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On the Equivalence of the Proportional and Damped Trend Order-Up-To Policies: An Eigenvalue Analysis

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Abstract

Our contribution is to show the equivalence of the order-up-to replenishment policy with damped trend forecasting (OUT-DT) to the proportional OUT (POUT) policy via an eigenvalue (zero-pole) analysis. We also investigate whether the OUT-DT policy has an always increasing in the lead time Bullwhip effect using the eigenvalues ordering approach of Gaalman, Disney and Wang (2018).

Keywords: Damped trend forecasting, Order-up-to policy, Bullwhip, Proportional order-up-to policy, Eigenvalue analysis

1. Introduction

We study the dynamic behaviour of the order-up-to (OUT) policy when the damped trend (DT) forecasting method predicts the lead-time demand. In particular, we investigate the *Bullwhip* and *NSAmp* generated by the system. *Bullwhip* is defined as the ratio of the variance replenishment orders, o_t, to the variance the demand, d_t. *NSAmp* is the ratio of the variance of the net stock levels ns_t to the variance of the demand:

$$Bullwhip = \frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]} \quad \text{and} \quad NSAmp = \frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]}. \tag{1}$$

Here $\mathbb{V}[\cdot]$ is the variance operator. Li, Disney and Gaalman (2014) show the Bullwhip effect can be avoided by using unconventional DT forecasting parameters within the OUT-DT policy. This analysis was extended by Li and Disney (2018) where the inventory implications of the OUT-DT policy were explored. By investigating the relationship between stability and invertibility, Li and Disney (2018) showed stable DT parameter sets produce invertible forecasts, justifying the use of unconventional DT parameter values. Li and Disney (2018) also characterized the frequency response of the inventory levels maintained by the OUT-DT policy, finding good inventory control when parameter values were selected from within a *Bullwhip Avoidance* ($\mathscr{BA}$) region.

In this paper, we study the eigenvalues (the poles and zeros) of the OUT-DT policy and show they are equivalent to the eigenvalues of the so-called proportional order-up-to (POUT) policy, Chen and Disney (2007). The POUT policy is well known to avoid the Bullwhip effect and is able to effectively balance the trade-off between inventory and capacity costs. If the two system have the same eigenvalues, they will have the same dynamic response to demand, Nise (2004). This is an interesting and practically useful insight. It means, we can incorporate the POUT policy into an enterprise resource planning system without creating user defined functions in the production planning module. Instead, we can get the same dynamic response by manipulating the forecasting parameters in the forecasting module. Potentially, this offers an easier implementation route for the POUT policy. We also investigate whether the Bullwhip effect is increasing in the lead time when ARIMA(1,1,2) demand is present. DT provides the optimal forecasting of this demand process.

As the OUT-DT and POUT policies operate on a discrete time basis, we use the z-transform in our study. The use of transform techniques to study forecasting problems has a long history (Brown 1963; Wikner 2006). We review the concepts of stability and invertibility of the DT method, (Jury 1974). We obtain expressions for the variance of the orders and the inventory (Tsypkin 1964) under i.i.d. demand. Finally, we conduct an eigenvalue (zero-pole) analysis to determine how the Bullwhip effect is influenced by the lead time under non-stationary ARIMA(1,1,2) demand.

2. Literature review

The DT forecasting method, often attributed to Gardner and McKenzie (1985), is an exponential smoothing based forecasting method based on three steps. The first step produces an exponential smoothing forecast of the level of the demand. The second step produces an exponential smoothing forecast of the rate of change in the demand, the trend. The third step produces a future projection. The projection could be linear, but it need not be. It could be *damped*, where the future projections flatten out to a constant level. The future projection could also exhibit linear or exponential growth (or decline) depending on the demand and the damping parameter selected. The future projections could also oscillate. DT is a generalisation of Holt's method, Roberts (1982).

DT outperformed many forecasting methods in the M3 competition (Makridakis and Hibon 2000). Only a few methods requiring additional effort and cost are able to consistently produce forecasts with better accuracy than DT. However, the improvements are small, and in most cases, not statistically significant (Makridakis and Hibon 2000). Using the monthly industry series from the M3 competition data, Petropoulos et al. (2019) explored the implications of various forecasting methods on both order and inventory variance in the OUT policy and confirmed DT's robust inventory performance.

DT is known to produce minimum mean squared error (MMSE) forecasts of demand k-periods ahead for the ARIMA(1,1,2) demand process, Roberts (1982). However, in principle, DT can be used to forecast any demand process. Just as exponential smoothing is optimal for IMA(0,1,1) demand processes but can be used (rightly or wrongly) to forecast other demand processes. Dejonckheere et al. (2003) found, for all lead times and all possible demand processes, the OUT policy with exponential smoothing forecasts always created the Bullwhip effect. Li et al. (2014) showed the OUT policy with Holt's forecasts also created Bullwhip for all lead times and all demand processes, but DT does not always create Bullwhip. Gaalman (2006) considers the closely related stationary ARMA(2,2) demand process where it was possible to obtain order and inventory variance expressions. Gaalman et al. (2018) present a novel method to determine whether the Bullwhip

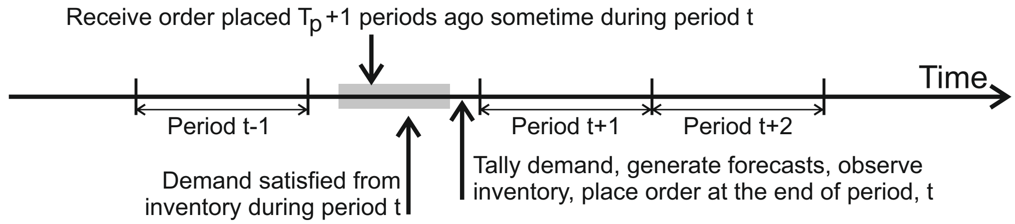


Figure 1: Sequence of event in the OUT policy.

generated by OUT policy reacting the ARMA(p,q) demand process is increasing in the lead-time effect or not. It was shown to depend upon the order of the eigenvalues. We will adapt this approach for the ARIMA(1,1,2) demand process.

3. The order-up-to policy

The OUT replenishment policy is frequently applied in industry, especially in high volume settings (Cannella et al. 2017; Li and Disney 2017). We follow the assumptions and notation used in Li et al. (2014) except now we consider a general lead-time: $T_p \in \mathbb{N}^0$, rather than $T_p = 1$. In each period t, the manufacturer receives the replenishment order placed $T_p + 1$ periods ago, and satisfies demand d_t from its finished goods inventory, or *net stock*, ns_t. The manufacturer sets production targets/replenishment orders o_t via

$$o_t = \hat{d}_{t,t+T_p+1} + ns^\star - ns_t + \sum_{i=1}^{T_p} \left(\hat{d}_{t,t+i} - o_{t-i} \right). \tag{2}$$

Here, $\hat{d}_{t,t+T_p+1}$ is a forecast of demand, made at time t in the period $t + T_p + 1$. That is, $\hat{d}_{t,t+T_p+1}$ is a forecast of demand in the period after the lead time. $\sum_{i=1}^{T_p} o_{t-i}$ is the work-in-progress. The time-varying target work-in-progress, $\sum_{i=1}^{T_p} \hat{d}_{t,i+t}$, is the sum of the demand forecasts made at time t for the periods from $t+1$ to $t+T_p$. The target net stock ($ns^\star$) is a safety stock used to ensure a strategic level of inventory availability; $ns^* = \sqrt{\mathbb{V}[ns_t]} F^{-1}[p]$. Here $F^{-1}[p]$ is the inverse of the cumulative inventory distribution evaluated at the target availability p (Hosoda and Disney 2009). If per unit, per period inventory holding (h) and backlog costs (b) exist and $p = b/(b+h)$, $ns^\star$ minimizes inventory costs, Churchman et al. (1957). The inventory balance equation,

$$ns_t = ns_{t-1} - d_t + o_{t-T_p-1}, \tag{3}$$

completes the OUT policy specification. Note, the sequence of events delay, the "-1" in the time index of the orders in (3). This is why $T_p = 0$ refers to a unit lead time as, following (Zipkin 2000, p404), the risk period includes the sequence of events delay. The sequence of events within each time period is illustrated in Figure 1. First, the order placed $T_p + 1$ periods ago is received sometime during the period; demand is also satisfied during the period. Demand is tallied, future forecasts determined, and replenishment orders are generated and placed at the end of the period.

In order to preserve linearity of the system and to allow for a tractable analysis, the following assumptions are made: Negative demand quantities indicate that customers are free to return products to suppliers (this can become negligible when the mean demand is sufficiently larger than the standard deviation of demand). Negative orders indicate that finished goods are disassembled into raw material (this can become negligible when the mean orders is sufficiently larger than the standard deviation of the orders). There are no capacity constraints in the system, and unmet demand is backlogged. We refer readers to Disney et al. (2021) for more discussion on these factors.

Boute et al. (2022) provide the following transfer function for the orders in the OUT policy,

$$\frac{O(z)}{\varepsilon(z)} = \frac{D(z)}{\varepsilon(z)}\left(1+\sum_{k=1}^{T_p}\frac{\hat{D}_k(z)}{\varepsilon(z)}\left(1-\frac{1}{z}\right)\right), \tag{4}$$

where $D(z)/\varepsilon(z)$ is the transfer function of the demand generation process and $\hat{D}_k(z)/\varepsilon(z)$ is the transfer function of the k-periods ahead forecast. These will be defined in later sections. A general form of the the net stock transfer function, $NS(z)/\varepsilon(z)$, is given by

$$\frac{NS(z)}{\varepsilon(z)} = \frac{z}{z-1}\left(z^{-T_p-1}\frac{O(z)}{\varepsilon(z)} - \frac{D(z)}{\varepsilon(z)}\right). \tag{5}$$

Here, $z/(z-1)$ is the z-transform of the integration operator and z^{-T_p-1} is the z-transform of the delay operator. By convention, lower case letters are used for variables in the time domain and equivalent upper case letters for the corresponding variables in the frequency domain.

4. The proportional order-up-to (POUT) policy under i.i.d. demand.

The POUT policy is the optimal linear replenishment rule for minimising the weighted sum of order and inventory variance and is an appropriate benchmark for this study. The POUT policy (Boute et al. 2009) is defined as

$$o_t = \hat{d}_{t,t+T_p+1} + \frac{1}{T_i}\left(ns^\star - ns_t + \sum_{i=1}^{T_p}\left(\hat{d}_{t,t+i} - o_{t-i}\right)\right). \tag{6}$$

T_i is a proportional feedback controller with which we can tune the dynamic behaviour of the OUT policy. When an i.i.d. demand is present, MMSE forecasts of future demands are given by

$$\forall i, \quad \hat{d}_{t,t+i} = \hat{d}_{t,t+1} = \mu_d. \tag{7}$$

Eqs (6) and (3) can be converted into a block diagram (omitted to save space in this short paper). The block diagram can be rearranged to yield the following z-transform of the POUT policy,

$$\left.\frac{O(z)}{\varepsilon(z)}\right|_{\text{POUT}} = \frac{\frac{1}{T_i}z}{z-\frac{T_i-1}{T_i}} \tag{8}$$

which is a zero-pole form.

The transfer function of the net stock levels maintained by the POUT policy can be written as

$$\left.\frac{NS(z)}{\varepsilon(z)}\right|_{\text{POUT}} = \frac{\sum_{i=0}^{T_p-1} z^i + z^{T_p}T_i}{z^{T_p-1}(T_i-1) - z^{T_p}T_i}. \tag{9}$$

4.1. Stability of the POUT policy.

Stability is concerned with a system's response to a bounded system input. If the system produces a bounded output, the system is considered to be stable. If the system's response diverges exponentially, or oscillate with ever increasing amplitude, the system is unstable. For stability, the eigenvalues (zeros and poles) of the POUT policy must lie within the unit circle. Eq. (8) shows the POUT policy has one real zero at $\lambda_1^\theta = 0$ and one real pole at $\lambda_1^\phi = (T_i - 1)/Ti$. The pole is inside the unit circle in the complex plane if $T_i > 0.5$, indicating the stability criteria, Disney (2008).

4.2. Variance ratio analysis of the POUT policy under i.i.d. demand

For a linear system reacting to an i.i.d. input ε_t, the long-run variance of the system's output x_t, can be calculated via Tsypkin's Relation, Disney and Towill (2003).

$$\frac{\mathbb{V}[\text{System output, } x_t]}{\mathbb{V}[\text{White noise input, } \varepsilon_t]} = \sum_{t=0}^{\infty} (\tilde{x}_t)^2 \tag{10}$$

where $\tilde{x}_t$ is the response of the system when demand is given by the impulse (Dirac delta) function; i.e. $\varepsilon_t = 1$ if $t = 0$, $\varepsilon_t = 0$ otherwise. Consider first the *Bullwhip* ratio. The relevant system output is the orders o_t, those impulse response $\tilde{o}_t$ can be obtained by taking the inverse z-transform of (8),

$$\tilde{o}_t = Z^{-1}\left[\frac{\frac{1}{T_i}z}{z - \frac{T_i-1}{T_i}}\right] = \frac{1}{T_i}\left(\frac{T_i - 1}{T_i}\right)^t. \tag{11}$$

Using (11) in (10) and yields the *Bullwhip* ratio for the POUT policy under i.i.d. demand:

$$Bullwhip = \frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]} = \sum_{t=0}^{\infty}\left(\frac{1}{T_i}\left(\frac{T_i - 1}{T_i}\right)^t\right)^2 = \frac{1}{2T_i - 1}. \tag{12}$$

Note, $Bullwhip = 1$ when $T_i = 1$, is decreasing convex in T_i and $Bullwhip = 0$ when $T_i \to \infty$. The *NSAmp* ratio can be obtained by first taking the inverse z-transform of (9) to yield,

$$\tilde{ns}_t = Z^{-1}\left[= \frac{\sum_{i=0}^{T_p-1} z^i + z^{T_p}T_i}{z^{T_p-1}(T_i - 1) - z^{T_p}T_i}\right] = \begin{cases} -1 & \text{if } t \leq T_p, \\ -\left(\frac{T_i-1}{Ti}\right)^{t-T_p} & \text{if } t > T_p. \end{cases} \tag{13}$$

Using (13) in (10) provides *NSAmp* for the POUT policy under i.i.d. demand:

$$NSAmp = \frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]} = \sum_{t=0}^{T_p}(-1)^2 + \sum_{t=T_p+1}^{\infty}\left(-\left(\frac{T_i - 1}{Ti}\right)^{t-T_p}\right)^2 = 1 + T_p + \frac{(T_i - 1)^2}{2T_i - 1}. \tag{14}$$

NSAmp is convex in T_i, with an asymptote to infinity when $T_i \downarrow 0.5$, and is increasing in T_i when $T_i > 1$; a minimum of $NSAmp = 1 + T_p$ at $T_i = 1$, The POUT policy represents the gold standard in linear replenishment rules for balancing inventory and capacity costs, Boute et al. (2022). The aim of the next section is to see if the OUT-DT policy can match, or better, this performance.

5. Damped trend forecasting

We now turn our attention to the DT forecasting mechanism. Gardner and McKenzie (1985) provide the following recurrence form of the DT forecasting method:

$$\hat{a}_t = \alpha d_t + (1-\alpha)\left(\hat{a}_{t-1} + \gamma \hat{b}_{t-1}\right), \tag{15}$$

$$\hat{b}_t = \beta\left(\hat{a}_t - \hat{a}_{t-1}\right) + (1-\beta)\gamma \hat{b}_{t-1}, \tag{16}$$

$$\hat{d}_{t,t+k} = \hat{a}_t + \varphi[k]\,\hat{b}_t. \tag{17}$$

Here, $\hat{d}_{t,t+k}$ is the forecast of the demand k periods ahead, d_{t+k}, made at time t. $\hat{d}_{t,t+k}$ is the sum of a level, $\hat{a}_t$, and a trend, $\hat{b}_t$, component and

$$\varphi[k] = \sum_{i=1}^{k} \gamma^i = \frac{\gamma\left(1-\gamma^k\right)}{1-\gamma}. \tag{18}$$

$\{\alpha, \beta, \gamma\}$ are the DT forecasting parameters. α is a smoothing constant applied to the level $\hat{a}_t$, β is a smoothing constant applied to the trend $\hat{b}_t$, and γ shapes the forecasts as they are projected into the future. The trend is damped for $0<\gamma<1$, although γ can take on other values. If $\gamma=0$, there is no trend and the forecasting system acts as exponential smoothing would react. If $\gamma=1$, the model is equivalent to Holt's method. When $\gamma>1$, the forecasts exhibit exponential growth.

Li et al. (2014) derived the following transfer function of (17), the k-period ahead DT forecast,

$$\hat{D}_k(z) = \frac{z^2\alpha\left(1+\beta\varphi[k]\right) - z\alpha\left(\gamma(1-\beta)+\beta\varphi[k]\right)}{z^2 - z(1+\gamma-\alpha-\alpha\beta\gamma) - \gamma(\alpha-1)}. \tag{19}$$

5.1. Invertibility and stability of damped trend forecasting

The concept of invertibility is concerned with the ability to identify the demand process structure from past demand observations. Invertibility is related to linear moving average (MA) models or the MA part of auto-regressive integrated moving average (ARIMA) models (Box et al. 2008). All exponential smoothing forecasting methods (of which DT is one) can be converted into an equivalent ARIMA model. If the MA part in an ARIMA model can be expressed as an autoregressive (AR) model of infinite order, the model is deemed invertible and implies all relevant state variables are directly observable (Box et al. 2008).

The stability region of DT has been previously studied by Li and Disney (2018) who showed the stability region was the same as the invertibility region. Gardner and McKenzie (1985, p. 1239) provided a stability region for DT, but it is only valid for $0 \le \gamma \le 1$; they do however acknowledge that stable parameters exist outside of their stated stability region. Hyndman et al. (2008, p. 412) studied the stability of the state space representation of the ETS(A,A_d,N) model, which is equivalent to DT after a suitable change in notation, and provided stability boundaries under the condition that $0<\gamma\le 1$. Li et al. (2014, pp. 5–6) studied the stability of DT via Jury's Inners approach (Jury 1974) and visualized the complete stability boundaries for all γ. For convenience we repeat them here; when $\gamma \neq 0$ the following relations must be satisfied for stability:

$$\left.\begin{array}{l} \gamma-1<\alpha\gamma<\gamma+1, \\ \alpha(\gamma-1)<\alpha\beta\gamma<(2-\alpha)(\gamma+1). \end{array}\right\} \tag{20}$$

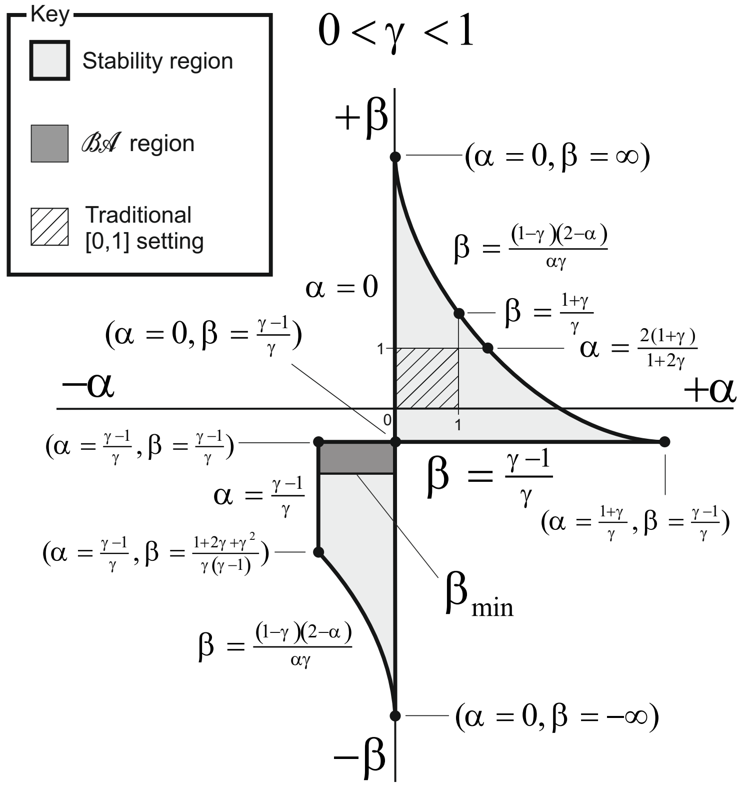


Figure 2: Characterisation of the DT parameter space when $0<\gamma<1$. Source: Adapted from Li and Disney (2018).

When $\gamma=0$, $0<\alpha<2$ is required for stability. Eq. (20) is equivalent to the result of Hyndman et al. (2008) when $0<\gamma\leq 1$ and we set their trend smoothing parameter equal to $\alpha\beta\gamma$.

Eq. (20) offers a much wider range of values to the parameter set $\{\alpha,\beta,\gamma\}$, compared to the traditional $[0,1]$ interval suggested in the literature (see for example, Winters (1960) and Gardner (1990)). Commercial software such as SAP and Forecast Pro® also selects α and β between 0 and 1 (SAP 2016; Stellwagen and Goodrich 2011), SAS/ETS® considers $0<\gamma<1$, $0<\alpha<2$, and $0<\gamma\beta<4/\alpha-2$, (SAS 2018). We emphasize, these do not include $\{\alpha,\beta\}<0$ and are only part of the complete stability region identified in (20) and characterised (when $0<\gamma<1$) in Figure 2.

6. The OUT policy under i.i.d. demand with damped trend forecasting

To study *Bullwhip* and *NSAmp* behaviour of the OUT-DT policy, we will need the transfer function of the replenishment orders and the net stock levels. The order transfer function (Li et al. 2014) is

$$\frac{O(z)}{\varepsilon(z)}=1+\frac{\alpha(z-1)\left(\beta\zeta(z-1)+(T_p+1)(z-(1-\beta)\gamma)\right)}{z^2+z(\alpha\beta\gamma+\alpha-\gamma-1)+(1-\alpha)\gamma},\tag{21}$$

where $\zeta=\Phi[T_p]+\varphi[T_p+1]$ and

$$\Phi[T_p]=\sum_{j=1}^{T_p}\varphi[j]=\frac{\gamma\left(\gamma^{T_p+1}-T_p\gamma+T_p-\gamma\right)}{(1-\gamma)^2}.\tag{22}$$

The net stock transfer function can be found by substituting (21) into (5) and simplifying to yield,

$$\frac{NS(z)}{D(z)} = \frac{\alpha\left(\beta\zeta(z-1)+(T_p+1)(z-(1-\beta)\gamma)\right)}{z^{T_p}\left(z^2+z(\alpha\beta\gamma+\alpha-\gamma-1)+(1-\alpha)\gamma\right)} - \frac{z^{-T_p}(z^{1+T_p}-1)}{z-1}. \tag{23}$$

These transfer functions will be used to study the *Bullwhip* and *NSAmp* behaviour under i.i.d. demand in §6.1; §6.2 considers the equivalence of the dynamic response of OUT-DT and POUT.

6.1. OUT-DT variance ratio analysis under i.i.d. demand

Although DT is the optimal forecast for ARIMA(1,1,2) demand, it is insightful to first investigate its performance under i.i.d. demand. Taking the inverse z-transform of (21), summing its square in (10), and dividing by the demand variance, provides the following expression for the *Bullwhip* ratio when i.i.d. demand is forecasted via the DT method:

$$\frac{\mathbb{V}[o_t]}{\mathbb{V}[d_t]} = \frac{\begin{array}{c}2\alpha^2\beta\zeta(1+\gamma(2-3\beta(1-\gamma)+\gamma)+2T_p(1-\gamma)(1+(1-\beta)\gamma))+4\alpha^2\beta^2\zeta^2(1-\gamma)+\\ 2-2\gamma^2-2\alpha^3(\beta\gamma+\gamma+1)(\beta\zeta+T_p+1)(\beta(\gamma-\zeta)-(1-\beta)\gamma T_p)+\\ \alpha\left(4\beta\left(1-\gamma^2\right)\zeta+\gamma(\beta\gamma-\beta-\gamma+2)+4\left(1-\gamma^2\right)T_p+3\right)+\\ \alpha^2\gamma^2\left(3+\beta+2T_p(\beta-T_p+1)+2\beta^2(T_p+1)^2\right)+\\ \alpha^2(\gamma(2(2-\beta)(T_p+1)-1)+2T_p(T_p+1))\end{array}}{(1-\gamma(1-\alpha))((2-\alpha)(\gamma+1)-\alpha\beta\gamma)}. \tag{24}$$

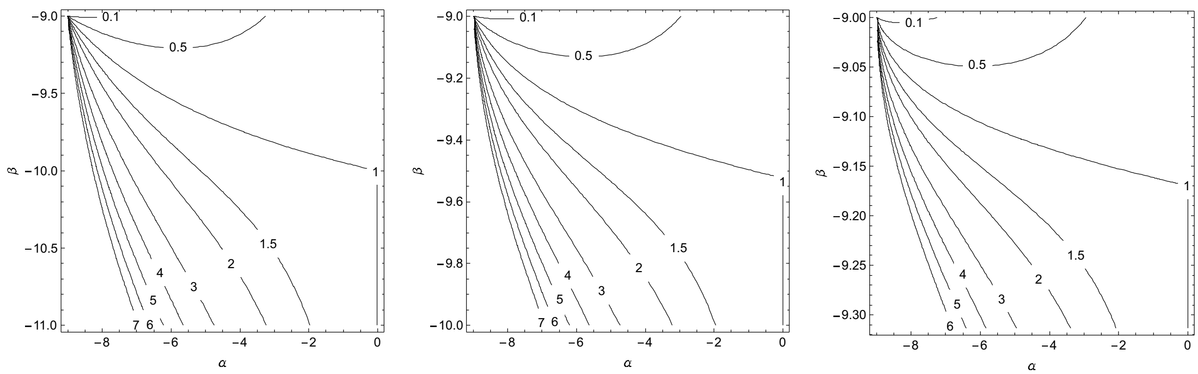
Li and Disney (2018) identified a region of the parametric plane where it is possible for the OUT-DT to avoid creating Bullwhip for any lead-time. The region was specified by:

$$\left.\begin{array}{l}0<\gamma<1,\\ \left(\alpha_{\min}=\frac{\gamma-1}{\gamma}\right)<\alpha<0,\\ \left(\beta_{\min}=\frac{-\left(T_p+1\right)(\gamma+1)(1-\gamma)^2}{\left(\gamma-\gamma^3\right)T_p+\gamma^2\left(2\gamma^{T_p+1}-\gamma-2\right)+\gamma}\right)\leq\beta\leq\left(\beta_{\max}=\frac{\gamma-1}{\gamma}\right).\end{array}\right\} \tag{25}$$

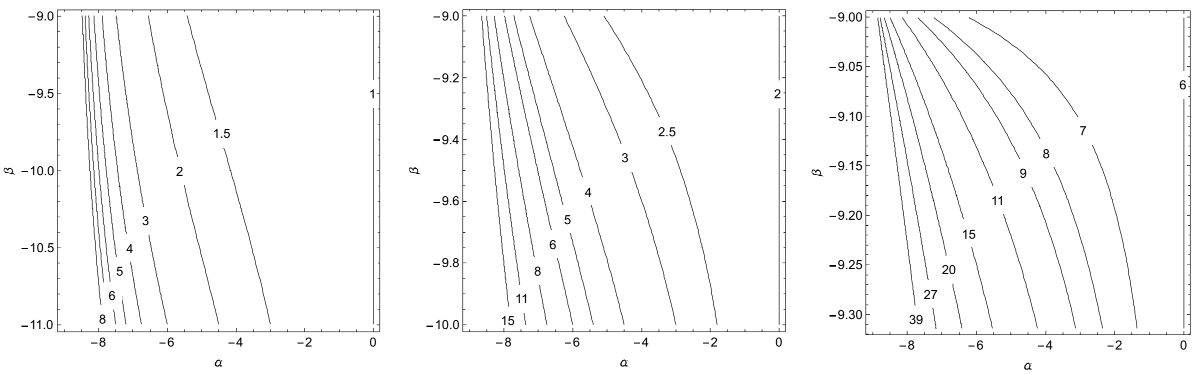
The lower bound, $\beta_{\min}$ in (25), is increasing in the lead-time. When (25) holds we say the parameter set is a member of the *Bullwhip Avoidance* ($\mathscr{BA}$) area, $\{\alpha,\beta,\gamma\}\in\mathscr{BA}$. The area was found by Li and Disney (2018) via a frequency response analysis that considered how the harmonic frequencies in demand were amplified by the OUT-DT policy. Note, (25) does not guarantee *Bullwhip* <1, only that there exists a demand pattern that could have *Bullwhip* <1. In this study, we are focusing on the characterisations of *Bullwhip* and *NSAmp* within the $\mathscr{BA}$ region unider i.i.d. demand and ARIMA(1,1,2) demand.

Studying the *Bullwhip* ratio of the OUT-DT policy (24), we found it has no stationary points within the stability region and is always differentiable within the $\mathscr{BA}$ region. Thus, any local minima and maxima must exist on the boundaries of the $\mathscr{BA}$ region. The same properties were found in the *NSAmp* ratio. Taking each boundary into consideration, we find (note, $\uparrow$ means *approach from below*, $\downarrow$ means *approach from above*, and $\rightarrow$ means *tends to*) a minimal *Bullwhip* of

$$Bullwhip = \frac{\begin{array}{c}\left(2\alpha^2\gamma^2\left(\gamma^{T_p}-1\right)\left(\gamma^{T_p+1}-1\right)+\\ \alpha(\gamma-1)\left(\gamma\left(2(\gamma+1)\gamma^{T_p}-3\right)-1\right)+(\gamma-1)\gamma-1\right)+1\end{array}}{(\gamma-1)^2(1+(1-\alpha)\gamma)} \tag{26}$$



(a) *Bullwhip* for $T_p = 0$ (b) *Bullwhip* for $T_p = 1$ (c) *Bullwhip* for $T_p = 5$



(d) *NSAmp* for $T_p = 0$ (e) *NSAmp* for $T_p = 1$ (f) *NSAmp* for $T_p = 5$

Figure 3: *Bullwhip* and *NSAmp* in the OUT-DT policy when $\gamma = 0.1$.

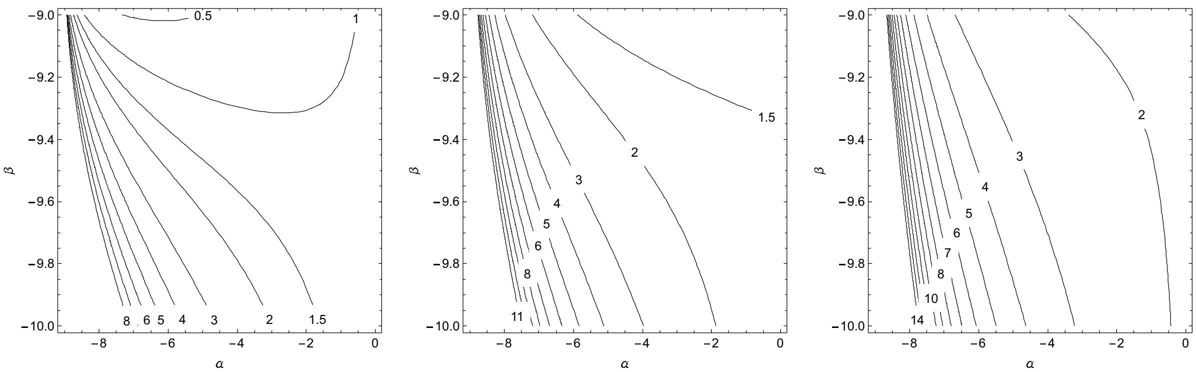
exists when $\beta \uparrow \beta_{max}$. (26) is a value between 0 and 1, $\forall \gamma$, $\alpha \in \mathscr{BA}$ and $T_p \in \mathbb{N}^0$. It is also interesting that $Bullwhip \to 1$ when $\alpha \uparrow 0$. This suggests when capacity costs dominate, a small negative α and/or a small negative β should be adopted due to its Bullwhip avoidance behaviour.

The *NSAmp* ratio maintained by the OUT-DT policy when reacting to i.i.d. demand is given by,

$$\frac{\mathbb{V}[ns_t]}{\mathbb{V}[d_t]} = 1 + T_p + \frac{\begin{array}{c}\left((\gamma-1)^2\left(-T_p^2\right)((\beta-1)\gamma+1)\left(\gamma\left((\alpha-1)\gamma^2-\alpha-2\alpha\beta\gamma+\gamma+1\right)-1\right)+\right.\\ \gamma\left(\alpha+\beta(\gamma+1)(\gamma-1)^4-\gamma^3((\gamma-4)\gamma+5)+\alpha\gamma\left(\gamma((\gamma-2)(\gamma-1)\gamma+2)+\right.\right.\\ 2\beta^2\gamma^2\left(\gamma^{T_p}-1\right)\left(\gamma^{T_p+2}+1-2\gamma\right)+\beta(\gamma-1)^2\left(\gamma\left(-\gamma+2(\gamma+1)\gamma^{T_p}-4\right)+\right.\\ \left.\left.\left.1\right)-3\right)-4+5\gamma\right)-2(\gamma-1)T_p((\beta-1)\gamma+1)\left(\gamma\left(\alpha-\left((\gamma-2)\gamma^2\right)+\right.\right.\\ \left.\left.\left.\alpha\gamma\left(\beta+(\gamma-1)\gamma+\beta\gamma\left((\gamma+1)\gamma^{T_p}-3\right)-1\right)-2\right)+1\right)+1\right)\end{array}}{(\gamma-1)^4((1-\alpha)\gamma-1)(\gamma(\alpha\beta+\alpha-2)+\alpha-2)}. \tag{27}$$

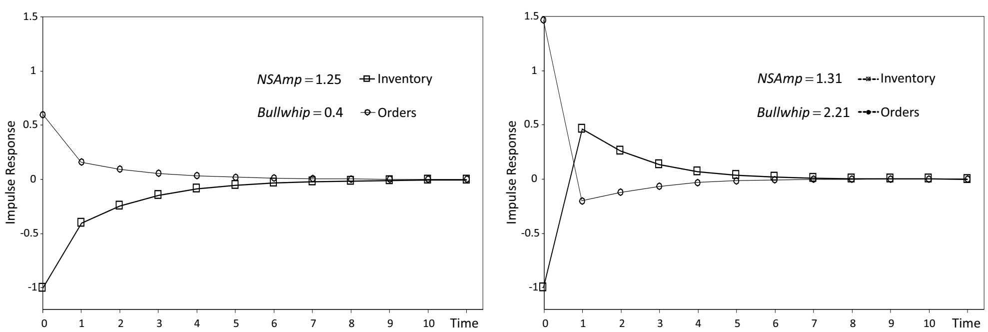
A minimal *NSAmp* of $1+T_p$ occurs when $\alpha \uparrow 0$. This means when i.i.d. demand is present, $NSAmp \geq 1+T_p$ in the OUT-DT system for $\{\alpha, \beta, \gamma\} \in \mathscr{BA}$. Further, when the inventory variance is minimized, $Bullwhip = 1$. We conclude, when inventory costs are significantly larger than capacity costs, and i.i.d. demand is present, $\alpha = 0$ is recommended. $\alpha = 0$ will result in MMSE forecasts of demand (i.e. all future forecasts equal the mean of the i.i.d. demand).

Note, when $\beta \uparrow \beta_{max}$ and $\alpha \downarrow \alpha_{min}$ and $\gamma \downarrow 0$, the Bullwhip effect is a global minimum, but



(a) $w = 0.1$ (b) $w = 0.5$ (c) $w = 0.9$

Figure 4: Contour plots for the weighted convex sum of the order and inventory variances, $w\mathbb{V}[ns] + (1 - w)\mathbb{V}[o]$, maintained by the OUT-DT policy when $\gamma = 0.1$ and $T_p = 1$.



(a) $\alpha = -2, \beta = -4, \gamma = 0.2.$ (b) $\alpha = 0.4, \beta = 0.8, \gamma = 0.2.$

Figure 5: The impulse response of inventory and order in the OUT-DT system, $T_p = 0$.

NSAmp $\to \infty$. In other words, the order variance can be reduced to zero for i.i.d. demand at the cost of increased inventory variance, indicating that a trade-off exists, just as it did in the POUT policy. When the business objective is to reduce both inventory and capacity costs, a γ close to 0 and a small negative β are recommended as it guarantees the elimination of the Bullwhip effect; we then only need to optimize α based on the balance between inventory and capacity costs.

Figure 3 illustrates examples of *Bullwhip* and *NSAmp* values in contour plots for various lead times. When $\beta \downarrow \beta_{min}$ and $\alpha \downarrow \alpha_{min}$, both *Bullwhip* and *NSAmp* ratios increase dramatically. Furthermore, the *Bullwhip* value is rather insensitive to the lead time, similar to the *Bullwhip* behaviour in the POUT policy, while *NSAmp* is significantly influenced by the lead time. These imply, in the long lead-time cases, $\beta \downarrow \beta_{min}$ and $\alpha \downarrow \alpha_{min}$ need to be avoided; the top right half of the $\mathscr{BA}$ plane is always superior–see Figure 4, where different weights w are placed on the order and inventory variances.

6.2. Comparison between the OUT-DT policy and the proportional OUT policy

Figure 5a shows the inventory and order impulse responses for $\gamma \downarrow 0$, $\beta \uparrow \beta_{max}$, and $\alpha_{min} < \alpha < 0$. They are quite different from the impulse responses from the conventional $[0, 1]$ parameter region as shown in Figure 5b. Our recommendation produces a smoothed, damped, and exponential increasing (or decreasing) impulse response, rather than an under-damped oscillatory response.

These are desirable properties dynamic properties which further strengthens our argument for selecting parameters from the $\mathscr{BA}$ region. *Bullwhip* and *NSAmp* ratios are also noted in Figure 4.

That the OUT-DT policy is capable of eliminating Bullwhip effect without using a proportional controller in the inventory position feedback loop is astonishing. Visually, the character of the impulse response in Figure 5a is similar to the POUT policies impulse response that we introduced in §4 (Dejonckheere et al. 2003; Gaalman and Disney 2009) and the closely related automatic pipeline inventory and order based production control system (Disney and Towill 2003). These facts motivate us to investigate the similarity between the OUT-DT policy and the POUT policy. In this section, we compare the poles and zeros of POUT and OUT-DT and show when they are equivalent. To avoid lengthy equations, we assume $\{\alpha, \beta, \gamma\}$ are selected from the $\mathscr{BA}$ region where $\beta \uparrow \beta_{\max} = (\gamma - 1)/\gamma$.

Writing OUT-DT order transfer function, (21), in pole-zero form to match (8) gives

$$\frac{O(z)}{\varepsilon(z)} = \frac{\frac{\alpha\gamma - \alpha\gamma^{T_p+2} - \gamma + 1}{1-\gamma}\left(z - \frac{\gamma\left(\alpha\gamma - \alpha\gamma^{T_p+1} - \gamma + 1\right)}{\alpha\gamma - \alpha\gamma^{T_p+2} - \gamma + 1}\right)}{z - (1-\alpha)\gamma}. \tag{28}$$

When $\alpha = (T_i(\gamma - 1) + 1)/(T_i\gamma)$, we may re-write (28) into the following form

$$\frac{O(z)}{\varepsilon(z)} = \frac{\frac{1-(1-T_i)\gamma^{T_p+1} - T_i\gamma^{T_p+2}}{T_i(1-\gamma)}\left(z - \frac{\gamma\left(1-(1-T_i)\gamma^{T_p} - T_i\gamma^{T_p+1}\right)}{1-(1-T_i)\gamma^{T_p+1} - T_i\gamma^{T_p+2}}\right)}{z - \frac{T_i - 1}{T_i}}. \tag{29}$$

Both (8) and (29) are first-order systems with a single pole at $z = (T_i - 1)/T_i$ and have a geometrically decreasing impulse response when $T_i > 1$. When $\gamma = 0$, the order transfer function of the OUT-DT policy has a zero at $z = 0$. Although γ cannot be 0 if we wish to select the DT parameters from the $\mathscr{BA}$ region, the zeros of the order transfer functions in OUT-DT and POUT policies can be very close to each other if $\gamma \downarrow 0$ (as when $\gamma > 0$, the $\mathscr{BA}$ region exists). Therefore, by letting $\alpha = (T_i(\gamma - 1) + 1)/(T_i\gamma)$, $\beta \uparrow \beta_{max}$ and $\gamma \downarrow 0$, the order transfer function in both the OUT-DT and POUT policies will have, for all intents and purposes, identical poles and zeros. If the poles and zeros are identical, the order transfer functions are identical, both systems respond to demand in exactly the same way, and their order and inventory responses will be identical. Figure 6 provides an example of the system impulse response when $T_p = 3$. Figure 6 confirms the order and inventory impulse responses in the OUT-DT system approximate the POUT's system responses. Note, we could have set γ closer to zero in Figure 6 and this would have resulted in impulse responses that were indistinguishable from each other. However, we elected to use $\gamma = 0.01$ to demonstrate how small the discrepancy is.

7. The OUT policy under ARIMA(1,1,2) demand with damped trend forecasting

Gardner and McKenzie (1985) show that the DT is the optimal forecasting method for predicting ARIMA(1,1,2) demand. In the section we explore the ARIMA(1,1,2) demand process further as it will allow us to understand the Bullwhip-lead time behaviour of the DT-OUT policy in §7.2.

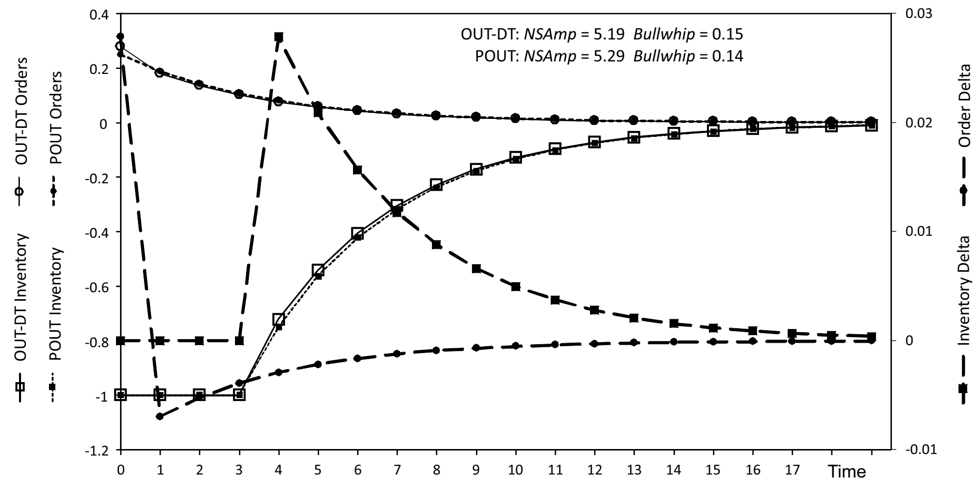


Figure 6: Impulse responses of the POUT (with $T_i = 4$) and OUT-DT (with $\{\gamma = 0.1,\ \alpha = -6.5,\ \beta = -9\}$) systems and their differences when $T_p = 3$. Note: Inventory Delta = $ns_t|_{\text{OUT-DT}} - ns_t|_{\text{POUT}}$, Order Delta = $o_t|_{\text{OUT-DT}} - o_t|_{\text{POUT}}$.

7.1. *Eigenvalue analysis of ARIMA(1,1,2) demand*

The ARIMA(1,1,2) demand process is given by,

$$d_t = d_{t-1} + \phi_1(d_{t-1} - d_{t-2}) - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} + \varepsilon_t. \tag{30}$$

Gardner and McKenzie (1985) also show the damped trend forecast produces a MMSE forecast of ARIMA(1,1,2) demand when

$$\left.\begin{array}{l} \theta_1 = 1 + \gamma - \alpha - \alpha\beta\gamma, \\ \theta_2 = \gamma(\alpha - 1), \\ \phi_1 = \gamma. \end{array}\right\} \tag{31}$$

Given a set of ARIMA(1,1,2) parameters, perhaps identified from a real time series, we can solve the simultaneous equations in (31) for the damped trend parameters:

$$\left.\begin{array}{l} \alpha = \frac{\theta_2 + \phi_1}{\phi_1}, \\ \beta = \frac{\phi_1^2 - \theta_2 - \theta_1\phi_1}{\theta_2\phi_1 + \phi_1^2}, \\ \gamma = \phi_1. \end{array}\right\} \tag{32}$$

Later, we will exploit the eigenvalues of the ARIMA(1,1,2) demand process to make some Bullwhip predictions. The eigenvalues can be identified from the z-transform transfer function of the ARIMA(1,1,2) demand process,

$$\frac{D_{ARIMA(1,1,2)}(z)}{\varepsilon(z)} = \frac{z^2 - z\theta_1 - \theta_2}{z^2 - z(1+\phi_1) + \phi_1}. \tag{33}$$

Eq. (33) has the following eigenvalues:

$$\lambda_1^\theta = \frac{1}{2}\left(\theta_1 - \sqrt{\theta_1^2 + 4\theta_2}\right), \quad \lambda_2^\theta = \frac{1}{2}\left(\theta_1 + \sqrt{\theta_1^2 + 4\theta_2}\right), \quad \lambda_1^\phi = \phi_1, \quad \text{and} \quad \lambda_2^\phi = 1. \tag{34}$$

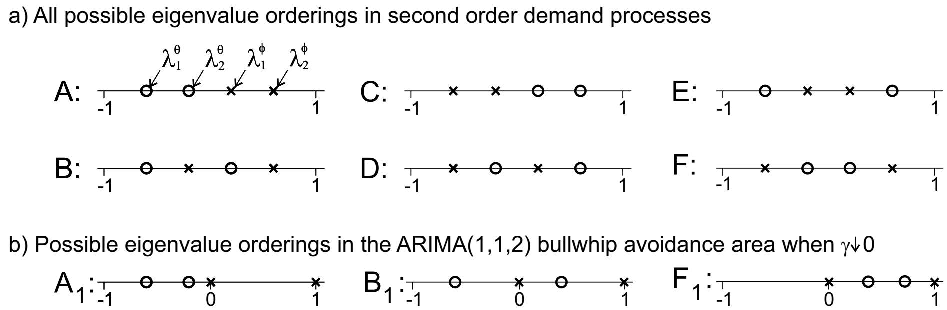


Figure 7: Possible eigenvalue orderings. Panel a) All possible eigenvalue ordering for second order demand processes. Panel b) Possible eigenvalue orderings under ARIMA(1,1,2) with parameters selected from the $\mathscr{BA}|_{\gamma\downarrow 0}$ region. Source: Adapted from Gaalman et al. (2018).

Here, $\lambda^{\theta}_{\{1,2\}}$ are the zeros, the roots of the numerator of (33) w.r.t. z; $\lambda^{\phi}_{\{1,2\}}$ are the poles, the roots of the denominator of (33) w.r.t. z. Note, the largest pole, $\lambda^{\phi}_{2} = 1$, implying the system is non-stationary. The impulse response, while it does not escape to infinity or oscillate with ever increasing amplitude, it does not return to zero. Rather it has an off-set, implying the demand has infinite variance, see (10).

7.2. *Bullwhip-lead time behaviour of the OUT-DT policy under ARIMA(1,1,2) demand*

The ARIMA(1,1,2) demand process is non-stationary and as such the demand and order variances are infinite; the *Bullwhip* ratio does not exist. However, Gaalman et al. (2018) and Gaalman et al. (2019) have investigated how the Bullwhip produced by the OUT policy is affected by the lead time and their study contains two important innovations that allow us to gain some insight into Bullwhip behaviour, despite the non-stationary nature of demand. First, they show how one may use the difference between the order variance and the demand variance to determine whether a Bullwhip effect is present or not under non-stationary demand. As the difference between the two infinite variances is finite, the positivity of this difference can indicate whether a Bullwhip effect is present or not. That is, they show Bullwhip effect is present if $\mathbb{V}(o_t) - \mathbb{V}(d_t) > 0$. They also reveal that if the demand impulse was always positive, $\forall i$, $\tilde{d}_{t+i} > 0$, the Bullwhip effect produced by the OUT policy increases in the lead time. The positivity of the demand impulse was determined by the order and location of the eigenvalues (poles and zeros) of the demand process.

For a second-order demand transfer function there are six possible eigenvalue orderings, see Figure 7. Then, within each ordering, a further three sub-cases were present depending on how many poles are positive or negative. Assume, from (34), the larger pole $\lambda^{\phi}_{2} =\uparrow 1$ (i.e. ϕ is very slightly smaller than unity) to ensure stability. This implies that cases C, D, and E cannot exist when we have ARIMA(1,1,2) demand, as those cases have a zero above the largest pole.

Let $\mathscr{BA}|_{\gamma\downarrow 0}$ denote the $\mathscr{BA}$ area defined by a small γ, i.e. $\gamma \downarrow 0$, the $\mathscr{BA}$ area where the DT-OUT policy mimics the POUT policy. In the $\mathscr{BA}|_{\gamma\downarrow 0}$ area, the smaller pole lies at $\lambda^{\phi}_{1} =\downarrow 0$. That is, the smaller pole is positive, implying that the $\mathscr{BA}|_{\gamma\downarrow 0}$ region contains only the sub-cases A_1, B_1, and F_1 which have two positive poles, see Gaalman et al. (2019).

Case A_1 exists if the largest zero λ^{θ}_{2} is less than $\lambda^{\phi}_{1} =\downarrow 0$, which is equivalent to

$$\theta_1 < 0 \wedge \theta_1^2 + 4\theta_2 \geq 0 \wedge \theta_2 < 0, \tag{35}$$

where $\wedge$ is the logical *and* operator. When case A_1 exists, Gaalman et al. (2019) shows the demand impulse, $\tilde{d}_{t+1} > 0$ and the Bullwhip always increases in the lead time. In the $\mathscr{BA}|_{\gamma\downarrow 0}$ region, the $\theta_1 < 0$ constraint is equivalent to $\beta \leq (1-\alpha+\gamma)/(\alpha\gamma)$. As the smallest $\beta = (\gamma-1)/\gamma$ is always greater than $(1-\alpha+\gamma)/(\alpha\gamma)$, case A_1 cannot exist in the $\mathscr{BA}|_{\gamma\downarrow 0}$ area.

Case B_1 exists if the largest zero λ_2^θ is greater than $\lambda_1^\phi =\downarrow 0$ and the smallest zero λ_1^θ is less than $\lambda_1^\phi =\downarrow 0$. This is equivalent to $\theta_2 > 0$. When case B_1 exists, Gaalman et al. (2018) show that the demand impulse, $\tilde{d}_{t+1} > 0$ and the Bullwhip always increases in the lead time. The constraint that $\theta_2 > 0$ is equivalent to $\gamma(\alpha-1) > 0$. This is not possible in the $\mathscr{BA}|_{\gamma\downarrow 0}$ region as $\alpha < 0$ and $\gamma > 0$.

Case F_1 exists if the smallest zero λ_1^θ is greater than $\lambda_1^\phi =\downarrow 0$, which is equivalent to

$$\theta_1 > 0 \wedge \theta_1^2 + 4\theta_2 \geq 0 \wedge \theta_2 < 0, \tag{36}$$

where $\wedge$ is the logical *and* operator. Case F_1 exists in the $\mathscr{BA}|_{\gamma\downarrow 0}$ region as it is the logical complement of case A_1 and B_1. Gaalman et al. (2018) show the demand impulse has two essential characters. **Case** F_{1a}**:** If $\tilde{d}_1 < 0$ then the demand impulse is initially negative (i.e. $\tilde{d}_{\text{small}\,t} < 0$) and *Bullwhip* does not increase in the lead time. However, when t becomes sufficiently large the demand impulse response turns, and remains, positive after one change of sign (i.e. $\tilde{d}_{\text{large}\,t} > 0$) and *Bullwhip* increases in the lead time. **Case** F_{1b}**:** The demand impulse is positive if $\tilde{d}_1 > 0$, which is equivalent to $\beta < -1/\gamma$. In this sub-case, *Bullwhip* is always increasing in the lead time.

8. Concluding remarks

By showing the invertibility and the stability regions of the DT forecasting mechanism were identical, we have offered theoretical support for exploring the performance of the OUT-DT policy over a wider range of parameter values than is usually recommended. While other evaluations of the utility of DT forecasts have chosen the parameter values from the [0, 1] interval, our work shows that if unconventional $\{\alpha, \beta, \gamma\}$ values are selected *Bullwhip* can be avoided without unduly increasing *NSAmp*, and these results hold for all lead-times.

We have shown that the OUT-DT policy has nearly identical poles and zeros as the POUT policy. The POUT policy, with its proportional feedback controller has long been known to avoid the Bullwhip effect, while maintaining reasonable inventory control. The OUT-DT policy has no such proportional feedback controller; yet despite this, it is able to perform–for all practical purposes–identically to the POUT policy. This provides a new implementation route for the Bullwhip reduction strategies. With only a change in the forecasting software one can obtain a smooth production rate without the need to make changes to an MRP system's planning book. This has practically important managerial implications as it allows the change to be easily implemented in only the forecasting module of popular ERP systems.

DT is optimal for the non-stationary ARIMA(1,1,2) demand process; as a result the demand and order variances are infinite. However, we were able to adopt the eigenvalue ordering approach of Gaalman et al. (2018) to investigate how the Bullwhip effect was influenced by the lead time. Within the $\mathscr{BA}|_{\gamma\downarrow 0}$ area we found that the Bullwhip effect could be either a) initial decreasing, and then increasing in the lead time, or b) always increasing in the lead time. Near the point of minimum Bullwhip, Bullwhip was found to be initially decreasing, and then increasing in the lead time.

References

Boute, R., Disney, S., Mieghem, J.V., 2022. Dual sourcing and smoothing under non-stationary demand time series: Re-shoring with speedfactories. Forthcoming in Management Science doi:10.1287/mnsc.2020.3951.

Boute, R.N., Disney, S.M., Lambrecht, M.R., Houdt, B.V., 2009. Designing replenishment rules in a two-echelon supply chain with a flexible or an inflexible capacity strategy. International Journal of Production Economics 119, 187–198.

Box, G.E.P., Jenkins, G.M., Reinsel, G.C., 2008. Time Series Analysis: Forecasting and Control. John Wiley & Sons, New Jersey.

Brown, R.G., 1963. Smoothing, Forecasting, and Prediction of Discrete Time Series. Prentice-Hall, Englewood Cliffs, NJ.

Cannella, S., Dominguez, R., Framinan, J.M., 2017. Inventory record inaccuracy—the impact of structural complexity and lead time variability. Omega 68, 123–138.

Chen, Y., Disney, S., 2007. The myopic order-up-to policy with a proportional feedback controller. International Journal of Production Research 45, 351–368.

Churchman, C.W., Ackoff, R.L., Arnoff, E.L., 1957. Introduction to Operations Research. John Wiley & Sons, London.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R., Towill, D.R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. European Journal of Operational Research 147, 567–590.

Disney, S.M., 2008. Supply chain aperiodicity, bullwhip and stability analysis with Jury's inners. IMA Journal of Management Mathematics 19, 101–116.

Disney, S.M., Ponte, B., Wang, X., 2021. Exploring the nonlinear dynamics of the lost-sales order-up-to policy. International Journal of Production Research 59, 5809–5830.

Disney, S.M., Towill, D.R., 2003. On the bullwhip and inventory variance produced by an ordering policy. Omega 31, 157–167.

Gaalman, G., 2006. Bullwhip reduction for arma demand: The proportional order-up-to policy versus the full-state-feedback policy. Automatica 42, 1283–1290.

Gaalman, G., Disney, S.M., Wang, X., 2018. Bullwhip behaviour as a function of the lead-time for the order-up-to policy under ARMA demand, in: Pre-prints of the 20th International Working Seminar of Production Economics, Innsbruck, Austria. p. 249–260.

Gaalman, G., Disney, S.M., Wang, X., 2019. When the bullwhip effect is an increasing function of the lead time. IFAC-PapersOnLine 52, 2297 – 2302.

Gaalman, G.J.C., Disney, S.M., 2009. On bullwhip in a family of order-up-to policies with ARMA(2,2) demand and arbitrary lead-times. International Journal of Production Economics 121, 454–463.

Gardner, E.S., 1990. Evaluating forecast performance in an inventory control system. Management Science 36, 490–499.

Gardner, E.S., McKenzie, E., 1985. Forecasting trends in time series. Management Science 31, 1237–1246.

Hosoda, T., Disney, S.M., 2009. Impact of market demand mis-specification on a two-level supply chain. International Journal of Production Economics 121, 739–751.

Hyndman, R.J., Akram, M., Archibald, B.C., 2008. The admissible parameter space for exponential smoothing models. Annals of the Institute of Statistical Mathematics 60, 407–426.

Jury, E., 1974. Inners and the Stability of Dynamic Systems. John Wiley & Sons, New Jersey.

Li, Q., Disney, S.M., 2017. Revisiting rescheduling: MRP nervousness and the bullwhip effect. International Journal of Production Research 55, 1992–2012.

Li, Q., Disney, S.M., 2018. Inventory performance of the damped trend forecasting method, in: Pre-prints of the 20th International Working Seminar of Production Economics, Innsbruck, Austria. pp. 249–260.

Li, Q., Disney, S.M., Gaalman, G.J.C., 2014. Avoiding the bullwhip effect using damped trend forecasting and the order-up-to replenishment policy. International Journal of Production Economics 149, 3–16.

Makridakis, S., Hibon, M., 2000. The M3-competition: results, conclusions and implications. International Journal of Forecasting 16, 451–476.

Nise, N.S., 2004. Control Systems Engineering. John Wiley & Sons, New Jersey.

Petropoulos, F., Wang, X., Disney, S.M., 2019. The inventory performance of forecasting methods: evidence from the M3 competition data. International Journal of Forecasting 35, 251–265.

Roberts, S.A., 1982. A general class of Holt-Winters type forecasting models. Management Science 28, 808–820.

SAP, 2016. SAP HANA Predictive Analysis Library.

SAS, 2018. SAS/ETS®15.1 User's Guide. SAS Institute Inc, Cary, NC.

Stellwagen, E., Goodrich, R., 2011. Forecast Pro®, Software Version 7. Business Forecast Systems Inc, MA.

Tsypkin, Y., 1964. Sampling Systems Theory and its Applications. Vol. 1. Pergamon Press, Oxford.

Wikner, J., 2006. Analysis of smoothing techniques: application to production-inventory systems. Kybernetes 35, 1323–1347.

Winters, P., 1960. Forecasting sales by exponentially weighted moving averages. Management Science 6, 324–342.

Zipkin, P., 2000. Foundations of Inventory Management. McGraw-Hill, New York, NY.

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