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Generalized polygons and star graphs of cyclic presentations of groups ☆

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A B S T R A C T

Groups defined by presentations for which the components of the corresponding star graph are the incidence graphs of generalized polygons are of interest as they are small cancellation groups that – via results of Edjvet and Vdovina – are fundamental groups of polyhedra with the generalized polygons as links and so act on Euclidean or hyperbolic buildings; in the hyperbolic case the groups are SQ-universal. A cyclic presentation of a group is a presentation with an equal number of generators and relators that admits a particular cyclic symmetry. We obtain a classification of the concise cyclic presentations where the components of the corresponding star graph are generalized polygons. The classification reveals that both connected and disconnected star graphs are possible and that only generalized triangles (i.e. incidence graphs of projective planes) and regular complete bipartite graphs arise as the components. We list the presentations that arise in the Euclidean case and show that at most two of the corresponding groups are not SQ-universal (one of which is not SQ-universal, the other is unresolved).

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We obtain results that show that many of the SQ-universal groups are large.

1. Introduction

An (m, k)-special presentation is a group presentation in which the relators have length $k \geq 3$ and whose star graph is the incidence graph of a generalized polygon, a generalized m-gon, where (necessarily) $m \in \{2, 3, 4, 6, 8\}$ [27]. (Definitions will be given in Section 2.) This generalizes the concept of special presentations (from [37]) which corresponds to the case $(m, k) = (3, 3)$, and so the star graph is a generalized triangle (or the incidence graph of a projective plane). Polygonal presentations (for pairs of natural numbers ν, k) were introduced in [60] (see also [61]) as a tool for constructing polyhedra with specified links. The concept was generalized in [27], where it was shown that a polyhedron (obtained by identifying edges of a set of k-gons) has ν vertices with links $\Gamma_0, \ldots, \Gamma_{\nu-1}$ if and only if the polyhedron corresponds to a polygonal presentation for ν, k over these links and that, given a concise (m, k)-special presentation with star graph Γ, there exists a polygonal presentation over Γ whose corresponding polyhedron has Γ as its link.

The $(3, 3)$-special presentations whose star graph is the smallest generalized triangle (the Heawood graph) were classified in [26]. It was shown in [37] that for any prime power $q - 1$ there is a $(3, 3)$-special presentation whose star graph is the incidence graph of the Desarguesian projective plane over the Galois field of order $q - 1$, and an example machine for their construction was provided. Polygonal presentations for $k = 3$, $\nu = 1$ and where Γ is the smallest or second smallest generalized triangle (or generalized 3-gon) were classified in [18,19] (where they are called triangle presentations). An example of a polygonal presentation for $k = 3, \nu = 1$ that corresponds to a non-Desarguesian projective plane (the Hughes plane) was given in [54]. Polygonal presentations for $k = 3, \nu = 1$ and where Γ is the smallest generalized quadrangle (or generalized 4-gon) were obtained in [42], and all such polygonal presentations were classified in [43,17] (subgroups of the groups acting on the corresponding polyhedron were studied in [44]). As noted in [27, page 924] these results therefore implicitly provide examples of $(3, 3)$ and $(4, 3)$-special presentations. Burger-Mozes groups [13–15] are groups that act properly and cocompactly on products of trees (see [16, Section 4] for a survey) and Burger-Mozes presentations furnish examples of $(2, 4)$-special presentations. Problem 1 of [27] asks if there are general methods for producing (m, k)-special presentations for $m \in \{4, 6, 8\}$ and it remains an open problem as to whether there are any examples of such presentations for $m \in \{6, 8\}$.

SQ-universality (a measure of the complexity of an infinite group) of groups defined by special presentations was studied in [27], where the problem of when such groups are large (another such measure) was posed [27, Problem 2]. We generalize the concept of

(m,k)-special presentations to (m,k,ν)-special presentations by replacing the condition that the star graph is a generalized m-gon with the condition that it has $\nu \geq 1$ isomorphic components, each of which is a generalized m-gon. We give the first examples of (m,k,ν)-special presentations with $\nu > 1$ (that are not unions of (m,k)-special presentations), from which explicit examples of polygonal presentations for ν, k with $\nu > 1$ can be readily obtained.

A cyclic presentation is a group presentation with an equal number of generators and relators that admits a particular cyclic symmetry and the corresponding group is a cyclically presented group ([40]). A $(3,3,1)$-special cyclic presentation, whose star graph is the Heawood graph was found in [26] and it was shown in [50, Theorem 3.5] and [38, Theorem 10] that (up to relabelling of generators) this is the only $(3,3,1)$-special cyclic presentation. A group presentation is concise if there is no redundancy amongst its relators. In this article we show that if a concise cyclic presentation is (m,k,ν)-special then $m=2$ or 3; we classify the concise $(3,k,\nu)$-special presentations in terms of perfect difference sets (in particular these only arise if the relators are positive or negative words), and we classify the concise $(2,k,\nu)$-special presentations. Except in one unresolved case, we determine which of the groups defined by these presentations are SQ-universal.

We now describe the structure of this article. In Section 2 we give definitions, terminology and background on group presentations, the star graph, special presentations, and polygonal presentations. In Section 3 we give examples of (m,k,ν)-special cyclic presentations, prove a theorem concerning the structure of the star graph of a cyclic presentation, and obtain corollaries that are needed for later results; in particular, we show that if $P_n(w)$ is concise and (m,k,ν)-special, where $\nu > 1$ and w is positive or negative then the corresponding group $G_n(w)$ is large. In Section 4 we show that if $P_n(w)$ is a concise cyclic presentation where w has length at least 3 then the girth of its star graph is at most 8 (with girth > 6 only attainable if w is a non-positive and non-negative word of length 3) and show that if such a presentation is (m,k,ν)-special then either $m=2$, or $m=3$ and w is a positive or negative word. In Section 5 we classify the concise $(3,k,\nu)$-special cyclic presentations in terms of perfect difference sets, and in particular the $(3,3,\nu)$-special cyclic presentations, and we show that a group defined by a concise $(3,k,\nu)$-special cyclic presentation is SQ-universal if and only if $(k,\nu) \neq (3,1)$ (and that there is precisely one such group that is not SQ-universal, which is just-infinite). In Section 6 we classify the concise $(2,k,\nu)$-special cyclic presentations, and in particular the $(2,4,\nu)$-special presentations; we show that, with one possible exception, the corresponding groups are SQ-universal, and we identify which of them define Burger-Mozes groups.

Many of the results from Sections 3, 5, 6 concerning cyclic presentations $P_n(w)$ will be expressed in terms of multisets of differences of subscripts in length 2 cyclic subwords of w. For convenience we define these multisets here:

$$\left.\begin{aligned} \mathcal{A} &= \{a \mid \text{there is a cyclic subword } x_i x_{i+a}^{-1} \text{ of } w, \text{with multiplicities}\}, \\ \mathcal{B} &= \{b \mid \text{there is a cyclic subword } x_i^{-1} x_{i+b} \text{ of } w, \text{with multiplicities}\}, \\ \mathcal{Q}^+ &= \{q \mid \text{there is a cyclic subword } x_i x_{i+q} \text{ of } w, \text{with multiplicities}\}, \\ \mathcal{Q}^- &= \{q \mid \text{there is a cyclic subword } x_{i+q}^{-1} x_i^{-1} \text{ of } w, \text{with multiplicities}\}, \\ \mathcal{Q} &= \mathcal{Q}^+ \cup \mathcal{Q}^-, \end{aligned}\right\} \qquad (1)$$

where $\cup$ denotes multiset sum and the entries are taken mod n.

2. Preliminaries

2.1. Presentations of groups

A word w in generators $x_0, \dots, x_{n-1}$ is said to be *positive* (respectively *negative*) if all of the exponents of generators are positive (respectively negative). We shall say that w is *cyclically alternating* if it has no cyclic subword of the form $(x_i x_j)^{\pm 1}$. A word $w = w(x_0, \dots, x_{n-1})$ is *freely reduced* if it does not contain a subword of the form $x_i x_i^{-1}$ or $x_i^{-1} x_i$; it is *cyclically reduced* if all cyclic permutations of it are freely reduced. If w is a cyclically reduced, non-empty, word in a free group $F(X)$ then the unique word $v \in F(X)$ such that $w = v^t$ with t maximal is called the *root* of w. We shall write $l(w)$ to denote the length of w in $F(X)$.

Following [10] given a group presentation $P = \langle X \mid R \rangle$, an element $r \in R$ is said to be *freely redundant* if it is freely trivial or if there exists $s \in R$ such that r and s are distinct elements of the free group with basis X and either r is freely conjugate to s or to s^{-1}. A presentation is said to be *redundant* if it contains a freely redundant relator and is *concise* otherwise ([21, page 8]). The *deficiency* of the presentation P is defined as $\mathrm{def}(P) = |X| - |R|$ and the *deficiency* of a group G, $\mathrm{def}(G)$, is defined to be the maximum of the deficiencies of all finite presentations defining G. A group G is *SQ-universal* if every countable group embeds in a quotient of G and it is *large* if it has a finite index subgroup that has a non-abelian free homomorphic image; every large group is SQ-universal [52]. A group is *just-infinite* if it is infinite and every proper quotient is finite; in particular, just-infinite groups are not SQ-universal.

2.2. Cyclic presentations and cyclically presented groups

For a positive integer n, let F_n be the free group with basis $X = \{x_0, \dots, x_{n-1}\}$ and let $\theta : F_n \to F_n$ be the *shift automorphism* given by $\theta(x_i) = x_{i+1}$ with subscripts taken modulo n. If w is a cyclically reduced word that represents an element in F_n then the presentation

$$P_n(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle$$

is called a *cyclic presentation* and the group $G_n(w)$ it defines is a *cyclically presented group*. Without loss of generality we may assume that the generator x_0 is a letter of w, and we make this assumption throughout this article. Then $P_n(w)$ is said to be *irreducible* if the greatest common divisor of n and the subscripts of the generators that appear in w are equal to 1 [25].

The shift automorphism θ of a cyclically presented group $G_n(w)$ satisfies $\theta^n = 1$ and the resulting $\mathbb{Z}_n$-action on $G_n(w)$ determines the *shift extension* $E_n(w) = G_n(w) \rtimes_\theta \mathbb{Z}_n$, which admits a presentation of the form $E_n(W) = \langle x, t \mid t^n, W(x,t)\rangle$ where $W = W(x,t)$ is obtained by rewriting w in terms of the substitutions $x_i = t^i x t^{-i}$, $0 \leq i < n$ (see, for example, [41, Theorem 4]). Thus there is a retraction $\nu^0 : E_n(W) \to \mathbb{Z}_n$ given by $\nu^0(t) = t$, $\nu^0(x) = t^0 = 1$ with kernel $G_n(w)$. Moreover, as shown in [10, Section 2], there may be further retractions ν^f for certain values of f $(0 \leq f < n)$. Specifically, by [10, Theorem 2.3] the kernel of a retraction $\nu^f : E_n(W) \to \mathbb{Z}_n$ given by $\nu^f(t) = t$, $\nu^f(x) = t^f$ is cyclically presented, generated by the elements $y_i = t^i x t^{-(i+f)}$ $(0 \leq i < n)$.

2.3. Star graph

Let $P = \langle X \mid R\rangle$ be a group presentation such that every relator $r \in R$ is a cyclically reduced word in the generators. Let $\tilde{R}$ denote the symmetrized closure of R, that is, the set of all cyclic permutations of elements in $R \cup R^{-1}$. The *star graph* of P is the undirected vertex-labelled graph Γ where the vertex set is in one-one correspondence with $X \cup X^{-1}$ and vertices are labelled by the corresponding element of $X \cup X^{-1}$ and where there is an edge joining vertices labelled x and y for each distinct word $xy^{-1}u$ in $\tilde{R}$ [47, page 61]. Such words occur in pairs, that is $xy^{-1}u \in \tilde{R}$ implies that $yx^{-1}u^{-1} \in \tilde{R}$. These pairs are called *inverse pairs* and the two edges corresponding to them are identified in Γ. It follows that replacing any relator of a presentation by its root, or removing a redundant relator from a presentation, leaves the star graph unchanged. We refer to vertices in X as *positive* vertices and vertices in X^{-1} as *negative* vertices.

We now set out our graph theoretic terminology. We allow graphs to have loops and to have more than one edge joining a pair of vertices. Given a graph Γ we write $V(\Gamma)$ to denote its vertex set. If Γ is bipartite with vertex partition $V(\Gamma) = V_1 \cup V_2$ where each edge connects a vertex in V_1 to a vertex in V_2 then V_1, V_2 are called the *parts* of $V(\Gamma)$. The *neighbours* of a vertex v, denoted by $N_\Gamma(v)$ is the set of all vertices that are adjacent to v in Γ. A graph Γ is *r-regular* if $|N_\Gamma(v)| = r$ for all $v \in V(\Gamma)$ and it is *regular* if it is r-regular for some r.

A *path* of *length* l in Γ is a sequence of vertices $(u = u_0, u_1, \ldots, u_l = v)$ with edges $u_i - u_{i+1}$ for each $0 \leq i < l$; it is a *closed path* if $u = v$. The path is *reduced* if the edge $u_{i+1} - u_{i+2}$ is not equal to the edge $u_{i+1} - u_i$ $(0 \leq i < l - 1)$. The *distance* $d_\Gamma(u, v)$ between vertices u, v of Γ is $l \geq 0$ if there is a path of length l from u to v, but no shorter path, and $d_\Gamma(u, v) = \infty$ if there is no path from u to v. The *girth*, girth(Γ) of a graph Γ is the length of a reduced closed path of minimal length, if Γ contains a reduced closed path, and girth$(\Gamma) = \infty$ otherwise. The *diameter*, diam(Γ) of a graph Γ is the greatest distance between any pair of vertices of the graph (which may be infinite). If Γ is a graph with finite girth then $\mathrm{girth}(\Gamma) \leq 2\mathrm{diam}(\Gamma) + 1$.

2.4. Special presentations

The concept of (m, k)-special presentations was introduced in [27], generalizing the concept of special presentations, introduced in [37] (which corresponds to the case $m = k = 3$). We extend this to define (m, k, ν)-special presentations, which reduces to (m, k)-special in the case $\nu = 1$.

Definition 2.1. Let $m \geq 2, k \geq 3, \nu \geq 1$. A finite group presentation $P = \langle X \mid R \rangle$ is said to be (m, k, ν)-*special* if the following conditions hold:

(a) the star graph Γ of P has ν isomorphic components, each of which is a connected, bipartite graph of diameter m and girth $2m$ in which each vertex has degree at least 3;
(b) each relator $r \in R$ has length k;
(c) if $m = 2$ then $k \geq 4$.

Note that the presentations considered in [27] are concise; however, our definition of (m, k, ν)-special (as with the definition of special in [37]) does not require the presentation to be concise. Note also that if a presentation is (m, k, ν)-special then it has at least 3 generators and the relators are cyclically reduced (for otherwise the star graph contains loops, so is not bipartite). We reiterate and expand on some remarks from [27] concerning (m, k, ν)-special presentations and their star graphs.

Remark 2.2. Let P be an (m, k, ν)-special presentation with star graph Γ.

1. By [59, Lemma 1.3.6] condition (a) is equivalent to Γ having ν isomorphic components, each of which is the incidence graph of a generalized m-gon and thus, by [29], $m \in \{2, 3, 4, 6, 8\}$. The incidence graph Λ of a generalized 2-gon is a complete bipartite graph ([59, page 11], [45, Section A.1]) so if, in addition, Λ is k-regular, then it is the complete bipartite graph $K_{k,k}$. The incidence graph of a generalized 3-gon (or projective plane) of order $q - 1$ (for some $q \geq 3$) is bipartite, q-regular, each part has $q^2 - q + 1$ vertices and any two vertices from the same part have exactly one common neighbour (see, for example, [8, page 373], [45, Section A.1]). Moreover, if Λ is a bipartite graph of girth greater than 2, in which every vertex has degree at least 3 and every pair of vertices from the same part have exactly one common neighbour then Λ has girth 6 and diameter 3. As described in [57], given a perfect difference set of order q, it is possible to construct a projective plane of order $q-1$. (A set of k integers $d_1, \ldots, d_k$ is called a *perfect difference set* (of order k) if among the $k(k-1)$ differences $d_i - d_j$ $(i \neq j)$ each of the residues $1, 2, \ldots, (k^2 - k)$ mod $k^2 - k + 1$ occurs

exactly once. For instance $\{1, 2, 4\}$ and $\{0, 1, 3, 9\}$ are perfect difference sets; further examples can be found in [58, Section 7].)

2. Since each vertex of Γ has degree > 1 each generator and its inverse is a piece and since girth$(\Gamma) > 2$ there are no pieces of length 2 (see, for example, [53, Section 5]) and therefore P satisfies the small cancellation condition $C(3)$ [47, Chapter V].
3. By [36], if a presentation satisfies $C(3)$ then the $T(q)$ condition is equivalent to the statement that its star graph does not contain any reduced closed path of length l where $3 \leq l < q$. Therefore the (m, k, ν)-special presentation P satisfies the small cancellation condition $C(k) - T(2m)$.

As in [27, Proof of Theorem 2], every group defined by an (m, k, ν)-special presentation with $1/m + 2/k < 1$ is non-elementary hyperbolic (by [32, Corollary 4.1], [22,26]), and hence SQ-universal (by [23, Théorème 3.5], [51, Theorem 1]). We shall refer to the cases $1/m + 2/k = 1$ (that is, the cases $(m, k) = (3, 3)$ and $(m, k) = (2, 4)$) as the *Euclidean* cases. It was shown in [27, Theorem 2] that groups defined by concise $(3, 3, 1)$-special presentations are just-infinite (and hence not SQ-universal) and that, by [22], groups defined by concise $(2, 4, 1)$-special presentations contain a non-abelian free subgroup and that there are examples (from [56,55]) of $(2, 4, 1)$-special presentations defining both SQ-universal and non-SQ-universal groups.

Since the direct product of two free groups $F_n \times F_n$ $(n \geq 2)$ contains a finitely generated subgroup that has undecidable membership problem [49] (or see [47, IV, Theorem 4.3]), groups that contain $F_2 \times F_2$ as a subgroup are of interest as they fail to satisfy certain properties, such as subgroup separability [48] and coherence [34]. Moreover, as hyperbolic groups cannot contain $\mathbb{Z} \times \mathbb{Z}$ (i.e. $F_1 \times F_1$) as a subgroup, and $F_2 \times F_2$ contains a wealth of $\mathbb{Z} \times \mathbb{Z}$ subgroups, groups that contain an $F_2 \times F_2$ subgroup are considered in [7] to "strongly fail" to be hyperbolic. Groups defined by $C(3) - T(6)$ presentations do not contain the subgroup $F_2 \times F_2$ ([6, Theorem 9.3.1]) so if a group defined by an (m, k, ν)-special presentation contains $F_2 \times F_2$ then $(m, k) = (2, 4)$. Groups defined by $(2, 4, \nu)$-special presentations, such as Burger-Mozes presentations, can contain this subgroup however.

If v is the root of a cyclically reduced word w and $l(u) \geq 2$, $w = v^p$ then the star graph Γ of $P_n(w)$ is equal to the star graph of $P_n(v)$, and if v has length 2 then the vertices of Γ have degree at most 2, so $P_n(w)$ is (m, pk, ν)-special if and only if $P_n(v)$ is (m, k, ν)-special. Thus, in classifying (m, k, ν)-special cyclic presentations $P_n(w)$ we can assume that w is not a proper power.

2.5. Polygonal presentations

In [60,27] *λ-polygonal presentations* were defined, and a process for obtaining a corresponding 2-complex K from such a presentation was given; we refer the reader to [27] for the precise definition. In that article, a *polyhedron* is defined to be a closed, connected 2-complex K obtained by identifying edges of a given set of k-gons $(k \geq 3)$; if the 2-complex K obtained from a λ-polygonal presentation $\mathcal{K}$ is a polyhedron, we say that K *corresponds* to $\mathcal{K}$. Lemma 1 of [27] constructs a λ-polygonal presentation, and a polyhedron that corresponds to it, from any concise (m,k)-special presentation. This can be readily extended to deal with (m,k,ν)-special presentations, as in Lemma 2.3, below. (The hypothesis that the presentation does not decompose as the disjoint union of two non-trivial sub-presentations ensures that the set of tuples $\mathcal{K}$ also does not properly decompose in such a manner, and so the resulting 2-complex K is connected.)

Lemma 2.3 *(Compare [27, Lemma 1]). Let $P = \langle X \mid R\rangle$ be an (m,k,ν)-special presentation that does not decompose as the disjoint union of two non-trivial sub-presentations, and suppose that $\Gamma_0,\dots,\Gamma_{\nu-1}$ are the components of the star graph Γ of P. Then there is a λ-polygonal presentation $\mathcal{K}$ over $\Gamma_0,\dots,\Gamma_{\nu-1}$ with corresponding polyhedron K having ν vertices $v_0,\dots,v_{\nu-1}$ such that the link of v_i is Γ_i $(0\le i<\nu)$.*

Let P, K be as in Lemma 2.3, let $\tilde{K}$ be the universal cover of K, and let G be the group defined by P. Then, as explained in [61,27], the following hold. The group G is the fundamental group of K, so G acts cocompactly on $\tilde{K}$. When equipped with the metric introduced in [1, page 165], $\tilde{K}$ is a complete metric space of non-positive curvature in the sense of [33], and it is a hyperbolic building if $1/m+2/k<1$ and a Euclidean building if $(m,k)=(2,4)$ or $(3,3)$ [30,2,4]. Hence G is non-hyperbolic if and only if $(m,k)=(3,3)$ or $(2,4)$ by [30, page 119], [12].

3. Star graphs of cyclic presentations

The following example shows that irreducible (m,k,ν)-special presentations are prevalent within the class of cyclic presentations.

Example 3.1.

(a) $P_7(x_0x_1x_3)$ is $(3,3,1)$-special ([26, Example 3.3], [37, Example 6.3]); its star graph is the Heawood graph (the incidence graph of a projective plane of order 2). The corresponding triangle presentation appears in [18, Section 4], [19, Section 4] and the group $G_7(x_0x_1x_3)$ also appears in [3, Section 3], and the proof of [27, Theorem 2]. In particular, it is known that $G_7(x_0x_1x_3)$ is just-infinite (so is not SQ-universal) and non-hyperbolic – see [50, Example 3.8] for a discussion.

(b) $P_{21}(x_0x_1x_5)$ is $(3,3,3)$-special; its star graph has 3 components, each of which is the Heawood graph – see Fig. 1. The group $G_{21}(x_0x_1x_5)$ is large (by [28, Lemma 2.3]) and is not hyperbolic see [20, Corollary 2.10]. The free product of three copies of $G_7(x_0x_1x_3)$ is the group $G_{21}(x_0x_3x_9)$ and the shift extension of $G_{21}(x_0x_3x_9)$ is isomorphic to the shift extension of $G_{21}(x_0x_1x_5)$, so the structures of $G_7(x_0x_1x_3)$ and $G_{21}(x_0x_1x_5)$ are related through this shift-extension. For example, in [20,

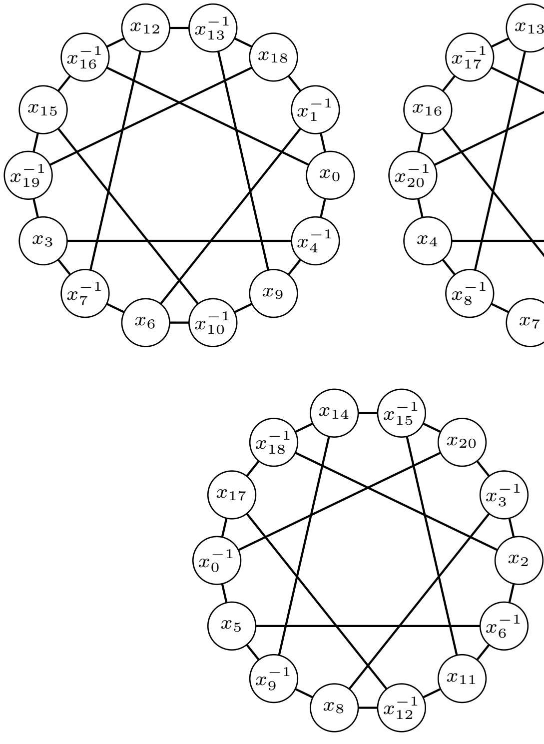

Fig. 1. The star graph of $P_{21}(x_0x_1x_5)$ (the disjoint union of three Heawood graphs).

Corollary 2.10] the fact that $G_7(x_0x_1x_3)$ is non-hyperbolic is used to prove that $G_{21}(x_0x_1x_5)$ is non-hyperbolic.

(c) $P_{13}(x_0^2x_1x_4)$ is $(3,4,1)$-special; its star graph is the $(4,6)$-cage [62] (the incidence graph of a projective plane of order 3) – see Fig. 2.

(d) $P_4(x_0x_1x_0^{-1}x_1^{-1})$ (defining $F_2 \times F_2$) is $(2,4,1)$-special; its star graph is the complete bipartite graph $K_{4,4}$.

(e) The presentation $P_7(x_0^2x_1x_4^2x_5x_1^2x_2x_5^2x_6x_2^2x_3x_6^2x_0x_3^2x_4)$ is redundant (as applying θ^4 to any relator produces a cyclic permutation of another relator) and $(3,21,1)$-special; its star graph is the Heawood graph.

We use Example 3.1(b) to give an example of a λ-polygonal presentation over a disconnected graph:

Example 3.2 *(A polygonal presentation over the union of three copies of the Heawood graph).* Let Γ be the graph in Fig. 1 and let $U_1 = \{x_i \mid 0 \leq i < 21\}$, $U_2 = \{x_i^{-1} \mid 0 \leq i < 21\}$, and $\lambda : U_1 \to U_2$ be the bijection given by $\lambda(x_i) = x_i^{-1}$. Then the set

$$\mathcal{K} = \{(x_i, x_{i+1}, x_{i+5}), (x_{i+1}, x_{i+5}, x_i), (x_{i+5}, x_i, x_{i+1}) \mid 0 \leq i < 21\}$$

(subscripts mod 21) is a λ-polygonal presentation over Γ.

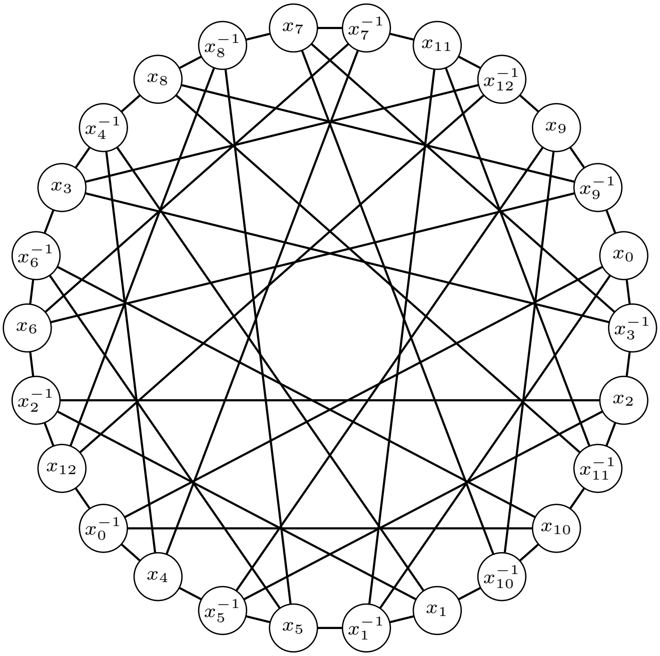

Fig. 2. The star graph of $P_{13}(x_0^2x_1x_4)$ (the $(4,6)$-cage).

Let δ denote the greatest common divisor of n and the subscripts of the generators that appear in w (recall that by our standing assumption x_0 is involved in w). Then the cyclic presentation $P = P_n(w(x_0, x_\delta, \ldots, x_{(n/\delta-1)\delta}))$ decomposes as the disjoint union of δ cyclic presentations $R = P_{n/\delta}(w(x_0, x_1, \ldots, x_{n/\delta-1}))$. Hence the star graph of P decomposes as the disjoint union of δ copies of the star graph of R, and so P is $(m, k, \delta\nu)$-special if and only if R is (m, k, ν)-special. Thus, it is often convenient to assume that $P_n(w)$ is irreducible (i.e. that $\delta = 1$).

The following theorem (compare [39, Lemma 2.3]) describes the components of the star graph of a concise cyclic presentation $P_n(w)$. For an integer $n \geq 2$ and subset $A \subseteq \{0, 1, \ldots, n-1\}$, the *circulant graph* $\mathrm{circ}_n(A)$ is the graph with vertices $v_0, \ldots, v_{n-1}$ and edges $v_i - v_{i+a}$ for all $0 \leq i < n$, $a \in A$ (subscripts mod n).

Theorem 3.3. *Let Γ be the star graph of a concise cyclic presentation $P_n(w)$ where w is cyclically reduced and is not a proper power, let $\mathcal{A}, \mathcal{B}, \mathcal{Q}$ be the multisets defined at (1), let $d_{\mathcal{A}} = \gcd(n, a\ (a \in \mathcal{A}))$, $d_{\mathcal{B}} = \gcd(n, b\ (b \in \mathcal{B}))$, and if w is not cyclically alternating let $q_0 \in \mathcal{Q}$ and set $d = \gcd(n, a\ (a \in \mathcal{A}), b\ (b \in \mathcal{B}), q - q_0\ (q \in \mathcal{Q}))$. Then Γ is $l(w)$-regular and has vertices x_i, x_i^{-1} $(0 \leq i < n)$ and edges $x_i - x_{i+a}, x_i^{-1} - x_{i+b}^{-1}, x_i - x_{i+q}^{-1}$ for all $a \in \mathcal{A}, b \in \mathcal{B}, q \in \mathcal{Q}$, $0 \leq i < n$.*

(a) If w is not cyclically alternating then Γ has d isomorphic connected components $\Gamma_0, \ldots, \Gamma_{d-1}$ where for $0 \leq j < d$ the graph Γ_j is the induced labelled subgraph of Γ

with vertex set $V(\Gamma_j) = V(\Gamma_j^+) \cup V(\Gamma_j^-)$ *where* Γ_j^+ *and* Γ_j^- *are the induced labelled subgraphs of* Γ *with vertex sets*

$$V(\Gamma_j^+) = \{x_{j+td} \mid 0 \le t < n/d\},$$
$$V(\Gamma_j^-) = \{x_{j+q_0+td}^{-1} \mid 0 \le t < n/d\}$$

(subscripts mod n*). In particular* $|V(\Gamma_j^+)| = |V(\Gamma_j^-)| = n/d$ *for all* $0 \le j < d$ *and the subscripts of the positive (respectively negative) vertices in any component are congruent mod* d*.*

(b) *If* w *is cyclically alternating then* Γ *has* $d_{\mathcal{A}}+d_{\mathcal{B}}$ *connected components* $\Gamma_0^+, \dots, \Gamma_{d_{\mathcal{A}}-1}^+$ *and* $\Gamma_0^-, \dots, \Gamma_{d_{\mathcal{B}}-1}^-$ *which are, respectively, the induced labelled subgraphs of* Γ *with vertex sets*

$$V(\Gamma_j^+) = \{x_{j+td_{\mathcal{A}}} \mid 0 \le t < n/d_{\mathcal{A}}\},$$
$$V(\Gamma_j^-) = \{x_{j+td_{\mathcal{B}}}^{-1} \mid 0 \le t < n/d_{\mathcal{B}}\}$$

(subscripts mod n*). Moreover each graph* Γ_j^+ *is isomorphic to the circulant graph* $\mathrm{circ}_{n/d_{\mathcal{A}}}(\{a/d_{\mathcal{A}}\ (a \in \mathcal{A})\})$ *and each graph* Γ_j^- *is isomorphic to the circulant graph* $\mathrm{circ}_{n/d_{\mathcal{B}}}(\{b/d_{\mathcal{B}}\ (b \in \mathcal{B})\})$*.*

Proof. Observe first that each positive vertex has the same degree and, since in the star graph of any finite presentation vertices corresponding to a generator and its inverse have the same degree (see [36, Section 2.3.3]), the graph Γ is regular. Moreover, the number of edges of the star graph of a concise presentation that has no proper power relators, and where the relators are cyclically reduced, is equal to the sum of the lengths of the relators, so the number of edges of Γ is equal to $nl(w)$, and hence Γ is $l(w)$-regular.

Let Γ^+, Γ^- denote the induced subgraphs of Γ with positive and negative vertices, respectively. Then Γ^+ has vertices $x_0, \dots, x_{n-1}$ and edges $x_i - x_{i+a}$ for each $a \in \mathcal{A}$, $0 \le i < n$, so is the circulant graph $\mathrm{circ}_n(\mathcal{A})$. Therefore its connected components are the graphs $\Gamma_0^+, \dots, \Gamma_{d_{\mathcal{A}}-1}^+$, each of which is isomorphic to the circulant graph $\mathrm{circ}_{n/d_{\mathcal{A}}}(\{a/d_{\mathcal{A}}\ (a \in \mathcal{A})\})$ (see, for example, [9], [35, page 154]). Similarly Γ^- is the circulant graph $\mathrm{circ}_n(\mathcal{B})$, and its connected components are the graphs $\Gamma_0^-, \dots, \Gamma_{d_{\mathcal{B}}-1}^-$, each of which is isomorphic to the circulant graph $\mathrm{circ}_{n/d_{\mathcal{B}}}(\{b/d_{\mathcal{B}}\ (b \in \mathcal{B})\})$. Thus part (b) is proved so consider part (a). Fix a j, $0 \le j < d$ and consider the graph Γ_j. Identifying the endpoints of the edges $x_{j+td} - x_{j+q_0+td}^{-1}$ $(0 \le t < n/d)$ of Γ_j leaves the circulant graph Λ_j with vertices $x_j, x_{j+d}, \dots, x_{j+n-d}$ and edges $x_{j+td} - x_{j+td}$, $x_{j+td} - x_{j+td+a}$, $x_{j+td} - x_{j+td+b}$, $x_{j+td} - x_{j+((q-q_0)/d+t)d}$ for all $a \in \mathcal{A}, b \in \mathcal{B}, q \in \mathcal{Q}$, $0 \le t < n/d$ (subscripts mod n). Setting $u_t = x_{j+td}$ for each $0 \le t < n/d$ the graph Λ_j has vertices $u_0, \dots, u_{n/d-1}$ and (multi-)edges $u_t - u_t, u_t - u_{t+a/d}, u_t - u_{t+b/d}, u_t - u_{t+(q-q_0)/d}$ (subscripts mod n/d), which is connected since $\gcd(n/d, a/d\ (a \in \mathcal{A}), b/d\ (b \in \mathcal{B}), (q-q_0)/d\ (q \in \mathcal{Q})) = 1$. Therefore Λ_j, and hence Γ_j, is connected, as required. $\square$

As an immediate corollary we have:

Corollary 3.4. *Let $P_n(w)$ be a concise cyclic presentation in which w is not a proper power. Then*

(a) $P_n(w)$ is $(3,k,\nu)$-special if and only if $k^2-k+1=n/\nu$ and each component of its star graph is the incidence graph of a projective plane of order $k-1$;
(b) $P_n(w)$ is $(2,k,\nu)$-special if and only if $k=n/\nu$ and each component of its star graph is the complete bipartite graph $K_{k,k}$.

Note that the 'concise' hypothesis cannot be removed from Corollary 3.4, as can be seen by Example 3.1(e). We also have the following:

Corollary 3.5. *Suppose that $P_n(w)$ is a concise (m,k,ν)-special cyclic presentation in which w is non-positive, non-negative, and not cyclically alternating, and is not a proper power. Then n/ν is even and, in particular, $P_n(w)$ is not $(3,k,\nu)$-special for any $k\geq 3$, $\nu\geq 1$.*

Proof. Since w is not cyclically alternating, in the notation of Theorem 3.3, $\nu=d$ so n/ν is an integer. Let $a\in\mathcal{A}$ (which is non-empty, since w is non-positive and non-negative) and let $r=\gcd(n/\nu,a)$. Then $x_0-x_a-x_{2a}-\cdots-x_{((n/\nu)/r-1)a}-x_0$ is a closed path in the star graph Γ of $P_n(w)$ of length $(n/\nu)/r$, which is even, since each component of Γ is bipartite. Hence n/ν is even. If $P_n(w)$ is $(3,k,\nu)$-special for some $k\geq 3$, $\nu\geq 1$ then by Corollary 3.4(a) $n/\nu=k^2-k+1$, which is odd, a contradiction. □

Corollary 3.6. *Let $P_n(w)$ be a concise cyclic presentation and let Δ be the number of components of the star graph of $P_n(w)$ and let σ be the exponent sum of w. If either*

(a) w is not cyclically alternating and $\Delta>1$ and $(\Delta,|\sigma|)\neq(2,2)$; or
(b) w is cyclically alternating and $\Delta>2$

then $G_n(w)$ is large. In particular, if $P_n(w)$ is (m,k,ν)-special where w is a positive or negative word and $\nu>1$, then $G_n(w)$ is large.

Proof. The values of σ and Δ do not depend on whether or not w is cyclically reduced, so we may assume that it is. Moreover, if v is the root of w then if $G_n(v)$ is large then so is $G_n(w)$, and the star graph of $P_n(w)$ is equal to that of $P_n(v)$ so we may assume that w is not a proper power.

Suppose first that w is cyclically alternating. Then, in the notation of Theorem 3.3, $d_{\mathcal{A}}>1$ or $d_{\mathcal{B}}>1$. By adjoining the relators $x_ix_{i+d_{\mathcal{A}}}^{-1}$ $(0\leq i<n)$ we see the group $G_n(w)$ maps onto the free group of rank $d_{\mathcal{A}}$, and adjoining the relators $x_ix_{i+d_{\mathcal{B}}}^{-1}$ $(0\leq i<n)$ it maps onto the free group of rank $d_{\mathcal{B}}$, and hence $G_n(w)$ is large.

Suppose then that w is not cyclically alternating, so that (in the notation of Theorem 3.3) $\Delta = d$. By inverting and cyclically permuting w, if necessary, we may assume that the exponent sum $\sigma \geq 0$ and the initial letter of w is x_0. Now $a \equiv 0, b \equiv 0, q \equiv q_0 \bmod d$ for all $a \in \mathcal{A}, b \in \mathcal{B}, q \in \mathcal{Q}$, so if $x_i^{\epsilon_i} u x_j^{\epsilon_j}$ is a cyclic subword of w, where u is cyclically alternating and $\epsilon_i, \epsilon_j \in \{\pm 1\}$ then $i \equiv j \bmod d$ if $\epsilon_i = -\epsilon_j$, $j \equiv i + q_0 \bmod d$ if $\epsilon_i = \epsilon_j = 1$, and $j \equiv i - q_0 \bmod d$ if $\epsilon_i = \epsilon_j = -1$. Therefore, by adjoining the relators $x_i x_{i+d}^{-1}$ $(0 \leq i < n)$ the group $G_n(w)$ maps onto $G_d(w')$, where $w' = x_0 x_{q_0} \ldots x_{(\sigma-1)q_0}$, which has shift extension $E = G_d(w') \rtimes \mathbb{Z}_d = \langle x, t \mid t^d, (xt^{q_0})^\sigma \rangle \cong \mathbb{Z}_d * \mathbb{Z}_\sigma$. Thus E, and hence $G_n(w)$, is large if $\sigma \neq 1$ and $(d, \sigma) \neq (2, 2)$. $\square$

Note that Corollary 3.6(a) cannot be directly extended to include the case $(\Delta, \sigma) = (2, 2)$ since, for example, for even n the group $G_n(x_0 x_1) \cong \mathbb{Z}$, and, similarly, part (b) cannot be directly extended to $\Delta = 2$ since, for example, $G_n(x_0 x_1^{-1}) \cong \mathbb{Z}$. In the following example we give an infinite family of $(2, k, 1)$-special cyclic presentations that define large groups.

Example 3.7. Let $w = x_0 x_{0+1} x_{0+1+2} \ldots x_{0+1+2+\cdots+n-1}$, where $n > 4$ is odd and composite. Then the star graph of $P_n(w)$ is the complete bipartite graph $K_{n,n}$. Let d be a proper divisor of n. Adjoining the relators $x_i x_{i+d}^{-1}$ $(0 \leq i < n)$ to $P_n(w)$ shows that $G_n(w)$ maps onto $G_d(w')$, where $w' = (x_0 x_{0+1} x_{0+1+2} \ldots x_{0+1+2+\cdots+d-1})^{n/d}$. Since either $d \geq 3$ or $d = 2$ and $n/d \geq 3$ the group $G_d(w')$ is large by [24].

4. Girth of the star graph of a cyclic presentation

In this section we prove:

Theorem A. *Let $P_n(w)$ be a concise cyclic presentation where w has length at least 3 and is not a proper power. Then,*

(a) if $P_n(w)$ satisfies $T(q)$ where $q \geq 7$ then $q \leq 8$ and w is a non-positive and non-negative word of length 3;
(b) if w is a non-positive and non-negative word of length $k = 3$ then $P_n(w)$ is not (m, k, ν)-special for any $m \geq 2$, $\nu \geq 1$.

Hence, if $P_n(w)$ is (m, k, ν)-special for some $m \geq 2$, $k \geq 3$, $\nu \geq 1$ then either $m = 2$ or ($m = 3$ and w is positive or negative).

The 'concise' hypothesis is necessary in Theorem A(a) since, for example, $P = P_6(x_0 x_1 x_3 x_4)$ is redundant and its star graph is a 12-cycle and so P satisfies $T(12)$.

Lemma 4.1. *Let Γ be the star graph of a concise cyclic presentation $P_n(w)$ where $w \in F_n$ is cyclically reduced and is not a proper power. Then*

(a) *if w contains non-overlapping cyclic subwords of the form $x_j x_{j+p}^{\epsilon}$, $(x_k x_{k+p}^{\epsilon})^{\pm 1}$ (for some $0 \leq j,k,p < n$), where $\epsilon = \pm 1$, then* $\mathrm{girth}(\Gamma) \leq 2$*;*
(b) *if w is a positive word of length 3 then* $\mathrm{girth}(\Gamma) \leq 6$*;*
(c) *if w contains a cyclic subword of the form $x_j x_{j+p} x_{j+p+q}^{-1} x_{j+p+q+r}^{-1}$ (for some $0 \leq j,p,q,r < n$) then* $\mathrm{girth}(\Gamma) \leq 6$*;*
(d) *if w contains a positive cyclic subword of length 4 then* $\mathrm{girth}(\Gamma) \leq 6$*;*
(e) *if w contains a cyclic subword of the form $x_j x_{j+p} x_{j+p+q} x_{j+p+q+r}^{-1}$ (for some $0 \leq j,p,q,r < n$) then* $\mathrm{girth}(\Gamma) \leq 6$*;*
(f) *if w contains non-overlapping cyclic subwords of the form $x_j x_{j+p}^{-1}$ and $x_k x_{k+q}^{-1}$ for some $0 \leq j,k,p,q < n$, then* $\mathrm{girth}(\Gamma) \leq 4$.

Proof. (a) Here the edges of Γ include the distinct edges $x_i - x_{i+p}^{-\epsilon}$, $x_i - x_{i+p}^{-\epsilon}$ $(0 \leq i < n)$ so Γ contains the reduced closed path $x_0 - x_p^{-\epsilon} - x_0$ of length 2.
(b) We may assume $w = x_0 x_p x_{p+q}$ for some $0 \leq p,q < n$ where p,q are not both 0 and $p \not\equiv q$, $p + 2q \not\equiv 0$, $2p + q \not\equiv 0$ mod n (by part (a)). Then Γ contains the reduced closed path $x_0 - x_p^{-1} - x_{p-q} - x_{-2q}^{-1} - x_{-p-2q} - x_{-p-q}^{-1} - x_0$ of length 6.
(c) By part (a) we may assume $p + r \not\equiv 0$ mod n. Then Γ contains the reduced closed path $x_0 - x_p^{-1} - x_{p+r} - x_{p+q+r} - x_{p+q}^{-1} - x_q - x_0$ of length 6.
(d) We may assume w contains the cyclic subword $x_0 x_p x_{p+q} x_{p+q+r}$ for some $0 \leq p,q,r < n$, where $p \not\equiv q$, $q \not\equiv r$, and $p \not\equiv r$ mod n (by part (a)). Then Γ contains the reduced closed path $x_0 - x_p^{-1} - x_{p-q} - x_{p-q+r}^{-1} - x_{-q+r} - x_r^{-1} - x_0$ of length 6.
(e) By part (a) we may assume $p \not\equiv q$ mod n. Then Γ contains the reduced closed path $x_0 - x_p^{-1} - x_{p-q} - x_{p-q+r} - x_{p+r}^{-1} - x_r - x_0$ of length 6.
(f) By part (a) we may assume $p \not\equiv \pm q$ mod n. Then Γ contains the reduced closed path $x_0 - x_p - x_{p+q} - x_q - x_0$ of length 4. □

Corollary 4.2. *Let Γ be the star graph of a concise cyclic presentation $P_n(w)$ where w is cyclically reduced of length at least 3 and is not a proper power. Then* $\mathrm{girth}(\Gamma) \leq 8$ *and if w is cyclically alternating then* $\mathrm{girth}(\Gamma) \leq 4$*; moreover, if* $\mathrm{girth}(\Gamma) > 6$ *then w is a non-positive, non-negative word of length 3.*

Proof. We refer to parts (a)–(f) of Lemma 4.1. By replacing w by its inverse, if necessary, we may assume that w is non-negative. If w is cyclically alternating then $\mathrm{girth}(\Gamma) \leq 4$ by part (f). If $l(w) = 3$ and w is positive the result follows from part (b). If $l(w) = 3$ and w is non-positive and non-negative, $w = x_0 x_p x_{p+q}^{-1}$, say, then $x_0 - x_p^{-1} - x_{-q}^{-1} - x_{-p-q} - x_{-p} - x_0^{-1} - x_{p+q}^{-1} - x_q - x_0$ is a reduced closed path in Γ of length 8. Thus $\mathrm{girth}(\Gamma) \leq 8$.

Suppose $l(w) = 4$. If w is positive the result follows from part (d) so assume that w is non-positive. By part (f) we may assume that w has exactly one cyclic subword of the form $x_j x_{j+p}^{-1}$ and therefore has a cyclic subword of the form of part (c) or (e), and the result follows.

Suppose then that $l(w) \geq 5$. If w is positive then the result follows from part (d) so assume that w is non-positive. By part (f) we may assume that w has exactly one cyclic subword of the form $x_j x_{j+p}^{-1}$, but then w or its inverse has a cyclic subword of the form given in part (e). $\square$

We note that girth 8 can be attained in Corollary 4.2; a presentation that demonstrates this is $P_{18}(x_0 x_8 x_1^{-1})$. We now prove Theorem A.

Proof of Theorem A. (a) If $P_n(w)$ satisfies $T(q)$ with $q \geq 7$ then every piece has length 1 so $P_n(w)$ satisfies $C(l(w))$, so it satisfies $C(3)$, and hence the star graph Γ of $P_n(w)$ contains no reduced closed path of length l where $3 \leq l < q$. Moreover, since every piece has length 1 there is no reduced closed path of length 2 in Γ. Then by Corollary 4.2 $q \leq 8$ and w is a non-positive and non-negative word of length 3.

(b) Suppose that w is a non-positive and non-negative word of length $k = 3$ and that $P_n(w)$ is (m, k, ν)-special for some $m \geq 2$, $\nu \geq 1$. Then, since $k = 3$ we have $m \neq 2$, and by part (a) $m \leq 4$. By Theorem 3.3, each component of Γ has $2n/\nu$ vertices and is 3-regular, and since it is the incidence graph of a generalized m-gon it has 14 vertices if $m = 3$ and 30 vertices if $m = 4$ (see [59, Corollary 1.5.5]). Therefore $n/\nu = 7$ or 15, a contradiction to Corollary 3.5.

If $P_n(w)$ is (m, k, ν)-special where $m \geq 4$ then it satisfies $T(8)$ so by part (a) w is a non-positive and non-negative word of length 3, in which case $P_n(w)$ is not (m, k, ν)-special by part (b). It remains to show that if $P_n(w)$ is $(3, k, \nu)$-special and $k > 3$ then w is positive or negative. This follows from Corollary 4.2 when w is cyclically alternating and from Corollary 3.5 otherwise. $\square$

We obtain our classifications of the concise (m, k, ν)-special cyclic presentations for $m = 3$ and $m = 2$ in Sections 5 and 6, respectively.

5. Classification of concise $(3, k, \nu)$-special cyclic presentations

In this section, in Theorem B, we classify the concise $(3, k, \nu)$-special cyclic presentations in terms of perfect difference sets of order k. In Corollary 5.1 we consider the Euclidean case and list explicitly the $(3, 3, \nu)$-special cyclic presentations. By Theorem A we may assume that w is a positive or negative word and, by replacing it by its inverse, we may assume that it is positive.

Theorem B. *Suppose that w is a positive word of length $k \geq 3$ that is not a proper power, let $\mathcal{Q}$ be the multiset defined at (1) and suppose that the cyclic presentation $P_n(w)$ is irreducible and concise. Then $P_n(w)$ is $(3, k, \nu)$-special if and only if*

(a) $n = \nu N$, where $N = k^2 - k + 1$; and
(b) $\mathcal{Q}$ is a perfect difference set; and
(c) $q \equiv q' \bmod \nu$, for all $q, q' \in \mathcal{Q}$.

Proof. Let $\mathcal{Q} = \{q_1, \dots, q_k\}$. Suppose first that $P_n(w)$ is $(3, k, \nu)$-special and let Γ denote the star graph of $P_n(w)$. Condition (a) holds by Corollary 3.4 and (c) holds by Theorem 3.3.

Let Λ be the component of Γ that contains x_0^{-1}. By Theorem 3.3 the subscripts of the negative vertices in Λ are all congruent to zero mod ν. Since there are N negative vertices in this component, they are precisely the negative vertices whose subscripts are the elements of the set $\{0, \nu, \dots, (N-1)\nu\}$. On the other hand, a vertex x_j^{-1} is a vertex of Λ if and only if $N_\Gamma(x_0^{-1}) \cap N_\Gamma(x_j^{-1}) = \{x_u\}$ for some vertex x_u, say. Then $j \equiv u + q_t$, $0 \equiv u + q_s$ mod n for some $1 \leq s, t \leq k$, so $j \equiv q_t - q_s$ mod n. That is, the subscripts of the negative vertices in Λ are precisely the elements of the set $\{(q_t - q_s) \bmod n \mid 1 \leq s, t \leq k\}$. Therefore,

$$\{j\nu \bmod N \mid 0 \leq j < N\} = \{(q_t - q_s) \bmod N \mid 1 \leq s, t \leq k\}.$$

By (c) for all $1 \leq i \leq k$ we have $\gcd(\nu, q_i) = \gcd(\nu, q_1, \dots, q_k)$ which divides $\gcd(n, q_1, \dots, q_k) = 1$, since $P_n(w)$ is irreducible. Thus $\gcd(\nu, q_i) = 1$ for all $1 \leq i \leq k$. Then, again by (c), $kq_i \equiv \sum_{\iota=1}^{k} q_\iota \equiv 0 \bmod \nu$, and since $\gcd(\nu, q_i) = 1$ we have $k \equiv 0 \bmod \nu$. Therefore $\gcd(N, \nu) = \gcd(k^2 - k + 1, \nu) = 1$, and hence

$$\{0, 1, \dots, N-1\} = \{(q_t - q_s) \bmod N \mid 1 \leq s, t \leq k\}.$$

Then, if $q_s \equiv q_t$ mod N for some $s \neq t$ the set $\{(q_t - q_s) \bmod N \mid 1 \leq s, t \leq k, s \neq t\}$, which has at most $k^2 - k$ elements, is equal to the set $\{(q_t - q_s) \bmod N \mid 1 \leq s, t \leq k\}$, which has $N = k^2 - k + 1$ elements, a contradiction. Therefore $q_s \equiv q_t$ mod N if and only if $s = t$ so

$$\{1, \dots, N-1\} = \{(q_t - q_s) \bmod N \mid 1 \leq s, t \leq k, s \neq t\}$$

and hence $\{q_1, \dots, q_k\}$ is a perfect difference set and so (b) holds.

For the converse, suppose that (a)–(c) hold and let Γ be the star graph of $P_n(w)$. We must show that Γ has ν components, each of which is bipartite, has girth 6 and diameter 3, and each vertex has degree at least 3.

Since w is positive the graph Γ is bipartite. Condition (b) implies that $q_1, \dots, q_k$ are distinct mod N and hence are distinct mod n, so by Theorem 3.3 Γ is k-regular (and hence each vertex has degree at least 3) and has no reduced closed path of length 2. By Theorem 3.3 Γ has $d = \gcd(n, q_1, \dots, q_k)$ isomorphic components $\Gamma_0, \dots, \Gamma_{d-1}$, each with $2n/d$ vertices. If we show that Γ_0 has girth 6 and diameter 3 then each component has $2(k^2 - k + 1)$ vertices, so $2n/d = 2(k^2 - k + 1)$ so $n/d = k^2 - k + 1$. But condition (a) implies that $n/\nu = k^2 - k + 1$ so $d = \nu$, i.e. Γ has ν components.

We now show this. It suffices to show that $|N_\Gamma(u) \cap N_\Gamma(v)| = 1$ whenever u, v are distinct vertices belonging to the same part of Γ_0 (see Remark 2.2(1)). Let $\epsilon = \pm 1$ and suppose $x_i^\epsilon, x_j^\epsilon \in V(\Gamma_0)$ $(i \neq j)$. Then

$$N_\Gamma(x_i^\epsilon) = \{x_{i+\epsilon q_s}^{-\epsilon} \mid 1 \le s \le k\},$$
$$N_\Gamma(x_j^\epsilon) = \{x_{j+\epsilon q_t}^{-\epsilon} \mid 1 \le t \le k\},$$

so $|N_\Gamma(x_i^\epsilon) \cap N_\Gamma(x_j^\epsilon)| = 1$ if and only if $i + \epsilon q_s \equiv j + \epsilon q_t$ mod n for some unique pair q_s, q_t ($1 \le s, t \le k$). By Theorem 3.3 $i - j \equiv p\nu$ mod n for some $1 \le p < N$. Now (a), (b), (c) imply that

$$\{(q_i - q_j) \bmod n \mid 1 \le i, j \le k,\ i \ne j\} = \{\nu, 2\nu, \ldots, (N-1)\nu \bmod n\}$$

so there exists a unique pair $q_s, q_t \in \{q_1, \ldots, q_k\}$ such that $\epsilon(q_t - q_s) \equiv p\nu$ mod n; i.e. $\epsilon(q_t - q_s) \equiv i - j$ mod n, or $i + \epsilon q_s \equiv j + \epsilon q_t$ mod n, as required. □

We remark that, as shown in the proof, a consequence of conditions (a), (c) of Theorem B is that ν divides k. Note that the 'concise' hypothesis cannot be removed from Theorem B; a presentation that demonstrates this is given in Example 3.1(e).

Corollary 5.1. *Let $w = x_0 x_{q_1} x_{q_1+q_2}$ where $0 \le q_1, q_2 < n$ and suppose that $P_n(w)$ is irreducible. Then $P_n(w)$ is $(3,3,\nu)$-special if and only if either:*

(a) $n = 7$, $\nu = 1$, and $\{q_1, q_2, (-q_1 - q_2) \bmod 7\} = \{1, 2, 4\}$ or $\{3, 5, 6\}$; or
(b) $n = 21$, $\nu = 3$, and $\{q_1, q_2, (-q_1 - q_2) \bmod 21\} = \{1, 4, 16\}$, $\{2, 8, 11\}$, $\{5, 17, 20\}$, or $\{10, 13, 19\}$.

Proof. If w is a proper power then $w = x_0^3$ and the star graph Γ of $P_n(w)$ consists of vertices x_i, x_i^{-1} and edges $x_i - x_i^{-1}$ ($0 \le i < n$). If $P_n(w)$ is redundant then $n = 3$ and $q_1 \equiv q_2 \equiv \pm 1$ mod 3, so Γ consists of vertices x_i, x_i^{-1} and edges $x_i - x_{i+1}^{-1}$ ($0 \le i < 3$). Therefore, in each case, $P_n(w)$ is not $(3,3,\nu)$-special so we may assume that w is not a proper power and $P_n(w)$ is concise.

By Theorem B the presentation $P_n(w)$ is $(3,3,\nu)$-special if and only if $n = 7\nu$, $\nu = 1$ or 3, $\{q_1, q_2, -q_1 - q_2\}$ is a perfect difference set and $q_1 \equiv q_2 \equiv -(q_1 + q_2)$ mod ν. The only possibilities for $q_1, q_2, -q_1 - q_2$ mod n such that $\gcd(n, q_1, q_2) = 1$ (for irreducibility) are those in the statement. □

The isomorphisms in [28, Lemma 2.1] show that the presentations in Corollary 5.1 (a) each define the group $G_7(x_0x_1x_3)$ while those in part (b) define the group $G_{21}(x_0x_1x_5)$. These groups are discussed in Example 3.1.

Note that if $P_n(w)$ is a concise $(3,k,\nu)$-special cyclic presentation where $\nu > 1$ then G is large by Corollary 3.6 and recall from Section 2.4 that a group defined by a $(3,k,1)$-special presentation is SQ-universal if and only if $k \ne 3$. Thus there is precisely one cyclically presented group defined by a concise $(3,k,\nu)$-special cyclic presentation that is not SQ-universal, namely $G_7(x_0x_1x_3)$, which is just-infinite.

6. Classification of concise $(2,k,\nu)$-special cyclic presentations

In this section, we classify the concise $(2,k,\nu)$-special cyclic presentations $P_n(w)$: in Theorem C we do this for positive words w, in Theorem D we do this for cyclically alternating words w, and in Theorem E for words w that are non-positive, non-negative, and not cyclically alternating. In Corollaries 6.1, 6.2, 6.4 we consider the Euclidean case, and classify the $(2,4,\nu)$-special cyclic presentations. Except in one case (given in Corollary 6.4(b)(iii)) we show the groups defined by these $(2,4,\nu)$-special presentations are large; it follows that, with that one possible exception, groups defined by concise $(2,k,\nu)$-special cyclic presentations are SQ-universal.

6.1. The positive case

Theorem C. *Let w be a positive word of length k that is not a proper power and let $\mathcal{Q}$ be the multiset defined at (1). Suppose also that the cyclic presentation $P_n(w)$ is irreducible and concise and let Γ be the star graph of $P_n(w)$. Then $P_n(w)$ is $(2,k,\nu)$-special if and only if either*

(a) $k \geq 5$ is odd, $n = k$, $\nu = 1$, $\mathcal{Q} = \{0, 1, 2, \dots, n-1\}$, in which case Γ is the complete bipartite graph $K_{n,n}$; or
(b) $k \geq 4$ is even, $n = 2k$, $\nu = 2$, $\mathcal{Q} = \{1, 3, \dots, n-1\}$, in which case Γ is the disjoint union of two copies of $K_{n/2,n/2}$.

Proof. Suppose that $P_n(w)$ is an irreducible and concise $(2,k,\nu)$-special cyclic presentation (so $k \geq 4$). By Corollary 3.4 $n = \nu k$ and the star graph Γ of $P_n(w)$ has ν components each of which is a complete bipartite graph $K_{k,k}$. Then by Theorem 3.3 $q \equiv q' \bmod \nu$ for all $q, q' \in \mathcal{Q}$. Note that for two elements q, q' of the multiset $\mathcal{Q}$ we have $q \not\equiv q' \bmod n$ for otherwise Γ has a reduced closed path of length 2. Thus $\mathcal{Q} = \{q_0, \nu + q_0, 2\nu + q_0, \dots, (k-1)\nu + q_0\}$ for some $q_0 \in \mathcal{Q}$. Now $\gcd(q_0, \nu)$ divides $\gcd(n, q\ (q \in \mathcal{Q})) = 1$ (since $P_n(w)$ is irreducible) so $\gcd(q_0, \nu) = 1$.

Summing the elements of $\mathcal{Q}$ gives $k(k\nu - \nu + 2q_0)/2$, so $k(k\nu - \nu + 2q_0)/2 \equiv 0 \bmod \nu k$. Hence $k(k\nu - \nu + 2q_0) = 2\nu k t$ for some integer t, and so $k = 2t + 1 - (2q_0/\nu)$ so ν divides $2q_0$. But $\gcd(\nu, q_0) = 1$ so ν divides 2. Therefore $\nu \in \{1, 2\}$ and k is odd if and only if $\nu = 1$. If $\nu = 1$ then $\mathcal{Q} = \{q_0, 1 + q_0, 2 + q_0, \dots, (n-1) + q_0\}$, that is, $\mathcal{Q} = \{0, 1, 2, \dots, n-1\}$, and if $\nu = 2$ then $\mathcal{Q} = \{q_0, 2 + q_0, 4 + q_0, \dots, 2(n/2 - 1) + q_0\}$, that is, $\mathcal{Q} = \{1, 3, \dots, n-1\}$.

For the converse suppose that (a) or (b) hold and let Λ be a connected component of Γ. Then since no two distinct elements of $\{1, \nu + 1, 2\nu + 1, \dots, (k-1)\nu + 1\}$ are congruent mod n Theorem 3.3 implies that the graph Λ is k-regular (so each vertex has degree ≥ 3) and contains no 2-cycles and Γ has ν components. When $\nu = 1$ the set $\mathcal{Q} = \{0, 1, \dots, n-1\}$ so every positive vertex is adjacent to every negative vertex and hence Γ is the complete bipartite graph $K_{n,n}$. Suppose then that $\nu = 2$. Then $k = n/2$ and

$\mathcal{Q} = \{1, 3, \dots, n-1\}$. By Theorem 3.3 two positive vertices are in the same component if and only if their subscripts are of the same parity. Moreover, the set of neighbours of any even (respectively odd) positive vertex consists of all odd (respectively even) negative vertices and hence each component is the complete bipartite graph $K_{n/2,n/2}$, as required. □

Corollary 3.6 implies that the groups $G_n(w)$ corresponding to part (b) are large while those in part (a) are SQ-universal (see Section 2.4); Example 3.7 gives examples of large groups $G_n(w)$ that correspond to part (a). Restricting to the Euclidean case, we now classify the concise $(2,4,\nu)$-special cyclic presentations $P_n(w)$, where w is positive. This classification will reveal that, up to isomorphism, such presentations define exactly three groups.

Corollary 6.1. *Let $P = P_n(x_0 x_{q_1} x_{q_1+q_2} x_{q_1+q_2+q_3})$ be irreducible and concise, where $0 \leq q_1, q_2, q_3 < n$ and let G be the group defined by P. Then P is $(2,4,\nu)$-special if and only if $n = 8$, $\nu = 2$, $q_1 \in \{1,3,5,7\}$ and one of the following holds:*

(a) $(q_2 \equiv 3q_1$ and $q_3 \equiv 5q_1$ mod $n)$ or $(q_2 \equiv 7q_1$ and $q_3 \equiv 5q_1$ mod $n)$ in which case $G \cong G_8(x_0x_1x_4x_1)$ which contains a subgroup isomorphic to $F_2 \times F_2$;
(b) $(q_2 \equiv 5q_1$ and $q_3 \equiv 3q_1$ mod $n)$ or $(q_2 \equiv 7q_1$ and $q_3 \equiv 3q_1$ mod $n)$ in which case $G \cong G_8(x_0x_1x_6x_1)$;
(c) $(q_2 \equiv 3q_1$ and $q_3 \equiv 7q_1$ mod $n)$ or $(q_2 \equiv 5q_1$ and $q_3 \equiv 7q_1$ mod $n)$ in which case $G \cong G_8(x_0x_1x_4x_3)$.

Proof. Let $q_4 = -(q_1 + q_2 + q_3)$ mod n. By Theorem C we have $\nu = 2$, $n = 8$, $\{q_1, q_2, q_3, q_4\} = \{1, 3, 5, 7\}$. Since $q_1 \in \{1, 3, 5, 7\}$ it has a multiplicative inverse mod n, $q_1^{-1} \in \{1, 3, 5, 7\}$. Define $Q_2 = q_1^{-1}q_2$, $Q_3 = q_1^{-1}q_3$, $Q_4 = q_1^{-1}q_4$ mod n. This implies that $(Q_2, Q_3) = (3, 5), (3, 7), (5, 3), (5, 7), (7, 3)$, or $(7, 5)$, and hence the stated values of q_2, q_3. Then by multiplying the subscripts of generators x_i by q_1^{-1} the group $G \cong G_n(x_0x_1x_{1+Q_2}x_{1+Q_2+Q_3})$. In the cases $(Q_2, Q_3) = (7, 5), (7, 3), (5, 7)$ the isomorphism to the stated group comes about by subtracting 1 from the subscripts of generators, multiplying them by 3 or 5 (mod 8), and cyclically permuting.

It remains to show that $G = G_8(x_0x_1x_4x_1)$ contains a subgroup isomorphic to $F_2 \times F_2$. A computation in GAP [31] shows that the subgroup of G generated by the set of elements

$$\{x_0^2, x_1^2, x_3^2, x_2, x_6, x_4x_0^{-1}, x_4^{-1}x_0^{-1}, x_5x_1^{-1}, x_7x_3^{-1}, x_0x_1x_3^{-1}, x_0x_1^{-1}x_3^{-1}\}$$

is of index 4 and has a presentation whose set of generators contains elements a, b, c, d and whose set of relators contains the commutators $[a, b], [b, c], [c, d], [d, a]$ and that the quotient obtained by killing all other generators has precisely these four generators and four relators, so defines $F_2 \times F_2$. Therefore a, b, c, d generate a subgroup of G that is isomorphic to $F_2 \times F_2$. □

The groups in Corollary 6.1 are pairwise non-isomorphic, as can be readily seen by computing their abelianisations, and they are large by Corollary 3.6. We have been unable to determine if the groups in parts (b), (c) contain a subgroup isomorphic to $F_2 \times F_2$.

6.2. The cyclically alternating case

Theorem D. *Let w be a cyclically reduced, cyclically alternating word of (even) length $k \geq 4$, let $\mathcal{A}, \mathcal{B}$ be the multisets defined at (1) and suppose that $P_n(w)$ is irreducible and concise. Then $P_n(w)$ is $(2, k, \nu)$-special if and only if $n = 2k$, $\nu = 2$, and $\mathcal{A}, \mathcal{B}$ are each sets of the form $\{\pm 1, \pm 3, \ldots, \pm(k-1)\}$.*

Proof. Let Γ be the star graph of $P_n(w)$ and let Γ^+, Γ^- be the induced subgraphs of Γ whose vertex sets are the positive and negative vertices of Γ, respectively, and let $d_{\mathcal{A}} = \gcd(n, a\ (a \in \mathcal{A})), d_{\mathcal{B}} = \gcd(n, b\ (b \in \mathcal{B}))$.

Suppose first that the given conditions hold. Then $\pm 1 \in \mathcal{A}$ so $d_{\mathcal{A}} = 1$ and hence, by Theorem 3.3, Γ^+ is the circulant graph $\mathrm{circ}_n(\mathcal{A})$, which is the complete bipartite graph $K_{n/2,n/2}$ with vertex partition $\{x_0, x_2, \ldots, x_{n-2}\} \cup \{x_1, x_3, \ldots, x_{n-1}\}$. Similarly Γ^- is the circulant graph $\mathrm{circ}_n(\mathcal{B})$, which is the complete bipartite graph $K_{n/2,n/2}$ with vertex partition $\{x_0^{-1}, x_2^{-1}, \ldots, x_{n-2}^{-1}\} \cup \{x_1^{-1}, x_3^{-1}, \ldots, x_{n-1}^{-1}\}$. Thus $P_n(w)$ is $(2, n/2, 2)$-special.

Suppose then that $P_n(w)$ is $(2, k, \nu)$-special. Since w is cyclically alternating of length k we have $|\mathcal{A}| = k/2$. Then by Corollary 3.4 each component of Γ is the complete bipartite graph $K_{k,k}$. Then Theorem 3.3 implies $d_{\mathcal{A}} = d_{\mathcal{B}}$, and since $P_n(w)$ is irreducible $1 = \gcd(n, a\ (a \in \mathcal{A}), b\ (b \in \mathcal{B})) = \gcd(d_{\mathcal{A}}, d_{\mathcal{B}})$ so $d_{\mathcal{A}} = d_{\mathcal{B}} = 1$. Thus Γ has 2 components so $\nu = 2$, and each of these components must therefore be the complete bipartite graph $K_{n/2,n/2}$, and hence $k = n/2$. If $a \equiv \pm a' \bmod n$ for some $a, a' \in \mathcal{A}$ then Γ contains a reduced closed path of length 2, contradicting the girth, so $a \not\equiv \pm a' \bmod n$ for all $a, a' \in \mathcal{A}$, and similarly $b \not\equiv \pm b' \bmod n$ for all $b, b' \in \mathcal{B}$. Since $d_{\mathcal{A}} = 1$ the set $\mathcal{A}$ contains an odd element, α, say. Suppose it also has an even element, a, say. Then the graph Γ^+ contains a closed path $x_0 - x_\alpha - x_{2\alpha} - \cdots - x_{(a-1)\alpha} - x_{a\alpha} - x_{a(\alpha-1)} - \cdots x_{2a} - x_a - x_0$ of length $a + \alpha$, which is odd, a contradiction (since Γ is bipartite). Therefore all elements of $\mathcal{A}$ are odd, so $\mathcal{A}$ consists of $k/2$ odd integers such that if $a \in \mathcal{A}$ then $n - a \notin \mathcal{A}$, so is of the form $\{\pm 1, \pm 3, \ldots, \pm(k-1)\}$. Similarly, $\mathcal{B}$ is of the same form. $\square$

Restricting to the Euclidean case, we now classify the concise $(2, 4, \nu)$-special cyclic presentations $P_n(w)$ where w is cyclically alternating.

Corollary 6.2. *Let $P = P_n(x_0 x_{a_1}^{-1} x_{a_1+b_1} x_{a_1+b_1+a_2}^{-1})$ be an irreducible and concise cyclic presentation, where $0 \leq a_1, b_1, a_2 < n$ and let G be the group defined by P. Then P is $(2, 4, \nu)$-special if and only if $n = 8$, $\nu = 2$ and one of the following holds:*

(a) $(a_2 \equiv 5a_1,\ b_1 \equiv 3a_1 \bmod n)$ *or* $(a_2 \equiv 5a_1,\ b_1 \equiv 7a_1 \bmod n)$*, in which case* $G \cong G_8(x_0x_1x_4x_1)$*;*
(b) $(a_2 \equiv 3a_1,\ b_1 \equiv 5a_1 \bmod n)$ *or* $(a_2 \equiv 3a_1,\ b_1 \equiv 7a_1 \bmod n)$*, in which case* $G \cong G_8(x_0x_1x_6x_1)$*;*
(c) $(a_2 \equiv 3a_1,\ b_1 \equiv a_1 \bmod n)$ *or* $(a_2 \equiv 3a_1,\ b_1 \equiv 3a_1 \bmod n)$*, in which case* $G \cong G_8(x_0x_1x_4x_3)$*.*

Proof. If $w = x_0x_{a_1}^{-1}x_{a_1+b_1}x_{a_1+b_1+a_2}^{-1}$ is a proper power then $w = (x_0x_{a_1}^{-1})^2$, in which case each vertex of the star graph of $P_n(w)$ has degree 2, so $P_n(w)$ is not $(2,4,\nu)$-special. Hence we may assume that w is not a proper power. Let $b_2 = -(a_1 + b_1 + a_2) \bmod n$. By Theorem D the presentation $P_n(w)$ is $(2,4,\nu)$-special if and only if $\nu = 2$, $n = 8$, a_1, a_2, b_1, b_2 are odd, and $a_1 \not\equiv \pm a_2$, $b_1 \not\equiv \pm b_2 \bmod 8$. Since $a_1 \in \{1,3,5,7\}$ it has a multiplicative inverse mod n, $a_1^{-1} \in \{1,3,5,7\}$. Define $B_1 = a_1^{-1}b_1$, $A_2 = a_1^{-1}a_2$, $B_2 = a_1^{-1}b_2 \bmod n$. By multiplying the subscripts of generators x_i by a_1^{-1} the group $G \cong G_n(x_0x_1^{-1}x_{1+B_1}x_{1+B_1+A_2}^{-1})$. Further $B_1, A_2, B_2 \in \{1,3,5,7\}$ and $A_2 \not\equiv \pm 1$, $B_1 \not\equiv \pm B_2$ and $1 + B_1 + A_2 + B_2 \equiv 0 \bmod n$. This implies $(A_2, B_1) = (3,1), (3,3), (3,5), (3,7), (5,3)$, or $(5,7)$, and hence the stated values of a_2, b_1. In parts (a), (b) isomorphisms $G_8(x_0x_1^{-1}x_6x_1^{-1}) \cong G_8(x_0x_1x_6x_1)$, $G_8(x_0x_1^{-1}x_0x_3^{-1}) \cong G_8(x_0x_1x_0x_3)$, $G_8(x_0x_1^{-1}x_4x_1^{-1}) \cong G_8(x_0x_1x_4x_1)$, and $G_8(x_0x_1^{-1}x_0x_5^{-1}) \cong G_8(x_0x_1x_0x_5)$ are obtained by replacing each odd numbered generator by its inverse. Then we have $G_8(x_0x_1x_6x_1) \cong G_8(x_0x_1x_0x_3)$ and $G_8(x_0x_1x_4x_1) \cong G_8(x_0x_1x_0x_5)$ as in Corollary 6.1. In part (c) the isomorphism $G_8(x_0x_1^{-1}x_2x_5^{-1}) \cong G_8(x_0x_1^{-1}x_4x_7^{-1})$ is obtained by inverting the relators and negating the subscripts. Then $G_8(x_0x_1^{-1}x_2x_5^{-1}) \cong G_8(x_0x_1x_6x_5)$ is obtained by inverting the odd numbered generators and interchanging x_2 and x_6 and interchanging x_3 and x_7 and then $G_8(x_0x_1x_6x_5) \cong G_8(x_0x_1x_4x_3)$ as in the proof of Corollary 6.1. □

Note that the groups appearing in Corollary 6.2 are the same as those in Corollary 6.1, in particular they are large and $G_8(x_0x_1x_4x_1)$ contains a subgroup isomorphic to $F_2 \times F_2$. We now show that all the groups defined by the $(2,k,\nu)$-special presentations in Theorem D are large.

Corollary 6.3. *Let* $P_n(w)$ *be a concise and irreducible* $(2,k,\nu)$*-special cyclic presentation where* w *is a cyclically alternating word of length at least 4. Then the cyclically presented group* $G_n(w)$ *defined by* $P_n(w)$ *is large.*

Proof. Let $w = x_0x_{a_1}^{-1}x_{a_1+b_1}x_{a_1+b_1+a_2}^{-1} \cdots x_{\sum_{i=1}^{k/2-1}(a_i+b_i)}x_{\sum_{i=1}^{k/2-1}(a_i+b_i)+a_{k/2}}^{-1}$ and let $b_{k/2} = -\sum_{i=1}^{k/2-1}(a_i + b_i) - a_{k/2} \bmod n$. Then the shift extension $E = G_n(w) \rtimes_\theta \mathbb{Z}_n$ has a presentation $E = \langle x, t \mid t^{2k}, \prod_{i=1}^{k/2}(xt^{a_i}x^{-1}t^{b_i})\rangle$ where each a_i, b_i is odd, so E maps onto the generalized triangle group $T = \langle x, t \mid x^7, t^2, (xtx^{-1}t)^{k/2}\rangle$ which, by [5, Theorem B], is large if $k/2 \geq 3$. Thus we may assume $k = 4$, in which case G is one of the groups in Corollary 6.2, which are large by Corollary 3.6. □

6.3. The non-positive, non-negative, and not cyclically alternating case

Theorem E. *Let w be a cyclically reduced word of length $k \geq 4$ that is non-positive, non-negative, not cyclically alternating and not a proper power and let $\mathcal{A}, \mathcal{B}, \mathcal{Q}$, $\mathcal{Q}^+$, $\mathcal{Q}^-$ be the multisets defined at (1) and suppose that $P_n(w)$ is irreducible and concise. Then $P_n(w)$ is $(2, k, \nu)$-special if and only if the following hold:*

(a) $n = \nu k$, and k is divisible by 4;
(b) $\mathcal{A}, \mathcal{B}$ are each sets of the form $\{\pm\nu, \pm 3\nu, \ldots, \pm(k/2-1)\nu\}$;
(c) $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$ and $\mathcal{Q} = \{q_0, q_0 + 2\nu, \ldots, q_0 + (k-2)\nu\}$ where $\gcd(q_0, \nu) = 1$.

Proof. Suppose first that $P_n(w)$ is $(2, k, \nu)$-special. Then by Corollary 3.4 $n = k\nu$ and each component of the star graph Γ of $P_n(w)$ is the complete bipartite graph $K_{k,k}$.

In the notation of Theorem 3.3, Γ has ν isomorphic components Γ_j $(0 \leq j < \nu)$ where, in particular, Γ_0 is the complete bipartite graph $K_{k,k}$ with vertex set $V(\Gamma_0) = V(\Gamma_0^+) \cup V(\Gamma_0^-)$ where Γ_0^+ and Γ_0^- are the induced labelled subgraphs of Γ with vertex sets

$$\begin{aligned} V(\Gamma_0^+) &= \{x_0, x_\nu, \ldots, x_{(k-1)\nu}\}, \\ V(\Gamma_0^-) &= \{x_{q_0}^{-1}, x_{q_0+\nu}^{-1}, \ldots, x_{q_0+(k-1)\nu}^{-1}\} \end{aligned}$$

for some $q_0 \in \mathcal{Q}$. Suppose for contradiction that $\nu, n - \nu \notin \mathcal{A}$. Then for each $0 \leq i < n$, vertices $x_i, x_{i+\nu}$ are not joined by an edge. Therefore the vertices of $V(\Gamma_0^+)$ all belong to the same part of Γ_0, and hence no two positive vertices of Γ are joined by an edge, and so $\mathcal{A} = \emptyset$, a contradiction, since w is non-positive and non-negative. Therefore ν or $n - \nu \in \mathcal{A}$ and, similarly, ν or $n - \nu \in \mathcal{B}$. Thus Γ_0 contains closed paths $x_0 - x_\nu - \cdots - x_{(k-1)\nu} - x_0$ and $x_{q_0}^{-1} - x_{q_0+\nu}^{-1} - \cdots - x_{q_0+(k-1)\nu}^{-1} - x_{q_0}^{-1}$ of length k which is even since Γ_0 is bipartite. Therefore the vertices $x_\nu, x_{3\nu}, \ldots, x_{(k-1)\nu}$ are precisely those positive vertices of Γ_0 that belong to a different part of Γ_0 to x_0 (and so are neighbours of x_0) and the vertices $x_{q_0+\nu}^{-1}, x_{q_0+3\nu}^{-1}, \ldots, x_{q_0+(k-1)\nu}^{-1}$ are precisely those negative vertices of Γ_0 that belong to a different part of Γ_0 to $x_{q_0}^{-1}$ (and so are neighbours of $x_{q_0}^{-1}$) and hence $\mathcal{A}, \mathcal{B} \subset \{\nu, 3\nu, \ldots, (k-1)\nu\}$. Moreover, for each odd t either $t\nu$ or $(k-t)\nu \in \mathcal{A}$ (resp. $\mathcal{B}$), and precisely one of $t\nu$ or $(k-t)\nu \in \mathcal{A}$ (resp. $\mathcal{B}$), for otherwise Γ_0 contains a reduced closed path of length 2, a contradiction. Therefore $\mathcal{A}, \mathcal{B}$ are each sets of the form $\{\pm\nu, \pm 3\nu, \ldots, \pm(k/2-1)\nu\}$, and as these have $k/4$ elements, k is divisible by 4 and (a), (b) are proved.

Since $x_{q_0}^{-1}$ and $x_{q_0+\nu}^{-1}$ belong in different parts of Γ_0, and (since $q_0 \in \mathcal{Q}$) x_0 and $x_{q_0}^{-1}$ belong in different parts, the negative neighbours of x_0 are $\{x_{q_0}^{-1}, x_{q_0+2\nu}^{-1}, \ldots, x_{q_0+(k-2)\nu}^{-1}\}$ so $\mathcal{Q} = \{q_0, q_0 + 2\nu, \ldots, q_0 + (k-2)\nu\}$. Also $\mathcal{Q}^+ \cap \mathcal{Q}^- = \emptyset$, for otherwise Γ_0 contains a reduced closed path of length 2, a contradiction. Finally, $\gcd(q_0, \nu)$ divides $\gcd(n, a\ (a \in \mathcal{A}), b\ (b \in \mathcal{B}), q\ (q \in \mathcal{Q})) = 1$, since $P_n(w)$ is irreducible, so $\gcd(q_0, \nu) = 1$, and (c) is proved.

Now suppose that the conditions in the statement hold. By Corollary 3.4(b) we must show that the star graph Γ of $P_n(w)$ consists of ν connected components, each of which is a complete bipartite graph $K_{k,k}$. By Theorem 3.3 the graph Γ has ν isomorphic components. Consider the component Γ_0 whose vertex set is $\{x_0, x_\nu, \ldots, x_{(k-1)\nu}\} \cup \{x_{q_0}^{-1}, x_{q_0+2\nu}^{-1}, \ldots, x_{q_0+(k-2)\nu}^{-1}\}$. The set of neighbours of $x_{j\nu}$ $(0 \leq j < k)$ is $N_\Gamma(x_{j\nu}) =$

$$\begin{cases} \{x_\nu, x_{3\nu}, \ldots, x_{(k-1)\nu}\} \cup \{x_{q_0}^{-1}, x_{q_0+2\nu}^{-1}, \ldots, x_{q_0+(k-2)\nu}^{-1}\} & \text{if } j \text{ is even,} \\ \{x_0, x_{2\nu}, \ldots, x_{(k-2)\nu}\} \cup \{x_{q_0+\nu}^{-1}, x_{q_0+3\nu}^{-1}, \ldots, x_{q_0+(k-1)\nu}^{-1}\} & \text{if } j \text{ is odd,} \end{cases}$$

and so Γ_0 is bipartite with vertex partition

$$\begin{gathered} \{x_0, x_{2\nu}, \ldots, x_{(k-2)\nu}, x_{q_0+\nu}^{-1}, x_{q_0+3\nu}^{-1}, \ldots, x_{q_0+(k-1)\nu}^{-1}\} \\ \dot{\cup}\{x_\nu, x_{3\nu}, \ldots, x_{(k-1)\nu}, x_{q_0}^{-1}, x_{q_0+2\nu}^{-1}, \ldots, x_{q_0+(k-2)\nu}^{-1}\}. \end{gathered}$$

Further, for each $0 \leq j < k$ the sets $N_\Gamma(x_{q_0+(j+1)\nu}^{-1})$ and $N_\Gamma(x_{j\nu})$ are equal so Γ_0 is a complete bipartite graph, as required. $\square$

Note that with $\mathcal{A}, \mathcal{B}, \mathcal{Q}^+, \mathcal{Q}^-$ as defined at (1)

$$\sum_{a\in\mathcal{A}} a + \sum_{b\in\mathcal{B}} b + \sum_{q\in\mathcal{Q}^+} q - \sum_{q\in\mathcal{Q}^-} q \equiv 0 \bmod n. \tag{2}$$

Recall (from Section 2.4) that if $k > 4$ then any group defined by a concise $(2, k, \nu)$-special presentation is SQ-universal. Restricting to the Euclidean case, we now classify the concise $(2, 4, \nu)$-special cyclic presentations $P_n(w)$ where w is non-positive, non-negative, and not cyclically alternating. By cyclically permuting and taking the inverse of w, if necessary, we may assume that either $w = x_0 x_p^{-1} x_q^{-1} x_r$ or $w = x_0 x_p^{-1} x_q x_r$ for some $0 \leq p, q, r < n$.

Corollary 6.4.

(a) *Let $P = P_n(x_0 x_p^{-1} x_q^{-1} x_r)$ be irreducible and concise and let G be the group defined by P. Then P is $(2, 4, \nu)$-special if and only if $n = 4\nu$, and $(p, q, r) = (\nu, -s, \nu - s)$ or $(3\nu, -s, 3\nu - s)$ (mod n) where $\gcd(s, \nu) = 1$, in which case G contains a subgroup isomorphic to $F_2 \times F_2$ and admits an epimorphism onto the free group of rank ν so is large if $\nu > 1$. Moreover, if $\nu = 1$ then one of the following holds:*

- *(i) $G \cong G_4(x_0 x_1^{-1} x_0^{-1} x_1) \cong F_2 \times F_2$; or*
- *(ii) $G \cong G_4(x_0 x_1^{-2} x_2)$ which has $F_5 \times F_5$ as an index 16 subgroup;*
- *(iii) $G \cong G_4(x_0 x_1^{-1} x_2^{-1} x_3)$ which has $F_3 \times F_5$ as an index 8 subgroup.*

(b) *Let $P = P_n(x_0 x_p^{-1} x_q x_r)$ be irreducible and let G be the group defined by P. Then P is $(2, 4, \nu)$-special if and only if $\nu = 1$, $n = 4$, and one of the following holds:*

(i) $(p,q,r) \in \{(1,0,1),(3,0,3)\}$, *in which case* $G \cong G_4(x_0x_1^{-1}x_0x_1)$ *and the (index 16) derived subgroup* $G' \cong F_5 \times F_5$;

(ii) $(p,q,r) \in \{(1,0,3),(3,0,1)\}$, *in which case* $G \cong G_4(x_0x_1^{-1}x_0x_3)$ *and* G *has* $F_5 \times F_5$ *as an index 16 subgroup;*

(iii) $(p,q,r) \in \{(1,2,0),(3,2,0),(1,2,2),(3,2,2)\}$, *in which case* $G \cong G_4(x_0^2x_1^{-1}x_2)$.

Proof. (a) With the notation (1) we have $\mathcal{A} = \{p\}$, $\mathcal{B} = \{r-q\}$, $\mathcal{Q}^+ = \{-r\}$, $\mathcal{Q}^- = \{p-q\}$. Then by Theorem E the presentation P is $(2,k,\nu)$-special if and only if $n = 4\nu$, $p = r - q \in \{\nu, 3\nu\}$, $\{p-q, -p-q\} = \{s+\nu, s+3\nu\}$ for some $\gcd(s,\nu) = 1$. That is, $(p,q,r) = (\nu, -s, \nu - s)$, $(3\nu, -s, 3\nu - s)$, $(\nu, 2\nu - s, 3\nu - s)$, $(3\nu, 2\nu - s, \nu - s)$. Replacing s by $2\nu + s$ in the last two cases transforms them to the first cases. If $\nu = 1$ then $\pm(p,q,r) = (1,0,1),(1,1,2),(1,2,3),(1,3,0) \pmod n$ and G is isomorphic to one of the stated groups. A computation in GAP reveals the subgroups claimed. Suppose then that $\nu > 1$. Then G maps onto $G_\nu(x_0x_p^{-1}x_q^{-1}x_r)$, which is free of rank ν, so G is large. Moreover, the shift extension of G is the group $E = \langle x, t \mid t^{4\nu}, xt^\nu x^{-1}t^{-(\nu+s)}x^{-1}t^\nu xt^{s-\nu}\rangle$ (by replacing s by $-s$, if necessary). Therefore G is the kernel the epimorphism $\phi_0 : E \to \langle t \mid t^8\rangle$ given by $\phi_0(t) = t$, $\phi_0(x) = 1$. On the other hand, the kernel of the epimorphism $\phi_{-s} : E \to \langle t \mid t^8\rangle$ given by $\phi_{-s}(t) = t$, $\phi_{-s}(x) = t^{-s}$ is the cyclically presented group $G_{4\nu}(y_0y_\nu^{-1}y_0^{-1}y_\nu)$, where $y_i = t^ixt^{-i+s}$, which is isomorphic to the free product of ν copies of $G_4(y_0y_1^{-1}y_0^{-1}y_1) \cong F_2 \times F_2$. In particular, the subgroup of $\ker(\phi_{-s})$ generated by $y_0, y_\nu, y_{2\nu}, y_{3\nu}$ is the group

$$\begin{aligned} H &= \langle y_0, y_\nu, y_{2\nu}, y_{3\nu} \mid y_0y_\nu^{-1}y_0^{-1}y_\nu, y_\nu y_{2\nu}^{-1}y_\nu^{-1}y_{2\nu}, y_{2\nu}y_{3\nu}^{-1}y_{2\nu}^{-1}y_{3\nu}, y_{3\nu}y_0^{-1}y_{3\nu}^{-1}y_0\rangle \\ &= \langle y_0, y_{2\mu} \mid \ \rangle \times \langle y_\nu, y_{3\mu} \mid \ \rangle \cong F_2 \times F_2. \end{aligned}$$

Therefore the subgroup K of H generated by $y_0^n, y_\nu^n, y_{2\nu}^n, y_{3\nu}^n$ is isomorphic to $F_2 \times F_2$, and since $\phi_0(y_i^n) = 1$, K is a subgroup of $\ker(\phi_0) = G$, as required.

(b) By Theorem E P is $(2,k,\nu)$-special if and only if $n = 4\nu$, $\mathcal{A} = \{p\}$, $\mathcal{B} = \{q-p\}$, $\mathcal{Q}^- = \emptyset$, $\mathcal{Q}^+ = \{r-q, -r\} = \{s+\nu, s+3\nu\}$, for some $0 \le s < n$ and $\gcd(s,\nu) = 1$, and $\mathcal{A} \cup \mathcal{B} \subset \{\nu, 3\nu\}$, so $p \equiv q - p$ or $p \equiv -(q-p) \bmod n$.

Suppose $p \equiv -(q-p) \bmod n$; then (2) implies $2s \equiv 0 \bmod n$ so $s \equiv 0$ or $2\nu \bmod n$, and then $\gcd(s,\nu) = 1$ implies $\nu = 1$, so $n = 4$ and $s \equiv 0$ or $s \equiv 2 \bmod 4$. Then (mod 4) $p \in \{1,3\}$, $\{r-q, -r\} = \{1,3\}$, which has solutions $(p,q,r) = (1,0,1),(1,0,3),(3,0,1),(3,0,3)$, as in parts (i), (ii). Computations in GAP reveal the $F_5 \times F_5$ subgroups. Suppose $p \equiv q - p \bmod n$; then (2) implies $2(s+\nu) \equiv 0 \bmod n$ so $s \equiv \nu$ or $3\nu \bmod n$, and then $\gcd(s,\nu) = 1$ implies that $\nu = 1$, so $n = 4$ and $s \equiv 1$ or $s \equiv 3 \bmod 4$. Then (mod 4) $p = q - p \in \{1,3\}$ and $\{r-q, -r\} = \{0,2\}$, the solutions of which are $(p,q,r) = (1,2,2),(1,2,0),(3,2,2),(3,2,0)$, in which case $G \cong G_4(x_0^2x_1^{-1}x_2)$, as in part (iii). □

The argument in the proof above for the existence of the $F_2 \times F_2$ subgroup in the groups arising in part (a) has its origins in [11, Example 3(b)]. We have been unable to determine if the group $G_4(x_0^2x_1^{-1}x_2)$ (from Corollary 6.4(b)(iii)) is SQ-universal or if it contains a subgroup isomorphic to $F_2 \times F_2$.

We now determine which groups from Corollaries 6.1, 6.2, 6.4 are Burger-Mozes groups, as defined in [46], whose notation $2 \times 2.j$ we use. Since these groups have deficiency at least zero, if they are Burger-Mozes groups then they have degree (4,4), by [56, Proposition 4.26]; the Burger-Mozes groups of degree (4,4) are classified in the Table in [46]. First consider the groups in Corollaries 6.1, 6.2. A comparison of abelianisations of these groups with those of [46] shows that the only possible pair of isomorphic groups is $G_8(x_0x_1x_4x_3)$ and the group 2×2.38, but these can be distinguished by comparing abelianisations of index 2 subgroups. We now turn to the groups in Corollary 6.4. Consider first the case $\nu > 1$, so G is a group from part (a). If $\nu > 3$ then G maps onto the free group of rank 4 so G^{ab} maps onto $\mathbb{Z}^4$, but the only group from the Table in [46] whose abelianisation maps onto $\mathbb{Z}^4$ is the group $2 \times 2.41 \cong F_2 \times F_2$, which does not map onto the free group of rank 4. When $\nu = 3$, a computation in GAP shows that the abelianisation G^{ab} is distinct from the abelianisations of the groups in [46]. When $\nu = 2$, a comparison of abelianisations shows that if G is isomorphic to a group H from the Table in [46] then H is the group 2×2.32; but then G can be distinguished from H by comparing abelianisations of index 2 subgroups. The groups in parts (a)(ii) and (b)(iii) can be distinguished from the groups in the Table in [46] by considering their abelianisations, or the abelianisations of their index 2 subgroups. The groups in parts (a)(i), (a)(iii), (b)(i), (b)(ii) are the Burger-Mozes groups $2 \times 2.41 \cong F_2 \times F_2$, 2×2.51, 2×2.12, 2×2.36, respectively. Thus, if a Burger-Mozes group is defined by a $(2, 4, \nu)$-special cyclic presentation, then it is one of these four groups.

The results of Sections 5, 6.1, 6.2 and Corollary 6.4 show that there are at most two groups defined by concise (m, k, ν)-special cyclic presentations that are not SQ-universal, namely $G_7(x_0x_1x_3)$ (which is just-infinite so is not SQ-universal) and $G_4(x_0^2x_1^{-1}x_2)$ (which remains unresolved).

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