{\gamma'}\right)\right) \,.
\end{aligned}$$

The result follows from Proposition 4.2–(3). □

Thanks to this result, the pairing $\langle\cdot,\cdot\rangle\colon \mathcal{L}_{+} \otimes \mathcal{L}_{-} \to \mathbf{k}$ descends to a pairing between the topological Lie bialgebras $\widetilde{\mathfrak{d}}_{\pm} := \mathcal{L}_{\pm}/\mathfrak{i}_{\pm}$.

Proposition 4.4. *Let $\mathfrak{s}_{\pm}$ be the ideal generated in $\widetilde{\mathfrak{d}}_{\pm}$ by the elements*

$$s^{\pm}_{\alpha\beta} := [x^{\pm}_{\alpha}, x^{\pm}_{\beta}] \mp \mathsf{b}_{\alpha\beta} x^{\pm}_{\alpha\oplus\beta} \,,$$

with $(\alpha, \beta) \in \mathsf{Serre}(X)$. Then, $\mathfrak{s}_{\pm}$ is a coideal and it is orthogonal to $\widetilde{\mathfrak{d}}_{\mp}$.

Proof. We proceed as before. Clearly, we have $\langle s^{\pm}_{\alpha\beta}, \xi^{\mp}_{\gamma}\rangle = 0$ and

$$\begin{aligned}
\langle s^{\pm}_{\alpha\beta}, x^{\mp}_{\gamma}\rangle &= \langle [x^{\pm}_{\alpha}, x^{\pm}_{\beta}], x^{\mp}_{\gamma}\rangle \mp \mathsf{b}_{\alpha\beta}\langle x^{\pm}_{\alpha\oplus\beta}, x^{\mp}_{\gamma}\rangle\\
&= \langle x^{\pm}_{\alpha} \wedge x^{\pm}_{\beta}, \delta_{\mp}(x^{\mp}_{\gamma})\rangle \mp \delta_{\gamma,\alpha\oplus\beta}\mathsf{b}_{\alpha\beta}\\
&= \pm\delta_{\gamma,\alpha\oplus\beta}\mathsf{b}_{\alpha\beta} \mp \delta_{\gamma,\alpha\oplus\beta}\mathsf{b}_{\alpha\beta} = 0 \,,
\end{aligned}$$

therefore the elements $s^{\pm}_{\alpha\beta}$ are orthogonal to $\widetilde{\mathfrak{d}}_{\mp}$. Finally, one checks by direct inspection that $\delta_{\pm}(s^{\pm}_{\alpha\beta}) = \sum_{\gamma\gamma'} s^{\pm}_{\gamma\gamma'} \wedge V^{\alpha\beta}_{\gamma\gamma'}$ for some vectors $V^{\alpha\beta}_{\gamma\gamma'} \in \widetilde{\mathfrak{d}}_{\pm}$. The result follows from Proposition 4.2–(3). □

4.3. **Continuum Kac–Moody algebras by duality.** We now show that the procedure described above realizes the continuum Kac–Moody algebra $\mathfrak{g}_X$ as a topological Lie bialgebras, endowed with a non–degenerate invariant bilinear form.

Set $\mathfrak{d}_\pm := \widetilde{\mathfrak{d}}_\pm/\mathfrak{s}_\pm$. Then, $\mathfrak{d}_\pm$ are topological Lie bialgebras endowed with a Lie bialgebra pairing $\langle\cdot,\cdot\rangle\colon \mathfrak{d}_+\otimes\mathfrak{d}_-\to\mathbf{k}$. In particular, $(\mathfrak{d}_+,\mathfrak{d}_-)$ is a matched pair of Lie algebras with respect to the coadjoint actions given by $\mathsf{ad}^*(d_\pm)(d'_\mp) := \pm\langle 1\otimes d_\pm, \delta_\mp(d'_\mp)\rangle$, $d_\pm, d'_\pm\in\mathfrak{d}_\pm$. Let $\mathfrak{d}^{(2)} := \mathfrak{d}_+\bowtie\mathfrak{d}_-$ be the double cross sum Lie bialgebra.

Proposition 4.5. *The following relations hold in $\mathfrak{d}^{(2)}$. For any $\alpha,\beta\in\mathsf{Int}(X)$*

$$[x_\alpha^+, x_\beta^-] = \delta_{\alpha\beta}\frac{\xi_\alpha^+ + \xi_\alpha^-}{2} + \mathsf{a}_{\alpha\beta}\left(x^+_{\alpha\ominus\beta} - x^-_{\beta\ominus\alpha}\right) ,$$

where $\mathsf{a}_{\alpha\beta} := (-1)^{\langle\alpha,\beta\rangle}\,(\alpha|\beta)$.

Proof. It is enough to observe that, by definition,

$$\mathsf{ad}^*(x_\alpha^+)(x_\beta^-) = \delta_{\alpha\beta}\frac{\xi_\alpha^-}{2} + \frac{\mathsf{b}_{\beta\ominus\alpha,\alpha} - \mathsf{b}_{\alpha,\beta\ominus\alpha}}{2}x^-_{\beta\ominus\alpha} ,$$
$$\mathsf{ad}^*(x_\beta^-)(x_\alpha^+) = \delta_{\alpha\beta}\frac{\xi_\alpha^+}{2} + \frac{\mathsf{b}_{\alpha\ominus\beta,\beta} - \mathsf{b}_{\beta,\alpha\ominus\beta}}{2}x^+_{\alpha\ominus\beta} .$$

Moreover, since $\mathsf{b}_{ab} = \mathsf{a}_{a,a\oplus b}$ and $\mathsf{a}_{a,a\oplus b} = -\mathsf{a}_{b,a\oplus b}$, we get

$$\frac{\mathsf{b}_{\beta\ominus\alpha,\alpha} - \mathsf{b}_{\alpha,\beta\ominus\alpha}}{2} = -\mathsf{a}_{\alpha\beta} , \qquad \frac{\mathsf{b}_{\alpha\ominus\beta,\beta} - \mathsf{b}_{\beta,\alpha\ominus\beta}}{2} = \mathsf{a}_{\alpha\beta} .$$

□

The combination of Propositions 4.3, 4.4, and 4.5 leads to the following.

Theorem 4.6. *Let $\mathcal{Q}_X$ be a continuum quiver and $\mathfrak{g}_X$ the corresponding continuum Kac–Moody algebras.*

(1) The Euler form (3.1) on $\mathfrak{f}_X$ uniquely extends to a non–degenerate invariant symmetric bilinear form $(\cdot|\cdot)\colon\mathfrak{g}_X\otimes\mathfrak{g}_X\to\mathbf{k}$ defined on the generators as follows:

$$(\xi_\alpha|\xi_\beta) := (\alpha|\beta) , \quad (x_\alpha^\pm|\xi_\beta) := 0 , \quad \left(x_\alpha^\pm|x_\beta^\pm\right) := 0 , \quad \left(x_\alpha^+|x_\beta^-\right) := \delta_{\alpha\beta} .$$

(2) There is a unique topological cobracket $\delta\colon\mathfrak{g}_X\to\mathfrak{g}_X\widehat{\otimes}\mathfrak{g}_X$ defined on the generators by

$$\delta(\xi_\alpha) := 0 \quad \text{and} \quad \delta(x_\alpha^\pm) := \xi_\alpha^\pm\wedge x_\alpha^\pm + \sum_{\beta\oplus\gamma=\alpha}\mathsf{b}_{\beta\gamma}x_\beta^\pm\wedge x_\gamma^\pm ,$$

and inducing on $\mathfrak{g}_X$ a topological Lie bialgebra structure, with respect to which the positive and negative Borel subalgebras $\mathfrak{b}_X^\pm$ are Lie sub-bialgebras.

(3) The Euler form restricts to a non–degenerate pairing of Lie bialgebras $(\cdot|\cdot)\colon\mathfrak{b}_X^+\otimes(\mathfrak{b}_X^-)^{\mathrm{cop}}\to\mathbf{k}$. Then, the canonical element $r_X\in\mathfrak{b}_X^+\widehat{\otimes}\mathfrak{b}_X^-$ corresponding to $(\cdot|\cdot)$ defines a quasi–triangular structure on $\mathfrak{g}_X$.

Proof. First, let $\mathfrak{c}$ be the ideal generated in $\mathfrak{d}^{(2)}$ by the elements $\xi_\alpha^+ - \xi_\alpha^-$, $\alpha\in\mathsf{Int}(X)$. It is clear that $\mathfrak{c}$ is central in $\mathfrak{d}^{(2)}$, is a coideal, and moreover it is contained in the kernel of the pairing $\langle\cdot,\cdot\rangle$ naturally extended to $\mathfrak{d}^{(2)}$. Therefore, $\mathfrak{d} := \mathfrak{d}^{(2)}/\mathfrak{c}$ is also Lie bialgebra endowed with a pairing, which we denote by $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$.

Set $\xi_\alpha := \frac{1}{2}(\xi_\alpha^+ + \xi_\alpha^-)$. In particular, we have

$$\langle\xi_\alpha,\xi_\beta\rangle_{\mathfrak{d}} = (\alpha|\beta) . \tag{4.3}$$

By Propositions 4.3, 4.4, and 4.5, there is an obvious identification $\mathfrak{g}_X = \mathfrak{d}$ as Lie algebras (cf. Theorem 3.11). This allows to define a cobracket and possibly degenerate pairing on $\mathfrak{g}_X$. However, it follows from (4.3) that the kernel of $\langle\cdot,\cdot\rangle_{\mathfrak{d}}$ is a two–sided graded ideal, which trivially intersects $\mathfrak{f}_X$. Therefore, by definition of $\mathfrak{g}_X$, it must hold $\ker\langle\cdot,\cdot\rangle_{\mathfrak{d}} = 0$. Therefore, (1), (2), (3) follows directly from the identification $\mathfrak{g}_X = \mathfrak{d}$. □

From the proof above, we also deduce the following

Corollary 4.7. *The Euler form* (3.1) *on* $\mathfrak{f}_X$ *uniquely extends to a non–degenerate invariant symmetric bilinear form* $(\cdot|\cdot) : \widetilde{\mathfrak{g}}_X \otimes \widetilde{\mathfrak{g}}_X \to \mathbf{k}$ *defined on the generators as follows:*

$$(\xi_\alpha|\xi_\beta) := (\alpha|\beta) \;, \quad (x_\alpha^\pm|\xi_\beta) := 0 \,, \quad \left(x_\alpha^\pm|x_\beta^\pm\right) := 0 \,, \quad \left(x_\alpha^+|x_\beta^-\right) := \delta_{\alpha\beta} \,.$$

Moreover, $\mathfrak{r}_X = \ker(\cdot|\cdot)$, i.e., $\ker(\cdot|\cdot)$ *is the maximal two–sided ideal trivially intersecting* $\mathfrak{f}_X$ *and it is generated by the Serre relations from Theorem 3.11.*

5. Continuum quantum groups

In this section we shall introduce the *continuum quantum groups*, which provide a quantization of the continuum Kac–Moody algebras. We will see that they can be similarly realized as uncountable colimits of Drinfeld–Jimbo quantum groups. Finally, when the underlying vertex space is the line or the circle, they coincide with the line quantum group and the circle quantum group of [SS17].

5.1. **Definition of continuum quantum groups.** Let $\mathcal{Q}_X := (\mathsf{Int}(X), \oplus, \ominus, \langle\cdot,\cdot\rangle, (\cdot|\cdot))$ be a *continuum quiver* with underlying vertex space X. In order to define the *continuum quantum group*, we need to introduce some new operations on intervals.

Definition 5.1. We define the following partial operations on $\mathsf{Int}(X)$:

(1) the *strict union of two intervals* α *and* β, whenever defined, is the smallest interval $\alpha\nabla\beta \in \mathsf{Int}(X)$ for which $(\alpha\nabla\beta)\ominus\alpha$ and $(\alpha\nabla\beta)\ominus\beta$ are both defined;

(2) the *strict intersection of two intervals* α *and* β, whenever defined, is the biggest interval $\alpha\,\triangle\,\beta \in \mathsf{Int}(X)$ for which $\alpha\ominus(\alpha\,\triangle\,\beta)$ and $\beta\ominus(\alpha\,\triangle\,\beta)$ are both defined.

⊘

Remark 5.2. Note that $\alpha\nabla\beta$ (resp. $\alpha\,\triangle\,\beta$) is defined and coincides with $\alpha\cup\beta$ (resp. $\alpha\cap\beta$) whenever it contains *strictly* α and β (resp. it is contained *strictly* in α and β). In particular, ∇ and $\triangle$ are clearly symmetric. △

Remark 5.3. Let $X = \mathbb{R}$ and $\alpha, \beta \in \mathsf{Int}(\mathbb{R})$.

- If $\alpha \to \beta$, then $\alpha\nabla\beta = \alpha\oplus\beta$ and $\alpha\,\triangle\,\beta$ is not defined.
- If $\alpha \pitchfork \beta$, then $\alpha\nabla\beta = \alpha\cup\beta$ and $\alpha\,\triangle\,\beta = \alpha\cap\beta$. Moreover,
$$((\alpha\nabla\beta)\ominus\beta)\oplus(\alpha\,\triangle\,\beta) = \alpha = (\alpha\nabla\beta)\ominus(\beta\ominus(\alpha\,\triangle\,\beta))$$
- If α and β are nested, then $\alpha\nabla\beta$ and $\alpha\,\triangle\,\beta$ are not defined. [9]

△

Definition 5.4. We shall use the following functions on $\mathsf{Int}(\mathbb{K}) \times \mathsf{Int}(\mathbb{K})$:

- $\mathsf{a}_{\alpha\beta} := (-1)^{\langle\alpha,\beta\rangle}\,(\alpha|\beta)$;
- $\mathsf{b}_{\alpha\beta} := \mathsf{a}_{\alpha,\alpha\nabla\beta}$, which generalizes the function $\mathsf{b}_{\alpha\beta}$ defined in (3.2);
- $\mathsf{c}^+_{\alpha\beta} := \dfrac{1}{2}\left(\mathsf{a}_{\beta,\alpha\ominus\beta} - 1\right)$, and $\mathsf{c}^-_{\alpha\beta} := \dfrac{1}{2}\left(\mathsf{a}_{\beta\ominus\alpha,\alpha} + 1\right)$;
- $\mathsf{r}_{\alpha\beta} := (1-\delta_{\alpha\beta})(-1)^{\langle\alpha,\beta\rangle}\,(\alpha|\beta)^2$;
- $\mathsf{s}^\pm_{\alpha\beta} := \dfrac{1}{2}\left(\mathsf{a}_{\beta,\alpha\oplus\beta} \pm 1\right)$.

[9] Recall that α and β are nested if they are perpendicular or one contained in the other.

⊘

Remark 5.5. Let $X = \mathbb{K}$, with $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. We summarize below all possible values of the functions above.

	$\alpha \star \beta$	$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	$\mathsf{a}_{\alpha\beta}$	$\mathsf{b}_{\alpha\beta}$	$\mathsf{c}^{+}_{\alpha\beta}$	$\mathsf{c}^{-}_{\alpha\beta}$	$\mathsf{r}_{\alpha\beta}$	$\mathsf{s}^{+}_{\alpha\beta}$	$\mathsf{s}^{-}_{\alpha\beta}$
(a)	$\alpha \to \beta$	-1	0	1	1	$n.d.$	$n.d.$	-1	0	-1
(b)	$\beta \to \alpha$	0	-1	-1	-1	$n.d.$	$n.d.$	1	1	0
(c)	$\alpha \pitchfork \beta$	-1	1	0	1	$n.d.$	$n.d.$	0	$n.d.$	$n.d.$
(d)	$\beta \pitchfork \alpha$	1	-1	0	-1	$n.d.$	$n.d.$	0	$n.d.$	$n.d.$
(e)	$\alpha \perp \beta$	0	0	0	$n.d.$	$n.d.$	$n.d.$	0	$n.d.$	$n.d.$
(f)	$\alpha < \beta$	0	0	0	$n.d.$	$n.d.$	$n.d.$	0	$n.d.$	$n.d.$
(g)	$\beta < \alpha$	0	0	0	$n.d.$	$n.d.$	$n.d.$	0	$n.d.$	$n.d.$
(h)	$\alpha \vdash \beta$	0	1	1	$n.d.$	$n.d.$	0	1	$n.d.$	$n.d.$
(i)	$\alpha \dashv \beta$	1	0	-1	$n.d.$	$n.d.$	1	-1	$n.d.$	$n.d.$
(j)	$\beta \vdash \alpha$	1	0	-1	$n.d.$	0	$n.d.$	-1	$n.d.$	$n.d.$
(k)	$\beta \dashv \alpha$	0	1	1	$n.d.$	-1	$n.d.$	1	$n.d.$	$n.d.$

(5.1)

△

Definition 5.6. Let $\mathcal{Q}_X$ be a continuum quiver. The *continuum quantum group of* X is the associative algebra $\mathbf{U}_q\mathfrak{g}_X$ generated by $\mathfrak{h}_X$ and the elements $X^{\pm}_{\alpha}$, $\alpha \in \mathsf{Int}(\mathbb{K})$, satisfying the following defining relations:

(1) **Diagonal action:** for any $\alpha, \beta \in \mathsf{Int}(X)$,

$$[\xi_\alpha, \xi_\beta] = 0 \qquad \text{and} \qquad [\xi_\alpha, X^{\pm}_{\beta}] = \pm\,(\alpha|\beta)\, X^{\pm}_{\beta}\ .$$

In particular, for $K_\alpha := \exp(\hbar/2 \cdot \xi_\alpha)$, it holds $K_\alpha X^{\pm}_{\beta} = q^{\pm(\alpha|\beta)} \cdot X^{\pm}_{\beta} K_\alpha$.

(2) **Quantum double relations:** for any $\alpha, \beta \in \mathsf{Int}(X)$,

$$[X^{+}_{\alpha}, X^{-}_{\beta}] = \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}} + \mathsf{a}_{\alpha\beta} \cdot \left(q^{\mathsf{c}^{+}_{\alpha\beta}}\, X^{+}_{\alpha\ominus\beta} K_\beta^{\mathsf{a}_{\alpha\beta}} - q^{\mathsf{c}^{-}_{\alpha\beta}}\, K_\alpha^{\mathsf{a}_{\alpha\beta}}\, X^{-}_{\beta\ominus\alpha} \right)$$
$$+ \,\mathsf{b}_{\beta\alpha}\, q^{\mathsf{b}_{\beta\alpha}}\, (q - q^{-1})\, X^{+}_{(\alpha\triangledown\beta)\ominus\beta}\, K^{\mathsf{b}_{\alpha\beta}}_{\alpha\vartriangle\beta}\, X^{-}_{(\alpha\triangledown\beta)\ominus\alpha}\ .$$

(3) **Quantum Serre relations:** for any $(\alpha, \beta) \in \mathsf{Serre}(X)$,

$$X^{\pm}_{\alpha} X^{\pm}_{\beta} - q^{\mathsf{r}_{\alpha\beta}} \cdot X^{\pm}_{\beta} X^{\pm}_{\alpha} = \pm \mathsf{b}_{\alpha\beta} \cdot q^{\mathsf{s}^{\pm}_{\alpha\beta}} \cdot X^{\pm}_{\alpha\oplus\beta} + \mathsf{b}_{\alpha\beta} \cdot (q - q^{-1}) \cdot X^{\pm}_{\alpha\triangledown\beta} X^{\pm}_{\alpha\vartriangle\beta}\ .$$

We assume that $X^{\pm}_{\alpha\odot\beta} = 0$ whenever $\alpha \odot \beta$ is not defined, for $\odot = \oplus, \ominus, \triangledown, \vartriangle$, and the functions $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{r}, \mathsf{s}$ are those introduced of Definition 5.4. ⊘

5.2. **Colimit structure.** In analogy with Section 3.4, we prove that the continuum quantum group $\mathbf{U}_q\mathfrak{g}_X$ is *covered* by an uncountable family of Drinfeld–Jimbo quantum groups. Let $\mathcal{J}$ be an irreducible family of intervals in $\mathsf{Int}(X)$ (cf. Section 3.4). We then consider two quantum algebras associated to $\mathcal{J}$:

- the Drinfeld–Jimbo quantum group $\mathbf{U}_q\mathfrak{g}^{\mathrm{BKM}}_{\mathcal{J}}$ with Cartan matrix $\mathsf{A}_{\mathcal{J}} = [(\alpha|\beta)]_{\alpha,\beta\in\mathcal{J}}$;
- the subalgebra $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$ generated in $\mathbf{U}_q\mathfrak{g}_X$ by the elements $\{\xi_\alpha, X^{\pm}_{\alpha} \mid \alpha \in \mathcal{J}\}$.

Proposition 5.7. *The assignment*

$$E_\alpha \mapsto X^{+}_{\alpha}\ , \quad F_\alpha \mapsto X^{-}_{\alpha} \quad \textit{and} \quad H_\alpha \mapsto \xi_\alpha$$

for any $\alpha \in \mathcal{J}$, defines a surjective homomorphism of algebras $\Phi_{\mathcal{J}} \colon \mathbf{U}_q\mathfrak{g}^{\mathrm{BKM}}_{\mathcal{J}} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$.

Proof. First, note that Proposition 3.13 follows from the result above by setting $\hbar = 0$. It is easy to check that, applying the quantum Serre relations (3) of Definition 5.6 corresponding to the elements $X_\alpha^\pm$, with $\alpha \in \mathcal{J}$, one recovers the *standard* quantum Serre relations of the Drinfeld–Jimbo quantum group $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}}$ (cf. Section 2.6). Thus, by mimicking the arguments of the proof of Proposition 3.13, the result follows. □

The following is straightforward.

Corollary 5.8. *Let $\mathcal{J}, \mathcal{J}'$ be two irreducible (finite) sets of intervals in X.*

(1) If $\mathcal{J}' \subseteq \mathcal{J}$, there is a canonical embedding $\phi'_{\mathcal{J},\mathcal{J}'}\colon \mathbf{U}_q\mathfrak{g}_{\mathcal{J}'} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$ sending $X_\alpha^\pm \mapsto X_\alpha^\pm$, $\xi_\alpha \mapsto \xi_\alpha$, $\alpha \in \mathcal{J}'$.

(2) If $\mathcal{J}$ is obtained from $\mathcal{J}'$ by replacing an element $\gamma \in \mathcal{J}'$ with two intervals α, β such that $\gamma = \alpha \oplus \beta$, there is a canonical embedding $\phi''_{\mathcal{J},\mathcal{J}'}\colon \mathbf{U}_q\mathfrak{g}_{\mathcal{J}'} \to \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}$, which is the identity on $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}'\setminus\{\gamma\}} = \mathbf{U}_q\mathfrak{g}_{\mathcal{J}\setminus\{\alpha,\beta\}}$ and sends

$$\xi_\gamma \mapsto \xi_\alpha + \xi_\beta\ , \qquad X_\gamma^\pm \mapsto \mp \mathsf{b}_{\alpha\beta}^{-1} \cdot q^{-\mathsf{s}_{\alpha\beta}^\pm} \cdot \left(X_\alpha^\pm X_\beta^\pm - q^{\mathsf{r}_{\alpha\beta}} \cdot X_\beta^\pm X_\alpha^\pm \right) \ .$$

(3) The collection of embeddings $\phi'_{\mathcal{J},\mathcal{J}'}, \phi''_{\mathcal{J},\mathcal{J}'}$, indexed by all possible irreducible sets of intervals in X, form a direct system. Moreover, there is a canonical surjective homomorphism

$$\mathsf{colim}_{\mathcal{J}}\, \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \to \mathbf{U}_q\mathfrak{g}(X)\ .$$

5.3. **Comparison with the quantum group of the line.** We will now show that the continuum quantum groups of $\mathbf{U}_q\mathfrak{g}_X$, $X = \mathbb{R}, S^1$, coincide with the quantum groups of the line and the circle introduced in [SS17]. Let us first recall the definition of the line quantum group $\mathbf{U}_q\mathfrak{sl}(\mathbb{R})$.

Definition 5.9. Let $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. The *quantum group of the line* is the associative algebra $\mathbf{U}_q\mathfrak{sl}(\mathbb{K})$ generated over $\mathbb{C}[\![\hbar]\!]$ by elements $E_\alpha, F_\alpha, H_\alpha$, with $\alpha \in \mathsf{Int}(\mathbb{K})$, with the following defining relations. Set $q := \exp(\hbar/2)$ and $K_\alpha := \exp(\hbar/2 \cdot H_\alpha)$.

- **Kac–Moody type relations:** for any two intervals α, β,

$$[H_\alpha, H_\beta] = 0\ , \qquad [H_\alpha, E_\beta] = (\alpha|\beta)\ E_\beta\ , \qquad [H_\alpha, F_\beta] = -\,(\alpha|\beta)\ F_\beta\ , \tag{5.2}$$

$$[E_\alpha, F_\beta] = \begin{cases} \dfrac{K_\alpha - K_\alpha^{-1}}{q - q^{-1}} & \text{if } \alpha = \beta\ , \\ 0 & \text{if } \alpha \perp \beta, \alpha \to \beta, \text{ or } \beta \to \alpha\ , \end{cases} \tag{5.3}$$

- **join relations:** for any two intervals α, β with $\alpha \to \beta$,

$$H_{\alpha\oplus\beta} = H_\alpha + H_\beta\ , \tag{5.4}$$

$$E_{\alpha\oplus\beta} = \quad q^{1/2} E_\alpha E_\beta - q^{-1/2} E_\beta E_\alpha\ , \tag{5.5}$$

$$F_{\alpha\oplus\beta} = -q^{1/2} F_\alpha F_\beta + q^{-1/2} F_\beta F_\alpha\ ; \tag{5.6}$$

- **nest relations:** for any nested $\alpha, \beta \in \mathsf{Int}(\mathbb{K})$ (that is, such that $\alpha = \beta$, $\alpha \perp \beta$, $\alpha < \beta$, $\beta < \alpha$, $\alpha \vdash \beta$, $\alpha \dashv \beta$, $\beta \vdash \alpha$, or $\beta \dashv \alpha$),

$$q^{\langle\alpha,\beta\rangle} E_\alpha E_\beta = q^{\langle\beta,\alpha\rangle} E_\beta E_\alpha \quad \text{and} \quad q^{\langle\alpha,\beta\rangle} F_\alpha F_\beta = q^{\langle\beta,\alpha\rangle} F_\beta F_\alpha\ . \tag{5.7}$$

⊘

It follows, in particular, that

$$K_\alpha K_\beta = K_\beta K_\alpha\ , \qquad K_\alpha E_\beta = q^{(\mathbb{1}_\alpha|\mathbb{1}_\beta)}\, E_\beta K_\alpha\ , \qquad K_\alpha F_\beta = q^{-(\mathbb{1}_\alpha|\mathbb{1}_\beta)}\, F_\beta K_\alpha\ .$$

As in the case of $\mathfrak{sl}(\mathbb{K})$, the Cartan subalgebra of $\mathbf{U}_q\mathfrak{sl}(\mathbb{K})$, namely $\mathbf{U}_q\mathfrak{h} := \langle H_\alpha \mid \alpha \in \mathsf{Int}(\mathbb{K})\rangle$, is canonically isomorphic to the symmetric algebra $S\mathfrak{f}_{\mathbb{K}}[\![\hbar]\!]$ generated by the characteristic functions $\{\xi_\alpha := \mathbb{1}_\alpha \mid \alpha \in \mathsf{Int}(\mathbb{K})\}$.

We have the following:

Proposition 5.10. *There is an isomorphism of algebras* $\mathbf{U}_q\mathfrak{g}_{\mathbb{K}} \to \mathbf{U}_q\mathfrak{sl}(\mathbb{K})$ *given by*

$$X_\alpha^+ \mapsto q^{\frac{1}{2}} E_\alpha\ , \quad X_\alpha^- \mapsto q^{-\frac{1}{2}} F_\alpha\ , \quad \xi_\alpha \mapsto H_\alpha\ ,$$

with $\alpha \in \mathsf{Int}(\mathbb{K})$.

Proof. First, we show that the relations (1)–(3) from Definition 5.6 imply those from Definition 5.9.

5.3.1. The Kac–Moody relations (5.2) and (5.3) follow immediately from (1) and (2), respectively. The join relation (5.4) is automatic, while (5.5) and (5.6) follow from (3). Namely, if $\alpha \to \beta$, then $\alpha\nabla\beta = \alpha \oplus \beta$, and $\alpha \vartriangle \beta$ is not defined (therefore the last summand on the RHS of (3) does not appear) and

$$\mathsf{r}_{\alpha\beta} = -1\ , \quad \mathsf{b}_{\alpha\beta} = 1\ , \quad \mathsf{s}_{\alpha\beta}^+ = 0\ , \quad \mathsf{s}_{\alpha\beta}^- = -1\ .$$

So that (3) reads $X_\alpha^+ X_\beta^+ - q^{-1} X_\beta^+ X_\alpha^+ = X_{\alpha\oplus\beta}^+$ (resp. $X_\alpha^- X_\beta^- - q^{-1} X_\beta^- X_\alpha^- = -q^{-1} X_{\alpha\oplus\beta}^-$). Then, since $X_\alpha^+ = q^{\frac{1}{2}} E_\alpha$ and $X_\alpha^- = q^{-\frac{1}{2}} F_\alpha$, one has

$$qE_\alpha E_\beta - E_\beta E_\alpha = q^{\frac{1}{2}} E_{\alpha\oplus\beta} \quad \text{and} \quad q^{-1} F_\alpha F_\beta - q^{-2} F_\beta F_\alpha = -q^{-\frac{3}{2}} F_{\alpha\oplus\beta}\ ,$$

which corresponds to (5.5) and (5.6), respectively. Assume now that α and β are nested and $\alpha \neq \beta$, so that $\alpha \oplus \beta$, $\alpha\nabla\beta$ and $\alpha \vartriangle \beta$ are not defined, and (3) reduces to $X_\alpha^\pm X_\beta^\pm = q^{\mathsf{r}_{\alpha\beta}} X_\beta^\pm X_\alpha^\pm$. Then, (5.7) follows by observing that, in case of nested intervals, $\mathsf{r}_{\alpha\beta} := (-1)^{\langle\alpha,\beta\rangle} (\alpha|\beta)^2 = \langle\beta,\alpha\rangle - \langle\alpha,\beta\rangle$, as one checks easily from the last seven rows (e–k) of the table (5.1) above.

5.3.2. Conversely, we shall show that the relations (1)–(3) holds in $\mathbf{U}_q\mathfrak{sl}(\mathbb{K})$. (1) follows from (5.2). By the previous discussion, (3) holds for the cases (a) and (e–k) listed in the table (5.1). It remains to prove it holds in the cases (b–d).

- *Case (b):* $\beta \to \alpha$. From (5.5) and (5.6), we get

$$q^{\frac{1}{2}} E_\beta E_\alpha - q^{-\frac{1}{2}} E_\alpha E_\beta = E_{\alpha\oplus\beta}\ , \qquad q^{\frac{1}{2}} F_\beta F_\alpha - q^{-\frac{1}{2}} F_\alpha F_\beta = -F_{\alpha\oplus\beta}\ .$$

Then, by $X_\alpha^+ = q^{\frac{1}{2}} E_\alpha$ and $X_\alpha^- = q^{-\frac{1}{2}} F_\alpha$, we get

$$q^{-\frac{3}{2}} X_\alpha^+ X_\beta^+ - q^{-\frac{1}{2}} X_\beta^+ X_\alpha^+ = -q^{-\frac{1}{2}} X_{\alpha\oplus\beta}^+\ , \qquad q^{\frac{1}{2}} X_\alpha^- X_\beta^- - q^{\frac{3}{2}} X_\beta^- X_\alpha^- = q^{\frac{1}{2}} X_{\alpha\oplus\beta}^-\ ,$$

Finally, we get

$$X_\alpha^+ X_\beta^+ - qX_\beta^+ X_\alpha^+ = -qX_{\alpha\oplus\beta}^+\ , \qquad X_\alpha^- X_\beta^- - qX_\beta^- X_\alpha^- = X_{\alpha\oplus\beta}^-\ ,$$

which agrees with (3), since for $\beta \to \alpha$ we have $\mathsf{r}_{\alpha\beta} = 1$, $\mathsf{b}_{\alpha\beta} = -1$, $\mathsf{s}_{\alpha\beta}^+ = 1$, and $\mathsf{s}_{\alpha\beta}^- = 0$.

- *Case (c):* $\alpha \pitchfork \beta$. Note that, in this case, $\alpha\nabla\beta$ and $\alpha \vartriangle \beta$ are both defined, $\mathsf{r}_{\alpha\beta} = 0$, $\mathsf{b}_{\alpha\beta} = 1$, and (3) reads

$$X_\alpha^\pm X_\beta^\pm - X_\beta^\pm X_\alpha^\pm = (q - q^{-1}) X_{\alpha\nabla\beta}^\pm X_{\alpha\vartriangle\beta}^\pm\ .$$

Set $a = \alpha \ominus (\alpha \vartriangle \beta)$, $b = \alpha \ominus (\alpha \vartriangle \beta)$, and $c = \alpha \vartriangle \beta$. Thus, $\alpha = a \oplus c$ with $a \to c$, $\beta = c \oplus b$ with $c \to b$, and $\alpha\nabla\beta = \alpha \oplus b$ with $\alpha \to b$. Since $c \dashv \alpha$ and $\alpha \to b$, we have

$$E_\beta = q^{\frac{1}{2}} E_c E_b - q^{-\frac{1}{2}} E_b E_c\ , \qquad E_\alpha E_c = qE_c E_\alpha\ , \qquad E_\alpha E_b = q^{-\frac{1}{2}} E_{\alpha\nabla\beta} + q^{-1} E_b E_\alpha\ .$$

Therefore,

$$\begin{aligned} E_\alpha E_\beta &= E_\alpha \left(q^{\frac{1}{2}} E_c E_b - q^{-\frac{1}{2}} E_b E_c \right) \\ &= q^{\frac{3}{2}} E_c \left(q^{-\frac{1}{2}} E_{\alpha\nabla\beta} + q^{-1} E_b E_\alpha \right) - q^{-\frac{1}{2}} \left(q^{-\frac{1}{2}} E_{\alpha\nabla\beta} + q^{-1} E_b E_\alpha \right) E_c \\ &= qE_c E_{\alpha\nabla\beta} + q^{\frac{1}{2}} E_c E_b E_\alpha - q^{-1} E_{\alpha\nabla\beta} E_c - q^{-\frac{1}{2}} E_b E_c E_\alpha\ . \end{aligned}$$

Since $c = \alpha \triangle \beta < \alpha \nabla \beta$, we get

$$E_\alpha E_\beta = E_\beta E_\alpha + (q - q^{-1}) E_{\alpha\nabla\beta} E_{\alpha\triangle\beta} \,,$$

which agrees with (3) under the identification $X^+_\alpha = q^{\frac{1}{2}} E_\alpha$. Similarly,

$$F_\beta = -q^{\frac{1}{2}} F_c F_b + q^{-\frac{1}{2}} F_b F_c \,, \qquad F_\alpha F_c = q F_c F_\alpha \,, \qquad F_\alpha F_b = -q^{-\frac{1}{2}} F_{\alpha\nabla\beta} + q^{-1} F_b F_\alpha \,.$$

Therefore,

$$\begin{aligned} F_\alpha F_\beta &= F_\alpha \left(-q^{\frac{1}{2}} F_c F_b + q^{-\frac{1}{2}} F_b F_c \right) \\ &= -q^{\frac{3}{2}} F_c \left(-q^{-\frac{1}{2}} F_{\alpha\nabla\beta} + q^{-1} F_b F_\alpha \right) + q^{-\frac{1}{2}} \left(-q^{-\frac{1}{2}} F_{\alpha\nabla\beta} + q^{-1} F_b F_\alpha \right) F_c \\ &= q F_c F_{\alpha\nabla\beta} - q^{\frac{1}{2}} F_c F_b F_\alpha - q^{-1} F_{\alpha\nabla\beta} F_c + q^{-\frac{1}{2}} F_b F_c F_\alpha \\ &= F_\beta F_\alpha + (q - q^{-1}) F_{\alpha\nabla\beta} F_{\alpha\triangle\beta} \,, \end{aligned}$$

which agrees with (3) under the identification $X^-_\alpha = q^{-\frac{1}{2}} F_\alpha$.

- *Case (d):* $\beta \pitchfork \alpha$. In this case, $\mathsf{r}_{\alpha\beta} = 0$, $\mathsf{b}_{\alpha\beta} = -1$, and (3) reads

$$X^\pm_\alpha X^\pm_\beta - X^\pm_\beta X^\pm_\alpha = -(q - q^{-1}) X^\pm_{\alpha\nabla\beta} X^\pm_{\alpha\triangle\beta} \,.$$

Thus, $\alpha = c \oplus a$ with $c \to a$, $\beta = b \oplus c$ with $b \to c$, and $\alpha\nabla\beta = b \oplus \alpha$ with $b \to \alpha$. Since $c \vdash \alpha$ and $b \to \alpha$, we have

$$E_\beta = q^{\frac{1}{2}} E_b E_c - q^{-\frac{1}{2}} E_c E_b \,, \qquad E_\alpha E_c = q^{-1} E_c E_\alpha \,, \qquad E_\alpha E_b = -q^{\frac{1}{2}} E_{\alpha\nabla\beta} + q E_b E_\alpha \,.$$

Therefore,

$$\begin{aligned} E_\alpha E_\beta &= E_\alpha \left(q^{\frac{1}{2}} E_b E_c - q^{-\frac{1}{2}} E_c E_b \right) \\ &= q^{\frac{1}{2}} \left(-q^{\frac{1}{2}} E_{\alpha\nabla\beta} + q E_b E_\alpha \right) E_c - q^{-\frac{3}{2}} E_c \left(-q^{\frac{1}{2}} E_{\alpha\nabla\beta} + q E_b E_\alpha \right) \\ &= -q E_{\alpha\nabla\beta} E_c + q^{\frac{1}{2}} E_b E_c E_\alpha + q^{-1} E_c E_{\alpha\nabla\beta} - q^{-\frac{1}{2}} E_c E_b E_\alpha \\ &= E_\beta E_\alpha - (q - q^{-1}) E_{\alpha\nabla\beta} E_{\alpha\triangle\beta} \,, \end{aligned}$$

which agrees with (3). Similarly,

$$F_\beta = -q^{\frac{1}{2}} F_b F_c + q^{-\frac{1}{2}} F_c F_b \,, \qquad F_\alpha F_c = q^{-1} F_c F_\alpha \,, \qquad F_\alpha F_b = q^{\frac{1}{2}} F_{\alpha\nabla\beta} + q F_b F_\alpha \,.$$

Therefore,

$$\begin{aligned} F_\alpha F_\beta &= F_\alpha \left(-q^{\frac{1}{2}} F_b F_c + q^{-\frac{1}{2}} F_c F_b \right) \\ &= -q^{\frac{1}{2}} \left(q^{\frac{1}{2}} F_{\alpha\nabla\beta} + q F_b F_\alpha \right) F_c + q^{-\frac{3}{2}} F_c \left(q^{\frac{1}{2}} F_{\alpha\nabla\beta} + q F_b F_\alpha \right) \\ &= -q F_{\alpha\nabla\beta} F_c - q^{\frac{1}{2}} F_b F_c F_\alpha + q^{-1} F_c F_{\alpha\nabla\beta} + q^{-\frac{1}{2}} F_c F_b F_\alpha \\ &= F_\beta F_\alpha - (q - q^{-1}) F_{\alpha\nabla\beta} F_{\alpha\triangle\beta} \,, \end{aligned}$$

which agrees with (3).

5.3.3. We now show that relations (2) hold in $\mathbf{U}_q\mathfrak{sl}(\mathbb{K})$. This is clear for the cases (a), (b), (e) in the table (5.1). We should prove it for all remaining cases. We start with the cases of a *boundary* subinterval (rows h–k).

- *Case (h):* $\alpha \vdash \beta$. In this case, we have $\mathsf{a}_{\alpha\beta} = 1$ and $\mathsf{c}^-_{\alpha\beta} = 0$, so that (2) reads

$$[X^+_\alpha, X^-_\beta] = -K_\alpha X^-_{\beta\ominus\alpha} \,.$$

Set $\gamma = \beta \ominus \alpha$. Thus, $\beta = \alpha \oplus \gamma$ with $\alpha \to \gamma$. We have

$$F_\beta = -q^{\frac{1}{2}} F_\alpha F_\gamma + q^{-\frac{1}{2}} F_\gamma F_\alpha \,, \quad [E_\alpha, F_\gamma] = 0 \,, \quad F_\gamma K_\alpha = q^{-1} K_\alpha F_\gamma \,, \quad F_\gamma K_\alpha^{-1} = q K_\alpha^{-1} F_\gamma \,.$$

Therefore,

$$\begin{aligned}[E_\alpha, F_\beta] &= -q^{\frac{1}{2}}[E_\alpha, F_\alpha]F_\gamma + q^{-\frac{1}{2}}F_\gamma[E_\alpha, F_\alpha] \\ &= -q^{\frac{1}{2}}\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}}F_\gamma + q^{-\frac{1}{2}}F_\gamma\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}} \\ &= \left(-q^{\frac{1}{2}}\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}} + q^{-\frac{1}{2}}\frac{q^{-1}K_\alpha - qK_\alpha^{-1}}{q-q^{-1}}\right)F_\gamma \\ &= \frac{-q^{\frac{1}{2}} + q^{-\frac{3}{2}}}{q-q^{-1}}K_\alpha F_\gamma = -q^{-\frac{1}{2}}\frac{q-q^{-1}}{q-q^{-1}}K_\alpha F_\gamma = -q^{-\frac{1}{2}}K_\alpha F_{\beta\ominus\alpha}\,,\end{aligned}$$

and we get $[X_\alpha^+, X_\beta^-] = -K_\alpha X^-_{\beta\ominus\alpha}$.

- *Case (i):* $\alpha \dashv \beta$.
  In this case, we have $\mathsf{a}_{\alpha\beta} = -1$ and $\mathsf{c}^-_{\alpha\beta} = 1$, so that (2) reads

$$[X_\alpha^+, X_\beta^-] = qK_\alpha^{-1}X^-_{\beta\ominus\alpha}\,.$$

Set $\gamma = \beta\ominus\alpha$. Thus, $\beta = \alpha\oplus\gamma$ with $\gamma\to\alpha$. We have

$$F_\beta = -q^{\frac{1}{2}}F_\gamma F_\alpha + q^{-\frac{1}{2}}F_\alpha F_\gamma\,,\quad [E_\alpha, F_\gamma] = 0\,,\quad F_\gamma K_\alpha = q^{-1}K_\alpha F_\gamma\,,\quad F_\gamma K_\alpha^{-1} = qK_\alpha^{-1}F_\gamma\,.$$

Therefore,

$$\begin{aligned}[E_\alpha, F_\beta] &= -q^{\frac{1}{2}}F_\gamma[E_\alpha, F_\alpha] + q^{-\frac{1}{2}}[E_\alpha, F_\alpha]F_\gamma \\ &= -q^{\frac{1}{2}}F_\gamma\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}} + q^{-\frac{1}{2}}\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}}F_\gamma \\ &= \left(-q^{\frac{1}{2}}\frac{q^{-1}K_\alpha - qK_\alpha^{-1}}{q-q^{-1}} + q^{-\frac{1}{2}}\frac{K_\alpha - K_\alpha^{-1}}{q-q^{-1}}\right)F_\gamma \\ &= \frac{q^{\frac{3}{2}} - q^{-\frac{1}{2}}}{q-q^{-1}}K_\alpha^{-1}F_\gamma = q^{\frac{1}{2}}K_\alpha^{-1}F_{\beta\ominus\alpha}\,,\end{aligned}$$

and we get $[X_\alpha^+, X_\beta^-] = qK_\alpha^{-1}X^-_{\beta\ominus\alpha} = X^-_{\beta\ominus\alpha}K_\alpha^{-1}$.

- *Case (j):* $\beta \vdash \alpha$.
  In this case, we have $\mathsf{a}_{\alpha\beta} = -1$ and $\mathsf{c}^+_{\alpha\beta} = 0$, so that (2) reads

$$[X_\alpha^+, X_\beta^-] = -X^+_{\alpha\ominus\beta}K_\beta^{-1}\,.$$

Set $\gamma = \alpha\ominus\beta$. Thus, $\alpha = \beta\oplus\gamma$ with $\beta\to\gamma$. We have

$$E_\alpha = q^{\frac{1}{2}}E_\beta E_\gamma - q^{-\frac{1}{2}}E_\gamma E_\beta\,,\quad [E_\gamma, F_\beta] = 0\,,\quad K_\beta E_\gamma = q^{-1}E_\gamma K_\beta\,,\quad K_\beta^{-1}E_\gamma = qE_\gamma K_\beta^{-1}\,.$$

Therefore,

$$\begin{aligned}[E_\alpha, F_\beta] &= q^{\frac{1}{2}}[E_\beta, F_\beta]E_\gamma - q^{-\frac{1}{2}}E_\gamma[E_\beta, F_\beta] \\ &= q^{\frac{1}{2}}\frac{K_\beta - K_\beta^{-1}}{q-q^{-1}}E_\gamma - q^{-\frac{1}{2}}E_\gamma\frac{K_\beta - K_\beta^{-1}}{q-q^{-1}} \\ &= E_\gamma\left(q^{\frac{1}{2}}\frac{q^{-1}K_\beta - qK_\beta^{-1}}{q-q^{-1}} - q^{-\frac{1}{2}}\frac{K_\beta - K_\beta^{-1}}{q-q^{-1}}\right) \\ &= E_\gamma K_\beta^{-1}\frac{-q^{\frac{3}{2}} + q^{-\frac{1}{2}}}{q-q^{-1}} = -q^{\frac{1}{2}}E_\gamma K_\beta^{-1}\end{aligned}$$

and we get $[X_\alpha^+, X_\beta^-] = -X^+_{\alpha\ominus\beta}K_\beta^{-1} = -q^{-1}K_\beta^{-1}X^+_{\alpha\ominus\beta}$.

- *Case (k):* $\beta \dashv \alpha$. In this case, we have $\mathsf{a}_{\alpha\beta} = 1$ and $\mathsf{c}^{+}_{\alpha\beta} = -1$, so that (2) reads

$$[X^{+}_{\alpha}, X^{-}_{\beta}] = q^{-1} X^{+}_{\alpha\ominus\beta} K_{\beta} \,.$$

Set $\gamma = \alpha \ominus \beta$. Thus, $\alpha = \gamma \oplus \beta$ with $\gamma \to \beta$. We have

$$E_{\alpha} = q^{\frac{1}{2}} E_{\gamma} E_{\beta} - q^{-\frac{1}{2}} E_{\beta} E_{\gamma} \,, \quad [E_{\gamma}, F_{\beta}] = 0 \,, \quad K_{\beta} E_{\gamma} = q^{-1} E_{\gamma} K_{\beta} \,, \quad K_{\beta}^{-1} E_{\gamma} = q E_{\gamma} K_{\beta}^{-1} \,.$$

Therefore,

$$\begin{aligned}
[E_{\alpha}, F_{\beta}] &= q^{\frac{1}{2}} E_{\gamma} [E_{\beta}, F_{\beta}] - q^{-\frac{1}{2}} [E_{\beta}, F_{\beta}] E_{\gamma} \\
&= q^{\frac{1}{2}} E_{\gamma} \frac{K_{\beta} - K_{\beta}^{-1}}{q - q^{-1}} - q^{-\frac{1}{2}} \frac{K_{\beta} - K_{\beta}^{-1}}{q - q^{-1}} E_{\gamma} \\
&= E_{\gamma} \left(q^{\frac{1}{2}} \frac{K_{\beta} - K_{\beta}^{-1}}{q - q^{-1}} - q^{-\frac{1}{2}} \frac{q^{-1} K_{\beta} - q K_{\beta}^{-1}}{q - q^{-1}} \right) \\
&= E_{\gamma} K_{\beta} \frac{q^{\frac{1}{2}} - q^{-\frac{3}{2}}}{q - q^{-1}} = q^{-\frac{1}{2}} E_{\gamma} K_{\beta}
\end{aligned}$$

and we get $[X^{+}_{\alpha}, X^{-}_{\beta}] = q^{-1} X^{+}_{\alpha\ominus\beta} K_{\beta} = K_{\beta} X^{+}_{\alpha\ominus\beta}$.

- *Case (c):* $\alpha \pitchfork \beta$. In this case, we have $\mathsf{b}_{\alpha\beta} = 1$ and $\mathsf{b}_{\beta\alpha} = -1$, so that (2) reads

$$[X^{+}_{\alpha}, X^{-}_{\beta}] = -q^{-1}(q - q^{-1}) X^{+}_{(\alpha\triangledown\beta)\ominus\beta} K_{\alpha\vartriangle\beta} X^{-}_{(\alpha\triangledown\beta)\ominus\alpha} \,.$$

Set $c = \alpha \vartriangle \beta$, $a = \alpha \ominus c$, $b = \beta \ominus c$. Thus, $\alpha = a \oplus c$ with $a \to c$, $\beta = c \oplus b$ with $c \to b$, and $c \vdash \beta$. Therefore,

$$E_{\alpha} = q^{\frac{1}{2}} E_{a} E_{c} - q^{-\frac{1}{2}} E_{c} E_{a} \,, \quad [E_{a}, F_{\beta}] = 0 \,, \quad [E_{c}, F_{\beta}] = -q^{-\frac{1}{2}} K_{c} F_{b} \,, \quad K_{c} E_{a} = q^{-1} E_{a} K_{c} \,,$$

and we have

$$\begin{aligned}
[E_{\alpha}, F_{\beta}] &= [q^{\frac{1}{2}} E_{a} E_{c} - q^{-\frac{1}{2}} E_{c} E_{a}, F_{\beta}] \\
&= q^{\frac{1}{2}} E_{a} [E_{c}, F_{\beta}] - q^{-\frac{1}{2}} [E_{c}, F_{\beta}] E_{a} \\
&= q^{\frac{1}{2}} E_{a} \left(-q^{-\frac{1}{2}} K_{c} F_{b} \right) - q^{-\frac{1}{2}} \left(-q^{-\frac{1}{2}} K_{c} F_{b} \right) E_{a} \\
&= -E_{a} K_{c} F_{b} + q^{-2} E_{a} K_{c} F_{b} = -(1 - q^{-2}) E_{a} K_{c} F_{b} \,.
\end{aligned}$$

Therefore, $[X^{+}_{\alpha}, X^{-}_{\beta}] = -q^{-1}(q - q^{-1}) X^{+}_{(\alpha\triangledown\beta)\ominus\beta} K_{\alpha\vartriangle\beta} X^{-}_{(\alpha\triangledown\beta)\ominus\alpha}$.

- *Case (d):* $\beta \pitchfork \alpha$. In this case, we have $\mathsf{b}_{\alpha\beta} = -1$ and $\mathsf{b}_{\beta\alpha} = 1$, so that (2) reads

$$[X^{+}_{\alpha}, X^{-}_{\beta}] = q(q - q^{-1}) X^{+}_{(\alpha\triangledown\beta)\ominus\beta} K^{-1}_{\alpha\vartriangle\beta} X^{-}_{(\alpha\triangledown\beta)\ominus\alpha} \,.$$

Set $c = \alpha \vartriangle \beta$, $a = \alpha \ominus c$, $b = \beta \ominus c$. Thus, $\alpha = c \oplus a$ with $c \to a$, $\beta = b \oplus c$ with $b \to c$, and $c \dashv \beta$. Therefore,

$$E_{\alpha} = q^{\frac{1}{2}} E_{c} E_{a} - q^{-\frac{1}{2}} E_{a} E_{c} \,, \quad [E_{a}, F_{\beta}] = 0 \,, \quad [E_{c}, F_{\beta}] = q^{\frac{1}{2}} K_{c}^{-1} F_{b} \,, \quad K_{c}^{-1} E_{a} = q E_{a} K_{c}^{-1} \,,$$

and we have

$$\begin{aligned}
[E_{\alpha}, F_{\beta}] &= [q^{\frac{1}{2}} E_{c} E_{a} - q^{-\frac{1}{2}} E_{a} E_{c}, F_{\beta}] \\
&= q^{\frac{1}{2}} [E_{c}, F_{\beta}] E_{a} - q^{-\frac{1}{2}} E_{a} [E_{c}, F_{\beta}] \\
&= q^{\frac{1}{2}} \left(q^{\frac{1}{2}} K_{c}^{-1} F_{b} \right) E_{a} - q^{-\frac{1}{2}} E_{a} \left(q^{\frac{1}{2}} K_{c}^{-1} F_{b} \right) \\
&= (q^{2} - 1) E_{a} K_{c}^{-1} F_{b} \,.
\end{aligned}$$

Therefore, $[X^{+}_{\alpha}, X^{-}_{\beta}] = q(q - q^{-1}) X^{+}_{(\alpha\triangledown\beta)\ominus\beta} K^{-1}_{\alpha\vartriangle\beta} X^{-}_{(\alpha\triangledown\beta)\ominus\alpha}$.

- *Case (f):* $\alpha < \beta$. Note that, in this case, $\beta \ominus \alpha$, $\alpha \ominus \beta$, $\alpha \nabla \beta$, $\alpha \triangle \beta$ are not defined. Let b, b'' be the two connected components of $\beta \setminus \alpha$, so that $\beta = b \oplus b'$ with $b' = \alpha \oplus b''$, $b \to b'$, $b \to \alpha$ and $\alpha \vdash b'$. Then,
$$F_\beta = -q^{\frac{1}{2}} F_b F_{b'} + q^{-\frac{1}{2}} F_{b'} F_b\,, \quad [E_\alpha, F_b] = 0\,, \quad [E_\alpha, F_{b'}] = -q^{-\frac{1}{2}} K_\alpha F_{b''}\,, \quad F_b K_\alpha = q^{-1} K_\alpha F_b\,.$$
and we get
$$\begin{aligned}[E_\alpha, F_\beta] &= [E_\alpha, -q^{\frac{1}{2}} F_b F_{b'} + q^{-\frac{1}{2}} F_{b'} F_b] \\ &= -q^{\frac{1}{2}} F_b [E_\alpha, F_{b'}] + q^{-\frac{1}{2}} [E_\alpha, F_{b'}] F_b \\ &= -q^{\frac{1}{2}} F_b \left(-q^{-\frac{1}{2}} K_\alpha F_{b''}\right) + q^{-\frac{1}{2}} \left(-q^{-\frac{1}{2}} K_\alpha F_{b''}\right) F_b \\ &= q^{-1} K_\alpha F_b F_{b''} - q^{-1} K_\alpha F_{b''} F_b = 0\,,\end{aligned}$$
where the last equality follows from (5.7), since $b \perp b''$ and therefore $F_b F_{b''} = F_{b''} F_b$. Thus, we get $[X_\alpha^+, X_\beta^-] = 0$.

- *Case (g):* $\beta < \alpha$. Note that, in this case, $\beta \ominus \alpha$, $\alpha \ominus \beta$, $\alpha \nabla \beta$, $\alpha \triangle \beta$ are not defined. Let a, a'' be the two connected components of $\alpha \setminus \beta$, so that $\alpha = a \oplus a'$ with $a' = \beta \oplus a''$, $a \to a'$, $a \to \beta$ and $\beta \vdash a'$. Then,
$$E_\alpha = q^{\frac{1}{2}} E_a E_{a'} - q^{-\frac{1}{2}} E_{a'} E_a\,, \quad [E_a, F_\beta] = 0\,, \quad [E_{a'}, F_\beta] = -q^{\frac{1}{2}} E_{a''} K_\beta^{-1}\,, \quad K_\beta^{-1} E_a = q E_a K_\beta^{-1}\,.$$
and we get
$$\begin{aligned}[E_\alpha, F_\beta] &= [q^{\frac{1}{2}} E_a E_{a'} - q^{-\frac{1}{2}} E_{a'} E_a, F_\beta] \\ &= q^{\frac{1}{2}} E_a [E_{a'}, F_\beta] - q^{-\frac{1}{2}} [E_{a'}, F_\beta] E_a \\ &= q^{\frac{1}{2}} E_a \left(-q^{\frac{1}{2}} E_{a''} K_\beta^{-1}\right) - q^{-\frac{1}{2}} \left(-q^{\frac{1}{2}} E_{a''} K_\beta^{-1}\right) E_a \\ &= -q E_a E_{a''} K_\beta^{-1} + q E_{a''} E_a K_\beta^{-1} = 0\,,\end{aligned}$$
where the last equality follows from (5.7), since $a \perp a''$ and therefore $E_b E_{b''} = E_{b''} E_b$. Thus, we get $[X_\alpha^+, X_\beta^-] = 0$.

□

5.4. **Quasi–triangular bialgebra structure on continuum quantum groups.** We now prove the second main result of the paper. Namely, we show that the continuum quantum group $\mathbf{U}_q\mathfrak{g}_X$ is naturally endowed with a topological quasi–triangular Hopf algebra structure, quantizing the topological quasi–triangular Lie bialgebra $\mathfrak{g}_X$.

More precisely, we prove the following.

Theorem 5.11. *Let $\mathcal{Q}_X$ be a continuum quiver and $\mathbf{U}_q\mathfrak{g}_X$ the corresponding continuum quantum group.*

(1) The algebra $\mathbf{U}_q\mathfrak{g}_X$ is a topological Hopf algebra with respect to the maps $\Delta\colon \mathbf{U}_q\mathfrak{g}_X \to \mathbf{U}_q\mathfrak{g}_X \widehat{\otimes} \mathbf{U}_q\mathfrak{g}_X$ and $\varepsilon\colon \mathbf{U}_q\mathfrak{g}_X \to \mathbb{C}[\![\hbar]\!]$ defined on the generators by $\varepsilon(\xi_\alpha) := 0 := \varepsilon(X_\alpha^\pm)$, $\Delta(\xi_\alpha) := \xi_\alpha \otimes 1 + 1 \otimes \xi_\alpha$, and
$$\Delta(X_\alpha^+) := X_\alpha^+ \otimes 1 + K_\alpha \otimes X_\alpha^+ + \sum_{\alpha=\beta\oplus\gamma} \mathsf{a}_{\gamma,\beta\oplus\gamma}\, \mathsf{s}_{\beta\gamma}^- \cdot q^{-1}(q-q^{-1})\, X_\beta^+ K_\gamma \otimes X_\gamma^+\,,$$
$$\Delta(X_\alpha^-) := 1 \otimes X_\alpha^- + X_\alpha^- \otimes K_\alpha^{-1} - \sum_{\alpha=\beta\oplus\gamma} \mathsf{a}_{\gamma,\beta\oplus\gamma}\, \mathsf{s}_{\beta\gamma}^- \cdot (q-q^{-1})\, X_\beta^- \otimes X_\gamma^- K_\gamma^{-1}\,.$$
In particular, $\varepsilon(K_\alpha) = 1$ and $\Delta(K_\alpha) = K_\alpha \otimes K_\alpha$. As usual, the antipode is given by the formula
$$S := \sum_n m^{(n)} \circ (\mathsf{id} - \iota \circ \varepsilon)^{\otimes n} \circ \Delta^{(n)}\,,$$
where $m^{(n)}$ and $\Delta^{(n)}$ denote the iterated product and coproduct, respectively.

(2) *Denote by* $\mathbf{U}_q\mathfrak{b}_X^{\pm}$ *the Hopf subalgebras generated by* $\mathfrak{f}_X$ *and* $X_\alpha^{\pm}$, $\alpha \in \mathsf{Int}(X)$. *Then, there exists a unique Hopf pairing* $(\cdot|\cdot) : \mathbf{U}_q\mathfrak{b}_X^{+} \otimes (\mathbf{U}_q\mathfrak{b}_X^{-})^{\mathrm{cop}} \to \mathbb{C}((\hbar))$, *defined on the generators by*

$$(1|1) := 1\,, \quad (\xi_\alpha|\xi_\beta) := \frac{1}{\hbar}\,(\alpha|\beta)\;, \quad \left(X_\alpha^+|X_\beta^-\right) := \frac{\delta_{\alpha\beta}}{q-q^{-1}}\,,$$

and zero otherwise. In particular, $\left(K_\alpha|K_\beta\right) = q^{(\alpha|\beta)}$.

(3) *Through the Hopf pairing* $(\cdot|\cdot)$, *the Hopf algebras* $(\mathbf{U}_q\mathfrak{b}_X^{+}, \mathbf{U}_q\mathfrak{b}_X^{-})$ *form a match pair. Therefore,* $\mathbf{U}_q\mathfrak{g}_X$ *can be realized as a quotient of the double cross product Hopf algebra* $\mathbf{U}_q\mathfrak{b}_X^{+} \bowtie \mathbf{U}_q\mathfrak{b}_X^{-}$ *obtained by identifying the two copies of* $\mathfrak{f}_X$. *In particular,* $\mathbf{U}_q\mathfrak{g}_X$ *is a topological quasi–triangular Hopf algebra.*

(4) *The topological quasi–triangular Hopf algebra* $\mathbf{U}_q\mathfrak{g}_X$ *is a quantization of the topological quasi–triangular Lie bialgebra* $\mathfrak{g}_X$.

The strategy of the proof is essentially identical to that of Theorem 4.6 and consists in showing that the continuum quantum group $\mathbf{U}_q\mathfrak{g}_X$ can be equivalently realized by duality. This is obtained by considering the quantum analogue of the techniques used earlier, generalizing the construction of Drinfeld–Jimbo quantum groups given by Lusztig (cf. [Lus10, Chapter 1]). We will schematically described the proof below, leaving the details to reader.

- Let $\mathcal{H}_\pm$ be the free associative algebras over $\mathbb{C}[\![\hbar]\!]$ with set of generators $\xi_\alpha^{\pm}$ and $X_\alpha^{\pm}$, $\alpha \in \mathsf{Int}(X)$. Then, the assigments $\varepsilon(\xi_\alpha^{\pm}) := 0 := \varepsilon(X_\alpha^{\pm})$, $\Delta_\pm(\xi_\alpha^{\pm}) := \xi_\alpha^{\pm} \otimes 1 + 1 \otimes \xi_\alpha^{\pm}$, and

$$\Delta_+(X_\alpha^+) := X_\alpha^+ \otimes 1 + K_\alpha \otimes X_\alpha^+ + \sum_{\alpha=\beta\oplus\gamma} \mathsf{a}_{\gamma,\beta\oplus\gamma}\,\mathsf{s}_{\beta\gamma}^{-} \cdot q^{-1}(q-q^{-1})\,X_\beta^+ K_\gamma \otimes X_\gamma^+\,,$$

$$\Delta_-(X_\alpha^-) := 1 \otimes X_\alpha^- + X_\alpha^- \otimes K_\alpha^{-1} - \sum_{\alpha=\beta\oplus\gamma} \mathsf{a}_{\gamma,\beta\oplus\gamma}\,\mathsf{s}_{\beta\gamma}^{-} \cdot (q-q^{-1})\,X_\beta^- \otimes X_\gamma^- K_\gamma^{-1}\,,$$

extend uniquely to two algebra maps $\Delta_\pm : \mathcal{H}_\pm \to \mathcal{H}_\pm \widehat{\otimes} \mathcal{H}_\pm$ and $\varepsilon\colon \mathcal{H}_\pm \to \mathbb{C}[\![\hbar]\!]$, defining on $\mathcal{H}_\pm$ a structure of topological bialgebra.

- There exists a unique pairing of bialgebras $(\cdot|\cdot) : \mathcal{H}_+ \otimes \mathcal{H}_- \to \mathbb{C}((\hbar))$ defined on the generators by

$$(1|1) := 1\,, \quad \left(\xi_\alpha^+|\xi_\beta^-\right) := \frac{1}{\hbar}\,(\alpha|\beta)\;, \quad \left(X_\alpha^+|X_\beta^-\right) := \frac{\delta_{\alpha\beta}}{q-q^{-1}}\,,$$

where $q := \exp(\hbar/2)$, and zero otherwise. In particular, $\left(K_\alpha|K_\beta\right) = q^{(\alpha|\beta)}$, where $K_\alpha := \exp \hbar/2 \cdot \xi_\alpha$.

- Let $\mathcal{I}_\pm$ be the ideal generated in $\mathcal{H}_\pm$ by the elements

$$\xi_{\alpha\oplus\beta}^{\pm} - \delta_{\alpha\oplus\beta}\left(\xi_\alpha^{\pm} + \xi_\beta^{\pm}\right)\,, \quad [\xi_\alpha^{\pm}, \xi_\beta^{\pm}]\,, \quad [\xi_\alpha^{\pm}, X_\beta^{\pm}] - \pm\,(\alpha|\beta)\,X_\beta^{\pm}$$

for any $\alpha, \beta \in \mathsf{Int}(X)$, and

$$X_\alpha^{\pm} X_\beta^{\pm} - q^{\mathsf{r}_{\alpha\beta}} \cdot X_\beta^{\pm} X_\alpha^{\pm} - \mp \mathsf{b}_{\alpha\beta} \cdot q^{\mathsf{s}_{\alpha\beta}^{\pm}} \cdot X_{\alpha\oplus\beta}^{\pm} - \mathsf{b}_{\alpha\beta} \cdot (q-q^{-1}) \cdot X_{\alpha\nabla\beta}^{\pm} X_{\alpha\Delta\beta}^{\pm}$$

for any $(\alpha, \beta) \in \mathsf{Serre}(X)$. Then, $\mathcal{I}_\pm$ is a coideal and it is orthogonal to $\mathcal{H}_\mp$.

- Set $\mathcal{B}_\pm := \mathcal{H}_\pm/\mathcal{I}_\pm$. Then, $(\mathcal{B}_+, \mathcal{B}_-)$ form a matched pair of topological bialgebras. Moreover, the quantum double relation (cf. Definition 5.6-(2)) holds in the double cross product bialgebra $\mathcal{D} = \mathcal{B}_+ \bowtie \mathcal{B}_- / \sim$, where the quotient is obtained by identifying the two copies of the commutative subalgebra generated by the elements $\xi_\alpha^{\pm}$, $\alpha \in \mathsf{Int}(X)$. In particular, there is a canonical algebra isomorphism $\mathbf{U}_q\mathfrak{g}_X \simeq \mathcal{D}$.

- Finally one observes that, for any irreducible set $\mathcal{J}$, the map $\mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \to \mathbf{U}_q\mathfrak{g}_X \simeq \mathcal{D}$ from Section 5.2 preserves the pairing. In particular, this implies that the pairing on $\mathcal{D}$, and therefore on $\mathbf{U}_q\mathfrak{g}_X$ is non–degenerate. The result follows.

Moreover we get the following.

Corollary 5.12. *The morphism* $\mathrm{colim}_{\mathcal{J}}\, \mathbf{U}_q\mathfrak{g}_{\mathcal{J}}^{\mathrm{BKM}} \to \mathbf{U}_q\mathfrak{g}(X)$ *from Corollary 5.8 is an algebra isomorphism.*

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